Recovering a Variable Exponent

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Abstract. We consider an inverse problem of recovering the nonlinearity in the one dimensional variable exponent $p(x)$-Laplace equation from the Dirichlet-to-Neumann map. The variable exponent can be recovered up to the natural obstruction of rearrangements. The main technique is using the properties of a moment problem after reducing the inverse problem to determining a function from its L^p-norms.

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1 Introduction

Calderón's fundamental inverse problem [11, 21] asks if a weight function γ can be recovered from Dirichlet and Neumann measurements on the boundary of a domain, when the data come from the weighted Laplace equation

$$-\operatorname{div}(\gamma\nabla u) = 0.$$

The weight function γ is considered as conductivity of electricity or heat, the Dirichlet boundary values as voltage or temperature, and the Neumann boundary values as current flux or heat flux through the boundary. The equation is derived from Ohm's law

$$-\gamma\nabla u = I$$

and Kirchhoff's law

$$\operatorname{div} I = 0,$$

or corresponding laws for heat conduction.

The problem has been generalized to many other equations, of which we are interested in non-linear and singular or degenerate elliptic ones. The physical motivation for these is that Ohm's law is only an approximation and many real-world systems exhibit highly non-linear IV (or current-voltage) patterns. Power law -type patterns lead to the p-conductivity equation

$$-\operatorname{div}\left(\gamma\,|\nabla u|^{p-2}\,\nabla u\right) = 0$$

first introduced by Salo and Zhong [33] and investigated further by Salo and others [2, 8, 4, 6, 22, 23, 26], with the triviality of the one-dimensional case explicitly treated in [3]. In these works the exponent $1 < p < \infty$ is assumed to be a known constant, whilst the linear factor in the conductivity γ is the unknown. Cârstea and Kar [14] investigate a combination of linear and power law type conductivity. Corbo Esposito and others [16] consider the widely used monotonicity principle for conductivities $a(x, \nabla u)$, which cover p-Laplace type problems among a wider class.

On the other hand, general, but typically non-degenerate, A-harmonic equations

$$-\operatorname{div}\left(a(x, u, \nabla u)\nabla u\right) = 0$$

have also been researched. A typical method is linearizing the equation or using Carleman estimates, hence relying on completely different techniques when compared to the present work. Quasilinearities depending on the solution, with $a(u)$, have been investigated first by Cannon [13] and then many others; we mention some more recent works [31, 18, 19]. Sun and Uhlmann [35] considered non-degenerate and fairly smooth dependence on u and x. A problem with similar dependency has also been studied by Chen, Chen and Wei [15]. Hervas and Sun [24] considered smooth coefficients $a(\nabla u)$. Muñoz and Uhlmann [32] considered non-degenerate elliptic $a(u, \nabla u)$. Lassas, Liimatainen and Salo considered general non-degenerate real-analytic conductivities [30], while Shankar [34] considered a non-degenerate and very smooth $a(x, u, \nabla u)$.

In the present work we consider the variable exponent $p(\cdot)$-Laplace equation

$$-\operatorname{div}\left(\gamma(x)\,|\nabla u|^{p(x)-2}\,\nabla u\right) = 0,$$

which can be both singular and degenerate at the same time. Also, we make no smoothness assumptions. The mathematical issues raised by the variable exponent in the forward problem have been covered in a monograph [17]. One physical motivation for such problems is the conductivity of electricity in certain almost-superconductive materials, where the exponent $p(\cdot)$ is a function of temperature [10, 25], which should not be assumed constant and might very well be unknown. Former work on inverse problems for the equation consists

of a boundary determination result with interior data [7], which does not make essential use of the variable exponent, and of a characterization of conductivities γ that can be recovered when the exponent $p(\cdot)$ is known [9].
In the present paper our aim is to recover the exponent $p(\cdot)$ from the Dirichlet-to-Neumann map. We use the results of Brander and Winterrose [9]; some basic facts about the problem are summarized as preliminaries in section 2, together with an unrelated lemma. Before the preliminaries we present the problem and our main results. In section 3, we investigate the behaviour of the Dirichlet-to-Neumann map as the difference of Dirichlet data goes to zero or infinity. This gives explicit information about the maximum and minimum of p with very few assumptions. Finally, in section 4, we prove the main injectivity result by reducing it to a moment problem.

1.1 The problem

As a forward model we consider the equation

$$-\left(\gamma(x)\left|u'(x)\right|^{p(x)-2}u'(x)\right)'=0, \tag{1}$$

with Dirichlet boundary data $u(a)=A$ and $u(b)=B$. We assume, for convenience, that $A\leq B$, so that the absolute value can be removed from the equation, and write $m=B-A$. We often neglect to write the argument x in $\gamma(x)$ and $u'(x)$, but, for emphasis, keep it in $p(x)$.
Brander and Winterrose [9, Section 3] observed that the Dirichlet-problem (1) can be solved almost explicitly in terms of an intermediate function. More precisely, the problem has a unique solution u which satisfies the equation

$$u(x)=u(a)+\int_a^x (K_m/\gamma)^{1/(p(s)-1)}\,\mathrm{d}s$$

for some non-negative constant K_m. By substituting $x=b$ and using that $m=u(b)-u(a)$, we obtain an implicit definition for K_m by the formula

$$m=\int_a^b (K_m/\gamma)^{1/(p(s)-1)}\,\mathrm{d}s. \tag{2}$$

Defined this way, the function $m\mapsto K_m$ is a well-defined continuous bijection from R_+ to itself with a continuous inverse [9, Lemma 7]. The Dirichlet-to-Neumann map (hereafter DN map) can then be defined as

$$\Lambda_\gamma^p(m)=\int_a^b \gamma^{-1/(p(x)-1)}K_m^{p(x)/(p(x)-1)}\,\mathrm{d}x=mK_m. \tag{3}$$

We provide additional details on this formulation and the setting in section 2.

