A Generalization of Synergistic Hybrid Feedback Control with Application to Maneuvering Control of Ships

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***Abstract*— This paper generalizes results on synergistic hybrid feedback control. Specifically, we propose a generalized definition of synergistic Lyapunov functions and feedbacks which allows the logic variable in traditional synergistic control, denoted the synergy variable, to be vector-valued and change during flows. Moreover, we introduce synergy gaps relative to components of product sets, which enables us to define jump conditions in the form of synergy gaps for different components of the synergy variable. In particular, this enables us to formulate existing hybrid output feedback control schemes within the synergistic control framework. Furthermore, we show that our generalized definition is amenable to backstepping. Finally, we give an example of how traditional synergistic control can be combined with ship maneuvering control with discrete path dynamics.**

I. Introduction

Continuous-time systems with non-contractible configuration manifolds cannot be globally asymptotically stabilized by continuous-time feedback [1]. This is referred to as a global topological obstruction to global asymptotic stability, and the most famous example is a mechanical system with rotational degrees of freedom constrained to the non-Euclidean space $\mathrm{SO}(3)$.

Global topological obstructions can be overcome by employing hybrid feedback. Specifically, the synergistic hybrid control framework has been applied to full state feedback trajectory tracking control of rigid body orientation in [2]. The work also presents a procedure to construct synergistic functions from modified trace functions, which were first employed for control of orientation in [3]. The paper [4] introduces a global synergistic tracking controller with integral action. A synergistic approach that does not utilize velocity measurements is presented in [5]. The work also provides a systematic procedure to construct synergistic functions by angular warping, an idea first introduced in [6]. Global hybrid tracking controllers for rigid body orientation explicitly exploiting the double cover property of the unit quaternion representation are presented in [7]. The aforementioned approach is utilized for trajectory tracking of translation-underactuated rigid vehicles in [8]. Synergistic control of rigid body planar and spherical orientation is presented in [9] and [10], respectively, while synergistic control barrier functions were introduced in [11]. The concept of synergistic Lyapunov function and feedback (SLFF) pairs were introduced in [12] and extended to adaptive control with matched uncertainties in [13], while an extension of the synergistic functions in [14] to a case where the logic variable is allowed to flow, is introduced in [15].

The main contribution of this paper is the extension of the SLFF definition from [12]. The proposed generalization allows the logic variable, now referred to as the synergy variable, to be vector-valued and possess flow dynamics. Moreover, since the synergy variable is vector-valued, we define synergy gaps relative to components of product sets. These synergy gaps enable us to define flow and jump sets and jump conditions in the form of synergy gaps for different components of the synergy variable. As a result, we can show that the output feedback control method for rigid-body orientation outlined in [16] is synergistic. The proposed generalization encompasses the results for $\mathrm{SO}(3)$ and $\mathrm{SE}(3)$ in [15], in which the scalar logic variable is also allowed to change during flows. However, our proposed framework also includes path-following control scenarios in which the path variable exhibits jump dynamics, such as instantaneously moving the desired state closer to the actual state. As a result, ship maneuvering control as outlined in [17] and [18] can be augmented with discrete path dynamics and combined with a traditional synergistic control approach such as [9] to ensure global asymptotic stability within the proposed framework.

This paper is organized as follows. In Section II, we extend the definition of SLFF pairs to SLFF triples, for which the synergy variables are allowed to have flow dynamics and be vector-valued. Moreover, we show how the hybrid feedback controller induced by an SLFF triple renders a given compact set globally pre-asymptotically stable. Section III introduces the notion of synergy gaps relative to components of product sets, which is a distinct feature of vector-valued synergy variables. Then, Section IV introduces a weaker notion of SLFF triples, and we show that if an affine control system admits a weak SLFF triple, then the same system augmented with an integrator at the input admits a (non-weak) SLFF triple. Section V presents a case study which combines the classical synergistic control approach in [6] using the synergistic Lyapunov functions in [9] with the ship maneuvering control in [17]. Finally, Section VI concludes the paper.

A. Preliminaries

The standard basis vectors in $\mathbb{R}^n$ are denoted $e_1, e_2, \ldots, e_n$. The special orthogonal group of dimension 2 is denoted $\mathrm{SO}(2)$ and defined by $\mathrm{SO}(2) \coloneqq \{R \in \mathbb{R}^{2\times 2} : \det R = 1, RR^\mathsf{T} =$

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$I\}$, where I is the identity matrix. A function $V : X \to \mathbb{R}_{\geq 0}$ is said to be proper if the set $\{x \in X : V(x) \leq c\}$ is compact for each $c > 0$. A function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is passive if it is continuous and $\varphi(x)^\mathsf{T} x \geq 0$ for all $x \in \mathbb{R}^n$. It is strongly passive if it is continuous and $\varphi(x)^\mathsf{T} x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. A double arrow denotes set-valued mappings, e.g. $G\colon X \rightrightarrows U$, where $X \subset \mathbb{R}^n$ is the domain of the mapping (the set where the mapping is not empty-valued), and $U \subset \mathbb{R}^m$ is its codomain (any set that contains all values G takes in its domain). The graph of G is the set defined as $\operatorname{gph} G \coloneqq \{(x, u) \in X \times U \colon u \in G(x)\}$. For a k-times continuously differentiable mapping $f : \mathbb{R} \to \mathbb{R}^n$, we denote the derivatives by $f', f'', f^{(3)}, \ldots, f^{(k)}$, with $f^{(0)} = f$. For a function $V : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ and a mapping $f : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^n$, we define the gradient operator with respect to the first argument of V and f by $\nabla_1 V(x, \theta) = \left(\frac{\partial V}{\partial x}(x, \theta)\right)^\mathsf{T}$ and $\nabla_1 f(x, \theta) \coloneqq (\nabla_1 f_1(x, \theta) \cdots \nabla_1 f_n(x, \theta))^\mathsf{T}$, respectively. The gradient with respect to the second argument is defined similarly. Moreover, let $x = (p, R) \in \mathrm{SE}(2)$, $v = (\zeta, \omega) \in \mathbb{R}^2 \times \mathbb{R}$, $V : \mathrm{SE}(2) \times \mathbb{R}^l \to \mathbb{R}$, $f : \mathrm{SE}(2) \times \mathbb{R}^l \to \mathbb{R}^n$ and define $\langle \mathrm{d}_1 V(x, \theta), v\rangle \coloneqq \langle\!\langle \nabla_1 V(x, \theta), x v_\wedge \rangle\!\rangle$ and $\mathrm{d}_1 f(x, \theta) \coloneqq (\mathrm{d}_1 f_1(x, \theta) \cdots \mathrm{d}_1 f_n(x, \theta))^\mathsf{T}$, where $\langle\!\langle a, b\rangle\!\rangle = \operatorname{tr}(a^\mathsf{T} b)$ is the Frobenius inner product and

