Proceedings

of the 45th conference of the international group for the psychology of mathematics education

July 18-23, 2022

EDITORS
Ceneida Fernández / Salvador Llinares
Ángel Gutiérrez / Núria Planas

VOLUME 1

Plenary Lectures, Plenary Panel, Research Forums, Working Groups, Seminar, National Presentation

PME 45
Alicante · Spain · 2022

Universitat d'Alacant
Universidad de Alicante

Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education

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Editors:

Ceneida Fernández

Salvador Llinares

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Volume 1

Plenary Lectures, Plenary Panel, Research Forums, Working Groups, Seminar, National Presentation

Psychology in Mathematics Education (PME) Conference
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PREFACE

We are pleased to welcome you to the 45th Annual Conference of the International Group for the Psychology of Mathematics Education. PME is one of the most important international conferences in mathematics education and draws educators, researchers, and mathematicians from all over the world. This year we have 390 presentations from 43 different countries, and around 530 people are expected to attend the conference.

PME 45 is hosted by the University of Alicante (Spain). This is the second time a PME conference has been organized in Spain since PME 20 was held at the University of Valencia in 1996. The campus of the University of Alicante stands out for its high quality and environmentally friendly urban design and its rich and varied green areas. Alicante, labelled as the "City of Light" and located on the Spanish Mediterranean coast, is an astonishing city full of life, contrasts and beauty. Participants can enjoy and experience the most varied landscapes and beaches. Alicante has unbeatable weather with more than 300 sunny days a year. The conference takes place during the Spanish summer, so participants must prepare for daily high temperatures of around 30°C (86ºF).

The theme of PME 45 is "Mathematics education research supporting practice: empowering the future". This theme stresses that practice supported by research is an important tool for improving the quality of mathematics teaching-learning since it allows the empowerment of the individuals and the groups. Mathematics education research provides opportunities for educators to exercise informed freedom of choice when making decisions in teaching and learning. The theme also has a special meaning for the mathematics education research community in the host country because the history of mathematics education in Spain was tightly knitted with the idea of empowering the future of the country by linking practice and research. In Spain, during the 80s, the concern about mathematics teaching and learning was linked to the care for improving practice. Several groups emerged at universities and schools which proposed and used innovation as a tool for effectively empowering teaching. Years later, mathematics education research broadened our understanding of mathematics teaching and learning. Multiple papers submitted to PME 45 indicate that this theme remains relevant in many parts of the world. We look forward to discussions across countries that can help us to highlight shared goals.

We are delighted to have plenary speakers and panellists from a wide range of countries representing different educational contexts and theoretical perspectives. We believe this diversity will strengthen our discussions through the conference on the theme. The program of PME 45 also includes different types of sessions as in previous PME conferences: Research Reports, Oral Communications, and Poster Presentations at the individual level, and Research Forums, Working Groups, and a Seminar at the group level. A National Presentation also provides insights into mathematics education research in Spain.

The four volumes of the proceedings are organized according to types of presentations. Volume 1 contains the Plenary Lectures, Plenary Panel, Research Forums, Working Groups, Seminar, and National Presentation. Volumes 2 and 3 contain Research Reports, while Volume 4 consists of Research Reports, Oral Communications and Poster Presentations.

The organization of PME 45 is a collaborative effort involving colleagues from the University of Alicante and other Spanish Universities. The conference is organized with the support of three committees: The International Program Committee for PME 45, the International Committee of PME together with the PME Administrative Manager, and the Local Organizing Committee. We acknowledge the support and effort of all involved in making the conference possible and thank all the people who have given their time and expertise. Finally, we also thank each PME participant for making your journey to PME 45 in Alicante and for your contributions to this conference.

We are also grateful for the support received to organize this conference from: Universidad de Alicante; Vicerrectorado de Infraestructuras, Sostenibilidad y Seguridad Laboral (Universidad de Alicante); Vicerrectorado de Investigación (Universidad de Alicante); Facultad de Educación (Universidad de Alicante); Departamento de Innovación y Formación Didáctica (Universidad de Alicante), vistitelche.com; Ayuntamiento de Benidorm; Ayuntamiento de Villajoyosa; and Ayuntamiento de Alicante.

After two years without an onsite PME conference because of the COVID-19 pandemic, we aim to make the PME 45 meeting scientifically and socially successful. Scientifically, we hope you can depart with new ideas and research directions. Socially, we hope this meeting provides you with new friendships and opportunities for future collaborations. We wish you an exciting, informative and inspiring participation.

Ceneida Fernández and Salvador Llinares

PME 45 International Program Committee Chairs

TABLE OF CONTENTS
VOLUME 1

PLENARY PANEL

RESEARCH FORUMS

WORKING GROUPS

TABLE OF CONTENTS
VOLUME 2

RESEARCH REPORTS (A-H)

TABLE OF CONTENTS

VOLUME 3

TABLE OF CONTENTS
VOLUME 4

RESEARCH REPORTS (Si – Z)

ORAL COMMUNICATIONS

POSTER PRESENTATIONS

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

HISTORY OF PME

The International Group for the Psychology of Mathematics Education (IGPME) is an autonomous body, governed as provided for in the constitution. It is an official subgroup of the International Commission for Mathematical Instruction (ICMI) and came into existence at the Third International Congress on Mathematics Education (ICME 3) held in Karlsruhe, Germany in 1976. Its former presidents have been:

Efraim Fischbein, Israel	Pearla Nesher, Israel
Stephen Lerman, UK	Fou-Lai Lin, Taiwan
Richard R. Skemp, UK	Nicolas Balacheff, France
Gilah Leder, Australia	João Filipe Matos, Portugal
Gerard Vergnaud, France	Kathleen Hart, UK
Rina Hershkowitz, Israel	Barbara Jaworski, UK
Kevin F. Collis, Australia	Carolyn Kieran, Canada
Chris Breen, South Africa	Peter Liljedahl, Canada

The current president is **Markku Hannula** (Finland). The president-elect is **Wim Van Dooren** (Belgium).