Question: Assuming that $m\mapsto\Lambda_\gamma^p(m)$ is known and that γ is fixed, can one recover p?

1.2 Results

Let $f\colon (X,\mu) \to [0,\infty]$ be a measurable function defined on a σ-finite measure space. Define the *distribution function* of f by

$$\mu^f(t) := \mu\left(\{x \in X; f(x) > t\}\right) \quad \text{for all } t \in [0,\infty]. \tag{4}$$

We say two functions are equimeasurable if and only if their distribution functions are equal. In the case of constant conductivity, we obtain the following theorem whose proof is postponed to Section 4.

Theorem 1. *Let $\gamma > 0$ be a constant, and the exponents p_1 and p_2 Lebesgue measurable and bounded away from one and infinity. Then $\Lambda_\gamma^{p_1} = \Lambda_\gamma^{p_2}$ if and only if p_1 and p_2 are equimeasurable with respect to the Lebesgue measure.*

A similar result should be reachable in one-dimensional multifrequency SPECT imaging [5]. There the corresponding question would be recovering the attenuation, given knowledge of the source term.
Theorem 1 gives a uniqueness result, but we also have a reconstruction procedure when $\gamma \equiv 1$. This procedure is described at the end of this article.

Theorem 2. *Let $\gamma \equiv 1$. Suppose that the variable exponent p is Lebesgue measurable and bounded away from one and infinity. Then a function equimeasurable with p can be obtained in a constructive way from the DN map and its derivatives with respect to m.*

The next theorem contains less information, but is computationally straightforward and assumes no a priori knowledge on γ. It is proven in section 3. Write $p^- = \operatorname{ess\,inf}_{a \le x \le b} p(x)$ and $p^+ = \operatorname{ess\,sup}_{a \le x \le b} p(x)$.

Theorem 3. *Let $\gamma \in L_+^\infty([a,b])$. The DN map determines the quantities p^+, p^-, and if these are reached in sets of positive measure also the integrals*

$$\int_{\{x\in[a,b];p(x)=p^-\}} \gamma^{-1/(p^- -1)}\,\mathrm{d}x \text{ and}$$
$$\int_{\{x\in[a,b];p(x)=p^+\}} \gamma^{-1/(p^+ -1)}\,\mathrm{d}x$$

in a constructive way.

This theorem can be used to estimate the sizes of the sets where the variable exponent reaches its maximum or minimum. More precisely, we have the following corollary.

Corollary 4. *Suppose that $c \le \gamma(x) \le C$ for some known constants $c, C > 0$. Suppose moreover that the set $\{x \in [a,b]; p(x) = p^\pm\}$ has a positive measure. Then this measure can be estimated in a constructive way via the DN map. In particular, the measure can be recovered if γ is a constant.*

Proof. By Theorem 3 the quantities p^- and

$$\lambda := \int_{\{x\in[a,b];p(x)=p^-\}} \gamma^{-1/(p^- -1)}\,\mathrm{d}x$$

can be constructively determined from the DN map. In particular, the quantities

$$l := \min_{c\le t\le C} t^{-1/(p^- -1)} \quad \text{and} \quad L := \max_{c\le t\le C} t^{-1/(p^- -1)}$$

are known. Then by the assumption $c \le \gamma(x) \le C$ we have

$$l\left|\left\{x\in[a,b];p(x)=p^-\right\}\right| \le \lambda \le L\left|\left\{x\in[a,b];p(x)=p^-\right\}\right|$$

Consequently we obtain the following estimate, where both the upper and lower bound can be recovered

$$\frac{\lambda}{L} \le \left|\left\{x\in[a,b];p(x)=p^-\right\}\right| \le \frac{\lambda}{l}.$$

In particular, if γ is a constant, then the precise measure of the set can be constructed. $\square$

2 Preliminaries

We give some results from the article of Brander and Winterrose [9], which build on known results for the variable exponent equation [17].
The present paper assumes the following standing assumptions, which guarantee that there are no undue complications in understanding the existence and uniqueness of solutions for the forward problem.

Assumption 5 (Standing assumptions). We consider a one-dimensional interval of positive, but finite, length, i.e. $-\infty < a < b < \infty$.
There exists $\varepsilon > 0$ with the following holding almost everywhere on the interval $[a,b]$: $0 < \varepsilon < \gamma(x) < 1/\varepsilon$ and $1+\varepsilon < p(x) < 1/\varepsilon$. We write the assumption that γ is essentially bounded and essentially bounded from below as $\gamma \in L^\infty_+$.