$$x \coloneqq \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix}, v_\wedge \coloneqq \begin{pmatrix} S\omega & \zeta \\ 0 & 0 \end{pmatrix}, S \coloneqq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1}$$

$$\nabla_1 V(x, \theta) = \begin{pmatrix} \frac{\partial V}{\partial x_{11}} & \cdots & \frac{\partial V}{\partial x_{1j}} \\ \vdots & \ddots & \vdots \\ \frac{\partial V}{\partial x_{i1}} & \cdots & \frac{\partial V}{\partial x_{ij}} \end{pmatrix} (x, \theta). \tag{2}$$

Finally, the adjoint mappings $\mathrm{ad} : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ and $\mathrm{Ad} : \mathrm{SE}(2) \to \mathbb{R}^3 \times \mathbb{R}^3$ are defined by

$$\mathrm{ad}_v \coloneqq \begin{pmatrix} S\omega & -S\zeta \\ 0 & 0 \end{pmatrix}, \quad \mathrm{Ad}_x \coloneqq \begin{pmatrix} R & -Sp \\ 0 & 1 \end{pmatrix} \tag{3}$$

II. Synergistic Lyapunov Function and Feedback

This section extends the definition of SLFF pairs from [12] by augmenting the SLFF definition with a feedback representing the flow dynamics of the synergy variables. Moreover, we show that the hybrid feedback control law induced by an SLFF triple renders a given compact set globally pre-asymptotically stable.

Consider the system

$$\dot{x} = f(x, v) \quad (x, v) \in X \times \mathbb{R}^k \tag{4}$$

with state $x \in X$, input $v \in \mathbb{R}^k$ and where $f : X \times \mathbb{R}^k \to \mathbb{R}^n$. Our goal is to design generalized synergistic controllers with state $\theta \in \Theta \subset \mathbb{R}^m$ of the form

$$\begin{aligned} \dot{\theta} &= \nu(x, \theta) \quad (x, \theta) \in C \\ \theta^+ &\in G(x, \theta) \quad (x, \theta) \in D \\ v &= \kappa(x, \theta) \end{aligned} \tag{5}$$

where $C \subset X \times \Theta$, $D \subset X \times \Theta$, $\nu : X \times \Theta \to \mathbb{R}^m$, and $G : X \times \Theta \rightrightarrows \Theta$ are the flow set, jump set, flow map and jump map of the controller, respectively. The controller state θ is also referred to as the synergy variable. We assume the following throughout the paper.

Assumptions.

1) *$X \subset \mathbb{R}^n$ is closed;*
2) *$f : X \times \mathbb{R}^k \to \mathbb{R}^n$ is continuous;*
3) *$\Theta \subset \mathbb{R}^m$ is closed.*

In the following, we generalize the notion of a synergy gap of a nonnegative and proper function V introduced in [12]. In particular, we evaluate the minimum of V over a set $\Psi \subset \Theta$ which need not be finite (or even compact).

Definition 1. *Let $V : X \times \Theta \to \mathbb{R}_{\geq 0}$ be continuous and proper, and let $\Psi \subset \Theta$ be closed and nonempty. The synergy gap of V with respect to Ψ is defined as*

$$\mu_{V,\Psi}(x, \theta) \coloneqq V(x, \theta) - \min_{\psi \in \Psi} V(x, \psi). \tag{6}$$

The set-valued solution mapping associated with $\mu_{V,\Psi}$ is $G_{V,\Psi} : X \times \Theta \rightrightarrows \Theta$, defined as

$$G_{V,\Psi}(x, \theta) \coloneqq \{\psi \in \Psi : \mu_{V,\Psi}(x, \psi) = 0\}. \tag{7}$$

The fact that V is nonnegative, continuous, and proper is sufficient for its synergy gap relative to any nonempty and closed set $\Psi \subset \Theta$ to be continuous. Moreover, the associated solution mapping has the key properties it has in traditional synergistic control. Specifically, nonemptiness, outer semicontinuity, and local boundedness. Consequently, even when Ψ is not compact, the set of points where $\theta \mapsto V(x, \theta)$ attains its minimum on Ψ is compact for each $x \in X$.

Proposition 1. *The synergy gap $\mu_{V,\Psi}$ is continuous. The associated set-valued solution mapping $G_{V,\Psi}$ is nonempty-valued, outer semicontinuous, and locally bounded.*

Proof. The claims follow from [19, Corollary 7.42]. □

The following definition extends the notion of SLFF pairs from [12]. In addition to utilizing the generalized notion of synergy gap from Definition 1, we allow the synergy variable θ to flow.

