GROUPOIDS AND HERMITIAN BANACH $*$-ALGEBRAS

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ABSTRACT. We study when the twisted groupoid Banach $*$-algebra $L^1(\mathcal{G},\sigma)$ is Hermitian. In particular, we prove that Hermitian groupoids satisfy the weak containment property. Furthermore, we find that for $L^1(\mathcal{G},\sigma)$ to be Hermitian it is sufficient that $L^1(\mathcal{G}_\sigma)$ is Hermitian. Moreover, if $\mathcal{G}$ is ample, we find necessary conditions for $L^1(\mathcal{G},\sigma)$ to be Hermitian in terms of the fibers $\mathcal{G}_x^x$.

1. Introduction

Due to the seminal paper [18], Hermitian Banach $*$-subalgebras of C^*-algebras have been intimately linked to Wiener's lemma and the notion of spectral invariance. As such, they appear in a variety of areas of mathematics, such as approximation theory, time-frequency analysis and signal processing and noncommutative geometry, see e.g. [10, 19, 20, 21, 22].

We say that a locally compact group G is Hermitian if the convolution algebra $L^1(G)$ is a Hermitian Banach $*$-algebra. In the realm of Hermitian locally compact groups the literature is very extensive (see for example [31, 32]). The class of Hermitian locally compact groups includes compact extensions of nilpotent groups [28] and famously also all compactly generated groups of polynomial growth [27]. In [36] it was proved that a Hermitian locally compact group must be amenable. In another, but related context, it was proved in [25] that if $\mathcal{A}$ is a Hermitian Banach $*$-algebra and Γ is a compact group acting on $\mathcal{A}$ by $*$-automorphisms, the generalized L^1-algebra $L^1(\Gamma,\mathcal{A})$ is Hermitian.

One of the main motivations for this article is the use of spectral invariance of noncommutative tori in time-frequency analysis as in e.g. [22]. Noncommutative tori can be viewed as the twisted group C^*-algebra $C^*(\mathbb{Z}^{2d},\sigma)$, where σ is the Heisenberg 2-cocycle. However, this can generalized to the group C^*-algebra $C^*(\Delta,\sigma)$ where Δ is a closed subgroup of the time-frequency plane $G\times\widehat{G}$, where G is a locally compact abelian group. This extension has been used to construct finitely generated modules over noncommutative solenoids [16, 24]. Spectral invariance of $L^1(\Delta,\sigma)$ was studied in [4]. Pushing forward this generalization, given a quasicrystal $\Lambda\subseteq\mathbb{R}^{2d}$ one can construct a twisted groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda,\sigma)$ where $\mathcal{G}_\Lambda$ is an étale groupoid. Finitely generated projective modules over $C^*(\mathcal{G}_\Lambda,\sigma)$ were constructed in [23].

This paper together with [6] is a starting point for a project that aims to extend the tools of operator algebras applied to Gabor analysis for lattices in the time-frequency plane (see for example in [5, 16, 29, 35]) to the more general context of quasicrystals [13, 23]. In particular, in this paper we focus our attention on when

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the Banach $*$-algebra $L^1(\mathcal{G},\sigma)$ associated to a locally compact groupoid $\mathcal{G}$ with a twist σ is Hermitian.

The paper is structured as follows. First, in Section 2 we give the necessary background and results about groupoids and Banach $*$-algebras.

In Section 3 we use twisted actions of groupoids on L^p-spaces to extend the main result of [36] to the setting of groupoids with 2-cocycle twists. Indeed, we show that if $L^1(\mathcal{G},\sigma)$ is (quasi-)Hermitian, then the full and the reduced twisted groupoid C^*-algebras coincide (also known as the weak containment property). When $\mathcal{G}$ is a group, this is equivalent to amenability. However, in the case of locally compact groupoids the situation is more subtle.

Then, in Section 4 we give a proof that for ample groupoids satisfying a minor technical assumption a necessary condition for $L^1(\mathcal{G},\sigma)$ to be quasi-Hermitian is that the "fibers" $L^1(\mathcal{G}_x^x,\sigma_x)$ are quasi-Hermitian for every $x\in\mathcal{G}^{(0)}$. Thus under assumptions appearing in applications, the "fibers" present obstructions to $L^1(\mathcal{G},\sigma)$ being quasi-Hermitian. Using this we give a simple example of an amenable étale groupoid such that $L^1(\mathcal{G},\sigma)$ is not quasi-Hermitian.

Finally, in Section 5 we give sufficient conditions for $L^1(\mathcal{G},\sigma)$ to be Hermitian. We construct the twisted groupoid $\mathcal{G}_\sigma$, and show that $L^1(\mathcal{G},\sigma)$ is Hermitian if the "untwisted" groupoid Banach $*$-algebra $L^1(\mathcal{G}_\sigma)$ is Hermitian. In particular, we prove that if Γ is a compact group or a locally compact abelian group acting by homeomorphisms on a locally compact Hausdorff space X, then $L^1(X\rtimes\Gamma,\sigma)$ is Hermitian for every group 2-cocycle σ of Γ, where $X\rtimes\Gamma$ is the transformation groupoid.

2. Preliminaries

2.1. **Hermitian Banach $*$-algebras.** If $\mathcal{A}$ is a unital Banach algebra, we denote by $\mathcal{A}^{-1}$ the set of invertible elements of $\mathcal{A}$. Given a unital Banach algebra $\mathcal{A}$ we denote by

$$\mathrm{Sp}_{\mathcal{A}}(a)=\{\lambda\in\mathbb{C}:\lambda 1_{\mathcal{A}}-a\notin\mathcal{A}^{-1}\}\,,$$

the *spectrum of* a in $\mathcal{A}$, and by

$$r_{\mathcal{A}}(a):=\sup\{|\lambda|:\lambda\in\mathrm{Sp}_{\mathcal{A}}(a)\}=\lim_{n\to\infty}\|f^n\|^{1/n}$$

the *spectral radius* of a. If $\mathcal{A}$ is not unital, given $a\in\mathcal{A}$ we define $\mathrm{Sp}_{\mathcal{A}}(a):=\mathrm{Sp}_{\mathcal{A}^+}(a)$ and $r_{\mathcal{A}}(a)=r_{\mathcal{A}^+}(a)$, where $\mathcal{A}^+$ is the minimal unitization of $\mathcal{A}$.

Now let $\mathcal{A}$ be a Banach $*$-algebra. Then $a\in\mathcal{A}$ is *Hermitian* if $a^*=a$. If $\mathcal{S}$ is a subset of $\mathcal{A}$, we denote by $\mathcal{S}_h=\{a\in\mathcal{S}:a^*=a\}$ the set of Hermitian elements in $\mathcal{S}$.

Definition 2.1. A Banach $*$-algebra $\mathcal{A}$ is *Hermitian* if

$$\mathrm{Sp}_{\mathcal{A}}(a)\subseteq\mathbb{R}$$

for every $a\in\mathcal{A}_h$. $\mathcal{A}$ is called *symmetric* if

$$\mathrm{Sp}_{\mathcal{A}}(aa^*)\subseteq[0,\infty)$$

for every $a\in\mathcal{A}$.

Remark 2.2. By the celebrated Shirali-Ford theorem, a Banach $*$-algebra is Hermitian if and only if it is symmetric.

A $*$-*representation* of a Banach $*$-algebra $\mathcal{A}$ is a $*$-homomorphism $\pi\colon \mathcal{A} \to B(\mathcal{H})$, where $B(\mathcal{H})$ denotes the bounded linear operators on a Hilbert space $\mathcal{H}$. We say $\mathcal{A}$ is *reduced* if $\mathcal{A}_{\mathcal{R}} = \{a \in \mathcal{A} : \pi(a) = 0 \text{ for every } * - \text{representation } \pi \text{ of } \mathcal{A}\} = \{0\}$. All Banach $*$-algebras we consider in the sequel will be *reduced*. The *enveloping* C^*-*algebra* of a reduced Banach $*$-algebra $\mathcal{A}$ is the completion of $\mathcal{A}$ with respect to the norm

$$\|a\| := \sup\{\|\pi(a)\| : \pi : \mathcal{A} \to B(\mathcal{H}) \text{ is a } *\text{-representation}\}$$

for every $a \in \mathcal{A}$, and it is denoted by $C^*(\mathcal{A})$. The enveloping C^*-algebra of a Banach $*$-algebra always exists [32, Section 10.1].

2.2. **Invariant spectral radius.**

Definition 2.3. We say that $\mathcal{A} \subseteq \mathcal{B}$ is a *nested pair* of reduced Banach $*$-algebras if $\mathcal{A}$ and $\mathcal{B}$ are reduced Banach $*$-algebras and $\mathcal{A}$ embeds continuously into $\mathcal{B}$ as a dense $*$-subalgebra. A *nested triple* of reduced Banach $*$-algebras is defined similarly.

Definition 2.4. Let $\mathcal{A} \subseteq \mathcal{B}$ be a nested pair of reduced Banach $*$-algebras and $\mathcal{S}$ a (not necessarily closed) $*$-subalgebra of $\mathcal{A}$. We say $\mathcal{S}_h$ has *invariant spectral radius* in $(\mathcal{A}, \mathcal{B})$ if

$$r_{\mathcal{A}}(a) = r_{\mathcal{B}}(a)\,,$$

for every $a \in \mathcal{S}_h$. If $\mathcal{A}_h$ has invariant spectral radius in $(\mathcal{A}, \mathcal{B})$, we say that $\mathcal{A}_h$ has *invariant spectral radius* in $\mathcal{B}$. Moreover, we say $\mathcal{S}$ is a spectrally invariant subalgebra of $(\mathcal{A}, \mathcal{B})$ if

$$\mathrm{Sp}_{\mathcal{A}}(a) = \mathrm{Sp}_{\mathcal{B}}(a)\,,$$

for every $a \in \mathcal{S}$. If $\mathcal{A}$ is a spectrally invariant subalgebra of $(\mathcal{A}, \mathcal{B})$, we say $\mathcal{A}$ is spectrally invariant in $\mathcal{B}$.

Clearly, if $\mathcal{S}$ is a spectrally invariant subalgebra of $(\mathcal{A}, \mathcal{B})$, then $\mathcal{S}_h$ has invariant spectral radius in $(\mathcal{A}, \mathcal{B})$. The Barnes-Hulanicki Theorem [7] provides a partial converse when $\mathcal{B}$ is a C^*-algebra.

Theorem 2.5. *Let $\mathcal{A}$ be a Banach $*$-algebra, $\mathcal{S}$ a $*$-subalgebra of $\mathcal{A}$, and $\pi : \mathcal{A} \to B(\mathcal{H})$ a faithful $*$-representation. If $\mathcal{A}$ is unital, we assume that $\pi(1_{\mathcal{A}}) = 1_{B(\mathcal{H})}$. If*

$$r_{\mathcal{A}}(a) = \|\pi(a)\|\,,$$

for all $a \in \mathcal{S}_h$, then

$$\mathrm{Sp}_{\mathcal{A}}(a) = \mathrm{Sp}_{\mathcal{B}(\mathcal{H})}(\pi(a))$$

for every $a \in \mathcal{S}$.

In [36, Proposition 2.7] they prove a very useful property related to invariant spectral radius.

