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The Nehari Problem for the Paley–Wiener Space of a Disc

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Abstract
There is a bounded Hankel operator on the Paley–Wiener space of a disc in $\mathbb{R}^2$ which does not arise from a bounded symbol.

Keywords Hankel operator · Paley–Wiener space · Several variables

Mathematics Subject Classification Primary 47B35 · Secondary 42B35

1 Introduction

Let $\mathbb{D}$ be the unit disc in $\mathbb{R}^2$. The Paley–Wiener space $\mathrm{PW}(\mathbb{D})$ is the subspace of $L^2(\mathbb{R}^2)$ comprised of functions f whose Fourier transforms $\widehat{f}$ are supported in $\overline{\mathbb{D}}$. For a tempered distribution φ, we consider the Hankel operator $\mathbf{H}_\varphi$ defined by the equation

$$\widehat{\mathbf{H}_\varphi f}(\eta) = \int_{\mathbb{D}} \widehat{f}(\xi)\widehat{\varphi}(\xi+\eta)\,\mathrm{d}\xi, \qquad \eta \in \mathbb{D}, \tag{1}$$

on the dense subset of $\mathrm{PW}(\mathbb{D})$ comprised of functions f such that $\widehat{f}$ is smooth and compactly supported in $\mathbb{D}$.

We are interested in the characterization of the symbols φ such that $\mathbf{H}_\varphi$ extends by continuity to a bounded operator on $\mathrm{PW}(\mathbb{D})$. If φ is in $L^\infty(\mathbb{R}^2)$, then clearly

$$\|\mathbf{H}_\varphi f\|_2 \leq \|f\|_2\|\varphi\|_\infty. \tag{2}$$

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Since $\xi + \eta$ is in $2\mathbb{D}$ whenever ξ and η are in $\mathbb{D}$, $\mathbf{H}_\varphi = \mathbf{H}_\psi$ for any ψ such that the restrictions of $\widehat{\psi}$ and $\widehat{\varphi}$ to $2\mathbb{D}$ coincide (as distributions in $2\mathbb{D}$). We thus find that

$$\|\mathbf{H}_\varphi\| \leq \inf\left\{\|\psi\|_\infty \,:\, \widehat{\psi}\,\big|_{2\mathbb{D}} = \widehat{\varphi}\,\big|_{2\mathbb{D}}\right\}. \tag{3}$$

We say that the Hankel operator $\mathbf{H}_\varphi$ has a bounded symbol if the quantity on the right hand side of (3) is finite. We have just demonstrated that if $\mathbf{H}_\varphi$ has a bounded symbol, then $\mathbf{H}_\varphi$ is bounded. We wish to explore the converse.

Question *Does every bounded Hankel operator on* $\mathrm{PW}(\mathbb{D})$ *have a bounded symbol?*

In the classical one-dimensional setting, where the role of $\mathbb{D}$ is played by the half-line $\mathbb{R}_+ = [0, \infty)$, Nehari [6] gave a positive answer to this question. We therefore refer to affirmative answers to analogous questions as Nehari theorems. Our question for $\mathrm{PW}(\mathbb{D})$ was first raised implicitly by Rochberg [9, Sec. 7], after he had proved that Nehari's theorem holds for the Paley–Wiener space $\mathrm{PW}(I)$ of a finite interval $I \subseteq \mathbb{R}$.

It was conditionally[1] shown in [1] that the Nehari theorem holds for the Paley–Wiener space $\mathrm{PW}(\mathbb{P})$ of any convex polygon $\mathbb{P}$. However, in view of C. Fefferman's negative resolution [3] of the disc conjecture for the Fourier multiplier of a disc, it would not be surprising to see differing results for $\mathrm{PW}(\mathbb{P})$ and $\mathrm{PW}(\mathbb{D})$.

The main purpose of the present note is to establish the following.