Definition 2. *Let $\mathcal{A} \subset X \times \Theta$ be compact. A continuously differentiable function $V : X \times \Theta \mapsto \mathbb{R}_{\geq 0}$ and continuous functions $\kappa : X \times \Theta \to \mathbb{R}^k$ and $\nu : X \times \Theta \to \mathbb{R}^m$ define a synergistic Lyapunov function and feedback triple (V, κ, ν) relative to $\mathcal{A}$ with synergy gap relative to Ψ exceeding $\rho > 0$ for the system* (4) *if*

1) *V is proper and positive definite with respect to $\mathcal{A}$;*
2) *The closed loop system*

$$\begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} = \underbrace{\begin{pmatrix} f(x, \kappa(x, \theta)) \\ \nu(x, \theta) \end{pmatrix}}_{F_c(x,\theta)} \quad (x, \theta) \in X \times \Theta \tag{8}$$

satisfies

$$\langle \nabla V(x, \theta), F_c(x, \theta)\rangle \leq 0, \quad \forall (x, \theta) \in X \times \Theta; \tag{9}$$

3) *$\mu_{V,\Psi}(x, \theta) > \rho$ for each $(x, \theta) \in \mathcal{I} \setminus \mathcal{A}$, where $\mathcal{I}$ is the largest weakly invariant subset for the system*

$$\begin{aligned} \dot{x} &= f(x, \kappa(x, \theta)) \\ \dot{\theta} &= \nu(x, \theta) \end{aligned} \quad (x, \theta) \in \mathcal{E} \tag{10}$$

and

$$\mathcal{E} := \{(x,\theta) \in X \times \Theta : \langle \nabla V(x,\theta), F_c(x,\theta)\rangle = 0\}. \quad (11)$$

We remark that if $\nu(x,\theta) = 0$ for all $(x,\theta) \in X \times \Theta$ and $\Theta = \Psi$ is finite, then Definition 2 reduces to the definition of an SLFF pair given in [12]. If $\Theta = \mathbb{R}$, and $\Psi \subset \mathbb{R}$ is finite, then Definition 2 encompasses the class of potential functions recently introduced in [15].

Analogous to SLFF pairs [12, Theorem 7], the existence of an SLFF triple relative to $\mathcal{A}$ with synergy gap relative to Ψ exceeding $\rho > 0$ guarantees global pre-asymptotic stability of $\mathcal{A}$ for a synergistic closed loop system resulting from (4).

Proposition 2. *Let (V,κ,ν) be an SLFF triple relative to $\mathcal{A}$ with synergy gap relative to Ψ exceeding $\rho > 0$. Then $\mathcal{A}$ is globally pre-asymptotically stable for the system*

$$\left.\begin{aligned} \dot{x} &= f(x,\kappa(x,\theta)) \\ \dot{\theta} &= \nu(x,\theta) \end{aligned}\right\} \quad (x,\theta) \in C \\ \theta^+ \in G(x,\theta) \qquad (x,\theta) \in D \quad (12)$$

where

$$\begin{aligned} C &:= \{(x,\theta) \in X \times \Theta : \mu_{V,\Psi}(x,\theta) \le \rho\}, \\ D &:= \{(x,\theta) \in X \times \Theta : \mu_{V,\Psi}(x,\theta) \ge \rho\}, \quad (13) \\ G(x,\theta) &:= G_{V,\Psi}(x,\theta). \end{aligned}$$

Proof. The sets C and D are closed since $\mu_{V,\Psi}$ is continuous by Proposition 1. Moreover, the closed-loop flow map is continuous, and G is nonempty-valued, outer semicontinuous, and locally bounded by Proposition 1. Consequently, the system (12) satisfies the hybrid basic conditions [20, Assumption 6.5] and is therefore well posed. From the definition of the jump set and jump map in (13), V decreases strictly across jumps by at least ρ. Since V is proper and positive definite with respect to $\mathcal{A}$ by 1) of Definition 2 and V does not grow along solutions to the system (12) by 2) of Definition 2 and the nonincrease of V across jumps, it follows that $\mathcal{A}$ is stable and that all solutions are bounded. Since V must vanish in $\mathcal{A}$ by 1) of Definition 2, it holds that $\mu_{V,\Psi}$ vanishes in $\mathcal{A}$ as well. Consequently, $\mathcal{A} \subset C$. From 3) of Definition 2, it then follows that $\mathcal{I} \cap C \subset \mathcal{A}$. The invariance principle [20, Corollary 8.4] then guarantees that complete solutions converge to $\mathcal{A}$. It follows that $\mathcal{A}$ is globally pre-asymptotically stable. ☐

Completeness of maximal solutions (and global asymptotic stability of $\mathcal{A}$ for (12)) is guaranteed if, in addition to the conditions of Proposition 2, it also holds that

$$F_c(x,\theta) \in T_{X\times\Theta}(x,\theta) \quad (14)$$

for all (x,θ) such that $\mu_{V,\Psi}(x,\theta) < \rho$, where $T_{X\times\Theta}(x,\theta)$ is the tangent cone to $X \times \Theta$ at (x,θ). Indeed, $T_{X\times\Theta}(x,\theta) = T_C(x,\theta)$ at these points, and the claim follows from [20, Proposition 6.10]. It should also be remarked that $T_{X\times\Theta} \neq T_X \times T_\Theta$ in general. See [19, Chapter 6], and in particular Proposition 6.41, for further results on this matter.

III. Synergy Gaps Relative to Components of Product Sets

The control approach covered in Proposition 2 updates the whole synergy variable θ when the instantaneous synergy gap is equal to or exceeds the threshold ρ. This approach offers relatively little flexibility in shaping the jump sets. When Θ is a product set, one can formulate the synergy gap and associated solution mapping relative to the components of Θ. For simplicity, it is assumed that Θ comprises two components, although the approach outlined in this section can be further generalized.

Assumptions (continued)**.**

4) *$\Theta = \Theta_a \times \Theta_b$, where Θ_a and Θ_b are closed.*

We now adapt Definition 1 to exploit the additional structure of Θ induced by this assumption.

Definition 3. *Let $V : X \times \Theta \to \mathbb{R}_{\ge 0}$ be continuous and proper, and let $\Psi = \Psi_a \times \Psi_b$ such that $\Psi_a \subset \Theta_a$ and $\Psi_b \subset \Theta_b$ are nonempty and closed. The synergy gap of V with respect to Ψ_a is defined as*

$$\mu_{V,\Psi_a}(x,\theta) := V(x,\theta) - \min_{\psi_a \in \Psi_a} V(x,(\psi_a,\theta_b)). \quad (15)$$

The synergy gap of V with respect to Ψ_b is defined as

$$\mu_{V,\Psi_b}(x,\theta) := V(x,\theta) - \min_{\psi_b \in \Psi_b} V(x,(\theta_a,\psi_b)). \quad (16)$$

The set-valued solution mapping associated with μ_{V,Ψ_a}, $G_{V,\Psi_a} : X \times \Theta \rightrightarrows \Theta$ is

$$G_{V,\Psi_a}(x,\theta) := \{\psi_a \in \Psi_a : \mu_{V,\Psi_a}(x,(\psi_a,\theta_b)) = 0\} \times \{\theta_b\}. (17)$$

The objects introduced in Definition 3 have similar properties as the ones introduced in Definition 1.