Proposition 2.6. *Let $\mathcal{A} \subseteq \mathcal{B}$ be a nested pair of reduced Banach $*$-algebras, and let $\mathcal{S}$ be a dense $*$-subalgebra of $\mathcal{A}$. Suppose that $\mathcal{S}_h$ has invariant spectral radius in $(\mathcal{A}, \mathcal{B})$. Then $\mathcal{A}$ and $\mathcal{B}$ have the same C^*-envelope.*

2.3. **Hermitian and quasi-hermitian Banach $*$-algebras.** Let $\mathcal{A}$ be a reduced Banach $*$-algebra. Then the following statements are equivalent:

(1) $\mathcal{A}$ is Hermitian,
(2) $\mathcal{A}$ is symmetric,
(3) $\mathcal{A}$ is spectrally invariant in $C^*(\mathcal{A})$,
(4) $r_{\mathcal{A}}(a) = r_{C^*(\mathcal{A})}(a)$ for every $a \in \mathcal{A}$.

(see [36, Lemma 2.8] and [26, p. 340]).

Definition 2.7. A dense $*$-subalgebra $\mathcal{S}$ of a Banach $*$-algebra $\mathcal{A}$ is called *quasi-Hermitian* in $\mathcal{A}$ if $\mathrm{Sp}_{\mathcal{A}}(a) \subseteq \mathbb{R}$ for every $a \in \mathcal{S}_h$. We say $\mathcal{S}$ is *quasi-symmetric* in $\mathcal{A}$ is $\mathrm{Sp}_{\mathcal{A}}(a^*a) \subseteq [0, \infty)$ for every $a \in \mathcal{S}$.

Let $\mathcal{S}$ be a dense $*$-subalgebra of a Banach $*$-algebra $\mathcal{A}$. If $\mathcal{A}$ is Hermitian, then $\mathcal{S}$ is automatically quasi-Hermitian in $\mathcal{A}$. The converse is not true in general, but holds whenever $\mathcal{A}$ is commutative [36, Proposition 2.10].

2.4. **Spectral interpolation.**

Definition 2.8. Suppose $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ is a nested triple of reduced Banach $*$-algebras and $\mathcal{S}$ is a dense $*$-subalgebra of $\mathcal{A}$. We say that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a *spectral interpolation triple* relative to $\mathcal{S}$ if there exists $\theta \in (0, 1)$ such that

$$r_{\mathcal{B}}(a) \leq r_{\mathcal{A}}(a)^{1-\theta} r_{\mathcal{C}}(a)^{\theta}$$

for every $a \in \mathcal{S}_h$.

A nice property of spectral interpolation triples of reduced Banach $*$-algebras was proved in [36, Propostion 3.4].

Proposition 2.9. *If $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a spectral interpolation triple of reduced Banach $*$-algebras relative to a quasi-Hermitian dense $*$-subalgebra $\mathcal{S}$ of $\mathcal{A}$, then $\mathcal{S}_h$ has invariant spectral radius in $(\mathcal{B}, \mathcal{C})$. In particular, $C^*(\mathcal{B}) = C^*(\mathcal{C})$ and*

$$r_{\mathcal{B}}(a) = r_{\mathcal{C}}(a)\,,$$

for every $a \in \mathcal{S}_h$.

2.5. **Groupoids, twists and associated algebras.** We now introduce the Banach $*$-algebras and C^*-algebras associated to a groupoid as described in [33]. Given a topological groupoid $\mathcal{G}$ we will denote by $\mathcal{G}^{(0)}$ its unit space and write $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ for the continuous range and source maps, respectively. We will also denote by $\mathcal{G}^{(2)} = \{(\alpha, \beta) \in \mathcal{G} \times \mathcal{G} : s(\alpha) = r(\beta)\}$ the set of *composable elements*. $\mathcal{G}^{(2)}$ inherits the subspace topology from $\mathcal{G} \times \mathcal{G}$. Given $x \in \mathcal{G}^{(0)}$ we define $\mathcal{G}_x := \{\gamma \in \mathcal{G} : s(\gamma) = x\}$ and $\mathcal{G}^x := \{\gamma \in \mathcal{G} : r(\gamma) = x\}$.

Let $\lambda = \{\lambda_x\}_{x \in \mathcal{G}^{(0)}}$ be a *Haar system for $\mathcal{G}$*, where λ_x are measures with support $\mathcal{G}_x$ such that

(1) for every $f \in C_c(\mathcal{G})$, the function $x \mapsto \int_{\mathcal{G}_x} f(\gamma) d\lambda_x(\gamma)$ is continuous,
(2) for every $\eta \in \mathcal{G}$ and $f \in C_c(\mathcal{G})$ we have that

$$\int_{\mathcal{G}_{r(\eta)}} f(\gamma\eta) d\lambda_{r(\eta)}(\gamma) = \int_{\mathcal{G}_{s(\eta)}} f(\gamma) d\lambda_{s(\eta)}(\gamma)\,.$$

If $\mathcal{G}$ is a locally compact groupoid, then there always exist Haar systems for $\mathcal{G}$.

A groupoid $\mathcal{G}$ is called *étale* if the range map, and hence also the source map, is a local homeomorphism. In this case the sets $\mathcal{G}^x$ and $\mathcal{G}_x$ are discrete sets for every $x \in \mathcal{G}^{(0)}$, and the Haar system consists of counting measures. A subset B of an étale groupoid $\mathcal{G}$ is called a *bisection* if there is an open set $U \subseteq \mathcal{G}$ containing B such that $r\colon U \to r(U)$ and $s\colon U \to s(U)$ are homeomorphisms onto open subsets of $\mathcal{G}^{(0)}$. Second-countable étale groupoids have countable bases consisting of open bisections.

The *orbit of* $x \in \mathcal{G}^{(0)}$ is defined to be $\mathrm{Orb}_{\mathcal{G}}(x) = \{r(\gamma) : \gamma \in \mathcal{G}_x\}$. The *isotropy group of* x is given by $\mathcal{G}_x^x := \mathcal{G}^x \cap \mathcal{G}_x = \{\gamma \in \mathcal{G} : s(\gamma) = r(\gamma) = x\}$, and the *isotropy subgroupoid of* $\mathcal{G}$ is the subgroupoid $\mathrm{Iso}(\mathcal{G}) := \bigcup_{x\in\mathcal{G}^{(0)}} \mathcal{G}_x^x$ with the relative topology from $\mathcal{G}$.

We will consider groupoid twists where the twist is implemented by a normalized continuous 2-cocycle. To be more precise, let $\mathcal{G}$ be a locally compact groupoid. A *normalized continuous* 2*-cocycle* is then a continuous map $\sigma\colon \mathcal{G}^{(2)} \to \mathbb{T}$ satisfying

$$\sigma(r(\gamma), \gamma) = 1 = \sigma(\gamma, s(\gamma)) \tag{2.1}$$

for all $\gamma \in \mathcal{G}$, and

$$\sigma(\alpha, \beta)\sigma(\alpha\beta, \gamma) = \sigma(\beta, \gamma)\sigma(\alpha, \beta\gamma) \tag{2.2}$$

whenever $(\alpha, \beta), (\beta, \gamma) \in \mathcal{G}^{(2)}$. The set of normalized continuous 2-cocycles on $\mathcal{G}$ will be denoted $Z^2(\mathcal{G}, \mathbb{T})$. Note that this is not the most general notion of a twist of a groupoid (see [37, Chapter 11]).

Let $\mathcal{G}$ be a locally compact groupoid with Haar system λ. We will define the σ-twisted convolution algebra $C_c(\mathcal{G}, \sigma)$ as follows: As a set it is just

$$C_c(\mathcal{G}, \sigma) = \{f : \mathcal{G} \to \mathbb{C} : f \text{ is continuous with compact support}\},$$

but equipped with σ-twisted convolution product

$$(f \star_\sigma g)(\gamma) = \int_{\mathcal{G}_{s(\gamma)}} f(\gamma\mu^{-1})g(\mu)\sigma(\gamma\mu^{-1}, \mu)\, d\lambda_{s(\gamma)}(\mu), \quad f, g \in C_c(\mathcal{G}, \sigma),\ \gamma \in \mathcal{G}, \tag{2.3}$$

and σ-twisted involution

$$f^{*_\sigma}(\gamma) = \overline{\sigma(\gamma^{-1}, \gamma)f(\gamma^{-1})}, \quad f \in C_c(\mathcal{G}, \sigma),\ \gamma \in \mathcal{G}. \tag{2.4}$$

We complete $C_c(\mathcal{G}, \sigma)$ in the "fiberwise 1-norm", also known as the I-norm, given by

$$\|f\|_I = \sup_{x\in\mathcal{G}^{(0)}} \left\{ \max\left\{ \int_{\mathcal{G}_x} |f(\gamma)|\, d\lambda_x(\gamma), \int_{\mathcal{G}_x} |f(\gamma^{-1})|\, d\lambda_x(\gamma) \right\} \right\} \tag{2.5}$$

$$= \sup_{x\in\mathcal{G}^{(0)}} \left\{ \max\left\{ \int_{\mathcal{G}_x} |f(\gamma)|\, d\lambda_x(\gamma), \int_{\mathcal{G}_x} |f^{*_\sigma}(\gamma)|\, d\lambda_x(\gamma) \right\} \right\} \tag{2.6}$$

for $f \in C_c(\mathcal{G}, \sigma)$. Denote by $L^1(\mathcal{G}, \sigma, \lambda)$ (or $L^1(\mathcal{G}, \sigma)$ when there is no ambiguity on the Haar system) the completion of $C_c(\mathcal{G}, \sigma)$ with respect to the I-norm. If $\mathcal{G}$ is an étale groupoid, then we denote it by $\ell^1(\mathcal{G}, \sigma)$.

The *(full) twisted groupoid C^*-algebra* $C^*(\mathcal{G},\sigma,\lambda)$ (or $C^*(\mathcal{G},\sigma)$ when there is no ambiguity on the Haar system) is the completion of $C_c(\mathcal{G},\sigma)$ in the norm

$$\|f\| := \sup\{\|\pi(f)\| : \pi \text{ is an } I\text{-norm bounded } *\text{-representation of } C_c(\mathcal{G},\sigma)\},$$

for $f \in C_c(\mathcal{G},\sigma)$. It was observed in [3, Lemma 3.3.19] that if $\mathcal{G}$ is a locally compact Hausdorff étale groupoid, then every $*$-representation of $C_c(\mathcal{G},\sigma)$ is bounded by the I-norm. In addition, if $\mathcal{G}$ is a transformation groupoid (see Example 2.12), every $*$-representation is bounded by the I-norm. In these cases, since we are completing with respect to a supremum over $*$-representations, $C^*(\mathcal{G},\sigma)$ is just the C^*-envelope of $L^1(\mathcal{G},\sigma)$.

Now we will construct a faithful $*$-representation of $L^1(\mathcal{G},\sigma)$ called the *σ-twisted left regular representation.* In particular, we have that $L^1(\mathcal{G},\sigma)$ is reduced. The completion of the image of $L^1(\mathcal{G},\sigma)$ under the σ-twisted left regular representation is called the *σ-twisted reduced groupoid C^*-algebra of $\mathcal{G}$* and will be denoted $C_r^*(\mathcal{G},\sigma,\lambda)$ (or $C_r^*(\mathcal{G},\sigma)$ when there is no ambiguity on the Haar system). Let $x \in \mathcal{G}^{(0)}$. Then there is a representation $L_x^{\sigma,2}\colon C_c(\mathcal{G},\sigma) \to B(L^2(\mathcal{G}_x))$ (here $L^2(\mathcal{G}_x) = L^2(\mathcal{G}_x,\lambda_x)$) which is given by

$$\left(L_x^{\sigma,2}(f)\xi\right)(\gamma) = \int_{\mathcal{G}_x} \sigma(\gamma\mu^{-1},\mu) f(\gamma\mu^{-1})\xi(\mu)\, d\lambda_{s(\gamma)}(\mu) \tag{2.7}$$

for $f \in C_c(\mathcal{G},\sigma)$, $\xi \in L^2(\mathcal{G}_x)$ and $\gamma \in \mathcal{G}_x$.