We then have:

1. The equation (1) has a unique solution $u(x)$ [9, p8, Lemma 5].
2. The map $m \mapsto K_m$ defined by equation (2) is well-defined, strictly increasing, continuous bijection (from $\mathbb{R}_+$ to itself) with a continuous inverse [9, p9, Lemma 7].
3. The DN map defined by equation (3) generalizes the usual DN map in Calderón's problem and Calderón's problem for p-Laplace equation with constant p – the DN maps are equal if $p(x)$ is a constant outside of a null set [9, p11, Lemma 8].

The paper [9] did not use the observation $\Lambda^p_\gamma = mK_m$; the observation would likely simplify the arguments therein.
We also use the notation

$$p^+ = \operatorname*{ess\,sup}_{a\le x\le b} p(x) \qquad \text{and}$$
$$p^- = \operatorname*{ess\,inf}_{a\le x\le b} p(x).$$

From the standing assumptions it follows that $1 < p^- \le p^+ < \infty$.
The following lemma states that if two functions are equimeasurable, and the same bijection acts on both of them, the composed functions are still equimeasurable. It is needed when proving some of our main results.

Lemma 6. *Suppose that $f, g : X \to [0,\infty)$ are equimeasurable with respect to the measures μ_1 and μ_2 in the sense that*

$$\mu_1(\{x \in X; f(x) > t\}) = \mu_2(\{x \in X; g(x) > t\}) \quad \textit{for all } t \in [0,\infty).$$

Suppose also that $\mu_1(X) = \mu_2(X)$. Let $h : I \to [0,\infty)$ be strictly monotonous and continuous, where I is an interval that contains the images of f and g. Then $h \circ f$ and $h \circ g$ are also equimeasurable with respect to the measures μ_1 and μ_2 (in the above sense).

Proof. Suppose first that h is strictly increasing. Let $\tau \ge 0$. If τ is in the image of h, there is t such that $\tau = h(t)$. Then, since h is strictly increasing, we have by the equimeasurability assumption

$$\begin{aligned}\mu_1(\{x \in X; h\circ f(x) > \tau\}) =&\mu_1(\{x \in X; h(f(x)) > h(t)\})\\ =&\mu_1(\{x \in X; f(x) > t\})\\ =&\mu_2(\{x \in X; g(x) > t\})\\ =&\mu_2(\{x \in X; h\circ g(x) > \tau\}).\end{aligned}$$

If τ is not in the image of h, then, since the image is a connected set, we have either $\tau > h(t)$ for all $t \in I$ or $\tau < h(t)$ for all $t \in I$. Consequently we have either

$$\mu_1(\{x \in X; h\circ f(x) > \tau\}) = \mu_1(X) = \mu_2(X) = \mu_2(\{x \in X; h\circ g(x) > \tau\})$$

or

$$\mu_1(\{x \in X; h\circ f(x) > \tau\}) = \mu_1(\emptyset) = \mu_2(\emptyset) = \mu_2(\{x \in X; h\circ g(x) > \tau\}).$$

This completes the proof for strictly increasing h.
Suppose now that h is strictly decreasing. Then, by the same argument as above, for all $\tau \ge 0$,

$$\mu_1\left(\{x \in X; h\circ f(x) < \tau\}\right) = \mu_2\left(\{x \in X; h\circ g(x) < \tau\}\right). \tag{5}$$

But now, by dominated convergence and for all $\tau \geq 0$,

$$\begin{aligned}\mu_1\left(\{x \in X; h \circ f(x) > \tau\}\right) &= \mu_1(X) - \mu_1\left(\{x \in X; h \circ f(x) \leq \tau\}\right)\\ &= \mu_1(X) - \lim_{\varepsilon \to 0} \mu_1\left(\{x \in X; h \circ f(x) < \tau + \varepsilon\}\right).\end{aligned}$$

The same reasoning applies to g with μ_2, and by equation (5), this finishes the proof. □

3 Limits of voltage difference

In this section we consider the situation where the difference of Dirichlet values m approaches zero or infinity. In this case we can learn something about the maximum or minimum of p, respectively. Recall that $m \mapsto K_m$ is a continuous bijection from $[0, \infty)$ to itself. Therefore by considering small or large enough m, we are able to assume that $K_m \leq 1$ or $K_m \geq 1$.

Proposition 7. *Suppose that $\gamma \in L^\infty_+$. Assume that $m \leq 1$ is so small that $K_m \leq 1$. Then for any $\varepsilon > 0$ there is a constant $C = C(\gamma, p, b-a, \varepsilon)$ such that*

$$\frac{1}{C} m^{p^+-1} \leq K_m \leq C m^{p^+-1-\varepsilon}$$

and

$$\frac{1}{C} m^{p^+} \leq \Lambda^p_\gamma(m) \leq C m^{p^+-\varepsilon}.$$

Moreover, if $p(x)$ reaches its essential supremum in a set of positive measure, then these estimates hold for $\varepsilon = 0$.