Proposition 3. *The synergy gaps μ_{V,Ψ_a} and μ_{V,Ψ_b} are continuous. The set-valued solution mapping G_{V,Ψ_a} is nonempty-valued, outer semicontinuous, and locally bounded.*

Proof. Apply [19, Corollary 7.42] with (x,θ_b) as parameters and θ_a as optimization variable to show the claims for μ_{V,Ψ_a} and G_{V,Ψ_a}. Continuity of μ_{V,Ψ_b} is shown similarly. ☐

Consequently, we may specialize the notion of an SLFF triple to the case where Θ is product set.

Definition 4. *Let $\mathcal{A} \subset X \times \Theta$ be compact, and (V,κ,ν) satisfy 1) and 2) in Definition 2 for the system* (4)*. We say that (V,κ,ν) is a synergistic Lyapunov function and feedback triple relative to $\mathcal{A}$ with synergy gap relative to Ψ_a exceeding $\rho_a > 0$ if*

3a) *$\mu_{V,\Psi_a}(x,\theta) > \rho_a$ for each $(x,\theta) \in \mathcal{I} \setminus \mathcal{A}$.*

We say that (V,κ,ν) is a synergistic Lyapunov function and feedback triple relative to $\mathcal{A}$ with synergy gap relative to (Ψ_a,Ψ_b) exceeding (ρ_a,ρ_b) with $\rho_a,\rho_b > 0$ if

3b) *$\mu_{V,\Psi_a}(x,\theta) > \rho_a$ or $\mu_{V,\Psi_b}(x,\theta) > \rho_b$ for each $(x,\theta) \in \mathcal{I} \setminus \mathcal{A}$. Moreover, there exist $(x,\theta) \in \mathcal{I} \setminus \mathcal{A}$ such that $\mu_{V,\Psi_a}(x,\theta) < \rho_a$ and $(x,\theta) \in \mathcal{I} \setminus \mathcal{A}$ such that $\mu_{V,\Psi_b}(x,\theta) < \rho_b$.*

In both cases, $\mathcal{I}$ is defined as in Definition 2.

It is clear that if (V, κ, ν) has synergy gap relative to Ψ_a exceeding $\rho_a > 0$, then it has synergy gap relative to Ψ exceeding ρ_a. If instead (V, κ, ν) has a synergy gap relative to to (Ψ_a, Ψ_b) exceeding (ρ_a, ρ_b), with $\rho_a, \rho_b > 0$, then it has a synergy gap relative to Ψ exceeding $\min(\rho_a, \rho_b) > 0$. The last part of item 3b) ensures that (V, κ, ν) is not an SLFF triple relative to $\mathcal{A}$ with synergy gap relative to Ψ_a or Ψ_b, and hence that ρ_a and ρ_b are well-defined.

A. Optional Jumps

When (V, κ, ν) is an SLFF triple relative to $\mathcal{A}$ with synergy gap relative to Ψ_a exceeding $\rho_a > 0$, it is not necessary to update θ_b to avoid the invariant sets where solutions may get stuck. Jumping θ_b may nonetheless increase the performance of the closed-loop system. We therefore define a closed-loop system in which jumps of θ_b are optional.

Proposition 4. *Let (V, κ, ν) be a synergistic Lyapunov function and feedback triple relative to $\mathcal{A}$ with synergy gap relative to Ψ_a exceeding $\rho_a > 0$. Then $\mathcal{A}$ is globally pre-asymptotically stable for the system* (12) *with*

$$\begin{aligned} C &:= \{(x,\theta) \in X \times \Theta : \mu_{V,\Psi_a}(x,\theta) \le \rho_a\}, \\ D &:= \left\{ (x,\theta) \in X \times \Theta : \begin{array}{l} \mu_{V,\Psi_a}(x,\theta) \ge \rho_a \\ \text{or } \mu_{V,\Psi}(x,\theta) \ge \rho \end{array} \right\}, \\ G(x,\theta) &:= \begin{cases} G_{V,\Psi_a}(x,\theta), & \begin{array}{l} \mu_{V,\Psi_a}(x,\theta) \ge \rho_a \\ \text{and } \mu_{V,\Psi}(x,\theta) < \rho, \end{array} \\ (G_{V,\Psi_a} \cup G_{V,\Psi})(x,\theta), & \begin{array}{l} \mu_{V,\Psi_a}(x,\theta) \ge \rho_a \\ \text{and } \mu_{V,\Psi}(x,\theta) \ge \rho, \end{array} \\ G_{V,\Psi}(x,\theta), & \begin{array}{l} \mu_{V,\Psi_a}(x,\theta) < \rho_a \\ \text{and } \mu_{V,\Psi}(x,\theta) \ge \rho, \end{array} \\ \emptyset & \text{otherwise,} \end{cases} \end{aligned} \tag{18}$$

where $\rho \ge \rho_a$.

Proof. It is clear that C and the sets

$$D_{\Psi_a} := \{(x,\theta) \in X \times \Theta : \mu_{V,\Psi_a}(x,\theta) \ge \rho_a\} \tag{19}$$
$$D_{\Psi} := \{(x,\theta) \in X \times \Theta : \mu_{V,\Psi}(x,\theta) \ge \rho\} \tag{20}$$

are closed since μ_{V,Ψ_a} and $\mu_{V,\Psi}$ are continuous. Therefore, $D = D_{\Psi_a} \cup D_{\Psi}$ is closed. The closed-loop flow map is continuous on $X \times \Theta$. We know that G_{V,Ψ_a} and $G_{V,\Psi}$ are nonempty-valued, outer semicontinuous, and locally bounded. Denote then by $\widetilde{G}_{V,\Psi_a}$ and $\widetilde{G}_{V,\Psi}$ the restrictions of G_{V,Ψ_a} and $G_{V,\Psi}$ to D_{Ψ_a} and D_{Ψ}, respectively. These restrictions are also outer semicontinuous and locally bounded. Now, G is defined such that $\operatorname{gph} G = \operatorname{gph} \widetilde{G}_{V,\Psi_a} \cup \operatorname{gph} \widetilde{G}_{V,\Psi}$. Thus, G is nonempty-valued on D. Since outer semicontinuity of a set-valued mapping is equivalent to its graph being closed, it also follows that G is outer semicontinuous. Moreover, the union of two locally bounded set-valued mappings is locally bounded. Consequently, G is locally bounded. Hence, the closed loop system (12) with data defined by (18) satisfies the hybrid basic conditions. The rest of the proof proceeds as the proof of Proposition 2, with the strict decrease of V across jumps now being at least ρ_a. □