THE CONSTITUTION OF PME

The constitution of PME was adopted by the Annual General Meeting (AGM) on August 17, 1980 and changed at the AGM on July 24, 1987, on August 10, 1992, on August 2, 1994, on July 18, 1997, on July 14, 2005, and on July 21, 2012. PME decided to seek charitable organization status under UK law and the new constitution related to this change was accepted by the Annual General Meetings in 2018 and 2019. The name of the Charitable Incorporated Organisation (“the CIO”) is the international Group for the Psychology of Mathematics Education. The objects of the CIO are to advance the field of mathematics education for the public benefit by:

- Promoting and stimulating research.
- Organizing regular educational conferences around the world.
- Supporting regional workshops around the world in general, and in under-represented regions of the world in particular.
- Collaborating with organizations with similar aims.
- Facilitating cross-disciplinary discussion and the sharing of information and research with an international emphasis.

- Helping scholars from different parts of the world establish collaborative networks to further our collective understanding of how to improve mathematics education in their respective countries.
- Providing grants to help bring scholars from under-represented regions of the world to our annual meeting.
- Disseminating our research for the benefit of improving mathematics education at the classroom, school, district, and national levels.
- Disseminating our research for the benefit of improving students' experiences with mathematics education at the classroom, school, district, and national levels.
- Disseminating our research for the benefit of improving the preparation of mathematics teachers at the university level around the world.
- Providing access to our research publications to the public.

All information concerning PME and its constitution can be found at the PME website: www.igpme.org

PME MEMBERSHIP AND OTHER INFORMATION

Membership is open to people involved in active research consistent with the aims of PME, or professionally interested in the results of such research. Membership is on an annual basis and depends on payment of the membership fees. PME has between 700 and 800 members from about 60 countries all over the world.

The main activity of PME is its yearly conference of about 5 days, during which members have the opportunity to communicate personally with each other during working groups, poster sessions and many other activities. Every year the conference is held in a different country.

There is limited financial assistance for attending conferences available through the Richard Skemp Memorial Support Fund.

A PME Newsletter can be found on the PME website. Occasionally PME issues a scientific publication, for example the result of research done in group activities.

WEBSITE OF PME

All information concerning PME, its constitution, and past conferences can be found at the PME website: www.igpme.org

HONORARY MEMBERS OF PME

Efraim Fischbein (Deceased)

Hans Freudenthal (Deceased)

Joop Van Dormolen (Retired)

PME ADMINISTRATIVE MANAGER

The administration of PME is coordinated by the Administrative Manager:

Dr. Khemduth Singh Angateeah

email: info@igpme.org

INTERNATIONAL COMMITTEE OF PME

Members of the International Committee (IC) are elected for four years. Every year, four members retire and four new members are elected. The IC is responsible for decisions concerning organizational and scientific aspects of PME. Decisions about topics of major importance are made at the Annual General Meeting (AGM) during the conference. The IC work is led by the PME president who is elected by PME members for three years.

President

Markku Hannula (Finland)

President-Elect

Wim Van Dooren (Belgium)

IC Members

Vice-President	Arindam Bose	India
Secretary	Jodie Hunter	New Zealand
Treasurer	Yasmine Abtahi	Norway
Policy	Anika Dreher	Germany
	Judy Anderson	Australia
	Ceneida Fernández	Spain
	Chiara Andrà	Italy
	Michal Ayalon	Israel
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	Ban Heng Choy	Singapore
	Maitree Inprasitha	Thailand
	Tracy Helliwell	United Kingdom
	Helena Johansson	Sweden
	Kotaro Kumatsu	Japan
	Christina Krause	US
	Daniel Sommerhoff	Germany

PROCEEDINGS OF PREVIOUS PME CONFERENCES

The table includes the ERIC numbers, links to download, ISBN/ISSN of the proceedings, and/or the website address of annual PME.

No.	Year	Location	ERIC number, ISBN/ISSN and/or website address
1	1977	Utrecht, The Netherlands	Not available in ERIC
2	1978	Osnabrück, Germany	ED226945, ISBN 3-922211-00-3
3	1979	Warwick, United Kingdom	ED226956
4	1980	Berkeley, USA	ED250186
5	1981	Grenoble, France	ED225809
6	1982	Antwerp, Belgium	ED226943, ISBN 2-87092-000-8
7	1983	Shoresh, Israel	ED241295, ISBN 965-281-000-2
8	1984	Sydney, Australia	ED306127
9	1985	Noordwijkerhout, The Netherlands	ED411130 (vol.1) ED411131 (vol. 2)
10	1986	London, United Kingdom	ED287715
11	1987	Montréal, Canada	ED383532, ISSN 0771-100X
12	1988	Veszprém, Hungary	ED411128 (vol. 1) ED411129 (vol. 2)
13	1989	Paris, France	ED411140 (vol. 1) ED411141 (vol. 2) ED411142 (vol. 3)
14	1990	Oaxtepex, Mexico	ED411137 (vol. 1) ED411138 (vol. 2)