We then obtain a faithful I-norm bounded $*$-representation of $C_c(\mathcal{G},\sigma)$ given by

$$\bigoplus_{x\in\mathcal{G}^{(0)}} L_x^{\sigma,2}\colon C_c(\mathcal{G},\sigma) \to \bigoplus_{x\in\mathcal{G}^{(0)}} B(L^2(\mathcal{G}_x)) \subset B(\bigoplus_{x\in\mathcal{G}^{(0)}} L^2(\mathcal{G}_x)).$$

$C_r^*(\mathcal{G},\sigma)$ is then the completion of $C_c(\mathcal{G},\sigma)$ with respect of the image of $C_c(\mathcal{G},\sigma)$ under the σ-twisted left regular representation, so given $f \in C_c(\mathcal{G},\sigma)$

$$\|f\|_{r,2} := \sup_{x\in\mathcal{G}^{(0)}} \left\{\|L_x^{\sigma,2}(f)\|_{B(L^2(\mathcal{G}_x))}\right\}.$$

As the representation is I-norm bounded, it extends to a $*$-representation of $L^1(\mathcal{G},\sigma)$, and $C_r^*(\mathcal{G},\sigma)$ is also the C^*-completion of $L^1(\mathcal{G},\sigma)$ in the extended $*$-representation. Moreover, since $C^*(\mathcal{G},\sigma)$ is the completion of $C_c(\mathcal{G},\sigma)$ with respect to the supremum of the I-bounded norms, the identity map on $C_c(\mathcal{G},\sigma)$ extends to a (surjective) $*$-homomorphism $\pi : C^*(\mathcal{G},\sigma) \to C_r^*(\mathcal{G},\sigma)$.

Definition 2.10. Let $\mathcal{G}$ be a locally compact groupoid with Haar system λ and let $\sigma \in Z^2(G,\mathbb{T})$. We say that $\mathcal{G}$ twisted by σ has the *weak containment property* when the natural map $\pi\colon C^*(\mathcal{G},\sigma) \to C_r^*(\mathcal{G},\sigma)$ is an isomorphism.

If $\mathcal{G}$ is an amenable groupoid with Haar measure λ [2], we have that $C_r^*(\mathcal{G},\sigma) = C^*(\mathcal{G},\sigma)$ for every $\sigma \in Z^2(\mathcal{G},\mathbb{T})$ [2, Proposition 6.1.8], and hence $\mathcal{G}$ twisted by σ has the weak containment property for every $\sigma \in Z^2(\mathcal{G},\mathbb{T})$. In [39] it was proved that amenability is not equivalent to having the weak containment property.

Remark 2.11. It was shown in [33] that the full and reduced C^*-algebras don't depend, up to Morita equivalence, on the Haar system. Suppose λ,λ' are two

Haar systems for a locally compact groupoid $\mathcal{G}$, let $\sigma \in Z^2(G, \mathbb{T})$, and suppose $C_r^*(\mathcal{G}, \sigma, \lambda) = C^*(\mathcal{G}, \sigma, \lambda)$. Then $C_r^*(\mathcal{G}, \sigma, \lambda)$ and $C_r^*(\mathcal{G}, \sigma, \lambda')$ are Morita equivalent, as are $C^*(\mathcal{G}, \sigma, \lambda)$ and $C^*(\mathcal{G}, \sigma, \lambda')$. However, it is not known to the authors if this also implies $C_r^*(\mathcal{G}, \sigma, \lambda') = C^*(\mathcal{G}, \sigma, \lambda')$, that is, if weak containment is independent of the Haar system.

Example 2.12. *Let Γ be a locally compact group with left Haar measure λ and with unit e, and let us consider an action of Γ by homeomorphisms on a locally compact Hausdorff space X. Then we define the* transformation groupoid $X \rtimes \Gamma$ *as the set $X \times \Gamma$ with the product topology, such that*

$$s(x, \gamma) = (x, e)\,, \qquad r(x, \gamma) = (\gamma \cdot x, e) \qquad \textit{and} \qquad (\gamma_1 \cdot x, \gamma_2)(x, \gamma_1) = (x, \gamma_2\gamma_1)\,.$$

Then $X \rtimes \Gamma$ is a locally compact groupoid. Moreover, if X and Γ are both second-countable, then so is $X \rtimes \Gamma$. One defines the Haar system $\{\delta_x \times \lambda\}_{x \in \mathcal{G}^{(0)}}$ where δ_x is the Dirac measure, and in this case, given $f \in C_c(X \rtimes \Gamma)$ we have that

$$\begin{aligned} \|f\|_I &= \sup\left\{\max\left\{\int_\Gamma |f(x,\gamma)|\,d\lambda(\gamma)\,, \int_\Gamma |f(\gamma\cdot x, \gamma^{-1})|\,d\lambda(\gamma)\right\} : x \in \mathcal{G}^{(0)}\right\} \\ &= \sup\left\{\int_\Gamma |f(x,\gamma)|\,d\lambda(\gamma) : x \in \mathcal{G}^{(0)}\right\}\,. \end{aligned}$$

Now let $\sigma \in Z^2(\Gamma, \mathbb{T})$. Then we can extend σ to a 2-cocycle of $X \rtimes \Gamma$, also denoted by σ, by defining

$$\sigma((x_1, \gamma_1), (x_2, \gamma_2)) := \sigma(\gamma_1, \gamma_2)\,,$$

for all $x_1, x_2 \in X$ and $\gamma_1, \gamma_2 \in \Gamma$ for which $((x_1, \gamma_1), (x_2, \gamma_2)) \in (X \rtimes \Gamma)^{(2)}$. Then

$$C^*(L^1(X \rtimes \Gamma, \sigma)) \cong C^*(X \rtimes \Gamma, \sigma) \cong C_0(X) \rtimes^\sigma \Gamma$$

is the full twisted crossed product C^*-algebra, and

$$C_r^*(X \rtimes \Gamma, \sigma) \cong C_0(X) \rtimes_r^\sigma \Gamma$$

is the reduced twisted crossed product C^*-algebra.

3. Quasi-Hermitian groupoids have the weak containment property.

In this section we prove the main result of the paper, namely that if $L^1(\mathcal{G}, \sigma)$ is Hermitian, then $C^*(\mathcal{G}, \sigma) = C_r^*(\mathcal{G}, \sigma)$. As a consequence, we also prove that if $L^1(\mathcal{G}, \sigma)$ is Hermitian, then $L^1(\mathcal{G}, \sigma)$ is spectrally invariant in $C_r^*(\mathcal{G}, \sigma)$. The general strategy is to follow [36, Section 4], but not all the steps trivially extend to our situation.

Definition 3.1. Let $\mathcal{G}$ be a locally compact groupoid with Haar system λ and let $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. We say that $\mathcal{G}$ is *σ-quasi-Hermitian* (resp. *σ-quasi-symmetric*) if $C_c(\mathcal{G}, \sigma)$ is quasi-Hermitian (resp. quasi-symmetric) in $L^1(\mathcal{G}, \sigma, \lambda)$.

Proposition 3.2. *Let $\mathcal{G}$ be a locally compact groupoid with Haar system λ and $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. If $\mathcal{G}$ is σ-quasi-symmetric, then $\mathcal{G}$ is σ-quasi-Hermitian.*

Proof. The proof of [36, Proposition 4.1] adapts trivially to give us the following result. Let $f \in C_c(\mathcal{G},\sigma)_h$, then by assumption $\mathrm{Sp}_{L^1(\mathcal{G},\sigma)}(f \star_\sigma f^{*_\sigma}) \subseteq [0,\infty)$. Therefore

$$\{\lambda^2 : \lambda \in \mathrm{Sp}_{L^1(\mathcal{G},\sigma)}(f)\} \subseteq \mathrm{Sp}_{L^1(\mathcal{G},\sigma)}(f \star_\sigma f) = \mathrm{Sp}_{L^1(\mathcal{G},\sigma)}(f \star_\sigma f^{*_\sigma}) \subseteq [0,\infty)\,.$$

Hence $\mathrm{Sp}_{L^1(\mathcal{G},\sigma)}(f) \subseteq \mathbb{R}$ for every $f \in C_c(\mathcal{G},\sigma)_h$. □

Let $\mathcal{G}$ be a locally compact groupoid with Haar system λ, $\sigma \in Z^2(\mathcal{G},\mathbb{T})$ and $1 \leq p \leq \infty$. Now fix $x \in \mathcal{G}^{(0)}$, so we define the representation $L_x^{\sigma,p} : C_c(\mathcal{G},\sigma) \to B(L^p(\mathcal{G}_x))$ by

$$L_x^{\sigma,p}(f)\xi(\gamma) = \int_{\mathcal{G}_x} \sigma(\gamma\mu^{-1},\mu) f(\gamma\mu^{-1})\xi(\mu)d\lambda_x(\mu),$$

for every $f \in C_c(\mathcal{G},\sigma)$ and $\gamma \in \mathcal{G}_x$. For $p = 2$ this is just (2.7).

Then we define the *L^p-reduced σ-representation of $\mathcal{G}$* as

$$L^{\sigma,p} := \bigoplus_{x\in\mathcal{G}^{(0)}} L_x^{\sigma,p} : C_c(\mathcal{G},\sigma) \to \bigoplus_{x\in\mathcal{G}^{(0)}} B(L^p(\mathcal{G}_x))\,.$$

The following lemma is a straightforward modification of [15, Lemma 4.6] to the situation of general L^p-spaces.

Lemma 3.3. *Let $\mathcal{G}$ be a second-countable locally compact groupoid with Haar system λ and $1 \leq p \leq \infty$. Then*

$$\|f\|_\infty := \sup_{\gamma\in\mathcal{G}}\{|f(\gamma)|\} \leq \|L^{\sigma,p}(f)\| = \sup_{x\in\mathcal{G}^{(0)}}\{\|L_x^{\sigma,p}(f)\|_{B(L^p(\mathcal{G}_x))}\} \leq \|f\|_I\,, \tag{3.1}$$

for every $f \in C_c(\mathcal{G},\sigma)$.

Definition 3.4. Let $\mathcal{G}$ be a second-countable locally compact groupoid with Haar system λ, $\sigma \in Z^2(\mathcal{G},\mathbb{T})$ and $1 \leq p \leq \infty$. The *reduced groupoid L^p-Banach algebra*, denoted by $F^p(\mathcal{G},\sigma,\lambda)$, is the completion of $C_c(\mathcal{G},\sigma)$ with respect to the norm

$$\|f\|_{r,p} := \sup_{x\in\mathcal{G}^{(0)}}\{\|L_x^{\sigma,p}(f)\|_{B(L^p(\mathcal{G}_x))}\}$$

for all $f \in C_c(\mathcal{G},\sigma)$. We will denote $F^p(\mathcal{G},\sigma,\lambda)$ by $F^p(\mathcal{G},\sigma)$ when there is no ambiguity on the Haar system λ.

