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Global Bifurcation of a Nonlocal Equation

Master's thesis in Mathematical Sciences
Supervisor: Mats Ehrnström
October 2022

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Abstract

We study a dispersive equation of fractional Korteweg–de Vrie type [38, 46] with a nonlocal nonlinearity which is a Coifman–Meyer operator [14], which when in steady variables takes the form

$$-\mu\varphi + \Lambda^s\varphi + \varphi\,\Lambda^r\varphi = 0$$

where Λ^s and Λ^r are Fourier multipliers of Bessel potential type with order parameters $r, s < 0$. Following the framework of the analysis laid out in Ehrnström–Wahlén [23] we establish a priori estimates, touching lemmata and a nodal property theorem in the case $r = s$, with comments on the case $r \neq s$ where partial results are achieved. A regularity theorem of solutions under the *ad hoc* condition $\Lambda^r\varphi < \mu$ is established. Furthermore, a setup of real-analytic global bifurcation analysis in the spirit of Buffoni–Toland [13] is established and used to perform local bifurcation analysis from which curves are extended into global continua of solutions over the bifurcation space of Hölder–Zygmund functions. Bifurcation formulas are established and analyzed for general $r, s < 0$. A theorem on the degenerating nature of solutions with $\Lambda^r\varphi(0) = 0$ is established and used to argue against the existence of singularities.

Sammendrag

Vi studerer en dispersiv likning av typen fraksjonell Korteweg–de Vrie [38, 46] med en ikke-lokal ikkelineæritet som er en Coifman–Meyer operator [14], som når betraktet i stabile variabler antar formen

$$-\mu\varphi + \Lambda^s\varphi + \varphi\,\Lambda^r\varphi = 0$$

hvor Λ^r og Λ^s er Fouriermultiplikatorer av Besselpotensiale med ordensparametre $r, s < 0$. I tråd med rammeverket til analysen lagt frem i Ehrnström–Wahlén [23] så etablerer vi *a priori* estimat , touching lemma og et teorem om nodalegenskaper i tilfellet $r = s$, hvor vi i tillegg kommenterer tilfellet $r \neq s$ hvor bare delvise resultat er oppnådd. Et teorem om regulariteten under betingelsen $\Lambda^r\varphi < \mu$, pålagt *ad hoc*. Dessuten etablerer vi et oppsett for reell-analytisk global bifurkasjon i tråd med Buffoni–Toland [13] og bruker det til å utføre lokal bifurkasjonsanalyse hvor bifurkasjonskurvene er utvidet til globale kontinuerlige kurver over bifurkasjonsrommet bestående av Hölder–Zygmund-funksjonene. Bifurkasjonsformler er etablert og analysert for generell $r, s < 0$. Et teorem angående den degenerative naturen av løsninger med $\Lambda^r\varphi(0) = 0$ er etablert og brukt til å argumentere mot eksistensen av singulære løsninger.

Preface

This thesis is in the fulfillment of the course MA3911, which marks the end of the two year long master's degree for students in the master's program in Mathematical Sciences (MSMNFMA) at the Norwegian University of Science and Technology.

I was presented this equation as an example a first example of a nonlinearity based on the so-called Coifman–Meyer operators [14]. Having written my bachelor's thesis on local bifurcation of a modified fractional Korteweg–de Vrie equation my supervisor Mats Ehrnström only thought it prudent to suggest global bifurcation of a generalization based off the very same equation as a thesis project. I gladly accepted, and this thesis is the result of roughly a year's worth of work from August 2021 till October 2022.

The thesis is structured as follows. First we review some distribution theory and define function spaces needed for the main analysis. We then move on to reviewing some functional analysis needed to make sense of the bifurcation analysis presented in Buffoni–Toland [13]. Following that we consider the main results used in the local and global bifurcation. Finally, the main analysis on global bifurcation of periodic solutions of the main equation is considered.

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Jonas Pedersen Vean
Trondheim October 2022

Contents

Chapter 1

Introduction

The theory of partial differential equations is grand and ever evolving. Bit by bit, mathematicians and physicists have been working to expand on equations modelling phenomena that occur in nature. In particular, the nonlinear nature of water waves, fluids, and gases have been especially eluding and difficult, leading to many important questions and subsequently interesting work concerning their properties and solutions – or, as importantly, lack thereof. In the field of fluid dynamics the Euler- and Navier–Stokes equations are notorious for their impenetrable difficulty to solve and analyze in generality, the latter of which even has its existence of smooth solutions set up as a Millennium problem [25] by the Clay Mathematics Institute in May 2000. We shall turn our attention to the former set of the Euler equations.

1.1 From Euler to Whitham – A timeline of water-wave equations

First introduced in a publication [24] from 1757, Leonhard Euler introduced the set of equations governing the flow of inviscid fluids which we now call the Euler equations. These equations in their utmost generality take a form where one struggles to get an analytical foothold. Hence, one usually has to specify the physical conditions in a way which simplifies the working conditions like for instance incompressibility, homogeneity, irrotationality, etc. See Lannes [39, Chapter 1] for details. Additionally, if these equations are supplemented with boundary conditions we obtain the so-called *free surface water-waves problem.* Solutions to these problems are usually characterized by their profile depth, amplitude, and their wavelength.

In 1834 the Scottish civil engineer John Scott Russell observed a long-wavelength, fast-moving wave in a boat canal which he called “the wave of translation” [50]. This wave, later called a *solitary wave*, along with its purported features, caused a problem for contemporary mathematicians as it could not be explained by the water-wave models that were commonplace at the time [35]. It took until 1871 when Joseph Valentin Boussinesq wrote a theoretical paper [6] mentioning Russell in name, explaining, at least in part, the phenomenon Russell had witnessed. Boussinesq would go on to write more on water waves and dispersive equations the years following in 1872 and 1877 [5, 7], meanwhile Lord Rayleigh famously published on the same phenomenon in 1876 [48]. Indeed, the analysis of Diederik Korteweg and Gustav de Vries [38] in 1895 laid to rest any doubts about the existence of solitary waves from a mathematical viewpoint [35].

Korteweg and de Vries also proposed in [38] what is known today as the Korteweg–de Vries (KdV) equation[1], modelling small amplitude, long wavelength waves under the influence of gravity in a shallow depth regime, which takes the form

$$\eta_t + c_0\eta_x + \frac{3c_0}{4h_0}(\eta^2)_x + \frac{1}{6}c_0 h_0 \eta_{xxx} = 0 \tag{1.1.1}$$

where $\eta(t,x)$ describes the deflection of the water surface at the point x and time t, $c_0 = \sqrt{gh_0}$ is the wave speed, h_0 is the water depth at flow rest compared to a flat reservoir bottom, and g is the gravitational constant. By inserting a wave of the form $\eta(t,x) = \varphi(kx - \omega t)$ into Equation (1.1.1) and ignoring the nonlinear term, we obtain the linearized phase velocity $c_{\mathrm{KdV}} = \omega/k$ by

$$c_{\mathrm{KdV}}(k) = c_0\Big(1 - \frac{1}{6}(kh_0)^2\Big)$$

where k is the wave number and ω is the angular frequency. The KdV equation is a faithful approximation in the shallow regime $h_0 \ll 1$, and is thought to have good accuracy for long wavelengths. Indeed, the KdV equation can be derived from the Euler equations In order to correct for higher values of kh_0, and thus shorter waves, one can instead analyze the Whitham equation given by

$$\eta_t + \frac{3c_0}{4h_0}(\eta^2)_x + K_{h_0} * \eta_x = 0 \tag{1.1.2}$$

proposed by Gerald B. Whitham in 1967 [58], where K_{h_0} in the convolution is the integral kernel given as the inverse Fourier transform

$$K_{h_0}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} c_{\mathrm{Euler}}(\xi)\, \exp\left(ix\xi\right) \mathrm{d}\xi$$

of the Eulerian phase velocity given by

$$c_{\mathrm{Euler}}(k) = \sqrt{\frac{g}{k}\tanh\left(kh_0\right)} \tag{1.1.3}$$

which relates the Whitham equation to the KdV equation by noting that $c_{\mathrm{KdV}}(k)$ is the second order expansion of the Mclaurin series of $c_{\mathrm{Euler}}(k)$. Whitham sought out after a water-wave model wherein the phenomenon of wave breaking was present – namely there are bounded solutions with tangent slopes that blow up in finite time [57, 58]. Waves that form a sharp crest exhibit a phenomenon we call *wave peaking.*

[1]The current consensus among scholars is that Boussinesq was the first to arrive at the KdV equation, c.f. de Jager [35].

Of special interest is the Stokes conjecture surrounding the cusped (peaking) angle of peaked waves with bounded derivatives [2, 52, 57].

By rescaling the variables of η in Equation (1.1.2) accordingly

$$x \mapsto h_0 x, \quad t \mapsto \frac{h_0}{c_0} t, \quad \eta \mapsto \frac{4h_0}{3}\eta = u,$$

and denoting $m(\xi) = \sqrt{\frac{\tanh \xi}{\xi}}$ and $K = \mathscr{F}^{-1} m(\xi)$ we can re-write the Whitham equation (1.1.2) in the form

$$u_t + Lu_x + (u^2)_x = 0 \tag{1.1.4}$$

where $L\colon f \mapsto K * f$ acts as a Fourier multiplier with symbol $m(\xi)$ [23]. Furthermore, we look at *travelling solutions*, also in this context called *steady solutions*, to Equation (1.1.4) of the form $\varphi(\tilde{x}) = u(x - \mu t)$ where $\mu \geq 0$ denotes the wave speed of the right-travelling wave. Equation (1.1.4) then becomes

$$-\mu\varphi + L\varphi + \varphi^2 = B \tag{1.1.5}$$

after having integrated over $\tilde{x}$ once, where B is a constant of integration. Following the introductory exposition of [31] one can view the integration constant B arising from a dispersive equation as above by the mean value integral

$$B = \fint n(\varphi(\tilde{x}))\,\mathrm{d}\tilde{x}$$

where $n(\varphi) = \varphi^2$ in the case of the Whitham equation (compare with the $n(\varphi)$ as in [31]), and the integral is taken over the set of $\tilde{x}$ considered for the problem. The constant B can be set to zero without loss of generality due to the Galilean transformations [23]

$$\varphi \mapsto \varphi + \gamma, \quad \mu \mapsto \mu + 2\gamma, \quad B \mapsto B + \gamma(1 - \mu - \gamma) \tag{1.1.6}$$

mapping solutions of Equation (1.1.7) to solutions of the transformed equation of the same form. Finally we arrive at

$$-\mu\varphi + L\varphi + \varphi^2 = 0 \tag{1.1.7}$$

which we will refer to as the Whitham equation in steady variables.

Whitham conjectured something similar to the Stokes conjecture but for the Whitham equation, positing in his work *Linear and nonlinear waves* [57, p. 479] that the Whitham equation achieves a highest, cusped, travelling-wave solution, where furthermore he writes (adapted notation to suit, see [23, p. 2])

> [...] it seems reasonable to assume that in fact a critical height is reached when $\varphi = \frac{\mu}{2}$. If $K(x)$ behaves like $|x|^p$ as $x \to 0$ and $\varphi(x)$ behaves like $\frac{\mu}{2} - |x|^q$, a local argument in (1.1.7) suggests that $2q - 1 = p + q$; hence $q = p+1$. According to this, the crest would be cusped with $\varphi \sim \frac{\mu}{2} - |x|^{1/2}$ for K.

It turned out that Whitham was correct, which was finally concluded by Mats Ehrnström announced in his Oberwolfach report from 2015 [18].

1.2 Historical notes on the Whitham equation

We note some historical points towards the development of the analysis regarding the Whitham equation, focusing in particular on its progress in proving the Whitham conjecture of a highest cusped wave using the framework of global bifurcation theory.

Whitham's formal argument for the presence of a cusped wave was rather striking, and along with it he noted some properties of the the kernel K and the calculus of variations methods that can be used to analyze and derive the equation itself [57, 58].

Some of the first analysis done on the Whitham equation was performed by the Soviet mathematicians S. A. Gabov [27] and A. A. Zaitsev [59] – both looking at travelling waves and doing rudimentary investigations of the dispersion relation associated with the Whitham equation. The phenomenon of wave breaking had been known since Seliger [51] and his investigations on a simplified Expanding on the idea behind weakening the dispersion of local equations, the book by Naumkin and Shishmarev [44] studied Equation (1.1.7) with replaced kernels for the operator L, and also looked at the periodic and solitary Cauchy problem for the resulting equations. The authors claimed to have positively resolved the question of wave breaking in both the periodic and solitary cases [43], however it would seem that their proof had a glitch as pointed out and resolved by Hur in 2015 [34].

Ehrnström *et al.* [19] studied some general properties of the Euler equations and found that horizontally symmetric solutions are necessarily travelling wave solutions, and additionally at the outset of their treatment mention in passing that a large class of problems also exhibit the converse statement *a priori*. The work of Brüll *et al.* from 2017 [10] establishes that all supercritical, solitary, steady waves of the Whitham equation are symmetric and monotone on each side of the crest, and the article of Brüll and Pei from 2021 [11] establishes a similar result for equations of similar form to that of Whitham but with different Fourier multipliers L.

The classical global bifurcation theory started with P. H. Rabinowitz who, in the general setting of the bifurcation problem

$$F(\lambda, x) = 0, \quad \lambda \in \mathbb{R}, \quad x \in X \setminus \{0\}$$

for $F\colon \mathbb{R} \times X \to Y$, where X, Y are Banach spaces and $F(\lambda, 0) \equiv 0$, proved that sets of solutions were connected[2], but not path-connected even if the operators involved were infinitely differentiable [13]. The work done by E. N. Dancer in the early 70's [15] showed that in the setting of real-analytic operators the solution sets of the bifurcation problem would become path-connected. This prompted further work by B. Buffoni and J. F. Toland jointly with Dancer [12], which eventually culminated in the monograph [13] by Buffoni and Toland in 2003.

[2]connected in a topologist's sense. Some authors in analysis do not make the distinction between connectedness and path-connectedness, see e.g. Munkres [42] for details.

The article *Travelling waves for the Whitham equation* [22] by Ehrnström and Kalisch in 2009 marked the hunt for a complete resolution of the Whitham conjecture, proving the existence of small amplitude periodic travelling wave solutions. Additionally, if one let the period of such solutions approach infinity the solutions, should they exist, would approach a solitary wave of long-wave speed c_0. Up until then there had been little attempts at including the full kernel K_{h_0} instead of coarse approximations like in Gabov's investigation [27].

Global bifurcation analysis was first employed by Ehrnström and Kalisch in 2013 [21] to deduce continuous global branches of 2π-periodic steady solutions from the global bifurcation results presented by Buffoni and Toland in [13], where furthermore the Hölder–Zygmund spaces of index $\alpha < 1/2$ were the bifurcation space considered. The authors were also able to characterize the blow-up as sequences of solutions $\{\varphi_n\}_n$ approach the supremum bound $\sup_x \varphi = \mu/2$ as $n \to \infty$.

The Oberwolfach report of Ehrnström [18], first announced in 2015, would lay down the final pieces needed to resolve Whitham's conjecture. The published paper [23] would end up in collaboration with E. Wahlén in 2019 and refined the ideas laid forth in [18]. What was left desired from the 2013 Ehrnström–Kalisch paper [21] were considerations on the properties of the kernel K and the periodized kernel K_P as well as successfully establishing a highest periodic wave solution with the conjectured properties. Indeed, a lot of the work of [23] surrounds the precise nature of the Whitham kernel and performing a more precise analysis of the $\frac{1}{2}$-Hölder-regularity of φ at $x = 0$ when a peaked crest forms as $\varphi(0) = \mu/2$.

The working framework of Ehrnström–Wahlén [23] provides a supple basis for global bifurcation analysis and proving the existence of highest peaked waves of solutions to equations that are Whitham-like in form, especially equations that share the same nonlinearity as the Whitham equation. Steady equations of the form

$$-\mu\varphi + m_s(\mathrm{D})\,\varphi + \varphi^2 = 0, \tag{1.2.1}$$

where the pseudodifferential operator $m_s(\mathrm{D})$ takes the form $m_s(\mathrm{D}) = (1 + |\mathrm{D}|^2)^{s/2}$ in the inhomogeneous case, and $m_s(\mathrm{D}) = |\mathrm{D}|^s$ in the homogeneous case, have been particularly successful when analyzed through the framework of Erhnström–Wahlén. We call the inhomogeneous case with $s < 0$ the *fractional Korteweg–de Vrie equation* (fKdV), given by

$$-\mu\varphi + \Lambda^s\varphi + \varphi^2 = 0 \tag{1.2.2}$$

which was studied for the case $-1 < s < 0$ as recently as early 2022 by M. C. Ørke [46] who looked at fKdV along with the fractional Degasperi–Procesis equation. Indeed, if one takes $p = 2$ from one of the cases covered in Hildrum–Xue [31], one obtains the homogeneous case with order $-1 < s < 0$. Included in Table 1.1 are other cases and examples from that of fKdV-like equations with symbols and orders listed.

Dispersive Operator	Order and Regularity		
	$s \in (-1,0)$ s-Hölder	$s=-1$ log-Lipschitz	$s<-1$ Lipschitz
Homogeneous: $m_s(\mathrm{D}) = \lvert\mathrm{D}\rvert^s$	[31]	N/A	[9]
Inhomogeneous: $m_s(\mathrm{D}) = (1+\lvert\mathrm{D}\rvert^2)^{s/2}$	[46],[1]	[20]	[40]

Table 1.1: Orders s and regularities at the crest for different dispersive operators m_s according to in-/homogeneity. Adapted from Hildrum–Xue [31], where the authors cite an article in preparation by Ehrnström *et al.* as their example for the homogeneous $s=-1$ case which is therefore dropped from this table. Adapted with permission from F. Hildrum.

1.3 The work at hand

Ehrnström and Wahlén [23] close out their abstract with the statement

> [...] Our methods may be generalized.

Indeed, as can be seen from the plethora of articles that sprung from Ehrnström–Wahlén (see Table 1.1) this is true, and often quite fruitful, for equations that generalize Whitham's equation in ways that modify the nonlinearity in ways in which it is still local. An example from Hildrum and Xue [31] is that of modifying the nonlinearity like $\varphi^2 \mapsto |\varphi|^p$ or $\varphi^2 \mapsto \varphi\,|\varphi|^{p-1}$ for some $p \geq 2$. In this thesis we instead look at a genuinely nonlocal nonlinearity in a dispersive equation of the form

$$-\mu\varphi + \Lambda^s\varphi + \varphi\,\Lambda^r\varphi = 0 \tag{1.3.1}$$

here stated in steady variables, where Λ^s and Λ^r are Bessel potential operators which look like Fourier multipliers with symbols $\mathscr{F}\Lambda^s\varphi(\xi) = \langle\xi\rangle^s\,\hat{\varphi}(\xi)$ for $\langle\xi\rangle^s = (1+|\xi|^2)^{s/2}$, similarly for Λ^r. This equation is stated *ad hoc*, and does not adjoin to any known physical phenomenology. However, it would be a first working example of a nontrivial Coifman–Meyer [14] operator used as a nonlinearity in the context of global bifurcation analysis of dispersive equations.

In this thesis we mainly follow the structure of the 2019 article of Ehrnström–Wahlén [23] in hopes of establishing a global bifurcation theory for P-periodic steady solutions of Equation (1.3.1). We prove crucial a priori estimates, regularity results, and furthermore prove that the setup for global bifurcation à la Buffoni–Toland [13, Chapter 8] is satisfied, and remark some curious features and obstacles that occur when regarding a nonlocal nonlinearity.

Of special note is the case $r = s$, where we are able to completely prove analogues of the touching lemmata and the nodal property theorem as encountered in [23], which are then used to further study properties of the solutions. The case $r \neq s$ is discussed in the same context, but only partial results are achieved vis-à-vis the touching lemmata, nodal property theorem. The nodal property theorem in the case $r = s$ is used to prove that periodic loop curves do not occur in the global bifurcation diagram. Bifurcation formulas in the most general setting are established from local bifurcation theory.

Due to the nonlocal properties of our considered nonlinearity we are not able to achieve a supremal bound on solutions φ in our setting. Indeed, most of the bounds considered are either from an a priori result considering the formal steady equation (see Lemma 4.3.1). The bound $\Lambda^r \varphi < \mu$ used throughout the analysis is found more-or-less *ad hoc* through the analysis of the regularity of our solutions when $r, s < 0$ (see Theorem 4.4.1).

Whether our equation admits a highest wave is discussed, although not concluded either positively or negatively due to complications owed to the nonlocal nonlinearity. Features regarding singularities of solutions as they approach a regularity-breaking limit is discussed and found to degenerate to a trivial solution (Theorem 4.4.2), and hence our setting produces no singularities.

Chapter 2

Function spaces, operators and classes

In this chapter we present some necessary background material and lay the foundation for the theory which will come later.

2.1 A primer on distribution theory

In this section we will provide a sufficient exposition to distribution theory required for understanding this thesis. A lot of this material follows the same expository flow and conventions of the book *Distributions, Sobolev spaces and elliptic equations* by D. Haroske and H. Triebel [30], along with some introductory material concerning integration theory from Tao [53] and functional analysis which is included for completeness.

We will have great use of tuples of integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ called *multi-indices.* We are able to introduce a partial ordering on $\mathbb{Z}^n_{\geq 0}$ by letting $\beta \leq \alpha$ mean that $\beta_i \leq \alpha_i$ for every $i = 1, 2, \ldots, n$. With the multi-index notation we have a short form of writing partial derivatives

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} \cdots \frac{\partial^{\alpha_n}}{\partial x_n}$$

for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where we usually suspend the notation of coordinates $x = (x_1, x_2, \ldots, x_n)$ whenever this is implicit from context. Given a vector $x \in \mathbb{R}^n$ we define x^α with $\alpha \in \mathbb{Z}^n_{\geq 0}$ as the scalar quantity

$$x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}. \tag{2.1.1}$$

For multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ we have the Leibniz rule given by

$$\partial^\alpha (u \cdot v) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha - \beta} u \cdot \partial^\beta v \tag{2.1.2}$$

for functions u, v with suitable regularity. The binomial coefficients are defined as usual through the factorial

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!\,(\alpha-\beta)!}$$

where the factorial is given by $\alpha! = \alpha_1! \cdots \alpha_n!$.

Throughout the rest of this section, a *domain* Ω will be an open subset of $\mathbb{R}^n$. We norm the set of m-times continuously differentiable functions (that can be extended continuously to $\overline{\Omega}$) by

$$\|f\|_{C^m(\Omega)} = \sum_{|\alpha|\le m} \sup_{x\in\Omega} |\partial^\alpha f(x)|. \tag{2.1.3}$$

With this in mind, the bounded continuous functions on Ω will be denoted $C(\Omega) = C^0(\Omega)$. The space of smooth functions on Ω, denoted by $C^\infty(\Omega)$, is given as the intersection

$$C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$$

as a set, and can be equipped with the family of seminorms $\|\cdot\|_{C^m(\Omega)}$.

2.1.1 Integration theory

There are crucial aspects of integration theory that need to be introduced in order to fully appreciate distribution theory. Our exposition will be rather barebones in accordance with what is needed later, however a more thorough treatment may be found in Tao [53] and Folland [26].

Definition 2.1.1. Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Then the space of Lebesgue p-integrable functions $L^p(\Omega, \mathbb{C})$ (or similarly for real-valued) is given by all $f\colon \Omega \to \mathbb{C}$ where

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \,\mathrm{d}x\right)^{1/p} < \infty$$

where the integral is taken to be interpreted in the Lebesgue sense. In the case $p = \infty$ we say that a function $f\colon \Omega \to \mathbb{C}$ is essentially bounded $f \in L^\infty(\Omega, \mathbb{C})$ (or similarly for real-valued) if

$$\|f\|_{L^\infty(\Omega)} = \inf\{M \ge 0 \mid |f(x)| \le M \text{ a.e. } x \in \Omega\} < \infty$$

where a.e. denotes "almost everywhere" in the Lebesgue sense.

We will not be discussing measure theory in-depth in this text but we do however need a few results pertaining to the measure-theoretic side of integration theory.

Definition 2.1.2. Let $\Omega \subseteq \mathbb{R}^n$ be a domain. A function $f\colon \Omega \to \mathbb{C}$ is called *locally integrable on* Ω, denoted $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{C})$, if f is $L^1(K, \mathbb{C})$-integrable on every compact subset $K \subset\subset \Omega$.

A standard theorem from integration theory is Fubini's Theorem, here adapted from a general case presented in Tao [53].

Theorem 2.1.1 (Fubini's Theorem).
Let $f \in L^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{C})$. Then we can exchange the order of integration

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

We present the following result without proof, adapted from Folland [26].

Theorem 2.1.2 (The Lebesgue Dominated Convergence Theorem).
Let $\{f_n\}$ be a sequence of $L^1(\Omega)$-integrable functions such that

(i) $f_n \to f$ almost everywhere on Ω,

(ii) there exists a non-negative function $g \in L^1(\Omega, \mathbb{C})$ such that $|f| \leq g$ almost everywhere on Ω for all n.

Then the function f is $L^1(\Omega)$-integrable and satisfies

$$\int_\Omega f(x)\,\mathrm{d}x = \lim_{n \to \infty} \int_\Omega f_n(x)\,\mathrm{d}x.$$

A *simple function* $g \colon \Omega \to \mathbb{C}$ is a function of the form

$$g(x) = \sum_j a_j\, \chi_{E_j}(x)$$

where $\{a_j\}$ is a sequence of complex coefficients and the sets $E_j \subseteq \Omega$ are measurable. The functions χ_{E_j} are *characteristic functions* on the sets E_j, defined as

$$\chi_{E_j}(x) = \begin{cases} 1, & x \in E_j \\ 0, & x \notin E_j \end{cases}. \tag{2.1.4}$$

Another important result, here adapted from Folland [26] and presented without proof, is the relationship between simple functions and integrable functions.

Theorem 2.1.3. *If $f \in L^1(\Omega)$ and $\varepsilon > 0$, there exists an integrable simple function $g = \sum_j a_j \chi_{E_j}$ for some Lebesgue measurable sets $E_j \subseteq \Omega$ such that*

$$\int_\Omega |f - g|\,\mathrm{d}x < \varepsilon.$$

As seen in Tao [53], this result can in some sense be used to define Lebesgue integrable functions. Indeed, the definition of the Lebesgue integral can be built and expressed through limits of simple functions.

The *Lebesgue set* L_f of a Lebesgue measurable function $f \colon \Omega \to \mathbb{C}$ is the set of points given by

$$L_f = \left\{ x \in \Omega \,\middle|\, \lim_{r \to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(x) - f(y)|\,\mathrm{d}y = 0 \right\}, \tag{2.1.5}$$

where m is the Lebesgue measure.

Theorem 2.1.4 (Lebesgue differentiation theorem).
Let $f \in L^1_{\text{loc}}(\Omega, \mathbb{C})$. *For almost every* $x \in \Omega$, *or for any* $x \in L_f$ *in particular, we have*

$$\lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(x) - f(y)| \,\mathrm{d}x \quad \text{and} \quad \lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y) \,\mathrm{d}y = f(x).$$

Proof. A detailed build-up of a more general version and proof of the Lebesgue differentiation theorem can be found in Folland [26]. □

Recall that a Banach space is a normed vector space $(X, \|\cdot\|_X)$ which is complete in the norm $\|\cdot\|_X$ - every Cauchy sequence in X converges in norm. A Hilbert space is an inner product space $(H, \langle\cdot,\cdot\rangle_H)$ such that H is complete with respect to the inner product.

Proposition 2.1.1. *The space* $(L^p(\Omega,\mathbb{C}), \|\cdot\|_{L^p(\Omega)})$ *is a Banach space for every* $1 \leq p \leq \infty$. *In particular, for the case* $p = 2$ *we have that* $(L^2(\Omega,\mathbb{C}), \langle\cdot,\cdot\rangle_{L^2(\Omega)})$ *is a Hilbert space with inner product*

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)\,\overline{g(x)}\,\mathrm{d}x \tag{2.1.6}$$

for $f, g \in L^2(\Omega,\mathbb{C})$, *where the bar denotes complex conjugation.*

Definition 2.1.3. The space of *test functions* on $\Omega \subseteq \mathbb{R}^n$ is defined as

$$\mathscr{D}(\Omega) = C_0^\infty(\Omega) = \{f \in C^\infty(\Omega) \mid \operatorname{supp} f \subset\subset \Omega\}.$$

We equip this space with the following topology: the sequence $\{\varphi_j\}_j$ converges to φ in $\mathscr{D}(\Omega)$ if all φ_j are all supported on some compact set $K \subset \mathbb{R}^n$ and we have uniform convergence

$$\|\varphi_j - \varphi\|_{C^m(\Omega)} \to 0$$

as $j \to \infty$ for all $m \in \mathbb{Z}_{\geq 0}$.

Example 2.1.1. An important example of a test function is the function

$$f(x) = \begin{cases} \exp\left(-\frac{1}{(1-|x|^2)}\right), & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \tag{2.1.7}$$

where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ for $x \in \mathbb{R}^n$. This function is compactly supported on the closed ball of unit radius and center at the origin. Through translation and rescaling we see that this function can be used to give an example of a compactly supported function on any closed ball in $\mathbb{R}^n$.

Definition 2.1.4. The collection $\mathscr{D}'(\Omega)$ of complex-valued $\mathbb{C}$-linear, continuous functionals $T\colon \mathscr{D}(\Omega) \to \mathbb{C}$ with the added property that for $T \in \mathscr{D}'(\Omega)$

$$T(\varphi_j) \to T(\varphi)$$

as $j \to \infty$ given that $\varphi_j \to \varphi$ in $\mathscr{D}(\Omega)$ are called the *distributions over the domain* Ω.

We equip the distributions $\mathscr{D}'(\Omega)$ with the *simple convergence topology*: $T_j \to T$ as $j \to \infty$ in $\mathscr{D}'(\Omega)$ if and only if $T_j(\varphi) \to T(\varphi)$ as $j \to \infty$ for all $\varphi \in \mathscr{D}(\Omega)$.

Proposition 2.1.2 (Actions of distributions, [26, p. 284]).
We have, to select just a few, the following properties for distributions.

(i) the derivative of a distribution $T \in \mathcal{D}'(\Omega)$ is given by $\partial^\alpha T(\varphi) = (-1)^{|\alpha|} T(\partial^\alpha \varphi)$ for functions $\varphi \in \mathcal{D}(\Omega)$ and multi-indices α,

(ii) multiplication of a distribution $T \in \mathcal{D}'(\Omega)$ by a smooth function $\psi \in \mathcal{D}(\Omega)$ is defined as $\psi T(\varphi) = T(\psi\varphi)$ for any $\varphi \in \mathcal{D}(\Omega)$,

(iii) translation by a vector $x \in \mathbb{R}^n$ in the sense of $\tau_x \varphi(y) = \varphi(y - x)$ is given by $\tau_x T(\varphi) = T(\tau_{-x}\varphi)$ for $\varphi \in \mathcal{D}(\Omega)$,

(iv) sign-reversal of a distribution T is given by $T_\sigma(\varphi) = T(\varphi_\sigma)$ where $\varphi_\sigma(x) = \varphi(-x)$

(v) convolution with a smooth $\psi \in \mathcal{D}(\Omega)$ is given by $T * \psi(x) = T(\tau_x(\psi_\sigma))$ and is a smooth function on Ω.

Example 2.1.2. An example of a distribution which is not a function in a classical sense is the *Dirac delta distribution* $\delta \colon \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$ whose action is defined by

$$\delta(\varphi) = \varphi(0) \tag{2.1.8}$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$. By composing with a translation operator τ_x, where $\tau_x \varphi(y) = \varphi(y + x)$ we may define $\delta_x \in \mathcal{D}'(\mathbb{R}^n)$ as

$$\delta_x(\varphi) = \tau_x \delta(\varphi) = \delta(\tau_{-x}\varphi) = \varphi(x). \tag{2.1.9}$$

Let $\omega \in \mathcal{D}(\mathbb{R}^n)$ be the radially symmetric function defined by

$$\omega(x) = \begin{cases} N \exp\left(-\frac{1}{(1-|x|^2)}\right), & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \tag{2.1.10}$$

where $N = N(n)$ is a positive constant ensuring the normalization of the integral $\int_{\mathbb{R}^n} \omega(x)\,\mathrm{d}x = 1$. This function is smooth and compactly supported on $|x| \leq 1$, and also satisfies the property that $\omega_h(x) = h^{-n}\,\omega(x/h)$ converges in the distributional sense to the Dirac delta distribution $\delta = \delta_0$ as $h \searrow 0$. With this function, we are able to take otherwise non-smooth functions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and mollify them to their "smoothened" counterparts, namely by taking the convolution of u with ω_h

$$\tilde{u}_h(x) = \int_{\mathbb{R}^n} \omega_h(x - y)\, u(y)\,\mathrm{d}y$$

which is a smooth function for all $h > 0$, along with the property that

$$\lim_{h \searrow 0} \tilde{u}_h(x) = u(x) \;\text{ a.e.}$$

We call ω_h from a family of smooth functions $\{\omega_h \mid h > 0\}$ satisfying the properties above a *Friedrich-Sobolev mollifier.* Bump functions are smooth compactly supported functions which resolve to the identity on their support.

Example 2.1.3. Denote the open ball of radius $r > 0$ and center $x \in \mathbb{R}^n$ by $B(r, x)$. Let χ_S be the characteristic function defined as in Equation (2.1.4) for sets $S \subseteq \mathbb{R}^n$. Let ω be as in Equation (2.1.10), and fix arbitrary $h > 0$. Then the characteristic function $\chi_{B(r,x)}(y)$ has a mollified version given by the convolution

$$\tilde{\chi}_{B(r,x),h}(y) = \int_{\mathbb{R}^n} \omega_h(y - z)\, \chi_{B(r,x)}(z)\, \mathrm{d}z = \int_{B(r,x)} \omega_h(y - z)\, \mathrm{d}z \qquad (2.1.11)$$

where $\omega_h(x) = h^{-n}\omega(x/h)$. We know by compactly supported distributions that

$$\mathrm{supp}(\tilde{\chi}_{B(r,x),h}) \subseteq \mathrm{supp}(\chi_{B(r,x)}) + \mathrm{supp}(\omega_h) = \overline{B(r,x)} + \overline{B(h,0)}.$$

Now let $x = 0$. If $h \leq r$ in Equation (2.1.11) we immediately get that $\tilde{\chi}_{B(r,0),h}(y) = 1$ if $|y| < r - h$ since we have $|y - z| \leq |y| + |z| < r - h + h = r$ which means that

$$\tilde{\chi}_{B(r,0),h}(y) = \int_{\mathbb{R}^n} \omega_h(z)\, \chi_{B(r,0)}(y - z)\, \mathrm{d}z = \int_{B(h,0)} \omega_h(z)\, \mathrm{d}z = 1$$

on $B(r-h, 0)$ and has support contained in $\overline{B(r+h,0)}$ since $\mathrm{supp}(\omega_h(y)) \subseteq \overline{B(h,0)}$ and $\mathrm{supp}(\chi_{B(r,0)}(y-z)) = \mathrm{supp}(\chi_{B(r,z)(y)}) \subseteq \overline{B(r+h,0)}$ so we have

$$\mathrm{supp}(\omega_h(y)) \subseteq \mathrm{supp}(\chi_{B(r,0)}(y - z)) \subseteq \overline{B(r+h,0)}.$$

Thus we see that the mollified function $\tilde{\chi}_{B(r,0),h}$ misses the criteria of being a bump function on $B(r, 0)$ by a "grazing set" $\overline{B(r+h,0)} \setminus B(r,0)$ which depends on the parameter $h > 0$ only. We can of course translate the sets in question and arrive at similar results for $B(r, x)$ instead.

Proposition 2.1.3. *Let $f \in L^1_{\mathrm{loc}}(\Omega)$ and $\varphi \in \mathscr{D}(\Omega)$ then $f * \varphi \in C^\infty(\Omega)$ and $\partial^\alpha(f * \varphi) = f * \partial^\alpha\varphi$.*

Proof. A proof of this fact can be found in Chapter 9 of Folland [26]. □

Another important result regarding test functions concerns their approximation properties with regards to functions in L^p-spaces.

Proposition 2.1.4. *Let $1 \leq p < \infty$. Then $\mathscr{D}(\Omega)$ is dense in $L^p(\Omega)$.*

Proof. Fix $1 \leq p < \infty$. Consider a function $f \in L^p(\Omega)$. Then we know that for every $\varepsilon > 0$ the function $f \in L^p(\Omega)$ can be approximated in norm by some simple function $g \in L^p(\Omega)$ of the form

$$g(x) = \sum_i a_i\, \chi_{U_i}(x)$$

for constant scalars $a_i \in \mathbb{C}$ and characteristic functions χ_{U_i} on some Lebesgue measurable sets $U_i \subseteq \Omega$, such that

$$\|f - g\|_{L^p(\Omega)} < \varepsilon.$$

A Lebesgue measurable set U_i can, by definition, be approximated by open balls $B(\delta; x)$, so we may assume without loss of generality that the U_i are open balls of the form $B(\delta_i; x_i)$. We have seen that by way of Friedrich-Sobolev mollifiers we have for every $\varepsilon > 0$ there exists some $h > 0$ such that

$$\left\|\chi_{B(\delta_i;x_i)} - \tilde{\chi}_{B(\delta_i;x_i),h}\right\|_{L^p(\Omega)} < \varepsilon$$

hence by the smoothing property of Friedrich–Sobolev mollifiers we have that this can be used to approximate f and is smooth in the limit $h \searrow 0$. □

It turns out that for every locally integrable function $f \in L^1_{\text{loc}}(\Omega)$ one can always define a distribution by way of the integral

$$T_f(\varphi) = \int_\Omega f(x)\,\varphi(x)\,\mathrm{d}x \tag{2.1.12}$$

whose action on any $\varphi \in \mathscr{D}(\Omega)$ defines it uniquely. Proving such a distribution is well-defined rests on the fact that we have continuity by

$$|T_f(\varphi)| \leq \|f\|_{L^1(K)}\,\|\varphi\|_{C(\Omega)}$$

for $\operatorname{supp}(\varphi) \subset\subset K$. Linearity is ensured by the linearity of complex-valued integrals. Uniqueness comes from the fact that one is able to characterize the zero-distribution $\Theta \in \mathscr{D}'(\Omega)$ precisely as such an integral

$$\Theta(\varphi) = \int_\Omega f_\Theta(x)\,\varphi(x)\,\mathrm{d}x = 0$$

for some $f_\Theta \in L^1_{\text{loc}}(\Omega)$. In fact, we have the following result.

Proposition 2.1.5 (Reymond–du Bois)**.**
Let $f_\Theta \in L^1_{\text{loc}}(\Omega)$. If we have the property

$$\int_\Omega f_\Theta(x)\,\varphi(x)\,\mathrm{d}x = 0 \tag{2.1.13}$$

for any $\varphi \in \mathscr{D}(\Omega)$, then $f_\Theta = 0$ a.e. on Ω.

Proof. We follow the idea of proof as presented in Hörmander [33]. First we prove the statement for f continuous. We may split f into real and imaginary parts and thus assume f to be real-valued. Assume $f(x_0) \neq 0$. Then we may take $\varphi \in \mathscr{D}(\Omega)$ non-negative such that $\varphi(x_0) \neq 0$ with support on $\overline{B(\delta, x_0)}$ for small $\delta > 0$. Since f is continuous, take an open neighborhood U of x_0 such that $f\varphi$ has constant sign. Then obviously the integral of $f\varphi$ cannot be zero, hence our assumption $f(x_0) \neq 0$ is not true.

Now consider $f_\Theta \in L^1_{\text{loc}}(\Omega)$. Recall that the Lebesgue differentiation theorem which states that

$$\lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(x) - f(y)|\,\mathrm{d}y = 0$$

holds almost everywhere for $x \in \Omega$. Now take $\varphi \in \mathscr{D}(\Omega)$ to have compact support on the unit ball with unit integral so that we may write

$$\begin{aligned} h(x) &= \int_\Omega h(x)\,\varphi\left(\frac{x-y}{r}\right)\frac{\mathrm{d}y}{r^n} \\ &= \int_\Omega (h(x)-h(y))\,\varphi\left(\frac{x-y}{r}\right)\frac{\mathrm{d}y}{r^n} + \int_\Omega h(y)\,\varphi\left(\frac{x-y}{r}\right)\frac{\mathrm{d}y}{r^n} \end{aligned}$$

where the first integral tends to zero as $r \to 0$ (r^{-n} is of the same order as $m(B(r,x))^{-1}$) and the latter integral is zero by assumption. This concludes the proof. □

Distributions of the form $T = T_f$ for $f \in L^1_{\mathrm{loc}}(\Omega)$ as described above are called *regular distributions*. We have shown that there is a one-to-one correspondence between functions $f \in L^1_{\mathrm{loc}}(\Omega)$ and distributions T_f of integral form as in Equation (2.1.12). Distributions that are not regular, so not formally expressible as $T = T_f$ for some $f \in L^1_{\mathrm{loc}}(\Omega)$, are called *singular*.