			ED411139 (vol. 3)
15	1991	Assisi, Italy	ED413162 (vol. 1) ED413163 (vol. 2) ED413164 (vol. 3)
16	1992	Durham, USA	ED383538
17	1993	Tsukuba, Japan	ED383536
18	1994	Lisbon, Portugal	ED383537
19	1995	Recife, Brazil	ED411134 (vol. 1) ED411135 (vol. 2) ED411136 (vol. 3)
20	1996	Valencia, Spain	ED453070 (vol. 1) ED453071 (vol. 2) ED453072 (vol. 3) ED453073 (vol. 4) ED453074 (addendum)
21	1997	Lahti, Finland	ED416082 (vol. 1) ED416083 (vol. 2) ED416084 (vol. 3) ED416085 (vol. 4)
22	1998	Stellenbosch, South Africa	ED427969 (vol. 1) ED427970 (vol. 2) ED427971 (vol. 3) ED427972 (vol. 4) ISSN 0771-100X
23	1999	Haifa, Israel	ED436403, ISSN 0771-100X
24	2000	Hiroshima, Japan	ED452301 (vol. 1) ED452302 (vol. 2) ED452303 (vol. 3) ED452304 (vol. 4)

			ISSN 0771-100X
25	2001	Utrecht, The Netherlands	ED466950, ISBN 90-74684-16-5
26	2002	Norwich, United Kingdom	ED476065, ISBN 0-9539983-6-3
27	2003	Honolulu, Hawai'i, USA	ED500857 (vol.1) ED500859 (vol.2) ED500858 (vol.3) ED500860 (vol.4) ISSN 0771-100X http://www.hawaii.edu/pme27
28	2004	Bergen, Norway	ED489178 (vol.1) ED489632 (vol.2) ED489538 (vol.3) ED489597 (vol.4) ISSN 0771-100X www.emis.de/proceedings/PME28
29	2005	Melbourne, Australia	ED496845 (vol. 1) ED496859 (vol. 2) ED496848 (vol. 3) ED496851 (vol. 4) ISSN 0771-100X
30	2006	Prague, Czech Republic	ED496931 (vol. 1) ED496932 (vol. 2) ED496933 (vol. 3) ED496934 (vol. 4) ED496939 (vol. 5) ISSN 0771-100X http://class.pedf.cuni.cz/pme30
31	2007	Seoul, Korea	ED499419 (vol. 1) ED499417 (vol. 2)

			ED499416 (vol. 3) ED499418 (vol. 4) ISSN 0771-100X
32	2008	Morelia, Mexico	ISBN 978-968-9020-06-6 ISSN 0771-100X http://www.pme32-na30.org.mx
33	2009	Thessaloniki, Greece	ISBN 978-960-243-652-3 ISSN 0771-100X
34	2010	Belo Horizonte, Brazil	ISSN 0771-100X http://pme34.lcc.ufmg.br
35	2011	Ankara, Turkey	ISBN 978-975-429-262-6 ISSN 0771-100X http://www.arber.com.tr/pme35.org
36	2012	Taipei, Taiwan	ISSN 0771-100X http://tame.tw/pme36
37	2013	Kiel, Germany	ISBN 978-3-89088-287-1 ISSN 0771-100X http:// http://www.pme2013.de/
38	2014	Vancouver, Canada	ISBN 978-0-86491-360-9 ISSN 0771-100X http://www.pme38.com/
39	2015	Hobart, Australia	ISBN 978-1-86295-829-6 ISSN 0771-100X http://www.pme39.com
40	2016	Szeged, Hungary	ISSN 0771-100X http://pme40.hu
41	2017	Singapore	ISBN 978-981-11-3742-6 http://math.nie.edu.sg/pme41
42	2018	Umeå, Sweden	ISSN 0771-100X ISBN (Vol 1) 978-91-7601-902-3

			ISBN (Vol 2) 978-91-7601-903-0 ISBN (Vol 3) 978-91-7601-904-7 ISBN (Vol 4) 978-91-7601-905-4 ISBN (Vol 5) 978-91-7601-906-1 http://www.pme42.se/
43	2019	Pretoria, South Africa	ISBN (Volume 1, print): 978-0-6398215-0-4 ISBN (Volume 1, electronic): 978-0-6398215-1-1 ISBN (Volume 2, print): 978-0-6398215-2-8 ISBN (Volume 2, electronic): 978-0-6398215-3-5 ISBN (Volume 3, print): 978-0-6398215-4-2 ISBN (Volume 3, electronic): 978-0-6398215-5-9 ISBN (Volume 4, print): 978-0-6398215-6-6 ISBN (Volume 4, electronic): 978-0-6398215-7-3 https://www.up.ac.za/pme43
44	2021	Khon Kaen, Thailand (Virtual)	ISBN (Volume 1, e-book): 978-616-93830-0-0 ISBN (Volume 2, e-book) : 978-616-93830-1-7 ISBN (Volume 3, e-book) : 978-616-93830-2-4 ISBN (Volume 4, e-book) : 978-616-93830-3-1 https://pme44.kku.ac.th/

THE PME 45 CONFERENCE

Two committees are responsible for the organization of the PME 45 Conference: The International Program Committee (IPC) and the Local Organizing Committee (LOC).