Let $1 \leq p \leq q \leq \infty$ with $1 = \frac{1}{p} + \frac{1}{q}$. Then, given any $x \in \mathcal{G}^{(0)}$, there is a duality relation $(L^p(\mathcal{G}_x))^* \cong L^q(\mathcal{G}_x)$ given by

$$\langle \xi, \zeta \rangle := \int_{\mathcal{G}_x} \xi(\gamma)\overline{\zeta(\gamma)}\, d\lambda_x\,,$$

for $\xi \in L^p(\mathcal{G}_x)$ and $\zeta \in L^q(\mathcal{G}_x)$.

Lemma 3.5. *Let $\mathcal{G}$ be a second-countable locally compact groupoid with Haar system λ, let $\sigma \in Z^2(\mathcal{G},\mathbb{T})$, and let $1 \leq p,q \leq \infty$ be such that $1 = \frac{1}{p} + \frac{1}{q}$. Then*

$$\langle L_x^{\sigma,p}(f^{*_\sigma})\xi, \zeta \rangle = \langle \xi, L_x^{\sigma,q}(f)\zeta \rangle\,,$$

for every $x \in \mathcal{G}^{(0)}$, $f \in L^1(\mathcal{G},\sigma,\lambda)$, $\xi \in L^p(\mathcal{G}_x,\lambda_x)$ and $\zeta \in L^q(\mathcal{G}_x,\lambda_x)$.

Proof. Fix $x \in \mathcal{G}^{(0)}$, $f \in L^1(\mathcal{G},\sigma)$, $\xi \in L^p(\mathcal{G}_x)$ and $\zeta \in L^q(\mathcal{G}_x)$. Then

$$\begin{aligned}
\langle L_x^{\sigma,p}(f^{*_\sigma})\xi,\zeta\rangle &= \int_{\mathcal{G}_x}\left(\int_{\mathcal{G}_x}\sigma(\gamma\mu^{-1},\mu)f^{*_\sigma}(\gamma\mu^{-1})\xi(\mu)d\lambda_x(\mu)\right)\overline{\zeta(\gamma)}\,d\lambda_x(\gamma)\\
&= \int_{\mathcal{G}_x}\left(\int_{\mathcal{G}_x}\sigma(\gamma\mu^{-1},\mu)\overline{\sigma((\gamma\mu^{-1})^{-1},\gamma\mu^{-1})f((\gamma\mu^{-1})^{-1})}\xi(\mu)d\lambda_x(\mu)\right)\overline{\zeta(\gamma)}\,d\lambda_x(\gamma)\\
&= \int_{\mathcal{G}_x}\xi(\mu)\left(\int_{\mathcal{G}_x}\overline{\overline{\sigma(\gamma\mu^{-1},\mu)}\sigma((\gamma\mu^{-1})^{-1},\gamma\mu^{-1})f(\mu\gamma^{-1})\zeta(\gamma)\,d\lambda_x(\gamma)}\right)d\lambda_x(\mu)\\
&= \int_{\mathcal{G}_x}\xi(\mu)\int_{\mathcal{G}_x}\left(\overline{\sigma(\mu\gamma^{-1},\gamma)f(\mu\gamma^{-1})\zeta(\gamma)\,d\lambda_x(\gamma)}\right)d\lambda_x(\mu)\\
&= \int_{\mathcal{G}_x}\xi(\mu)\overline{L_x^{\sigma,q}(f)\zeta(\mu)}\,d\lambda_x(\mu) = \langle\xi, L_x^{\sigma,q}(f)\zeta\rangle.
\end{aligned}$$

Here we used that $\overline{\sigma(\gamma\mu^{-1},\mu)}\sigma((\gamma\mu^{-1})^{-1},\gamma\mu^{-1}) = \sigma(\mu\gamma^{-1},\gamma)$, or equivalently, that

$$\sigma(\mu\gamma^{-1},\gamma\mu^{-1}) = \sigma(\mu\gamma^{-1},\gamma)\sigma(\gamma\mu^{-1},\mu)$$

for every $\mu,\gamma\in\mathcal{G}_x$. We get this from (2.2) using $\mu\gamma^{-1}$, $(\mu\gamma^{-1})^{-1}$, μ instead of α, β, γ, and then applying (2.1). □

Proposition 3.6. *Let $\mathcal{G}$ be a second-countable locally compact groupoid with Haar system λ and $\sigma\in Z^2(\mathcal{G},\mathbb{T})$, and suppose $1\le p,q\le\infty$ with $1=\frac{1}{p}+\frac{1}{q}$. The algebra $L^1(\mathcal{G},\sigma,\lambda)$ is a normed ∗-algebra with norm*

$$\|f\|_{\sharp,p} := \max\{\|f\|_{r,p},\ \|f\|_{r,q}\}\,,$$

for $f\in C_c(\mathcal{G},\sigma)$, with the convolution and involution in $L^1(\mathcal{G},\sigma,\lambda)$.

Proof. We will adapt the proof of [36, Proposition 4.2] to our setting. Given $f\in C_c(\mathcal{G},\sigma)$ we have that

$$\begin{aligned}
\|f\|_{\sharp,p} &= \max\{\sup_{x\in\mathcal{G}^{(0)}}\{\|L_x^{\sigma,p}(f)\|\},\ \sup_{x\in\mathcal{G}^{(0)}}\{\|L_x^{\sigma,q}(f)\|\}\}\\
&= \sup_{x\in\mathcal{G}^{(0)}}\{\|L_x^{\sigma,p}(f)\|,\ \|L_x^{\sigma,q}(f)\|\}\le\|f\|_I,
\end{aligned}$$

by (3.1), so $\|f\|_{\sharp,p}$ is well defined. It is easy to check that $(L^1(\mathcal{G},\sigma),\|\cdot\|_{\sharp,p})$ is a normed algebra. We only have to prove that the involution is an isometry with respect to $\|\cdot\|_{\sharp,p}$. Let $f\in L^1(\mathcal{G},\sigma)$. Given $x\in\mathcal{G}^{(0)}$, by Lemma 3.5 we have that

$$\begin{aligned}
\|L_x^{\sigma,p}(f^{*_\sigma})\|_{B(L^p(\mathcal{G}_x))} &= \sup\{|\langle L_x^{\sigma,p}(f^{*_\sigma})\xi,\zeta\rangle| : \|\xi\|_p\le 1,\ \|\zeta\|_q\le 1\}\\
&= \sup\{|\langle\xi, L_x^{\sigma,q}(f)\zeta\rangle| : \|\xi\|_p\le 1,\ \|\zeta\|_q\le 1\}\\
&= \|L_x^{\sigma,q}(f)\|_{B(L^q(\mathcal{G}_x))}\,.
\end{aligned}$$

Similarly, switching p and q we obtain that

$$\|L_x^{\sigma,q}(f^{*_\sigma})\|_{B(L^q(\mathcal{G}_x))} = \|L_x^{\sigma,p}(f)\|_{B(L^p(\mathcal{G}_x))}\,,$$

and therefore $\|f^{*_\sigma}\|_{\sharp,p} = \|f\|_{\sharp,p}$, as desired. □

Definition 3.7. Let $\mathcal{G}$ be a second-countable locally compact groupoid with Haar system λ and $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$, and let $1 \leq p \leq \infty$. The Banach $*$-algebra $F^p_\sharp(\mathcal{G}, \sigma, \lambda)$ ($F^p_\sharp(\mathcal{G}, \sigma)$ when there is no ambiguity on the Haar system) is defined to be the completion of $L^1(\mathcal{G}, \sigma, \lambda)$ with respect to the norm $\| \cdot \|_{\sharp,p}$.

Given $1 \leq p, q \leq \infty$ with $1 = \frac{1}{p} + \frac{1}{q}$, the Banach $*$-algebras $F^p_\sharp(\mathcal{G}, \sigma, \lambda)$ and $F^q_\sharp(\mathcal{G}, \sigma, \lambda)$ are isometrically isomorphic. Moreover, $F^2_\sharp(\mathcal{G}, \sigma, \lambda) = C^*_r(\mathcal{G}, \sigma, \lambda)$.

The following result is a combination of [9, Theorem 5.1.1] and [14, Section 10.1].

Lemma 3.8. *Let $\mathcal{G}$ be a second-countable locally compact groupoid with Haar system λ, let $x \in \mathcal{G}^{(0)}$, and let $1 \leq p_1 \leq p_2 \leq p_3 \leq \infty$. Suppose that T is a bounded operator defined on $L^{p_1}(\mathcal{G}_x, \lambda_x) \cap L^{p_3}(\mathcal{G}_x, \lambda_x)$ such that it extends continuously to bounded operators on both $L^{p_1}(\mathcal{G}_x, \lambda_x)$ and $L^{p_3}(\mathcal{G}_x, \lambda_x)$: Then T extends continuously on $L^{p_2}(\mathcal{G}_x, \lambda_x)$. Furthermore, if M_i is the norm of the extension of T on $L^{p_i}(\mathcal{G}_x, \lambda_x)$ for $i = 1, 2, 3$, then*

$$M_2 \leq M_1^{1-\theta} M_3^{\theta}\,,$$

for $0 < \theta < 1$ satisfying

$$\frac{1}{p_2} = \frac{1-\theta}{p_1} + \frac{\theta}{p_3}\,.$$

Proposition 3.9. *Let $\mathcal{G}$ be a second-countable locally compact groupoid with Haar system λ and $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$, and suppose $1 < p < 2$. Then*

$$(L^1(\mathcal{G}, \sigma, \lambda), F^p_\sharp(\mathcal{G}, \sigma, \lambda), C^*_r(\mathcal{G}, \sigma, \lambda))$$

is a spectral interpolation triple of reduced Banach $*$-algebras relative to $L^1(\mathcal{G}, \sigma, \lambda)$.

Proof. Let $q \in (2, \infty)$ such that $1 = \frac{1}{p} + \frac{1}{q}$, and let

$$\theta = \frac{2p-2}{p} \in (0, 1)\,,$$

and hence

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}\,.$$

Then, for $x \in \mathcal{G}^{(0)}$ and $f \in L^1(\mathcal{G}, \sigma)$, using Lemma 3.8 and (2.5) we have that

$$\begin{aligned} \|L_x^{\sigma,p}(f)\|_{B(L^p(\mathcal{G}_x))} &\leq \|L_x^{\sigma,1}(f)\|^{1-\theta}_{B(L^1(\mathcal{G}_x))} \|L_x^{\sigma,2}(f)\|^{\theta}_{B(L^2(\mathcal{G}_x))} \\ &\leq \|f\|^{1-\theta}_{\sharp,1} \|f\|^{\theta}_{r,2} \leq \|f\|^{1-\theta}_{I} \|f\|^{\theta}_{r,2}\,, \end{aligned}$$

and therefore $\|f\|_{r,p} \leq \|f\|^{1-\theta}_I \|f\|^{\theta}_{r,2}$. On the other hand,

$$\begin{aligned} \frac{1}{q} = 1 - \frac{1}{p} &= 1 - \left(\frac{1-\ \theta}{1} + \frac{\theta}{2}\right) \\ &= 1 - (1-\theta) - \theta\left(1 - \frac{1}{2}\right) \\ &= 0 + \frac{\theta}{2} = \frac{1-\theta}{\infty} + \frac{\theta}{2}\,, \end{aligned}$$

so we can apply the same above argument to show that $\|f\|_{r,q} \leq \|f\|_I^{1-\theta}\|f\|_{r,2}^{\theta}$. Hence,

$$\|f\|_{\sharp,p} \leq \|f\|_I^{1-\theta}\|f\|_{r,2}^{\theta}\,, \tag{3.2}$$

for every $f \in L^1(\mathcal{G},\sigma)$.