Theorem 1 *There is a bounded Hankel operator on* $\mathrm{PW}(\mathbb{D})$ *which does not have a bounded symbol.*

Minor modifications of our proof show that if $\mathbb{P}_n$ is an n-sided regular polygon, then the optimal constant in the inequality

$$\inf\left\{\|\psi\|_\infty \,:\, \widehat{\psi}\,\big|_{2\mathbb{P}_n} = \widehat{\varphi}\,\big|_{2\mathbb{P}_n}\right\} \leq C_n \|\mathbf{H}_\varphi\|_{\mathrm{PW}(\mathbb{P}_n)}$$

satisfies $C_n \geq c_\varepsilon n^{1/2-\varepsilon}$ for any fixed $\varepsilon > 0$. Here, $c_\varepsilon > 0$ denotes a constant which depends only on ε. Conversely, the conditional argument of [1] yields that $C_n \leq cn$ for some absolute constant $c > 0$. Analogous estimates for Fourier multipliers associated with polygons were considered in [2].

Finally, let us remark that Ortega-Cerdà and Seip [7] have shown that Nehari's theorem also fails for (small) Hankel operators on the infinite-dimensional torus. However, Helson [4] proved that if the Hankel operator is in the Hilbert–Schmidt class S_2, then it is induced by a bounded symbol. We are led to the following.

Question *Does every Hankel operator on* $\mathrm{PW}(\mathbb{D})$ *in* S_2 *have a bounded symbol?*

In this context, we mention that Peng [8] has characterized when $\mathbf{H}_\varphi$ is in the Schatten class S_p, for $1 \leq p \leq 2$, in terms of the membership of φ in certain Besov spaces adapted to $2\mathbb{D}$. In particular, $\mathbf{H}_\varphi$ is in S_2 if and only if

$$\int_{2\mathbb{D}} |\widehat{\varphi}(\xi)|^2 (2 - |\xi|)^{3/2}\, \mathrm{d}\xi < \infty.$$

[1] The arguments in [1] rely on Nehari's theorem for $\mathbb{R}_+ \times \mathbb{R}_+$ as a black box. It was long believed that the Nehari theorem had been proven in this setting, but a significant flaw was recently observed in the available reasoning. We refer to [5, Sect. 10] for a detailed discussion.

2 Proof of Theorem 1

If the Nehari theorem were to hold for $\mathrm{PW}(\mathbb{D})$, there would by the closed graph theorem exist an absolute constant $C < \infty$ such that

$$\inf \left\{ \|\psi\|_\infty \,:\, \widehat{\psi}\,\big|_{2\mathbb{D}} = \widehat{\varphi}\,\big|_{2\mathbb{D}} \right\} \leq C \|\mathbf{H}_\varphi\| \tag{4}$$

for every bounded Hankel operator on $\mathrm{PW}(\mathbb{D})$. To prove Theorem 1, we will construct a sequence of symbols which demonstrates that no such $C < \infty$ can exist.

We begin with an upper bound for $\|\mathbf{H}_\varphi\|$. Guided by the following lemma, our plan is to construct φ such that $\mathbf{H}_\varphi$ admits an orthogonal decomposition. For a symbol φ, define

$$D_\varphi = \left\{ \eta \in \mathbb{D} \,:\, \xi + \eta \in \operatorname{supp} \widehat{\varphi} \text{ for some } \xi \in \mathbb{D} \right\}.$$

Lemma 2 *Suppose that $\varphi = \varphi_1 + \varphi_2$ and that $D_{\varphi_1} \cap D_{\varphi_2} = \emptyset$. Then,*

$$\mathbf{H}_\varphi = \mathbf{H}_{\varphi_1} \oplus \mathbf{H}_{\varphi_2}.$$

Proof Let f be any function in $\mathrm{PW}(\mathbb{D})$ such that $\widehat{f}$ is smooth and compactly supported in $\mathbb{D}$. Since $\mathbf{H}_\varphi f = \mathbf{H}_{\varphi_1} f + \mathbf{H}_{\varphi_2} f$ by linearity of the integral (1), it is sufficient to demonstrate that $\mathbf{H}_{\varphi_1} f \perp \mathbf{H}_{\varphi_2} f$. It follows directly from the definition of the Hankel operator (1) that

$$\operatorname{supp} \widehat{\mathbf{H}_{\varphi_1} f} \subseteq D_{\varphi_1} \qquad \text{and} \qquad \operatorname{supp} \widehat{\mathbf{H}_{\varphi_2} f} \subseteq D_{\varphi_2}.$$