THE INTERNATIONAL PROGRAM COMMITTEE (IPC)

Ceneida Fernández	Universidad de Alicante (Spain) Co-chair, PME and LOC representative
Salvador Llinares	Universidad de Alicante (Spain) Co-chair, LOC representative
Núria Planas	Universitat Autònoma de Barcelona (Spain) LOC representative
Ángel Gutiérrez	Universidad de Valencia (Spain) LOC representative
Markku Hannula	University of Helsinki (Finland) PME President
Wim Van Dooren	KU Leuven (Belgium) PME President-elect
Ban Heng Choy	Nanyang Technological University (Singapore) PME representative
Michal Ayalon	University of Haifa (Israel) PME representative
Arindam Bose	Tata Institute of Social Sciences, Mumbai (India) PME representative

THE LOCAL ORGANISING COMMITTEE (LOC)

Ceneida Fernández (Conference Chair)	Universidad de Alicante (Spain)
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Núria Planas	Universitat Autònoma de Barcelona (Spain)
Ángel Gutiérrez	Universidad de Valencia (Spain)
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ACKNOWLEDGMENTS

We thank all of the reviewers and the IPC for the detailed review work that has led to the presentations in these proceedings.

REVIEW PROCESS OF PME 45

RESEARCH REPORTS (RR)

Research Reports are intended to present empirical or theoretical research results on a topic that relates to the major goals of PME. Reports should state what is new in the research, how the study builds on past research, and/or how it has developed new directions and pathways. Some level of critique must exist in all papers.

The number of submitted RR proposals was 263, and 124 of them were accepted. Of those not accepted as RR proposals, 72 were invited to be re-submitted as Oral Communication (OC) and 60 as Poster Presentation (PP).

As in previous years, every RR submission underwent a fully independent double-blind peer review by three international experts in the field in order to decide acceptance for the conference.

ORAL COMMUNICATIONS (OC)

Oral Communications are intended to present smaller studies and research that is best communicated by means of a shorter oral presentation instead of a full Research Report. They should present empirical or theoretical research studies on a topic that relates to the major goals of PME.

The number of submitted OC proposals was 161, and 125 of them were accepted. Of those not accepted as OC proposals, 5 were invited to be re-submitted as Poster Presentation (PP). In the end, considering re-submissions of Research Reports as Oral Communications, 165 OCs were accepted for presentation at PME 45.

POSTER PRESENTATIONS (PP)

Poster Presentations are intended for information/research that is best communicated in a visual form rather than an oral presentation. They should present empirical or theoretical research studies on a topic that relates to the major goals of PME.

The number of submitted PP proposals was 70, and 58 of them were accepted. In the end, considering re-submissions of Research Reports and Oral Communications as Poster Presentations, 91 PPs were accepted for presentation at PME 45.

COLLOQUIUM (CQ)

The goal of a Colloquium is to provide the opportunity to present a set of three papers that are interrelated in a particular way (e.g., they are connected through related or contrasting theoretical stances, use identical instruments or methods, or focus on closely related research questions), and to initiate a discussion with the audience on the interrelated set.

The number of submitted CQ proposals was 1, and it was rejected.

RESEARCH FORUMS (RF)

The goal of a Research Forum is to create dialogue and discussion by offering PME members more elaborate presentations, reactions, and discussions on topics on which substantial research has been undertaken in the last 5-10 years and which continue to hold the active interest of a large subgroup of PME. A Research Forum is not supposed to be a collection of presentations but instead is meant to convey an overview of an area of research and its main current questions, thus highlighting contemporary debates and perspectives in the field.

The number of submitted RF proposals was 4, and all of them were accepted.

WORKING GROUPS (WG)

The aim of Working Group is that PME participants are offered the opportunity to engage in exchange or to collaborate in respect to a common research topic (e.g., start a joint research activity, share research experiences, continue or engage in academic discourse). A Working Group may deal with emerging topics (in the sense of newly developing) as well as topics that are not new but possibly subject to changes. It must provide opportunities for contributions of the participants that are aligned with a clear goal (e.g. share materials, work collaboratively on texts, and discuss well-specified questions). A Working Group is not supposed to be a collection of individual research presentations (see Colloquium format), but instead is meant to build a coherent opportunity to work on a common research topic. In contrast to the Research Forum format that is meant to present the state of the art of established research topics, Working Groups are considered to involve fields where research topics are evolving.

The number of submitted WG proposals was 8 and 7 of them were accepted.

SEMINARS (SE)

The goal of a Seminar is the professional development of PME participants, especially new researchers and/or first comers, in different topics related to scientific PME activities. This encompasses, for example, aspects like research methods, academic writing, or reviewing. A Seminar is not intended to be only a presentation but should involve the participants actively.

The number of submitted SE proposals was 1, and this was accepted.

LIST OF PME 45 REVIEWERS

The International Program Committee of PME 45 thanks the following people for their help in the review process.