Definition 2.1.5. Let $T \in \mathscr{D}'(\Omega)$. Then we define the *support* of T as the closed set

$$\operatorname{supp} T = \{x \in \overline{\Omega} \mid T\big|_{\Omega\cap B(\delta,x)} \neq 0 \text{ for any } \delta > 0\} \tag{2.1.14}$$

where the restriction $T\big|_U$ is defined by restricting $T\colon \mathscr{D}(\Omega) \to \mathbb{C}$ to $T\big|_U\colon \mathscr{D}(U) \to \mathbb{C}$ for $U \subseteq \Omega$ open. The set of compactly supported distributions on Ω is denoted $\mathscr{E}'(\Omega)$.

We state two very useful results on distributions, as found in Haroske–Triebel [30], without proof.

Theorem 2.1.5 (Localization Property)**.**
Let $T \in \mathscr{D}(\Omega)$ and let $K \subset \Omega$ be a compact set. Then there exists $C > 0$ and $N \in \mathbb{Z}_{\geq 0}$ dependent on K such that

$$|T\big|_K(\varphi)| \leq C \sum_{|\alpha|\leq N} \sup_{x\in K} |\partial^\alpha \varphi(x)|$$

holds for all $\varphi \in \mathscr{D}(K)$.

Proposition 2.1.6. *Let $T \in \mathscr{E}'(\Omega)$ be supported on $a \in \Omega$, so* $\operatorname{supp} T = \{a\}$. *Then T takes the form*

$$T = \sum_{|\alpha|\leq N} a_\alpha\, \partial^\alpha \delta_a \tag{2.1.15}$$

where N is the same N as that of the localization property. A similar result holds if T is supported on countably many points.

Proposition 2.1.7. *Let $T, S \in \mathscr{E}'(\Omega)$ be compactly supported distributions. Then $T * S$ is also compactly supported and we have*

$$\operatorname{supp}(T * S) \subseteq \operatorname{supp}(T) + \operatorname{supp}(S) \tag{2.1.16}$$

2.2 The Fourier transform, Schwartz spaces and Sobolev spaces

Definition 2.2.1. The *Schwartz space* $\mathscr{S}(\mathbb{R}^n)$ is the space of rapidly decreasing functions defined by

$$\mathscr{S}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) \mid \|\varphi\|_{k,\ell} < \infty,\ \forall k, \ell \in \mathbb{Z}_{\geq 0}\} \tag{2.2.1}$$

where the seminorms $\|\varphi\|_{k,\ell}$ are defined by

$$\|\varphi\|_{k,\ell} = \sup_{x\in\mathbb{R}^n} (1 + |x|^2)^{k/2} \sum_{|\alpha|\leq \ell} |\partial^\alpha \varphi(x)|. \tag{2.2.2}$$

A sequence $\{\varphi_j\}_j$ in $\mathscr{S}(\mathbb{R}^n)$ is said to converge to $\varphi \in \mathscr{S}(\mathbb{R}^n)$ if $\|\varphi_j - \varphi\|_{k,\ell} \to 0$ as $j \to \infty$ for all $k, \ell \in \mathbb{Z}_{\geq 0}$.

Remark. Observe that $\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$. In particular, this means that $\mathscr{S}(\mathbb{R}^n)$ sits densely in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ since $\mathscr{D}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. An equivalent family of seminorms to $\|\cdot\|_{k,\ell}$ can be made from the restriction $k = \ell$. Furthermore, one can also characterize the Schwartz space by a different albeit equivalent estimate

$$\mathscr{S}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) \mid |x^\beta \partial^\alpha \varphi| < \infty \text{ for all } \alpha, \beta \in \mathbb{Z}^n_{\geq 0}\}.$$

For Schwartz functions $f \in \mathscr{S}(\mathbb{R}^n)$ we can define the Fourier transform in our convention as

$$\mathscr{F} f(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, \mathrm{d}x \tag{2.2.3}$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$, along with the formal inverse Fourier transform

$$\mathscr{F}^{-1} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\xi)\, e^{ix\cdot\xi}\, \mathrm{d}\xi \tag{2.2.4}$$

where both of which can be extended to functions $f \in L^1(\mathbb{R}^n)$ since $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$. It should be noted that Equations (2.2.3) and (2.2.4) make sense formally. Looking at the Schwartz space one might ask the question of whether there is a more natural space for which the Fourier transform makes sense *a priori.* Looking at $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ one can show that the Fourier transform is meaningful for functions in $L^2(\mathbb{R}^n)$, and hence one has the following classical result

Theorem 2.2.1 (Plancherel,[26, Theorem 8.29])**.**
If $f \in L^1 \cap L^2$, then the Fourier transform $\mathscr{F} f \in L^2$, and furthermore $\mathscr{F}[(L^1 \cap L^2)]$ extends uniquely to a unitary isomorphism on L^2.

Example 2.2.1. One variant of the Gaussian function is given by

$$f(x) = e^{-a|x|^2} \tag{2.2.5}$$

for $a > 0$, and is a standard example of a Schwartz function that is not compactly supported, so indeed $\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$ is a proper inclusion. In particular, this function enjoys nice properties on the Fourier transform. Consider first the one-dimensional integral

$$\int_{\mathbb{R}} e^{-ax^2+bx}\,\mathrm{d}x = \int_{\mathbb{R}} e^{-a(x-b/(2a))^2+b^2/4a}\,\mathrm{d}x = e^{b^2/4a}\int_{\mathbb{R}} e^{-a(x-b/2a)^2}\,\mathrm{d}x$$

where $a > 0$, then substitute $y = (x - b/2a)/\sqrt{a}$

$$e^{b^2/4a}\int_{\mathbb{R}} e^{-a(x-b/2a)^2}\,\mathrm{d}x = \frac{e^{b^2/4a}}{\sqrt{a}}\int_{\mathbb{R}} e^{-y^2}\,\mathrm{d}y =: \frac{e^{b^2/4a}}{\sqrt{a}}\,I$$

so then by Fubini's theorem and switching to polar coordinates

$$\begin{aligned} I^2 &= \int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x \int_{\mathbb{R}} e^{-y^2}\,\mathrm{d}y = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= -\frac{2\pi}{2}\int_{\mathbb{R}} (e^{-r^2})'\,\mathrm{d}r = -\pi e^{-r^2}\Big|_{r=0}^{\infty} = \pi \end{aligned}$$

which then implies that

$$I = \int_{\mathbb{R}^n} e^{-y^2}\,\mathrm{d}y = \sqrt{\pi}$$

and furthermore we have shown the integral identity

$$\int_{\mathbb{R}} e^{-ax^2+bx}\,\mathrm{d}x = e^{b^2/4a}\sqrt{\frac{\pi}{a}}.$$

We can use this to compute the Fourier transform

$$\begin{aligned} \mathscr{F}(e^{-a|x|^2})(\xi) &= \int_{\mathbb{R}^n} e^{-a|x|^2-ix\cdot\xi}\,\mathrm{d}x = \prod_{j=1}^{n}\int_{\mathbb{R}} e^{-ax_j^2-ix_j\xi_j}\,\mathrm{d}x_j \\ &= \prod_{j=1}^{n}\sqrt{\pi}\,e^{-\xi_j^2/a} = \left(\frac{\pi}{4a}\right)^{n/2} e^{-|\xi|^2/4a}. \end{aligned}$$

By setting $a = \pi$ we see that

$$\mathscr{F}(e^{-\pi|x|^2})(\xi) = e^{-\pi|\xi|^2} \tag{2.2.6}$$

which in other words means that the Fourier transform acts as the identity on the Gaussian function when properly scaled.

Proposition 2.2.1. *For functions $f \in L^1(\mathbb{R}^n)$ we have that the Fourier transform $\mathscr{F}f$ is continuous and bounded on $\mathbb{R}^n$ with estimate*

$$\|\mathscr{F}f\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}\,.$$

Proof. Boundedness is trivial, so it remains to check continuity. Let $f \in L^1(\mathbb{R}^n)$. Then we know that

$$\begin{aligned} |\mathscr{F}f(\xi) - \mathscr{F}f(\zeta)| &= \left|\int_{\mathbb{R}^n} f(x)\,e^{-ix\cdot\xi}\,\mathrm{d}x - \int_{\mathbb{R}^n} f(x)\,e^{-ix\cdot\zeta}\,\mathrm{d}x\right| \\ &= \left|\int_{\mathbb{R}^n} f(x)\,(e^{-ix\cdot\xi} - e^{-ix\cdot\zeta})\,\mathrm{d}x\right| \\ &\leq \|f\|_{L^1(\mathbb{R}^n)}\left\|e^{-i*\cdot\xi} - e^{-i*\cdot\zeta}\right\|_{L^\infty(\mathbb{R}^n)} \end{aligned}$$

where we immediately see that we may bound $\left\|e^{-i*\cdot\xi} - e^{-i*\cdot\zeta}\right\|_{L^\infty(\mathbb{R}^n)}$ by arbitrary $\varepsilon > 0$ given that $|\xi - \zeta| < \delta$ for some $\delta > 0$. This establishes continuity, and we are done. $\square$

For functions $f, g \in L^1(\mathbb{R}^n)$ we have that

$$\int_{\mathbb{R}^n} |\mathscr{F} f(\xi)\, g(\xi)| \,\mathrm{d}\xi \leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x)\, e^{-ix\cdot\xi}| \,\mathrm{d}x\right) |g(\xi)| \,\mathrm{d}\xi = \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} < \infty$$

and similarly for $f \cdot \mathscr{F} g$, so both $\mathscr{F} f \cdot g$ and $f \cdot \mathscr{F} g$ are integrable and thus the following calculations make sense

$$\begin{aligned}\int_{\mathbb{R}^n} \mathscr{F} f(\xi) g(\xi) \,\mathrm{d}\xi &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi} \,\mathrm{d}x\, g(\xi) \,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} g(\xi)\, e^{-ix\cdot\xi} \,\mathrm{d}\xi \,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\, \mathscr{F} g(x) \,\mathrm{d}x \qquad (2.2.7)\end{aligned}$$

by Fubini's theorem (c.f. Tao [53]). This equation is called *the multiplication formula.*

For $f, g \in \mathscr{S}(\mathbb{R}^n)$ we have that $\mathscr{F} f$, $\mathscr{F} g$ along with f, g are all integrable and bounded, and thus we may use Equation (2.2.7) to obtain

$$\begin{aligned}\langle \mathscr{F} f, \mathscr{F} g\rangle_{L^2} &= \int_{\mathbb{R}^n} \mathscr{F} f(\xi)\, \overline{\mathscr{F} g}(\xi) \,\mathrm{d}\xi = (2\pi)^n \int_{\mathbb{R}^n} \mathscr{F} f(\xi) \mathscr{F}^{-1}\overline{g}(\xi) \,\mathrm{d}\xi \\ &= (2\pi)^n \int_{\mathbb{R}^n} f(x)\, \mathscr{F}\, \mathscr{F}^{-1}\overline{g}(x) \,\mathrm{d}x = (2\pi)^n \langle f, g\rangle_{L^2}\end{aligned}$$

which can then be extended to L^2-functions by using the fact that $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.

Remark. We should note that our convention of the Fourier transform is non-unitary, which means that in particular we have

$$\langle \mathscr{F} f, \mathscr{F} g\rangle_{L^2} = (2\pi)^n \langle f, g\rangle_{L^2}$$

for $f, g \in L^2(\mathbb{R}^n)$. In particular this means that $\|\mathscr{F} f\|_{L^2(\mathbb{R}^n)} = (2\pi)^n \|f\|_{L^2(\mathbb{R}^n)}$, hence we call our convention of the Fourier transform *non-unitary.*

Lemma 2.2.1. *(Fourier Inversion)*
Let $f \in L^1(\mathbb{R}^n)$ and assume also that $\mathscr{F} f \in L^1(\mathbb{R}^n)$. Then the Fourier inversion property holds

$$\mathscr{F}^{-1}(\mathscr{F} f)(x) = f(x), \quad \mathscr{F}(\mathscr{F}^{-1} f)(\xi) = f(\xi) \qquad (2.2.8)$$

almost everywhere for x, ξ, where $\mathscr{F}^{-1}(\cdot)$ is formally expressed as Equation (2.2.4).

Proof. Recall the Gauss function $e^{-\pi|x|^2}$ from Equation (2.2.6). By the Lebesgue Dominated Convergence theorem we ascertain

$$\begin{aligned}\mathscr{F} f(\xi) &= \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi} \,\mathrm{d}x = \int_{\mathbb{R}^n} \lim_{\varepsilon\to 0} f(x)\, e^{-\varepsilon|\xi|^2}\, e^{-ix\cdot\xi} \,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} f(x)\, e^{-\varepsilon|\xi|^2}\, e^{-ix\cdot\xi} \,\mathrm{d}x\end{aligned}$$

then we may move the Fourier transform onto $e^{-\varepsilon^2|\xi|^2}$ by using the multiplication formula

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} f(x)\, e^{-\varepsilon^2|\xi|^2}\, e^{-ix\cdot\xi}\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} f(x)\int_{\mathbb{R}^n}\left(\frac{\pi}{\varepsilon^2}\right)^{n/2} e^{-|y|^2/4\varepsilon}\, e^{iy\cdot\xi}$$

□

Remark. Indeed, requiring that $f, g \in L^1(\mathbb{R}^n)$ is not enough for Fourier inversion, since the (absolute) integrability of $\mathscr{F} f$ and $\mathscr{F} g$ is not guaranteed if f, g themselves are integrable.

Theorem 2.2.2 (Convolution theorem for non-unitary Fourier transform)**.**
Let $f, g \in L^p(\mathbb{R}^n)$ for $p \in \{1, 2\}$. Then we have

$$\mathscr{F}(f \cdot g) = \frac{1}{(2\pi)^n}\,\{\mathscr{F}(f)\} * \{\mathscr{F}(g)\}. \tag{2.2.9}$$

Proof. Step 1. We first prove the convolution theorem for L^1-functions f, g. Writing out the convolution

$$\begin{aligned}
\mathscr{F}(f) * \mathscr{F}(g)(\xi) &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(y)\, e^{-iy\cdot(\xi-\zeta)}\,\mathrm{d}y\right)\left(\int_{\mathbb{R}^n} g(x)\, e^{-ix\cdot\zeta}\,\mathrm{d}x\right)\mathrm{d}\zeta\\
&= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)\, g(x)\, e^{-i(y\cdot(\xi-\zeta)-x\cdot\zeta)}\,\mathrm{d}y\,\mathrm{d}x\right)\mathrm{d}\zeta\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)\, g(x)\, e^{-iy\cdot\xi}\, e^{i(y+x)\cdot\zeta}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\zeta
\end{aligned}$$

by Fubini's theorem. Now consider the shifting property of the Fourier transform, namely that for $h \in \mathbb{R}^n$ one has

$$\mathscr{F}(g(x+h))(\xi) = \int_{\mathbb{R}^n} g(x+h)\, e^{-ix\cdot\xi}\,\mathrm{d}x = \int_{\mathbb{R}^n} g(x)e^{-i(x-h)\cdot\xi}\,\mathrm{d}x = \mathscr{F} g(\xi)\, e^{i\xi\cdot h}.$$

Introduce the variable $z = x + y$, so by replacing $x = z - y$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)\, g(x)\, e^{-iy\cdot\xi}\, e^{i(y+x)\cdot\zeta}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\zeta\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)\, g(z-y)\, e^{-iy\cdot\xi}\, e^{iz\cdot\zeta}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\zeta\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)\, e^{-iy\cdot\xi}\left(\int_{\mathbb{R}^n} g(z-y)\, e^{-iz\cdot(-\zeta)}\,\mathrm{d}z\right)\mathrm{d}y\,\mathrm{d}\zeta\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)\, e^{-iy\cdot\xi}\,\mathscr{F} g(-\zeta)\, e^{i\zeta\cdot y}\,\mathrm{d}y\,\mathrm{d}\zeta\\
&= \int_{\mathbb{R}^n} f(y)\, e^{-iy\cdot\xi}\left(\int_{\mathbb{R}^n}(2\pi)^n\,\mathscr{F}^{-1} g(\zeta)\, e^{-iy\cdot\zeta}\,\mathrm{d}\zeta\right)\mathrm{d}y\\
&= (2\pi)^n\int_{\mathbb{R}^n} f(y)\, g(y)\, e^{-iy\cdot\xi}\,\mathrm{d}y\\
&= (2\pi)^n\,\mathscr{F}(f\cdot g)(\xi)
\end{aligned}$$

as desired, again by using Fubini, Fourier inversion and the shifting property along with the fact that $\mathscr{F}g(-\zeta) = (2\pi)^n \mathscr{F}^{-1}g(\zeta)$ for all $\zeta \in \mathbb{R}^n$.

Step 2. For the case $f, g \in L^2(\mathbb{R}^n)$, we use the fact that $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and the boundedness of the Fourier transform in what follows. Let $\{f_n\}_n \subset \mathscr{S}(\mathbb{R}^n)$ be a sequence such that $f_n \to f$ as $n \to \infty$ and furthermore assume $g \in \mathscr{S}(\mathbb{R}^n)$. Then by approximating in norm we get the desired result, and it remains to do the same process having used what we have just shown but now approximating g also. Details can be found in Haroske–Triebel [30]. □

It turns out that one can always take the Fourier inverse of functions $f \in \mathscr{S}(\mathbb{R}^n)$, which is classically stated as the following.

Theorem 2.2.3 ([26, Corollary 8.28]).
The Fourier transform is an automorphism on $\mathscr{S}$.

Definition 2.2.2. The collection of all complex-valued linear continuous functionals $T\colon \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ is called the space of *tempered distributions* $\mathscr{S}'(\mathbb{R}^n)$. We furnish this space with the weak star topology, namely

$$T_j \to T \text{ as } j \to \infty \text{ in } \mathscr{S}'(\mathbb{R}^n) \iff T_j(\varphi) \to T(\varphi) \text{ as } j \to \infty \text{ for all } \varphi \in \mathscr{S}(\mathbb{R}^n).$$

Remark (Notation). Sometimes we denote the action of a (tempered) distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ ($T \in \mathscr{S}'(\mathbb{R}^n)$) by the angled-bracket notation

$$T(\varphi) = \langle T, \varphi \rangle.$$

Proposition 2.2.2 ([26, p. 295]).
The Fourier transform extends by duality onto the tempered distributions, defined by $\mathscr{F}T(\varphi) = T(\hat{\varphi})$ where $T \in \mathscr{S}'(\mathbb{R}^n)$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$. One can do the same for the inverse Fourier transform.

Note that we have the canonical inclusion $\mathscr{S}'(\mathbb{R}^n) \subseteq \mathscr{D}'(\mathbb{R}^n)$, so everything that could be said about distributions hold in particular for the tempered distributions. We finally note an important result of compactly supported distributions which will be important later.

Proposition 2.2.3 ([26, Proposition 9.11]).
If $T \in \mathscr{E}'$ is a compactly supported distribution, then $\mathscr{F}T$ is a slowly increasing, smooth function with

$$|\partial^\alpha \mathscr{F}T(\xi)| \leq C_\alpha (1 + |\xi|)^{N(\alpha)}$$

such that furthermore $\mathscr{F}(T)(\xi) = \langle T, E_{-\xi} \rangle$ where $E_\xi(x) = e^{\pi i \xi \cdot x}$.

Definition 2.2.3. Let $m \in \mathbb{Z}_{\geq 0}$ and $1 < p < \infty$. We define the Sobolev space $W^{m,p}(\mathbb{R}^n)$ as all of the functions $f \in L^p(\mathbb{R}^n)$ whose distributional derivatives $\partial^\alpha f$ also lie in $L^p(\mathbb{R}^n)$ for all multiindices α satisfying $|\alpha| \leq m$. We norm $W^{m,p}(\mathbb{R}^n)$ by

$$\|f\|_{W^{m,p}(\mathbb{R}^n)} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}.$$

In this case one often calls the distributional derivatives of f as *weak derivatives of f*.

Let $\xi \in \mathbb{R}^n$. We introduce the *Japanese bracket* defined by

$$\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}.$$

Definition 2.2.4. Let $s \in \mathbb{R}$ and let $1 < p < \infty$. We define the inhomogeneous Sobolev space $W^{s,p}(\mathbb{R}^n)$ as the space of all tempered distributions $u \in \mathscr{S}'(\mathbb{R}^n)$ with the property that

$$\mathscr{F}^{-1}(\langle \xi \rangle^s \hat{u}(\xi))$$

is an element of $L^p(\mathbb{R}^n)$. We norm $W^{s,p}(\mathbb{R}^n)$ by

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} = \left\| \mathscr{F}^{-1}(\langle \xi \rangle^s \hat{u}(\xi)) \right\|_{L^p(\mathbb{R}^n)}.$$

In the case $p = 2$ we denote $W^{s,p}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$.

2.3 Notes on topology and compactness

Recall that a *metric space* (M, d) is a set M with a distance function $d\colon M \times M \to \mathbb{R}$ such that

(i) (non-negativity) $d(x, y) \geq 0$ for all $x, y \in M$,

(ii) (symmetry) $d(x, y) = d(y, x)$ for all $x, y \in M$,

(iii) (triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in M$.

Using this formalism, we can make precise what we mean by open and closed sets in metric spaces.

Definition 2.3.1. Let (M, d) be a metric space. A subset $U \subseteq M$ is said to be

(i) *open* if for every $x \in U$ there exists $\varepsilon > 0$ such that if $d(x, y) < \varepsilon$ for $y \in M$, then $y \in U$

(ii) *closed* if for any $\{x_n\}_n \subseteq U$ convergent implies that $\lim_{n\to\infty} x_n \in U$.

Remark. It is clear that any normed space $(X, \|\cdot\|_X)$ inherits a metric space structure from the norm via the distance function $d(x, y) = \|x - y\|_X$.

An *open cover of* $N \subseteq M$ for a metric space (M, d) is a collection of open sets $\{U_\alpha\}_{\alpha\in\mathscr{A}}$ from M such that

$$N \subseteq \bigcup_{\alpha\in\mathscr{A}} U_\alpha.$$

We have already seen the definition of an open cover for $\mathbb{R}^n$, this is just the generalized version. An *ε-net* is a collection of open balls $\{B(\varepsilon, x_j)\}_j$ where $\varepsilon > 0$ and $x_j \in M$.

Definition 2.3.2. Let (M, d) be a metric space. We say that M is

- *compact* if every open cover admits a finite open subcover,

- *sequentially compact* if every infinite sequence in M has a convergent subsequence,
- *totally bounded* if for any $\varepsilon > 0$ there exists an ε-net defined by finitely many points $\{x_1, \ldots, x_n\}$ in M covering all of M.

A subset $N \subseteq M$ is called *precompact* if the closure of N is compact.

Theorem 2.3.1. *A metric space is compact if and only if it is totally bounded and complete.*

Proof. A proof is given in DiBenedetto [16]. □

Proposition 2.3.1. *A subset of a complete metric space is precompact if and only if it is totally bounded.*

Proof. We refer again to DiBenedetto [16] for a proof of this. □

Proposition 2.3.2. *A subset of a metric space is precompact if and only if the space is sequentially compact.*

Proof. Follows readily from Proposition 2.3.1. □

Definition 2.3.3. Let X be a Banach space, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ a field. Let $\mathscr{F} \subseteq C(X, \mathbb{F})$ be a family of continuous functions. We say that $\mathscr{F}$ is *equicontinuous at a point* $x \in X$ if for given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ holds for all $f \in \mathscr{F}$ simultaneously whenever $|x - y| < \delta$. Furthermore, $\mathscr{F}$ is *equicontinuous* if equicontinuity holds for all $x \in X$. We call $\mathscr{F}$ *pointwise bounded* if $\sup_{x \in X} |f(x)| < \infty$ for all $f \in \mathscr{F}$.

We borrow the Arzelà–Ascoli Theorem as in Driver [17], specialized to the case of metric spaces, which reads as follows.

Theorem 2.3.2 (Arzelà–Ascoli)**.**
Let (M, d) be a compact metric space. And let $\mathscr{F} \subset C(M, \mathbb{F})$ be a family of functions. Then $\mathscr{F}$ is precompact in $C(M, \mathbb{F})$ if and only if $\mathscr{F}$ is pointwise bounded and equicontinuous. In the case of a compact metric space we additionally have that pointwise

Proof. This is a classic result. See the proof of Theorem 2.86 in Driver [17]. □

2.4 Partitions of unity, Besov spaces and Hölder–Zygmund spaces

One motivation for the concept regarding resolutions of unity comes from wanting to locally study something which is a priori globally defined on a space, domain, manifold, or similar object. Now, for differential equations and differential geometry, we often work from the perspective of topology. To this end, a resolution of unity will necessarily deal with open sets of whatever object we are trying to cover. We will only consider open subsets of $\mathbb{R}^n$ or the P-torus $\mathbb{T}^n_P$ as our "objects" here.

Definition 2.4.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. An *open cover* of Ω is a collection $\{U_\alpha\}_{\alpha\in\mathscr{A}}$ of open sets relative Ω such that

$$\Omega = \bigcup_{\alpha\in\mathscr{A}} U_\alpha.$$

A *(locally finite) partition of unity* of Ω subordinate to the open cover $\{U_\alpha\}_{\alpha\in\mathscr{A}}$ is a collection of smooth functions $\{\rho_\alpha\colon \Omega \to \mathbb{R}_{\geq 0}\}_{\alpha\in\mathscr{A}}$ such that

(i) $\sum_{\alpha\in\mathscr{A}} \rho_\alpha(x) = 1$, with only finitely many $\rho_\alpha(x)$ non-zero on a given open neighborhood of $x \in \Omega$, for all $x \in \Omega$,

(ii) ρ_α is compactly supported on U_α for all $\alpha \in \mathscr{A}$.

There are various ways of constructing partitions of unity, for example by using Sobolev/Friedrich mollifiers. For our purposes we would like to construct a partition of unity for $\mathbb{R}^n$ itself. Let $\Psi \in \mathscr{S}(\mathbb{R}^n)$ be a radially symmetric function. Our construction will rely on the fact that since $\mathbb{R}^n$ is σ-finite as a measure space when endowed with the Lebesgue measure, we can decompose $\mathbb{R}^n$ into a union of concentric annuli. We wish to make the Fourier transform of Ψ into a *dyadic partition of unity* by constructing a compactly supported smooth function on a particular annulus in $\mathbb{R}^n$ and extrapolating this to the other annuli. Let $r, R > 0$ be the small and large radii of an annulus in $\mathbb{R}^n$ and let $\delta, \varepsilon > 0$ be positive constants. Our desired properties are

(1) $\hat{\Psi}$ is supported on $r - \delta \leq |\xi| \leq R$,

(2) equal to 1 on $r - \delta \leq |\xi| \leq R - \varepsilon$,

(3) satisfies $\hat{\Psi}(\xi) + \hat{\Psi}(\xi/2) = 1$ on $r < |\xi| \leq R$.

Fix $R = 2$, $r = 1$ and $\delta = 1/2$, $\varepsilon = 1$. To obtain condition (1), let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ be a non-negative radial function which is supported on the closed ball $0 \leq |\xi| \leq 2$ and equal to one on $0 \leq |\xi| \leq 1$. We have seen how such a function is constructed as a mollified characteristic function on the open ball. Then we obtain a smooth function $\hat{\Psi}$ supported on the annulus $1/2 \leq |\xi| \leq 2$ by taking

$$\hat{\Psi}(\xi) = \varphi(\xi) - \varphi(2\xi) \tag{2.4.1}$$

since $\varphi(\xi) = \varphi(2\xi)$ if $|\xi| < 1/2$, and both are zero for $|\xi| > 2$. For $1/2 < |\xi| \leq 1$ we have $\phi(\xi) = 1$ and $\phi(2\xi) = 0$ and thus we have condition (2) with $\hat{\Psi}(\xi) = 1$ on $1/2 < |\xi| \leq 1$. Finally, the function $\hat{\Psi}$ satisfied condition (3) since

$$\hat{\Psi}(\xi) + \hat{\Psi}(\xi/2) = \varphi(\xi) - \varphi(2\xi) + \varphi(\xi/2) - \varphi(\xi) = \varphi(\xi/2) - \varphi(2\xi) = 1$$

for $1 < |\xi| \leq 2$.

Remark. It should be noted that different authors opt for different constructions of the dyadic partition of unity for Littlewood-Paley theory, hence one might have different choices for r, R and δ, ε as above. The principle behind this partition of unity, however, remains the same across conventions.

A key observation now is that we can translate the annulus $1/2 \leq |\xi| \leq 2$ by scaling the input variable for $\hat{\Psi}(c\xi)$ to cover the rest of $\mathbb{R}^n$. The functions $\hat{\Psi}(2^{-j}\xi)$ for $j \in \mathbb{Z}$ are smooth functions supported on $2^{j-1} \leq |\xi| \leq 2^{j+1}$. Along with condition (3) above we see that

$$\sum_{j\in\mathbb{Z}} \hat{\Psi}(2^{-j}\xi) = \hat{\Psi}(2^{-\ell}\xi) + \hat{\Psi}(2^{-(\ell+1)}\xi) = \varphi(2^{-(\ell+1)}\xi) - \varphi(2^{-(\ell-1)}\xi) = 1 \tag{2.4.2}$$

whenever $2^\ell < |\xi| \leq 2^{\ell+1}$. Here we note that more care is needed to cover the case $\xi = 0$, but this is not really necessary in our case. Write $\Psi_{2^{-j}} = \mathscr{F}^{-1}(\hat{\Psi}(2^{-j}\xi))$.

Define the annulus C_j by

$$C_j = \{\xi \in \mathbb{R}^n \mid 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$$

for $j \in \mathbb{Z}$. Then we note that $C_j \cap C_{j'} = \emptyset$ if $|j - j'| > 1$, and that their union satisfies

$$\mathbb{R}^n = \bigcup_{j\in\mathbb{Z}} C_j.$$

Having defined Ψ as above we can introduce the *Littlewood-Paley operator* $\Delta_j = \Delta_j^\Psi$ associated to the dyadic partition as defined/characterized by the function Ψ

$$\Delta_j(f) = \Delta_j^\Psi(f) = \Psi_{2^{-j}} * f = (2\pi)^{-n}\mathscr{F}^{-1}(\hat{\Psi}(2^{-j}\xi) \cdot \hat{f}(\xi)). \tag{2.4.3}$$

It is customary to introduce a Schwartz function Φ defined such that its Fourier transform satisfies

$$\hat{\Phi}(\xi) = \begin{cases} \sum_{j\leq 0} \hat{\Psi}(2^{-j}\xi) & \xi \neq 0 \\ 1 & \xi = 0 \end{cases} \tag{2.4.4}$$

which in turn means that $\hat{\Phi}(\xi)$ is equal to 1 for $|\xi| \leq 1$ and vanishes whenever $|\xi| \geq 2$. Let $S_0 = S_0^\Psi$ be the operator defined by

$$S_0(f) = S_0^\Psi(f) = \Phi * f.$$

Then we have the operator identity

$$S_0 + \sum_{j=1}^{\infty} \Delta_j = I$$

where I is the identity operator on $\mathscr{S}'(\mathbb{R}^n)$. This series makes sense and converges in $\mathscr{S}'(\mathbb{R}^n)$. Equivalently, we may introduce the operator Δ_0 given by

$$\Delta_0(f) = \mathscr{F}^{-1}(\varphi) * f$$

where φ is the function as in Equation (2.4.1). This serves a similar purpose since we may show that in the same vein as for S_0 we have

$$\sum_{j=0}^{\infty} \Delta_j = I$$

in $\mathscr{S}'(\mathbb{R}^n)$.

We follow the convention set by Grafakos [28] and define the Besov–Lipschitz spaces thusly.

Definition 2.4.2. (Besov–Lipschitz Spaces on $\mathbb{R}^n$)
Let $s \in \mathbb{R}$ be a parameter and suppose $1 \leq p, q \leq \infty$. For $f \in \mathscr{S}'(\mathbb{R}^n)$ we define the *inhomogeneous Besov–Lipschitz norm* by the quantity

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \|S_0(f)\|_{L^p(\mathbb{R}^n)} + \left(\sum_{j=1}^{\infty}(2^{js}\|\Delta_j(f)\|_{L^p(\mathbb{R}^n)})^q\right)^{\frac{1}{q}} \tag{2.4.5}$$

which is a valid expression for $1 \leq p \leq \infty$ and $1 \leq q < \infty$, and for the $q = \infty$ case we write

$$\|f\|_{B^s_{p,\infty}(\mathbb{R}^n)} = \sup\{\|S_0(f)\|_{L^p(\mathbb{R}^n)}, \sup_{j\in\mathbb{Z}_{\geq 1}} 2^{js}\|\Delta_j(f)\|_{L^p(\mathbb{R}^n)}\} \tag{2.4.6}$$

The *(inhomogeneous) Besov–Lipschitz space* $B^s_{p,q}(\mathbb{R}^n)$ with fixed parameters s, p, q is the collection of all functions with finite $\|\cdot\|_{B^s_{p,q}}$-norm.

Remark. In the definition above we employ the inhomogeneous description as seen in Grafakos, however with the operator Δ_0 we achieve equivalent norms given by

$$\|f\|^*_{B^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{j=0}^{\infty}(2^{js}\|\Delta_j(f)\|_{L^p(\mathbb{R}^n)})^q\right)^{1/q}$$

for $1 \leq p \leq \infty$, $1 \leq q < \infty$ and $s \in \mathbb{R}$, and for the $q = \infty$ case we obtain

$$\|f\|^*_{B^s_{p,\infty}(\mathbb{R}^n)} = \sup_{j\in\mathbb{Z}_{\geq 0}} 2^{js}\|\Delta_j(f)\|_{L^p(\mathbb{R}^n)}.$$

Additionally, one can extend the definition of L^p-spaces to that of $0 < p \leq \infty$ by way of *quasi-Banach spaces*, where one sacrifices the triangle inequality in norm for a weaker inequality giving rise to the concept of a *quasi-norm.* Authors like Grafakos [28] and Triebel [56] both concern themselves with Besov–Lipschitz spaces with $0 < p, q \leq \infty$, whereas we have no use for this added generality.

Theorem 2.4.1 (Besov–Lipschitz spaces are Banach, [56, p. 48])**.**
Let $1 \leq p, q \leq \infty$ and let $s < 0$. Then $B^s_{p,q}(\mathbb{R}^n)$ is a Banach space.

Theorem 2.4.2 (Lifting Property of Besov–Lipschitz Spaces)**.**
Let $s, \sigma \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Define $\Lambda^\sigma \colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ by

$$\Lambda^\sigma f = \mathscr{F}^{-1}(\langle\xi\rangle^\sigma \mathscr{F} f).$$

Then Λ^σ induces an isomorphism of Besov–Lipschitz spaces $\Lambda^\sigma \colon B^s_{p,q}(\mathbb{R}^n) \to B^{s-\sigma}_{p,q}(\mathbb{R}^n)$. Furthermore, $\|\Lambda^\sigma(\cdot)\|_{B^{s-\sigma}_{p,q}(\mathbb{R}^n)}$ is an equivalent norm to $\|\cdot\|_{B^s_{p,q}(\mathbb{R}^n)}$.

Proof. See the main theorem from Section 2.3.8 in Triebel [56]. □

Definition 2.4.3. Let $(X, \|\cdot\|_X)$ be a (quasi-) Banach space. We call X a *multiplication algebra* if there exists $C \geq 0$ such that

$$\|f \cdot g\|_X \leq C\|f\|_X\|g\|_X$$

for every $f, g \in X$. For function spaces we always interpret the multiplication as pointwise multiplication of functions. Given $C = 1$ we call X a *(quasi-) Banach algebra.*

The following result from Triebel [56] makes it clear for which choices of the parameters (s, p, q) the space $B^s_{p,q}(\mathbb{R}^n)$ becomes a multiplication algebra, and in particular when this space becomes a Banach algebra.

Theorem 2.4.3 (Besov–Lipschitz Spaces as Multiplication Algebras).
Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. The following three statements are pairwise equivalent

(i) $B^s_{p,q}(\mathbb{R}^n)$ *is a multiplication algebra,*

(ii) $B^s_{p,q}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$,

(iii) $s > n/p$ *(or* $s = n/p$ *if* $q = 1$ *and* $1 \leq p < \infty$*).*

Furthermore, in the case $p = \infty$ with $1 \leq q \leq \infty$ and $s > 0$ we have the even sharper result that $B^s_{\infty,q}(\mathbb{R}^n)$ is a Banach algebra.

Proof. A proof of the original statement as in Triebel [56] can be found in Peetre [47]. However, an article by Nguyen and Sickel [45] points out that the statement in Triebel's text is incorrect - the corrected version is stated here. The last statement is Proposition 3 from Bourdaud *et al.* [4]. □

Definition 2.4.4 (Hölder–Zygmund spaces).
Let $s \in \mathbb{R}$ be a parameter. The space defined by restricting the Besov–Lipschitz space $B^s_{p,q}(\mathbb{R}^n)$ to the case $p = q = \infty$,

$$\mathscr{C}^s(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n),$$

is called the *Hölder–Zygmund space with index s*. Functions in the space $\mathscr{C}^s(\mathbb{R}^n)$ are normed by

$$\|f\|_{\mathscr{C}^s(\mathbb{R}^n)} = \sup_{j \in \mathbb{Z}_{\geq 0}} 2^{js} \|\Delta_j(f)\|_{L^\infty(\mathbb{R}^n)}.$$

Remark. Theorem 2.4.3 implies that the Hölder–Zygmund space $\mathscr{C}^s(\mathbb{R}^n)$ for $s > 0$ is a Banach algebra.

Theorem 2.4.4 (Embedding theorems for $B^s_{p,q}(\mathbb{R}^n)$).
Let $1 \leq p_0 \leq p_1 \leq \infty$, $1 \leq q \leq \infty$, $s_0, s_1 \in \mathbb{R}$ with $s_0 \leq s_1$ and $\varepsilon > 0$. Then the following embeddings are continuous

$$B^{s+\varepsilon}_{p,q}(\mathbb{R}^n) \subset B^s_{p,q}(\mathbb{R}^n) \quad \textit{for} \quad s \in \mathbb{R}, 1 \leq p \leq \infty \tag{2.4.7}$$

$$B^{s_0}_{p_0,q}(\mathbb{R}^n) \subset B^{s_1}_{p_1,q}(\mathbb{R}^n) \quad \textit{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1} \tag{2.4.8}$$

$$B^m_{\infty,1}(\mathbb{R}^n) \subset C^m(\mathbb{R}^n) \quad \textit{for} \quad m \in \mathbb{Z}_{\geq 0} \tag{2.4.9}$$

$$B^{s+\frac{n}{p}}_{p,q}(\mathbb{R}^n) \subset \mathscr{C}^s(\mathbb{R}^n) \quad \textit{for} \quad s > 0, 1 \leq p, q \leq \infty. \tag{2.4.10}$$

Proof. See Section 2.7 in Triebel [56]. □

In particular we have the embedding

$$\mathscr{C}^s(\mathbb{R}^n) \longrightarrow C(\mathbb{R}^n) \tag{2.4.11}$$

whenever $s > 0$.

Recall that in our convention functions of class $C^k(\mathbb{R}^n)$ have bounded C^k-norm as in Equation (2.1.3). We then state the definition of a Hölder–Lipschitz space as follows.

Definition 2.4.5 (Hölder–Lipschitz spaces)**.**
Let $k \in \mathbb{Z}_{\geq 0}$ and $\Omega \subseteq \mathbb{R}^n$ be a domain. A function $f \in C^k(\Omega)$ is said to be *Hölder k-times continuously differentiable* with exponent (or index) $0 < \alpha \leq 1$ if it has finite $C^{k,\alpha}$-norm given by

$$\|f\|_{C^{k,\alpha}(\Omega)} = \|f\|_{C^k(\Omega)} + \sum_{|\beta|=k} [\partial^\beta f]_\alpha, \quad [g]_\alpha := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|g(x) - g(y)|}{|x-y|^\alpha} \tag{2.4.12}$$

We denote the space of functions with finite $C^{k,\alpha}$-norm as above $C^{k,\alpha}(\Omega)$.