By the assumption that $D_{\varphi_1} \cap D_{\varphi_2} = \emptyset$, we therefore conclude that

$$\langle \mathbf{H}_{\varphi_1} f, \mathbf{H}_{\varphi_2} f \rangle = \langle \widehat{\mathbf{H}_{\varphi_1} f}, \widehat{\mathbf{H}_{\varphi_2} f} \rangle = 0.$$

□

In particular, if $D_{\varphi_1} \cap D_{\varphi_2} = \emptyset$, then

$$\|\mathbf{H}_\varphi\| = \max(\|\mathbf{H}_{\varphi_1}\|, \|\mathbf{H}_{\varphi_2}\|).$$

Let us next explain the construction of φ. Consider a radial smooth bump function $\widehat{b}$ which is bounded by 1, equal to 1 on $\frac{1}{2}\mathbb{D}$ and compactly supported in $\mathbb{D}$. For a real number $0 < r < 1/2$, set $\widehat{b}_r(\xi) = \widehat{b}(\xi/r)$. Note that

$$\|\widehat{b}_r\|_1 \leq \pi r^2. \tag{5}$$

For $j = 1, 2, \ldots, n$, we let $\widehat{\varphi}_j$ be the function obtained by translating $\widehat{b}_r$ by $2 - r$ units in the direction $\theta_j = 2\pi(j-1)/n$, as measured with respect to the positive ξ_1-axis in the $\xi_1\xi_2$-plane. We set

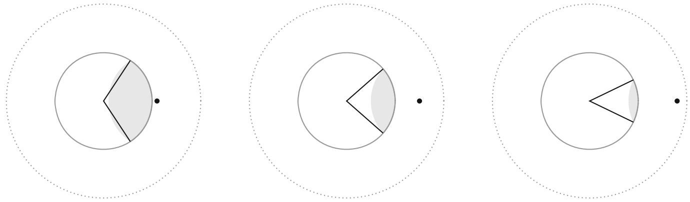

Fig. 1 Plots of $D(w)$ and the corresponding disc sector from the proof of Lemma 3, for $w = 1.1$, $w = 1.5$, and $w = 1.8$

$$\varphi = \varphi_1 + \varphi_2 + \cdots + \varphi_n. \tag{6}$$

Since $0 < r < 1/2$, it is clear that $\operatorname{supp} \widehat{\varphi} \subseteq 2\mathbb{D} \setminus \mathbb{D}$. Let $r_0 = 1 - \frac{1}{\sqrt{2}} = 0.29\ldots$.

Lemma 3 *If $n \geq 2$ and $r = \min(r_0, (2/n)^2)$, then*

$$D_{\varphi_j} \cap D_{\varphi_k} = \emptyset$$

for every $1 \leq j \neq k \leq n$.

Proof Throughout this proof, we identify $\mathbb{R}^2$ with $\mathbb{C}$. We consider first a simpler situation. For a point w in $2\mathbb{D} \setminus \mathbb{D}$, let

$$D(w) = \big\{\eta \in \mathbb{D} \,:\, \xi + \eta = w \text{ for some } \xi \in \mathbb{D}\big\}.$$

In other words, $D(w)$ is the intersection of the discs defined by $|\xi| < 1$ and $|w - \xi| < 1$. To find the intersection of the corresponding circles, we set $\xi = e^{i\theta}$ and let $\theta^{\pm}$ denote the solutions of the equation

$$1 = |w - e^{i\theta}| \qquad \iff \qquad \theta^{\pm} = \arg w \pm \arccos\left(\frac{|w|}{2}\right).$$

Let P_0 denote the origin, $P_{\pm}$ the points $e^{i\theta^{\pm}}$, and P_w the point w. The law of cosines implies that the angle $\angle P_0 P_{\pm} P_w$ is greater than or equal to $\pi/2$ if and only if $|w| \geq \sqrt{2}$. If this holds, then the intersection of the two discs is contained in the disc sector defined by the origin and the two points $P_{\pm}$. See Fig. 1.

