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On Understanding in Mathematics

Hans Georg Schaathun
Department of ICT and Natural Sciences, NTNU — Norwegian University of Science and Technology, Postboks 1517, N-6025 Ålesund, Norway

hasc@ntnu.no

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It is often difficult to teach mathematics for understanding. Many students seem to prefer to learn computational skills by rote, and seem to remember very little of the course contents the following semester. The reasons for this are surely manifold. This paper reviews the concept of understanding, looking particularly towards the meaning it has in hermeneutic philosophy. The known distinction between hermeneutical and epistemological understanding is resituated in mathematics, and we argue that it is the hermeneutic understanding which is most relevant when mathematics is taught as a support subject for engineering and other professions.

1. Introduction

There seems to be wide agreement that one ought to teach for understanding, rather than for rote learning. Teaching a mathematics degree, the students are expected to understand why each technique works and why any given statement is true. In service teaching, it is not obvious that the students need understanding in this sense, and many students show little motivation for understanding. Given how difficult it appears to be to understand mathematics, one may be content if the students acquire a modest selection of computational skills. However, the German mathematics didactician Wolfram Meyerhöfer[1] claims that also students who struggle with mathematics can learn to understand, and moreover they *can only learn computational skills if* they understand.

Understanding seems to be sought, one way or another, by most people. As one of my students put it,[2] 'can't you just give us a formula, so that we can start to understand?' While this seemingly contradictory question puzzled me at the time, there may still be some sense behind it. The student, presumably, was only looking for meaning relevant to his studies in business administration. Understanding why the formula is true may be irrelevant to his life and prospective career. Whether he wanted to understand how to pass the exam, or how to carry out his prospective profession, he was just aiming for what Skemp (1976) calls *instrumental understanding*.

This paper reviews what is or could be meant by understanding when we disregard instrumental or procedural understanding, considering primarily service teaching of mathematics in higher education. We shall see that the ambiguity of the word 'understanding' is not at all captured by the single

1 See 'Ich will, dass jedes Kind rechnen lernt', interview in *Der Spiegel*, 27.10.2013
2 Translated from Norwegian, see Schaathun & Moe (2021) for the original text and context.

instrumental/relational dichotomy of Skemp's (ibid.). In the initial review, in Section 2–4, we investigate, in turn, three different positions on and approaches to teaching and learning of mathematics, and compare them to prevailing theory from cognitive psychology, education, and philosophy. Readers with a background in teacher education may find this material more well-known than many readers who teach in higher education. In Section 5, we turn to a fourth position, which is less frequently seen in the context of mathematics, namely the perspective of hermeneutic philosophy as known from Gadamer (2004) and his followers. Finally, in Section 6–7, we discuss the implication to learning and teaching of mathematics in higher education and for engineering in particular. The methodology is philosophical and hermeneutical.

2. Epistemological Understanding

Understanding is a difficult concept to define. The *Stanford Encyclopedia of Philosophy* (Grimm, 2021) calls it a protean concept. The literature on education often refers to a dichotomy between rote learning and learning through understanding (e.g. Holm, 2012, p. 44).

Mathematics is a study of *necessary truths*. New propositions are deduced with certainty from known premises. As Leibnitz put it, the proposition is true because the negation would be impossible.[3] It is possible to take mathematical propositions and procedures on trust, and simply learn them by rote. If trust is not sufficient, it may be necessary to see why the negation would be impossible. This is a reasonable interpretation of understanding a result in mathematics; to know not just the result, but also how to be certain that it is a necessary truth. In this sense, understanding supports critical thinking, while rote learning does not. A critical thinker is a free individual, not dependent on the word of authority.

This kind of understanding is the concern of *epistemology*, that is, the study of how we can know that we know. Such epistemological understanding in mathematics would entail the processing of each deductive step, so that new knowledge is gained from the premises. As Gallagher (1992, p. 40) writes,

> In epistemology the word understanding usually signifies a mental process which takes place in the mind (the soul or consciousness). It is an intellectual process whereby a knower gains knowledge about something.