Proof. Since γ is bounded away from zero and $m \leq 1$ is so small that $K_m \leq 1$, the definition of K_m in (2) implies

$$\begin{aligned} m &= \int_a^b \gamma^{-1/(p(x)-1)}(x) K_m^{\frac{1}{p(x)-1}}\,\mathrm{d}x \\ &\leq C(\gamma, p) \int_a^b K_m^{\frac{1}{p^+-1}}\,\mathrm{d}x \\ &\leq C(\gamma, p)(b-a) K_m^{\frac{1}{p^+-1}}, \end{aligned}$$

so that

$$K_m \geq C(\gamma, p)(b-a) m^{p^+-1}. \tag{6}$$

For the other direction, let $\varepsilon > 0$. Then, since γ is bounded and K_m is nonnegative, we have

$$\begin{aligned} m &\geq C(\gamma, p) \int_a^b K_m^{\frac{1}{p(x)-1}}\,\mathrm{d}x \\ &\geq C(\gamma, p) \int_{\{x \in [a,b]; p(x) \geq p^+-\varepsilon\}} K_m^{\frac{1}{p^+-1-\varepsilon}}\,\mathrm{d}x \\ &= C(\gamma, p) \left|\{x \in [a,b]; p(x) \geq p^+-\varepsilon\}\right| K_m^{\frac{1}{p^+-1-\varepsilon}}. \end{aligned}$$

By the definition of p^+, the set $\{x \in [a,b]; p(x) \geq p^+ - \varepsilon\}$ has a positive measure for all $\varepsilon > 0$. Hence the above implies that

$$K_m \leq C(\gamma, p, \varepsilon) m^{p^+ - 1 - \varepsilon}. \tag{7}$$

Combining (6) and (7) we arrive at the first estimate of the proposition.
The second estimate follows immediately from the identity $\Lambda^p_\gamma(m) = mK_m$. To prove the final claim, simply repeat the above proof with $\varepsilon = 0$. □

Lemma 8. *Assume that $\gamma \in L^\infty_+$. Then*

$$p^+ = \sup\left\{q > 0; \lim_{m \to 0} m^{-q} \Lambda^p_\gamma = 0\right\} = \inf\left\{q > 0; \lim_{m \to 0} m^{-q} \Lambda^p_\gamma(m) = \infty\right\}.$$

Proof. If $q > p^+$, then $q = p^+ + \varepsilon$ for some $\varepsilon > 0$, and so by proposition 7 we have

$$m^{-q} \Lambda^p_\gamma(m) \geq C m^{-p^+ - \varepsilon} m^{p^+} = C m^{-\varepsilon} \to \infty \text{ as } m \to 0.$$

If $q < p^+$, then $q = p^+ - \varepsilon$ for some $\varepsilon > 0$, and so by Proposition 7 we have

$$m^{-q} \Lambda^p_\gamma(m) \leq C m^{-p^+ + \varepsilon} m^{p^+ - \varepsilon/2} = C m^{\varepsilon/2} \to 0 \text{ as } m \to 0.$$ □

We get similar results with m large, but with p^-.

Proposition 9. *Suppose that $\gamma \in L^\infty_+$. Assume that $m \geq 1$ is so big that $K_m \geq 1$. Then for any $\varepsilon > 0$ there is a constant $C = C(\gamma, p, b-a, \varepsilon)$ such that*

$$\frac{1}{C} m^{p^- - 1} \leq K_m \leq C m^{p^- - 1 + \varepsilon}$$

and

$$\frac{1}{C} m^{p^-} \leq \Lambda^p_\gamma(m) \leq C m^{p^- + \varepsilon}.$$

Moreover, if $p(x)$ reaches its essential infimum in a set of positive measure, then these estimates hold for $\varepsilon = 0$.

Proof. As the proof of proposition 7, but with m large and p estimated by p^-. □

The proof of the following lemma, too, is similar to previous proofs.

Lemma 10. *Assume that $\gamma \in L^\infty_+$. Then*

$$p^- = \inf\left\{q > 0; \lim_{m \to \infty} m^{-q} \Lambda^p_\gamma = 0\right\} = \sup\left\{q > 0; \lim_{m \to \infty} m^{-q} \Lambda^p_\gamma(m) = \infty\right\}.$$

Now we can prove one of the inverse problem theorems.

Proof of theorem 3. Lemmas 10 and 8 provide p^- and p^+.
Suppose now $\{x \in [a,b]; p(x) = p^-\}$ has positive measure, with the intention of taking $m \to \infty$. The other case is similar.

$$\begin{aligned}
m^{-p^-}\Lambda_\gamma^p(m) &= m^{-p^-+1}K_m \\
&= K_m \left(\int_a^b \gamma^{-1/(p(x)-1)} K_m^{1/(p(x)-1)} \,\mathrm{d}x \right)^{-p^-+1} \\
&= \left(\int_a^b \gamma^{-1/(p(x)-1)} K_m^{-1/(p^--1)+1/(p(x)-1)} \,\mathrm{d}x \right)^{-p^-+1} \\
&= \left(\int_{\{x\in[a,b];p(x)=p^-\}} \gamma^{-1/(p^--1)} \,\mathrm{d}x \right)^{-p^-+1} \\
&\quad + \left(\int_{\{x\in[a,b];p(x)>p^-\}} \gamma^{-1/(p(x)-1)} K_m^{-1/(p^--1)+1/(p(x)-1)} \,\mathrm{d}x \right)^{-p^-+1}.
\end{aligned}$$