A

Adler, Jill (South Africa)
Akkoc, Hatice (Turkey)
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C

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Choy, Ban Heng (Singapore)
Coles, Alf (United Kingdom)
Cooper, Jason (Israel)
Csíkos, Csaba (Hungary)
Cusi, Annalisa (Italy)

D

Da Ponte, Joao-Pedro (Portugal)
Di Martino, Pietro (Italy)
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Doerfler, Willi (Austria)
Dole, Shelley (Australia)
Downton, Ann Patricia (Australia)
Dreher, Anika (Germany)
Dreyfus, Tommy (Israel)
Dyrvold, Anneli (Sweden)

E

Edmonds-Wathen, Cris (Australia)
Edwards, Laurie (United States)
Ekol, George L. (South Africa)
El Mouhayar, Rabih Raif (Lebanon)
Erens, Ralf (Germany)

Eriksson, Anna Helena (Sweden)
Essien, Anthony (South Africa)
Even, Ruhama (Israel)

F

Fellus, Olga (Canada)
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Font, Vicenç (Spain)
Francisco, John (United States)
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Furinghetti, Fulvia (Italy)

G

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Gates, Peter (United Kingdom)
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Getenet, Seyum Tekeher (Australia)
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H

Halverscheid, Stefan (Germany)
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Heinze, Aiso (Germany)
Hernandez-Martinez, Paul (Australia)
Heyd-Metzuyanim, Einat (Israel)
Hino, Keiko (Japan)
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Horne, Marj (Australia)

I

Inglis, Matthew (United Kingdom)
Inprasitha, Maitree (Thailand)
Ivars, Pedro (Spain)

J

Jiang, Chunlian (China)

K

Kageyama, Kazuya (Japan)
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Khan, Steven K. (Canada)
Kieran, Carolyn (Canada)
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Krzywacki, Heidi (Finland)
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Kwon, Oh Nam (South Korea)

L

Lavy, Ilana (Israel)
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Liljedahl, Peter (Canada)
Lim, Kien (United States)
Lin, Fou-Lai (Taiwan)
Lin, Pi-Jen (Taiwan)
Lin, Yung-Chi (Taiwan)

Lindmeier, Anke Maria (Germany)
Liu, Minnie (Canada)
Llinares, Salvador (Spain)

M

Magajna, Zlatan (Slovenia)
Makonye, Judah Paul (South Africa)
Mamona-Downs, Joanna (Greece)
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Moutsios-Rentzos, Andreas (Greece)
Mulligan, Joanne (Australia)
Murphy, Carol (Australia)

N

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Nortvedt, Guri (Norway)
Novotná, Jarmila (Czech Republic)
Ntow, Forster (Ghana)

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Oates, Greg (Australia)
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Oesterle, Susan Michele (Canada)
Ofri, Ofra (Israel)
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Olteanu, Lucian (Sweden)
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PLENARY LECTURES

STORIES OF OBSERVING, INTERVIEWING AND RESEARCHING IN COLLABORATIVE GROUPS TO DEVELOP MATHEMATICS TEACHING AND LEARNING

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Using the context of comparing and contrasting three papers from early in my career with three recent papers, I tell stories of my developing research practices, interviewing, observing and researching in collaborative groups focused on learning. Teachers supporting students to develop their stories of mathematics looks to the future with children working on solving real world problems, such as, climate change.

BEGINNINGS

I attended my first PME (Lisbon, 1994) without presenting a paper. I was a mature professional, having recently moved from being a mathematics teacher to being a mathematics teacher educator at the University of Bristol, School of Education (SoE). Since I still thought of myself as a teacher, it seemed natural for me to attend the *Teachers as researcher in mathematics education* working group led by Judy Mousley (RIP), Vicki Zack, and Chris Breen. They invited me to contribute in a small way to the following year's conference in Recife, Brazil, where I also presented my first research report. So began many fruitful academic collaborations and friendships with colleagues interested in the development of teachers of mathematics and, later, the development of mathematics teacher educators. In Recife, I found a theoretical position, enactivism, at a discussion group run by Rafael Núñez and Laurie Edwards, meeting another participant, David Reid (Goodchild, 2014), who became a close collaborator. There is not space to do justice to a development of the theory of enactivism in this plenary. From my present enactivist position, knowing is doing. What I do brings forth my world of knowing and my world of knowing is what I do. In enactivist terms, our history of structural coupling with our environment leads to patterned actions. Chairs may look quite different, but we recognise them as such. Basic-level categories are not actual behaviours, but they are how we label the same actions, such as a chair being for "sitting-on". The theme of this conference, mathematics education research, supporting practice, is central to my own research. In this plenary presentation, I look back over a life spent observing and interviewing mathematics teacher educators and teachers and students of mathematics, researching in collaborative groups and attending PME, asking what any implications there might be to take forward into the future of mathematics teaching and learning.

The title of this talk includes the word, "stories". What comes to mind when you think of this word? Many years ago, on reading a draft of a proposed PME paper, written by myself and Alf Coles, *The story of silence*: *Teacher as researcher – researcher as*

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teacher (Brown & Coles, 1996), a mentor of mine commented that we needed to change the title because readers would think the paper was fictional. For me the word "story" had a technical meaning. Bateson (1979) talks about the "pattern which connects" (p. 8), a story being "a little knot or complex of that species of connectedness which we call *relevance*" (p. 13). That paper, with story in its title, became the first in a series of PME papers documenting the research journey with Alf Coles. This presentation looks back over my research looking for distillations that are relevant, within this "*context,* of *pattern through time*" (p. 14). I do this by focusing on three aspects of my research practice that have been present across my work through time, interviewing, observing and researching in collaborative groups.

I will begin by focusing on three papers from early in my career to identify themes across all three aspects in each paper. I will then focus on three recent papers, each with insights mainly from one of the aspects to address where I am now. I will then look forward, briefly, to the future.