Some teachers may say that to understand a result, the student has to be able to reconstruct a sound argument. Others may be content if the student is able critically to validate the argument, and thereby independently ascertain the result. We shall leave that debate for another time, and instead turn to why understanding makes for better learning than rote learning.

Learning means that new information is transferred from working memory (aka. short-term memory), where it is processed, into long-term memory to be retained (more or less) permanently. Several theories exist on how this transfer is secured, both within learning theories (see e.g. Tetzchner, 2012) and in cognitive psychology (see e.g. Anderson, 2015). Common for most (if not all) of the dominant theories is that memory depends on the knowledge fitting into a meaningful context.

> Memory is better for material if we are able to meaningfully interpret that material. *(Anderson, 2015, p. 103)*

Rote learning leads to fragmented knowledge (Holm, 2012, p. 44), where each result is memorised independently and context is ignored. Empirical research shows that time spent on such passive rehearsal does not improve memory. The depth of processing theory holds that what matters is that the material is processed in a 'deep and meaningful' way (Anderson, 2015, p. 128). Mayer (2004) argues that this is

3 For a more complete introduction to the philosophy of mathematics, one can read e.g. Körner (1960).

what 'active learning' has to mean. Behavioural activity, mindlessly carrying out procedures, does not foster learning. Learning comes from cognitive activity, where the student meaningfully processes the information. To the extent that deductive proofs are meaningful to the student, such in-depth processing which supports memory can be achieved by working through the proof to validate it and convince oneself that the result is true.

Another aspect of meaningful interpretation of new material is consolidation with prior knowledge. This is central to constructivist learning theories. Knowledge in long-term memory are organised in so-called cognitive schemata, that is mental models of the world. New material is learnt through *assimilation* in, or, in the case of inconsistency, *accommodation* of, existing cognitive schemata.[4] When the student goes through the process of deduction, each new result is immediately linked to prior knowledge which serves as premises. Thus they can construct a connected system of knowledge. To the extent that the premises of the mathematical argument are mentally encoded as cognitive schemata, one should expect the conclusion, that is the new knowledge, to be assimilated in these schemata.

Researchers of mathematics construct new knowledge through deductive logic. Admittedly, they do not necessarily *discover* the results deductively, but the deductive proof is key to elevating a result from conjecture to knowledge. One can imagine a learner constructing their knowledge in the same way, by following the same deductive process, by cognitively processing each step of the proof in relation to their own prior schemata. This approach has theoretical merit, but we have to ask if the students have the background and the motivation truly to engage in the proof. Proofs too can be learnt by rote, and that is not the same thing.

3. Induction versus Deduction

In the 1820s in Boston, Warren Colburn published a very successful textbook in arithmetics, based on the ideas of the Swiss educational reformer Pestalozzi. In the preface, Colburn (1822) tells the story of the boy without a genius. He has given up arithmetics because he could not do it, but yet he has no problem calculating how many marbles he can buy for ha'penny or for tuppence, using his experience that a penny buys a dozen. There may be many interpretations of the story, and it serves as illustration for many pedagogical concepts found in more recent literature.

Playing marbles, and collecting marbles, is obviously important for this boy. Numbers and arithmetic operations gain purpose and meaning when they are understood in terms of concrete objects which the boy needs or wants. Through years of pre-school experience, he has built complex cognitive schemata of marbles, pennies, and trade. Within these schemata, arithmetic operations have meaning, and through lived experience and practice, they have been firmly embedded in his mind, not as propositional rules, but as tacit *know-how*.[5] This enables him to answer with great confidence the questions which relate to his daily life.

In addition to experience, the boy may be motivated by the calculations of pennies and marbles. Admittedly, motivation or intent is not as important to learning as one may think, but because motivated students are inclined to process the material more deeply, it still has an important and positive, though indirect, effect on learning (Anderson, 2015, p. 144). NFS Grundtvig put this in terms of love. No one

4 Details and elaboration can probably be found in any textbook on developmental psychology or learning theories. The author has relied on Tetzchner (2012, p. 212ff), in Norwegian.