The second integral vanishes by dominated convergence as $m \to \infty$, since then also $K_m \to \infty$. ☐

4 Proof of the main theorem

We know [9, proposition 26] that, for $n \in \mathbb{N} \cup \{0\}$, the quantities

$$\int_a^b \gamma^{-1/(p(x)-1)} \left(\frac{1}{p(x)-1} \right)^n \mathrm{d}x,$$

can be recovered constructively from the DN map and its derivatives with respect to m. Suppose $\gamma \equiv 1$. Then what can be recovered are essentially the L^n-norms $\left\| \frac{1}{p(x)-1} \right\|_{L^n([a,b])}^n$. We write the weighted L^n-space with weight f as $L^n([a,b], f(x)\,\mathrm{d}x)$, and omit the weight when $f \equiv 1$ almost everywhere.

Proposition 11. *The following L^n-norms are determined constructively by the DN map:*

$$\left\| \frac{1}{p(x)-1} \right\|_{L^n([a,b],\gamma^{-1/(p(x)-1)}\,\mathrm{d}x)}^n. \tag{8}$$

If $\gamma \equiv 1$, we get instead

$$\left\| \frac{1}{p(x)-1} \right\|_{L^n([a,b])}^n.$$

In lemma 13, we show that the L^n-norms of a function uniquely determine its distribution function with respect to the underlying measure. Combined with the previous proposition, this allows us to recover the distribution function of $p(x)$, given a constant γ. If γ is not constant, then the previous result is still true, but the distribution function will be with respect to a measure that depends on the unknown power p. This still gives a restatement of the original problem, but not a satisfactory characterization of the exponents p which give the same DN map.

Recently Klun [27] and Erdélyi [20] proved that the equality of L^n norms implies the equimeasurability of the functions. However, if γ is not identically one, then we only know the weighted L^n norms (8), where the weight depends on the unknown power p. For this reason we need the slightly more general statement of lemma 11 with the two different weights. Moreover, since Klun makes use of a Müntz-Szász theorem, he is able to prove that one only needs suitably many Lebesgue norms to characterize a function. Since we know all the Lebesgue norms, we can avoid the Müntz-Szász theorem and instead rely on the properties of a moment problem. Our proof resembles Klun's, but we have nevertheless included it for the benefit of the reader.

To prove lemma 13 we use the following theorem about the uniqueness of solutions to a moment problem [37, p61, Corollary 6.1b].

Theorem 12. *Suppose that h is Lebesgue integrable in $(0, M)$, $M > 0$, and*

$$\int_0^M t^n h(t)\,\mathrm{d}t = 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Then $h(t) = 0$ almost everywhere in $(0, M)$.

Lemma 13. *Let $f, g : X \to [0, \infty)$ be μ_1 and μ_2 measurable, respectively, where μ_i are finite measures on X. Suppose also that f and g are bounded, and*

$$\int_X f^n \,\mathrm{d}\mu_1(x) = \int_X g^n \,\mathrm{d}\mu_2(x) \quad \text{for all } n \in \mathbb{N}.$$

Then $\mu_1^f = \mu_2^g$ in $[0, \infty]$, where μ_1^f and μ_2^g denote distribution functions as defined in (4).

Proof. Let $M := \max(\|f\|_{L^\infty}, \|g\|_{L^\infty})$. Then by Tonelli's theorem and a change of variables we have

$$\begin{aligned}
\int_X f^n \,\mathrm{d}\mu_1(x) &= \int_0^\infty \mu_1\left(\{x \in X; f(x)^n > t\}\right) \mathrm{d}t \\
&= \int_0^\infty \mu_1\left(\left\{x \in X; f(x) > t^{\frac{1}{n}}\right\}\right) \mathrm{d}t \\
&= \int_0^\infty \mu_1^f(t^{\frac{1}{n}})\,\mathrm{d}t \\
&= n \int_0^\infty t^{n-1} \mu_1^f(t)\,\mathrm{d}t \\
&= n \int_0^M t^{n-1} \mu_1^f(t)\,\mathrm{d}t
\end{aligned}$$

and similarly for g. Thus

$$0 = \int_X (f^n - g^n)\,\mathrm{d}x = n \int_0^M t^{n-1} \left(\mu_1^f(t) - \mu_2^g(t)\right) \mathrm{d}t$$

for all $n \in \mathbb{N}$. Hence, denoting $h(t) := \mu_1^f(t) - \mu_2^g(t)$, we have h integrable on $(0, M)$ and

$$\int_0^M t^{n-1} h(t)\,\mathrm{d}t = 0 \quad \text{for all } n \in \mathbb{N}.$$

It follows now from theorem 12 that $h(t) = 0$ for almost every $t \in (0, M)$. Since μ_f and μ_g are continuous from the right [1, p37], so is h, and consequently $h \equiv 0$ on $[0, M]$. It follows that $\mu_1^f = \mu_2^g$ in $[0, \infty)$. □

Define weighted measures on $[a, b]$ by

$$\mu_i(E) = \int_E \gamma^{-1/(p_i(x)-1)}\,\mathrm{d}x \quad \text{for all } E \subset [a, b].$$

Theorem 14. *Under the standing assumptions, if* $\Lambda_\gamma^{p_1} = \Lambda_\gamma^{p_2}$, *then* $\mu_1^{p_1} = \mu_2^{p_2}$.