I am not seeking to share patterns from my own awarenesses, developed through my research journey, thinking that everyone needs to work in this way or do what I do. The intention is to interrogate how I see what I see and do what I do, perhaps expanding your space of the possible through what you might notice in the future. I map territory, asking questions about doing and being or how what I do informs what I think and believe, paraphrasing Bruner's (1990) "culturally sensitive psychology" (p. 16). I have been helped in the process of interrogating my early experiences through hour-long on-line meetings with Alf Coles, focused each time on a particular one of the first three papers. When we worked together on a one-year course with a group of prospective teachers at the University of Bristol, talking together after sessions became ritualised in what we came to call *reciprocal narrative interviewing* (Brown & Coles, 2019; 2020). We would take turns in questioning each other to uncover new awarenesses. It seemed natural to use this process to capture the origins of themes that have been with us since 1995 when we started to work together. Having both read the paper before meeting, we spent about 20 minutes talking through what we had been struck by. The rest of the hour was spent with Alf narratively interviewing me. Some extracts from the narrative interviews are included in the discussions of the papers that follow.

PAPER 1: "STOPPERS"

Firstly, I remembered a paper that was not written by me but by Joan Yates (1983), who worked at the University of Bristol, SoE as a mathematics teacher educator when I was a young teacher. In 1979, I was a teacher of mathematics in a school for students aged 11-18. After studying mathematics at university, in 1973-4 I completed a one-year post-graduate course to obtain a teaching qualification. I had been teaching for 5 years when I was invited to join a small group, three members, facilitated by Joan Yates. The work of this group was written up in a *For the Learning of Mathematics* (FLM) article (Yates, 1983) entitled *"Stoppers"*. "Stoppers", a label suggested by one member of the group, was an "aspect of our experience that had grown in importance, in that it had engaged and captured our interest" (p. 35). We defined a stopper as "the moment when a pupil is no longer able to "cope" […] there is an observable breakdown" (p. 35). Although my focus in this presentation is on the processes of doing research, inevitably, given my interests, it is worth stressing that these processes are in the service of developing mathematics teaching and learning.

Joan talks about her work with a group of three teachers, who wished to remain anonymous. I was one of those teachers. Reading this paper again, strands of the researcher who developed out of this young teacher are apparent.

Joan had been to a conference run by Stenhouse and was now trying out the principles of teacher as researcher she had encountered there. Stenhouse (1996) wrote "curriculum research and development ought to belong to the teacher" (p. 142). He believed that "critical characteristics" (p. 144) of such a teacher could be:

> The commitment to systematic questioning of one's own teaching as a basis for development; The commitment and the skills to study one's own teaching; The concern to question and to test theory in practice by the use of those skills. (p. 144)

As Joan Yates writes in the paper, "our aim was to examine our own practice critically and systematically" (p. 35) echoing Stenhouse's words. In so doing we, the teachers, also acquired research skills. A particularly memorable, for me, article by Ginsburg (1981) looking at clinical interviewing included the sentence:

> Verbilisation can be misleading since the child may not have direct access to his [*sic*] cognitive processes or may have poor command over language. (p. 35)

In the paper, a substantial amount of the teachers' reflections on their children's mathematical thinking is included. My way of collecting data from 26 children, "To compare understanding of decimals without and with a calculator", is described. I ask the children to show me what they do in written form before any verbilisation. Here is the task for you try first, Order the following set of decimals from smallest to largest:

> 2.19, .888, 1.699, 2.2, 1.8989

Ask yourself whether you ask children to do this sort of task. In the UK at that time, most children would work at calculations, such as adding and subtracting, with decimals, but would not have been asked to put them in order of size. What did our children understand about the numbers they were calculating with?

> I asked [them] to space out and gave them a piece of rough paper each. I requested that they work on their own. On the blackboard I wrote: 2.19, .888, 1.699, 2.2, 1.8989. I gave the instruction: On one side of the paper I'd like you to put these numbers in order of size, from smallest to biggest and explain very clearly how you did it. They had ten minutes…most only needed five minutes. (p. 37)

Having collected the results together on the board (see Table 1, one child missed out 1.699) "to set up a visible reminder of different possibilities […] the ones where there were discrepancies were pointed out" (p. 37). What takes your attention in Table 1?

The students were asked to turn over the piece of paper and could “use a calculator to try to convince themselves, if necessary, and could talk” (p. 37).

		2.19	.888	1.699	2.2	1.8989
Smallest	1		26			
	2			21	1	3
	3			4		23
	4	17			9	
Biggest	5	9			16	

Table 1: Results of ordering the decimals.

The discussions that followed led to some students changing their minds about their previous answers. Is this process teaching or researching? Here, I recognise what I would now see as my need to base teaching and researching on something that the students or, later, prospective teachers do. I might interview teachers about their practices in the classroom, always, where possible, linked with observing lessons. Stenhouse (1975, p. 154) quotes Hamilton’s eight propositions “which are of interest to all who are concerned to observe teaching”, the second of which is “Students (or for that matter teachers) are never ignorant or know nothing”. I believe passionately that children’s and teachers’ talk makes sense, from their perspectives. My task as teacher researcher is to learn their world. As a teacher, about the time of the “stoppers” research, I had some awareness of this when Dave Pratt and I, then both teachers, began a conversation at an Association of Teachers of Mathematics (ATM) weekend meeting at the start of a free evening. We were sharing our practices and began exploring differences heatedly. As we continued talking, we started sharing the details of our practices rather than labels, such as, structured, or open-ended. Observers, friends who kept returning to see if they could attract our attention, reported, later the same evening, being surprised that we each now seemed to be talking heatedly from the opposite position to earlier. Teachers in a particular context cannot be expected to see their classroom and report on it in the way that an experienced observer of many classrooms would. What is important is to hear what teachers say and see what they say through observing their classroom practice. I started to talk about the language a teacher uses about their practice being in the direction of their development or movement, what they were working on, talking in vectors rather than placing themselves on a continuum of practice that an observer could do.