5 The distinction between knowing how and knowing that was famously discussed by Ryle (1945). Polanyi (1966) introduced the concept of tacit knowing.

has ever lived to comprehend anything which they did not first hold dear, he said.[6] Interestingly, Colburn does not try to motivate the student at all. That is, he does not explain why the subject matter is worth learning. Instead, like Søren Kierkegaard, he meets the students where they are, with problems which they are likely to find motivating and soluble. According to Sotto (2007, p. 21), it is impossible to create motivation. The best a teacher can hope for is to avoid impeding the motivation the student already has. The default assumption should be that the student *is* motivated, why else would they attend the course? Motivation is easily lost when the contents of the course does not appear to serve the student's purpose of attendance. Only by addressing that pre-existing motivation will teaching remain purposeful, motivating, and meaningful.

Colburn called his method the *inductive mode of instruction*. In the conventional, deductive mode of instruction, the teacher explains the rules of mathematics, and the students are expected to deduce the answers to particular problems from the rules. Colburn, in contrast, gave them lots and lots of concrete problems which could be solved more or less empirically. The general rules can then be inferred inductively from the experience. This inductive method was affirmed empirically in a series of psychological experiments by Gertrude Hendrix in the 1940s and 50s. She showed that students who learn from examples, without verbalising any rules or procedures, are better equipped to transfer learning to new problems (Hendrix, 1947). Even formulating the propositional rule before it is mastered has a negative effect on the transfer ability. The students ought to learn to carry out the calculations and gain experience with these calculations, before verbalising their knowledge.

In all of these accounts, from Colburn's 200 years ago to Sotto's in our own century, new understanding depends on prior understanding, but on a kind of understanding which is very different from the axiomatic and propositional knowledge that is assumed in a deductive approach to mathematics. Both inductive and deductive learning are compatible with constructivist and depth of processing theories. Both may lead to what we may call relational understanding; should 'relational' refer to internal relations between mathematical concepts or external relations between mathematics and lived experience? More importantly, either method will fail if the student lacks the appropriate background, but this does not make the student unfit to learn from the other.

4. Different Understandings of Mathematics

Both deductive and inductive approaches to learning depend on prior understanding and meaning. New material is interpreted in relation to what is already in the mind. When the new and existing knowledge are compatible enough to facilitate such interpretation, we can call it meaningful. This meaning is subjective, in the sense that each learner needs to find what is meaningful *for them*. Any meaning explained by the teacher is worthless, if it cannot be accommodated or assimilated into the cognitive schemata held by the learner. Depending on the student, meaning can derive from either spontaneous or scientific concepts. Hence, learners with different experience and different motivation may not always be able to learn mathematics in the same way. To understand the differences, let us consider what it means to *be* a mathematician.

There are at least three distinct roles where distinct forms of mathematical competency are required. Firstly, there is the *pure mathematician*, who explores mathematics in its abstract form, to reveal new insights. This is a creative role, which requires deep understanding of mathematical concepts and their

6 Quoted by Korsgaard (2007) in Danish: 'Den har aldrig levet, som klog på det er blevet, han først ej hadde kjær.' Apologies for not being able to preserve the poetry in translation.

relations. In contrast, no understanding of the physical world or its relations to mathematics is required. Secondly, there is the *computer*. Two generations ago, many engineers took their first job as a computer, a highly esteemed profession requiring great skill and meticulous accuracy. To become a computer, one would need to know a range of computational algorithms and be able to execute them correctly. The computer is not concerned with the meaning of the problems and their solutions; interpretation belongs to a different role.

Finally, there is the *mathematical modeller*, be they engineers, applied mathematicians, or otherwise engaged in comprehending the physical world by means of mathematics. To the modeller, the understanding of mathematics and the understanding of the physical world have to be one. They have to see mathematical patterns and relationships in the world which they sense, and it is through these mathematical patterns that they make sense of the physical world. Calculations can be left to computers, whether human or digital, so only cursory knowledge of computation is required. Pure mathematics is only useful insofar as it improves the understanding of the sense-world.