Suppose therefore that $|w| \geq \sqrt{2}$ and set $I(w) = (\theta^-, \theta^+)$. If ξ is in $D(w)$, we have just seen that $\arg \xi$ is in $I(w)$. It follows that if w_1 and w_2 are points in $2\mathbb{D} \setminus \sqrt{2}\mathbb{D}$, then

$$I(w_1) \cap I(w_2) = \emptyset \qquad \Longrightarrow \qquad D(w_1) \cap D(w_2) = \emptyset. \tag{7}$$

Our goal is now to estimate

$$I_{\varphi_j} = \bigcup_{w \in \operatorname{supp} \widehat{\varphi}_j} I(w).$$

Since $\operatorname{supp}\widehat{\varphi}_j$ is contained in a disc with center $(2-r)e^{i\theta_j}$ and radius r, straightforward geometric arguments show that if w is in $\operatorname{supp}\widehat{\varphi}_j$, then

$$|w| \geq 2(1-r) \qquad \text{and} \qquad |\arg w - \theta_j| \leq \arctan\left(\frac{r}{2-r}\right).$$

To ensure that $|w| \geq \sqrt{2}$ we require that $r \leq r_0 = 1-\frac{1}{\sqrt{2}}$. Moreover, if $\theta^{\pm}$ correspond to the point w as above, then

$$|\theta^{\pm} - \theta_j| \leq \arccos(1-r) + \arctan\left(\frac{r}{2-r}\right) \leq 2\sqrt{r} + r \leq 3\sqrt{r}.$$

Here, we used that $2-r \geq 1$ and that $\arctan r \leq r$ for $0 \leq r \leq 1$. This shows that

$$I_{\varphi_j} \subseteq \left(\theta_j - 3\sqrt{r}, \theta_j + 3\sqrt{r}\right).$$

Since $|\theta_j - \theta_k| \geq 2\pi/n$ for every $1 \leq j \neq k \leq n$ and since $\pi > 3$, it follows that if we choose $r = \min(r_0, (\frac{2}{n})^2)$, then we guarantee that $I_{\varphi_j} \cap I_{\varphi_k} = \emptyset$ for every $1 \leq j \neq k \leq n$. The proof is completed by appealing to (7). □

Let φ be as in (6), with $n \geq 2$ and $r = \min(r_0, (2/n)^2)$. It then follows from Lemmas 2, 3, (2), and (5) that

$$\|\mathbf{H}_\varphi\| = \|\mathbf{H}_{\varphi_j}\| \leq \|\varphi_j\|_\infty \leq \|\widehat{\varphi}_j\|_1 = \|\widehat{b}_r\|_1 \leq \pi r^2. \tag{8}$$

A lower bound for the left hand side in (4) will be established through duality.

Lemma 4 *Suppose that $\widehat{f}$ is smooth and compactly supported in $2\mathbb{D}$. Then,*

$$\frac{|\langle \widehat{f}, \widehat{\varphi}\rangle|}{\|f\|_1} \leq \inf\left\{\|\psi\|_\infty \,:\, \widehat{\psi}\,\big|_{2\mathbb{D}} = \widehat{\varphi}\,\big|_{2\mathbb{D}}\right\}.$$

Proof Obviously,

$$\frac{|\langle f, \psi\rangle|}{\|f\|_1} \leq \|\psi\|_\infty,$$

and when $\widehat{f}$ is supported in $2\mathbb{D}$ and $\widehat{\psi}|_{2\mathbb{D}} = \widehat{\varphi}|_{2\mathbb{D}}$, we have that

$$\langle f, \psi\rangle = \langle \widehat{f}, \widehat{\psi}\rangle = \langle \widehat{f}, \widehat{\varphi}\rangle.$$ □