Denote the second-order difference of f by

$$(D_h^2 f)(x) = (D_h(D_h f))(x) = f(x+2h) - 2f(x+h) + f(x)$$

then one can equivalently define the space of *Zygmund functions* $\mathscr{C}^\alpha(\mathbb{R}^n)$ [60] by the seminorm

$$[f]^*_\alpha = \sup_{0\neq h\in\mathbb{R}^n} \frac{\left\| D_h^2 f^{(\lfloor\alpha\rfloor)} \right\|_{C(\mathbb{R})}}{|h|^{\alpha - \lfloor\alpha\rfloor}} \tag{2.4.13}$$

so then $\mathscr{C}^\alpha(\mathbb{R}^n) = \{f \in C^{\lfloor\alpha\rfloor} \mid [f]^*_\alpha < \infty\}$ for $\alpha > 0$. Details of this equivalence can be found in Triebel [56], and from the same book one can find the following result.

Proposition 2.4.1 ([56])**.** *Let $s > 0$, $s \notin \mathbb{Z}$. Then the Hölder–Zygmund space $\mathscr{C}^s(\mathbb{R}^n)$ and Hölder space $C^{\lfloor s\rfloor, s-\lfloor s\rfloor}(\mathbb{R}^n)$ coincide exactly*

$$\mathscr{C}^s(\mathbb{R}^n) = C^{\lfloor s\rfloor, s-\lfloor s\rfloor}(\mathbb{R}^n)$$

and have equivalent norms. Here $\lfloor s\rfloor$ denotes the integer part of s rounded down.

2.5 Periodic functions, Fourier series and the tori $\mathbb{T}^n_P$

In this section we collect the needed theory to understand periodic solutions to equations as well as how we would define the spaces encountered so far in the setting of tori $\mathbb{T}^n_P$.

Periodicity and tori $\mathbb{T}^n_P$

Assume $\varphi(x)$ is some function involving the variable x. Then φ is said to be *P-periodic in x* if for some $P \in \mathbb{R}$ the relation

$$\varphi(x) = \varphi(x + nP), \quad n \in \mathbb{Z} \tag{2.5.1}$$

holds for all applicable x. In the setting of evolution equation one might have a function of the form $\varphi(t,x)$ and thus assume $x \in \mathbb{R}$ in order for Equation (2.5.1) to make sense, but the temporal variable t is allowed to have arbitrary domain. In the case of steady equations we only have one variable, so this distinction does not matter. Recall from basic algebra that we can form a quotient space $\mathbb{R}/P\mathbb{Z}$ from the equivalence relation $\sim$ defined by

$$x \sim y \quad \text{if and only if} \quad x = y + nP \quad \text{for some } n \in \mathbb{Z}$$

for $x, y \in \mathbb{R}$. We think of this as x *wrapping around* to y whenever they are distanced by an integer multiple of P. Denote the circle of circumference P as $\mathbb{S}_P$. It turns out that the space $\mathbb{R}/\mathbb{Z}$ defined by the equivalence relation

$$x \sim y \quad \text{if and only if} \quad x - y \in \mathbb{Z}$$

is homeomorphic to the unit circle $S^1 = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$ via the map

$$g\colon \mathbb{R}/\mathbb{Z} \longrightarrow S^1,\ x \longmapsto g(x) = e^{ix}.$$

Likewise, we can do similarly for $\mathbb{R}/P\mathbb{Z}$ and $\mathbb{S}_P$ by simply rescaling, so there exists a homeomorphism h from $\mathbb{R}/P\mathbb{Z}$ to the circle $\mathbb{S}_P$ of circumference P given by

$$h\colon \mathbb{R}/P\mathbb{Z} \longrightarrow \mathbb{S}_P,\ x \longmapsto h(x) = e^{2\pi i x/P}.$$

Of course, there is nothing stopping us from generalizing this construction from one copy of S^1 or $\mathbb{S}_P$ to several dimensions. Thus we call

$$\mathbb{T}^n = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i \sim y_i \text{ iff } x_i - y_i = 2\pi n, \quad n \in \mathbb{Z}\} \tag{2.5.2}$$

the n-dimensional torus (of period 2π). One can similarly construct a version with period P defined by

$$\mathbb{T}^n_P = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i \sim y_i \text{ iff } x_i - y_i = Pn, \quad n \in \mathbb{Z}\}. \tag{2.5.3}$$

Integration on tori

Recall from multivariable calculus, or indeed manifold theory like in Lee that for compact smooth manifolds M we can consider open sets $U \subseteq M$, $V \subseteq \mathbb{R}^n$ and their corresponding C^∞-diffeomorphisms $F\colon V \subseteq \mathbb{R}^n \xrightarrow{\simeq} U \subseteq M$ by definition of a smooth manifold. In this setup, one can integrate on manifolds locally via the change of variables integral transformation rule

$$\int_U f(x)\,\mathrm{d}x_1 \dots \mathrm{d}x_n = \int_V f \circ F(y)\ |\det J_F(y)|\ \mathrm{d}y_1 \dots \mathrm{d}y_n \tag{2.5.4}$$

where $x = (x_1, \dots, x_n) = F(y_1, y_2, \dots, y_n) \in U$ and $y = (y_1, \dots, y_n) \in V$. The Jacobian $J_F(y)$ of F is given by the matrix

$$J_F(y) = \left[\frac{\partial F_i}{\partial y_j}(y)\right]_{i,j}$$

where $F(y) = (F_1, F_2, \dots, F_n)$. Assuming that the manifold is orientable, we are able to patch together open sets $U \subseteq M$ to the whole of M by using a partition of unity in the integral

$$\int_M f(x)\,\mathrm{d}x = \sum_i \int_{U_i} \rho_i(x)\, f(x)\,\mathrm{d}x. \tag{2.5.5}$$

Since the tori $\mathbb{T}^n$, $\mathbb{T}^n_P$ are compact, smooth manifolds that are also orientable, we are able to calculate integrals over the tori through Equation (2.5.5). It is of course enough to do this for $\mathbb{T}^n_P$. Our map F as above is given by $F(y_1, y_2, \ldots, y_n) = (e^{\frac{2\pi i}{P}y_1}, e^{\frac{2\pi i}{P}y_2}, \ldots, e^{\frac{2\pi i}{P}y_n})$. Then we can compute the integral

$$\int_{\mathbb{T}^n_P} f(x)\,\mathrm{d}x = \int_0^P \cdots \int_0^P f(e^{\frac{2\pi i}{P}y_1}, \ldots, e^{\frac{2\pi i}{P}y_n})\,|\det J_F(y)|\,\mathrm{d}y_1 \ldots \mathrm{d}y_n$$

where since x_j is given as $x_j = e^{2\pi i y_j/P}$ the determinant of the Jacobian becomes

$$\det J_F(y) = \left(\frac{2\pi i}{P}\right)^n e^{\frac{2\pi i}{P}(y_1+y_2+\cdots+y_n)}$$

so taking the (complex) absolute value we furthermore obtain

$$\int_{\mathbb{T}^n_P} f(x)\,\mathrm{d}x = \left(\frac{2\pi}{P}\right)^n \int_0^P \cdots \int_0^P f(e^{\frac{2\pi i}{P}y_1}, \ldots, e^{\frac{2\pi i}{P}y_n})\,\mathrm{d}y_1 \ldots \mathrm{d}y_n.$$

Here comes the critical insight, namely that to each function $f\colon \mathbb{T}^n_P \to \mathbb{C}$ there exists a unique corresponding function $\tilde{f}\colon \mathbb{R}^n \to \mathbb{C}$ which is P-periodic. These pairs $f, \tilde{f}$ exist in bijection through the relation

$$f(e^{\frac{2\pi i}{P}y_1}, e^{\frac{2\pi i}{P}y_2}, \ldots, e^{\frac{2\pi i}{P}y_n}) = \tilde{f}(y_1, y_2, \ldots, y_n)$$

which is valid for all $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. This then results in the integral formula

$$\int_{\mathbb{T}^n_P} f(x)\,\mathrm{d}x = \left(\frac{2\pi}{P}\right)^n \int_{[0,P]^n} \tilde{f}(y)\,\mathrm{d}y.$$

We will henceforth keep the bijection between $f, \tilde{f}$ as above implicit, and abuse notation to write

$$\int_{\mathbb{T}^n_P} f(x)\,\mathrm{d}x = \left(\frac{2\pi}{P}\right)^n \int_{[0,P]^n} f(y)\,\mathrm{d}y.$$

Furthermore, following conventions set in measure theory (c.f. Folland [26, p. 239]), we will rescale the measure $\mathrm{d}y$ to get rid of the prefactor, thus we take

$$\int_{\mathbb{T}^n_P} f(x)\,\mathrm{d}x = \int_{[0,P]^n} f(y)\,\mathrm{d}y = \int_{[-P/2,P/2]^n} f(y)\,\mathrm{d}y \tag{2.5.6}$$

as our definition of the integral on the tori $\mathbb{T}^n_P$.

Definition 2.5.1. Let $1 \le p \le \infty$. Let $P > 0$. Then we define $L^p(\mathbb{T}^n_P)$ as the collection of all functions $f\colon \mathbb{T}^n_P \to \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{T}^n_P)} = \left(\int_{\mathbb{T}^n_P} |f(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} < \infty$$

for $1 \le p < \infty$, along with the case $p = \infty$

$$\|f\|_{L^\infty(\mathbb{T}^n_P)} = \inf\{M \ge 0 \mid |f(x)| \le M \text{ a.e. on } \mathbb{T}^n_P\} < \infty.$$

Remark. Note that since $\mathbb{T}^n_P$ is compact in $\mathbb{R}^n$, the spaces $\mathscr{D}(\mathbb{T}^n_P)$ and $\mathscr{S}(\mathbb{T}^n_P)$ have to coincide. Furthermore, looking at distributions over $\mathbb{T}^n_P$ it is not necessary by default that we should require that the distributions are defined over the test functions defined as $\mathscr{D}(\mathbb{T}^n_P) = C^\infty_0(\mathbb{T}^n_P)$. The reason for this is due to our heuristic analysis of regular distributions. If we want to make sense of regular distributions $f \in L^1_{\text{loc}}(\mathbb{T}^n_P)$ there is no need for decay conditions on our test functions. There is however a need for our space of test functions to be dense in $L^p(\mathbb{T}^n_P)$ should our arguments carry over to the setting of distributions over the tori. Since $\mathbb{T}^n_P$ is compact we can use $\mathscr{D}(\mathbb{T}^n_P) = C^\infty(\mathbb{T}^n_P)$ as our test functions.

Assuming $f \in L^1(\mathbb{T}^n_P)$ we may formally write the Fourier transform of f over $\mathbb{T}^n_P$ as an integer-tuple function

$$\hat{f}_k = \frac{1}{P^n} \int_{\mathbb{T}^n_P} f(x)\, e^{-2\pi i x\cdot k/P}\, \mathrm{d}x \tag{2.5.7}$$

where as the notation suggests we have $k \in \mathbb{Z}^n$, so indeed $\hat{f}_{(\cdot)} \colon \mathbb{Z}^n \to \mathbb{C}$. The reason we do not consider a continuous variable $\xi \in \mathbb{R}^n$ is because in order for the integral to be defined we want the integrand to be P-periodic, and $e^{-2\pi i x\cdot\xi/P}$ is only P-periodic for $\xi = k \in \mathbb{Z}^n$. We call $\hat{f}(k)$ the k-th *Fourier coefficient of* f. With this we can define the Fourier series

$$\sum_{k\in\mathbb{Z}^n} \hat{f}_k\, e^{2\pi i x\cdot k/P}. \tag{2.5.8}$$

We will not spend time on the results of when Fourier series converge uniformly to their function representative, however surface-level details can be found in Folland [26] and Haroske–Triebel [30].

One can define an inner product on $L^2(\mathbb{T}^n_P)$ to obtain the following result.

Theorem 2.5.1 (Orthonormal basis for $L^2(\mathbb{T}^n_P)$, [26, Theorem 8.20])**.**
The set $\{\exp(2\pi i k x/P)\}_{k\in\mathbb{Z}^n}$ *constitutes an orthonormal basis of* $L^2(\mathbb{T}^n_P)$.

2.6 Periodic distributions and function spaces on $\mathbb{T}^n$

The following proposition and definition are adapted from Triebel [56].

Definition 2.6.1. (Periodic distributions on $\mathbb{R}^n$)
A tempered distribution $T \in \mathscr{S}'(\mathbb{R}^n)$ is called a *P-periodic distribution on* $\mathbb{R}^n$ if the periodicity condition

$$T(\varphi(\cdot)) = T(\varphi(\cdot + Pk)), \quad k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \tag{2.6.1}$$

holds for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and all $k \in \mathbb{Z}^n$. The space of all P-periodic distributions will be denoted by $\mathscr{D}'(\mathbb{T}^n_P)$.

Remark. Define the translation operator $\tau_h\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ by $\tau_h\varphi(x) = \varphi(x+h)$. This operator can be extended to tempered distributions $T \in \mathscr{S}'(\mathbb{R}^n)$ via the rule

$$\tau_h T(\varphi) = T(\tau_{-h}\varphi)$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and all $h \in \mathbb{R}^n$. We can therefore restate Equation (2.6.1) as

$$T - \tau_{-Pk}T = 0. \tag{2.6.2}$$

Proposition 2.6.1 ([56, p. 264])**.**
A distribution $T \in \mathscr{S}'(\mathbb{R}^n)$ is P-periodic if and only if it can be represented as

$$T(\varphi) = \sum_{k\in\mathbb{Z}^n} a_k \mathscr{F}(\varphi)(2\pi k/P) \tag{2.6.3}$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$, where the complex coefficients $\{a_k\}_{k\in\mathbb{Z}^n}$ have at most polynomial growth in k: there exists constants $C \geq 0$ and $N > 0$ such that

$$|a_k| \leq C(1 + |k|)^N, \quad k \in \mathbb{Z}^n. \tag{2.6.4}$$

Remark. There is a refinement to the formulation as in Equation (2.6.3), namely that $T \in \mathscr{S}'(\mathbb{R}^n)$ is P-periodic as a distribution on $\mathbb{R}^n$ if and only if it takes the form

$$T = \sum_{k\in\mathbb{Z}^n} a_k\, e^{2\pi i x\cdot k/P} \tag{2.6.5}$$

where $\{a_k\}_{k\in\mathbb{Z}^n}$ is of polynomial growth as above and the sum converges in the topology of $\mathscr{S}'(\mathbb{R}^n)$.

Definition 2.6.2. Let $s \in \mathbb{R}$ and $\{\Delta_j\}_{j\in\mathbb{Z}_{\geq 0}}$ be the operators coming from a Littlewood–Paley partition of unity Ψ.

The space $B^s_{p,q}(\mathbb{T}^n_P)$ of P-periodic distributions $f \in \mathscr{D}'(\mathbb{T}^n_P)$ with

$$\|f\|_{B^s_{p,q}(\mathbb{T}^n_P)} = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\|\Delta_j\left(\sum_{k\in\mathbb{Z}^n} a_k\, e^{2\pi i x\cdot k/P}\right)\right\|^q_{L^p(\mathbb{T}^n_P)}\right)^{1/q} \tag{2.6.6}$$

whenever $\{a_k\}_k$ is a sequence of complex numbers of polynomial growth such that the representation

$$f = \sum_{k\in\mathbb{Z}^n} a_k\, e^{2\pi i x\cdot k/P} \in \mathscr{D}'(\mathbb{T}^n_P) \tag{2.6.7}$$

holds and converges in a distributional sense. The usual modifications for the case $p = q = \infty$ which leads to the P-periodic versions of the Hölder–Zygmund spaces are

$$\|f\|_{\mathscr{C}^s(\mathbb{T}^n_P)} = \sup_{j\in\mathbb{Z}_{\geq 0}} 2^{js} \left\|\Delta_j\left(\sum_{k\in\mathbb{Z}^n} a_k\, e^{2\pi i x\cdot k/P}\right)\right\|_{L^\infty(\mathbb{T}^n_P)}. \tag{2.6.8}$$

We state a refinement of the previous embedding theorem, Theorem 2.4.4, in the case of periodic Hölder–Zygmund spaces. This embedding result is lifted from Taylor [54, A. 39, p. 100].

Theorem 2.6.1 (Embeddings of periodic Hölder–Zygmund spaces)**.**
Let $r < s$. Then the continuous embedding

$$\mathscr{C}^s(\mathbb{T}^n_P) \hookrightarrow \mathscr{C}^r(\mathbb{T}^n_P) \tag{2.6.9}$$

as encountered in Theorem 2.4.4 is compact. Furthermore, there are strict inclusions

$$C^s(\mathbb{T}^n_P) \subsetneq \mathscr{C}^s(\mathbb{T}^n_P) \subsetneq C^{s,s-1}(\mathbb{T}^n_P), \quad s = 1, 2, \ldots \tag{2.6.10}$$

where $C^s(\mathbb{T}^n_P)$ are the bounded s-times continuously differentiable functions on $\mathbb{T}^n_P$.

Proof. A proof of the compact embedding is found in the preceding analysis to (A.39) of Taylor [54]. The second statement is lifted from an observation of Hildrum–Xue [31]. □

2.6.1 Periodization of operators

Operators $K\colon X \to Y$ on spaces of periodic functions X, Y ought to map to periodic functions to be well-defined. We have already encountered the Fourier kernel $e^{-2\pi i k\cdot x/P}$ in Equation (2.5.7) which is P-periodic for $k \in \mathbb{Z}^n$ as a function and maps periodic functions to periodic functions as an integral transform.

Consider $K\colon L^\infty(\mathbb{T}^n) \to L^\infty(\mathbb{R}^n)$ given by the convolutional transform

$$Kf(x) = \int_{\mathbb{R}^n} h(x-y)\, f(y)\, \mathrm{d}y$$

for some kernel $h \in L^1(\mathbb{R}^n)$. Then we may write

$$\begin{aligned} Kf(x) &= \int_{\mathbb{R}^n} h(x-y)\, f(y)\, \mathrm{d}y = \int_{[0,P]^n} \sum_{k\in\mathbb{Z}^n} h(x-y+Pk)\, f(y)\, \mathrm{d}y \\ &= \int_{[0,P]^n} h_P(x-y) f(y)\, \mathrm{d}y \end{aligned}$$

where h_P is the P-periodization of h. By our canonical identification of periodic functions we obtain (by slightly abusing notation) that

$$Kf(x) = \int_{\mathbb{T}^n_P} h_P(x-y)\, f(y)\, \mathrm{d}y,$$

in which case $K\colon L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ maps periodic functions to periodic functions. We may extend this periodization to operators on P-periodic tempered distributions $f \in \mathscr{D}'(\mathbb{T}^n_P)$ by the construction as given in Folland [26, p. 298]. We can define a *P-periodization map* $\mathscr{P}$ given by

$$\mathscr{P}\phi = \sum_{k\in\mathbb{Z}^n} \tau_{Pk}\phi$$

which maps $\mathscr{D}(\mathbb{R}^n)$ into $C^\infty(\mathbb{T}^n_P)$. This map induces another map under duality $\mathscr{P}'\colon \mathscr{D}'(\mathbb{T}^n_P) \to \mathscr{D}'(\mathbb{R}^n)$ by the rule

$$\langle \mathscr{P}'T, \phi\rangle = \langle T, \mathscr{P}\phi\rangle$$

for $\phi \in \mathscr{D}(\mathbb{R}^n)$. Clearly, since $\mathscr{P} \circ \tau_{Pk} = \mathscr{P}$ for $k \in \mathbb{Z}^n$ we also have $\tau_{Pk} \circ \mathscr{P}' = \mathscr{P}'$. Denote set of P-periodized distributions by

$$\mathscr{D}'(\mathbb{R}^n)_{\mathrm{per}} = \{T \in \mathscr{D}'(\mathbb{R}^n) \mid \tau_{Pk}T = T \text{ for } k \in \mathbb{Z}^n\}.$$

Then the range of $\mathscr{P}'$ lies in $\mathscr{D}'(\mathbb{R}^n)$, and it is a striking fact that one can prove that $\mathscr{P}' \colon \mathscr{D}'(\mathbb{T}_P^n) \to \mathscr{D}'(\mathbb{R}^n)_{\mathrm{per}}$ is a bijection. Hence if $f \in L^1(\mathbb{T}_P^n)$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$, then

$$\begin{aligned}\langle \mathscr{P}'f, \phi\rangle = \langle f, \mathscr{P}\phi\rangle &= \int_{\mathbb{T}_P^n} f(x) \sum_{k\in\mathbb{Z}^n} \phi(x - Pk)\,\mathrm{d}x \\ &= \sum_{k\in\mathbb{Z}^n} \int_{\mathbb{T}_P^n + kP} f(x)\,\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\,\phi(x)\,\mathrm{d}x = \langle f, \phi\rangle\end{aligned}$$

so the two descriptions of f and $\mathscr{P}'f$ coincide. Let $T \in \mathscr{D}'(\mathbb{T}_P^n)$, then the Fourier series representation $\sum_k \hat{T}_k\, e^{2\pi i k\cdot x/P}$ as encountered in Equation (2.6.7) converges distributionally to T hence also in $\mathscr{S}'(\mathbb{R}^n)$. Therefore we have $\mathscr{D}'(\mathbb{R}^n)_{\mathrm{per}} \subset \mathscr{S}'(\mathbb{R}^n)$ as expected, and one can also show that (see Folland [26])

$$\mathscr{F}(\mathscr{P}'T) = \sum_{k\in\mathbb{Z}^n} \hat{T}_k\, \mathscr{F}(e^{2\pi i k\cdot x/P}) = \sum_{k\in\mathbb{Z}^n} \hat{T}_k\, \tau_{Pk}\delta.$$

This yields a relation between the Fourier transform on $\mathbb{R}^n$ and that of $\mathbb{T}_P^n$ for periodic distributions. One can recover the Poisson summation formula in the relation above.

2.7 Pseudodifferential operators and symbol classes

Differential operators can come in various different forms, and there are several ways of generalizing what we mean by a differential operator to its broadest meaning. To get a feel for how general such a concept might become, given enough abstraction, consider the algebraic notion of a *derivation*, namely an R-bilinear map $D \colon A \to A$ from an associative algebra $(A, \cdot\,)$ (as an R-module for a commutative ring R, c.f. Lee [41]) to itself satisfying Leibniz' rule

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$$

for all $a, b \in A$. An example of this could be $(C^\infty(\mathbb{R}), \cdot\,)$ where the multiplication is given pointwise $(f \cdot g)(x) = f(x) \cdot g(x)$, with a derivation $D = \frac{\mathrm{d}}{\mathrm{d}x}$. This notion of a derivation is not sharp enough for our purposes, but it illustrates the level of generality with which one can work.

For our purposes, the concepts that we will be looking at will revolve around the Fourier transformation acting on *symbols* from their corresponding *symbol classes*. In the section to come we will mainly be basing our theory of pseudodifferential operators from Taylor [55].

Define $D_j = -i\partial_{x_j}$, so then for a multi-index $\alpha \in \mathbb{Z}^n_{\geq 0}$ we have $D^\alpha = D_1^{\alpha_1} \cdot D_2^{\alpha_2} \cdots D_n^{\alpha_n}$. This means that for a function $u \in \mathscr{S}(\mathbb{R}^n)$, say, we have

$$D^\alpha u(x) = \int_{\mathbb{R}^n} \xi^\alpha \, \hat{u}(\xi) \, e^{ix\cdot\xi} \, \mathrm{d}\xi$$

by Fourier inversion on $\mathscr{S}(\mathbb{R}^n)$. Introduce a differential operator of order k by the action

$$p(x, D) \, u(x) = \int_{\mathbb{R}^n} p(x, \xi) \, \hat{u}(\xi) \, e^{ix\cdot\xi} \, \mathrm{d}\xi \tag{2.7.1}$$

where on the Fourier side the *symbol* $p(x, \xi)$ is given by

$$p(x, \xi) = \sum_{|\alpha| \leq k} a_k(x) \, \xi^\alpha$$

for some coefficient functions $a_k(x)$. We say that the symbol $p(x, \xi)$ comes from a class of symbols. It is natural to describe the decay/growth of such integral kernels like our symbol $p(x, \xi)$ since it is imperative for making sense of our integral formulation of $p(x, D)$.

Definition 2.7.1. Let $0 \leq \rho, \delta \leq 1$ and $m \in \mathbb{R}$. Define the symbol class $S^m_{\rho,\delta}$ to be all functions $p(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that there exists $C_{\alpha,\beta} \geq 0$ fulfilling the estimate

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m - \rho|\alpha| - \delta|\beta|} \tag{2.7.2}$$

for multi-indices $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. The differential operator $p(x, D)$ corresponding to the symbol $p(x, \xi)$ is said to be in the operator symbol class denoted $\mathscr{O}S^m_{\rho,\delta}$ as in Taylor [55].

Example 2.7.1. The symbols $m_s(\xi) = \langle \xi \rangle^s$ define pseudodifferential operators $m_s(D)$ with action

$$m_s(D)u(x) = \int_{\mathbb{R}^n} m_s(\xi) \, \hat{u}(\xi) \, e^{ix\cdot\xi} \, \mathrm{d}\xi.$$

Pseudodifferential operators which act like multiplying a smooth symbol $m_s(\xi)$ on the Fourier side with no x-dependency are often called *Fourier multipliers*. There is a rich field dedicated to studying the ramifications of Fourier multipliers called microlocal analysis (see e.g. Gigris–Sjöstrand [29]). Indeed, it turns out that we have

$$|\partial_\xi^k \langle \xi \rangle^s| \lesssim_k (1 + |\xi|^2)^{s-k}$$

by estimating the k-th derivative, see Folland's treatise on Sobolev spaces in [26].

Remark (Notation).
In this thesis we will write upright D whenever total derivatives are considered, for instance in the case of ordinary derivatives $\mathrm{D}_t f(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t)$. Other authors like to reserve the upright D for pseudodifferential operators, however we follow Taylor's [55] conventions, with exception of the introductory chapter where we use the more commonly established upright D.

2.8 Pseudoproducts

Let $u, v \in \mathscr{D}'(\mathbb{R}^n)$ be distributions, and recall Leibniz' rule Equation (2.1.2) for multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$

$$\partial^\alpha (u \cdot v) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} u \cdot \partial^\beta v.$$

We would like to consider $u, v \in \mathscr{S}(\mathbb{R}^n)$ and take the Fourier transform of the Leibniz rule:

$$\mathscr{F}(\partial^\alpha(u \cdot v))(\xi) = \mathscr{F}\left(\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} u \cdot \partial^\beta v \right)$$

but since the Fourier transform is $\mathbb{C}$-linear we only need to consider the individual terms of the form

$$\mathscr{F}(\partial^{\alpha-\beta} u \cdot \partial^\beta v)$$

where $\beta \leq \alpha$. Keeping the Fourier convolution theorem in mind, we readily have

$$\begin{aligned}
(2\pi)^n \mathscr{F}(\partial^{\alpha-\beta} u \cdot \partial^\beta v) &= \{\mathscr{F}(\partial^{\alpha-\beta} u)\} * \{\mathscr{F}(\partial^\beta v)\} \\
&= \{(i\xi)^{|\alpha-\beta|} \mathscr{F}(u)\} * \{(i\xi)^{|\beta|} \mathscr{F}(v)\} \\
&= \int_{\mathbb{R}^n} (i\eta)^{|\alpha-\beta|} \hat{u}(\eta) (i(\xi-\eta))^{|\beta|} \, \hat{v}(\xi-\eta) \, \mathrm{d}\eta \\
&= \int_{\mathbb{R}^n} i^{|\alpha|} \eta^{|\alpha-\beta|} (\xi-\eta)^{|\beta|} \, \hat{u}(\eta) \, \hat{v}(\xi-\eta) \, \mathrm{d}\eta
\end{aligned}$$

where we have used that $i^{|\alpha|} = i^{|\alpha-\beta|} \cdot i^{|\beta|}$ for any multiindex β.

Thus we have shown that

$$\mathscr{F}(\partial^{\alpha-\beta} u \cdot \partial^\beta v)(\xi) = \frac{i^{|\alpha|}}{(2\pi)^n} \int_{\mathbb{R}^n} \eta^{|\alpha-\beta|} (\xi-\eta)^{|\beta|} \, \hat{u}(\eta) \, \hat{v}(\xi-\eta) \, \mathrm{d}\eta$$

which furthermore implies that

$$\mathscr{F}(\partial^\alpha(u \cdot v))(\xi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left[\frac{i^{|\alpha|}}{(2\pi)^n} \int_{\mathbb{R}^n} \eta^{|\alpha-\beta|} (\xi-\eta)^{|\beta|} \, \hat{u}(\eta) \, \hat{v}(\xi-\eta) \, \mathrm{d}\eta \right].$$

This kind of action under the Fourier transformation motivates the definition of a class of symbols $\mathscr{M}$ which we will refer to as the class of *pseudoproducts* $m \in \mathscr{M}$.

Definition 2.8.1. Let u, v be two functions and let D_1, D_2 be pseudodifferential operators. A *pseudoproduct* (or *Coifman–Meyer operator* [14]) $m = m(D_1, D_2) \in \mathscr{M}$ is a bilinear symbol $m(D_1, D_2)$ which acts on the pair (u, v) through

$$\mathscr{F}(m(D_1, D_2)(u, v)) = \int_{\mathbb{R}^n} m(\xi-\eta, \eta) \, \hat{u}(\xi-\eta) \, \hat{v}(\eta) \, \mathrm{d}\eta. \tag{2.8.1}$$

Recall that a function $f\colon X \to Y$ is, more abstractly, just a rule of assignment between the sets X and Y, that is for every $x \in X$ the map f assigns to x one and only one value which we call $f(x)$. In contrast to this, the lesser used *multifunctions* can give several outputs to just one input. Define the collection of functions between the sets X, Y as $F(X,Y)$. We see that if Y is a vector space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, then $F(X,Y)$ becomes a vector space with the pointwise defined scalar multiplication and additive map. In this case, we call $F(X,Y)$ a *function space* over X and Y respectively. See Munkres [42] for further discussion.

We include an informal definition of a nonlocal operator on function spaces.

Definition 2.8.2 (Nonlocal operators - informal definition)**.**
Let $A\colon F \to F'$ be a map of function spaces $F = F(X,Y)$ and $F' = F(X',Y)$, where X is equipped with a topology (norm or metric topology for instance). If for every $x \in X$ the function $Af(x)$ is fully determinable by the information of A on some proper open subset $U \subset X$, then A is said to be a *local operator* of function spaces. If this is not the case, then $Af(x)$ is only determinable on X, and we call A a *nonlocal operator.*

Example 2.8.1. Indeed, many integral operators $A\colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ of the form

$$Af(x) = \int_{\mathbb{R}} K(x,y)f(y)\,\mathrm{d}x,$$

with sufficient assumptions on the integral kernel K, are nonlocal operators.

Chapter 3

Functional analysis and bifurcation theory

3.1 Background on bifurcation theory

Bifurcation theories arise from trying to solve non-linear equations of the form

$$F(\lambda, x) = 0 \tag{3.1.1}$$

where $F\colon \mathbb{F} \times X \to Y$ is a prescribed non-linear function, λ is a parameter variable from $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and x is a variable from a Banach space X. Here we also assume Y is a Banach space. The difficulty then becomes finding which pairs (λ, x) solve Equation (3.1.1) in general. To this end, one generally needs to give more information on F. Our development of bifurcation theory and what is needed in accordance to make sense of the theory is mainly based on the text of Buffoni–Toland [13], however we will use some material from Kielhöfer [37] at the very end. Some references on functional analysis are Rudin [49], Brezis [8] and we also borrow some results from the aforementoined book Buffoni–Toland.

3.2 Functional analysis

Definition 3.2.1. The space of *bounded linear functionals* X^* on a Banach space X over $\mathbb{F}$ is given by all linear mappings $f\colon X \to \mathbb{F}$ such that f has finite operator norm

$$\|f\|_{\mathscr{L}(X,\mathbb{F})} = \inf\{M \geq 0 \mid |f(x)| \leq M \|x\|_X \text{ for all } x \in X\}. \tag{3.2.1}$$

Definition 3.2.2. The space of all bounded linear operators $\mathscr{L}(X, Y)$ of Banach spaces X, Y is that of linear mappings $A\colon X \to Y$ with finite operator norm in the sense that there exists $M > 0$ such that

$$\|Ax\|_Y \leq M \|x\|_X \tag{3.2.2}$$

for all $x \in X$.

Proposition 3.2.1. *Assuming X and Y are Banach spaces over $\mathbb{F}$, then the space $\mathscr{L}(X,Y)$ of bounded linear operators $X \to Y$ is a Banach space when endowed with the norm given by*

$$\|A\|_{\mathscr{L}(X,Y)} = \inf\{M \geq 0 \mid \|Ax\|_Y \leq M\|x\|_X \text{ for all } x \in X\}. \tag{3.2.3}$$

Remark. An equivalent norm to Equation (3.2.3) is the one given by

$$\|A\|^*_{\mathscr{L}(X,Y)} = \sup_{\substack{x \in X \\ \|x\|=1}} \|Ax\|_Y. \tag{3.2.4}$$

For $\mathscr{L}(X,Y)$ to be Banach we only need Y to be Banach, although throughout this text we will assume X is Banach as well.

Proof. The equivalence between the norms Equation (3.2.3) and Equation (3.2.4) is easily seen by the elementary ε-definitions of the supremum and infimum. As for proving that $\mathscr{L}(X,Y)$ is a Banach space, assume that $\{A_i\}_i$ is a Cauchy sequence in $\mathscr{L}(X,Y)$. Then for all $\varepsilon > 0$ there exists $N = N(i,j)$ such that given $i, j \geq N$ we have

$$\|A_i - A_j\|_{\mathscr{L}(X,Y)} < \varepsilon \iff \|A_i x - A_j x\|_Y < \varepsilon \quad \text{for all } x \in X.$$

We see that $\{A_i x\}_i$ becomes a Cauchy sequence in Y. Therefore we propose a candidate for the limit $A_i \to A$ as $n \to \infty$ by way of the pointwise limit $A_i x \to Ax$ as $n \to \infty$ for all $x \in X$, which always exists since $A_i x$ converges in Y due to Y being Banach. It is clear that this pointwise limit Ax is linear in its input $x \in X$. It is also bounded due to

$$\|Ax\|_Y \leq \|(A - A_i)x\|_Y + \|A_i x\|_Y < \varepsilon + K\|x\|_Y$$

where since any Cauchy sequence is bounded we have that $K \geq 0$ bounds the Cauchy sequence $\{A_i\}_i$ in operator norm. For sufficiently large $\tilde{N} = \tilde{N}(i,j)$ we can bound

$$\|(A - A_i)x\|_Y \leq \|(A - A_j)x\|_Y + \|(A_j - A_i)x\|_Y < 2\varepsilon$$

which furthermore implies $A_i \to A$ in $\mathscr{L}(X,Y)$. □

For every bounded linear operator $A \in \mathscr{L}(X,Y)$ we define the kernel and range as usual by

$$\begin{aligned}\ker(A) &= \{x \in X \mid Ax = 0 \in Y\},\\ \operatorname{ran}(A) &= \{y \in Y \mid y = Ax \text{ for some } x \in X\}.\end{aligned}$$

Note that both of these sets are subspaces of their respective spaces, and that $\ker(A) \subseteq X$ is in particular closed in the topological sense since A is continuous as a mapping.

Definition 3.2.3. Let $A \in \mathscr{L}(X,Y)$. If A is bijective as a mapping with $A^{-1} \in \mathscr{L}(Y,X)$ we say that A is a *homeomorphism* of X and Y.

If a map $f\colon M \to N$ between metric spaces M and N maps open sets in M to open sets in N, then we call f an *open mapping.*

Theorem 3.2.1 (Open Mapping Theorem)**.**
Let X and Y be Banach spaces, and let $A \in \mathscr{L}(X, Y)$ be a surjective linear operator. Then $A\colon X \to Y$ is an open mapping.

Proof. A proof, relying in large part on Baire's category theorem, is presented in Rudin [49]. □

An immediate corollary from the open mapping theorem is a statement on homeomorphisms.

Corollary 3.2.1 (Homeomorphism–bijection equivalence)**.**
Let X and Y be Banach spaces, and let $A \in \mathscr{L}(X, Y)$ be bijective. Then A is a homeomorphism of the spaces X and Y.

Proof. For linear operators we have an equivalence between continuity and boundedness. From the Open Mapping Theorem we know that A with the assumptions of the Corollary must be an open mapping, and since A is bijective we know that the inverse $A^{-1}\colon Y \to X$ exists. Clearly, A^{-1} needs to be linear as A is linear. It remains to show that A^{-1} is bounded. For linear operators we have equivalence between continuity and boundedness, so we prove that A^{-1} is continuous - pre-images of open sets V in Y under A^{-1} are open in X. Without loss of generality we can assume $V \subseteq Y$ has the form $V = \{y \in Y \mid \|y\|_Y < \delta\}$ for some $\delta > 0$. Let $y \in V$ and consider $A^{-1}y = x$ for some $x \in X$. Then $\|y\|_Y = \|Ax\|_Y < \delta$ which implies $\|A\| \, \|x\|_X < \delta$, so then

$$\left\|A^{-1}y\right\|_X = \|x\|_X < \frac{\delta}{\|A\|_{\mathscr{L}(X,Y)}}$$

holds for all $y \in V$, so $U = \{x \in X \mid \|x\|_X < \delta / \|A\|\}$ is an open set and A^{-1} is therefore continuous. This finishes the proof. □

Unfortunately, it is not the case that the norm of an inverse operator A^{-1} is the reciprocal of the norm of the operator A in general. Assume A is bijective, so we know that for any $y \in Y$ there exists $x \in X$ such that $Ax = y$, or $A^{-1}y = x$. Using the operator norm as in Equation (3.2.4)

$$\left\|A^{-1}\right\|_{\mathscr{L}(Y,X)} = \sup_{\substack{y \in Y \\ \|y\|=1}} \left\|A^{-1}y\right\|_X = \sup_{\substack{Ax \in Y \\ \|Ax\|=1}} \|x\|_X$$

where in the final equality we note that $1 = \|Ax\|_Y \leq \|A\|_{\mathscr{L}(X,Y)} \|x\|_X$, which rearranged gives $\|x\|_X \geq 1/\|A\|_{\mathscr{L}(X,Y)}$ and hence

$$\left\|A^{-1}\right\|_{\mathscr{L}(Y,X)} \geq \|A\|^{-1}_{\mathscr{L}(X,Y)} .$$

Proposition 3.2.2 (Cauchy Summability Criterion)**.**
Let $(X, \|\cdot\|_X)$ be a Banach space. Then the series $\sum_{i=1}^{\infty} x_i$ converges if it is absolutely summable in norm

$$\left\|\sum_{i=1}^{\infty} x_i\right\|_X \leq \sum_{i=1}^{\infty} \|x_i\|_X < \infty.$$

Proof. We give a brief sketch of the proof. The sequence of partial sums $\sum_{i=1}^{N} x_i$ can be shown to be a Cauchy sequence in X, and therefore ought to have a limit. Then one employs a standard argument to show that the infinite series is the right candidate for this limit. □

Proposition 3.2.3. *(Von Neumann Series)*
Let $A \in \mathscr{L}(X,X)$ with $\|A\|_{\mathscr{L}(X,X)} < 1$. Then the k-fold composition $A^k = A \circ A \circ \cdots \circ A$ satisfies the von Neumann series identity

$$\sum_{k=0}^{\infty} A^k = (I - A)^{-1}$$

where $I \in \mathscr{L}(X,X)$ is the identity operator.

Proof. This result follows from the Cauchy Summability Criterion since $\sum_{k=0}^{N} \left\|A^k\right\|_{\mathscr{L}(X,X)}$ is geometric as a sequence and therefore converges as $N \to \infty$. Furthermore, the actual identity follows from the fact that $A^0 = I$ and

$$(I - A)\left(\sum_{k=0}^{\infty} A^k\right) = \left(\sum_{k=0}^{\infty} A^k\right)(I - A) = I.$$

□

Definition 3.2.4. Let X be a Banach space. We call an operator $P \in \mathscr{L}(X,X)$ a *projection operator on X* if it is idempotent:

$$P(Px) = Px \quad \text{for all } x \in X. \tag{3.2.5}$$

Projections will be pivotal in our analysis, and we briefly study their general properties here. First note that if P is a projection, then $I - P$ is also a projection.

Proposition 3.2.4. *Let X be a Banach space and $P \in \mathscr{L}(X,X)$ be a projection. Then* $\ker(P)$ *and* $\operatorname{ran}(P)$ *are both closed,* $\ker(P) \oplus \operatorname{ran}(P) = X$ *and*

$$\ker(P) = \operatorname{ran}(I - P), \quad \ker(I - P) = \operatorname{ran}(P).$$

Proof. The most important point here is that $\operatorname{ran}(P)$ is closed, as $\ker(P)$ will be closed for any $P \in \mathscr{L}(X,X)$. This readily follows from the fact that $I - P$ is a projection, and hence $\ker(I - P) = \operatorname{ran}(P)$ is closed. □

Lemma 3.2.1. *Assume $X' \subseteq X$ is a finite dimensional subspace of a Banach space X. Then there exists a projection $P \in \mathscr{L}(X,X)$ such that* $\operatorname{ran}(P) = X'$.