Mathematical modelling fills, to the engineer, a role similar to that of sketching to the architect. Both engineers and architects are designers in the sense that they devise sequences of actions to change a situation into a preferred one (Simon, 1969), and models and sketches are representations of the situation in question. It is well known that designers see more in the sketch than what was designed in their making. Schön & Wiggins (1992) relate this to Wittgenstein's concept of *seeing-as* (Wittgenstein, 1986). The representation is seen *as* the real thing. This idea can conceivably carry over to mathematical models. When engineers make models, they are not interested in models for the sake of models. Models are the representation in which they *see* the real physical artefact that they create. Mathematics is not sufficiently understood (for the purpose of their profession) until they can see through the model, and see it as the real thing.

The time for human computers is past, and machine learning is out of scope for this paper. Therefore the analysis focuses on the other two roles. The pure mathematician and the modeller can have quite different conceptions of what is meaningful in mathematics. To the modeller, meaning derives from the physical world. Mathematics is meaningful when it explains the sense-world. The pure mathematician generalises beyond the sense-world, and meaning has, at least in many cases, to derive from within the abstract mathematics. Cognitive schemata are also likely to be different, as the modeller may incorporate physical concepts in the schemata of mathematics. Mathematical concepts can be understood in terms of the real-world phenomena which they explain. This is not the case in pure mathematics, where the concepts need no correspondence in the sensed reality and axioms may be hypothetical only.

The distinction between the concrete sense-world and the abstract world of mathematical forms is different from Vygotsky's and Piaget's dichotomy of spontaneous and scientific concepts. Most of their studies considered children in the early stages of forming scientific concepts. When the students reach higher education, they already have a rich repertoire of scientific concepts across a range of subjects. While it may possibly suffice in primary school, to relate the scientific concepts to everyday or spontaneous concepts, higher education will have to relate different scientific conceptualisations to each other. Mathematical concepts, in particular, are not only scientific as opposed to spontaneous, but also abstract as opposed to concrete scientific concepts. Applied mathematics relates not only to spontaneous concepts, but at least as much to scientific concepts studied in other subjects.

The different roles mentioned are not necessarily exhaustive, but they suffice to illustrate the contrast between the various approaches to learning and understanding under discussion. There is nothing to say that a learner could not be both pure and applied mathematician, but neither is there reason to say that they have to. A learner may find meaning in applied mathematics, and learn that, even if no meaning is found in pure mathematics, and vice versa. Requiring engineering students to construct mathematical knowledge in terms of pure mathematics is neither necessary nor sufficient. What is necessary is a knowledge construction which incorporates mathematics with the physical reality studied in engineering. As we argued in Section 3, constructivist learning in mathematics can build just as well on real world experience as it can on axiomatic theory. Students reading mathematics as a support subject are much more likely to find meaning in real world problems.

5. Hermeneutic Understanding

In spite of the differences, the views and theories reviewed so far all have one thing in common. They see understanding as the *acquisition* of knowledge, and the debate focuses on *what* is acquired and *how* to acquire it. These views are also well known in pedagogical literature. Hermeneutic philosophy offers a more radical shift, seeing understanding not as the acquisition of knowledge, but as a *transformation* of the learner (knower) (Kerdeman, 1998). While there is a rich literature on hermeneutics, it rarely features in the context of mathematics education, let alone engineering mathematics.

Modern hermeneutics is commonly attributed to Gadamer (2004), building on Heidegger (with whom he had been a student). To them, understanding is existential, and an essential part of being human. Gallagher (1992, p. 42) writes:

> For Heidegger, understanding is essentially a way of being, the way of being which belongs to human existence .... Being-in-the-world is not primarily a cognitive relation between subject and object, although being-in-the-world is a way of existing which allows there to be cognition. Human existence discloses the world, or is in-the-world by way of an understanding that functions on all levels of behavior, conscious or unconscious. Thus, Heidegger, contends, understanding is "a basic determination of [human] existence itself"

At first sight, this is just abstract philosophy which says very little about real students and mathematics, so we need to dig a little deeper. What is this *being-in-the-world* that the students aspire to?