Therefore given $f \in L^1(\mathcal{G},\sigma)$, we have that

$$\|f^n\|_{\sharp,p}^{\frac{1}{n}} \leq \|f^n\|_I^{\frac{1-\theta}{n}}\|f^n\|_{r,2}^{\frac{\theta}{n}}\,,$$

for every $n \in \mathbb{N}$. Then taking the limit as $n \to \infty$ we have that

$$r_{F^p_\sharp(\mathcal{G},\sigma)}(f) \leq r_{L^1(\mathcal{G},\sigma)}(f)^{1-\theta} r_{C^*_r(\mathcal{G},\sigma)}(f)^{\theta}\,, \tag{3.3}$$

for every $f \in L^1(\mathcal{G},\sigma)$.

To finish the proof we only need to prove that $(L^1(\mathcal{G},\sigma), F^p_\sharp(\mathcal{G},\sigma), C^*_r(\mathcal{G},\sigma))$ is a nested triple of reduced Banach $*$-algebras.

Let $\theta \in (0,1)$ be such that

$$\frac{1}{2} = \frac{1-\theta}{p} + \frac{\theta}{q}\,.$$

Then for $f \in L^1(\mathcal{G},\sigma)$,

$$\begin{aligned}
\|f\|_{\sharp,p} &= \sup_{x\in\mathcal{G}^{(0)}} \{\|L_x^{\sigma,p}(f)\|_{B(L^p(\mathcal{G}_x))},\, \|L_x^{\sigma,q}(f)\|_{B(L^q(\mathcal{G}_x))}\} \\
&\geq \sup_{x\in\mathcal{G}^{(0)}} \{\|L_x^{\sigma,p}(f)\|_{B(L^p(\mathcal{G}_x))}^{1-\theta}\|L_x^{\sigma,q}(f)\|_{B(L^q(\mathcal{G}_x))}^{\theta}\} \\
&\geq \sup_{x\in\mathcal{G}^{(0)}} \{\|L_x^{\sigma,2}(f)\|_{B(L^2(\mathcal{G}_x))}\} = \|f\|_{r,2}\,,
\end{aligned}$$

by Lemma 3.8. Therefore the identity map on $L^1(\mathcal{G},\sigma)$ extends to a contractive $*$-homomorphism

$$\pi_{p,2} : F^p_\sharp(\mathcal{G},\sigma) \to C^*_r(\mathcal{G},\sigma)\,.$$

Now we want to see that $\pi_{p,2}$ is injective. Let $f \in \ker(\pi_{p,2})$. Then there exists a sequence $\{f_n\}_{n=1}^\infty$ in $L^1(\mathcal{G},\sigma)$ such that $\lim f_n = f$ in $F^p_\sharp(\mathcal{G},\sigma)$. Then given $x \in \mathcal{G}^{(0)}$ and $\xi \in C_c(\mathcal{G}_x)$ we have that

$$\lim_{n\to\infty} L_x^{\sigma,2}(f_n)\xi = 0 \qquad \text{in } L^2(\mathcal{G}_x)\,,$$

that is

$$\begin{aligned}
0 &= \|\lim_{n\to\infty} L_x^{\sigma,2}(f_n)\xi\|_{L^2(\mathcal{G}_x)} \\
&= \left(\int_{\mathcal{G}_x} \lim_{n\to\infty} |L_x^{\sigma,2}(f_n)\xi(\gamma)|^2\, d\lambda_x(\gamma)\right)^{1/2} \\
&= \left(\int_{\mathcal{G}_x} \lim_{n\to\infty} \left|\int_{\mathcal{G}_x} \sigma(\gamma\mu^{-1},\mu) f_n(\gamma\mu^{-1})\xi(\mu)\, d\lambda_x(\mu)\right|^2 d\lambda_x(\gamma)\right)^{1/2}\,,
\end{aligned}$$

which forces

$$\lim_{n\to\infty} \left|\int_{\mathcal{G}_x} \sigma(\gamma\mu^{-1},\mu) f(\gamma\mu^{-1})\xi(\mu)\, d\lambda_x(\mu)\right| = 0\,. \tag{3.4}$$

Now observe that the map

$$\Psi : L^1(\mathcal{G},\sigma) \to B\left(\oplus_{x\in\mathcal{G}^{(0)}}\left(L^p(\mathcal{G}_x)\oplus L^q(\mathcal{G}_x)\right)\right) ,$$

defined by $f \mapsto \bigoplus_{x\in\mathcal{G}^{(0)}}(L_x^{\sigma,p}(f)\oplus L_x^{\sigma,q}(f))$, extends isometrically to a map

$$\Psi : F_\sharp^p(\mathcal{G},\sigma) \to B\left(\oplus_{x\in\mathcal{G}^{(0)}}\left(L^p(\mathcal{G}_x)\oplus L^q(\mathcal{G}_x)\right)\right) .$$

Now fix $x\in\mathcal{G}^{(0)}$ and $i\in\{p,q\}$. Then given $\xi\in C_c(\mathcal{G}_x)$ we have that

$$\Psi(f)\xi = \lim_{n\to\infty}\Psi(f_n)\xi\,,$$

but

$$\begin{aligned}\|\Psi(f)\xi\|_{L^i(\mathcal{G}_x)} &= \|\lim_{n\to\infty}\Psi(f_n)\xi\|_{L^i(\mathcal{G}_x)}\\ &= \|\lim_{n\to\infty}L_x^{\sigma,i}(f_n)\xi\|_{L^i(\mathcal{G}_x)}\\ &= \lim_{n\to\infty}\left(\int_{\mathcal{G}_x}\left|\int_{\mathcal{G}_x}\sigma(\gamma\mu^{-1},\mu)f_n(\gamma\mu^{-1})\xi(\mu)\,d\lambda_x(\mu)\right|^i d\lambda_x(\gamma)\right)^{1/i}\\ &= 0\end{aligned}$$

because of (3.4). Thus, $\Psi(f)=0$ and since Ψ is isometric we have that $f=0$. Hence $\pi_{p,2}$ is injective.

Now, by (3.2) and the fact that the regular representation is I-norm bounded, we get

$$\|f\|_{\sharp,p}\le\|f\|_I^{1-\theta}\|f\|_{r,2}^{\theta}\le\|f\|_I^{1-\theta}\|f\|_I^{\theta}=\|f\|_I\,,$$

for every $f\in L^1(\mathcal{G},\sigma)$. Hence, the identity map on $L^1(\mathcal{G},\sigma)$ extends to a contraction

$$\Phi_{p,I} : L^1(\mathcal{G},\sigma)\to F_\sharp^p(\mathcal{G},\sigma)\,.$$

Observe that then $(\pi_{p,2}\circ\Phi_{p,I}) : L^1(\mathcal{G},\sigma)\to C_r^*(\mathcal{G},\sigma)$ is the regular representation of $L^1(\mathcal{G},\sigma)$, which is injective. It follows that $\Phi_{p,I} : L^1(\mathcal{G},\sigma)\to F_\sharp^p(\mathcal{G},\sigma)$ is injective too. □

Proposition 3.10. *Let $\mathcal{G}$ be a second-countable locally compact groupoid with Haar system λ and $\sigma\in Z^2(\mathcal{G},\mathbb{T})$. Then the following statements are equivalent:*

1. *$\mathcal{G}$ is σ-quasi-symmetric,*
2. *$\mathcal{G}$ is σ-quasi-Hermitian,*
3. *$r_{L^1(\mathcal{G},\sigma,\lambda)}(f) = r_{C_r^*(\mathcal{G},\sigma,\lambda)}(f)$ for every $f\in C_c(\mathcal{G},\sigma)_h$,*
4. *$\mathrm{Sp}_{L^1(\mathcal{G},\sigma,\lambda)}(f) = \mathrm{Sp}_{C_r^*(\mathcal{G},\sigma,\lambda)}(f)$ for every $f\in C_c(\mathcal{G},\sigma)$.*

Proof. (1) ⇒ (2) was proved in Proposition 3.2. (3) ⇒ (4) is proved in Theorem 2.5, and (4) ⇒ (1) is clear. So we only need to prove (2) ⇒ (3).

Suppose that $\mathcal{G}$ is σ-quasi-Hermitian and $1<p<2$. Then

$$(L^1(\mathcal{G},\sigma), F_\sharp^p(\mathcal{G},\sigma), C_r^*(\mathcal{G},\sigma))$$

is a spectral interpolation triple of reduced Banach $*$-algebras relative to $L^1(\mathcal{G},\sigma)$ by Proposition 3.9. Hence, by Proposition 2.9 we have that

$$r_{F_\sharp^p(\mathcal{G},\sigma)}(f) = r_{C_r^*(\mathcal{G},\sigma)}(f)\,,$$

for every $f\in C_c(\mathcal{G},\sigma)_h$.

Now fix $f \in C_c(\mathcal{G},\sigma)_h$. Then the sets $U = \mathrm{Supp}(f)$ and $s(U)$ are compact sets. Replacing U by $U \cup U^{-1}$ we can assume that $r(U) = s(U)$, and that given any $x \in s(U)$, the map $\mathcal{G}_x \cap U \to \mathcal{G}^x \cap U$ given by $\gamma \mapsto \gamma^{-1}$ is a bijection. Then since $\mathcal{G}$ is locally compact, using a partition of the unit, there exists a function $g_1 \in C_c(\mathcal{G},\sigma)$ such that $g_1(\gamma) = 1$ for every $\gamma \in U$. Then since the map $x \mapsto \int_{\mathcal{G}_x} g_1(\gamma)\, d\lambda_x(\gamma)$ is continuous we have that

$$K := \sup\left\{\int_{\mathcal{G}_x} g_1(\gamma)\, d\lambda_x(\gamma) : x \in s(U)\right\} < \infty\,. \tag{3.5}$$

Observe that given $x \in s(U)$ we have that

$$\int_{\mathcal{G}_x} g_1(\gamma)\, d\lambda_x(\gamma) = \lambda_x(\mathcal{G}_x \cap U) \leq K\,.$$

Now given $n \in \mathbb{N}$, we denote by f^n the n'th convolution power $f \star_\sigma \cdots \star_\sigma f$. Then we have that

$$\mathrm{Supp}(f^n) \subseteq U^{(n)} = \{\gamma_1 \cdots \gamma_n : \gamma_i \in U \text{ such that } r(\gamma_i) = s(\gamma_{i+1}) \text{ for } i = 1,\dots,n{-}1\}\,.$$

By continuity of the groupoid product $U^{(n)}$ is a compact subset of $\mathcal{G}$. Now given $x \in s(U^{(n)})$ we define $U_x^{(n)} := U^{(n)} \cap \mathcal{G}_x$, so

$$\begin{aligned}
\lambda_x(U_x^{(n)}) &= \int_{U_x^{(n-1)}} \lambda_{s(\gamma)}(U_{r(\gamma)}\gamma)\, d\lambda_x(\gamma) \leq \int_{U_x^{(n-1)}} \lambda_{r(\gamma)}(U_{r(\gamma)})\, d\lambda_x(\gamma)\\
&\leq \int_{U_x^{(n-1)}} K\, d\lambda_x(\gamma) = K\lambda_x(U_x^{(n-1)})\\
&\leq K^2\lambda_x(U_x^{(n-2)}) \leq \cdots \leq K^{n-1}\lambda_x(\mathcal{G}_x \cap U)\\
&\leq K^n\,,
\end{aligned}$$

by using the invariance of the Haar measures.