Proof. A proof of this fact, relying mainly on the Hahn–Banach theorem, can be found in the proof of Lemma 4.21 in Rudin [49]. □

Definition 3.2.5. (Compact operators, [17, Definition 16.1]).
A bounded operator $K \colon X \to Y$ is said to be compact if K maps bounded sets in X to precompact sets in Y. Equivalently, all bounded sequences in X admit a convergent subsequence in Y when mapped by K.

For Banach spaces X, Y we denote the collection of compact linear operators $K\colon X \to Y$ by $\mathscr{K}(X,Y)$, which when equipped with the operator norm is a closed subspace of $\mathscr{L}(X,Y)$ [17].

Lemma 3.2.2. *Let X, Y be Banach spaces over $\mathbb{F}$. If there is a sequence $\{K_n\} \subset \mathscr{K}(X,Y)$ all with finite-dimensional ranges (finite rank) and an operator $K \in \mathscr{L}(X,Y)$ such that $\|K_n - K\|_{\mathscr{L}(X,Y)} \to 0$ as $n \to \infty$, then K is a compact operator.*

Recall from linear algebra that for a vector space V with subspace $W \subseteq V$, the concept of a *codimension* is defined by

$$\operatorname{codim}(W) = \operatorname{codim}_V(W) = \dim(V) - \dim(W)$$

and holds whenever V and W are both finite-dimensional. To generalize this to an infinite-dimensional setting we look at the *quotient space* V/W and set the codimension as the dimension of this quotient space. In any case we are looking at the *algebraic* dimension of these spaces, as opposed to the *geometric* dimension – see Appendix C of Haroske–Triebel [30] for the differences of both in the context of infinite dimensional spaces.

Definition 3.2.6 (Fredholm operator)**.**
We say an operator $A \in \mathscr{L}(X,Y)$, where X, Y are both Banach spaces over the field $\mathbb{F}$, is a *Fredholm operator of index p* if

(i) $\dim(\ker(A)) = n < \infty$,

(ii) $\operatorname{ran}(A)$ is closed in Y with $\operatorname{codim}(\operatorname{ran}(A)) = r < \infty$,

(iii) the index p satisfies $p = n - r$.

Remark. Note that from the preceding definition that any linear homeomorphism $A\colon X \to Y$ of Banach spaces is a Fredholm operator of index zero.

It is clear that one should be able to tie Fredholm operators to compact operators given the 'finiteness' conditions as per the definition of a Fredholm operator. One way of doing so is encoded in the Fredholm alternative.

Theorem 3.2.2 (Fredholm alternative, [13, Theorem 2.7.5])**.**
Let X be a Banach space and let $K \in \mathscr{K}(X,X)$ be a compact operator. Then both $I - K \in \mathscr{L}(X,X)$ and $I - K^* \in \mathscr{L}(X^*, X^*)$ are Fredholm of index zero. Furthermore, these satisfy

$$\dim\ker(I-K) = \operatorname{codim}\operatorname{ran}(I-K) = \dim\ker(I-K^*) = \operatorname{codim}\operatorname{ran}(I-K^*) \quad (3.2.6)$$

which in a finite-dimensional setting reduces to the rank-nullity theorem. Here, X^* denotes the continuous linear dual of X and K^* denotes the adjoint. We will not be needing the adjoint, however details on adjoint mappings can be found in Brezis [8].

Proof. The statement above is from Buffoni–Toland [13], however the authors do not provide a proof of the Fredholm alternative. A slightly different albeit equivalent version of the Fredholm alternative stated with proof can be found in Rudin [49]. □

Yet another important result from Buffoni–Toland is a sufficient criterion for operators of index zero when operators map between different spaces X and Y.

Theorem 3.2.3 ([13, Theorem 2.7.6])**.**
Let X, Y be Banach spaces. Assume that $K \in \mathscr{K}(X, Y)$ is a compact operator, $T \in \mathscr{L}(X, Y)$ is a homeomorphism of Banach spaces. Then the operator $S = K + T$ is Fredholm of index zero.

Proof. Note that $S = T + K = T(I + T^{-1}K)$ and hence S is Fredholm index zero if and only if $I + T^{-1}K \in \mathscr{L}(X, X)$ is Fredholm index zero due to the boundedness of T (and conversely T^{-1}). The rest follows from the Fredholm alternative since $T^{-1}K$ is compact due to T^{-1} being bounded, and we are done. □

3.3 Banach space calculus

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Consider a mapping $F\colon X \to Y$. We say that F is continuous at x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\|x - x_0\|_X < \delta$ we have $\|F(x) - F(x_0)\|_Y < \varepsilon$. If F is continuous at every $x_0 \in X$, then we say F is continuous and we denote $F \in C(X, Y) = C^0(X, Y)$. Continuity is a local property, meaning one may in particular consider $F\colon U \to Y$ on open sets $U \subseteq X$. In what follows we will consider the local situation.

We turn to the Banach space analogue of the derivative.

Definition 3.3.1 (Fréchet derivative)**.**
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces, $U \subseteq X$ be an open set, and $F\colon U \to Y$ a map. The map F is called *Fréchet differentiable* at $x_0 \in U$ if there exists a bounded linear operator $A \in \mathscr{L}(X, Y)$ such that

$$\lim_{h \to 0} \frac{\|F(x_0 + h) - F(x_0) - Ah\|_Y}{\|h\|_X} = 0.$$

If the above holds we call A the *Fréchet derivative* of F at x_0 and we denote $A = \mathrm{d}F[x_0]$. Assuming $\mathrm{d}F[x]$ exists for every $x \in U$ we define the map $\mathrm{d}F\colon U \to \mathscr{L}(X, Y)$; $x \mapsto \mathrm{d}F[x]$ and say that F is Fréchet differentiable on U.

Definition 3.3.2 (Partial Fréchet derivative)**.**
Let X,Y and Z be Banach spaces, $W \subseteq X \times Y$ be an open set, $F\colon W \to Z$ a map, then consider $(x_0, y_0) \in W$ with $x_0 \in U$, $y_0 \in V$ such that $U \times V \subseteq W$, U open in X, V open in Y. We say $F(\cdot, y_0)$ has a partial Fréchet derivative with respect to x at x_0 denoted by $\partial_x F[(x_0, y_0)] \in \mathscr{L}(X, Z)$ if $\mathrm{d}F(x_0, y_0)$ exists, in which case we set $\partial_x F[(x_0, y_0)] = \mathrm{d}F(x_0, y_0)$. This partial derivative extends to a map $\partial_x F[(\cdot, y_0)]\colon U \to \mathscr{L}(X, Z)$ if $F(\cdot, y_0)$ has a partial derivative for all $x \in U$.
A similar definition holds for the y-variable, with x_0 kept fixed instead.

Remark. It can be checked that if the map $F\colon W \to Z$, $W \subseteq X \times Y$ open, is Fréchet differentiable, then both partial derivatives exist. Partial derivatives with respect to more variables are defined inductively from the definition as stated above.

In order to define higher order Fréchet derivatives of $F\colon U \subseteq X \to Y$ inductively like

$$\mathrm{d}^k F[x](x_1, x_2, \ldots, x_k) = \mathrm{d}(\mathrm{d}^{k-1}F[x])(x_1, x_2, \ldots, x_k)$$

we require multilinearity with respect to its input tuple $(x_1, x_2, \ldots, x_k)$ and the existence of $\mathrm{d}^{k-1}F[x]$.

Definition 3.3.3 (Gateaux derivative)**.**
Let X and Y be Banach spaces, $U \subseteq X$ open, $F\colon U \to Y$ a map. The *Gateaux derivative* of F at $u \in U$ along the direction $v \in X$ is the map $\mathrm{d}_v F[u]$ given by

$$\mathrm{d}_v F[u] = \lim_{h \to 0} \frac{F(u + hv) - F(u)}{h} = \frac{\mathrm{d}}{\mathrm{d}h} F(u + hv)\bigg|_{h=0}. \tag{3.3.1}$$

Note that this map need not be linear *a priori*.

It turns out that whenever $F\colon U \subseteq X \to Y$ is Fréchet differentiable, it is also Gateaux differentiable and the maps $\mathrm{d}F[u]v = \mathrm{d}_v F[u]$ coincide for all $v \in X$. See e.g. Driver [17] for a proof. Obviously in that case the Gateaux derivative has to be linear.

Theorem 3.3.1 (Implicit function theorem)**.**
Let X, Y, Z be Banach spaces and let $U \subset X \times Y$ be an open set. Assume $F \in C^k(U, Z)$ for some $k \in \mathbb{N}$, and let $(x_0, y_0) \in U$ be a distinguished pair of points for which $F(x_0, y_0) = z_0$ and $\partial_x F[(x_0, y_0)] \in \mathscr{L}(X, Z)$ is a homeomorphism.

Then there exists an open ball $B(\delta, y_0) \subset Y$, a connected open set $\tilde{U} \subset U$ and a function $\phi \in C^k(B(\delta, y_0), X)$ such that

$$(x_0, y_0) \in \tilde{U} \text{ along with } F^{-1}(z_0) \cap \tilde{U} = \{(\phi(y), y) \mid y \in B(\delta, y_0)\}.$$

If F is analytic, then the above result holds with ϕ also analytic.

3.4 Local bifurcation theory

One established global bifurcation theory as an extension of local bifurcation theory through results which make these extensions possible. Local bifurcation theory treats bifurcation equations $F(\lambda, x) = 0$ only in the local sense. First let us discuss what we mean by a *bifurcation point.* In the equation $F(\lambda, x) = 0$ for $(\lambda, x) \in \mathbb{F} \times X$ we call (λ_0, x_0) a bifurcation point if $F\colon \mathbb{F} \times X \to Y$ is continuously differentiable at $(\lambda_0, 0)$ and $\partial_x F[(\lambda_0, x_0)]\colon X \to Y$ is not a homeomorphism – which in this setting simply means not bijective by Corollary 3.2.1. A necessary condition for this is that given F is Fredholm of index zero, where the kernel of $\partial_x F[(\lambda_0, x_0)]$ is nontrivial.

Theorem 3.4.1 (Lyapunov–Schmidt reduction, [13, Theorem 8.2.1])**.**
Let X and Y be Banach spaces, U an open set of $\mathbb{F} \times X$, $F \in C^k(U, Y)$ for some $k \in \mathbb{N}$. Suppose that $(\lambda_0, x_0) \in U$ satisfies $F(\lambda_0, x_0) = 0$ and furthermore that $\partial_x F[(\lambda_0, x_0)]\colon X \to Y$ is a Fredholm operator such that $\ker(\partial_x F[(\lambda_0, x_0)])$ is nontrivial and with range satisfying $\operatorname{codim} \operatorname{ran}(\partial_x F[(\lambda_0, x_0)]) = r$ in Y.

Then there exists open sets $\tilde{U} \subset U$ *and* $V \subset \mathbb{F} \times \ker(\partial_x F[(\lambda_0, x_0)])$ *along with functions* $\psi \in C^k(V, X)$ *with* $\psi(\lambda_0, 0) = x_0$, $h \in C^k(V, \mathbb{F}^r)$ *such that we have* $(\lambda_0, x_0) \in \tilde{U}$ *and* $(\lambda_0, 0) \in V$*, and moreover the functions* ψ, h *satisfy*

$$F(x, \lambda) = 0 \text{ for } (\lambda, x) \in \tilde{U} \text{ iff } \psi(\lambda, \xi) = x \text{ for some } (\lambda, \xi) \text{ satisfying } h(\lambda, \xi) = 0.$$

Proof. Our assumption that $J = \partial_x F[(\lambda_0, x_0)]$ is Fredholm means that there exists a finite-dimensional subspace $Z \subset Y$ complement to $\operatorname{ran}(J)$ with $\dim(Z) = r$ such that $Y = Z \oplus \operatorname{ran}(J)$. Similarly, there exists a complemented subspace W to $\ker(J)$ such that $X = \ker(J) \oplus W$. Define a bounded projection $P \in \mathscr{L}(Y, Y)$ with $\operatorname{ran}(P) = Z$. By Proposition 3.2.4 we have $\ker(P) = \operatorname{ran}(J)$. Also, $I - P\colon \operatorname{ran}(J) \to \operatorname{ran}(J)$ is a projection. By the decomposition $X = \ker(J) \oplus W$ we see that the operator $(I - P)J\colon W \to \operatorname{ran}(J)$ is a homeomorphism.

Going further we introduce the function $G\colon \mathbb{F} \times \ker(J) \times W \to \operatorname{ran}(J)$ given by

$$G(\lambda, \xi, \eta) = (I - P)F(\lambda, x_0 + \xi + \eta).$$

Observe that with the datum $(\lambda, \xi, \eta) = (\lambda_0, 0, 0)$ inserted into G and its derivative with respect to η we obtain

$$\begin{aligned} G(\lambda_0, 0, 0) &= (I - P)F(\lambda_0, x_0) = 0 \\ \partial_\eta G[(\lambda_0, 0, 0)]\eta &= (I - P)\partial_x F[(\lambda_0, x_0)]\eta = (I - P)J\eta \end{aligned}$$

which implies that $\partial_\eta G[(\lambda_0, 0, 0)]\colon W \to \operatorname{ran}(J)$ is a homeomorphism, which by the Implicit Function Theorem 3.3.1 there exists open sets $\tilde{U} \subset U$, $V \subset \mathbb{F} \times \ker(J)$ and a function $\phi \in C^k(V, W)$ such that $(\lambda_0, 0) \in V$ with $\phi(\lambda_0, 0) = 0$, $(\lambda_0, x_0) \in \tilde{U}$ and $G(\lambda, \xi, \phi(\lambda, \xi)) = 0$ whenever $(\lambda, \xi) \in V$. Moreover, this implies that

$$\begin{aligned} &\{(\lambda, x_0 + \xi + \eta) \in \tilde{U} \mid (I - P)F(\lambda, x_0 + \xi + \eta) = 0\} \\ &= \{(\lambda, x_0 + \xi + \eta) \mid (\lambda, \xi) \in V \text{ and } \eta = \phi(\lambda, \xi)\} \end{aligned}$$

hence we put $\psi(\lambda, \xi) = x_0 + \xi + \phi(\lambda, \xi)$ and $h(\lambda, \xi) = PF(\lambda, \psi(\lambda, \xi))$. Furthermore, for all $(\lambda, \xi) \in V$ we have that $h(\lambda, \xi) = 0$ if and only if $PF(\lambda, x_0 + \xi + \phi(\lambda, \xi)) = 0$ which happens if and only if $F(\lambda, x_0 + \xi + \phi(\lambda, \xi)) = 0$ because $(I - P)F(\lambda, x_0 + \xi + \phi(\lambda, \xi)) = 0$. Since $\operatorname{codimran}(J) = r$ we have $Z \cong \mathbb{F}^r$. □

Theorem 3.4.2 (Crandall–Rabinowitz transversality theorem, [13, Theorem 8.3.1])**.** *Let* X, Y *be Banach spaces over* $\mathbb{F}$*, let* $F \in C^k(\mathbb{F} \times X, Y)$ *with* $k \geq 2$ *satisfy* $F(\lambda, 0) = 0$ *in* Y *for all* $\lambda \in \mathbb{F}$*. Suppose furthermore that*

(i) $\partial_x F[(\lambda_0, 0)]$ *is a Fredholm operator of index zero,*

(ii) $\ker(\partial_x F[(\lambda_0, 0)])$ *is one-dimensional and furthermore is given by*

$$\ker(\partial_x F[(\lambda_0, 0)]) = \{\xi \in X \mid \xi = s\xi_0 \text{ for some } s \in \mathbb{F}\}$$

for a given $0 \neq \xi_0 \in X$,

(iii) the transversality condition holds:

$$\partial^2_{\lambda,x} F[(\lambda_0, 0)](1, \xi_0) \notin \operatorname{ran}(\partial_x F[(\lambda_0, 0)]).$$

Given (i)-(iii), then $(\lambda_0, 0)$ *is a bifurcation point. There exists* $\varepsilon > 0$ *and a branch of solutions to* $F(\lambda, x) = 0$ *given by*

$$\{(\lambda, x) = (\Lambda(s), s\,\chi(s)) \mid s \in \mathbb{F}, |s| < \varepsilon\} \subset \mathbb{F} \times X \tag{3.4.1}$$

such that $\Lambda(0) = \lambda_0$, $\chi(0) = \xi_0$, Λ *and* $\kappa(s) = s\chi(s)$ *are both functions of class* C^{k-1} *on* $(-\varepsilon, \varepsilon)$. *In addition to this branch of solutions, there exists an open set* $\tilde{U} \subset \mathbb{F} \times X$ *such that* $(\lambda_0, 0) \in \tilde{U}$ *and*

$$\{(\lambda, x) \in \tilde{U} \mid F(\lambda, x) = 0, x \neq 0\} = \{\Lambda(s), s\,\chi(s)) \mid 0 < |s| < \varepsilon\}.$$

If F *is analytic, then both* Λ *and* $\kappa(s) = s\chi(s)$ *are analytic on* $(-\varepsilon, \varepsilon)$.

The following theorem turns out to be very useful in the event of local bifurcation on a trivial solution curve in the bifurcation space. First we define the concept of a simple eigenvalue, and also state a result connected to that of eigenvectors of simple eigenvalues.

Definition 3.4.1 ([13, Definition 2.7.8])**.** Let $\iota\colon X \to Y$ be the continuous embedding of X into Y for Banach spaces X and Y. Let $\lambda_0 \in \mathbb{F}$ and $A \in \mathscr{L}(X, Y)$ such that $\lambda_0\iota - A$ is Fredholm index zero with $\ker(\lambda_0\iota - A)$ one-dimensional over $\mathbb{F}$ such that $(\lambda_0\iota - A) \cap \iota(\ker(\lambda_0\iota - A)) = \{0\}$. Then we call λ_0 a *simple eigenvalue* of A. An element $\xi_0 \in X \setminus \{0\}$ is called an eigenvector corresponding to the eigenvalue λ_0 if $A\xi_0 = \lambda_0\iota\xi_0$.

Proposition 3.4.1 ([13, Lemma 2.7.9])**.** *If* λ_0 *is a simple eigenvalue of* A *with eigenvector* ξ_0, *then if*

$$Y_1 = \operatorname{ran}(\lambda_0\iota - A), \quad X_1 = \iota^{-1}(Y_1)$$

then $X = X_1 \oplus \operatorname{span}(\xi_0)$ *and furthermore* $\lambda_0\iota - A$ *is a homeomorphism from* X_1 *to* Y_1.

Theorem 3.4.3 (Bifurcation from a simple eigenvalue, [13, Theorem 8.4.1])**.** *Suppose that the space* X *is continuously embedded in* Y *where* X *and* Y *are real Banach spaces, and assume that* $F(\lambda, 0) = 0$ *for all* $\lambda \in \mathbb{R}$ *with* $(\lambda, 0) \subset U \subset \mathbb{R} \times X$ *for some* U *open. Assume also that* $F \in C^k(U, Y)$ *with* $k \geq 2$, *and* $\partial_x F[(\lambda, 0)] = \lambda\iota - A$. *Then every simple eigenvalue* λ_0 *of* A *is a bifurcation point, and the conclusion of the Crandall–Rabinowitz theorem 3.4.2 holds.*

3.5 Global bifurcation of real-valued analytic functions

Setup.
In what follows we will assume that X, Y are real Banach spaces, $U \subset \mathbb{R} \times X$ is an open set, $F\colon U \to Y$ is real analytic. Furthermore, assume that $F(\lambda, 0) = 0$ and $(\lambda, 0) \in U$ for all $\lambda \in \mathbb{R}$, so $(\lambda, 0)$ is a solution curve in U. Also, whenever

$F(\lambda, x) = 0$ for $(\lambda, x) \in U$ we impose that $\partial_x F[(\lambda, x)]$ is Fredholm of index zero. Finally, we need that $(\lambda_0, 0)$ is a bifurcation point for some $\lambda_0 \in \mathbb{R}$ and that

$$\ker(\partial_x F[(\lambda_0, 0)]) = \{s\,\xi_0 \mid s \in \mathbb{R}\}$$
$$\partial^2_{\lambda,x} F[(\lambda_0, 0)](1, \xi_0) \notin \operatorname{ran}(\partial_x F[(\lambda_0, 0)])$$

By the Crandall–Rabinowitz transversality Theorem 3.4.2, there exists an analytic pair of functions $(\Lambda, \kappa)\colon (-\varepsilon, \varepsilon) \to \mathbb{R} \times X$, where $\kappa(s) = s\chi(s)$ as in Theorem 3.4.2, such that $(\lambda, x) = (\Lambda(s), \kappa(s))$ parametrizes a solution curve $F(\lambda, x) = 0$ with $\Lambda(0) = \lambda_0$ and $\kappa'(0) = \xi_0$. We introduce the notation

$$\mathscr{R}^+ = \{(\Lambda(s), \kappa(s)) \mid s \in (0, \varepsilon)\} \qquad \text{(the positive-parametrized curve)}$$
$$S = \{(\lambda, x) \in U \mid F(\lambda, x) = 0\} \qquad \text{(the solution set contained in } U)$$
$$\mathscr{T} = \{(\lambda, x) \in S \mid x \neq 0\} \qquad \text{(the nontrivial set of solutions in } U)$$

which will be used in discussion to follow. Suppose for consistency's sake that $\varepsilon > 0$ is small enough so that $\kappa'(s) \neq 0$ for $s \in (-\varepsilon, \varepsilon)$, and additionally such that $\mathscr{R}^+ \subset \mathscr{T}$. With this setup, we are now ready to state a vital result from Buffoni–Toland [13], which gives a global extension of the pair (Λ, κ) from $(0, \varepsilon)$ to $(0, \infty)$ and relates their extension to solutions of $F(\lambda, x) = 0$.

Theorem 3.5.1 (Global Bifurcation – One Dimensional Branches, [13, Theorem 9.1.1])**.**
Assume the preceding setup holds, along with $\Lambda'(0) \not\equiv 0$ on $(-\varepsilon, \varepsilon)$ and S has every closed and bounded subset being compact in $\mathbb{R} \times X$. If these conditions hold then there is a continuous curve (also called a branch) $\mathfrak{R}$ extending $\mathscr{R}^+$ as follows

- *(a) (Extendability)*
  We have $\mathfrak{R} = \{(\Lambda(s), \kappa(s)) \mid s \in [0, \infty)\} \subset U$ with the pair $(\Lambda, \kappa)\colon [0, \infty) \to \mathbb{R} \times X$ being continuous.

- *(b) (Consistency in solvability)*
  The inclusions $\mathscr{R}^+ \subset \mathfrak{R} \subset S$ hold.

- *(c) (Consistency in nullity)*
  The set $\{s \geq 0 \mid \ker(\partial_x F[(\Lambda(s), \kappa(s))]) \neq \{0\}\}$ has no accumulation points.

- *(d) (Re-parametrization)*
  Every point in $\mathfrak{R}$ admits a local, analytic re-parametrization. For $0 < s^* \leq \varepsilon$ a local re-parametrization $(\Lambda(t), \kappa(t)) \in \mathfrak{R}$ for t close to s^* leads to a re-parametrization of $\mathscr{R}^+$, which obviously exists and is real analytic. More generally, for all $s^* \in (0, \infty)$ there exists a continuous and injective $\rho\colon (-1, 1) \to \mathbb{R}$ such that

  $$\rho(0) = s^*, \quad t \mapsto (\Lambda(\rho(t), \rho(t))), \;\; \text{for } t \in (-1, 1), \;\; \text{is analytic.}$$

  Additionally, Λ is injective locally around $s^* > 0$ in the sense that there exists $\varepsilon > 0$ such that Λ is injective on both $[s^*, s^* + \varepsilon]$ and $[s^* - \varepsilon, s^*]$.

(e) *(Asymptotics of branches)*
Let $\|(\lambda, x)\|_{\mathbb{R}\times X} = (|\lambda|^2 + \|x\|_X^2)^{1/2}$. *The branches take one of the following modes.*

(i) $\|(\Lambda(s), \kappa(s))\|_{\mathbb{R}\times X} \to \infty$ *as* $s \to \infty$.

(ii) $(\Lambda(s), \kappa(s))$ *approaches* ∂U *as* $s \to \infty$.

(iii) $\mathfrak{R}$ *is a closed loop in the sense that there exists* $T > 0$ *(and we consider the smallest such* T*) such that* $\mathfrak{R} = \{(\Lambda(s), \kappa(s)) \mid 0 \le s \le T\}$ *and* $(\Lambda(s+T), \kappa(s+T)) = (\Lambda(s), \kappa(s))$ *for all* $s \ge 0$.

(f) *(Periodicity implies closed loop)*
If there exists a pair $s_1 \neq s_2$ *such that*

$$(\Lambda(s_1), \kappa(s_1)) = (\Lambda(s_2), \kappa(s_2)) \text{ along with } \ker(\partial_x F[(\Lambda(s_1), \kappa(s_1))]) = \{0\}$$

then $|s_1 - s_2|$ *is an integer multiple of a period* $T > 0$ *and* $\mathfrak{R}$ *is a closed loop.*

3.6 Global bifurcation in a cone

The following section follows Chapter 9, Section 2 of Buffoni–Toland [13] closely. The aim is to eliminate the possibility of the extended global branch $\mathfrak{R}$ being a closed loop by imposing sufficient conditions on the bifurcation diagram.

Definition 3.6.1. Let X be a real Banach space. A closed set $\mathcal{K}$ is called a *(non-convex) cone* if $\gamma x \in \mathcal{K}$ for all $\gamma \ge 0$ and $x \in \mathcal{K}$.

Theorem 3.6.1 (Global Analytic Bifurcation in Cones, [13, Theorem 9.2.2])**.** *Assume the setup preceding Theorem 3.5.1 and the theorem's hypotheses are satisfied. Assume also that the following criteria hold*

(i) $\mathcal{K}$ *is a cone in a real Banach space* X.

(ii) $\mathscr{R}^+ \subset \mathbb{R} \times \mathcal{K}$ *provided* ε *is small enough.*

(iii) *If* $\lambda \in \mathbb{R}$ *and* $\hat{\xi} \in \ker(\partial_x F[(\lambda, 0)]) \cap \mathcal{K}$, *then* $\hat{\xi} = \alpha\xi_0$ *for some* $\alpha \ge 0$, *in which case we necessarily have* $\lambda = \lambda_0$. *Also,* $-\xi_0 \notin \mathcal{K}$.

(iv) *Each point of* $\mathfrak{R} \cap \mathcal{T} \cap (\mathbb{R} \times \mathcal{K})$ *is an interior point of* $\mathcal{T} \cap (\mathbb{R} \times \mathcal{K}) \subset S$.

Then $\kappa(s) \in \mathcal{K} \setminus \{0\}$ *for all* $s > 0$ *and* $\mathfrak{R}$ *cannot be a closed loop as in Theorem 3.5.1 Item (e)(iii).*

We prove Theorem 3.6.1 in the applied case we are considering in the final chapter. The original proof, though, will be quite similar.

3.7 Modes of bifurcation

This section briefly reviews some concepts and results from Kielhöfer's book [37] on bifurcation theory. We adapt the notation to suit that of Buffoni–Toland. In particular, we are interested in the material from Section I.6 of [37]. Proofs of the presented formulae in the following can be found in the aforementioned section, and details can be found in the preceding sections I.4 and I.5 of [37].

Assume $F\colon \mathbb{R} \times X \to Y$ is the bifurcation map in question and assume F is analytic (not necessary, but assumed for simplicity), and let Z be the complement of $\operatorname{ran}(\partial_x F[(\lambda_0, x_0)])$ and assume Z is spanned by ξ_0 with

$$Z = \operatorname{span}(\{\xi_0\}), \quad \|\xi_0\|_Y = 1.$$

Then by the Hahn–Banach theorem (c.f. Brezis [8]) there exists $\xi_0^* \in Z^*$, where Z^* is the dual of Z, such that $\langle \xi_0, \xi_0^* \rangle = 1$ and $\langle z, \xi_0^* \rangle = 0$ for all $z \in \operatorname{ran}(\partial_x F[(\lambda_0, x_0)])$. Here $\langle z, \cdot \rangle\colon Z^* \to \mathbb{R}$ denotes the action of elements in the dual. Define the projection $Q\colon Y \to Z$ by

$$Qy = \langle y, \xi_0^* \rangle \, \xi_0, \quad y \in Y$$

then we have, for $\Lambda(s)$ as in Crandall–Rabinowitz 3.4.2, with $x_0 = 0$

$$\dot{\Lambda}(0) = -\frac{1}{2} \frac{\langle \partial_{xx}^2 F[(\lambda_0, 0)](\zeta_0, \zeta_0), \xi_0^* \rangle}{\langle \partial_\lambda F[(\lambda_0, 0)], \xi_0^* \rangle} \tag{3.7.1}$$

where ζ_0 spans the kernel of $\partial_x F[(\lambda_0, 0)]$ with $\|\zeta_0\|_X = 1$. This formula can be used to deduce the local behaviour around points for which $\dot{\Lambda}(0) \neq 0$. In the event that $\dot{\Lambda}(0) = 0$ we have to turn to analysis of the second derivative, which turns out to take the form

$$\ddot{\Lambda}(0) = -\frac{1}{3} \frac{\langle \partial_{xxx}^3 \Phi[(\lambda, 0)](\zeta_0, \zeta_0, \zeta_0), \xi_0^* \rangle}{\langle \partial_{x\lambda}^2 F[(\lambda, 0)]\zeta_0, \xi_0^* \rangle} \tag{3.7.2}$$

where $\Phi(\lambda, x) = QF(\lambda, x + \psi(\lambda, x))$ for some smooth ψ as given by the Lyapunov–Schmidt reduction, and Q is the projection as above.

Remark. In the setting of Hilbert spaces, and in the event one could continuously embed Y or $Z \subset Y$ into a Hilbert space, the actions of members of the dual spaces discussed above can be characterized through the inner product in the Hilbert space by the Riesz representation theorem (see Brezis [8]), in which case we could, practically speaking, replace ξ_0^* as above with $c\xi_0$ for some $c \in \mathbb{R}$. In the formulae these constants c drop out.

Definition 3.7.1. If $\dot{\Lambda}(0) \neq 0$ as above, then we say that the bifurcation is *transcritical.* If $\dot{\Lambda}(0) = 0$, then the bifurcation mode $\ddot{\Lambda}(0) > 0$ is called *supercritical* and if $\ddot{\Lambda}(0) < 0$ it is called *subcritical.*

Pictures of the above bifurcation modes are included at the end of Section I.6 of Kielhöfer [37].

Chapter 4

Global bifurcation of a nonlocal equation

Consider the following equation in its most general form

$$u_t + Lu_x + N(u,u)_x = 0 \tag{4.0.1}$$

where the operator L is a Fourier multiplier and N is a bilinear form yet to be determined. First of all, we impose that our solutions are *travelling solutions* of the form

$$u(t,x) = \varphi(x - \mu t)$$

for some function φ and a parameter $\mu \geq 0$. We regard the parameter μ as a wave speed, as is convention in dispersive equations, such as e.g. the Whitham equation, Korteweg–de Vrie equation, see for instance [23] where indeed μ would be a wave speed in the physical sense, and therefore we keep this parameter non-negative for the remainder of this chapter. In addition we will be looking for even solutions φ, which does not delimit the generality of our potential solutions by much given that dispersive water-wave equations usually feature that symmetric waves are travelling (see [19]), and in certain cases one has equivalence between symmetric solutions and periodic travelling wave solutions under mild criteria on the wave profile (see e.g. [10] for the Whitham case).

Inserting the travelling wave ansatz into Equation (4.0.1) the first term transforms as $\varphi_t \to -\mu\varphi_x$ and thus

$$-\mu\varphi_x + L\varphi_x + N(\varphi,\varphi)_x = 0 \tag{4.0.2}$$

where furthermore we integrate with respect to x, normalize the integration constant to obtain our integrated version of Equation (4.0.1)

$$-\mu\varphi + L\varphi + N(\varphi,\varphi) = B. \tag{4.0.3}$$

In the language of bifurcation theory and in keeping our conventions from the previous chapter we note that Equation (4.0.3) can be rewritten as

$$F(\mu,\varphi) = -\mu\varphi + L\varphi + N(\varphi,\varphi) - B = 0, \tag{4.0.4}$$

which will become the basis of our bifurcation problem.

4.1 Local bifurcation analysis

Our local bifurcation analysis will lay the groundwork for the global bifurcation considered later in the chapter, while also forcing us to explore what kind of theoretical limitations and boundaries that may arise from the local bifurcation analysis alone.

We define the Fourier multiplier of *homogeneous Bessel type of order s* by the action

$$\Lambda^s \varphi = \mathscr{F}^{-1}\{(1+\xi^2)^{s/2}\,\hat{\varphi}(\xi)\} = \mathscr{F}^{-1}\{\langle\xi\rangle^s\,\hat{\varphi}(\xi)\}$$

for $s \in \mathbb{R}$ – extended by duality onto the tempered distributions $\Lambda^s \colon \mathscr{S}' \to \mathscr{S}'$ if need be. Note for instance the case $\Lambda^2\varphi = (1-\partial_x^2)\varphi$ with $s = 2$, where if $\varphi \in C^k(\mathbb{R})$, for $k \geq 2$, the operator Λ^2 explicitly lowers the regularity of the function φ to that of class C^{k-2}. Similarly, if $s > 0$ we see from Theorem 2.4.2 that Λ^s acting on $\varphi \in B^t_{p,q}$ lowers the regularity in the scale of Besov–Lipschitz spaces.

For our equation we wish to consider, define L as the symbol acting as

$$L\varphi = \Lambda^s\varphi$$

for a parameter $s \in \mathbb{R}$, and the symbol N as

$$N(\varphi, \varphi) = \varphi\,\Lambda^r\varphi$$

for yet another parameter $r \in \mathbb{R}$, not necessarily related to the parameter $s \in \mathbb{R}$ *prima facie*. Substituting in these for Equation (4.0.4) we obtain the equation in steady variables

$$-\mu\varphi + \Lambda^s\varphi + \varphi\,\Lambda^r\varphi = B. \tag{4.1.1}$$

As in the introductory chapter we wish to perform the Galilean transformation

$$\varphi \mapsto \varphi + \gamma, \quad \mu \mapsto \mu + 2\gamma, \quad B \mapsto B + \gamma(1-\mu-\gamma)$$

which when applied to a solution pair (μ, φ) amounts to

$$\begin{aligned}
&-(\mu+2\gamma)(\varphi+\gamma) + \Lambda^s(\varphi+\gamma) + (\varphi+\gamma)\Lambda^r(\varphi+\gamma)\\
&= -(\mu+2\gamma-\gamma)\varphi + \Lambda^s\varphi + \varphi\,\Lambda^r\varphi - (\mu\gamma + 2\gamma^2 - \gamma - \gamma^2) + \gamma\Lambda^r\varphi\\
&= -\mu\varphi + \Lambda^s\varphi + \varphi\,\Lambda^r\varphi - (\mu+\gamma-1)\gamma + \gamma\Lambda^r\varphi - \gamma\varphi\\
&= (1+\mu-\gamma)\gamma + \gamma\Lambda^r\varphi - \gamma\varphi.
\end{aligned}$$

Immediately we see that this fails to resolve to a constant unless $r = 0$, hence this Galilean transform only works for the fractional Korteweg–de Vrie equation where indeed $r = 0$. Hence we will artificially set $B = 0$ in an *ad hoc* fashion. In other words, our problem now looks like

$$F(\mu, \varphi) = -\mu\varphi + \Lambda^s\varphi + \varphi\,\Lambda^r\varphi = 0 \tag{4.1.2}$$

clarifying the assumed form of the map $F \colon \mathbb{R} \times X \to Y$ in the context of performing local and global bifurcation analysis, from which it will now be understood that we will treat $\mu \geq 0$ as our bifurcation parameter. We want to examine local bifurcation and global bifurcation over the space $X = \mathscr{C}^t_{\text{even}}(\mathbb{S}_P)$, the P-periodic, even functions in the Hölder–Zygmund space of index $t \geq 0$.

Remark. In the extremal case $r = 0$ with $s < 0$ we recover the fKdV equation as studied by e.g. Ørke [46]. However, if we also set $s = 0$ along with $r = 0$ we actually recover Burgers' equation from the theory of hyperbolic equations, which is a classic example of an equation that is solvable via methods of characteristics – see the first chapter of Holden–Risebro [32] for details. We do not cover either of these cases in this thesis, however we will be making comments in passing on comparisons with the $r = 0$ Whitham/fKdV cases.

The space Y as for the map $F\colon \mathbb{R} \times X \to Y$ in Equation (4.1.2) can be chosen to be $Y = \mathscr{C}^t_{\text{even}}(\mathbb{S}_P)$ since if $\varphi \in \mathscr{C}^t_{\text{even}}(\mathbb{S}_P)$ then all of the terms of the F-map also lie in $\mathscr{C}^t_{\text{even}}(\mathbb{S}_P)$. Indeed, due to the lifting theorem 2.4.2 and the Banach algebra property one can deduce that

$$\|\varphi\,\Lambda^r\varphi\|_{\mathscr{C}^t(\mathbb{S}_P)} \leq \|\varphi\|_{\mathscr{C}^t(\mathbb{S}_P)}\,\|\Lambda^r\varphi\|_{\mathscr{C}^t(\mathbb{S}_P)} = \|\varphi\|_{\mathscr{C}^t(\mathbb{S}_P)}\,\|\varphi\|_{\mathscr{C}^{t+r}(\mathbb{S}_P)} \lesssim \|\varphi\|^2_{\mathscr{C}^t(\mathbb{S}_P)}$$

where the final estimate is provided by the compact embedding from Theorem 2.6.9. Similarly, one proves that $\Lambda^s\varphi$ lies in $\mathscr{C}^t(\mathbb{S}_P)$, so our choice of Y makes sense in terms of regularity. For preservation of evenness under Λ^s and Λ^r when $r, s < 0$, refer to Lemma 4.2.4. Finally, the map $F\colon \mathbb{R} \times \mathscr{C}^t_{\text{even}}(\mathbb{S}_P) \to \mathscr{C}^t_{\text{even}}(\mathbb{S}_P)$ is real analytic in both arguments (μ, φ) in the sense of Buffoni–Toland [13, Section 4.3] since $\Lambda^s(\cdot)$ is analytic by a result owing to smoothing operators in the scale of Hölder–Zygmund spaces, see Grigis–Sjöstrand [29].

Proposition 4.1.1. *The kernel of the Fréchet derivative* $\partial_\varphi F[(\mu^*, 0)]$ *is one-dimensional for* $0 < \mu^* < 1$*, and furthermore is spanned by*

$$\varphi^* = \cos(2\pi \cdot /P).$$

Additionally, we have that the transversality condition holds

$$\partial^2_{\mu,\varphi}F[(\mu^*, 0)](1, \varphi^*) \notin \operatorname{ran}(\partial_\varphi F[(\mu^*, 0)]).$$

Proof. We compute by the Gateaux derivative that

$$\begin{aligned}
\partial_\varphi F[(\mu,\varphi)]\psi &= \frac{\mathrm{d}}{\mathrm{d}h} F(\mu, \varphi + h\psi)\bigg|_{h=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}h}\left(-\mu(\varphi + h\psi) + \Lambda^s(\varphi + h\psi) + (\varphi + h\psi)\,\Lambda^r(\varphi + h\psi)\right)\bigg|_{h=0} \\
&= -\mu\psi + \Lambda^s\psi + \psi\,\Lambda^r\varphi + \varphi\,\Lambda^r\psi + 2h\,\psi\,\Lambda^r\psi\bigg|_{h=0} \\
&= -\mu\psi + \Lambda^s\psi + \psi\,\Lambda^r\varphi + \varphi\,\Lambda^r\psi
\end{aligned}$$

which all-in-all culminates to

$$\partial_\varphi F[(\mu,\varphi)] = (\Lambda^r\varphi - \mu)\,\mathrm{Id} + \varphi\,\Lambda^r \circ \mathrm{Id} + \Lambda^s \tag{4.1.3}$$

where $\varphi\,\Lambda^r \circ \mathrm{Id}$ is to be understood as φ multiplied by $\Lambda^r(\cdot)$. Inserting $\varphi = 0$ with $\mu = \mu^*$ amounts to

$$\partial_\varphi F[(\mu^*, 0)]\psi = -\mu^*\psi + \Lambda^s\psi \tag{4.1.4}$$

which when set to zero for the kernel reduces to the eigenvalue equation

$$\Lambda^s \psi = \mu^* \psi.$$

Taking the Fourier transform of both sides and isolating on the left-hand side amounts to

$$((1 + |\xi|^2)^{s/2} - \mu^*)\hat{\psi} = 0 \tag{4.1.5}$$

where we take $\hat{\psi} \in \mathscr{S}'$ in the distributional sense if need be since we can technically work with $\psi \in L^\infty \subset B^0_{\infty,\infty}$. We note that $0 < \langle \xi \rangle^s \leq 1$, and hence unless $\text{supp}(\hat{\psi})$ is trivial, meaning $\hat{\psi} \equiv 0$, it is necessary that $0 < \mu^* \leq 1$. When $\mu^* = 1$ one has that $\xi = 0$ solves Equation (4.1.5), and hence $\psi \equiv D$ for some constant $D \in \mathbb{R}$ since in that case $\hat{\psi} = D\delta_0 \in \mathscr{S}'$. In the case $0 < \mu^* < 1$ we deduce that $\text{supp}(\hat{\psi}) = \{\pm\xi_0\}$ for some ξ_0 satisfying $(1 + |\xi_0|^2)^{s/2} = \mu^*$. In the spirit of P-periodic distributions, the function ψ has to satisfy

$$\psi = C \cos(\xi_0 \,\cdot)$$

for some constant $C \in \mathbb{R}$. This follows from Proposition 2.1.6 and using the form of P-periodic distributions as in Equation 2.6.5. Requiring that the solution is even amounts to ψ being a cosine. P-periodicity furthermore reduces $\xi_0 = 2\pi k/P$ for some $k \in \mathbb{Z} \setminus \{0\}$ and thus setting $C = 1$ for the sake of deriving a basis we get a (P, k)–dependent eigenvalue $\mu^* = \mu_{P,k}$. We usually take $k = 1$ as our choice for main bifurcation branch, and thus declare the main bifurcation direction to be given by, for $\mu^* = \mu_{P,1}$,

$$\varphi^* = \cos(2\pi \cdot /P) \tag{4.1.6}$$

which therefore implies that

$$\ker(\partial_\varphi F[(\mu^*, 0)]) = \text{span}(\varphi^*). \tag{4.1.7}$$

Alternatively, $\partial_\varphi F[(\mu^*, 0)]$ is Fredholm index zero by the Fredholm alternative 3.2.2 since Λ^s is compact by Theorem 2.6.1. It follows that $\mu_{P,k}$ as above are simple eigenvalues and thus by Proposition 3.4.1 we have that $\ker(\partial_\varphi F[(\mu_{P,k}, 0)])$ is spanned by, for instance, $\varphi = \cos(2\pi k \cdot /P)$ as an eigenvector.