In narratively interviewing me for this presentation, Alf drew attention to questions where my “thoughts were stimulated in the direction of” (Yates, 1983, p. 38):

AC You offer two questions – When do I know something in mathematics? When do I know that another knows? Do those feel like questions that stayed with you?

LB How do I know that they know? We have talked about thata question in our writing. All it takes is a disturbance down the line and I need to question what I know. How do prospective teachers know what their children know? What do they do to find out? I think it’s a motivating question for which you don’t have to have an answer.

AC A shift was in recognising that question as one of a class of helpful questions that a prospective teacher can ask in the process of starting to accrue a range of behaviours in their teaching. Then the recognition, later on, was labelling that as a purpose, what we now see as a basic-level category. Here it’s you using a purpose before you had that label.

LB When I first met you, I was inarticulate about my practice as a teacher educator. If I was going to be able to talk about what I did then I had to have words to talk with. In those early papers, whenever we gave an example of a purpose it was always “how do I know that they know?”

AC Here’s you as teacher using that purpose – quite a precious example because quite quickly you go on to use it as teacher educator. Quite quickly you’re supporting others to find their own purposes. Here’s Joan having supported you to find that purpose for yourself.

Through Alf narratively interviewing me, we found an early example of a motivating question, “When do I know that another knows?”, for my teaching that had emerged from the critical and systematic work in the group. At the time, it was a particular question but, over time, working with prospective mathematics teachers to identify their own motivating questions arising from the interrogation of their practice led to the labelling of such questions as purposes. These purposes became linked for me with basic-level categories in the enactivist literature in that they lead to patterned actions in the world of the classroom. I now recognise “stoppers” as a motivating label for the research group in the paper.

So, the three themes of the title are all represented in this first research paper: interviewing influenced by Ginsburg (1981), giving children something to do to support their verbalisation; close observation of children in classrooms as a teacher researcher and close observation of my own actions as a teacher in a critical and systematic way; and, researching in a collaborative research group with Stenhouse influences, research as exploration and personal transformation, using each other as critical friends to identify concepts with a direct link with practice to interrogate, in this case, “stoppers”.

PAPER 2: THE INFLUENCE OF TEACHERS ON CHILDREN’S IMAGE OF MATHEMATICS

In 1988, I completed an MEd in Mathematics Education at the University of Bristol. I was also making the transition from being a teacher in a classroom to working in curriculum development at the Resources for Learning Development Unit (RLDU) (Llinares, Krainer & Brown, 2014, p. 602), supporting groups of teachers to develop resources for mathematics classrooms. I was now a facilitator of collaborative teacher research groups, although the size of each group was at most ten teachers, in line with RLDU policy, rather than three. In 1995, I was full-time at the University of Bristol, SoE and wanted to submit a PME paper for the January deadline. I had written a paper for FLM (Brown, 1992), *The influence of teachers on children's image of mathematics*, which is the second paper that I have chosen, reporting the findings from my master's dissertation. Reporting on this research and extending it slightly provided the focus of my first PME paper (Brown, 1995).

The motivation for the dissertation research was "devising a set of instruments which might allow me to explore whether a particular teacher did, in fact, influence the image of mathematics of their pupils in the same way" (1992, p. 30). The individual's image of mathematics was the "personal theory (Kelly, 1955; Claxton, 1984) which an individual holds about mathematics at the present time" (p. 30), including "feelings, expectations, experiences and confidences" (p. 30). What I was looking for as an influence of the teacher on children's image of mathematics was a "common change or adaption of the pupils through working with the teacher" (p. 30). I worked with four teachers chosen as being effective by advisory teachers and heads of department, with contrasting structures in which they worked, for example, individualised learning or whole-class interactive. I did not believe that there would be a common image so, in limiting the number of children in the project to six for each teacher, I suggested they choose two who did respond to whatever they did with the class, two who did not respond and two to make up any imbalance in, say, gender.

Looking back, from a perspective of "knowing is doing", I find it hard to imagine why I so strongly thought that I would not see any common strand across the students and their teachers. Before giving my present position on the findings of this project, I want to say a little more about its design. I knew that I wanted to interview the students and the teachers, making the process as similar as possible. In trialling various protocols for the student interviews, it became apparent that simply asking what mathematics is to them did not work. We needed to do some mathematics together given potential issues with verbalisation. The Ginsburg (1981) article used in Paper 1 was important for interview strategies for doing mathematics. Each student chose one from 5 activities presented to them for us to work on together. In telling me about lessons they had experienced, the "work of Hoyles (1985) in *asking pupils to recall particular episodes"* (p. 31) was influential. In trialling the various protocols, I settled on two practices that seemed to support students in answering directly without the need to clarify. In early trials of questions, I had often been met by a blank face and "What do you mean?". The first practice was asking what appear to be long-winded questions, where the precise wording developed over time, such as:

> I am going to make some statements and, for each one, see what is brought to mind by what I say. Try to remember the event so clearly that you can tell me a story about what happened (a) Tell me about an activity you have done recently in a maths lesson, and, although you probably did not think so at the time, it is brought to mind now when I say,

> there you are, sitting in a maths lesson and what you are doing does feel like mathematics. (p. 31, italicised in orginal)

The second practice led to energised responses, what in the 1995 PME paper I called "provoked articulation" (Brown, 1995, p. 148) from the students:

> What I am interested in is your image of mathematics. So far you have indicated in your responses to the various statements and activities that maths is … Is there anything else you'd like to add that has not been covered so far to the question: What is mathematics to you? (p. 31, italicised in original)

As interviewer, I am attending carefully to what the interviewee is saying with the intention of, using as close to their words and phrasing as possible, feeding back to them my thoughts of, in this case, what they think mathematics is. The responses are what I, as researcher, see as robust evidence. Provoked articulation is energetic, "Yes, yes, yes, and …" or "No, no, no, what I think is …" and so on.