Now, fix $n \in \mathbb{N}$ and $x \in s(U^{(n)})$. Let $1_{U_x^{(n)}} \in L^1(\mathcal{G}_x)$ be the characteristic function on $U_x^{(n)}$. Then we have that

$$\begin{aligned}
\|f^n_{|\mathcal{G}_x}\|_{L^1(\mathcal{G}_x)} &= \|f^n_{|\mathcal{G}_x} 1_{U_x^{(n)}}\|_{L^1(\mathcal{G}_x)}\\
&\leq \|f^n_{|\mathcal{G}_x}\|_{L^p(\mathcal{G}_x)} \|1_{U_x^{(n)}}\|_{L^q(\mathcal{G}_x)} \qquad (\text{where } 1 = \frac{1}{p} + \frac{1}{q})\\
&\leq \|L_x^{p,\sigma}(f^{n-1})\|_{B(L^p(\mathcal{G}_x))} \|f_{|\mathcal{G}_x}\|_{L^p(\mathcal{G}_x)} K^{\frac{n}{q}}\\
&\leq \|f^{n-1}\|_{\sharp,p} \|f_{|\mathcal{G}_x}\|_{L^p(\mathcal{G}_x)} K^{\frac{n}{q}}\,.
\end{aligned}$$

In a similar way we have that

$$\|(f^{*_\sigma})^n_{|\mathcal{G}_x}\|_{L^1(\mathcal{G}_x)} \leq \|(f^{*_\sigma})^{n-1}\|_{\sharp,p} \|f^{*_\sigma}_{|\mathcal{G}_x}\|_{L^p(\mathcal{G}_x)} K^{\frac{n}{q}} = \|f^{n-1}\|_{\sharp,p} \|f^{*_\sigma}_{|\mathcal{G}_x}\|_{L^p(\mathcal{G}_x)} K^{\frac{n}{q}}\,.$$

Now using (3.5) but replacing g_1 with $|f|^p$ and $|f^{*_\sigma}|^p$, we have that

$$\sup_{x \in \mathcal{G}^{(0)}} \{\max\{\|f_{|\mathcal{G}_x}\|_{L^p(\mathcal{G}_x)}, \|f^{*_\sigma}_{|\mathcal{G}_x}\|_{L^p(\mathcal{G}_x)}\} = C < \infty\,,$$

and then

$$\begin{aligned}\|f^n\|_I &= \sup_{x\in\mathcal{G}^{(0)}}\{\max\{\|f^n_{|\mathcal{G}_x}\|_{L^1(\mathcal{G}_x)}, \|(f^{*_\sigma})^n_{|\mathcal{G}_x}\|_{L^1(\mathcal{G}_x)}\}\} \\ &\leq \|f^{n-1}\|_{\sharp,p}\sup_{x\in\mathcal{G}^{(0)}}\{\max\{\|f_{|\mathcal{G}_x}\|_{L^p(\mathcal{G}_x)}, \|(f^{*_\sigma})_{|\mathcal{G}_x}\|_{L^p(\mathcal{G}_x)}\}\}K^{\frac{n}{q}} \\ &\leq \|f^{n-1}\|_{\sharp,p}CK^{\frac{n}{q}}\,.\end{aligned}$$

Therefore,

$$\|f^n\|_I^{\frac{1}{n}} \leq \|f^{n-1}\|_{\sharp,p}^{\frac{1}{n}}C^{\frac{1}{n}}K^{\frac{1}{q}}\,,$$

and then when $n\to\infty$, we have that

$$r_{L^1(\mathcal{G},\sigma)}(f) \leq r_{F^p_\sharp(\mathcal{G},\sigma)}(f)K^{\frac{1}{q}} = r_{C^*_r(\mathcal{G},\sigma)}(f)K^{\frac{1}{q}}\,.$$

Then, taking the limit for $p\to 1^+$, we have that $q\to\infty$, so $r_{L^1(\mathcal{G},\sigma)}(f)\leq r_{C^*_r(\mathcal{G},\sigma)}(f)$. But we always have that $r_{C^*_r(\mathcal{G},\sigma)}(f)\leq r_{L^1(\mathcal{G},\sigma)}(f)$, and hence

$$r_{C^*_r(\mathcal{G},\sigma)}(f) = r_{L^1(\mathcal{G},\sigma)}(f)\,.$$

□

Theorem 3.11. *Let $\mathcal{G}$ be a second-countable locally compact groupoid with Haar system λ and $\sigma\in Z^2(\mathcal{G},\mathbb{T})$. If $\mathcal{G}$ is σ-quasi-Hermitian, then $C^*_r(\mathcal{G},\sigma)$ is the C^*-envelope of $L^1(\mathcal{G},\sigma)$. In particular, $\mathcal{G}$ with Haar system λ and the twist σ has the weak containment property.*

Proof. By Proposition 3.10, if $C_c(\mathcal{G},\sigma)$ is quasi-Hermitian in $L^1(\mathcal{G},\sigma,\lambda)$, then for every $f\in C_c(\mathcal{G},\sigma)_h$ we have $r_{C^*_r(\mathcal{G},\sigma)}(f) = r_{L^1(\mathcal{G},\sigma)}(f)$. Therefore by Proposition 2.6 we have that $C^*_r(\mathcal{G},\sigma,\lambda)$ is the C^*-envelope of $L^1(\mathcal{G},\sigma,\lambda)$. But this means that the reduced norm is the maximal norm, and so $C^*_r(\mathcal{G},\sigma,\lambda) = C^*(\mathcal{G},\sigma,\lambda)$. □

Finally, we address the problem of spectral invariance. Recall that spectra in non-unital algebras are defined in terms of the spectra in their minimal unitizations. Since a Banach $*$-algebra $\mathcal{A}$ is Hermitian if and only if its minimal unitization is Hermitian [34, Theorem (4.7.9)], we obtain the following corollary.

Corollary 3.12. *Let $\mathcal{G}$ be a second-countable locally compact groupoid with Haar system λ and $\sigma\in Z^2(\mathcal{G},\mathbb{T})$. Then $L^1(\mathcal{G},\sigma,\lambda)$ is Hermitian if and only if $L^1(\mathcal{G},\sigma,\lambda)$ is spectrally invariant in $C^*_r(\mathcal{G},\sigma,\lambda)$.*

Proof. Suppose that $L^1(\mathcal{G},\sigma)$ is Hermitian. Then $C^*(L^1(\mathcal{G},\sigma)) = C^*_r(\mathcal{G},\sigma)$ by Theorem 3.11. But as $L^1(\mathcal{G},\sigma)$ is Hermitian, it must be spectrally invariant in its enveloping C^*-algebra, hence it is spectrally invariant in $C^*_r(\mathcal{G},\sigma)$. The converse implication is trivial.

□

4. Non-Hermitian ample amenable groupoids

In the previous section we proved that if $\mathcal{G}$ is a second-countable locally compact quasi-Hermitian groupoid, then $\mathcal{G}$ satisfies the weak containment property, i.e. $C^*_r(\mathcal{G}) \cong C^*(\mathcal{G})$. In the case $\mathcal{G}$ is a group, this translates to the fact that quasi-Hermitian groups are amenable. In the case of groupoids the situation is more subtle, since the weak containment property is not equivalent to $\mathcal{G}$ being amenable. There are examples of non-amenable groupoids with the weak containment property in [1, 39]. In this section we will see that when $\mathcal{G}$ is an ample groupoid, an obstruction to $\mathcal{G}$ being quasi-Hermitian is that the isotropy groups are not quasi-Hermitian. We can then easily construct an amenable ample groupoid that is not Hermitian.

Definition 4.1. A locally compact Hausdorff étale groupoid is called *ample* if it has a basis consisting of open and compact bisections.

Let $\mathcal{G}$ be an ample groupoid and let $\sigma \in Z^2(\mathcal{G},\mathbb{T})$. Then, given $x \in \mathcal{G}^{(0)}$, the restriction of σ to the isotropy group $\mathcal{G}_x^x$ is a group 2-cocycle. We denote this restricted 2-cocycle by σ_x.

Proposition 4.2. *Let $\mathcal{G}$ be a second-countable ample groupoid and $\sigma \in Z^2(\mathcal{G},\mathbb{T})$, with $\mathcal{G}^{(0)}$ compact. Suppose that for every $\gamma \in \mathrm{Iso}(\mathcal{G})$ there exists a clopen bisection $U \subseteq \mathcal{G}$ such that $\gamma \in U$ and $r(U) = s(U) = \mathcal{G}^{(0)}$. If $\mathcal{G}$ is σ-quasi-Hermitian, then $\mathcal{G}_x^x$ is a σ_x-quasi-Hermitian group for every $x \in \mathcal{G}^{(0)}$.*

Proof. Let us suppose that there exists $x \in \mathcal{G}^{(0)}$ such that $\mathcal{G}_x^x$ is not σ_x-quasi-Hermitian. Then there exists $f \in C_c(\mathcal{G}_x^x,\sigma_x)_h$ such that $\mathrm{Sp}_{\ell^1(\mathcal{G}_x^x,\sigma_x)}(f) \nsubseteq \mathbb{R}$. Observe that f must be of the form

$$f = \sum_{i=1}^{m} \lambda_i \delta_{\gamma_i} + \sum_{i=1}^{m} \overline{\lambda_i \sigma(\gamma_i^{-1},\gamma_i)}\delta_{\gamma_i^{-1}}\,,$$

where $\lambda_i \in \mathbb{C}$ and $\gamma_i \in \mathcal{G}_x^x$. Let $\lambda \in \mathrm{Sp}_{\ell^1(\mathcal{G}_x^x,\sigma_x)}(f) \setminus \mathbb{R}$. By assumption, for every $\gamma \in \mathcal{G}_x^x$ there exists a bisection $U_\gamma \subseteq \mathcal{G}$ such that $\gamma \in U_\gamma$ and $r(U_\gamma) = s(U_\gamma) = \mathcal{G}^{(0)}$. Then $\hat{f} = \sum_{i=1}^m \lambda_i 1_{U_{\gamma_i}} + \sum_{i=1}^m \overline{\lambda_i \sigma(\gamma_i^{-1},\gamma_i)} 1_{U_{\gamma_i}^{-1}} \in C_c(\mathcal{G},\sigma)_h$. We claim that $\lambda \in \mathrm{Sp}_{\ell^1(\mathcal{G},\sigma)}(\hat{f})$, that is $\lambda 1_{\mathcal{G}^{(0)}} - \hat{f}$ is not invertible in $\ell^1(\mathcal{G},\sigma)$. Suppose that $\lambda 1_{\mathcal{G}^{(0)}} - \hat{f}$ is invertible in $\ell^1(\mathcal{G},\sigma)$, so there exists $\hat{g} \in \ell^1(\mathcal{G},\sigma)$ such that $(\lambda 1_{\mathcal{G}^{(0)}} - \hat{f}) \star_\sigma \hat{g} = 1_{\mathcal{G}^{(0)}}$. Let $\{\hat{g}_n\}_{n=1}^\infty$ be a sequence in $C_c(\mathcal{G},\sigma)$ such that $\hat{g}_n \to \hat{g}$ in $\ell^1(\mathcal{G},\sigma)$, and hence $(\lambda 1_{\mathcal{G}^{(0)}} - \hat{f}) \star_\sigma \hat{g}_n \to 1_{\mathcal{G}^{(0)}}$ in $\ell^1(\mathcal{G},\sigma)$. In particular, $((\lambda 1_{\mathcal{G}^{(0)}} - \hat{f}) \star_\sigma \hat{g}_n)(x) \to 1$ and $((\lambda 1_{\mathcal{G}^{(0)}} - \hat{f}) \star_\sigma \hat{g}_n)(\gamma) \to 0$ for every $\gamma \in \mathcal{G}_x^x \setminus \{x\}$. Let $\hat{g}_n = \sum_{j=1}^{l_n} \beta_{j,n} 1_{V_{j,n}}$ where the $V_{j,n}$'s are compact open bisections. Then

$$\begin{aligned}(\lambda 1_{\mathcal{G}^{(0)}} - \hat{f}) \star_\sigma \hat{g}_n &= (\lambda 1_{\mathcal{G}^{(0)}} - \hat{f}) \star_\sigma (\sum_{j=1}^{l_n} \beta_{j,n} 1_{V_{j,n}}) \\ &= \sum_{j=1}^{l_n} \lambda\beta_{j,n} 1_{V_{j,n}} - \sum_{j=1}^{l_n} \beta_{j,n}(\hat{f} \star_\sigma 1_{V_{j,n}})\,.\end{aligned}$$