For the last part, note that we have

$$\partial^2_{\mu,\varphi} F[(\mu, \varphi)](\lambda, \psi) = \frac{\mathrm{d}}{\mathrm{d}t} \partial_\varphi F[(\mu + \lambda t, \varphi)]\psi \bigg|_{t=0} = -\lambda\varphi$$

which when evaluated with $\lambda = 1$ and $\psi = \varphi^*$ with $\varphi = 0$, $\mu = \mu^*$ becomes

$$\partial^2_{\mu,\varphi} F[(\mu^*, 0)](1, \varphi^*) = -\varphi^*$$

which is not in the range of $\partial_\varphi F[(\mu^*, 0)] = -\mu^* \text{Id} + \Lambda^s$ by Proposition 3.4.1. □

Lemma 4.1.1 (Fredholm).
Let $r, s < 0$. The Fréchet derivative $\partial_\varphi F[(\mu, \varphi)]$ is a Fredholm operator of index zero when $\varphi \in \mathscr{C}^t_{\text{even}}(\mathbb{S}_P)$ is a solution of Equation (4.1.2) *and satisfies $\Lambda^r \varphi < \mu$.*

Proof. From previous calculations we have

$$\partial_\varphi F[(\mu,\varphi)] = (\Lambda^r\varphi - \mu)\,\mathrm{Id} + \varphi\,\Lambda^r \circ \mathrm{Id} + \Lambda^s$$

where we note that $(\Lambda^r - \mu)\,\mathrm{Id} \in \mathscr{L}(\mathscr{C}^t_{\text{even}}(\mathbb{S}_P))$ is an isomorphism due to Theorem 2.4.2 and both $\varphi\,\Lambda^r \circ \mathrm{Id}$ and Λ^s are compact as linear operators on $\mathscr{C}^t_{\text{even}}(\mathbb{S}_P)$ due to Theorem 2.6.9 and that since φ is bounded and if V is a bounded set in $\mathscr{C}^t_{\text{even}}(\mathbb{S}_P)$ then for a bounded sequence $\psi_n \in V$ there exists a convergent subsequence $(\Lambda^r\psi_{n_k})$ in Y with $\Lambda^r\psi_{n_k} \to \eta$ for some $\eta \in Y$ since Λ^r is a compact operator, and hence

$$\|\varphi\,\Lambda^r\psi_{n_k} - \varphi\,\eta\|_{\mathscr{C}^t(\mathbb{S}_P)} \leq \|\varphi\|_{\mathscr{C}^t(\mathbb{S}_P)}\,\|\Lambda^r\psi_{n_k} - \eta\|_{\mathscr{C}^t(\mathbb{S}_P)} \to 0$$

as $k \to \infty$ due to the Banach algebra property of Hölder–Zygmund spaces. Thus $\varphi\,\Lambda^r\psi_n$ has a convergent subsequence, and hence $\varphi\,\Lambda^r \circ \mathrm{Id}(V)$ is precompact in Y, so finally $\varphi\,\Lambda^r \circ \mathrm{Id}$ is compact. Theorem 3.2.3 then implies that $\partial_\varphi F[(\mu,\varphi)]$ is Fredholm with index zero. □

Remark. The proof presented in Ehrnström–Kalisch [21] makes use of the property that, when adapted to our case, the operator $\partial_\varphi F[(\mu,0)] = -\mu\,\mathrm{Id} + \Lambda^s$ is Fredholm index zero along the trivial curve $(\mu,0)$, $\mu > 0$, and that by mapping

$$\tau \mapsto (\tau\Lambda^r\varphi - \mu)\mathrm{Id} + \tau\varphi\,\Lambda^r \circ \mathrm{Id} + \Lambda^s \in C([0,1], \mathscr{L}(\mathscr{C}^t(\mathbb{S}_P)))$$

one can continuously map the Fredholm index along a homotopy curve in the bifurcation plane, thus concluding that $\partial_\varphi F[(\mu,\varphi)]$ necessarily has index zero. However, this is excessive since Theorem 3.2.3 readily guarantees that the Fredholm index is equal to zero with no extra work, see [13, Theorem 2.7.6].

Remark (Coifman–Meyer operator)*.* Note that taking the Fourier transform of the nonlinearity $\psi\,\Lambda^r\varphi$ explicitly gives

$$\mathscr{F}(\psi\,\Lambda^r\varphi)(\xi) = (2\pi)^{-1}(\hat{\psi} * \langle\xi\rangle^r\hat{\varphi})(\xi) = (2\pi)^{-1}\int_{\mathbb{R}} \langle\xi-\zeta\rangle^r\,\hat{\psi}(\zeta)\,\hat{\varphi}(\xi-\zeta)\,\mathrm{d}\zeta \quad (4.1.8)$$

which is therefore clearly a pseudoproduct, or Coifman–Meyer operator, as in Equation (2.8.1).

We rephrase the theorem of Lyapunov–Schmidt 3.4.1 in the context of our spaces, following the notation and spirit of Ehrnström–Kalisch [21]. To this end, let $\mu^* = \mu_1$ be the bifurcation point and let φ^* be the main bifurcation direction given by

$$\varphi^* = \cos(2\pi\cdot/P).$$

In the spirit of viewing the Wiener algebra [36] of series of cosines as a subalgebra of $\mathscr{C}^t_{\text{even}}(\mathbb{S}_P)$, we introduce as in [21] the subspaces

$$M := \left\{ \sum_{k\neq 1} a_k \cos(2\pi kx/P) \in \mathscr{C}^t_{\text{even}}(\mathbb{S}_P) \right\}$$

and the kernel subspace

$$N := \ker(\partial_\varphi F[(\mu^*,0)]) = \operatorname{span}(\varphi^*).$$

Then indeed we have $\mathscr{C}^t_{\text{even}}(\mathbb{S}_P) = M \oplus N$ and can use the canonical embedding $\mathscr{C}^t_{\text{even}}(\mathbb{S}_P) \hookrightarrow L^2(\mathbb{S}_P)$ to define the continuous projection

$$\Pi\varphi = \langle \varphi, \varphi^* \rangle_{L^2(\mathbb{S}_P)}\, \varphi^*,$$

where the inner product is given by $\langle u, v\rangle_{L^2(\mathbb{S}_P)} = \frac{2}{P}\int_0^P uv\,\mathrm{d}x$ in the finite period case.

Theorem 4.1.1 (Lyapunov–Schmidt, rephrased)**.**
There exists a neighborhood $Y \times \mathscr{O} \subset U$ around the bifurcation point $(\mu^*, 0)$ on which the bifurcation problem

$$F(\mu,\varphi) = 0$$

becomes exactly equivalent to

$$\Phi(\mu, \varepsilon\varphi^*) := \Pi\, F(\mu, \varepsilon\phi^* + \psi(\mu, \varepsilon\varphi^*)) = 0 \tag{4.1.9}$$

for functions $\psi \in C^\infty(Y \times \mathscr{O}_N, M)$, $\Phi \in C^\infty(Y \times \mathscr{O}_N, N)$, and some open neighborhood $\mathscr{O}_N \subseteq N$ around the zero function in N. Since Φ projects onto the kernel of $\partial_\varphi F[(\mu^*,0)]$ we have $\psi(\mu^*,0) = 0$ and $\partial_\varphi\psi[(\mu^*,0)] = 0$, also choosing $\varepsilon = 0$ one gets $\Phi(\mu^*,0) = 0$ as required. Solving Equation (4.1.9) *amounts to a solution $\varphi = \varepsilon\varphi^* + \psi(\mu,\varepsilon\varphi^*)$ of the bifurcation problem, which is now a finite dimensional problem.*

Proof. It follows from the definition of U that $F(\mu,\varphi) = 0$ whenever $(\mu,\varphi) \in U$, also $\ker \partial_\varphi F[(\mu,\varphi)]$ is nontrivial along with $\operatorname{codim}\operatorname{ran}(\partial_\varphi F[(\mu,\varphi)]) = 1$ since $\partial_\varphi F[(\mu,\varphi)])$ is Fredholm with index zero by Lemma 4.1.1.

The rest comes from rephrasing the language of the original Lyapunov–Schmidt reduction, Theorem 3.4.1, namely that the map h as in the statement is just the projection $\Pi F(\mu, \varepsilon\phi^* + \psi(\mu,\varepsilon\varphi^*))$ with the notation changed and ε introduced. Alternatively, one can find this version of the Lyapunov–Schmidt theorem in Kielhöfer [37]. □

4.2 Properties of Λ^s and the kernel K^s

The action of Λ^s on functions can be described through the Fourier multiplier $m(\xi) = \langle\xi\rangle^s$

$$\mathscr{F}(\Lambda^s\varphi)(\xi) = \langle\xi\rangle^s\, \hat{\varphi}(\xi)$$

but we need to find a periodization of this operator. Therefore we write the action of Λ^s as a convolution by way of the convolution theorem 2.2.2 for the inverse Fourier transform. If f is a function, let $f_\sigma(x) = f(-x)$ be the sign-reversed version of f. Then we have for the inverse version of Theorem 2.2.2

$$\mathscr{F}^{-1}(f_\sigma \cdot g_\sigma) = \mathscr{F}^{-1} f_\sigma * \mathscr{F}^{-1} g_\sigma \tag{4.2.1}$$

which follows immediately from the Convolution Theorem keeping in mind that the inverse Fourier transform satisfies $\mathscr{F} f = 2\pi \mathscr{F}^{-1} f_\sigma$. Now choose $g_\sigma = \hat{\varphi}$ and observe that

$$\mathscr{F}^{-1}(f_\sigma \cdot \hat{\varphi}) = \mathscr{F}^{-1} f_\sigma * \varphi.$$

Choose $f_\sigma = \langle \xi \rangle^s$ so that $f = \langle -\xi \rangle^s = \langle \xi \rangle^s$ and establish the convolutional form

$$\Lambda^s \varphi = K^s * \varphi$$

where the convolution kernel K^s is given by

$$K^s = \mathscr{F}^{-1} \langle \xi \rangle^s = (2\pi)^{-1} \mathscr{F} \langle \xi \rangle^s. \tag{4.2.2}$$

Given this, we are able to periodize K^s as

$$\begin{aligned} \Lambda^s \varphi(x) &= \int_{\mathbb{R}} K^s(x-y)\, \varphi(y)\, \mathrm{d}x = \sum_{n \in \mathbb{Z}} \int_{-P/2}^{P/2} K^s(x - y + nP)\, \varphi(y)\, \mathrm{d}x \\ &= \int_{-P/2}^{P/2} K_P^s(x-y)\, \varphi(y)\, \mathrm{d}x \end{aligned}$$

for any period $P > 0$. Notice here the abuse of notation regarding φ from the section on periodic functions on tori. This then allows us to characterize the action of Λ^s on P-periodic functions and functions on $\mathbb{R}$ similarly with the appropriate choice of convolution kernels K_P^s and K^s respectively.

Lemma 4.2.1. *For $r \leq s < 0$ the symbol $\Lambda^{(\cdot)}$ satisfies*

$$\Lambda^r \varphi = \Lambda^{r-s} \circ \Lambda^s \varphi = \Lambda^s \circ \Lambda^{r-s} \varphi$$

for all even functions $\varphi \in L^\infty(\mathbb{R})$.

Proof. Consider the formal expression for $\Lambda^r \varphi$ written like

$$\begin{aligned} \Lambda^r \varphi &= \int_{\mathbb{R}} \langle \xi \rangle^r \hat{\varphi}(\xi)\, e^{ix \cdot \xi}\, \mathrm{d}\xi \\ &= \int_{\mathbb{R}} \langle \xi \rangle^{r-s} \langle \xi \rangle^s \hat{\varphi}(\xi)\, e^{ix \cdot \xi}\, \mathrm{d}\xi \\ &= \mathscr{F}^{-1}(\langle \xi \rangle^{r-s} \langle \xi \rangle^s \hat{\varphi}(\xi)) \end{aligned}$$

regarding $\hat{\varphi}$ in a distributional sense if needed. By the inverse Fourier transform 4.2.1 (assuming we can apply this) and the observation that if φ is even and real-valued then $\hat{\varphi}$ is also even and real-valued:

$$\begin{aligned} \hat{\varphi}(-\xi) &= \int_{\mathbb{R}} \varphi(x)\, e^{-ix \cdot (-\xi)}\, \mathrm{d}x = \overline{\int_{\mathbb{R}} \varphi(x)\, e^{-ix \cdot \xi}\, \mathrm{d}x} = \overline{\hat{\varphi}(\xi)} \\ &= \int_{\mathbb{R}} \varphi(-x)\, e^{-i(-x) \cdot \xi}\, \mathrm{d}x = \int_{\mathbb{R}} \varphi(y)\, e^{-iy \cdot \xi}\, \mathrm{d}y = \hat{\varphi}(\xi) \end{aligned}$$

where the bar denotes complex conjugation, and furthermore leads us to conclude that

$$\mathscr{F}^{-1}(\langle\xi\rangle^{r-s}\cdot\langle\xi\rangle^{s}\,\hat{\varphi}(\xi))=\mathscr{F}^{-1}\langle\xi\rangle^{r-s}*\mathscr{F}^{-1}(\langle\xi\rangle^{s}\,\hat{\varphi}(\xi))=\mathscr{F}^{-1}\langle\xi\rangle^{s}*\mathscr{F}^{-1}(\langle\xi\rangle^{r-s}\,\hat{\varphi}(\xi)).$$

Using the convolutional interpretation this can be re-written as

$$\Lambda^r\varphi=K^{r-s}*\Lambda^s\varphi=K^s*\Lambda^{r-s}\varphi$$

which is equivalent with the statement we set out to prove, and we are done. The inverse Fourier transform only ensures commuting inverses for functions $\varphi\in L^1$ with $\hat{\varphi}\in L^1$ and vice versa, so to achieve the above in a fully distributional setting one has to simply note that for $\hat{\varphi}\in\mathscr{S}'(\mathbb{R})$ we can establish

$$\langle(\hat{\varphi})_\sigma,f\rangle=\langle\mathscr{F}(\varphi_\sigma),f\rangle=\langle\varphi_\sigma,\hat{f}\rangle=\langle\varphi,\hat{f}\rangle=\langle\hat{\varphi},f\rangle$$

where $f\in\mathscr{S}(\mathbb{R})$ and σ is the sign-reversal of a distribution. Similarly, one can show that $\hat{\varphi}$ is real-valued when $\varphi\in L^\infty$ is real-valued and even by doing a similar computation with the complex conjugate. The rest follows as above. □

The following result is adapted from Proposition 6.1.5 of Grafakos [28] and concerns Bessel potential operators Λ^s for $s<0$ as convolution operators with kernel K^s. We state the result for functions on $\mathbb{R}^n$. Recall that the *Gamma function* Γ is given by

$$\Gamma(z)=\int_0^\infty t^{z-1}\,e^{-t}\,\mathrm{d}t,\quad z>0.$$

Lemma 4.2.2. *Let $s<0$. Then K^s is smooth on $\mathbb{R}^n\setminus\{0\}$ and satisfies $K^s(x)>0$ for all $x\in\mathbb{R}^n$. The explicit formula for K^s is given by*

$$K^s(x)=\frac{(2\sqrt{\pi})^{-n}}{\Gamma(-\frac{s}{2})}\int_0^\infty e^{-t}e^{-\frac{|x|^2}{4t}}\,t^{-\frac{s+n}{2}}\,\frac{\mathrm{d}t}{t},\tag{4.2.3}$$

where Γ is the standard Gamma function. The kernel has unit integral

$$\int_{\mathbb{R}^n}K^s(x)\,\mathrm{d}x=1\tag{4.2.4}$$

owing to the fact that the integral is equal to $\mathscr{F}K^s(0)=\langle 0\rangle^s=1$.

Additionally, there are positive, finite constants $c_{n,s}$, $C_{n,s}$, $\tilde{C}_{n,s}$ such that

$$K^s(x)\le C_{s,n}\,e^{-|x|/2},\quad \textit{when } |x|\ge 2,\tag{4.2.5}$$

and furthermore such that

$$\frac{1}{c_{s,n}}\le\frac{K^s(x)}{K^s_{\mathrm{sing}}(x)}\le c_{s,n},\quad \textit{when } |x|\le 2.\tag{4.2.6}$$

Here, the singular part K^s_{sing} is equal to

$$K^s_{\mathrm{sing}}(x)=\begin{cases}|x|^{-s-n}+1+O(|x|^{-s-n+2}) & \textit{if } -n<s<0,\\ \log\frac{2}{|x|}+1+O(|x|^2) & \textit{if } s=-n,\\ 1+O(|x|^{-s-n}) & \textit{if } s<-n,\end{cases}\tag{4.2.7}$$

where $O(t)$ is a function such that $|O(t)|\le\tilde{C}_{s,n}|t|$ for $0\le t\le 4$.

Remark. It should be noted that different authors use different conventions regarding the sign of the index s as pertains the lifting operator $\Lambda^s\colon B^t_{p,q}(\Omega) \to B^{t\pm s}_{p,q}(\Omega)$. In this thesis we adopt the convention that $\Lambda^s\colon B^t_{p,q}(\Omega) \to B^{t-s}_{p,q}(\Omega)$ with $s < 0$ *raises regularity* in the scale of Besov–Lipschitz spaces, mainly for the sake of notational brevity, whereas authors like Grafakos [28] would denote Λ^{-s} with $s > 0$ for the same operator.

By rescaling the x-variable, setting $n = 1$ and noting a property of the derivative, we obtain the following corollary.

Corollary 4.2.1. *Let $-1 < s < 0$. Then the kernel K^s on $\mathbb{R}$ has unit integral, is smooth on $\mathbb{R} \setminus \{0\}$, is even and positive, and there exist positive constants C_s and $\tilde{C}_s$ such that*

$$\begin{cases} K^s(x) \lesssim_s e^{-|x|} & |x| \geq 1, \\ K^s(x) = C_s|x|^{-s-1} + H^s(x) & |x| < 1, \end{cases} \tag{4.2.8}$$

where the regular part H^s satisfies $H^s(x) = \tilde{C}_s + O(|x|^{-s+1})$ with derivatives satisfying

$$|\mathrm{D}_x H^s(x)| = O(|x|^{-s}), \quad |\mathrm{D}_x^2 H^s(x)| = O(|x|^{-s-1}). \tag{4.2.9}$$

Furthermore, if $0 < |x| \ll 1$ we have $\mathrm{D}_x K^s(x) \gtrsim_s |x|^{-s-2}$.

Proof. The proof follows readily from Lemma 4.2.2. □

We call a function $g\colon (0,\infty) \to \mathbb{R}$ *completely monotone* if the condition

$$(-1)^n g^{(n)}(\lambda) \geq 0$$

holds for all $n \in \mathbb{Z}_{\geq 0}$ and all $\lambda > 0$. We borrow a result from Ørke [46] who follows the exposition of monotone functions based on a theorem of Bernstein laid out by Ehrnström–Wahlén [23, Section 2] to prove properties of the kernel K^s.

Proposition 4.2.1 (Complete monotonicity of K^s, [46, Proposition 2.2])**.**
Let $-1 < s < 0$. Then the kernel K^s is completely monotone. Furthermore, it is strictly decreasing and strictly convex on $(0,\infty)$.

Proof. The proof presented by Ørke utilizes the Stieltjes function $\lambda \mapsto (1+\lambda)^{-1}$ and the results on Stieltjes functions laid out in Ehrnström–Wahlén [23, Section 2]. □

We also prove the same for the periodized kernel K^s_P, using a trick from Ehrnström–Wahlén [23, Remark 3.4].

Corollary 4.2.2. *The periodized kernel K^s_P is even, P-periodic and strictly increasing on $(-P/2, 0)$.*

Proof. Evenness and periodicity follow trivially from the properties of K^s. Consider the derivative

$$\mathrm{D}_x K^s_P(x) = \sum_{k\in\mathbb{Z}} \mathrm{D}_x K^s(x + kP) = \sum_{k=0}^{\infty} (\mathrm{D}_x K^s(x + kP) + \mathrm{D}_x K^s(x - (k+1)P)).$$

Whenever $x \in (0, P/2)$ and $k \in \mathbb{Z}_{\geq 0}$ one has that $|x + kP| < |x - (k+1)P|$ holds. Since K^s is even and strictly convex on $(-P/2, 0)$ in particular, we get $|\mathrm{D}_x K^s(x + kP)| > |\mathrm{D}_x K^s(x - (k+1)P)|$ and thus we have the inequality

$$\mathrm{D}_x K^s(x + kP) + \mathrm{D}_x K^s(x - (k+1)P) < 0$$

on $(0, P/2)$ for every $k \in \mathbb{Z}_{\geq 0}$. As such, K^s_P is strictly increasing on $(-P/2, 0)$. □

We prove two important lemmata regarding the operator Λ^s (likewise Λ^r) which follow from the same proofs of the equivalent results Lemma 3.5 and Lemma 3.6 of Ehrnström–Wahlén [23].

Lemma 4.2.3 (Strict monotonicity)**.** *Let $s < 0$. If two bounded and continuous functions f and g satisfy $f \gneq g$, then $\Lambda^s f > \Lambda^s g$ holds everywhere.*

Proof. Recall that both K^s and its periodization K^s_P are completely monotone and strictly positive everywhere. Assume that x_0 is a point such that $f(x_0) > g(x_0)$. Then due to the continuity of f and g there exists an open neighborhood U around x_0 for which $f(y) > g(y)$ and hence

$$\begin{aligned}\Lambda^s f(x) - \Lambda^s g(x) &= \int_{\mathbb{R}} K^s(x-y)(f(y) - g(y))\,\mathrm{d}y \\ &\geq \int_U K^s(x-y)(f(y) - g(y))\,\mathrm{d}y > 0\end{aligned}$$

which completes the proof for both the case of $P = \infty$ and P finite case. □

Lemma 4.2.4 (Parity preservation under Λ^s, odd monotonicity)**.** *The operator Λ^s for $s < 0$ is a parity-preserving operator for any period $P > 0$ (including the case $P = \infty$) and furthermore satisfies $\Lambda^s f(x) > 0$ for $x \in (-P/2, 0)$ where f is P-periodic, odd and continuous with $f \gneq 0$.*

Remark. In the proof below one obtains uniformity in the period $P > 0$, and hence one can extend to the case $P = \infty$ and take $K^s_P \to K^s$ in the limit as $P \to \infty$.

Proof. For the parity preservation of Λ^s, observe that

$$\begin{aligned}\Lambda^s f(x) \pm \Lambda^s f(-x) &= \int_{-P/2}^{P/2} K^s_P(x-y) f(y)\,\mathrm{d}y \pm \int_{-P/2}^{P/2} K^s_P(-x-y) f(y)\,\mathrm{d}y \\ &= \int_{-P/2}^{P/2} K^s_P(x-y)(f(y) \pm f(-y))\,\mathrm{d}y\end{aligned}$$

by changing the variables $y \mapsto -y$ in the second integral and using the evenness of K^s_P. Thus $\Lambda^s f(x) + \Lambda^s f(-x) = 0$ if f is odd and $\Lambda^s f(x) - \Lambda^s f(-x) = 0$ if f is even.

Assume now that f is P-periodic, odd and continuous with some point x_0 such that $f(x_0) > 0$ and $f(x) \geq 0$ for $x \in (-P/2, 0)$. Then we employ a symmetrization trick of the form

$$\begin{aligned}\Lambda^s f(x) &= \int_{-P/2}^{P/2} K^s_P(x-y) f(y)\,\mathrm{d}y \\ &= \int_{-P/2}^{0} \left(K^s_P(x-y) - K^s_P(x+y)\right) f(y)\,\mathrm{d}y. \qquad (4.2.10)\end{aligned}$$

Assume first that $P = \infty$ and consider the kernel K^s. Fix a point $x \in (-P/2, 0)$ and note that $|x+y| = |x| + |y| > |x-y|$ for any $y \in (-P/2, 0)$ which ensures that the distance from the origin to the point $x+y$ is larger than that of $x-y$. Since K^s is even and strictly increasing we have $K^s(x-y) > K^s(x+y)$ on $(-P/2, 0)$, and hence the integral in Equation (4.2.10) is positive.

When P is finite we can similarly fix $x \in (-P/2, 0)$ and let y satisfy

$$-P < x+y \leq x-y < P/2$$

aiming for a similar method of proof as in the case $P = \infty$. Since $x \in (-P/2, 0)$ we have by P-periodicity that all values of $x+y$ and $x-y$ are covered in Equation (4.2.10). The periodized kernel K_P^s is P-periodic and decreases when moving away from the origin in the period $(-P/2, P/2)$. Indeed, all that remains is to check that

$$\operatorname{dist}(x-y, 0) < \min\{\operatorname{dist}(x+y, 0), \operatorname{dist}(x+y, -P)\}.$$

This is obviously true for $x = y$. Assuming $x \neq y$ share signs then $|x-y| < |x+y|$, and similarly for $|x-y| < P + x + y$ whenever $x, y > -P/2$. Hence $K_P^s(x-y) > K_P^s(x+y)$ almost everywhere for $y \in (-P/2, 0)$. It follows that $\Lambda^s f(x) > 0$ for $x \in (-P/2, 0)$. $\square$

Conjecture. Let $r \neq s$ with $r, s < 0$. Then the difference of operators $\Lambda^s - \Lambda^r$ applied to P-periodic, non-negative functions f is neither strictly positive or strictly negative for any period P and any pair (r, s) not equal to each other.

Indeed, one can write

$$(\Lambda^s - \Lambda^r) f(x) = \int_{-P/2}^{P/2} (K_P^s(y) - K_P^r(y)) f(x-y) \, \mathrm{d}y$$

where we can therefore look at the difference of kernels as given by Poisson's summation formula

$$K_P^s(x) - K_P^r(x) = \frac{2}{P} \sum_{k=1}^{\infty} (\langle k/P \rangle^s - \langle k/P \rangle^r) \cos\left(\frac{2\pi k x}{P}\right).$$

One can numerically experiment with this sum for different r and s and various $P > 0$, both small and large, to verify that this difference fails to be uniformly positive or negative on $(-P/2, P/2)$, so it is sign changing for at least some choices of r, s and P. Verifying this by analytical means has been difficult, which therefore relegates this supposition to conjecture.

4.3 A priori estimates, touching lemmata and a nodal property theorem

Having established the possibility to perform a local bifurcation, we now wish to extend solutions from the local case to that of a global solution curve in the bifurcation space by way of Theorem 3.6.1. In order to carry out this analysis it is necessary to define the sets U and $\mathscr{K}$ as in the setting of global bifurcation in cones as means to an end for our goals - namely we want $U \subseteq \mathbb{R} \times \mathscr{C}^s_{\text{even}}(\mathbb{S}_P)$ to be some subset which carries certain *a priori* properties of our solutions φ. Our first task is to establish the essential estimates for φ - namely how small and how big solutions can get.

Remark. (Notation). For functions f, g we write $f \geq g$ whenever $f(x) \geq g(x)$ holds for all x. Likewise, $f > g$ when $f(x) > g(x)$ for all x. We denote $f \gneq g$ whenever $f(x) \geq g(x)$ for all x and $f(x_0) > g(x_0)$ for some x_0. The same conventions apply for the symbols $\leq$, $<$ and $\lneq$.

We are interested in deriving a priori estimates for the equation

$$-\mu\varphi + \Lambda^s\varphi + \varphi\,\Lambda^r\varphi = 0$$

given impositions on φ and $\Lambda^r\varphi$ with respect to the bifurcation parameter μ.

Note that $L^\infty(\mathbb{S}_P) \subset B^s_{\infty,\infty}(\mathbb{S}_P) = \mathscr{C}^s(\mathbb{S}_P)$ and therefore the estimate

$$\varphi\,\Lambda^r\varphi \leq \|\varphi\,\Lambda^r\varphi\|_{L^\infty(\mathbb{S}_P)} \leq \|\varphi\|^2_{L^\infty(\mathbb{S}_P)}$$

holds everywhere by the embedding result Theorem 2.4.4 and the lifting property of Theorem 2.4.2. This observation is key to understanding how the *a priori* estimates to come share such striking similarities to that of the analysis of the Whitham equation [23] and fKdV equations [46], namely since the nonlinearities $N(\varphi)$ in $-\mu\varphi + L\varphi + N(\varphi) = 0$ can be approximated as $N(\varphi) \leq \|\varphi\|^2_{L^\infty}$.

Note that in any event we also have the approximation

$$\Lambda^s\varphi \leq \|\Lambda^s\varphi\|_{L^\infty(\mathbb{S})} \leq \sup_x |\varphi(x)|$$

which follows from the fact that $\|K^s\|_{L^1(\mathbb{R})} = 1$. In fact, due to the positivity of K^s and the unit integral of K^s one can show that

$$\Lambda^r\varphi \leq \sup_x \varphi(x)$$

holds regardless of the sign of φ, so indeed we have the additional estimate

$$\varphi\,\Lambda^r\varphi \leq \sup_x(\varphi(x)\,\sup_y \varphi(y))$$

whenever $\Lambda^r\varphi$ is non-negative. Likewise one can show that

$$\Lambda^s\varphi \geq \inf_x \Lambda^s\varphi(x) \geq \inf_x \varphi(x)$$

holds regardless of the signs of φ or $\Lambda^s\varphi$.

In the following we suspend the notation for $\inf_x$ and $\sup_x$ to just inf and sup respectively. We will make use of the following observation: let f, g be real valued functions that are bounded and assume that $f \geq 0$ uniformly. Then the following inequalities hold regardless of the sign of g:

$$f \cdot g \leq f \cdot \sup g, \quad f \cdot g \geq f \cdot \inf g.$$

If instead $f \le 0$ uniformly, the inequalities would be reversed. Also note that

$$\sup(-\varphi) = -\inf \varphi, \quad \inf(-\varphi) = -\sup \varphi$$

for bounded functions φ.

We call φ_1 a supersolution of our equation given that

$$-\mu\varphi_1 + \Lambda^s \varphi_1 + \varphi_1 \Lambda^r \varphi_1 \le 0, \tag{4.3.1}$$

and likewise we call φ_2 a subsolution given that

$$-\mu\varphi_2 + \Lambda^s \varphi_2 + \varphi_2 \Lambda^r \varphi_2 \ge 0. \tag{4.3.2}$$

A solution to our equation is both a subsolution and a supersolution. For consistency's sake we will retain the notation φ_1, φ_2 for super- and subsolutions respectively throughout the remainder of this discussion.

Lemma 4.3.1. *Let I_μ be the closed interval with endpoints $\mu - 1$ and 0. Then supersolutions φ_1 and subsolutions φ_2 of Equation (4.1.2) both satisfy*

$$\inf \varphi_1 \in I_\mu \quad \textit{and} \quad \sup \varphi_2 \notin \operatorname{int}(I_\mu).$$

Furthermore, if φ is a solution, then either $\mu - 1 \le \inf \varphi \le 0 \le \sup \varphi$ or $\varphi \equiv \mu - 1$ for $\mu < 1$, or either $0 \le \inf \varphi \le \mu - 1 \le \sup \varphi$ or $\varphi \equiv 0$ for $\mu \ge 1$.

Remark. The constant solutions φ to our equation happen to be $\varphi \equiv \mu - 1$ or $\varphi \equiv 0$ since Λ^s and Λ^r map constants to constants identically. Furthermore, if $\varphi(x) = 0$ for some x the function φ necessarily changes sign or vanishes everywhere. This is due to the strict monotonicity of Λ^s, and furthermore the sign of $\Lambda^s\varphi(x)$ can be deduced from the sign of $\varphi(x)$ when we control the sign of $\mu - \Lambda^r\varphi(x)$.

Proof. Cases 1-4 cover the supersolutions φ_1, and Cases 5-8 cover subsolutions φ_2.

Case 1: Assume the following

$$\mu - \Lambda^r \varphi_1 \le 0, \quad \inf \varphi_1 \ge 0.$$

Then our supersolution inequality reads

$$(\Lambda^r \varphi_1 - \mu)\varphi_1 \le -\Lambda^s \varphi_1 \le -\inf \varphi_1$$

and noting that $\Lambda^r\varphi_1 - \mu \ge 0$ we use the fact that $(\Lambda^r\varphi_1 - \mu)\varphi_1 \ge (\Lambda^r\varphi_1 - \mu) \inf \varphi_1$ to obtain

$$(\Lambda^r \varphi_1 - \mu) \inf \varphi_1 \le -\inf \varphi_1.$$

Now we observe that for constants c one has $\Lambda^r c = c$, which leads to the estimate

$$\Lambda^r \varphi_1 - \mu = \Lambda^r(\varphi_1 - \mu) \ge \inf(\varphi_1 - \mu) = \inf \varphi_1 - \mu$$

which when combined with the assumption that $\inf \varphi_1 \ge 0$ leads to

$$(\inf \varphi_1 - (\mu - 1)) \inf \varphi_1 \le 0. \tag{4.3.3}$$

Clearly, since $\inf \varphi_1 \geq 0$ one must have $\inf \varphi_1 - (\mu - 1) \leq 0$, in other words $\inf \varphi_1 \leq \mu - 1$. For this to make sense we have to impose that $\mu > 1$ or $\inf \varphi_1 = 0$.

Case 2: Assume the following

$$\mu - \Lambda^r \varphi_1 \leq 0, \quad \inf \varphi_1 \leq 0.$$

Then due to $\Lambda^r \varphi_1 - \mu \geq 0$ we have as in Case 1

$$(\Lambda^r \varphi_1 - \mu) \inf \varphi_1 \leq - \inf \varphi_1.$$

Collecting terms we obtain

$$(\Lambda^r \varphi_1 - (\mu - 1)) \inf \varphi_1 \leq 0$$

However now we have $\inf \varphi_1 \leq 0$, which means that necessarily

$$\Lambda^r \varphi_1 - (\mu - 1) \geq 0$$

so then $\Lambda^r \varphi_1 \geq \mu - 1$. Since $\Lambda^r \varphi_1 \geq \inf \varphi_1$ we have $\Lambda^r \varphi_1 - (\mu - 1) \geq \inf \varphi_1 - (\mu - 1)$ we have

$$(\Lambda^r \varphi_1 - (\mu - 1)) \inf \varphi_1 \leq (\inf \varphi_1 - (\mu - 1)) \inf \varphi_1.$$

It is clear that since $\inf \varphi_1 \leq 0$ if $\inf \varphi_1 - (\mu - 1) \leq 0$ then $(\Lambda^r \varphi_1 - (\mu - 1)) \inf \varphi_1 = 0$ so either $\inf \varphi_1 = 0$ or $\Lambda^r \varphi_1 \equiv \mu - 1$ which is equivalent with $\varphi_1 \equiv \mu - 1$ (using the inverse Λ^{-r} of Λ^r given by Theorem 2.4.2). If instead $\inf \varphi_1 - (\mu - 1) \geq 0$ then obviously $\inf \varphi_1 \geq \mu - 1$, in which case one needs $\mu < 1$ or $\inf \varphi_1 = 0$.

Case 3: Assume the following

$$\mu - \Lambda^r \varphi_1 \geq 0, \quad \inf \varphi_1 \geq 0.$$

Starting with, as before,

$$(\Lambda^r \varphi_1 - \mu) \varphi_1 \leq - \inf \varphi_1$$

which after negating both sides becomes

$$(\mu - \Lambda^r \varphi_1) \varphi_1 \geq \inf \varphi_1.$$

We see that $\mu - \inf \varphi_1 \geq \mu - \Lambda^r \varphi_1 \geq 0$ by assumption and by Equation (ii). Furthermore, since $\mu - \inf \varphi_1 \geq 0$ is constant we have $\inf((\mu - \inf \varphi_1)\varphi_1) = (\mu - \inf \varphi_1) \inf \varphi_1$. Clearly, if a function $f \geq M$ is bounded below by a constant M, then $\inf f \geq M$ as well, which leads us to conclude

$$(\mu - \inf \varphi_1) \inf \varphi_1 \geq \inf \varphi_1$$

which after collecting everything becomes

$$((\mu - 1) - \inf \varphi_1) \inf \varphi_1 \geq 0.$$

Negating both sides we end up with

$$(\inf \varphi_1 - (\mu - 1)) \inf \varphi_1 \leq 0$$

which is the same result as in Case 1, with the same conclusion.

CASE 4: Assume the following

$$\mu - \Lambda^r \varphi_1 \geq 0, \quad \inf \varphi_1 \leq 0.$$

Indeed, we proceed as in Case 3 with the inequality

$$(\mu - \Lambda^r \varphi_1)\varphi_1 \geq \inf \varphi_1.$$

It is apparent that we will have to proceed as in Case 2, namely with $\mu - \inf \varphi_1 \geq \mu - \Lambda^r \varphi_1$, so $(\mu - \inf \varphi_1) \inf \varphi_1 \leq (\mu - \Lambda^r \varphi_1) \inf \varphi_1$ and since $(\mu - \Lambda^r \varphi_1)\varphi_1 \geq (\mu - \Lambda^r \varphi_1) \inf \varphi_1 \geq (\mu - \inf \varphi_1) \inf \varphi_1$ we can bound

$$(\Lambda^r \varphi_1 - (\mu - 1)) \inf \varphi_1 \leq (\inf \varphi_1 - (\mu - 1)) \inf \varphi_1$$

which is exactly the same as for Case 2, hence also the same conclusions.

CASE 5: Assume the following

$$\mu - \Lambda^r \varphi_2 \leq 0, \quad \sup \varphi_2 \geq 0.$$

Now φ_2 is a subsolution satisfiying

$$(\Lambda^r \varphi_2 - \mu)\varphi_2 \geq -\Lambda^r \varphi_2 \geq -\sup \varphi_2.$$

Clearly $\sup \varphi_2 - \mu \geq \Lambda^r \varphi_2 - \mu$, so since $(\Lambda^r \varphi_2 - \mu) \geq 0$ we have

$$(\sup \varphi_2 - \mu) \sup \varphi_2 \geq (\Lambda^r \varphi_2 - \mu) \sup \varphi_2 \geq (\Lambda^r \varphi_2 - \mu)\varphi_2 \geq -\sup \varphi_2$$

which after collecting the terms yields

$$(\sup \varphi_2 - (\mu - 1)) \sup \varphi_2 \geq 0.$$

The condition $\sup \varphi_2 \geq 0$ then forces $\sup \varphi_2 \geq \mu - 1$ for $\mu > 1$ or $\sup \varphi_2 = 0$.

CASE 6: Assume the following

$$\mu - \Lambda^r \varphi_2 \leq 0, \quad \sup \varphi_2 \leq 0.$$

Then by the supremal property as in the remarks preceding this lemma $\sup \varphi_2 \geq \Lambda^r \varphi_2 \geq \mu \geq 0$ but $\sup \varphi_2 \leq 0$ so $\sup \varphi_2 = 0$. Hence we have $\varphi_2 \equiv 0$.