The purpose of engineering is to understand, manipulate, and design technical artefacts. Being an engineer is thus a relationship with these artefacts. Mathematics is not something to be understood, but a way to understand artefacts. However, the mathematical understanding does not replace concrete understanding. When the engineer proposes a change to the artefact, they change at the same time the mathematical model and the real system. The change can never be fully appreciated in the mathematical model alone, because the intention is always a function to real users in the real world. The engineer has to *see* the real artefact in the model and vice versa. In this sense, understanding is transformational. It changes how the student sees the sense-world.

The significance of the unconscious, as mentioned in Gallagher's quote above, can be seen when we look at the problem solving models of Schön (1983) and Simon (1972)[7]. Whether we consider abstract problems of mathematics or concrete problems from the real world, problem solving is an iterative process of trial and error. Moves are generated (discovered) and then tested (evaluated). Move testing ought to be rigorous, using the slow and methodological reasoning that is well understood in mathematics. Move generation, in contrast, is often intuitive. Except for the rare, obvious cases where the solution is obvious, it has to be quick, to give time to consider a large number of ideas. As Boaler (2015) writes, it is necessary to make a large number of errors on the way. In a hermeneutic sense,

[7] See also Schaathun (2022) for a recent comparison of Simon's and Schön's models.

mathematical understanding guides and changes the student's intuition, and it enables new ways to generate spontaneous ideas.

Intuition, as we use the word here, involves no magic. It corresponds to a fast mode of thinking, also known as System 1. Kahneman (2011), in his book *Thinking, fast and slow*, explains how our two modes of thinking work. System 2 is rigorous and reliable, but slow. System 1 is subconscious and fallible, but critically important because it is fast and sufficiently accurate often enough. Understanding which only operates in System 2, is not understanding in the hermeneutic sense.

It is well-known that even high-performing students are slow to learn to solve non-routine problems in mathematics (e.g. Selden *et al.*, 2000). One plausible explanation for this is found in the common misconceptions documented by Schoenfeld (1988). Students are led to believe that 'one succeeds in school by performing the tasks, to the letter, as described by the teacher'. This false belief encourages procedural understanding, learning well-defined procedures for well-defined problems. Facing a problem not falling into any of the known categories, the student is left without a strategy. Either they guess or they give up, and they make no attempt to understand the problem.

Real world problems are rarely solved by knowing the solution. Instead the problem has to be interpreted, and sometimes transformed into a different problem. The iterative problem solving process is not only aiming to find the solution. Often the main task is to make sense of the problem. Understanding is not (just) what one knows about mathematics, but (also) what one does mathematically in relation to a new problem. Learning and problem solving are very similar activities, in that new concepts are related to old concepts in order to understand. In hermeneutic theory understanding is is not a deliberate and methodological effort, but rather the natural state of the human mind (Gadamer, 2004; Kerdeman, 1998). We cannot help but engage in understanding of the world wherein we live. As Davis (1992) puts it in the context of mathematics education,

> Students are determined to understand, and they create their own ways of understanding.

It may be reasonable to assume that hermeneutic understanding entails conceptual understanding and constructivist learning, but they are not the same. The student who writes an excellent analysis relating mathematical concepts may demonstrate advanced conceptual or relational understanding without any evidence of subconscious behaviour, neither the subconscious mastery of known concepts, nor the activity of making sense of new problems. Thus, the hermeneutic understanding of 'understanding' brings new ideas into mathematics education.

6. Learning Mathematics for Understanding

Having seen what hermeneutic understanding may mean in mathematics, let's turn to the question of how it can be promoted. Two concepts will be central to this discussion, namely freedom and imitation.

If we accept that understanding is the result of a natural urge, rather than a methodological effort, it follows that the student needs the freedom to pursue their urges. *Freedom to learn* was the famous catch-phrase of Carl Rogers in the 1960s, and he pointed out the paradox of school discipline (Rogers & Freiberg, 1994). In working life, one generally succeeds as unique individuals, through original ideas and innovation. Many taught courses value instead the students who do exactly as they are told, and top grades are given for solving the same problems as everybody else, using standardised solution techniques. While this could be a useful skill in the role of computer, it hardly promotes the understanding and creativity needed for professional problem solving.