Proof. Since γ and p are suitably bounded, $\gamma^{-1/(p_i(x)-1)}$ are bounded from below and above by positive numbers. Hence, $([a, b], \mu_i)$ are finite measure spaces. By proposition 11 we get

$$\int_a^b \left(\frac{1}{p_i(x) - 1}\right)^n \mathrm{d}\mu_i(x)$$

for all $n \in \mathbb{N}$ from the DN map. The functions $1/(p_i(x) - 1)$ are bounded. Now lemma 13 gives

$$\mu_1^{1/(p_1-1)}(t) = \mu_2^{1/(p_2-1)}(t) \quad \text{for all } t \in [0, \infty). \tag{9}$$

Using this with $t = 0$ and noticing that $[a, b] = \{x \in [a, b] : 1/(p_i - 1) > 0\}$, we obtain

$$\mu_1([a, b]) = \mu_1^{1/(p_1-1)}(0) = \mu_2^{1/(p_2-1)}(0) = \mu_2([a, b]). \tag{10}$$

Let $h(t) = 1 + 1/t$. Then $h : (0, \infty) \to (1, \infty)$ is strictly decreasing and we have $h \circ (1/(p_i - 1)) = p_i$. It now follows from (9), (10) and Lemma 6 that

$$\mu_1^{p_1}(t) = \mu_1^{h\circ(1/(p_1-1))}(t) = \mu_2^{h\circ(1/(p_2-1))}(t) = \mu_2^{p_2}(t) \quad \text{for all } t \in [0, \infty). \quad \square$$

If $\gamma \equiv 1$, then the previous theorem immediately yields the equimeasurability of p_1 and p_2 in the Lebesgue measure. If γ is some other constant, then the statement is seemingly different, but below we show that this is only apparent. This is natural as a constant $\gamma \neq 0$ plays no role in equation (1), though it affects the DN map.

Theorem 15. *Suppose that $\gamma > 0$ is a constant and that the assumptions of theorem 14 hold. Then p_1 and p_2 are equimeasurable with respect to the Lebesgue measure.*

Proof. We begin with the case $\gamma < 1$. By lemma 6 it suffices to show the equimeasurability of the functions $f := 1/(p_1 - 1)$ and $g := 1/(p_2 - 1)$. By (9) we have

$$\int_{\{x \in [a,b]; f(x) > t\}} \gamma^{-f(x)} \,\mathrm{d}x = \int_{\{x \in [a,b]; g(x) > t\}} \gamma^{-g(x)} \,\mathrm{d}x \quad \text{for all } t \in \mathbb{R}. \tag{11}$$

By denoting $\tilde{f} := f \log \gamma^{-1}$ and $\tilde{g} := g \log \gamma^{-1}$, this implies that

$$\int_{\left\{x \in [a,b]; \tilde{f}(x) \leq t\right\}} e^{\tilde{f}(x)} \,\mathrm{d}x = \int_{\{x \in [a,b]; \tilde{g}(x) \leq t\}} e^{\tilde{g}(x)} \,\mathrm{d}x \quad \text{for all } t \in \mathbb{R}. \tag{12}$$

We compute for $t \geq 0$

$$\begin{aligned}
\int_{\left\{x \in [a,b]; \tilde{f}(x) \leq t\right\}} e^{\tilde{f}(x)} \,\mathrm{d}x &= \int_0^\infty \left| \left\{ x \in [a,b]; \tilde{f}(x) \leq t \text{ and } e^{\tilde{f}(x)} > s \right\} \right| \mathrm{d}s \\
&= \int_0^\infty \left| \left\{ x \in [a,b]; \log s < \tilde{f}(x) \leq t \right\} \right| \mathrm{d}s \\
&= \int_{-\infty}^\infty \left| \left\{ x \in [a,b]; u < \tilde{f}(x) \leq t \right\} \right| e^u \,\mathrm{d}u, \qquad (13)
\end{aligned}$$

where in the last identity we did a change of variables $s = e^u$. We define for all $u \in \mathbb{R}$ the function $F(u) := \left| \left\{ x \in [a,b]; \tilde{f}(x) \leq u \right\} \right|$. Since F is non-decreasing, continuous from the right and $F(0) = 0$, there exists an associated Stieltjes measure [28, chapter 6, section 8] (still denoted by F) such that

$$F(c,d] = F(d) - F(c) \quad \text{whenever } c < d.$$

Then for any $u \in \mathbb{R}$ we have

$$\left| \left\{ x \in [a,b]; u < \tilde{f}(x) \leq t \right\} \right| = \chi_{\{u<t\}} (F(t) - F(u)) = \chi_{\{u<t\}} \int_{(u,t]} \mathrm{d}F(y).$$