Completeness of maximal solutions to the closed-loop system with data (18) is guaranteed if the tangent cone condition (14) holds for all (x, θ) such that $\mu_{V,\Psi_a}(x,\theta) < \rho_a$. In this case, the system always admits complete solutions over the course of which θ_b does not jump.

B. Independently Triggered Jumps

The following proposition introduces the concept of independently triggered jumps, where both components of θ jump when either of their jump conditions are met.

Proposition 5. *Let (V, κ, ν) be a synergistic Lyapunov function and feedback triple relative to $\mathcal{A}$ with synergy gap relative to (Ψ_a, Ψ_b) exceeding (ρ_a, ρ_b), with $\rho_a, \rho_b > 0$. Then $\mathcal{A}$ is globally pre-asymptotically stable for the system* (12) *with*

$$\begin{aligned} C &:= \left\{ (x,\theta) \in X \times \Theta : \begin{array}{l} \mu_{V,\Psi_a}(x,\theta) \le \rho_a \\ \text{and } \mu_{V,\Psi_b}(x,\theta) \le \rho_b \end{array} \right\}, \\ D &:= \left\{ (x,\theta) \in X \times \Theta : \begin{array}{l} \mu_{V,\Psi_a}(x,\theta) \ge \rho_a \\ \text{or } \mu_{V,\Psi_b}(x,\theta) \ge \rho_b \end{array} \right\}, \\ G(x,\theta) &:= G_{V,\Psi}(x,\theta). \end{aligned} \tag{21}$$

The proof of Proposition 5 is very similar to the proofs of Proposition 2 and Proposition 4 and is therefore omitted. An example where independently triggered switching is used is furnished by the quaternion output feedback control scheme for rigid-body orientation in [7, Section V-B]. In this work, θ_a corresponds to a traditional synergy variable for a feedback controller, and θ_b corresponds to a traditional synergy variable for an observer, and $\nu(x,\theta) = 0$ for all $(x, \theta) \in X \times \Theta$.

IV. Backstepping

This section begins by introducing a weaker notion of SLFF triples for affine control systems. Then, given a system that admits a weak SLFF triple, we construct a (non-weak) SLFF triple for the same system augmented with an integrator at the input.

By assuming that (4) is affine in the control input v, we obtain the system

$$\dot{x} = f_0(x) + g_0(x)v \quad (x, v) \in X \times \mathbb{R}^k \tag{22}$$

Definition 5. *Let $\mathcal{A} \subset X \times \Theta$ be compact. A continuously differentiable function $V : X \times \Theta \mapsto \mathbb{R}_{\ge 0}$ and continuous functions $\kappa : X \times \Theta \to \mathbb{R}^k$ and $\nu : X \times \Theta \to \mathbb{R}^m$ define a weak synergistic Lyapunov function and feedback triple (V, κ, ν) relative to $\mathcal{A}$ with a weak synergy gap relative to Ψ exceeding $\rho > 0$ for the system* (22) *if*

1) *V is proper and positive definite with respect to $\mathcal{A}$;*
2) *The closed loop system*

$$\begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} = \underbrace{\begin{pmatrix} f_0(x) + g_0(x)\kappa(x,\theta) \\ \nu(x,\theta) \end{pmatrix}}_{F_0(x,\theta)} \quad (x,\theta) \in X \times \Theta \tag{23}$$

satisfies

$$\langle \nabla V(x,\theta), F_0(x,\theta) \rangle \le 0, \quad \forall (x,\theta) \in X \times \Theta; \tag{24}$$

3) *$\mu_{V,\Psi}(x,\theta) > \rho$ for each $(x,\theta) \in \mathcal{I} \setminus \mathcal{A}$, where $\mathcal{I}$ is the largest weakly invariant subset for the system*

$$\left.\begin{aligned}\dot{x} &= f_0(x) + g_0(x)\kappa(x,\theta)\\ \dot{\theta} &= \nu(x,\theta)\end{aligned}\right\} (x,\theta) \in \mathcal{E} \cap \mathcal{W} \quad (25)$$

where $\mathcal{E}$ is given in Definition 2, and

$$\mathcal{W} := \{(x,q) \in X \times \Theta : g_0(x)^\top \nabla_1 V(x,\theta) = 0\}. \quad (26)$$

Augmenting the system (22) with an integrator at the input results in the control system

$$\dot{z} = f_1(z) + g_1(z)u \quad (z,u) \in Z \times \mathbb{R}^k \quad (27)$$

where $z = (x,v) \in Z := X \times \mathbb{R}^k$, $u \in \mathbb{R}^k$ is the control input and

$$f_1(z) = \begin{pmatrix} f_0(x) + g_0(x)v \\ 0 \end{pmatrix}, g_1(z) = \begin{pmatrix} 0 \\ I \end{pmatrix}. \quad (28)$$