We now need to choose a test function f adapted to the symbol φ of (6). It turns out that $f = f_1 + f_2 + \cdots + f_n$, where $f_j = \varphi_j$ for $j = 1, 2, \ldots, n$, will do. By our choice of $n \geq 2$ and $r = \min(r_0, (2/n)^2)$, it is clear that $\operatorname{supp}\widehat{f_j} \cap \operatorname{supp}\widehat{f_k} = \emptyset$ for every $1 \leq j \neq k \leq n$, since the converse statement would contradict Lemma 3.

Exploiting this, we find that

$$|\langle f, \varphi\rangle| = \|f\|_2^2 = \|\widehat{f}\|_2^2 = n\|\widehat{b_r}\|_2^2 \geq \frac{\pi}{4}nr^2. \tag{9}$$

To get an upper bound for $\|f\|_1$, we split the integral at some $R > 0$,

$$\|f\|_1 = \int_{|x|\leq R} |f(x)|\,\mathrm{d}x + \int_{|x|>R} |f(x)|\,\mathrm{d}x = I_1 + I_2.$$

For the first integral, we use the Cauchy–Schwarz inequality,

$$I_1 \leq \sqrt{\pi}R\left(\int_{|x|\leq R} |f(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \leq \sqrt{\pi}R\|f\|_2 = \sqrt{\pi}R\|\widehat{f}\|_2 \leq \pi R\sqrt{n}r,$$

where we again exploited that $\operatorname{supp}\widehat{f_j} \cap \operatorname{supp}\widehat{f_k} = \emptyset$ for $1 \leq j \neq k \leq n$. For the second integral, we note that b is rapidly decaying, since $\widehat{b}$ is smooth and compactly supported. In particular, for every $\kappa \geq 1$, there is a constant A_κ such that

$$\int_{|x|>\varrho} |b(x)|\,\mathrm{d}x \leq \frac{A_\kappa}{\varrho^{\kappa-1}}, \tag{10}$$

holds for every $\varrho > 0$. We constructed $\widehat{f_j}$ by translating $\widehat{b_r}$ by $2 - r$ units in direction θ_j, so there is a unimodular function g_j such that

$$f_j(x) = g_j(x)b_r(x) = g_j(x)r^2b(rx).$$

Thus $|f(x)| \leq nr^2b(rx)$ and (10), with $\varrho = Rr$, yields

$$I_2 \leq n\int_{|x|>R} r^2|b(rx)|\,\mathrm{d}x = n\int_{|x|>rR} |b(x)|\,\mathrm{d}x \leq A_\kappa\frac{n}{(Rr)^{\kappa-1}}.$$

Combining our estimates for I_1 and I_2 and choosing $R = n^{1/(2\kappa)}/r$, we find that

$$\|f\|_1 = I_1 + I_2 \leq (\pi + A_\kappa)n^{1/2+1/(2\kappa)}. \tag{11}$$

Inserting the estimates (9) and (11) into Lemma 4, we obtain

$$\frac{\pi r^2 n^{1/2-1/(2\kappa)}}{4(\pi + A_\kappa)} \leq \inf\left\{\|\psi\|_\infty \,:\, \widehat{\psi}\,\big|_{2\mathbb{D}} = \widehat{\varphi}\,\big|_{2\mathbb{D}}\right\}. \tag{12}$$

Final part of the proof of Theorem 1 To finish the proof of Theorem 1, we combine (8) and (12) to conclude that the constant C in (4) must satisfy

$$\frac{n^{1/2-1/(2\kappa)}}{4(\pi + A_\kappa)} \leq C$$

for any fixed $\kappa \geq 1$ and every integer $n \geq 2$. Choosing some $\kappa > 1$ and letting $n \to \infty$, we obtain a contradiction. □

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