We combine this with (13) and continue the computation by using Tonelli's theorem to obtain

$$
\begin{aligned}
\int_{\{x\in[a,b];\tilde f(x)\le t\}} e^{\tilde f(x)}\,\mathrm{d}x &= \int_{-\infty}^{\infty} \chi_{\{u<t\}} e^u \int_{(u,t]} \mathrm{d}F(y)\,\mathrm{d}u \\
&= \int_{-\infty}^{\infty}\int_{(u,t]} e^u\,\mathrm{d}F(y)\,\mathrm{d}u \\
&= \int_{-\infty}^{\infty}\int_{\mathbb{R}} \chi_{\{u<y\le t\}} e^u\,\mathrm{d}F(y)\,\mathrm{d}u \\
&= \int_{\mathbb{R}}\int_{-\infty}^{\infty} \chi_{\{u<y\le t\}} e^u\,\mathrm{d}u\,\mathrm{d}F(y) \\
&= \int_{(0,t]}\int_{-\infty}^{y} e^u\,\mathrm{d}u\,\mathrm{d}F(y) \\
&= \int_{(0,t]} e^y\,\mathrm{d}F(y). \qquad (14)
\end{aligned}
$$

Integrating by parts (see e.g. [28, p344]), we obtain

$$\int_{(0,t]} e^y\,\mathrm{d}F(y) = e^t F(t) - e^0 F(0) - \int_0^t e^y F(y)\,\mathrm{d}y.$$

Combining this with (14), we have

$$\int_{\{x\in[a,b];\tilde f(x)\le t\}} e^{\tilde f(x)}\,\mathrm{d}x = e^t F(t) - F(0) - \int_0^t e^y F(y)\,\mathrm{d}y \quad \text{for all } t \ge 0.$$

Of course, an analogical identity holds for $\tilde g$. Thus by applying (12) and using that $F(0) = 0 = G(0)$, we obtain

$$e^t(F(t) - G(t)) = \int_0^t e^y (F(y) - G(y))\,\mathrm{d}y \quad \text{for all } t \ge 0.$$

Denoting $h(t) = e^t\,|F(t) - G(t)|$, this implies

$$h(t) \le \int_0^t h(y)\,\mathrm{d}y \quad \text{for all } t \ge 0.$$

By iterating the above inequality or applying Grönwall's lemma, it follows that $h \equiv 0$, which implies the equimeasurability of $\tilde f$ and $\tilde g$, and thereby of f and g. Next we consider the case $\gamma > 1$. The outline of the proof is similar to above, but there are some technical differences. By dominated convergence, it follows from (11) that

$$\int_{\{x\in[a,b];f(x)\ge t\}} \gamma^{-f(x)}\,\mathrm{d}x = \int_{\{x\in[a,b];g(x)\ge t\}} \gamma^{-g(x)}\,\mathrm{d}x \quad \text{for all } t \in \mathbb{R}.$$

Denoting $\tilde{f} := f \log \gamma^{-1}$ and $\tilde{g} := g \log \gamma^{-1}$, where now $\log \gamma^{-1} < 0$, we have

$$\int_{\{x \in [a,b]; f(x) \geq t\}} e^{\tilde{f}(x)} \, \mathrm{d}x = \int_{\{x \in [a,b]; g(x) \geq t\}} e^{\tilde{g}(x)} \, \mathrm{d}x \quad \text{for all } t \in \mathbb{R}. \tag{15}$$

We compute for $t \geq 0$ (denote $\tilde{t} := t \log \gamma^{-1} \leq 0$)

$$\begin{aligned} \int_{\{x \in [a,b]; f(x) \geq t\}} e^{\tilde{f}(x)} \, \mathrm{d}x &= \int_0^\infty \left| \left\{ x \in [a,b]; f(x) \geq t \text{ and } e^{\tilde{f}(x)} > s \right\} \right| \, \mathrm{d}s \\ &= \int_0^\infty \left| \left\{ x \in [a,b]; \tilde{f}(x) \leq \tilde{t} \text{ and } \tilde{f}(x) > \log s \right\} \right| \, \mathrm{d}s \\ &= \int_{-\infty}^\infty \left| \left\{ x \in [a,b]; u < \tilde{f}(x) \leq \tilde{t} \right\} \right| e^u \, \mathrm{d}u. \end{aligned} \tag{16}$$

We define for all $u \in \mathbb{R}$ the function $F(u) := \left| \left\{ x \in [a,b]; \tilde{f}(x) \leq u \right\} \right| - (b-a)$. Since F is non-decreasing, continuous from the right and $F(0) = 0$, there exists an associated Stieltjes measure. Then

$$\left| \left\{ x \in [a,b]; u < \tilde{f}(x) \leq \tilde{t} \right\} \right| = \chi_{\{u < \tilde{t}\}} (F(\tilde{t}) - F(u)) = \chi_{\{u < \tilde{t}\}} F(u, \tilde{t}].$$

Using this and continuing the computation (16), we get

$$\begin{aligned} \int_{\{x \in [a,b]; f(x) \geq t\}} e^{\tilde{f}(x)} \, \mathrm{d}x &= \int_{-\infty}^\infty \chi_{\{u < \tilde{t}\}} F(u, \tilde{t}] e^u \, \mathrm{d}u \\ &= \int_{-\infty}^\infty \chi_{\{u < \tilde{t}\}} e^u \int_{(u, \tilde{t}]} \mathrm{d}F(y) \, \mathrm{d}u \\ &= \int_{-\infty}^{\tilde{t}} \int_{\mathbb{R}} \chi_{\{u < y \leq \tilde{t}\}} e^u \, \mathrm{d}F(y) \, \mathrm{d}u \\ &= \int_{\mathbb{R}} \int_{-\infty}^{\tilde{t}} \chi_{\{u < y\}} \chi_{\{y \leq \tilde{t}\}} e^u \, \mathrm{d}u \, \mathrm{d}F(y) \\ &= \int_{(-\infty, \tilde{t}]} \int_{-\infty}^{y} e^u \, \mathrm{d}u \, \mathrm{d}F(y) \\ &= \int_{(-\infty, \tilde{t}]} e^y \, \mathrm{d}F(y). \end{aligned}$$