Hermeneutic philosophy sees education both as a free process and as a pursuit of goals put forth by others (Gustavsson, 1996; Kemp, 2006). Whitehead (1929) too advocates a balance between freedom and discipline, and Vygotsky (2012, p. 114) writes,

> to introduce a new concept means just to start the process of its appropriation. Deliberate introduction of new concepts does not preclude spontaneous development, but rather charts the new paths for it.

The teacher may point the direction, but the student has to walk down the path for themself. Appropriation is personal, and cannot be instructed.

Many mathematics textbooks and exam papers are full of problems with a well-defined 'right answer'. These problems encourage the students to copy template solutions, and thus reinforce the misconceptions identified by Schoenfeld (1988). Schön (1987) and Kemp (2006) cast learning instead as *creative imitation*,[8] as opposed to copying with neither creativity nor originality. Truly to imitate the teacher, the student would look, not at the teacher's solution, but at their approach to finding the solution. Learning problem solving, the goal is not to recall the teacher's solution, which applied to some past problem. Instead the learner would imagine what the teacher might have done in relation to the new problem at hand. This is the creative part. The learner has to interpret the teacher's example in their own situation, in order to make sense of their own problems. Again we see the need for freedom. To solve genuine real world problems, the student has to free themself from rigid obedience to the teacher's example, or as Vygotsky (2012, p. 112) puts it

> developmental progress reveals itself [...] in the achievement of a certain freedom of thinking in scientific concepts.

Understanding of mathematics has to entail this freedom of thinking in mathematical terms.

Creative imitation gives an alternative way to think about examples. Textbooks in mathematics often use examples to illustrate a procedure which has already been explained, giving the correct solution only. However, the subconscious behaviour needed in move generation can hardly be explained, and to solve real problems the student has to learn to make, detect, and correct mistakes. Thus they need examples of genuine problem solving in the context of discovery, including all the failed attempts, rather than polished arguments from the context of justification.

7. Teaching Engineering Mathematics

In the last two sections we have argued that understanding of mathematics cannot be separated from understanding of the real world, at least not for students of engineering or other professions that use mathematics as a tool. This raises the question if there is still scope, in an engineering degree, to separate mathematics modules from engineering modules. There are several reasons why such modularisation is challenging.

The first issue is motivation. Many authors have acknowledged the students' determination to understand, but the determination does not always extend unconditionally to anything the teacher might present. It is fair to assume that engineering students are determined to understand engineering, and at some point that will inevitably involve a lot of mathematics. However, mathematics may be both meaningless and demotivating, until it appears in the *lived experience* of engineering. Toy examples

[8] Kemp uses the Greek word *mimesis* for imitation, as known from Aristotle's *Poetics*, and he further relates education to the three-fold mimesis of Ricœur (1984).

illustrating relevance may not suffice. Motivation is a *feeling* and relevance must be experienced, not merely known and explained. Genuine engineering problems are more likely to provoke the deep and meaningful processing that leads to learning. An example which is passively read does not have the same learning effect, no matter how relevant it might be. Thus there are two insights to take from Søren Kierkegaard's oft-quoted secret of helping:

> If One Is Truly to Succeed in Leading a Person to a Specific Place, One Must First and Foremost Take Care to Find Him Where He is and Begin There. This is the secret in the entire art of helping.[9]

The most obvious insight is that we need to find where the student is in terms of background knowledge, upon which they can build. Additionally, the educator needs to find where the student is in terms of motivation. Where are they headed? What kind of problems would engage them in deep and meaningful processing?