Now, let (V_0, κ_0, ν_0) be a weak SLFF triple relative to the compact set $\mathcal{A}_0 \subset X \times \Theta$, define the set

$$\mathcal{A}_1 = \{(z,\theta) \in Z \times \Theta : (x,\theta) \in \mathcal{A}_0, v = \kappa_0(x,\theta)\}, \quad (29)$$

and consider the following SLFF triple

$$V_1(z,\theta) = V_0(x,\theta) + \frac{1}{2}|v - \kappa_0(x,\theta)|^2_\Gamma, \quad (30a)$$

$$\begin{aligned}\kappa_1(z,\theta) = \nabla_1\kappa_0(x,\theta)\,(f_0(x) + g_0(x)v)&\\ + \nabla_2\kappa_0(x,\theta)\nu_0(x,\theta) - \gamma_1(v - \kappa_0(x,\theta))&\\ - \Gamma^{-1}g_0(x)^\top\nabla_1 V_0(x,\theta),&\end{aligned} \quad (30b)$$

$$\nu_1(z,\theta) = \nu_0(x,\theta) - \vartheta_1(\nabla_2 V_1(z,\theta)), \quad (30c)$$

where $\Gamma \in \mathbb{R}^{k\times k}$ is positive definite, and $\gamma_1 : \mathbb{R}^k \to \mathbb{R}^k$ and $\vartheta_1 : \mathbb{R}^m \to \mathbb{R}^m$ are strongly passive and passive functions, respectively. The following proposition establishes that (V_1, κ_1, ν_1) is an SLFF triple for the system (27) with synergy gap exceeding $\rho > 0$ relative to Ψ.

Proposition 6. *If (V_0, κ_0, ν_0) is a weak synergistic Lyapunov function and feedback triple for the system (22) relative to $\mathcal{A}_0$, with a weak synergy gap relative to Ψ exceeding $\rho > 0$, then (V_1, κ_1, ν_1) is a (non-weak) synergistic Lyapunov function and feedback triple for the system (27) relative to $\mathcal{A}_1$ with a (non-weak) synergy gap relative to Ψ exceeding $\rho > 0$.*

Proof. The derivative of V_1 along the solutions of (22) is

$$\begin{aligned}\dot{V}_1 &= \langle \nabla_1 V_1(z,\theta), f_1(z) + g_1(z)\kappa_1(z,\theta)\rangle\\ &\quad + \langle \nabla_2 V_1(z,\theta), \nu_1(z,\theta)\rangle\\ &= \langle \nabla_1 V_0(x,\theta), f_0(x) + g_0(x)\kappa_0(x,\theta)\rangle\\ &\quad - \langle v - \kappa_0(x,\theta), \Gamma\gamma_1(v - \kappa_0(x,\theta))\rangle\\ &\quad - \langle v - \kappa_0(x,\theta), \Gamma\nabla_2\kappa_0(x,\theta)(\nu_1(z,\theta) - \nu_0(x,\theta))\rangle\\ &\quad + \langle \nabla_2 V_0(x,\theta), \nu_1(z,\theta)\rangle\\ &= \langle \nabla_1 V_0(x,\theta), f_0(x) + g_0(x)\kappa_0(x,\theta)\rangle\\ &\quad - \langle v - \kappa_0(x,\theta), \Gamma\gamma_1(v - \kappa_0(x,\theta))\rangle\\ &\quad + \langle \nabla_2 V_0(x,\theta), \nu_0(x,\theta)\rangle\\ &\quad - \langle \nabla_2 V_1(z,\theta), \vartheta_1(\nabla_2 V_1(z,\theta))\rangle \le 0.\end{aligned} \quad (31)$$

Define $\mathcal{E}_0, \mathcal{W}_0$ and $\mathcal{E}_1, \mathcal{W}_1$ according to (11) and (26) for the systems (22) and (27), respectively. It follows from (31) that

$$\begin{aligned}\mathcal{E}_1 = \{(z,\theta) \in X \times \Theta : (x,\theta) \in \mathcal{E}_0,&\\ v = \kappa_0(x,\theta), \vartheta_1(\nabla_2 V_1(z,\theta)) = 0\} \subset \mathcal{W}_1.&\end{aligned} \quad (32)$$

Let $\mathcal{I}_1 \subset \mathcal{E}_1$ denote the largest weakly invariant subset for the system

$$\left.\begin{aligned}\dot{z} &= f_1(z) + g_1(z)\kappa_1(z,\theta)\\ \dot{\theta} &= \nu_1(z,\theta)\end{aligned}\right\} (z,\theta) \in \mathcal{E}_1 \quad (33)$$

It follows that

$$\begin{aligned}\mathcal{I}_1 = \big\{(z,\theta) \in Z \times \Theta : (x,\theta) \in \Omega_0, v = \kappa_0(x,\theta),&\\ \vartheta_1(\nabla_2 V_1(z,\theta)) = 0\big\}.&\end{aligned} \quad (34)$$

From Definition 1 and 3) in Definition 5 it holds that

$$\begin{aligned}\mu_{V_1,\Psi} &\ge \mu_{V_0,\Psi}(x,\theta) + \frac{1}{2}|v - \kappa_0(x,\theta)|^2_\Gamma\\ &\quad - \min_{\psi\in\Psi}\frac{1}{2}|v - \kappa_0(x,\psi)|^2_\Gamma\\ &\ge \mu_{V_0,\Psi}(x,\theta) > \rho.\end{aligned}$$

Consequently, (V_1, κ_1, ν_1) is an SLFF triple with synergy gap relative to Ψ exceeding $\rho > 0$. □

V. Synergistic Maneuvering for Ships

In this section, the proposed theory is exemplified by combining the traditional synergistic control approach of [6], [9] with the ship maneuvering control of [17], [18], where we augment the path variable with jump dynamics. The configuration space of a ship can be reasonably described by $\mathrm{SE}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$. Configurations of the ship are then represented as $x = (p, R)$, where $p \in \mathbb{R}^2$ represents the ship position and $R \in \mathrm{SO}(2)$ represents the ship heading.

The desired position of the ship is described in terms of a sufficiently smooth planar path.