Writing $M := \operatorname{ess\,inf}_{[a,b]} \tilde f = \operatorname{ess\,inf}_{[a,b]} \tilde g$ and integrating by parts we obtain

$$\begin{aligned}\int_{(-\infty,\tilde t\,]} e^y \,\mathrm{d}F(y) &= \lim_{k\to-\infty} \int_{(k,\tilde t\,]} e^y \,\mathrm{d}F(y) \\ &= \lim_{k\to-\infty} e^{\tilde t} F(\tilde t) - e^k F(k) - \int_k^{\tilde t} e^y F(y)\,\mathrm{d}y \\ &= e^{\tilde t} F(\tilde t) - \int_M^{\tilde t} e^y F(y)\,\mathrm{d}y - \int_{-\infty}^M e^y (a-b)\,\mathrm{d}y. \\ &= e^{\tilde t} F(\tilde t) - \int_M^{\tilde t} e^y F(y)\,\mathrm{d}y + (b-a)e^M.\end{aligned}$$

Combining the last two displays and recalling that $\tilde t = t\log\gamma^{-1}$, we obtain for any $\tilde t \in [M,0]$

$$\int_{\{x\in[a,b];f(x)\geq t\}} e^{\tilde f(x)}\,\mathrm{d}x = e^{\tilde t}F(\tilde t) - \int_M^{\tilde t} e^y F(y)\,\mathrm{d}y + (b-a)e^M$$

and a similar identity for g. Since the left-hand side is by (15) the same for f and g, we get

$$e^{\tilde t}(F(\tilde t) - G(\tilde t)) = \int_M^{\tilde t} e^y (F(y)-G(y))\,\mathrm{d}y \quad \text{for all } \tilde t \in [M,0].$$

Grönwall's lemma now implies that $F(\tilde t) - G(\tilde t) \equiv 0$ and the equimeasurability of f and g follows. □

If γ were not constant, we could try following the same proof, but the sets in equations (12) and (13) would have additional x-dependencies, making the next step infeasible.
With theorem 15 at hand, we are ready to finish the proof of our main theorem.

Proof of Theorem 1. Suppose first that $\Lambda_\gamma^{p_1} = \Lambda_\gamma^{p_2}$. Then theorem 15 implies that that the exponents p_1 and p_2 are equimeasurable.
Suppose then that p_1 and p_2 are equimeasurable and consider the definition of K_m in equation (2). Since γ is constant, the functions $x \mapsto (K/\gamma)^{1/(p_i(x)-1)}$ are equimeasurable by lemma 6, given any fixed constant $K/\gamma \neq 1$, and in case of $K/\gamma = 1$ they are equal. But then their integrals agree by Tonelli's theorem, whence K_m takes the same value for p_1 and p_2 for every $m > 0$ (note that $m \mapsto K_m$ is injective, given any fixed γ and p [9, lemma 7]).
We now consider equation (3) written as

$$\Lambda_\gamma^{p_i}(m) = K_m \int_a^b (K_m/\gamma)^{1/(p_i(x)-1)}\,\mathrm{d}x.$$

Consider a fixed $m > 0$, whence K_m takes the same value for p_1 and p_2. It was already established that the integrals take the same value independent of i, which gives the equality of the DN maps. □

Finally, we describe a procedure that can be used to obtain a rearrangement of the variable exponent, as stated in theorem 2. We begin with the reconstruction of the distribution function μ^f, where $f(x) = 1/(p(x)-1)$. First, the maximum M of f can be obtained from lemma 10. Thus by proposition 11 and the proof of lemma 13 we know that the moments

$$\lambda_n = \int_0^1 t^n \mu^f(tM)\,\mathrm{d}t, \quad n \in \mathbb{N},$$

can be constructively recovered. It is now possible to obtain $\mu^f(tM)$ (and thus also μ^f) using λ_n. One such way has been described for example in [36] where the function is recovered as the $L^2(0,1)$ limit of a series of certain polynomials depending on λ_n. The numerics of the reconstruction are also considered therein.

With the distribution function μ^f recovered, we can obtain one special rearrangement of f by the formula

$$f^*(x) = \inf\{t \in [0,\infty]; \mu^f(t) \le x\}.$$

The function f^* is called the non-symmetric decreasing rearrangement of f. It is equimeasurable with f and continuous from the right [1, 29]. Consequently, by lemma 6, also the functions $1+1/f$ and $1+1/f^*$ are equimeasurable. Since $1+1/f(x) = p(x)$, it follows that a rearrangement r of the variable exponent can be recovered from μ^f by the formula

$$r(x) = 1 + 1/f^*(x).$$

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