The second issue is the fact that learning is not a linear course, building brick upon brick. New concepts are not learnt once and for all, but reinterpreted over and over again, as they are encountered in new contexts. This is known as the hermeneutic circle (cf. Gadamer, 2004). The path from first encounter to appropriation of a concept is 'long and complex', writes Vygotsky (2012, p. 114), quoting Tolstoy

> When he has heard or read an unknown word in an otherwise comprehensible sentence, and another time in another sentence, he begins to have a hazy idea of the new concept; sooner or later he will... feel the need to use that word—and once he has used it, the word and the concept are his.

Luntley (2018) puts this in terms of play. By playing with partial concepts and placeholders, more complete conceptualisations are gradually developed. It is rather optimistic to expect students to complete this cycle and understand new mathematical concepts within a one-semester maths module. Consequently, it is unfair to examine them in mathematics early in the degree programme.

A third point is made by Whitehead. The separation of mathematics from the application domain is often based on the idea that the mind is an instrument, which has to be sharpened before it is used. Whitehead (1929, p. 18) contends, in contrast, that

> The mind is never passive [...] You cannot postpone its life until you have sharpened it.

This is in line with Gadamer (2004). The mind will always engage in understanding, because this is the natural state of its life. As teachers, we have to give the students room to live. One may have sympathy with those mathematics lecturers who see their role in sharpening the students, referring application to a different department. Yet, one has to realise that such an approach is needlessly hard on the student. As is learnt from cognitive psychology, memory is much improved by meaningful interpretation of the material, and interpretation and meaning depend on context. For many students, when mathematics is separated from application, it is also separated from meaning, and dedicated mathematics modules become unnecessarily hard.

8. Conclusion

Exploring different understandings of understanding, we do not aim to prescribe any particular method for teaching. Every method may be right for the right kind of student. However, few educators have the

9 Translation quoted by Oscar Tranvåg in an interview published by the University of Bergen at https://www.uib.no/en/news/103133/secret-behind-art-helping.

privilege to find the ideal students for their own conception of the subject. Most of us are expected to help the students we happen to have to get the education they need. This means that each and every one of us needs to reflect upon what mathematics is to *our own* students, and not only meet them where they are but also see where they are headed. It is the student who is at the centre of education, not the method and not the subject.

We have focused on students of engineering. Mathematics is meaningful when it changes how the students think about engineering not only consciously but also subconsciously or intuitively, i.e. when it leads to the transformational understanding advocated by hermeneutic philosophy. For some engineering students this may be the only meaning mathematics ever has. We should stress that the meaning is not found in computation, for which we have computers, but in the relational understanding of mathematical models and their relation to the sense world. The findings are likely transferable to other degree programmes where mathematics has a supporting role, but this is for the lecturers within each degree programme to decide.

Understanding remains a protean concept, and we may have to accept that understanding means different things to different learners. If we insist on an instrumental definition, we could consider this: *A learner has understood a concept, when and only when they can recognise and apply it, intuitively and subconsciously, as well as consciously, in novel contexts and non-routine problems within their own life and vocation.* To be more formal, 'intuitively and subconsciously' refers to our fast mode of thinking, the so-called System 1 (Kahneman, 2011).

There are a couple of important consequences of this analysis. The students need the freedom to interpret new material in their own situation. Copying of template solutions has to be replaced by creative imitation of solution processes. Again this is an interpretative endeavour. Understanding is subconscious as well as conscious, and subconscious thinking cannot be explained.

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Hans Georg Schaathun received the cand.mag. degree in mathematics, economics, and informatics, the cand.scient. degree in industrial and applied mathematics and informatics, and the dr.scient. degree in informatics–coding theory from the University of Bergen, Norway, in 1996, 1999, and 2002, respectively. He was a Lecturer in coding and cryptography at the University of Bergen, in 2002, and a Postdoctoral Researcher, from 2003 to 2006. From 2006 to 2010, he was a Lecturer and a Senior Lecturer in computer science at the University of Surrey, U.K., with research activities in multimedia security, including applications of coding theory and machine learning in steganalysis. He became a Professor of computing at Ålesund University College, in 2011, which merged into NTNU, in 2016. His current research interests include artificial intelligence, machine learning, applications in health care, epistemology, and the didactics of mathematics and engineering.