CASE 7: Assume the following

$$\mu - \Lambda^r \varphi_2 \geq 0, \quad \sup \varphi_2 \geq 0.$$

As per the remark preceding the proof we assume that there is some x_0 for which $\varphi_2(x_0) > 0$, otherwise we may as well assume $\sup \varphi_2 \leq 0$. Then we immediately get

$$(\mu - \sup \varphi_2)\varphi_2(x_0) \leq (\mu - \Lambda^r \varphi_2(x_0))\varphi_2(x_0) \leq \sup \varphi_2$$

from our equation defining the supersolution φ_2. Noting that if $f \leq M$ for some constant $M \geq 0$ then $\sup f \leq M$, we obtain

$$(\mu - \sup \varphi_2) \sup \varphi_2 \leq \sup \varphi_2$$

since $\varphi_2(x_0) \leq \sup \varphi_2$, which after collecting terms becomes

$$((\mu - 1) - \sup \varphi_2) \sup \varphi_2 \leq 0$$

where since $\sup \varphi_2 \geq 0$ we necessarily have $\mu - 1 \leq \sup \varphi_2$ for $\mu > 1$.

CASE 8: Assume the following

$$\mu - \Lambda^r \varphi_2 \geq 0, \quad \sup \varphi_2 \leq 0.$$

Assume first that $\mu \geq 1$, so $\mu - 1 \geq 0$ and hence $\mathrm{Int}(I_\mu) = (0, \mu - 1)$. Then there is nothing to prove since $\sup \varphi_2 \leq 0$ and $\sup \varphi_2 \notin \mathrm{Int}(I_\mu)$. Now assume instead that $\mu < 1$. We have to prove $\sup \varphi_2 \leq \mu - 1 < 0$. We start out with

$$(\mu - \Lambda^r \varphi_2)\varphi_2 \leq \sup \varphi_2$$

where furthermore we add by zero through $-\sup \varphi_2 + \sup \varphi_2$ in the parentheses of the left-hand side and write

$$(\mu - \Lambda^r \varphi_2)\varphi_2 + (\sup \varphi_2 - \Lambda^r \varphi_2)\varphi_2 \leq \sup \varphi_2.$$

Notice that $\mu - \Lambda^r \varphi_2 \geq 0$ and $\sup \varphi_2 - \Lambda^r \varphi_2 \geq 0$. We turn our attention to a particular description of the supremum, namely that the supremum is the smallest upper bound such that for every $\varepsilon > 0$ there exists x_0 such that $\varphi_2(x_0) + \varepsilon > \sup \varphi_2$. Then we obtain from the previous inequality that

$$(\mu - \sup \varphi_2)(\sup \varphi_2 - \varepsilon) + (\sup \varphi_2 - \Lambda^r \varphi_2(x_0))(\sup \varphi_2 - \varepsilon) < \sup \varphi_2.$$

Let $m > 0$ be such that $\sup \varphi_2 - \Lambda^r \varphi(x_0) < m\varepsilon$, then we have

$$(\mu - \sup \varphi_2)(\sup \varphi_2 - \varepsilon) + m\varepsilon\,(\sup \varphi_2 - \varepsilon) < \sup \varphi_2$$

which after taking the limit $\varepsilon \to 0$ amounts to

$$(\mu - \sup \varphi_2) \sup \varphi_2 \leq \sup \varphi_2$$

and hence by collecting terms we get

$$(\mu - 1 - \sup \varphi_2) \sup \varphi_2 \leq 0$$

from which we deduce that $\sup \varphi_2 \leq \mu - 1$ since $\sup \varphi_2 \leq 0$, and we are done.

We now consider a solution φ which is simultaneously a supersolution and subsolution. Let $\mu < 1$. Then our interval $I_\mu = [\mu - 1, 0]$ dictates that since $\inf \varphi \in I_\mu$ and $\inf \varphi \leq \sup \varphi \notin \mathrm{Int}(I_\mu) = (\mu - 1, 0)$ then either $\mu - 1 \leq \inf \varphi \leq 0 \leq \sup \varphi$ or $\sup \varphi \leq \mu - 1 \leq \inf \varphi$ and hence $\varphi \equiv \mu - 1$. Likewise, let $\mu \geq 1$ so that $I_\mu = [0, \mu - 1]$. Then either $0 \leq \inf \varphi \leq \mu - 1 \leq \sup \varphi$ or $\sup \varphi \leq 0 \leq \inf \varphi$ implying $\varphi \equiv 0$. This completes the proof. $\square$

Remark. The splittings of the form $\mu - \Lambda^r \varphi_i(x) \geq 0$ or $\mu - \Lambda^r \varphi_i(x) \leq 0$ for $i = 1, 2$ need not hold uniformly for all x in the proof above, as the estimates hold when $\mu - \Lambda^r \varphi(x)$ is controlled pointwise. Going forward we shall see that the uniform estimate $\mu - \Lambda^r \varphi > 0$ will be crucial to our analysis.

We turn our attention to proving two lemmata concerning the touching of solution curves $\varphi_1 \gneq \varphi_2$. Indeed, these results echo similar premises to that of the strong maximum results of elliptic theory, as noted at the offset of Section 4 in Ehrnström–Wahlén [23]. The proofs and setup of these are entirely original, as some more care has to be applied compared to the equivalent lemmata from Ehrnström–Wahlén. Indeed, we have to apply the a priori estimates from Lemma 4.3.1 in order to achieve our results, which suggests that the touching inequalities exhibit a sharpness that is rather special to our case.

We start by noting a simple observation. If φ is a solution to Equation (4.1.2) and there is some point x such that $\varphi(x) = 0$, then φ is either identically zero everywhere or changes sign at the point x. If $\varphi(x) = 0$ then Equation (4.1.2) reduces to $\Lambda^s \varphi(x) = 0$, which can only be the case if $\varphi \equiv 0$ or φ changes sign at x since Λ^s is strictly monotone.

Remark (Notation). We sometimes do not reduce the notation in the case $r = s$ from Λ^r to that of Λ^s simply to make it clear which term comes from the nonlinearity, and also to help illustrate where the arguments go wrong in the $r \neq s$ case as we shall see.

Lemma 4.3.2 (Touching lemma)**.** *Let $s < 0$. Let φ_1, φ_2 be solutions to our equation with $\varphi_1 \geq \varphi_2$ and $\Lambda^r \varphi_1 \leq \mu$. Then either*

(i) $\varphi_1 = \varphi_2$, *or*

(ii) $\varphi_1 > \varphi_2$ *when* $\Lambda^r \varphi_1 + \Lambda^r \varphi_2 < 2\mu$ *for* $r < 0$ *and* $r = s$, *or*

(iii) $\varphi_1(x_0) > \varphi_2(x_0)$ *when* $\Lambda^r \varphi_1 + \Lambda^r \varphi_2 < 2\mu$ *whenever* $\varphi_1(x_0) \geq 0$ *for* $r \neq s$ *with* $r < 0$, *or*

(iv) $\varphi_1 > \varphi_2$ *when* $\varphi_1 + \varphi_2 < \mu$ *for* $r = 0$.

Proof. Since φ_1 and φ_2 are both solutions, we have that

$$(\mu - \Lambda^s)(\varphi_1 - \varphi_2) = \varphi_1 \Lambda^r \varphi_1 - \varphi_2 \Lambda^r \varphi_2. \tag{4.3.4}$$

If $\varphi_1 = \varphi_2$ there is nothing to prove, so assume instead $\varphi_1 \gneq \varphi_2$ in which case there exists some x such that $\varphi_1(x) > \varphi_2(x)$. First note the identity

$$(\Lambda^r \varphi_1 + \Lambda^r \varphi_2)(\varphi_1 - \varphi_2) = \varphi_1 \Lambda^r \varphi_1 + \varphi_1 \Lambda^r \varphi_2 - \varphi_2 \Lambda^r \varphi_1 - \varphi_2 \Lambda^r \varphi_2 \tag{4.3.5}$$

which we subtract from both sides of Equation (4.3.4) resulting in

$$(\mu - \Lambda^s - (\Lambda^r \varphi_1 + \Lambda^r \varphi_2))(\varphi_1 - \varphi_2) = \varphi_2 \Lambda^r \varphi_1 - \varphi_1 \Lambda^r \varphi_2. \tag{4.3.6}$$

We add another $(\mu - \Lambda^s)(\varphi_1 - \varphi_2)$ to the left-hand side and correspondingly add $\varphi_1 \Lambda^r \varphi_1 - \varphi_2 \Lambda^r \varphi_2$ to the right-hand side to obtain

$$(2\mu - 2\Lambda^s - (\Lambda^r\varphi_1 + \Lambda^r\varphi_2))(\varphi_1 - \varphi_2) = \varphi_1 \Lambda^r \varphi_1 - \varphi_2 \Lambda^r \varphi_2 + \varphi_2 \Lambda^r \varphi_1 - \varphi_1 \Lambda^r \varphi_2$$

which we rearrange and simplify to the identity of interest

$$(2\mu - (\Lambda^r\varphi_1 + \Lambda^r\varphi_2))(\varphi_1 - \varphi_2) = 2\Lambda^s(\varphi_1 - \varphi_2) + (\varphi_1 + \varphi_2)(\Lambda^r\varphi_1 - \Lambda^r\varphi_2). \quad (4.3.7)$$

In effect we have to prove that the left-hand side is strictly positive. To this end we assume there is a point x_0 such that $\varphi_1(x_0) = \varphi_2(x_0)$, in which case Equation (4.3.7) reduces to

$$2\Lambda^s(\varphi_1 - \varphi_2) + (\varphi_1 + \varphi_2)(\Lambda^r\varphi_1 - \Lambda^r\varphi_2) = 0,$$

where since $\Lambda^s\varphi_1 > \Lambda^s\varphi_2$ and $\Lambda^r\varphi_1 > \Lambda^r\varphi_2$ both hold uniformly due to strict monotonicity of Λ^s, Λ^r for $r, s < 0$, we observe that $\varphi_1(x_0) = \varphi_2(x_0)$ are both (strictly) negative. Indeed, any other combination of signs of φ_1 and φ_2 yields the desired conclusion.

Assume first that $r < 0$, and furthermore that $r = s$. Then we have that $\Lambda^s = \Lambda^r$, and the right-hand side of Equation (4.3.7) becomes

$$2\Lambda^s(\varphi_1 - \varphi_2) + (\varphi_1 + \varphi_2)(\Lambda^s\varphi_1 - \Lambda^s\varphi_2)$$

where we now assume $\mu > 0$ and use the bound $\varphi_i \geq \mu - 1 > -1$ for $i = 1, 2$ obtained from Lemma 4.3.1 and therefore

$$2\Lambda^s(\varphi_1 - \varphi_2) + (\varphi_1 + \varphi_2)(\Lambda^s\varphi_1 - \Lambda^s\varphi_2) > 2\Lambda^s(\varphi_1 - \varphi_2) - 2(\Lambda^s\varphi_1 - \Lambda^s\varphi_2) = 0$$

where the strict monotonicity of Λ^s from Lemma 4.2.3 ensures the strict inequality in the previous line since $\Lambda^s(\varphi_1 - \varphi_2) > 0$. Then we have from Equation (4.3.7) that

$$(2\mu - (\Lambda^r\varphi_1 + \Lambda^r\varphi_2))(\varphi_1 - \varphi_2) > 0$$

and therefore we have $\varphi_1 > \varphi_2$ given that $2\mu > \Lambda^r\varphi_1 + \Lambda^r\varphi_2$.

In the case of $\mu = 0$, we see that our equations become $\Lambda^s\varphi_i + \varphi_i \Lambda^r \varphi_i = 0$ for $i = 1, 2$. By imposing the condition $\Lambda^r\varphi_1 + \Lambda^r\varphi_2 < 2\mu = 0$ it follows that $\Lambda^r\varphi_2 < 0$ everywhere and hence $\varphi_2 \equiv -1 = \mu - 1$ by Lemma 4.3.1. By assumption there exists at least one point x_0 for which $\varphi_1(x_0) > \varphi_2(x_0)$, then it is clear that if $\varphi_1 > 0$ everywhere it cannot be a solution, similarly if $\varphi_1 < 0$ everywhere. Hence there has to be a point y_0 where $\varphi_1(y_0) = 0$, but then φ_1 is either sign-changing in y_0 or identically equal to zero. In either case φ_1 does not touch $\varphi_2 \equiv -1$ assuming $r = s$. Hence there is no contradiction if we impose the same condition $\Lambda^r\varphi_1 + \Lambda^r\varphi_2 < 2\mu$ uniformly on $\mu \geq 0$.

For the case $r \neq s$ we cannot conclude the full touching lemma as in the case $r = s$, however everything works whenever $\varphi_1(x_0) \geq 0$ for a point x_0 regardless of the sign of $\varphi_2(x_0)$.

Assume now that $r = 0$. Then the mixed terms drop out and Equation (4.3.6) reduces to

$$(\mu - \Lambda^s - (\varphi_1 + \varphi_2))(\varphi_1 - \varphi_2) = 0$$

which by the strict monotonocity of Λ^s implies that

$$(\mu - (\varphi_1 + \varphi_2))(\varphi_1 - \varphi_2) > 0.$$

We see that both the strict inequality $\varphi_1 > \varphi_2$ and the condition $\varphi_1 + \varphi_2 < \mu$ hold everywhere simultaneously. ☐

Remark. Establishing a better result for the case $r \neq s$ as in the previous touching lemma happens to be very difficult owing to the precise nature of the estimates involved. We include the $r = 0$ case to illustrate how coarse one can be given that the mixed terms like $\varphi_1 \Lambda^r \varphi_2$ do not have to be accounted for. Indeed, for $r = 0$ we state the version found in Ehrnström–Wahlén [23] since bounding $\varphi < \mu$ in that case does not make sense vis-à-vis the supremum of φ which is found to be $\sup \varphi = \frac{\mu}{2}$.

Corollary 4.3.1. *Any non-trivial solution φ of Equation (4.1.2) with $r \leq 0$ satisfies*

$$\begin{aligned} \mu - 1 < \varphi, &\quad \mu < 1, \\ 0 < \varphi, &\quad \mu > 1. \end{aligned}$$

If $\mu = 1$, then the unique integrable solution $\varphi \in L^1(\mathbb{S}_P)$ has to identically vanish due to Proposition 4.3.1 as stated below. In the special case $r = 0$ one also can bound

$$\begin{aligned} \mu - 1 < \varphi < 1, &\quad \mu < 1, \\ 0 < \varphi < \mu, &\quad \mu > 1. \end{aligned}$$

Proof. First note that $x \mapsto 0$ and $x \mapsto \mu - 1$, for any $\mu \in \mathbb{R}$, are the only constant solutions

$$-\mu(\mu - 1) + \Lambda^s(\mu - 1) + (\mu - 1)\Lambda^r(\mu - 1) = -(\mu - 1)^2 + (\mu - 1)^2 = 0.$$

For $\mu < 1$ we have $\varphi \geq \mu - 1$ 4.3.1 for supersolutions φ. Using Lemma 4.3.2 with $\varphi_1 = \varphi$ and $\varphi_2 = \mu - 1$ assures the strict inequality $\varphi > \mu - 1$. Also if $r = 0$ one has the added condition that $\varphi + \mu - 1 < \mu$ so then $\varphi < 1$.

Similarly, if $\mu > 1$ Lemma 4.3.1 assures that $\varphi \geq 0$, so choosing $\varphi_1 = \varphi$ and $\varphi_2 = 0$ in Lemma 4.3.2 assures the strict inequality $\varphi > 0$. Furthermore, if $r = 0$ we have $\varphi + 0 < \mu$ so $\varphi < \mu$. ☐

Remark. Note that in the fKdV/Whitham case of $r = 0$ collates infimum and supremum bounds of φ in the preceding corollary, however the $r < 0$ case only produces the infimum bound of φ with a supremum bound on $\Lambda^r \varphi$. This asymmetry is to be expected, however the disconnect between working with $\Lambda^r \varphi$ and φ in terms of bounds adds a layer of complexity which shall be interesting in the analysis to follow.

Proposition 4.3.1. *Let $r, s < 0$. If $\mu = 1$, then the corresponding integrable and even solution $\varphi \in L^1(\mathbb{S}_P)$ for any $P \in (0, \infty]$ has to be the zero solution.*

Proof. We prove this for the finite period $P < \infty$ case. Let $\varphi \in L^1(\mathbb{S}_P)$ and integrate the equation $(\mu - \Lambda^s)\varphi = \varphi\, \Lambda^r \varphi$ over a whole period to obtain

$$\mu \int_{-P/2}^{P/2} \varphi(x)\, \mathrm{d}x - \int_{-P/2}^{P/2} \Lambda^s \varphi(x)\, \mathrm{d}x = \int_{-P/2}^{P/2} \varphi(x)\, \Lambda^r \varphi(x)\, \mathrm{d}x.$$

Now recall that the periodization K_P^s is given by

$$K_P^s(z) = \sum_{n \in \mathbb{Z}} K^s(z + nP)$$

which we use to compute the integral

$$\begin{aligned} \int_{-P/2}^{P/2} \Lambda^s \varphi(x)\, \mathrm{d}x &= \int_{-P/2}^{P/2} \int_{-P/2}^{P/2} K_P^s(x - y)\, \varphi(y)\, \mathrm{d}y\, \mathrm{d}x \\ &= \int_{-P/2}^{P/2} \int_{-P/2}^{P/2} K_P^s(x - y)\, \mathrm{d}x\, \varphi(y)\, \mathrm{d}y \end{aligned}$$

where we have used Fubini's theorem, and thus due to the unit integral of the periodization

$$\int_{-P/2}^{P/2} K_P^s(x - y)\, \mathrm{d}x = \sum_{n \in \mathbb{Z}} \int_{-P/2}^{P/2} K^s(x - y + nP)\, \mathrm{d}x = \int_{\mathbb{R}} K^s(x - y)\, \mathrm{d}x = 1$$

we finally obtain

$$\int_{-P/2}^{P/2} \Lambda^s \varphi(x)\, \mathrm{d}x = \int_{-P/2}^{P/2} \varphi(x)\, \mathrm{d}x.$$

Then our integral over $(\mu - \Lambda^s)\varphi = \varphi\, \Lambda^r \varphi$ becomes

$$(\mu - 1) \int_{-P/2}^{P/2} \varphi(x)\, \mathrm{d}x = \int_{-P/2}^{P/2} \varphi(x)\, \Lambda^r \varphi(x)\, \mathrm{d}x$$

inserting $\mu = 1$ and noting that the integrand on the right-hand side is even for φ even, then necessarily the integrand has to vanish on $\mathbb{S}_P$ and hence $\varphi \equiv 0$ due to continuity.

The proof for the solitary $P = \infty$ case is proven similarly, or can be seen from the limiting case $P \to \infty$ since we have uniformity in P where it is clear that the periodization dependent on P drops out of our calculations due the unit integral of K^s. □

Remark. Proposition 4.6 from Ehrnström–Wahlén [23], which is in correspondence to the preceding result in the case of the Whitham equation, includes estimates of the integral mean of φ, where if $\mu < 1$ then φ has negative mean and if $\mu > 1$ then φ has positive mean. Additionally, the authors are able to establish square integrability of φ whenever φ is integrable. Both the integral mean and square integrability is, in generality, not possible for us to establish due to the nonlocal nature of the nonlinearity $\varphi\, \Lambda^r \varphi$.

Lemma 4.3.3 (Touching lemma for derivatives)**.**
Let φ_1, φ_2 be even and continuously differentiable solutions to the Equation (4.1.2)*, where we impose $\varphi_1 \geq \varphi_2$ and $\varphi_1' \gneq \varphi_2' \geq 0$ on $(-P/2, 0)$. Then*

(i) $\varphi_1' > \varphi_2'$ when $\Lambda^r\varphi_1 + \Lambda^r\varphi_2 < 2\mu$ in $(-P/2, 0)$ for $r < 0$ and $r = s$,

(ii) $\varphi_1'(x_0) > \varphi_2'(x_0)$ when $\Lambda^r\varphi_1 + \Lambda^r\varphi_2 < 2\mu$ in $(-P/2, 0)$ whenever $\varphi_1(x_0) \geq 0$ for $r \neq s$ with $r < 0$,

(iii) $\varphi_1' > \varphi_2'$ when $\varphi_1 + \varphi_2 < \mu$ in $(-P/2, 0)$ for $r = 0$ and $s < 0$.

Remark. From Lemma 4.3.3 one can readily establish that $\varphi_1 > \varphi_2$ on the whole period $(-P/2, P/2)$ since φ_1, φ_2 are both even and satisfy $\varphi_1' > \varphi_2'$ on the half-interval $(-P/2, 0)$.

Proof. The functions φ_1, φ_2 are solutions to $(\mu - \Lambda^s)\varphi = \varphi\,\Lambda^r\varphi$, which after differentiating amounts to

$$(\mu - \Lambda^s)\varphi' = \varphi'\,\Lambda^r\varphi + \varphi\,\Lambda^r\varphi'. \tag{4.3.8}$$

Substituting φ for φ_1 and φ_2 and subtracting the latter equation from the former we obtain

$$(\mu - \Lambda^s)(\varphi_1' - \varphi_2') = (\varphi_1\,\Lambda^r\varphi_1 - \varphi_2\,\Lambda^r\varphi_2)' = \varphi_1'\,\Lambda^r\varphi_1 + \varphi_1\,\Lambda^r\varphi_1' - \varphi_2'\,\Lambda^r\varphi_2 - \varphi_2\,\Lambda^r\varphi_2'$$

and also note the identity

$$(\Lambda^r\varphi_1 + \Lambda^r\varphi_2)(\varphi_1' - \varphi_2') = \varphi_1'\,\Lambda^r\varphi_1 + \varphi_1'\,\Lambda^r\varphi_2 - \varphi_2'\,\Lambda^r\varphi_1 - \varphi_2'\,\Lambda^r\varphi_2$$

where as in Lemma 4.3.2 we rearrange to the form

$$(\mu - \Lambda^s - (\Lambda^r\varphi_1 + \Lambda^r\varphi_2))(\varphi_1' - \varphi_2') = \varphi_1\,\Lambda^r\varphi_1' - \varphi_2\,\Lambda^r\varphi_2' + \varphi_2'\,\Lambda^r\varphi_1 - \varphi_1'\,\Lambda^r\varphi_2 \tag{4.3.9}$$

to which we add $(\mu - \Lambda^s)(\varphi_1' - \varphi_2')$ to both sides, resulting in

$$\begin{aligned}(2\mu - 2\Lambda^s &- (\Lambda^r\varphi_1 + \Lambda^r\varphi_2))(\varphi_1' - \varphi_2') \\ &= 2\varphi_1\,\Lambda^r\varphi_1' - 2\varphi_2\,\Lambda^r\varphi_2' + (\varphi_1' + \varphi_2')(\Lambda^r\varphi_1 - \Lambda^r\varphi_2)\end{aligned} \tag{4.3.10}$$

$$(2\mu - 2\Lambda^s - (\Lambda^r\varphi_1 + \Lambda^r\varphi_2))(\varphi_1' - \varphi_2') = 2\varphi_1\,\Lambda^r\varphi_1' - 2\varphi_2\,\Lambda^r\varphi_2' + (\varphi_1' + \varphi_2')(\Lambda^r\varphi_1 - \Lambda^r\varphi_2)$$

by using the identity for $(\mu - \Lambda^s)(\varphi_1' - \varphi_2')$ as above. Again, we need to prove positivity of the right-hand side case when both φ_1 and φ_2 are negative at a point. When $\varphi_1 \geq 0$ and $\varphi_2 \leq 0$ or both φ_1, $\varphi_2 \geq 0$, then there is nothing to check. Assuming $\mu > 0$ we can establish the inequality

$$2\varphi_1\,\Lambda^r\varphi_1' - 2\varphi_2\,\Lambda^r\varphi_2' \geq 2\varphi_2(\Lambda^r\varphi_1' - \Lambda^r\varphi_2') > -2(\Lambda^r\varphi_1' - \Lambda^r\varphi_2')$$

owing to Lemma 4.3.1 and the strict monotonicity of Λ^r which in this instance yields $\Lambda^r(\varphi_1' - \varphi_2') > 0$. Assume $r = s$. Moving $2\Lambda^s(\varphi_1' - \varphi_2')$ to the right-hand side of Equation (4.3.10) we then readily have that

$$\begin{aligned}&(2\mu - (\Lambda^r\varphi_1 + \Lambda^r\varphi_2))(\varphi_1' - \varphi_2') \\ &= 2\Lambda^s(\varphi_1' - \varphi_2') + 2\varphi_1\,\Lambda^r\varphi_1' - 2\varphi_2\,\Lambda^r\varphi_2' + (\varphi_1' + \varphi_2')(\Lambda^r\varphi_1 - \Lambda^r\varphi_2) \\ &> 2\Lambda^s(\varphi_1' - \varphi_2') - 2\Lambda^r(\varphi_1' - \varphi_2') + (\varphi_1' + \varphi_2')(\Lambda^r\varphi_1 - \Lambda^r\varphi_2) \\ &\geq 0.\end{aligned}$$

Then we have $(2\mu - (\Lambda^r\varphi_1 - \Lambda^r\varphi_2))(\varphi_1' - \varphi_2') > 0$ and hence we have to conclude $\varphi_1' > \varphi_2'$ given that the condition $\Lambda^r\varphi_1 - \Lambda^r\varphi_2 < 2\mu$ holds. Note that the passage from the second to third line is strict since $2\varphi_1 - 2\varphi_2 > 2(\mu - 1) > -2$ since $\mu > 0$.

If $\mu = 0$ the function φ_2 reduces to a constant as in the proof of Lemma 4.3.2, hence there is nothing to prove.

In the case of $r = 0$ the mixed terms drop out and Equation (4.3.10) reduces to

$$(\mu - (\varphi_1 + \varphi_2))(\varphi_1' - \varphi_2') = \Lambda^s(\varphi_1' - \varphi_2') > 0$$

from which one infers that $\varphi_1' > \varphi_2'$ given that $\varphi_1 + \varphi_2 < \mu$. □

We note that the touching lemmata in the case of fKdV with $r = 0$ are much simpler to prove, but are also in some sense less sharp in their estimates. Indeed, one does not have to invoke the a priori estimates at all in the case of $r = 0$. In the case $r \neq s$ with $r, s < 0$ one struggles to prove anything more than the partial results gathered in the preceding two lemmata, there would have to be some uniformly satisfied relationship between $\Lambda^s(\cdot)$ and $\varphi\,\Lambda^r(\cdot)$, which seems implausible to exist, or in any case difficult to concoct. The conjecture regarding the difference $\Lambda^s - \Lambda^r$ implies that we cannot uniformly bound $2\Lambda^s(\varphi_1 - \varphi_2) + (\varphi_1 + \varphi_2)\Lambda^r(\varphi_1 - \varphi_2) > 2\Lambda^s(\varphi_1 - \varphi_2) - 2\Lambda^r(\varphi_1 - \varphi_2) \geq 0$ since one does not have $\Lambda^s f \geq \Lambda^r f$, for non-negative functions f, in general. The conjecture states that this bound is always impossible for any r, s and P.

Theorem 4.3.1 (Nodal property theorem)**.** *Let $P \in (0, \infty]$ and $r, s < 0$. Then a P-periodic, non-constant and even solution $\varphi \in C^1(\mathbb{R})$ of Equation (4.1.2) that is non-decreasing on $(-P/2, 0)$ satisfies*

$$\varphi' > 0 \text{ and } \Lambda^r\varphi < \mu \text{ on } (-P/2, 0)$$

for any period P when $r = s$, and whenever $x \in (\tilde{x}_{r,s}, 0)$ where $\tilde{x}_{r,s}$ has $0 > \varphi(\tilde{x}_{r,s})$ and $|\varphi(\tilde{x}_{r,s})|$ small in the case $r \neq s$. Furthermore, for a solution φ as above one necessarily has $\mu > 0$. Moreover, if $\varphi \in C^2(\mathbb{R})$ and $r = s$ with $-1 < s < 0$, then $\Lambda^r\varphi < \mu$ holds everywhere and

$$\varphi''(0) < 0.$$

Furthermore, for finite periods $P < \infty$ one has $\varphi''(\pm P/2) > 0$.

Remark. We do not suspend the notation of Λ^r to that of Λ^s when $r = s$ to illustrate how it would look if $r \neq s$ symbolically.

Proof. We first set out to prove $\varphi' > 0$ and $\Lambda^r\varphi < \mu$ on the half-interval $(-P/2, 0)$. From our assumptions it is clear that φ' must be odd, non-trivial since φ is non-constant, and non-negative on $(-P/2, 0)$ since φ is non-decreasing on $(-P/2, 0)$. Since φ is assumed non-constant and φ' non-negative we have $\varphi' \gneq 0$ on $(-P/2, 0)$. Thus strict monotonicity of Λ^s implies that $\Lambda^s\varphi' > 0$, likewise $\Lambda^r\varphi' > 0$.

Note that in the proof of both touching lemmas we split in terms of $\mu > 0$ and $\mu = 0$, where the latter condition only yields a constant solution. Hence we necessarily have $\mu > 0$ owing to our assumptions.

In the spirit of Lemma 4.3.3 one can write

$$(\mu - \Lambda^r\varphi)\varphi' = \Lambda^s\varphi' + \varphi\,\Lambda^r\varphi' > \Lambda^s\varphi' - \Lambda^r\varphi'$$

which is strict due to Lemma 4.3.1 and $\Lambda^r\varphi' > 0$. Indeed, for $r = s$ we are done since $\mu > \Lambda^r\varphi$ is assumed. For $r \neq s$ everything works when $\varphi(x) \geq 0$, and we can by continuity extend to some $\tilde{x}_{r,s}$ such that $\Lambda^s\varphi'(\tilde{x}_{r,s}) > -\varphi(\tilde{x}_{r,s})\Lambda^r\varphi'(\tilde{x}_{r,s})$ where $0 > \varphi(\tilde{x}_{r,s})$ is small since both $\Lambda^s\varphi'$ and $\Lambda^r\varphi'$ are strictly increasing.

Now assume $\varphi \in C^2(\mathbb{R})$. Then we may write

$$(\mu - \Lambda^s)\varphi'' = (\varphi'\,\Lambda^r\varphi + \varphi\,\Lambda^r\varphi')' = \varphi''\,\Lambda^r\varphi + 2\varphi'\,\Lambda^r\varphi' + \varphi\,\Lambda^r\varphi''$$

and thus

$$(\mu - \Lambda^r\varphi)\varphi'' = \Lambda^s\varphi'' + 2\varphi'\,\Lambda^r\varphi' + \varphi\,\Lambda^r\varphi''.$$

We evaluate this equality at $x = 0$ and use the evenness of K^s, K^r and φ to obtain

$$\begin{aligned}
(\mu - \Lambda^r\varphi(0))\varphi''(0) &= 2\int_0^{P/2} (K_P^s(y) + \varphi(0)K_P^r(y))\,\varphi''(y)\,\mathrm{d}y \\
&= 2\int_0^{\varepsilon} (K_P^s(y) + \varphi(0)K_P^r(y))\,\varphi''(y)\,\mathrm{d}y + 2\int_{\varepsilon}^{P/2} (K_P^s(y) + \varphi(0)K_P^r(y))\,\varphi''(y)\,\mathrm{d}y \\
&= 2\int_0^{\varepsilon} (K_P^s(y) + \varphi(0)K_P^r(y))\,\varphi''(y)\,\mathrm{d}y + [(K_P^s(y) + \varphi(0)K_P^r(y))\varphi'(y)]_{y=\varepsilon}^{y=P/2} \\
&\quad - 2\int_{\varepsilon}^{P/2} (\mathrm{D}_yK_P^s(y) + \varphi(0)\,\mathrm{D}_yK_P^r(y))\varphi'(y)\,\mathrm{d}y.
\end{aligned}$$

Since φ'' is continuous and both K_P^s and K_P^r are integrable with $K_P^s \sim |x|^{-s-1}$ and $K_P^r \sim |x|^{-r-1}$ for $|x| \ll 1$, the first integral in the final line vanishes as $\varepsilon \to 0$. The boundary term $(K_P^s(P/2) + \varphi(0)K_P^r(P/2))\varphi'(P/2)$ vanishes since $\varphi'(P/2) = 0$ for finite periods P, and also vanishes if $P = \infty$ since $K^s(x)$, $K^r(x)$ both tend to 0 as $x \to \infty$ with φ' bounded. The regularity and evenness of φ ensures that $\varphi'(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$ and therefore $\varphi'(\varepsilon)(K_P^s(\varepsilon) + \varphi(0)K_P^r(\varepsilon)) = O(\varepsilon^{\min\{-r,-s\}}) \to 0$ as $\varepsilon \to 0$ provided that $0 < -s = -r < 1$ as assumed.

By Corollary 4.2.2 and what we have just proved, we see that $\mathrm{D}_xK_P^s$, $\mathrm{D}_xK_P^r$ and φ' are all strictly negative on the half-interval $(0, P/2)$. Noting that $\varphi(0) > 0$ by Lemma 4.3.1, the term $-\int_{\varepsilon}^{P/2}(\mathrm{D}_yK_P^s(y) + \varphi(0)\,\mathrm{D}_yK_P^r(y))\varphi'(y)\,\mathrm{d}y$ is negative for any $\varepsilon > 0$, and is strictly decreasing as $\varepsilon \searrow 0$. Thus, letting $\varepsilon \to 0$ we obtain

$$(\mu - \Lambda^r\varphi(0))\varphi''(0) = -\lim_{\varepsilon \searrow 0}\int_{\varepsilon}^{P/2} (\mathrm{D}_yK_P^s(y) + \varphi(0)\mathrm{D}_yK_P^r(y))\varphi'(y)\,\mathrm{d}y < 0$$

and hence we infer that $\varphi''(0) < 0$ and $\mu > \Lambda^r\varphi$ everywhere on $\mathbb{S}_P$ by continuity.

Let $P < \infty$ and observe that $K_P^s(P/2 - y) = K_P^s(-P/2 - y) = K_P^s(y + P/2)$ due to the evenness and periodicity of K_P^s. Likewise for K_P^r. Using this, we calculate

$$
\begin{aligned}
(\mu - \Lambda^r\varphi(P/2))\varphi''(P/2) &= \int_0^{P/2} (K_P^s(y+P/2) + \varphi(P/2)K_P^r(y+P/2))\varphi''(y)\,\mathrm{d}y \\
&= \left(\int_0^{P/2-\varepsilon} + \int_{P/2-\varepsilon}^{P/2}\right)(K_P^s(y+P/2) + \varphi(P/2)K_P^r(y+P/2))\varphi''(y)\,\mathrm{d}y \\
&= [(K_P^s(y+P/2) + \varphi(P/2)K_P^r(y+P/2))\varphi'(y)]_{y=0}^{y=P/2-\varepsilon} \\
&\quad + \int_{P/2-\varepsilon}^{P/2} (K_P^s(y+P/2) + \varphi(P/2)K_P^r(y+P/2))\varphi''(y)\,\mathrm{d}y \\
&\quad - \int_0^{P/2-\varepsilon} (\mathrm{D}_y K_P^s(y+P/2) + \varphi(P/2)\mathrm{D}_y K_P^r(y+P/2))\varphi'(y)\,\mathrm{d}y.
\end{aligned}
$$

The first two terms in the final line vanish as $\varepsilon \searrow 0$ by the same arguments as before, however we need to examine the last integral a bit more closely. It is clear from Lemma 4.3.1 that $\varphi(P/2) < 0$ if $\mu \leq 1$. If $r = s$ we can simply bound $\varphi(P/2) > -1$ by Lemma 4.3.2, and hence the last integral is strictly positive and increasing as $\varepsilon \searrow 0$. From there one can conclude $\varphi''(P/2) > 0$ as desired. □

Remark. The touching lemma with derivatives 4.3.3 and nodal property theorem are very closely interlinked, and are perhaps in some sense equivalent. In Ehrnström–Wahlén [23] they append this theorem with a convexity result on the second derivative of φ, however the equivalent of this result is not needed for our analysis, and is also not used in e.g. Ørke [46] or Hildrum–Xue [31].

4.4 Regularity analysis and singularities

Theorem 4.4.1 (Regularity)**.** *Let $\varphi \in L^\infty(\mathbb{R})$ be an even solution to the steady equation* (4.1.2) *and let $r, s < 0$. Then:*

(i) If $\Lambda^r\varphi < \mu$ uniformly on all of $\mathbb{R}$, then $\varphi \in C^\infty(\mathbb{R})$.

(ii) If $\Lambda^r\varphi < \mu$ uniformly on all of $\mathbb{R}$ and $\varphi \in L^2(\mathbb{R})$ then $\varphi \in H^\infty(\mathbb{R})$.

(iii) φ is smooth on any open set where $\Lambda^r\varphi < \mu$.

Proof. Assume first that $\Lambda^r\varphi < \mu$ uniformly on all of $\mathbb{R}$ and assume $r \leq s$. The operator $\Lambda^s\varphi$ maps $B^t_{p,q}(\mathbb{R})$ into $B^{t-s}_{p,q}(\mathbb{R})$ isomorphically, raising the regularity since we have assumed $s < 0$. Also, by the embedding theorem of Besov–Lipschitz spaces 2.4.4 we have that Λ^s maps $L^\infty(\mathbb{R}) \subset \mathscr{C}^0(\mathbb{R}) = B^0_{\infty,\infty}(\mathbb{R})$ into $\mathscr{C}^{-s}(\mathbb{R}) = B^{-s}_{\infty,\infty}(\mathbb{R}) \subset L^\infty(\mathbb{R})$ isomorphically. Theorem 2.87 as in Bahouri et al. [3] states that for every smooth function f vanishing at 0 and $u \in B^t_{p,q}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ then it follows that $f \circ u \in B^t_{p,q}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ also. Consider now the relation

$$\varphi\,(\mu - \Lambda^r\varphi) = \Lambda^s\varphi$$

where if we insist that $\mu - \Lambda^r\varphi > 0$ uniformly on all of $\mathbb{R}$ we can algebraically invert this factor and write

$$\varphi = (\mu - \Lambda^r\varphi)^{-1}\,\Lambda^s\varphi.$$

We now wish to define a smooth function f vanishing at 0 such that $f(\Lambda^s\varphi) = \varphi$. One way of doing this is to define f as

$$f\colon u \mapsto u\,(\mu - \Lambda^{r-s}u)^{-1}$$

where since $\Lambda^{r-s} = \Lambda^r \circ \Lambda^{-s}$ by Lemma 4.2.1 we have that $f(\Lambda^s\varphi) = \varphi$. Recall that the operator Λ^{-s} is interpreted as the inverse operator of Λ^s as given by the Lifting Theorem 2.4.2. Also note that we never divide by zero since we have to impose the bound $\mu - \Lambda^{r-s}u > 0$. Thus we are justified in taking the algebraic inverse $(\mu - \Lambda^{r-s}u)^{-1}$ and we immediately see that f vanishes at 0 uniquely. The reciprocal of a non-vanishing smooth function is smooth, and hence it follows that f is smooth since in particular $\Lambda^{r-s}(\cdot)$ is real analytic in its argument when $r \leq s$. It follows that f maps $B^t_{p,q}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ onto itself.

If $\varphi \in B^t_{p,q} \cap L^\infty$ then $\Lambda^s\varphi \in B^{t-s}_{p,q} \cap L^\infty$, and thus if $\mu - \Lambda^r\varphi > 0$ then $f(\Lambda^s\varphi) = (\mu - \Lambda^r\varphi)^{-1}\Lambda^s\varphi = \varphi \in B^{t-s}_{p,q}$. So we started out with $\varphi \in B^t_{p,q} \cap L^\infty$ and ended up with $\varphi \in B^{t-s}_{p,q}$, so the composition of maps

$$[u \mapsto u\,(\mu - \Lambda^{r-s}u)^{-1}] \circ [\varphi \mapsto \Lambda^s\varphi]\colon B^t_{p,q}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \hookrightarrow B^{t-s}_{p,q}(\mathbb{R})$$

holds for all $t \geq 0$ and raises the regularity of φ by $-s > 0$ in the scale of Besov–Lipschitz spaces.

In the event that $r \geq s$ we instead define the map

$$[u \mapsto \Lambda^{s-r}u\,(\mu - u)^{-1}] \circ [\varphi \mapsto \Lambda^r\varphi]\colon B^t_{p,q}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \hookrightarrow B^{t-r}_{p,q}(\mathbb{R}).$$

Choosing $p = q = \infty$ we can bootstrap the regularity of $\varphi \in L^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$ by repeated iteration of the map f as above, proving Item (i). Choosing $p = q = 2$ similarly proves Item (ii).