Definition 6. *A C^r-path in $X \subset \mathbb{R}^n$ is a C^r-mapping $\eta : [0,1] \to X$. If $r \ge 1$, we say that a C^r-path is regular if $\eta'(s) \neq 0$ for all $s \in [0,1]$.*

Given a regular C^3-path η in $\mathbb{R}^2$, we synthesize a C^2-path in $\mathrm{SE}(2)$ by requiring that the heading of the ship is tangential to the path. Such a path has the form $s \mapsto (p_d(s), R_d(s))$, where

$$\begin{aligned}p_d(s) &:= \eta(s)\\ R_d(s) &:= \frac{1}{|\eta'(s)|}\begin{pmatrix}\eta'(s) & S\eta'(s)\end{pmatrix}.\end{aligned} \quad (35)$$

A desired speed assignment for $\dot{s}$ along the path, $u_d : [0,1] \to \mathbb{R}$, is chosen as

$$u_d(s) := \frac{U_d(s)}{|p_d'(s)|}, \quad (36)$$

where $U_d : [0,1] \to \mathbb{R}$ is a continuously differentiable signed desired ship speed along the path. In particular, u_d is defined such that if $\dot{s} = u_d(s)$, then $\dot{p}_d(s) = \frac{p_d'(s)}{|p_d'(s)|}U_d(s)$. A two times continuously differentiable path in the configuration space $x_d : [0,1] \to \mathrm{SE}(2)$ can now be defined as $x_d(s) \coloneqq (p_d(s), R_d(s))$.

The general ship maneuvering problem comprises a geometric task that represents convergence to the desired path, and a dynamic task that represents the attainment of the speed assignment u_d on this path. It was presented in [18] as follows under the assumption of maximal solutions being complete.

Problem Statement (Maneuvering Problem [18])**.**

- ***Geometric Task:*** *Force the position and heading of the ship to converge to the desired path,*

$$\lim_{(t+j)\to\infty} \|x(t,j) - x_d(s(t,j))\| = 0. \tag{37}$$

- ***Dynamic Task:*** *Force the path speed to converge to the desired speed assignment:*

$$\lim_{(t+j)\to\infty} |\dot{s}(t,j) - u_d(s(t,j))| = 0. \tag{38}$$

We denote by $v = (\zeta, \omega) \in \mathbb{R}^3$ the velocity of the ship, where $\zeta \in \mathbb{R}^2$ is its linear velocity and $\omega \in \mathbb{R}$ is its angular velocity. A model for the ship kinematics and dynamics is [21, Chapter 6.5]

$$\left.\begin{aligned} \dot{x} &= x v_\wedge \\ \dot{v} &= -\gamma(v) + M^{-1}(d(v) + u) \end{aligned}\right\} (x,v) \in \mathrm{SE}(2) \times \mathbb{R}^3, \tag{39}$$

where $M = M^\top > 0$ is the ship inertia tensor (including hydrodynamic inertia), $\gamma : \mathbb{R}^3 \to \mathbb{R}^3$ describes the Coriolis and centripetal accelerations associated with M, $d : \mathbb{R}^3 \to \mathbb{R}^3$ describes the hydrodynamic drag forces acting on the ship, and u are idealized input forces produced by the actuators.

A. Backstepping Controller

We set $X = \mathrm{SE}(2)$ and $\Theta = \Theta_a \times \Theta_b$, where $\Theta_a = \{-1, 1\}$, $\Theta_b = [0,1]$ and $\theta = (\theta_a, \theta_b) = (q, s)$. In particular, q is a classical synergistic logic variable and s is a path variable utilized in the ship maneuvering control problem. Then, the kinematics of the ship and the flow of q and s may be cast as a system of the form (22),

$$\dot{x} = x v_\wedge \quad (x, v) \in \mathrm{SE}(2) \times \mathbb{R}^3. \tag{40}$$

The set $\mathcal{A}_0 \subset X \times \Theta$ is now chosen as

$$\mathcal{A}_0 = \{(x, \theta) \in X \times \Theta : x = x_d(s)\}. \tag{41}$$

Compactness of $\mathcal{A}_0$ holds because the mapping $(q, s) \mapsto x_d(s)$ is continuous and Θ is compact.

We now introduce a synergistic potential function which is similar to [9] for the heading control of the ship. In particular, let $P : \mathrm{SO}(2) \times [0,1] \to \mathbb{R}$ and, with $\delta > 0$, the mapping $T : \mathrm{SO}(2) \times \Theta \to \mathrm{SO}(2)$

$$P(R, s) \coloneqq (1 - \langle e_1, R_d(s)^\top R e_1 \rangle), \tag{42}$$

$$T(R, \theta) \coloneqq \exp(\delta q P(R, s) S) R_d(s)^\top R. \tag{43}$$

Let $k_0 > 0$ and let $K_0 = K_0^\top$ be a positive definite matrix. Then, (V_0, κ_0, ν_0) defined as

$$V_0(x, \theta) = \frac{1}{2} |R_d^\top (p - p_d(s))|^2_{K_0} + k_0 P(T(R, \theta), s) \tag{44a}$$

$$\begin{aligned} \kappa_0(x, \theta) = {} & \mathrm{Ad}^{-1}_{x_d(s)^{-1}x} (x_d(s)^{-1} x_d'(s))_\vee u_d(s) \\ & - K \, \mathrm{d}_1 V_0(x, \theta) \end{aligned} \tag{44b}$$

$$\nu_0(x, \theta) = \begin{pmatrix} 0 \\ u_d(s) \end{pmatrix}, \tag{44c}$$

where $K = K^\top$ is a positive definite matrix, is an SLFF triple for (40) with synergy gap relative to $\{-1, 1\}$ exceeding $\frac{1}{2}$.

We now augment (40) with the ship dynamics

$$\left.\begin{aligned} \dot{x} &= x v_\wedge \\ \dot{v} &= -\gamma(v) + M^{-1}(d(v) + u) \end{aligned}\right\} (z, u) \in (\mathrm{SE}(2) \times \mathbb{R}^3) \times \mathbb{R}^3, \tag{45}$$

and define

$$\mathcal{A}_1 = \{(x, v, \theta) : (x, \theta) \in \mathcal{A}_0, v = \kappa_0(x, \theta)\}. \tag{46}$$