Therefore, defining $\eta_{j,n} := xV_{j,n}x \in \mathcal{G}_x^x$ and $g_n = \sum_{j=1}^{l_n} \beta_{j,n}\delta_{\eta_{j,n}} \in \ell^1(\mathcal{G}_x^x, \sigma_x)$, we have that the sequence $\{g_n\}_{n=1}^\infty$ converges in $\ell^1(\mathcal{G}_x^x, \sigma_x)$ because $\{\hat{g}_n\}_{n=1}^\infty$ converges in $\ell^1(\mathcal{G}, \sigma)$, and

$$(\lambda 1 - f) \star_{\sigma_x} g_n \to (\lambda 1 - f) \star_{\sigma_x} g = 1\,.$$

Therefore $\lambda \notin \mathrm{Sp}_{\ell^1(\mathcal{G}_x^x, \sigma_x)}(f)$, a contradiction. □

Remark 4.3. It was observed in [30, Lemma 4.9] that the condition that for every $\gamma \in \mathrm{Iso}(\mathcal{G})$ there exists a clopen bisection $U \subset \mathcal{G}$ such that $\gamma \in U$ and $r(U) = s(U) = \mathcal{G}^{(0)}$ is satisfied if $|\mathrm{Orb}_{\mathcal{G}}(x)| \geq 2$ for every $x \in \mathcal{G}^{(0)}$.

Example 4.4. *Willett constructed in* [39] *a second-countable locally compact ample groupoid $\mathcal{G}$ with $\mathcal{G}^{(0)}$ compact that satisfies the weak-containment property. $\mathcal{G}$ is a group bundle so it clearly satisfies the assumptions in Proposition 4.2. Moreover, $\mathcal{G}$ has an isotropy group isomorphic to the free non-abelian group with two generators, which is not quasi-Hermitian by* [36, Corollary 4.8]. *Therefore, by Proposition 4.2 we have that $\mathcal{G}$ is not quasi-Hermitian.*

Example 4.5. *Let Γ be a countable discrete group with unit e, and let us consider an action of Γ on a second-countable compact Hausdorff space X. Then $X \rtimes \Gamma$ is a second-countable locally compact Hausdorff étale groupoid. Let us suppose that Γ contains a free semigroup on two generators z, t. Given $\gamma \in \Gamma$ we define the bisection $U_\gamma = (X, \gamma)$ of $X \rtimes \Gamma$. Observe that given $\gamma, \gamma' \in \Gamma$ we have that $U_\gamma U_{\gamma'} = U_{\gamma\gamma'}$. Let us consider*

$$f = a_0 1_{U_e} + a_1 1_{U_z} + a_2 1_{U_{z^2}} \in C_c(X \rtimes \Gamma)\,,$$

where $a_0, a_1, a_2 \in \mathbb{C}$ satisfy $|a_0| = |a_1| = |a_2| = \frac{1}{3}$ and

$$\sup\{|a_0 + a_1 x + a_2 x^2| : x \in \mathbb{T}\} < 1\,.$$

Observe that 1_{U_e} is the unit of $\ell^1(X \rtimes \Gamma)$. We then have

$$r_{\ell^1(X \rtimes \Gamma)}(f) < 1\,,$$

by using the spectral mapping theorem and maximum modulus principle, and since f is normal, i.e. $ff^* = f^*f$, we obtain

$$\|f\|_{C_r^*(X \rtimes \Gamma)} = r_{C_r^*(X \rtimes \Gamma)}(f) < 1\,.$$

Now, since 1_{U_t} is a unitary in $C^*(X \rtimes \Gamma)$ it follows that

$$\|f \star_\sigma 1_{U_t}\| = \|a_0 1_{U_t} + a_1 1_{U_{zt}} + a_2 1_{U_{z^2 t}}\|\,.$$

Then since t, z generate a free non-abelian group we have that

$$(a_0 1_{U_t} + a_1 1_{U_{zt}} + a_2 1_{U_{z^2 t}})^n$$

has 3^n linearly independent terms. Hence, since $U_\gamma \cap U_{\gamma'} = \emptyset$ if and only if $\gamma \neq \gamma'$, we have that

$$\|(a_0 1_{U_t} + a_1 1_{U_{zt}} + a_2 1_{U_{z^2 t}})^n\|_{\ell^1(X \rtimes \Gamma)} = 1$$

for every $n \in \mathbb{N}$, and so $r_{\ell^1(X \rtimes \Gamma)}(a_0 1_{U_t} + a_1 1_{U_{zt}} + a_2 1_{U_{z^2 t}}) = 1$. Thus we have that

$$\mathrm{Sp}_{\ell^1(X \rtimes \Gamma)}(a_0 1_{U_t} + a_1 1_{U_{zt}} + a_2 1_{U_{z^2 t}}) \neq \mathrm{Sp}_{C_r^*(X \rtimes \Gamma)}(a_0 1_{U_t} + a_1 1_{U_{zt}} + a_2 1_{U_{z^2 t}})\,,$$

and hence by Proposition 3.10 we have that $X \rtimes \Gamma$ is not quasi-Hermitian.

Then, let $\mathbb{F}_2$ be the non-abelian free group with two generators. It is known that there exists a locally compact space X and an amenable free action of $\mathbb{F}_2$ on X (see for example [38]). Hence, $X \rtimes \mathbb{F}_2$ is an amenable groupoid but not quasi-Hermitian and with no non-trivial isotropy groups.

5. Hermitian twisted groupoid Banach $*$-algebras

Let $\mathcal{G}$ be a second-countable locally compact groupoid with Haar system λ and let $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. In this section we give a sufficient condition for the Banach $*$-algebra $L^1(\mathcal{G}, \sigma, \lambda)$ to be Hermitian. As a consequence we are able to give conditions for twisted transformation groupoids so that the associated twisted transformation groupoid Banach $*$-algebras are Hermitian.

Definition 5.1. Given a second-countable locally compact groupoid $\mathcal{G}$ with Haar system λ and $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$, we define the *twisted groupoid* $\mathcal{G}_\sigma$ to be the groupoid $\mathcal{G} \times \mathbb{T}$ with product topology and operations defined by

$$(\gamma_1, z_1) \cdot (\gamma_2, z_2) = (\gamma_1\gamma_2, z_1 z_2 \overline{\sigma(\gamma_1, \gamma_2)}) \qquad \text{if } (\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}\,,$$

and

$$(\gamma, z)^{-1} = (\gamma^{-1}, \overline{z}\sigma(\gamma, \gamma^{-1}))\,.$$

The Haar system of $\mathcal{G}_\sigma$ is the one given by $\lambda \times \eta = \{\lambda_x \times \eta\}_{x \in \mathcal{G}^{(0)}}$, where η is the normalized Lebesgue measure on $\mathbb{T}$.

Proposition 5.2. *Let $\mathcal{G}$ be a second-countable locally compact groupoid with Haar system λ and $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. The map*

$$j : L^1(\mathcal{G}, \sigma, \lambda) \to L^1(\mathcal{G}_\sigma, \lambda \times \eta)\,,$$

given by $j(f)(\gamma, z) = zf(\gamma)$ is an isometric $*$-homomorphism.

Proof. First we prove that j is a $*$-homomorphism. Fix $f, g \in L^1(\mathcal{G}, \sigma)$, $\gamma \in \mathcal{G}$ and $z \in \mathbb{T}$. Then we have that

$$\begin{aligned}
j(f \star_\sigma g)(\gamma, z) = z(f \star_\sigma g)(\gamma) &= z \int_{\mathcal{G}_{s(\gamma)}} \sigma(\gamma\mu^{-1}, \mu) f(\gamma\mu^{-1}) g(\mu)\, d\lambda_{s(\gamma)}(\mu) \\
&= \int_{\mathcal{G}_{s(\gamma)}} z\overline{\sigma(\gamma, \mu^{-1})}\sigma(\mu, \mu^{-1}) f(\gamma\mu^{-1}) g(\mu)\, d\lambda_{s(\gamma)}(\mu) dt \\
&= \int_{\mathbb{T}} \int_{\mathcal{G}_{s(\gamma)}} z\overline{\sigma(\gamma, \mu^{-1})}\sigma(\mu, \mu^{-1}) f(\gamma\mu^{-1}) g(\mu)\, d\lambda_{s(\gamma)}(\mu) dt \\
&= \int_{\mathbb{T}} \int_{\mathcal{G}_{s(\gamma)}} \overline{t\sigma(\gamma, \mu^{-1})} z\sigma(\mu, \mu^{-1}) f(\gamma\mu^{-1}) t g(\mu)\, d\lambda_{s(\gamma)}(\mu) dt \\
&= \int_{\mathbb{T}} \int_{\mathcal{G}_{s(\gamma)}} j(f)(\gamma\mu^{-1}, \overline{t\sigma(\gamma, \mu^{-1})} z\sigma(\mu, \mu^{-1})) j(g)(\mu, t)\, d\lambda_{s(\gamma)}(\mu) dt \\
&= \int_{\mathbb{T}} \int_{\mathcal{G}_{s(\gamma)}} j(f)((\gamma, z)(\mu, t)^{-1}) j(g)(\mu, t)\, d\lambda_{s(\gamma)}(\mu) dt \\
&= (j(f) \star j(g))(\gamma, z)\,,
\end{aligned}$$

where, at the third equality, we have used $\sigma(\gamma\mu^{-1},\mu)\sigma(\gamma,\mu^{-1}) = \sigma(\mu^{-1},\mu)$. This identity follows from (2.2) by using γ, μ^{-1}, μ instead of α, β, γ and then applying (2.1). We then use that $\sigma(\mu^{-1},\mu) = \sigma(\mu,\mu^{-1})$, which follows from (2.2) using μ, μ^{-1}, μ instead of α, β, γ and then applying (2.1).