Let $\mathscr{C}^t_{\mathrm{loc}}(\mathbb{R})$ be the set of all φ such that for any open set $U \subseteq \mathbb{R}$ there exists $\psi \in C^\infty_0(U)$ such that $\psi\,\varphi \in \mathscr{C}^t(\mathbb{R})$. Then, if $\varphi \in \mathscr{C}^t_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we can prove that $\Lambda^s\varphi \in \mathscr{C}^{t-s}_{\mathrm{loc}}(\mathbb{R})$. To this end, assume a particular open set U and corresponding $\psi \in C^\infty_0(U)$ such that $\psi\,\varphi \in \mathscr{C}^t(\mathbb{R})$ we let $\tilde{\psi}$ be a smooth cut-off function with $\tilde{\psi} = 1$ on a compact neighborhood $V \subset\subset U$ of $\operatorname{supp}\psi$. Then

$$\psi\,\Lambda^s\varphi = \psi\,\Lambda^s(\tilde{\psi}\,\varphi) + \psi\,\Lambda^s((1 - \tilde{\psi})\varphi).$$

The first term is of class $\mathscr{C}^{t-s}$ on V. The latter term is given by the convolution

$$\psi\,\Lambda^s((1 - \tilde{\psi})\varphi)(x) = \int_{-\infty}^{\infty} K^s(x - y)\,\psi(x)\,(1 - \tilde{\psi}(y))\,\varphi(y)\,\mathrm{d}y.$$

The integrand vanishes on V since $1 - \tilde{\psi}$ vanishes on V, and therefore the integrand ought to vanish as y approaches x, since if $x \in \operatorname{supp}\psi \subseteq V$ we have $1 - \tilde{\psi}(y) \to 0$ as $y \to x$ or if $x \notin \operatorname{supp}\psi$ then $\psi(x) = 0$. Recall that $K^s(x - y)$ is smooth on $\mathbb{R} \setminus \{0\}$ by Lemma 4.2.2, so $K^s(x - y)\,\psi(x)\,(1 - \tilde{\psi}(y))$ is smooth on $\mathbb{R}$. Then by Proposition 2.1.3 we have that $\psi\,\Lambda^s((1 - \tilde{\psi})\varphi)$ is smooth on $\mathbb{R}$. Therefore we have $\Lambda^s\varphi \in \mathscr{C}^{t-s}_{\mathrm{loc}}$ in U so if in addition $\Lambda^r\varphi < \mu$ on U we see that we can bootstrap to $\varphi \in C^\infty(U)$. Thus we have proven (iii). $\square$

Lemma 4.4.1. *Let $-1 < r, s < 0$ and $\mu > 0$. Assume a finite period $P < \infty$ and let φ be an even, non-constant solution to Equation (4.1.2) that is non-decreasing on $(-P/2, 0)$ with $\Lambda^r \varphi < \mu$ on $(-P/2, P/2)$. Then there exists a universal constant $\lambda_{r,s,P,\mu} > 0$ depending only on the kernels K^s and K^r, the period P and the parameter $\mu > 0$ such that*

$$\mu - \Lambda^r \varphi(P/2) \geq \lambda_{r,s,P,\mu}. \tag{4.4.1}$$

In fact, we can establish the more general estimate

$$\mu - \Lambda^r \varphi(x) \gtrsim_{r,s,P,\mu} |x_0|^{\min\{-r,-s\}} \tag{4.4.2}$$

which holds uniformly for all $x \in [-P/2, x_0]$ with $x_0 < 0$ small. Additionally, when $r = s$, these estimates hold also when $\Lambda^s \varphi(0) = \mu$.

Proof. We first prove the result given that $\Lambda^r \varphi(0) < \mu$ for all $r, s < 0$ and will treat the $\Lambda^s \varphi(0) = \mu$ case for $r = s$ later. We prove a partial result for $r \neq s$ and note that the case for $r = s$ holds similarly, and furthermore one can prove the estimate when $\Lambda^s \varphi(0) = \mu$ in the $r = s$ case, but not in the $r \neq s$ case due to the same reason as for the partial results in the nodal pattern theorem.

Since $\Lambda^r \varphi' > 0$ we have $\Lambda^r \varphi(P/2) < \Lambda^r \varphi(x)$ for $x \in [-\frac{3P}{8}, -\frac{P}{8}]$ and furthermore that

$$\begin{aligned}
(\mu - \Lambda^r \varphi(P/2))\varphi'(x) &\geq (\mu - \Lambda^r \varphi(x))\varphi'(x) \\
&= \Lambda^s \varphi'(x) + \varphi(x)\, \Lambda^r \varphi'(x) \\
&= \int_{-P/2}^{P/2} K_P^s(x-y)\, \varphi'(y)\, \mathrm{d}y + \int_{-P/2}^{P/2} \varphi(x)\, K_P^r(x-y)\, \varphi'(y)\, \mathrm{d}y.
\end{aligned} \tag{4.4.3}$$

Next we note that symmetrization yields

$$\begin{aligned}
\int_{-P/2}^{P/2} K_P^s(x-y)\, \varphi'(y)\, \mathrm{d}y &= \int_{-P/2}^{0} (K_P^s(x-y) - K_P^s(x+y))\, \varphi'(y)\, \mathrm{d}y \\
&\geq \int_{-3P/8}^{-P/8} (K_P^s(x-y) - K_P^s(x+y))\, \varphi'(y)\, \mathrm{d}y
\end{aligned}$$

in view of that $K_P^s(x-y) > K_P^s(x+y)$ for $x, y \in (-P/2, 0)$. Since φ is nonconstant we see that $\varphi > \mu - 1$ because of Lemma 4.3.2, and therefore

$$\begin{aligned}
(\mu - \Lambda^r \varphi(P/2))\varphi'(x) &\geq (\mu - \Lambda^r \varphi(x))\varphi'(x) \\
&\geq \int_{-P/2}^{P/2} K_P^s(x-y)\, \varphi'(y)\, \mathrm{d}y \\
&\quad + \varphi(x) \int_{-P/2}^{P/2} K_P^r(x-y)\, \varphi'(y)\, \mathrm{d}y \\
&> \int_{-3P/8}^{-P/8} (K_P^s(x-y) - K_P^s(x+y))\, \varphi'(y)\, \mathrm{d}y \\
&\quad + (\mu - 1) \int_{-3P/8}^{-P/8} (K_P^r(x-y) - K_P^r(x+y))\, \varphi'(y)\, \mathrm{d}y.
\end{aligned} \tag{4.4.4}$$

There exists a universal constant $\tilde{\lambda}_{K^s,P} > 0$ depending on the kernel K^s and period P satisfying

$$\min\{K_P^s(x-y) - K_P^s(x+y)\colon x, y \in [-\tfrac{3P}{8}, -\tfrac{P}{8}]\} \geq \tilde{\lambda}_{K^s,P},$$

similarly with $\tilde{\lambda}_{K^r,P}$ for K_P^r. Let $\tilde{\lambda}_{r,s,P} = \min\{\tilde{\lambda}_{K^s,P}, \tilde{\lambda}_{K^r,P}\}$.

Integrating Equation (4.4.4) with respect to x over the interval $[-\frac{3P}{8}, -\frac{P}{8}]$ we obtain

$$\begin{aligned}
(\mu - \Lambda^r\varphi(P/2))&(\varphi(-\tfrac{P}{8}) - \varphi(-\tfrac{3P}{8})) \\
&> \int_{-3P/8}^{-P/8} \left(\int_{-3P/8}^{-P/8} K_P^s(x-y) - K_P^s(x+y)\,\mathrm{d}x \right) \varphi'(y)\,\mathrm{d}y \\
&\quad + (\mu - 1) \int_{-3P/8}^{-P/8} \left(\int_{-3P/8}^{-P/8} K_P^r(x-y) - K_P^r(x+y)\,\mathrm{d}x \right) \varphi'(y)\,\mathrm{d}y \\
&\geq \tfrac{P}{4}\tilde{\lambda}_{K^s,P}\,(\varphi(-\tfrac{P}{8}) - \varphi(-\tfrac{3P}{8})) + \tfrac{P}{4}\,(\mu-1)\,\tilde{\lambda}_{K^r,P}\,(\varphi(-\tfrac{P}{8}) - \varphi(-\tfrac{3P}{8})) \\
&\geq \tfrac{P}{4}\mu\,\tilde{\lambda}_{r,s,P}\,(\varphi(-\tfrac{P}{8}) - \varphi(-\tfrac{3P}{8}))
\end{aligned}$$

By the nodal property theorem 4.3.1 we know that $\varphi(-P/8) > \varphi(-3P/8)$ in the case of $r = s$, but we can also use the partial result from Lemma 4.3.2 by modifying the interval $[-\frac{3P}{8}, -\frac{P}{8}]$ to ensure that φ is not negative at both of the endpoints of the interval, and thus we can divide both sides of the latter inequality by $\varphi(-\frac{P}{8}) - \varphi(-\frac{3P}{8})$ to finally arrive at our desired estimate

$$\mu - \Lambda^r\varphi(P/2) \geq \tfrac{P}{4}\mu\,\tilde{\lambda}_{r,s,P} =: \lambda_{r,s,P,\mu}. \tag{4.4.5}$$

For the x-dependent estimate we fix x_1, x_2 with $-P/4 < x_2 < x_1 < 0$ and let $x \in (x_2, x_1)$. Write $\xi \in [-P/2, x_2]$ as the input variable in $\Lambda^r\varphi(\cdot)$ as in Equation (4.4.3). Then we have

$$\begin{aligned}
(\mu - \Lambda^r\varphi(\xi))\,\varphi'(x) &\geq \int_{x_2}^{x_1} (K_P^s(x-y) - K_P^s(x+y))\,\varphi'(y)\,\mathrm{d}y \\
&\quad + \varphi \int_{x_2}^{x_1} (K_P^r(x-y) - K_P^r(x+y))\,\varphi'(y)\,\mathrm{d}y \\
&\geq \int_{x_2}^{x_1} (-2y)\,\mathrm{D}_x K_P^s(y+\zeta)\,\varphi'(y)\,\mathrm{d}y \\
&\quad + (\mu - 1) \int_{x_2}^{x_1} (-2y)\,\mathrm{D}_x K_P^r(y+\zeta)\,\varphi'(y)\,\mathrm{d}y \\
&\geq -2x_1\,(\varphi(x_1) - \varphi(x_2))\,(\mathrm{D}_x K_P^s(2x_2) + (\mu-1)\mathrm{D}_x K_P^r(2x_2))
\end{aligned}$$

where $|\zeta| < |x|$ arises from the mean value theorem. Now we integrate over (x_2, x_1) with respect to x and obtain

$$\begin{aligned}
(\mu - &\Lambda^r\varphi(\xi))(\varphi(x_1) - \varphi(x_2)) \\
&\geq (\mathrm{D}_x K_P^s(2x_2) + (\mu-1)\mathrm{D}_x K_P^r(2x_2))\,|2x_1(x_2 - x_1)|(\varphi(x_1) - \varphi(x_2))
\end{aligned}$$

where again we are justified dividing by $\varphi(x_1) - \varphi(x_2)$ due to Theorem 4.3.1, or by the trick outlined above in the case $r \neq s$, thus yielding

$$\mu - \Lambda^r\varphi(\xi) \geq (\mathrm{D}_x K_P^s(2x_2) + (\mu - 1)\mathrm{D}_x K_P^r(2x_2))|2x_1(x_2 - x_1)|. \tag{4.4.6}$$

Let $x_2 = x_0$ and $x_1 = x_0/2$ to obtain

$$\mu - \Lambda^r\varphi(\xi) \geq \tfrac{1}{2}x_0^2\,(\mathrm{D}_x K_P^s(2x_2) + (\mu - 1)\mathrm{D}_x K_P^r(2x_2)) \gtrsim_{r,s,P} |x_0|^{\min\{-r,-s\}} \tag{4.4.7}$$

since $\mathrm{D}_x K_P^s(x) \gtrsim_{s,P} |x|^{-s-2}$, likewise for $\mathrm{D}_x K_P^r(x)$, when $0 < |x| \ll 1$ according to Corollary 4.2.1.

Now consider the case where $\Lambda^r\varphi(0) = \mu$ when $r = s$. Then we do not necessarily have that φ is C^1 and therefore need to be more careful in our analysis. Indeed, we cannot use the nodal property theorem, so we have to prove strict monotonicity some other way. The double symmetrization formula reads

$$\begin{aligned}(\Lambda^s\varphi)&(x+h) - (\Lambda^s\varphi)(x-h)\\ &= \int_{-P/2}^{0} (K_P^s(y-x) - K_P^s(y+x))(\varphi(y+h) - \varphi(y-h))\,\mathrm{d}y\end{aligned}$$

which follows from the fact that φ and K_P^s are both even and periodic. It will then follow from Theorem 4.4.1 that φ is smooth away from $x = kP$ for $k \in \mathbb{Z}$. Note the identity

$$\begin{aligned}(2\mu - \Lambda^r\varphi(x) - \Lambda^r\varphi(y))(\varphi(x) - \varphi(y)) = 2(&\Lambda^s\varphi(x) - \Lambda^s\varphi(y))\\ &+ (\varphi(x) + \varphi(y))(\Lambda^r\varphi(x) - \Lambda^r\varphi(y))\end{aligned}$$

which when $r = s$ contracts to

$$(2\mu - \Lambda^r\varphi(x) - \Lambda^r\varphi(y))(\varphi(x) - \varphi(y)) = (2 + \varphi(x) + \varphi(y))(\Lambda^s\varphi(x) - \Lambda^s\varphi(y)).$$

Note that we obtain $\Lambda^s\varphi(x) = \Lambda^s\varphi(x)$ whenever $\varphi(x) = \varphi(y)$ holds.[1] If we take $x \mapsto x+h$, $y \mapsto x-h$ for $x \in (-P/2, 0)$ and $h \in (0, P/2)$, which after inserting into the previous identity and dividing by $2h$ amounts to

$$\begin{aligned}(2\mu - \Lambda^r\varphi(x+h) - \Lambda^r\varphi(x-h))&\frac{(\varphi(x+h) - \varphi(x-h))}{2h}\\ = (2 + \varphi(x+h) + \varphi(x-h))&\frac{(\Lambda^s\varphi(x+h) - \Lambda^s\varphi(x-h))}{2h}\end{aligned}$$

to which we can apply Fatou's lemma as $h \searrow 0$, and obtain in the limit that

$$(\mu - \Lambda^r\varphi(x))\varphi'(x) \geq (1 + \varphi(x)) \int_{-P/2}^{0} (K_P^s(y-x) + K_P^s(y+x))\varphi'(y)\,\mathrm{d}y$$

from which one can complete the remainder of the proof as before. □

[1] This property fails in the $r \neq s$ case, thus rendering us unable to justify the proof in said case.

We note a curious feature of our equation, which is at odds with the potential of a singularity of solutions satisfying $\Lambda^r\varphi(0) = \mu$. Indeed, the only solution with $\Lambda^r\varphi(0) = \mu$ with the assumptions of the nodal property theorem, except allowing for constant solutions, is the degenerated solutions $(\mu, \varphi) = (0, 0)$. The following result and its accompanying proof are original.

Theorem 4.4.2. *Let $r, s < 0$. Assume that φ is an even, continuous solution to Equation (4.1.2) that is non-decreasing on $(-P/2, 0)$ with $\Lambda^r\varphi < \mu$ on $(-P/2, 0)$ and $\Lambda^r\varphi(0) = \mu$. Then φ is identically equal to zero, and moreover μ has to be zero.*

Proof. We work with a period $P > 0$, however the following proof also works in the solitary case $P = \infty$. If φ satisfies the assumptions, then $\Lambda^r\varphi(0) = \mu$ implies that $\Lambda^s\varphi(0) = 0$ by the identity $\Lambda^s\varphi = (\mu - \Lambda^r\varphi)\varphi$. Assume φ is non-constant. Then the function $\Lambda^s\varphi(x)$ is strictly increasing on $(-P/2, 0)$ by the assumption $\varphi' \gneq 0$ and Item (iii) of Theorem 4.4.1 since $\Lambda^r\varphi < \mu$ on $(-P/2, 0)$ which ensures that φ' exists on $(-P/2, 0)$. Hence $\Lambda^s\varphi < 0$ on $(-P/2, 0)$. However then $\varphi(x) \neq 0$ for all $x \in (-P/2, 0)$ by $\Lambda^s\varphi = (\mu - \Lambda^r\varphi)\varphi$, which by the continuity of φ implies either $\varphi > 0$ or $\varphi < 0$ on $(-P/2, 0)$. Note that if $\varphi(0) = 0$ then one has that φ has to change signs in $x = 0$ since φ is assumed non-constant, which conflicts with the assumed evenness of φ. Hence we either have $\varphi > 0$ or $\varphi < 0$ everywhere. In either which case it is impossible that $\Lambda^s\varphi(0) = 0$, hence a contradiction, and thus φ has to be constant. Indeed, in order for $\Lambda^s\varphi(0) = 0$ we need $\varphi \equiv 0$. However, this also forces $\Lambda^r\varphi \equiv 0$ and hence $\mu = 0$. This completes the proof. □

In other related works like Ehrnström–Wahlén [23], Ørke [46], and Hildrum–Xue [31] where one has a non-trivial singularity analysis, the main focus is that of using first- and second order differences to infer properties like for instance the Hölder–regularity of the singularity at $x = 0$. For local nonlinearities like polynomials of the form φ^p, $p \geq 2$, this is readily achievable by employing Taylor's theorem, as is prominently done in Hildrum–Xue [31], to obtain an expression like

$$(\varphi(0) - \varphi(x))^2 \lesssim L\varphi(0) - L\varphi(x) \tag{4.4.8}$$

where L is the dispersive operator in consideration. The articles of Ehrnström–Wahlén [23], Ørke [46], and indeed any of the articles cited in Table 1.1, have some similar setup to Equation (4.4.8).

For the sake of discussion, let $-1 < r = s < 0$ and let us pretend as if we do not know that $\Lambda^s\varphi(0) = 0$ in the case of $\Lambda^r\varphi(0) = \mu$. Assume furthermore that $\varphi' \gneq 0$ and that φ is even. Then we have

$$(\mu - \Lambda^r\varphi(y))(\varphi(x_0) - \varphi(y)) = \Lambda^s\varphi(x_0) - \Lambda^s\varphi(y) + (\varphi(x_0) + \varphi(y))(\mu - \Lambda^r\varphi(y))$$

which estimates to

$$\begin{aligned}(\Lambda^r\varphi(x_0) - \Lambda^r\varphi(y))(\varphi(x_0) - \varphi(y)) &\leq \Lambda^s\varphi(x_0) - \Lambda^s\varphi(y) \\ &\quad + (\varphi(x_0) + \varphi(y))(\Lambda^r\varphi(x_0) - \Lambda^r\varphi(y)) \\ &\leq (1 + 2\|\varphi\|_{L^\infty})(\Lambda^s\varphi(x_0) - \Lambda^s\varphi(y))\end{aligned}$$

Now choose a $\delta > 0$ such that for $|y| < \delta$ we have $\Lambda^r\varphi(y) \geq 0$, then we are able to approximate the following

$$\begin{aligned}\mu(\varphi(x_0) - \varphi(y)) &= \Lambda^s\varphi(x_0) - \Lambda^s\varphi(y) + \varphi(x_0)\,\Lambda^r\varphi(x_0) - \varphi(y)\,\Lambda^r\varphi(y)\\ &\geq (1+\varphi(x_0))\Lambda^r\varphi(x_0) - (1+\varphi(y))\Lambda^r\varphi(y)\\ &\geq (1+\varphi(x_0))(\Lambda^r\varphi(x_0) - \Lambda^r\varphi(y))\\ &> \mu(\Lambda^r\varphi(x_0) - \Lambda^r\varphi(y))\end{aligned}$$

having used $\varphi(x_0) > \mu - 1$ from Lemma 4.3.2. Assuming $\mu > 0$ we are able to conclude $\Lambda^r\varphi(x_0) - \Lambda^r\varphi(y) < \varphi(x_0) - \varphi(y)$ and furthermore

$$(\Lambda^s\varphi(x_0) - \Lambda^s\varphi(y))^2 < (1 + 2\,\|\varphi\|_{L^\infty})(\Lambda^s\varphi(x_0) - \Lambda^s(y)). \tag{4.4.9}$$

Note the identity

$$\Lambda^s\varphi(0) - \Lambda^s(x) = \int_{\mathbb{R}} (K^s(x+y) + K^s(x-y) - 2K^s(y))(\varphi(0) - \varphi(y))\,\mathrm{d}y$$

which when inserted into Equation (4.4.9) with $x_0 = 0$ and $y = x$ yields

$$(\mu - \Lambda^r\varphi(x))^2 < (1 + 2\,\|\varphi\|_{L^\infty}) \int_{\mathbb{R}} (K^s(x+y) + K^s(x-y) - 2K^s(y))(\varphi(0) - \varphi(y))\,\mathrm{d}y \tag{4.4.10}$$

where it remains to approximate the integral on the right-hand side. From Corollary 4.2.1 we have that

$$K^s(x) = C_s|x|^{-s-1} + H^s(x), \tag{4.4.11}$$

where H_s is continuously differentiable with the estimate

$$|\mathrm{D}_x H^s(x)| \lesssim (1 + |x|)^{-s-2} \tag{4.4.12}$$

and the second derivative satisfies (see Ørke [46])

$$\begin{cases} |\mathrm{D}_x^2 H^s(x)| = O(|x|^{-s-1}) & \text{when } |x| < 1,\\ |\mathrm{D}_x^2 H^s(x)| \lesssim (1+|x|)^{-s-3} & \text{when } |x| \geq 1. \end{cases} \tag{4.4.13}$$

Applying the mean value theorem we obtain

$$|H^s(x+y) - H^s(x)| \leq |y| \int_0^1 |\mathrm{D}_x H^s(x+ty)|\,\mathrm{d}t = |y|\,\mathscr{H}^1(x;y)$$

where $\mathscr{H}^1(x;y)$ denotes the integral part of the latter equation. We also approximate

$$|H^s(x+y) - H^s(x-y) - 2H^s(x)| \leq |y|^2 \int_0^1 \int_0^1 2t|\mathrm{D}_x^2 H^s(x - ty + 2sty)|\,\mathrm{d}s\mathrm{d}t = |y|^2 \mathscr{H}^2(x;y).$$

Using Equation (4.4.11) to further estimate the right-hand side Equation (4.4.10) we first obtain for the singular part that

$$\begin{aligned}&\int_{\mathbb{R}} ||x+y|^{-s-1} + |x-y|^{-s-1} - 2|y|^{-s-1}|(\varphi(0) - \varphi(y))\,\mathrm{d}y\\ &\leq 2\,\|\varphi\|_{L^\infty}\,|x|^{-s} \int_{\mathbb{R}} ||1+t|^{-s-1} + |1-t|^{-s-1} - 2|t|^{-s-1}|\,\mathrm{d}t\\ &\lesssim |x|^{-s}\end{aligned}$$

which follows by changing the variable of integration and observing that the final estimate holds since the integral form of K^s ought to converge for all $-1 < s < 0$ in particular. Similarly, one estimates the regular part by

$$\begin{aligned}
&\int_{\mathbb{R}} |H^s(x+y) - H^s(x-y) - 2H^s(y)|(\varphi(0) - \varphi(y))\,\mathrm{d}y \\
&\lesssim \|\varphi\|_{L^\infty} |x|^2 \int_{\mathbb{R}} \mathscr{H}^2(y;x)\,\mathrm{d}x \\
&\lesssim |x|^2
\end{aligned}$$

which is owed in part by changing the variables of integration and also that the integral part $\mathscr{H}^2(y;x)$ converges which can be surmised from Equation (4.4.13). Putting the estimates together yields that $(\mu - \Lambda^s\varphi(x))^2 \lesssim |x|^{-s}$. Hence $\Lambda^r\varphi(x)$ is at least $-s/2$–Hölder continuous in the point $x = 0$. This is rather obvious since in any event $\varphi \in L^\infty$ and hence $\mu - \Lambda^r\varphi$ is $-r$–Hölder continuous by virtue of Λ^r raising the regularity of φ. However, we see that with this method we cannot bootstrap this process further since the two preceding integral inequalities use $\varphi(0) - \varphi(y)$ and not something like $\Lambda^s\varphi(0) - \Lambda^s\varphi(y)$. It would be a real challenge to find some estimate wherein $(\varphi(x) - \varphi(y))^2$ is propped up against an expression with only $\Lambda^s\varphi(x) - \Lambda^s\varphi(y)$, or something to that effect, in order to analyze singularities.

4.5 Global bifurcation analysis

In this section we will perform the global bifurcation analysis tied to travelling solutions of Equation (4.0.1). In this section we let $\alpha \in (\min\{|s|, |r|\}, 1)$ where $-1 < r, s < 0$, which means $\mathscr{C}^\alpha_{\text{even}}(\mathbb{S}_P) = C^{0,\alpha}_{\text{even}}(\mathbb{S}_P)$ as a Hölder–Zygmund space and its corresponding Hölder space. For brevity we denote $C^\alpha_{\text{even}}(\mathbb{S}_P) := C^{0,\alpha}_{\text{even}}(\mathbb{S}_P)$ in this section. Furthermore, we define the map $F\colon \mathbb{R} \times C^\alpha_{\text{even}}(\mathbb{S}_P) \to C^\alpha_{\text{even}}(\mathbb{S}_P)$ by the operator

$$F(\mu, \varphi) = \mu\varphi - \Lambda^s\varphi - \varphi\,\Lambda^r\varphi. \tag{4.5.1}$$

The bound on α is to ensure the convergence of Fourier series, and for the sake of argument since we need not surpass index 1 in the scale of Hölder–Zygmund spaces to carry out the analysis below.

Theorem 4.5.1 (Sub-, super- and transcritical bifurcation).

(i) Sub- and supercritical bifurcation.
For each finite period $P > 0$ and each integer $k \geq 1$ there exists $\mu^*_{P,k} = \langle 2\pi k/P\rangle^s = m_s(2\pi k/P)$ and a local, analytic curve given by

$$t \mapsto (\mu_{P,k}(t), \varphi_{P,k}(t)) \in \mathbb{R} \times C^\alpha_{\text{even}}(\mathbb{S}_P)$$

of nontrivial P/k-periodic solutions to Equation (4.1.2) with tangent $\mathrm{D}_t\varphi_{P,k}(0) = \cos(2\pi k \cdot /P)$ which bifurcates from the trivial solution curve the point $\mu \mapsto (\mu, 0)$ at $(\mu_{P,k}(0), \varphi_{P,k}(0)) = (\mu^*_{P,k}, 0)$. Furthermore, the curve admits a parametrization such *that the map $t \mapsto \mu_{P,k}(t)$ is even, and there exists positive numbers $P_1 < P_2$ with the property that*

$$\mu''_{P<kP_1,k}(0) > 0, \quad \mu''_{P>kP_1,k}(0) < 0.$$

Therefore a subcritical pitchfork bifurcation takes place at $(\mu^*_{P,k}, 0)$ for $P > kP_2$, and a supercritical pitchfork bifurcation happens at the same point for $P < kP_1$.

(ii) Transcritical bifurcation. *At $\mu = 1$ the trivial solution curve $\mu \mapsto (\mu, 0)$ intersects the curve $\mu \mapsto (\mu, \mu - 1)$ of constant solutions $\varphi_0 \equiv \mu - 1$.*

Thus, all nonzero solutions of $F(\mu, \varphi) = 0$ in a neighborhood of the trivial solution curve $\{(\mu, 0) \mid \mu \in \mathbb{R}\}$ in $\mathbb{R} \times C^\alpha_{\text{even}}(\mathbb{S}_P)$ are either of the category (i) or (ii).

Remark. As is evident from the proof of the theorem, only the quotient of k and P makes a material difference in distinguishing solutions. Two solution branches with similar quotients $k/P = \tilde{k}/\tilde{P}$ have to coincide locally near the bifurcation points from which they emanate. However, it is possible for these curves to deviate when extended by global continuation. We do not consider bifurcations from $\mu > 1$ here, due to lack of Galilean transformations as noted at the beginning of this chapter.

Remark. The following proof is heavily inspired by Ehrnström–Wahlén [23, Theorem 6.1], however we are dealing with an equation wherein the nonlocal nonlinearity adds a bit of complexity to the formulae involved.

Proof. It suffices to consider the case $k = 1$, whereas one can retrieve the general case from rescaling the period P. Denote $(\mu_{P,1}(t), \varphi_{P,1}(t)) = (\mu(t), \varphi(t))$ for the sake of brevity. We start by showing that $t \mapsto \mu(t)$ admits a parametrization such that

$$\mu(t) = \mu(-t). \tag{4.5.2}$$

Denote the cosine-coefficients of a P-periodic function φ by

$$[\varphi]_j = \frac{2}{P} \int_{-P/2}^{P/2} \varphi(x) \cos\left(\frac{2\pi j x}{P}\right) \mathrm{d}x$$

for $j = 0, 1, 2, \ldots$ such that we can represent φ through the even Fourier series

$$\varphi(x) = \frac{[\varphi]_0}{2} + \sum_{j=1}^{\infty} [\varphi]_j \cos\left(\frac{2\pi j x}{P}\right).$$

Now we wish to parametrize the local bifurcation curve such that $[\varphi(t)]_1 = t$. This corresponds with the parametrization given in the Lyapunov–Schmidt reduction. Since (μ, φ) is an even P-periodic solution, the curve $(\mu, \varphi(\cdot + P/2))$ is also a P-periodic even solution satisfying

$$[\varphi(\cdot + P/2)]_1 = -[\varphi]_1.$$

hence $[\varphi(t)(\cdot + P/2)]_1 = -[\varphi(t)]_1 = -t$ and so it follows by uniqueness of the cosine-expansion that

$$(\mu(t), \varphi(t)) = (\mu(-t), \varphi(-t))$$

which yields that $\mu(t)$ is indeed even in t.

Since μ is analytic, and hence analytic in t, we therefore write

$$\mu(t) = \sum_{n=0}^{\infty} \mu_{2n} t^{2n}$$

with uniform convergence in t in a neighborhood of the origin. We also represent the function φ through the series

$$\varphi(t) = \sum_{n=1}^{\infty} \varphi_n t^n$$

with convergence in $C^{\alpha}_{\mathrm{even}}(\mathbb{S}_P)$. Then we apply the operator Λ^s to φ to obtain

$$\Lambda^s \varphi(t) = \sum_{n=1}^{\infty} \Lambda^s \varphi_n \, t^n$$

and applying the operator Λ^r to φ and subsequently multiplying by φ from the left we have that the nonlinearity reads as

$$\varphi(t)\, \Lambda^r \varphi(t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varphi_m \, \Lambda^r \varphi_n \, t^{m+n}$$

and also we should note that

$$\mu(t)\varphi(t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \mu_{2m} \, \varphi_n \, t^{2m+n}.$$

Because of uniqueness we may compute the coefficients by substituting the series expansions as above into Equation (4.1.2) and collecting terms of equal order in t. Then we have

$$0 = (-\mu_0 \varphi_1 + \Lambda^s \varphi_1) t + (-\mu_0 \varphi_2 + \Lambda^s \varphi_2 + \varphi_1 \, \Lambda^r \varphi_1) t^2 + \dots$$

which after comparing coefficients we have, up to and including fifth order

$$\Lambda^s \varphi_1 - \mu_0 \varphi_1 = 0, \tag{4.5.3}$$
$$\Lambda^s \varphi_2 - \mu_0 \varphi_2 = -\varphi_1 \, \Lambda^r \varphi_1, \tag{4.5.4}$$
$$\Lambda^s \varphi_3 - \mu_0 \varphi_3 = \mu_2 \varphi_1 - \varphi_1 \, \Lambda^r \varphi_2 - \varphi_2 \, \Lambda^r \varphi_1, \tag{4.5.5}$$
$$\Lambda^s \varphi_4 - \mu_0 \varphi_4 = \mu_2 \varphi_2 - \varphi_1 \, \Lambda^r \varphi_3 - \varphi_3 \, \Lambda^r \varphi_1 - \varphi_2 \, \Lambda^r \varphi_2, \tag{4.5.6}$$
$$\Lambda^s \varphi_5 - \mu_0 \varphi_5 = \mu_2 \varphi_3 + \mu_4 \varphi_1 - \varphi_1 \, \Lambda^r \varphi_4 - \varphi_4 \, \Lambda^r \varphi_1 - \varphi_2 \, \Lambda^r \varphi_3 - \varphi_3 \, \Lambda^r \varphi_2. \tag{4.5.7}$$

As in the local bifurcation case we have that $\varphi_1(x) = \cos(\xi x)$ and $\mu_0 = \langle \xi \rangle^s = m_s(\xi)$ where in this case $\xi = 2\pi/P$. The remaining coefficients in the power series determining μ can be deduced from the requirement that the right-hand side has to lie in the range of the linear operator defined by the left-hand side. It is then clear how we can iterate this process to compute the functions φ_n. By our choice of parametrization we have that $[\varphi_n]_1 = 0$ for each $n \geq 2$.

Applying Λ^r to φ_1 amounts to multiplying φ_1 by $m_r(\xi)$ as follows from the computation $\mathscr{F}(\cos(\xi x))(\zeta) = \pi(\delta_\xi + \delta_{-\xi})$ and so

$$\begin{aligned}
\Lambda^r\varphi_1(x) &= \mathscr{F}^{-1}(\langle\zeta\rangle^r\mathscr{F}(\cos(\xi x))) = \pi\mathscr{F}^{-1}(\langle\zeta\rangle^r(\delta_\xi + \delta_{-\xi})) \\
&= \frac{1}{2}\int_{\mathbb{R}} \langle\zeta\rangle^r\delta(\zeta - \xi)\, e^{ix\zeta}\,\mathrm{d}\zeta + \frac{1}{2}\int_{\mathbb{R}} \langle\zeta\rangle^r\delta(\zeta + \xi)\, e^{ix\zeta}\,\mathrm{d}\zeta \\
&= \frac{1}{2}(m_r(\xi)e^{ix\xi} + m_r(-\xi)e^{-ix\xi}) \\
&= m_r(\xi)\cos(\xi x),
\end{aligned}$$

where we have used the physicist's notation for the Dirac delta distribution as an integral kernel and the evenness of $m_r(\xi)$. This implies that the right hand side of Equation (4.5.3) reduces to

$$-\varphi_1\,\Lambda^r\varphi_1 = -\frac{m_r(\xi)}{2} - \frac{m_r(\xi)}{2}\cos(2\xi x),$$

due to the angle-doubling formula for $\cos(x)$. Then we immediately obtain

$$\varphi_2(x) = -\frac{m_r(\xi)}{2(m_s(0) - m_s(\xi))} - \frac{m_r(\xi)}{2(m_s(2\xi) - m_s(\xi))}\cos(2\xi x)$$

where we note that the action of Λ^s on $\cos(2\xi x)$ is just multiplying by $m_s(2\xi)$ as

$$\Lambda^s\cos(2\xi x) = m_s(2\xi)\cos(2\xi x).$$

which can be used to verify that φ_2 assumes the form as above. This generalizes to $\Lambda^s\cos(k\xi x) = m_s(k\xi)\cos(k\xi x)$ as is readily seen from the computation of $\Lambda^r\varphi_1$. Recall that cosine has a recurrence relation given by

$$\cos(k\xi x) = 2\cos((k-1)\xi x)\cos(\xi x) - \cos((k-2)\xi x).$$

Next we compute the mixed terms

$$\begin{aligned}
-\varphi_1\,\Lambda^r\varphi_2 &= -\cos(\xi x)\left(-\frac{m_r(\xi)}{2(m_s(0) - m_s(\xi))} - \frac{m_r(\xi)\,m_r(2\xi)}{2(m_s(2\xi) - m_s(\xi))}\cos(2\xi x)\right) \\
&= \left(\frac{m_r(\xi)}{2(m_s(0) - m_s(\xi))} + \frac{m_r(\xi)\,m_r(2\xi)}{4(m_s(2\xi) - m_s(\xi))}\right)\cos(\xi x) + \frac{m_r(\xi)\,m_r(2\xi)}{4(m_s(2\xi) - m_s(\xi))}\cos(3\xi x)
\end{aligned}$$

and also

$$\begin{aligned}
-\varphi_2\,\Lambda^r\varphi_1 &= \left(\frac{m_r(\xi)}{2(m_s(0) - m_s(\xi))} + \frac{m_r(\xi)}{2(m_s(2\xi) - m_s(\xi))}\cos(2\xi x)\right)m_r(\xi)\cos(\xi x) \\
&= \left(\frac{m_{2r}(\xi)}{2(m_s(0) - m_s(\xi))} + \frac{m_{2r}(\xi)}{4(m_s(2\xi) - m_s(\xi))}\right)\cos(\xi x) + \frac{m_{2r}(\xi)}{4(m_s(2\xi) - m_s(\xi))}\cos(3\xi x).
\end{aligned}$$

The right-hand side of Equation (4.5.5) becomes

$$\begin{aligned}
&\left(\mu_2 + \frac{m_{2r}(\xi) + m_r(\xi)}{2(m_s(0) - m_s(\xi))} + \frac{m_r(\xi)(m_r(\xi) + m_r(2\xi))(\xi)}{4(m_s(2\xi) - m_s(\xi))}\right)\cos(\xi x) \\
&\quad + \frac{m_r(\xi)(m_r(\xi) + m_r(2\xi))}{4(m_s(2\xi) - m_s(\xi))}\cos(3\xi x).
\end{aligned}$$

Hence we see that since the range of $\Lambda^s\varphi_3 - \mu_0\varphi_3$ consists of functions like $C(\xi)\cos(3\xi x)$, then μ_2 is determined by the relation

$$\mu_2 = \frac{m_{2r}(\xi) + m_r(\xi)}{2(m_s(\xi) - m_s(0))} + \frac{m_r(\xi)(m_r(\xi) + m_r(2\xi))(\xi)}{4(m_s(\xi) - m_s(2\xi))}.$$

Noting that $m_{2r}(\xi) + m_r(\xi) \geq 2$ and $m_r(\xi)(m_r(\xi) + m_r(2\xi)) \geq 2$ for all ξ we have that the sign of μ_2 is controlled via the expression

$$\frac{1}{m_s(\xi) - m_s(0)} + \frac{1}{2(m_s(\xi) - m_s(2\xi))}$$

which is negative as $\xi \searrow 0$ and positive as $\xi \nearrow \infty$. Thus, the numbers P_1 and P_2 as in the statement of the theorem exist. □

Remark. We include the expansion up to fifth order in order to illuminate the nature of the power series, and also to appeal to the following calculation for the fourth-order derivative $\mu^{(4)}(0)$ as in Remark 6.3 in Ehrnström–Wahlén [23]. Solving Equation (4.5.5) amounts to

$$\varphi_3(x) = \frac{m_r(\xi)(m_r(\xi) + m_r(2\xi))}{4(m_s(2\xi) - m_s(\xi))(m_s(3\xi) - m_s(\xi))} \cos(3\xi x).$$