It then follows directly from Proposition 6 that

$$V_1(z, \theta) = V_0(x, \theta) + \frac{1}{2} |v - \kappa_0(x, \theta)|^2_M, \tag{47a}$$

$$\begin{aligned} \kappa_1(z, \theta) = {} & M \, \mathrm{d}_1 \kappa_0(x, \theta) v + M \nabla_2 \kappa_0(x, \theta) \nu_0(x, \theta) \\ & + M \gamma(v) - d(v) \\ & - \gamma_1 (v - \kappa_0(x, \theta)) \\ & - \mathrm{d}_1 V_0(x, \theta), \end{aligned} \tag{47b}$$

$$\nu_1(z, \theta) = \nu_0(x, \theta), \tag{47c}$$

is an SLFF triple for the system (45) relative to $\mathcal{A}_1$ with synergy gap relative to $\{-1, 1\}$ exceeding $\frac{1}{2}$. Consequently, the synergistic controller

$$\dot{\theta} = \nu_1(z, \theta) \qquad (z, \theta) \in C, \tag{48}$$

$$\theta^+ \in G(z, \theta) \qquad (z, \theta) \in D, \tag{49}$$

$$u = \kappa_1(z, \theta), \tag{50}$$

where (C, D, G) are given by (18), renders $\mathcal{A}_1$ globally pre-asymptotically stable for the resulting closed-loop system by Proposition 4. Moreover, if $u_d(s) \in T_{[0,1]}(s)$ for all $s \in [0,1]$, then all maximal solutions are complete and $\mathcal{A}_1$ is globally asymptotically stable for the resulting closed-loop system, which implies that the problem statement is solved.

B. Simulations

Simulation results are presented in Figs. 1 to 5. The model parameters can be found in [22]. In the simulations, we have chosen $\delta = 0.1$, $\rho_a = \delta k_0$, $\rho = 1.2 \rho_a$, $k_0 = 5$, $K_0 = 5 I_2$, $K = 0.05 I_3$ and $\gamma_1 = \mathrm{diag}(10, 10, 7)$. The chosen path is given by $p_d(s) \coloneqq 5(\cos(\pi s), \sin(\pi s))$. The ship is initialized at $p = (5, 2)$ with an initial heading of $\psi = -85°$, an initial velocity of $v = 0$ and a desired speed of $U_d = 0.3\,\mathrm{m/s}$.

From Fig. 1 we observe that the position references are successfully tracked after an initial transient phase. An optional jump is immediately triggered such that q is mapped to -1 and s is mapped to approximately 0.08. An optional

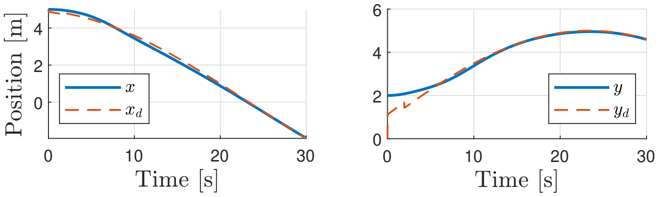

Fig. 1. The position $p = (x, y)$ and desired position $p_d = (x_d, y_d)$.

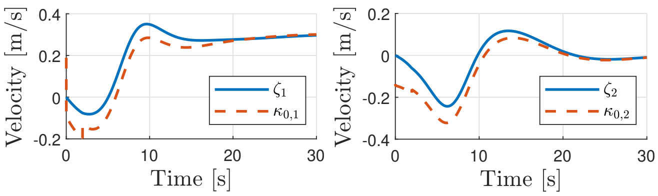

Fig. 2. The body linear velocity ζ_1 and ζ_2 and the first and second component of κ_0.

jump is triggered around $t \approx 2\,\mathrm{s}$ as seen in Fig. 1. The error in the x-direction is slightly decreased while the error in the y-direction is slightly increased. Moreover, from Fig. 2 we note that the difference between ζ_1 and $\kappa_{0,1}$ decreases over the jump in s. In Fig. 4, we observe that s is decreased over the jump, while q remains the same. Moreover, from Fig. 5, we observe a discontinuity in u_2 at the time of the jump.

VI. Conclusions

In this paper, we have generalized the definition of synergistic Lyapunov functions and feedbacks to allow the traditional logic variable of synergistic control to be vector-valued and change during flows. Since the logic variable is allowed to be vector-valued, we have introduced the notion of synergy gaps relative to components of product sets, which enables existing hybrid output feedback control laws to be reformulated within the synergistic framework. Furthermore, we have shown that the properties of an SLFF triple are preserved through integrator backstepping. Finally, we have given an example in which a classical synergistic control apporach is combined with a ship maneuvering control approach to enable discrete path dynamics and global asymptotic stability properties.

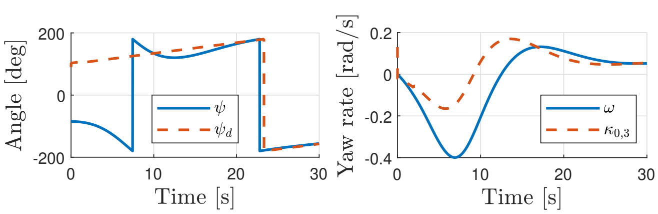

Fig. 3. The heading angle $\psi = \operatorname{atan2}(R_{21}, R_{11})$, desired heading angle $\psi_d = \operatorname{atan2}(R_{d,21}, R_{d,11})$, angular velocity ω and the third component of κ_0.

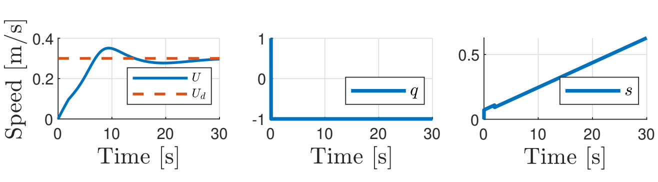

Fig. 4. The speed $U = (\zeta_1^2 + \zeta_2^2)^{\frac{1}{2}}$, desired speed U_d and synergistic variables q and s.

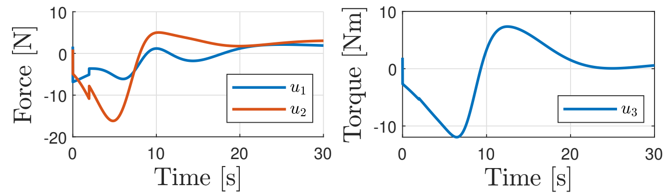

Fig. 5. The control forces and moment u.

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