Moreover,

$$\begin{aligned} j(f^{*_\sigma})(\gamma,z) = zf^{*_\sigma}(\gamma) &= z\overline{\sigma(\gamma,\gamma^{-1})f(\gamma^{-1})} \\ &= \overline{j(f)(\gamma^{-1},\overline{z}\sigma(\gamma,\gamma^{-1}))} \\ &= \overline{j(f)((\gamma,z)^{-1})} \\ &= j(f)^*(\gamma,z)\,. \end{aligned}$$

Finally,

$$\begin{aligned} \int_{\mathcal{G}_x} |f(\gamma)|\,d\lambda_x(\gamma) &= \int_{\mathbb{T}} |z| \int_{\mathcal{G}_x} |f(\gamma)|\,d\lambda_x(\gamma)\,dz \\ &= \int_{\mathbb{T}}\int_{\mathcal{G}_x} |zf(\gamma)|\,d\lambda_x(\gamma)\,dz \\ &= \int_{\mathbb{T}}\int_{\mathcal{G}_x} |j(f)(\gamma,z)|\,d\lambda_x(\gamma)\,dz \end{aligned}$$

and

$$\begin{aligned} \int_{\mathcal{G}_x} |f(\gamma^{-1})|\,d\lambda_x(\gamma) &= \int_{\mathbb{T}} |\overline{z}| \int_{\mathcal{G}_x} |f(\gamma^{-1})|\,d\lambda_x(\gamma)\,dz \\ &= \int_{\mathbb{T}}\int_{\mathcal{G}_x} |\overline{z}\sigma(\gamma,\gamma^{-1})f(\gamma^{-1})|\,d\lambda_x(\gamma)\,dz \\ &= \int_{\mathbb{T}}\int_{\mathcal{G}_x} |j(f)(\gamma^{-1},\overline{z}\sigma(\gamma,\gamma^{-1}))|\,d\lambda_x(\gamma)\,dz \\ &= \int_{\mathbb{T}}\int_{\mathcal{G}_x} |j(f)((\gamma,z)^{-1}))|\,d\lambda_x(\gamma)\,dz\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \|f\|_I &= \sup_{x\in\mathcal{G}^{(0)}} \max\left\{\int_{\mathcal{G}_x} |f(\gamma)|\,d\lambda_x(\gamma)\,,\int_{\mathcal{G}_x} |f(\gamma^{-1})|\,d\lambda_x(\gamma)\right\} \\ &= \sup_{x\in\mathcal{G}^{(0)}} \max\left\{\int_{\mathbb{T}}\int_{\mathcal{G}_x} |j(f)(\gamma,z)|\,d\lambda_x(\gamma)\,dz\,,\int_{\mathbb{T}}\int_{\mathcal{G}_x} |j(f)((\gamma,z)^{-1})|\,d\lambda_x(\gamma)\,dz\right\} \\ &= \|j(f)\|_I\,. \end{aligned}$$

□

Proposition 5.3. *Let $\mathcal{G}$ be a second-countable locally compact groupoid with Haar system λ and $\sigma \in Z^2(\mathcal{G},\mathbb{T})$. Suppose that $L^1(\mathcal{G}_\sigma)$ is Hermitian. Then $L^1(\mathcal{G},\sigma)$ is Hermitian.*

Proof. By Proposition 5.2 $L^1(\mathcal{G},\sigma)$ is a closed Banach $*$-subalgebra of $L^1(\mathcal{G}_\sigma)$. Then by [10, Proposition 7.10] $L^1(\mathcal{G},\sigma)$ is Hermitian. □

Example 5.4. *Let X be a second-countable locally compact Hausdorff space, and let Γ be a second-countable locally compact group acting on X by homeomorphisms. Let $\mathcal{G} = X \rtimes \Gamma$ be the transformation groupoid, which is locally compact. Recall that $\mathcal{G}$ is étale if and only if Γ is discrete. Let $\sigma \in Z^2(\Gamma, \mathbb{T})$, and extend σ to a 2-cocycle of $X \rtimes \Gamma$ as shown in Example 2.12. We define Γ_σ to be the group that is $\Gamma \times \mathbb{T}$ with the product topology, and*

$$(\gamma_1, z_1)(\gamma_2, z_2) = (\gamma_1\gamma_2, z_1 z_2 \overline{\sigma(\gamma_1, \gamma_2)})\,,$$

for every $z_1, z_2 \in \mathbb{T}$ and $\gamma_1, \gamma_2 \in \Gamma$. Now if we define the action of Γ_σ on X by

$$(\gamma, z) \cdot x := \gamma \cdot x\,,$$

then the transformation groupoid $X \rtimes \Gamma_\sigma$ is isomorphic to $(X \rtimes \Gamma)_\sigma$.

Let X be a second-countable locally compact Hausdorff space, and let Γ be a second-countable locally compact group with modular function Δ, acting on X by homeomorphisms. Further, let $\sigma \in Z^2(\Gamma, \mathbb{T})$, and let $C_0(X)$ be the Banach $*$-algebra of continuous functions on X that vanish at infinity with the supremum norm $\|\cdot\|_\infty$. Then Γ acts on $C_0(X)$ by $\gamma \cdot f(x) = f(\gamma^{-1} \cdot x)$ for every $f \in C_0(X)$ and $\gamma \in \Gamma$. Let us define the *generalized L^1-algebra* $L^1(\Gamma, C_0(X), \sigma)$ to be the completion of

$$C_c(\Gamma, C_0(X), \sigma) = \{f : \Gamma \to C_0(X) : f \text{ is continuous with compact support}\}$$

with respect to the norm

$$\|f\| := \int_\Gamma \|f(\gamma)\|_\infty \, d\lambda(\gamma)\,,$$

where λ is a left Haar measure of Γ. Then $L^1(\Gamma, C_0(X), \sigma)$ becomes a Banach $*$-algebra with the operations

$$(f \star_\sigma g)(\gamma)(x) = \int_\Gamma \sigma(\gamma\mu^{-1}, \mu) f(\gamma\mu^{-1})(\mu \cdot x) g(\mu)(x) \, d\lambda(\mu)\,,$$

and

$$(f^{*_\sigma})(\gamma)(x) = \Delta(\gamma^{-1}) \overline{\sigma(\gamma, \gamma^{-1}) f(\gamma^{-1})(\gamma \cdot x)}\,,$$

for every $f, g \in L^1(\Gamma, C_0(X), \sigma)$, $\gamma \in \Gamma$ and $x \in X$.

If σ is the trivial twist, we denote $L^1(\Gamma, C_0(X), \sigma)$ by $L^1(\Gamma, C_0(X))$. The C^*-envelope of $L^1(\Gamma, C_0(X), \sigma)$ is the twisted crossed product C^*-algebra $C_0(X) \rtimes^\sigma \Gamma$.

Lemma 5.5. *Let X be a second-countable locally compact Hausdorff space, and let Γ be a second-countable locally compact unimodular group ($\Delta \equiv 1$) acting on X by homeomorphisms. Let $\sigma \in Z^2(\Gamma, \mathbb{T})$. Then there exists a surjective $*$-homomorphism $\Phi : L^1(\Gamma, C_0(X), \sigma) \to L^1(X \rtimes \Gamma, \sigma)$. Consequently, if $L^1(\Gamma, C_0(X), \sigma)$ is Hermitian, then so is $L^1(X \rtimes \Gamma, \sigma)$.*

Proof. First observe that $C_c(\Gamma, C_c(X), \sigma)$ is a dense $*$-subalgebra of $L^1(\Gamma, C_0(X), \sigma)$. The map $\Phi : C_c(\Gamma, C_c(X), \sigma) \to C_c(X \rtimes \Gamma, \sigma)$ given by $\Phi(f)(x, \gamma) = \Phi(f)(\gamma)(x)$ defines a $*$-homomorphism. Indeed, given $f, g \in C_c(\Gamma, C_c(X), \sigma)$, $\gamma \in \Gamma$ and $x \in X$

$$\begin{aligned}\Phi(f \star_\sigma g)(x,\gamma) &= (f \star_\sigma g)(\gamma)(x) \\ &= \int_\Gamma \sigma(\gamma\mu^{-1},\mu) f(\gamma\mu^{-1})(\mu\cdot x) g(\mu)(x)\, d\lambda(\mu) \\ &= \int_\Gamma \sigma(\gamma\mu^{-1},\mu)\Phi(f)(\mu\cdot x,\gamma\mu^{-1})\Phi(g)(x,\mu)\, d\lambda(\mu) \\ &= (\Phi(f)\star_\sigma \Phi(g))(x,\gamma)\,,\end{aligned}$$

and

$$\begin{aligned}\Phi(f^{*_\sigma})(x,\gamma) = f^{*_\sigma}(\gamma)(x) &= \overline{\sigma(\gamma,\gamma^{-1}) f(\gamma^{-1})(\gamma\cdot x)} \\ &= \overline{\sigma(\gamma,\gamma^{-1})\Phi(f)(\gamma\cdot x,\gamma^{-1})} = \Phi(f)^{*_\sigma}(x,\gamma)\,.\end{aligned}$$

Observe that clearly Φ is a bijection. Now given $f \in C_c(\Gamma, C_c(X),\sigma)$ we have that

$$\begin{aligned}\|\Phi(f)\|_I &= \sup\left\{\int_\Gamma |\Phi(f)(x,\gamma)|\, d\lambda(\gamma) : x\in X\right\} \\ &= \sup\left\{\int_\Gamma |f(\gamma)(x)|\, d\lambda(\gamma) : x\in X\right\} \\ &\le \int_\Gamma \sup\{|f(\gamma)(x)| : x\in X\}\, d\lambda(\gamma) \\ &= \int_\Gamma \|f(\gamma)\|_\infty\, d\lambda(\gamma) = \|f\|\,.\end{aligned}$$

Thus, $\|\Phi(f)\|_I \le \|f\|$ for every $f\in C_c(\Gamma, C_c(X),\sigma)$, and so Φ extends to a continuous surjective $*$-homomorphism $\Phi : L^1(\Gamma, C_0(X),\sigma) \to L^1(X\rtimes\Gamma,\sigma)$.

Finally, the last statement follows by the fact that quotients of Hermitian $*$-algebras are Hermitian [32, Theorem 10.4.4]. ☐

Corollary 5.6. *Let X be a second-countable locally compact Hausdorff space, and let Γ be a second-countable compact group or a locally compact abelian group acting on X by homeomorphisms. Let $\sigma\in Z^2(\Gamma,\mathbb{T})$. Then $L^1(X\rtimes\Gamma,\sigma)$ is Hermitian, and, in particular, $L^1(X\rtimes\Gamma,\sigma)$ is spectrally invariant in $C^*_r(X\rtimes\Gamma,\sigma)$.*

Proof. By Example 5.4 we have that $(X\rtimes\Gamma)_\sigma \cong X\rtimes\Gamma_\sigma$. Then by Proposition 5.3 $L^1(X\rtimes\Gamma,\sigma)$ is Hermitian if $L^1(X\rtimes\Gamma_\sigma)$ is Hermitian. By Lemma 5.5 $L^1(X\rtimes\Gamma_\sigma)$ is Hermitian if $L^1(\Gamma_\sigma, C_0(X))$ is Hermitian. Now if Γ is compact, then Γ_σ is compact too, and if Γ is abelian, then Γ_σ is nilpotent, and hence $L^1(\Gamma_\sigma, C_0(X))$ is Hermitian [8, pg. 1285]. So in both cases $L^1(X\rtimes\Gamma,\sigma)$ is Hermitian, and it then follows by Theorem 3.11 that $X\rtimes\Gamma$ has the weak containment property with respect to σ. Finally, by Corollary 3.12 it follows that $L^1(X\rtimes\Gamma,\sigma)$ is spectrally invariant in $C^*_r(X\rtimes\Gamma,\sigma)$. ☐

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