We readily compute the mixed term

$$\begin{aligned}
&-\varphi_1 \Lambda^r \varphi_3 = -\cos(\xi x)\left(\frac{m_r(\xi)(m_r(\xi) + m_r(2\xi))}{4(m_s(2\xi) - m_s(\xi))(m_s(3\xi) - m_s(\xi))} m_r(3\xi)\cos(3\xi x)\right) \\
&= -\frac{m_r(3\xi)\, m_r(\xi)(m_r(\xi) + m_r(2\xi))}{8(m_s(2\xi) - m_s(\xi))(m_s(3\xi) - m_s(\xi))} \cos(2\xi x) \\
&\quad - \frac{m_r(3\xi)\, m_r(\xi)(m_r(\xi) + m_r(2\xi))}{8(m_s(2\xi) - m_s(\xi))(m_s(3\xi) - m_s(\xi))} \cos(4\xi x)
\end{aligned}$$

and likewise the term

$$\begin{aligned}
&-\varphi_3 \Lambda^r \varphi_1 = -\left(\frac{m_r(\xi)(m_r(\xi) + m_r(2\xi))}{4(m_s(2\xi) - m_s(\xi))(m_s(3\xi) - m_s(\xi))} \cos(3\xi x)\right) m_r(\xi)\cos(\xi x) \\
&= -\frac{m_{2r}(\xi)(m_r(\xi) + m_r(2\xi))}{8(m_s(2\xi) - m_s(\xi))(m_s(3\xi) - m_s(\xi))} \cos(2\xi x) \\
&\quad - \frac{m_{2r}(\xi)(m_r(\xi) + m_r(2\xi))}{8(m_s(2\xi) - m_s(\xi))(m_s(3\xi) - m_s(\xi))} \cos(4\xi x)
\end{aligned}$$

where $m_{2r}(\xi) = (m_r(\xi))^2 = \langle\xi\rangle^{2r}$. Using the identity $2\cos(2\xi x)^2 = 1 + \cos(4\xi x)$ we obtain

$$
\begin{aligned}
-\varphi_2\,\Lambda^r\varphi_2 &= \left(\frac{m_r(\xi)}{2(m_s(0)-m_s(\xi))}+\frac{m_r(\xi)}{2(m_s(2\xi)-m_s(\xi))}\cos(2\xi x)\right)\\
&\quad\times\left(-\frac{m_r(\xi)}{2(m_s(0)-m_s(\xi))}-\frac{m_r(2\xi)\,m_r(\xi)}{2(m_s(2\xi)-m_s(\xi))}\cos(2\xi x)\right)\\
&= -\frac{m_{2r}(\xi)}{4(m_s(0)-m_s(\xi))^2}-\frac{m_{2r}(\xi)\,m_r(2\xi)}{8(m_s(2\xi)-m_s(\xi))^2}\\
&\quad-\left(\frac{m_{2r}(\xi)(m_r(2\xi)+1)}{4(m_s(2\xi)-m_s(\xi))(m_s(0)-m_s(\xi))}\right)\cos(2\xi x)\\
&\quad-\frac{m_{2r}(\xi)\,m_r(2\xi)}{8(m_s(2\xi)-m_s(\xi))^2}\cos(4\xi x),
\end{aligned}
$$

and the remaining term to compute becomes

$$
\begin{aligned}
\mu_2\varphi_2 &= \left(\frac{m_{2r}(\xi)+m_r(\xi)}{2(m_s(\xi)-m_s(0))}+\frac{m_r(\xi)(m_r(\xi)+m_r(2\xi))}{4(m_s(\xi)-m_s(2\xi))}\right)\\
&\quad\times\left(-\frac{m_r(\xi)}{2(m_s(0)-m_s(\xi))}-\frac{m_r(\xi)}{2(m_s(2\xi)-m_s(\xi))}\cos(2\xi x)\right)\\
&= \frac{m_r(\xi)(m_{2r}(\xi)+m_r(\xi))}{4(m(0)-m_s(\xi))^2}-\frac{m_{2r}(\xi)(m_r(\xi)+m_r(2\xi))}{8(m_s(0)-m_s(\xi))(m_s(\xi)-m_s(2\xi))}\\
&\quad+\left(\frac{m_{2r}(\xi)(m_r(\xi)+m_r(2\xi))}{8(m_s(\xi)-m_s(2\xi))^2}+\frac{m_r(\xi)(m_{2r}(\xi)+m_r(\xi))}{4(m_s(0)-m_s(\xi))(m_s(2\xi)-m_s(\xi))}\right)\cos(2\xi x)
\end{aligned}
$$

Putting it all together, the right-hand side of Equation (4.5.6) now becomes

$$
\begin{aligned}
&\frac{m_r(\xi)(m_{2r}(\xi)+m_r(\xi))}{4(m(0)-m_s(\xi))^2}-\frac{m_{2r}(\xi)(m_r(\xi)+m_r(2\xi))}{8(m_s(0)-m_s(\xi))(m_s(\xi)-m_s(2\xi))}\\
&\quad+\left(\frac{m_{2r}(\xi)(m_r(\xi)+m_r(2\xi))}{8(m_s(\xi)-m_s(2\xi))^2}+\frac{m_r(\xi)(m_{2r}(\xi)+m_r(\xi))}{4(m_s(0)-m_s(\xi))(m_s(2\xi)-m_s(\xi))}\right)\cos(2\xi x)\\
&-\frac{m_r(3\xi)\,m_r(\xi)(m_r(\xi)+m_r(2\xi))}{8(m_s(2\xi)-m_s(\xi))(m_s(3\xi)-m_s(\xi))}\cos(2\xi x)\\
&\quad-\frac{m_r(3\xi)\,m_r(\xi)(m_r(\xi)+m_r(2\xi))}{8(m_s(2\xi)-m_s(\xi))(m_s(3\xi)-m_s(\xi))}\cos(4\xi x)\\
&-\frac{m_{2r}(\xi)(m_r(\xi)+m_r(2\xi))}{8(m_s(2\xi)-m_s(\xi))(m_s(3\xi)-m_s(\xi))}\cos(2\xi x)\\
&\quad-\frac{m_{2r}(\xi)(m_r(\xi)+m_r(2\xi))}{8(m_s(2\xi)-m_s(\xi))(m_s(3\xi)-m_s(\xi))}\cos(4\xi x)\\
&-\frac{m_{2r}(\xi)}{4(m_s(0)-m_s(\xi))^2}-\frac{m_{2r}(\xi)\,m_r(2\xi)}{8(m_s(2\xi)-m_s(\xi))^2}\\
&\quad-\left(\frac{m_{2r}(\xi)(m_r(2\xi)+1)}{4(m_s(2\xi)-m_s(\xi))(m_s(0)-m_s(\xi))}\right)\cos(2\xi x)\\
&\quad-\frac{m_{2r}(\xi)\,m_r(2\xi)}{8(m_s(2\xi)-m_s(\xi))^2}\cos(4\xi x),
\end{aligned}
$$

which after collecting terms and simplifying where possible becomes

$$\frac{m_r(\xi)(m_{2r}(\xi)+m_r(\xi))-m_{2r}(\xi)}{4(m(0)-m_s(\xi))^2}-\frac{m_{2r}(\xi)(m_r(\xi)+m_r(2\xi))}{8(m_s(0)-m_s(\xi))(m_s(\xi)-m_s(2\xi))}$$
$$-\frac{m_{2r}(\xi)\,m_r(2\xi)}{8(m_s(2\xi)-m_s(\xi))^2}$$
$$+\Bigg(\frac{m_{2r}(\xi)(m_r(\xi)+m_r(2\xi))}{8(m_s(\xi)-m_s(2\xi))^2}+\frac{m_{2r}(\xi)(m_r(\xi)-m_r(2\xi))}{4(m_s(0)-m_s(\xi))(m_s(2\xi)-m_s(\xi))}$$
$$-\frac{(m_r(3\xi)\,m_r(\xi)+m_{2r}(\xi))(m_r(\xi)+m_r(2\xi))}{8(m_s(2\xi)-m_s(\xi))(m_s(3\xi)-m_s(\xi))}\Bigg)\cos(2\xi x)$$
$$-\Bigg(\frac{m_{2r}(\xi)\,m_r(2\xi)}{8(m_s(2\xi)-m_s(\xi))^2}+\frac{(m_r(3\xi)\,m_r(\xi)+m_{2r}(\xi))(m_r(\xi)+m_r(2\xi))}{8(m_s(2\xi)-m_s(\xi))(m_s(3\xi)-m_s(\xi))}\Bigg)\cos(4\xi x)$$

where we note that, as a sanity check for correspondence with [23], if $r=0$ the second term in the bracket of the $\cos(2\xi x)$ term vanishes – as it should. We then calculate

$$\varphi_4(x)=\frac{m_{3r}(\xi)}{4(m(0)-m_s(\xi))^3}-\frac{m_{2r}(\xi)(m_r(\xi)+m_r(2\xi))}{8(m_s(0)-m_s(\xi))^2(m_s(\xi)-m_s(2\xi))}$$
$$-\frac{m_{2r}(\xi)\,m_r(2\xi)}{8(m_s(0)-m_s(\xi))(m_s(2\xi)-m_s(\xi))^2}$$
$$+\Bigg(\frac{m_{2r}(\xi)(m_r(\xi)+m_r(2\xi))}{8(m_s(2\xi)-m_s(\xi))^3}+\frac{m_{2r}(\xi)(m_r(\xi)-m_r(2\xi))}{4(m_s(0)-m_s(\xi))(m_s(2\xi)-m_s(\xi))^2}$$
$$-\frac{(m_r(3\xi)\,m_r(\xi)+m_{2r}(\xi))(m_r(\xi)+m_r(2\xi))}{8(m_s(2\xi)-m_s(\xi))^2(m_s(3\xi)-m_s(\xi))}\Bigg)\cos(2\xi x)$$
$$-\frac{1}{m_s(4\xi)-m_s(\xi)}$$
$$\times\Bigg(\frac{m_{2r}(\xi)\,m_r(2\xi)}{8(m_s(2\xi)-m_s(\xi))^2}+\frac{(m_r(3\xi)\,m_r(\xi)+m_{2r}(\xi))(m_r(\xi)+m_r(2\xi))}{8(m_s(2\xi)-m_s(\xi))(m_s(3\xi)-m_s(\xi))}\Bigg)\cos(4\xi x).$$

We need to compute the prefactor of $\cos(\xi x)$ from the right-hand side of Equation (4.5.7). To achieve this, we first remark the identities $2\cos(4\xi x)\cos(\xi x)=\cos(5\xi x)+\cos(3\xi x)$, $2\cos(2\xi x)\cos(\xi x)=\cos(3\xi x)+\cos(\xi x)$ and $2\cos(3\xi x)\cos(2\xi x)=\cos(5\xi x)+\cos(\xi x)$, hence the prefactor of the $\cos(\xi x)$ term is

$$\mu_4-(m_r(\xi)+m_r(4\xi))\Bigg(\frac{m_{3r}(\xi)}{4(m(0)-m_s(\xi))^3}$$
$$-\frac{m_{2r}(\xi)(m_r(\xi)+m_r(2\xi))}{8(m_s(0)-m_s(\xi))^2(m_s(\xi)-m_s(2\xi))}-\frac{m_{2r}(\xi)\,m_r(2\xi)}{8(m_s(0)-m_s(\xi))(m_s(2\xi)-m_s(\xi))^2}\Bigg)$$
$$-\frac{m_r(\xi)+m_r(4\xi)}{2}\Bigg(\frac{m_{2r}(\xi)(m_r(\xi)+m_r(2\xi))}{8(m_s(2\xi)-m_s(\xi))^3}+\frac{m_{2r}(\xi)(m_r(\xi)-m_r(2\xi))}{4(m_s(0)-m_s(\xi))(m_s(2\xi)-m_s(\xi))^2}$$
$$-\frac{(m_r(3\xi)\,m_r(\xi)+m_{2r}(\xi))(m_r(\xi)+m_r(2\xi))}{8(m_s(2\xi)-m_s(\xi))^2(m_s(3\xi)-m_s(\xi))}\Bigg)$$
$$+\frac{m_{2r}(\xi)m_r(3\xi)(m_r(\xi)+m_r(2\xi))}{16(m_s(2\xi)-m_s(\xi))^2(m_s(3\xi)-m_s(\xi))}+\frac{m_{2r}(\xi)m_r(2\xi)(m_r(\xi)+m_r(2\xi))}{16(m_s(2\xi)-m_s(\xi))^2(m_s(3\xi)-m_s(\xi))}$$

which after simplifying and isolating for μ_4 in recognition of how μ_n's are calculated, we obtain

$$\begin{aligned}\mu_4 &= \frac{m_r(\xi)+m_r(4\xi)}{4(m_s(0)-m_s(\xi))^2}\left(\frac{m_{3r}(\xi)}{m_s(0)-m_s(\xi)}+\frac{m_{2r}(\xi)(m_r(\xi)+m_r(2\xi))}{2(m_s(2\xi)-m_s(\xi))}\right)\\ &\quad-\frac{1}{8(m_s(2\xi)-m_s(\xi))^2}\Bigg(\frac{m_{2r}(\xi)(m_r(2\xi)-m_r(\xi))}{m_s(0)-m_s(\xi)}+\frac{m_{2r}(\xi)\,m_r(2\xi)}{m_s(0)-m_s(\xi)}+\\ &\quad\frac{(2(m_r(\xi)+m_r(4\xi))m_r(3\xi)\,m_r(\xi)+m_{2r}(\xi)\,m_r(3\xi)+m_{2r}(\xi)\,m_r(2\xi))(m_r(\xi)+m_r(2\xi))}{2(m_s(3\xi)-m_s(\xi))}\Bigg)\\ &\quad+\frac{(m_r(\xi)+m_r(4\xi))(m_{2r}(\xi)(m_r(\xi)+m_r(2\xi)))}{16(m_s(2\xi)-m_s(\xi))^3}.\end{aligned}$$

Note in particular that, as a sanity check, the formulae all reduce to the case of the fKdV equation [46] or, barring some notation, the Whitham equation when $r=0$. In view of [23, Remark 6.3], there is probably some suited choice $P_{r,s}$ for every pair (r,s) such that one obtains a similar result to that of the Whitham equation however it is difficult to see what such a value would be for every choice of (r,s). Again, the result of Ehrnström–Wahlén is only numerically verified and not analytically verified. We shall not attempt analytical verification here either.

Remark (Correspondence between Lyapunov–Schmidt and the method of comparing orders). The method of calculating bifurcation formulas as in the proof of Theorem 4.5.1 is equivalent to that of Lyapunov–Schmidt. Following the footsteps of Ehrnström–Kalisch [21] we have that as in the rephrased statement one has

$$\varphi(t)=t\varphi^*+\psi(\mu(t),t\varphi^*).$$

Furthermore we have that the main branch bifurcation direction is as expected $\varphi'(0)=\cos(\xi x)=\varphi^*$, which is seen by the chain rule noting that

$$\mathrm{D}_t\psi(\mu(t)),t\varphi^*)=\partial_\mu\psi[(\mu(t)),t\varphi^*)]+\partial_\varphi\psi[(\mu(t),t\varphi^*)]\varphi^*$$

vanishes when evaluated at $t=0$ since $\partial_\varphi\psi[(\mu^*,0)]=0$ and $\partial_\mu\psi[(\mu^*,0)]=0$ since $\psi(\mu,0)\equiv 0$. Computing the second derivative we obtain

$$\begin{aligned}\varphi''(t)&=\partial^2_{\varphi\varphi}\psi[(\mu(t),t\varphi^*)](\varphi^*,\varphi^*)+2\partial^2_{\varphi\mu}\psi[(\mu(t),t\varphi^*)](\mu'(t),\varphi^*)\\ &\quad+\partial^2_{\varphi\varphi}\psi[(\mu(t),t\varphi^*)](\mu'(t),\mu'(t))+\partial_\varphi\psi[(\mu(t),t\varphi^*)]\mu'(t)\end{aligned}$$

which evaluated at $t=0$ combined with the relations $\mu'(0)=0$, $\partial_\varphi\psi[(\mu^*,0)]=0$ and $\partial_\mu\psi[(\mu^*,0)]=0$ yields

$$\varphi''(0)=\partial^2_{\varphi\varphi}\psi[(\mu^*,0)](\varphi^*,\varphi^*).$$

We have by formulae established in [37, Section I.6] that

$$\partial^2_{\varphi\varphi}\psi[(\mu^*,0)]=-(\partial_\varphi F[(\mu^*,0)])^{-1}\,(\mathrm{Id}-\Pi)\partial^2_{\varphi\varphi}F[(\mu^*,0)](\varphi^*,\varphi^*)$$

which is well-defined since $\partial_\varphi F[(\mu^*,0)]$ is an isomorphism on M as in the setup to Theorem 4.1.1. Hence we calculate

$$\partial^2_{\varphi\varphi}F[(\mu^*,0)](\varphi^*,\varphi^*)=\varphi^*\Lambda^r\varphi^*$$

which then means that

$$\begin{aligned}\partial^2_{\varphi\varphi}\psi[(\mu^*,0)] &= -(\partial_\varphi F[(\mu^*,0)])^{-1}\,(\mathrm{Id}-\Pi)(\varphi^*\Lambda^r\varphi^*)\\ &= (\partial_\varphi F[(\mu^*,0)])^{-1}(-\varphi^*\Lambda^r\varphi^*)\\ &= -\frac{m_r(\xi)}{2(1-\mu^*)} - \frac{m_r(\xi)}{2(m_s(2\xi)-\mu^*)}\cos(2\xi x),\end{aligned}$$

noting $\langle \varphi^*\Lambda^r\varphi^*, \varphi^*\rangle_{L^2(\mathbb{S}_P)} = 0$, where $m_s(\xi) = \mu_0 = \mu^*$ as in the proof of Theorem 4.5.1. Hence one has equality up to second order, and one can continue like this.

We define the set U by

$$U = \{(\mu,\varphi) \in \mathbb{R}\times C^\alpha_{\text{even}}(\mathbb{S}_P) \mid \Lambda^r\varphi < \mu\},$$

and the solution set S as in the setup for global bifurcation theory

$$S = \{(\mu,\varphi) \in U \mid F(\mu,\varphi) = 0\}$$

where F is given as in Equation (4.5.1).

The following compactness proof follows the spirit of Hildrum–Xue [31], as they follow a less stringent method of proof than that of e.g. Ehrnström–Kalisch [21].

Lemma 4.5.1. *(Compactness). Let $r, s < 0$. Then bounded and closed sets of S are compact in $\mathbb{R}_{\geq 0}\times C^\alpha_{\text{even}}(\mathbb{S}_P)$.*

Proof. We regard the fixed-point equation as in the proof of Theorem 4.4.1

$$\varphi = G(\varphi;\mu) = (\mu - \Lambda^r\varphi)^{-1}\Lambda^s\varphi$$

which sends (μ,φ) to $\varphi \in S$, and also boundedly maps S into C^m for any $m \geq 1$ in light of the proof of Theorem 4.4.1. Since $C^\alpha_{\text{even}}(\mathbb{S}_P)$ is embedded compactly in $C^\beta_{\text{even}}(\mathbb{S}_P)$ whenever $\beta < \alpha$ by Theorem 2.6.1 we see that G maps bounded sets of S into precompact subsets of $C^\alpha_{\text{even}}(\mathbb{S}_P)$ by composing maps. Hence by the Bolzano–Weierstrass theorem (c.f. DiBenedetto [16]) if $\{(\mu_j,\varphi_j)\}_j$ is a sequence of a bounded subset $K \subseteq S$, then a subsequence of $\{\varphi_j\}_j$ converges in $C^\alpha_{\text{even}}(\mathbb{S}_P)$ due to precompactness, so Bolzano–Weierstrass implies that a subsequence of the original sequence $\{\mu_j,\varphi_j\}_j$ has to converge in the topology of $\mathbb{R}_{\geq 0}\times C^\alpha_{\text{even}}(\mathbb{S}_P)$. If K is also closed then K has to be compact, finishing the proof. □

Theorem 4.5.2 (Global bifurcation)**.**
Let $r, s < 0$. Whenever $\mu^{(j)}_{P,1}(0) \neq 0$ for some $j \in \mathbb{N}$ for $\mu_{P,1}(t)$ as in the map $t \mapsto (\mu_{P,1}(t), \varphi_{P,1}(t))$ of solutions in Theorem 4.5.1, then these solution curves extend to continuous, global curves of solutions $\mathfrak{R}_P\colon \mathbb{R}_{\geq 0} \to S$, which are locally real-analytically reparametrizable around every $t > 0$. Furthermore, one of the following alternatives are true:

(i) $\|(\mu_{P,1}(t), \varphi_{P,1}(t))\|_{\mathbb{R}\times C^\alpha(\mathbb{S}_P)} \to \infty$ *as* $t \to \infty$*.*

(ii) $\operatorname{dist}(\mathfrak{R}_P, \partial U) = 0$.

(iii) The map $t \mapsto (\mu_{P,1}(t), \varphi_{P,1}(t))$ is T-periodic for some finite $T > 0$.

Proof. Indeed, this follows from Theorem 3.5.1 having proven everything needed in the setup. We have seen that $\mu''_{P,1}(0) \neq 0$ in certain cases by the local bifurcation analysis from Theorem 4.5.1, however we need the equivalent condition $\mu^{(j)}_{P,1}(0) \neq 0$ to hold in general, thus we have to assume this. □

In the article of Ehrnström and Wahlén [23] the analysis concerning the nodal property theorem culminates in proving that no bifurcation branch returns to itself in a periodic fashion as in Item (iii) in Theorem 4.5.2. To the same end, we define

$$\mathscr{K} = \{\varphi \in C^{\alpha}_{\text{even}}(\mathbb{S}_P) \mid \varphi \text{ nondecreasing on } (-P/2, 0)\}$$

as a closed cone in $C^{\alpha}(\mathbb{S}_P)$. For brevity, let $\varphi(t) = \varphi_{P,1}(t)$ and $\mu(t) = \mu_{P,1}(t)$, $\mathfrak{R} = \mathfrak{R}_P$, and also let $\mathfrak{R}^2$ and S^2 denote the φ-components of the branch main branch $\mathfrak{R}$ and solution set S respectively. Additionally, define $\mu^* = \mu^*_{P,1}$ as the primary bifurcation point along with

$$\varphi^* = \cos(2\pi \cdot /P)$$

as the direction of bifurcation in the space $C^{\alpha}(\mathbb{S}_P)$.

Remark. We declare the condition for the cone $\mathscr{K}$ as all $(\mu, \varphi) \in U$ with φ nondecreasing instead of simply writing $\varphi' > 0$ since we want to include the constant solutions in $\mathscr{K}$. Obviously, if φ is nonconstant and nondecreasing on $(-P/2, 0)$ we have $\varphi' > 0$ on the same interval.

Theorem 4.5.3. *Let $-1 < r = s < 0$. Then Item (iii) in Theorem 4.5.2 cannot occur.*

Remark. This proof is heavily inspired by the proof of Theorem 6.7 in Ehrnström–Wahlén [23], which in turn is based on the proof of the conic bifurcation result as in Theorem 3.6.1 as stated in this thesis, borrowed from Buffoni–Toland [13]. The main difference between the two aforementioned proofs lies in the added treatment needed of the transcritical curve of constant solutions $\mu \mapsto (\mu, \mu - 1)$ which crosses the line of trivial solutions at $\mu = 1$. The proof below also adds a detail noted in Ørke [46] concerning the second derivative estimates achieved in Theorem 4.3.1 which is not included in Ehrnström–Wahlén [23].

Remark. As is remarked in Ehrnström–Wahlén, one can show that the wave speed has a uniform bound $\mu(t) < 1$ for all t. Indeed, the bound $\mu(t) < 1$ has to hold locally for small t, although Proposition 4.3.1 implies that the only way to reach $\mu = 1$ is by approaching a curve with $\varphi = 0$. The transcritical bifurcations of Theorem 4.5.1, Item (ii) imply that the solutions locally in a neighborhood of $(\mu, \varphi) = (0, 1)$ are the constant solutions $\varphi \equiv 0$ and $\varphi \equiv \mu - 1$. It should be possible to prove that solutions bifurcating from the line with $(\mu, \mu - 1)$, where $\mu > 1$, do not cross the critical line with $\mu = 1$, however the defective Galilean transform as noted and discussed at the beginning of this chapter was not known until much later than when this was initially written and thusly corrected. The consequence of this is that we have to disregard all bifurcations from said line in this proof, thus we are in a way limiting ourselves to wave speeds $0 \leq \mu \leq 1$.

Proof. In the event that $\varphi(t) \in \mathscr{K}\setminus\{0\}$ for all $t > 0$ we do not have to prove anything, since the corresponding bifurcation branch cannot possibly return to the $(\mu, 0)$-axis in a loop. Aiming for a proof by contradiction, assume that there exists a largest t_* for which $\varphi(t) \in \mathscr{K} \setminus \{0\}$ for all $t < t_*$. The cone $\mathscr{K} \setminus \{0\}$ is closed in $C^\alpha_{\text{even}}(\mathbb{S}_P)$, so necessarily we have that $\varphi(t_*) \in \mathscr{K}$ by the closure property $\lim_{t\to t_*} \varphi(t) \in \overline{\mathscr{K}} = \mathscr{K}$. The plan is to show that we have $\varphi(t_*) = \text{const.}$ by proving that if $\varphi \in \mathfrak{R}^2 \cap \mathscr{K}$ is nonconstant then it follows that $(\mu, \varphi) \in \text{int}(S \cap (\mathbb{R}_{\geq 0} \times \mathscr{K}))$ as an interior point when looking at the topology with respect to the $\mathbb{R}_{\geq 0} \times C^\alpha$-metric relative to S induced by the norm.

Let φ be a nonconstant solution along the main branch $\mathfrak{R}$ that is nondecreasing on $(-P/2, 0)$. The regularity theorem 4.4.1 then implies that φ is smooth, and hence by the nodal property theorem 4.3.1 we have $\varphi' > 0$ on $(-P/2, 0)$ along with $\varphi''(0) < 0$ and $\varphi''(-P/2) > 0$ when $r = s$. Now consider another solution $(\lambda, \phi) \in S$ within distance $\delta \ll 1$ away from (μ, φ) in the metric of the $\mathbb{R}_{\geq 0} \times C^\alpha$-norm such that $\Lambda^r\phi < \mu$, which is possible owing to the continuity of Λ^r as an operator on C^α. Consider the map as in Theorem 4.4.1

$$G(\varphi; \mu) = [u \mapsto u\,(\mu - \Lambda^{r-s}u)^{-1}] \circ [\varphi \mapsto \Lambda^s\varphi] \colon C^\alpha(\mathbb{S}_P) \cap L^\infty(\mathbb{R}) \hookrightarrow \mathscr{C}^{\alpha-s}(\mathbb{S}_P)$$

which was used to bootstrap the regularity of φ and prove smoothness of solutions when $\Lambda^r\varphi < \mu$. This map yields a continuous map such that

$$\|\phi - \varphi\|_{C^2(\mathbb{S}_P)} = \|G(\phi; \lambda) - G(\varphi; \mu)\|_{C^2(\mathbb{S}_P)} < \tilde{\delta} \ll 1$$

where furthermore $\tilde{\delta}$ can be made arbitrarily small by making δ smaller. Choosing δ small enough ensures that ϕ is strictly increasing on $(-P/2, 0)$. If $\tilde{\delta}$ is small then $\phi' > 0$ on some closed interval $[x_2, x_1] \subset (-P/2, 0)$ by the following estimates

$$\begin{aligned}
\phi'(x) &= \varphi'(x) - (\varphi'(x) - \phi'(x)) \\
&\geq \varphi'(x) - |\varphi'(x) - \phi'(x)| \\
&\geq \tilde{\delta} - \sup_{x\in[x_2,x_1]} |\varphi'(x) - \phi'(x)| \\
&> 0
\end{aligned}$$

given that we bound

$$\sup_{x\in[x_2,x_1]} \varphi'(x) > \tilde{\delta}$$

which is guaranteed possible by making $\tilde{\delta}$ even smaller by shrinking δ if necessary. Similarly, one can prove that $\phi''(0) < 0$ and $\phi''(-P/2) > 0$ analogously. If $\phi' \leq 0$ on $(x_1, 0)$ we have $\phi'(0) < \phi'(x) \leq 0$ for $x \in (x_1, 0)$ due to $\phi''(0) < 0$ which cannot happen if ϕ is even. Similarly one achieves the same contradiction on $(-P/2, x_2)$, and we conclude that $\phi' > 0$ on $(-P/2, 0)$. Then we have shown $\phi \in \mathscr{K}$, furthermore this implies that $\|(\lambda, \phi) - (\mu, \varphi)\|_{\mathbb{R}_{\geq 0}\times C^\alpha(\mathbb{S}_P)} < \delta$ with $\phi \in \mathscr{K}$ which implies that φ is an interior point of $S \cap (\mathbb{R}_{\geq 0} \times \mathscr{K})$. Hence $\varphi(t_*) = C$ for some constant function C otherwise we would have a contradiction to the definition of t_*.

Assume now that $\varphi(t_*) = 0$, so $C = 0$. Theorem 4.5.1 then tells us that $(\mu(t_*), 0)$ is a bifurcation point, which we will show is the only admissible option. On the other hand if $\varphi(t_*) = C \neq 0$, a possibility which we will exclude, it is clear that $\varphi(t_*) = \mu - 1$ with $\mu \neq 1$ and furthermore if $\varphi(t_*)$ is a non-zero constant we necessarily have $\mu(t) < 1$ since if $\mu(t_0) = 1$ for some t_0 we necessarily have $\varphi(t_0) = 0$ by Proposition 4.3.1 and thus we surpass a trivial curve at this point, thus one concludes $\varphi(t_*) = 0$. We wish to prove that for $\mu < 1$, the trivial curve $\mu \mapsto (\mu, \mu-1)$ is locally unique in the sense that no other solution curve in S connects to said curve.

The case $\mu(t_*) = 1$ combined with $\varphi(t_*) = 0$ cannot occur simultaneously. Proposition 4.3.1 implies that given $t < t_*$ the wave speed is bounded from above as $\mu(t) < 1$. The transcritical part of Theorem 4.5.1 tells us that the curves in S connecting to the point $(1, 0) \in S$ are the curves $\{(\mu, 0) \mid 0 \leq \mu \leq 1\}$ and $\{(\mu, \mu - 1) \mid 0 \leq \mu \leq 1\}$, however the latter curve has been neglected as part of our analysis – see the preceding remarks. Hence, assuming $\mu(t_*) = 1$, the remaining possibility is that of the main branch $\mathfrak{R}$ connecting the bifurcation point $(\mu^*, 0)$ to $(1, 0)$ by a zero curve $\{((1 - h)\mu^* + h, 0) \mid 0 \leq h \leq 1\}$. The definition of t_* precludes this, as $\varphi(t) = 0$ on this entire segment. Indeed, the only option left is for $\mu(t_*) = \mu^*_{P,k}$ for some $k \geq 1$.

The preceding analysis justifies the assumption that $(\mu(t_*), \varphi(t_*))$ is a local bifurcation point, so from our global bifurcation result Theorem 4.5.2 we can choose a real-analytic reparametrization of $\mathfrak{R}$ around the point $(\mu(t_*), \varphi(t_*))$ in such a way that

$$\varphi(t) = \mathrm{D}_t^n \varphi(t_*) \frac{(t - t_*)^n}{n!} + O(|t - t_*|^{n+1})$$

since $\varphi(t_*) = 0$, where the integer $n \geq 1$ is chosen as large as possible. By the definition of t_* we ought to have

$$(-1)^n \, \mathrm{D}_t^n \varphi(t_*) \in \mathscr{K} \setminus \{0\}$$

whenever $t < t_*$. Correspondingly, by taking the n-th (Gateaux) derivative of the bifurcation map $F(\mu(t), \varphi(t))$ with respect to the parameter t we obtain

$$(-1)^n \, \partial_\varphi F[(\mu^*_{P,k}, 0)] \, \mathrm{D}_t^n \varphi(t_*) = 0$$

since in particular F evaluates to zero on $(\mu(t_*), \varphi(t_*)) = (\mu^*_{P,k}, 0)$. Take $\psi = (-1)^n \, \mathrm{D}_t^n \varphi(t_*)$, which by the above has to satisfy $(\Lambda^s - \mu^*_{P,k})\psi = 0$ following the logic of Lyapunov–Schmidt reduction. Clearly we have $\psi \in \mathscr{K} \setminus \{0\} \subset C^\alpha_{\text{even}}(\mathbb{S}_P)$, and hence P-periodicity and evenness implies $\psi(x) = \tau \cos(2\pi k x/P)$ for some $\tau \in \mathbb{R}_{\geq 0}$. If $k \geq 2$ then $\psi'(x) = -C\tau \sin(2\pi k x/P)$, $C > 0$ constant, fails to be non-negative everywhere on $(-P/2, 0)$, hence $\psi \notin \mathscr{K}$ if $k \geq 2$. Furthermore, we immediately have $-\varphi^* \notin \mathscr{K}$. Let $\mathscr{R}^+$ be the bifurcation curve parametrized by t emanating from the bifurcation point $(\mu^*_{P,k}, 0)$, which is guaranteed to be in $\mathbb{R}_{\geq 0} \times \mathscr{K}$ given that $0 < t \ll 1$ in view of Theorem 4.5.1 which yields that $\varphi(t) = t\cos(2\pi k x/P) + O(t^2)$ and is also smooth since small solutions are smooth by Theorem 4.4.1. Now observe that if $t < t_*$ with $|t - t_*|$ small, then local uniqueness ensures that the curve $\mathscr{R}^+$ coincides with the primary bifurcation branch $\mathfrak{R}$ in such a way that $\mathscr{R}^+ \subseteq \mathfrak{R}$. Because of this we may take sequences $\{t_k\}_k$, $\{\tilde{t}_k\}_k$ such that for all k

$$(\mu(t_k), \varphi(t_k)) = (\mu(t_* - \tilde{t}_k), \varphi(t_* - \tilde{t}_k), \quad \text{with} \quad t_k \searrow 0, \, \tilde{t}_k \searrow 0.$$

We now invoke Theorem 3.5.1 Item (f), where necessarily $T > 0$ divides $t_* - t_k - \tilde{t}_k$ for all k which clearly cannot happen. Hence such a t_* cannot exist. Finally, the point $(\mu^*_{P,k}, 0) = (\mu^*, 0)$ for $k = 1$ is a bifurcation point which defines the curve $\mathscr{R}^+$, and thus $\mathfrak{R}$ when extended globally, so therefore $\varphi(t) \in \mathscr{K} \setminus \{0\}$ for all $t > 0$. □

Remark. As is pointed out in Remark 4.10 in the article [31] by Hildrum and Xue, one should be careful to work in the appropriate topologies as regards the local behaviour around solutions $\varphi \in S^2$. Indeed, the fixed point iteration map $G(\varphi; \tilde{\mu})$ need not equal φ unless $\tilde{\mu} = \mu$ for the appropriate parameter μ, thus rationalizing us working with pairs $(\lambda, \phi) \in \mathbb{R}_{\geq 0} \times C^\alpha(\mathbb{S}_P)$ and doing estimates in the accompanying norm of $\mathbb{R}_{\geq 0} \times C^\alpha(\mathbb{S}_P)$ in the proof.

4.6 Discussion on the existence of a highest wave

The final piece of analysis from Ehrnström–Wahlén [23] culminates in showing that Items (i) and (ii) in Theorem 4.5.2 have to occur simultaneously. What this means more explicitly is that solutions blow up in C^α-norm if and only if the bifurcation curves touch the boundary ∂U for which $\max \varphi = \frac{\mu}{2}$, hence one can characterize the blow-up of solutions exactly as sequences of solutions approach a solution φ with $\varphi(0) = \frac{\mu}{2}$.

Ehrnström–Wahlén are able to bound both $\mu \leq 2$ and $\varphi \leq \mu/2$ from above *a priori* simply by looking at the steady equation $-\mu\varphi + L\varphi + \varphi^2 = 0$. We are not so fortunate, and roadblocks occur when trying to derive similar bounds in our case. Indeed, the maximal curve in the articly by Hildrum and Xue [31] is derived by similar a method in the end.

Lemma 6.10 of Ehrnström–Wahlén proves that φ is bounded whenever μ is bounded and from there uses Arzelà–Ascoli to prove that bounded sequences of $\{(\mu_j, \varphi_j)\}_j \in S$ necessarily converge uniformly to some solution φ. In our case, considering the bound

$$\|\varphi\, \Lambda^r \varphi\|_{L^\infty} \leq (|\mu| + 1)\, \|\varphi\|_{L^\infty}$$

unfortunately does not go anywhere, however we would need to be able to bound $\|\varphi\|_{L^\infty}$ whenever μ is bounded, so it becomes rather difficult to find a workaround to this defect. The nonlocality of our nonlinearity seems to disallow further conclusions unless there is another way of proving something equivalent to the aforementioned Lemma 6.10 of Ehrnström–Wahlén.

Corollary 6.11 of Ehrnström–Wahlén asserts that the wave speed is bounded with $\mu(s) \gtrsim 1$ uniformly for all $s \geq 0$ along the global branch. If we try to follow the proof the authors present, then we are not able to prove the same contradiction. They are able to reach a contradiction of the following form, for sequences $(\mu_{n_k}, \varphi_{n_k})$ where $\mu_{n_k} \to 0$ as $k \to \infty$,

$$0 = \lim_{k\to\infty} \left(\tfrac{\mu_{n_k}}{2} - \varphi_{n_k}(P/2)\right) \geq \lambda_{K,P} > 0$$

where $\lambda_{K,P}$ is a universal constant dependent on the Whitham kernel K and period P. Our problem arises in the realization that our universal constant $\lambda_{r,s,P,\mu}$ from Lemma 4.4.1 is dependent on μ, so this limit does not yield a contradiction in the same way.

The final crux is that of Theorem 4.4.2. There is no non-trivial, non-decreasing solution with the desired property that $\Lambda^r\varphi(0) = \mu$, indeed even the parameter μ has to degenerate to zero. The condition $\Lambda^r\varphi < \mu$ came out as a *ad hoc* natural bound due to our regularity theorem and nodal property theorem, however Theorem 4.4.2 might point to suggest that $\Lambda^r\varphi < \mu$ is the best possible bound that we could ask for. Without a supremal bound on φ we cannot ascertain for certain which bound is the most optimal in the sense of the regularity, nodal property and supremum altogether equally. The articles in Table 1.1 that establish a highest wave with singularity exhibit a property that these properties and their natural bounds coincide in a particular sense – for instance the regularity and nodal property break down at the supremal limit $\max\varphi \to \frac{\mu}{2}$ in the case of Whitham [23], and the same holds true for Hildrum–Xue [31] and their supremal bound.

Concluding remarks

We have established local bifurcation formulas, touching lemmata, a nodal property theorem, and have used these to extend local bifurcation curves to global continua of smooth, non-degenerative P-periodic solutions (μ, φ) of Equation (4.1.2) in $\mathbb{R}_{\geq 0} \times \mathscr{C}^t(\mathbb{S}_P)$ whenever $-1 < r = s < 0$. The degeneracy result of Theorem 4.4.2 seemingly disallows singularities in our case.

4.7 Suggested further work

- Develop *a priori* estimates of steady equations of the form $-\mu\varphi + L\varphi + N(\varphi,\varphi) = 0$ for a larger family of Coifman–Meyer operators $N(\varphi,\varphi)$.

- Develop nodal property theorems out of a larger family of Coifman–Meyer operators and determine when it is possible or impossible to fully achieve such a result.

- Attempt to find a workaround to the defects caused by the lack of Galilean transformation of $-\mu\varphi+\Lambda^s\varphi+\varphi\,\Lambda^r\varphi = 0$ and extrapolate to general Coifman–Meyer nonlinearities $N(\varphi,\varphi)$.

- Find an example of a Coifman–Meyer operator $N(\varphi,\varphi)$ for which waves with singularity is a possibility.

Notation

Relational symbols
$\cong$ – indicates a bijection, homeomorphism or diffeomorphism depending on the context
$\equiv$ – identically equal to, uniformly equal to (a constant)
$f \lesssim g$ - there exists $M \geq 0$ such that $f \leq Mg$ for all f, g in a space
$f \gtrsim g$ – there exists $M \geq 0$ such that $Mf \geq g$ for all f, g in a space
$f \eqsim g$ – if $f \lesssim g \lesssim f$ or $g \lesssim f \lesssim g$
$U \subset V$ – U is strictly contained (or continuously embedded) in V
$U \subseteq V$ – U is nonstrictly contained in V
$U \subset\subset V$ – U is compactly contained (or compactly embedded) in V

$\mathbb{N}$ – the natural numbers
$\mathbb{Z}_{\geq 0}$ – the non-negative integers, denoted by some authors as $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$L^p(X)$ – the set of p-integrable functions on a set X. The measure is taken to be the Lebesgue measure unless otherwise stated
$\mathscr{F}(\cdot)$ – the operator associated with the Fourier transform
$B(r, x)$ – open ball of radius r and center x
$\mathscr{D}(\Omega)$ – the space of test functions on a domain Ω
$\mathscr{S}(\mathbb{R}^n)$ – the space of Schwartz functions
$\mathscr{D}'(\Omega)$ – the space of distributions on Ω
$\mathscr{S}'(\mathbb{R}^n)$ – the space of tempered distributions
$W^{m,p}(\cdot)$ – the Sobolev space with integer indices m
$W^{s,p}(\cdot)$ – the inhomogeneous Sobolev space with real-valued indices s
$B^s_{p,q}(\mathbb{R}^n)$ – the inhomogeneous Besov–Lipschitz space with parameters s, p, q
$\mathscr{C}^s(\mathbb{R}^n)$ – the Hölder–Zygmund space of index s
$C^{k,\alpha}(\mathbb{R}^n)$ – the Hölder space of order k and exponent α
$\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^n$. In the last chapter we also denote $\langle\xi\rangle^s = m_s(\xi)$
$\Gamma(z)$ – the standard Gamma function for $z > 0$
$\|\cdot\|_X$ – norm on the space X
$|\cdot|$ – absolute value, or Euclidean norm on $\mathbb{R}^n$ or $\mathbb{C}^n$, tuple norm on indices $\alpha \in \mathbb{Z}^n_{\geq 0}$
$\ker(\cdot)$ – the kernel of a linear map
$\operatorname{ran}(\cdot)$ – the range of a linear map
$\mathscr{L}(X, Y)$ – the space of bounded linear operators with domain X and target Y
$\mathscr{K}(X, Y)$ – the space of compact operators from $\mathscr{L}(X, Y)$

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