I was surprised at the time that whether the student responded to what their teacher did or not their image of mathematics was influenced by the teacher. I share statements from Teacher C and the six children (Figure 1) from the class as an example. There were other patterns for the other three teachers and their children. I was struck in all the interviews by the question, "Where are the children who don't like mathematics?" What did these teachers do that led to engagement? My answer now would be that the teachers had conviction in what they were doing.

I also wanted to observe lessons and planned one observation before the interviews and one after. In fact, I gained little from the post-interview observations since the children interviewed were intent on carrying on the interactions! In observing in Teacher C's classroom, I was struck by seeing a teaching strategy reportedly used previously in a primary school:

> A pupil offered an explanation of how they had begun to tackle a problem. The other pupils were invited to close their eyes and put up their hand if they had started in the same way. An alternative start was requested and the pupils again closed their eyes and put up their hands if this was their way of starting. The process continued with more information being collected and these different starts were then used for further exploration: *What was the aim of the people who drew the radius?* (Brown, 1992, p. 32)

These children experience mathematics through expressing their ideas and hearing other's ideas. The teacher is not the expert, "even Teacher C doesn't know all the answers" (Pupil C6), supporting students in "building their own frameworks, that are not necessarily your frameworks" (Teacher C). How could there not be a common strand of overlapping experience? The teaching strategy described above means that students hear multiple ways of starting to engage with a complex problem. They are not being presented with one method to learn. It is their task to build their own framework. Their experience is tackling complex problems but with the support of a range of solution strategies used by others. One student said, in the interview,

Pupil C2
I prefer having to solve it myself. It gives you that satisfaction of not having to take it from a book. I enjoy mathematics. I find it more of a challenge than a chore. The problem-solving exercises would help me because I could imagine how I felt and go logically through the steps.

Pupil C3
Mathematics is problem-solving. In Connect-4 I'd start by experimenting on a smaller grid to see if there's any pattern and be able to predict: maybe changing the number of counters which you have to make a row.

Pupil C1
You've got to actually solve things for yourself which aren't in a book. That's not really what I thought maths was going to be in the earlier years because that was just numeral sort of maths. You can relate it more to things outside, it's not just like a picture on the board, you can imagine it.

Teacher C: Influence through philosophy
I think the whole idea of a problem is that you model it and make it solvable. Mathematics is a framework and mathematics teaching is fun. Fun when you see the children building their own frameworks which are not necessarily your frameworks.

Pupil C4
I think maths is just applying stuff that you have learned in the lesson in reality.

Pupil C5
Maths is using what I already know like trigonometry and measurement.

Pupil C6
There was a real problem there. I understood what was happening and there were so many different types of maths used to find the final answer. Maths was numbers to me. I felt that in maths everyone knew the answer but as time's gone on I've discovered that even Teacher C doesn't know all the answers – so maths has changed – you can experiment.

Figure 1: Commonality of image *building frameworks* between Teacher C and pupils.

> There are several solutions to one problem. If you go round the class asking, you'll come up with six or seven. You can experiment […] like a real mathematician. (p. 32, italicised in original)

This strategy was one that I have used in my own teaching and told other people about. Thinking of questions for future research at the end of the paper, I thought I might look for what I called "transferable strategies" in future research. What has happened is that my focus has been on the meta-level to the actual strategies, motivating labels such as "how do I know that they know?" that lead to a range of teaching behaviours being employed flexibly, or "building frameworks" for Teacher C.

In our on-line discussion, Alf reported noticing, in the paper, me using a purpose, "Is this a classroom in which it's all right to be wrong?", as a "way of analysing teacher and pupil behaviours" in the early days of moving from being a teacher to visiting other people's classrooms, which was "a continual source of surprise, recognition, disturbance (resonance and dissonance) and consequently, through reflection, personal learning" (p. 29).

In the same way as Teacher C gets pleasure from his students building their own frameworks, as a mathematics teacher educator I watched my prospective teachers developing through identifying purposes, encouraging them to talk and write about stories from their classroom illustrating resonance or dissonance. Here is one such story as an example:

> I wrote the question on the board, asking for hands up for the answer. The first child gave me the answer to which I said "Correct". Belatedly, I asked if anyone else had anything different, but of course the children were then unwilling to offer an alternative answer that they now knew was definitely wrong. I realised immediately that I could not now see what the rest of the children had done. Since that occasion I have been attempting to gain answers from several members of the class […] An advantage in listing the variety of answers to a question is to show children that they are not alone in making a mistake and that others have had the same (or different) problems. Similarly, multiple equivalent answers can be highlighted whereas otherwise a child may feel that their answer is wrong just because it does not look identical or is in a different form. Hence the art, as a teacher, of "being expressionless" as a variety of answers are given to a problem appears to be a very useful one. (Brown, 2004, p. 6)

The identification of a purpose such as "being expressionless" is energising for the prospective teacher, supporting them in learning from their own experiences. I learnt that there is not one way to teach from this project, but that what is important is a conviction in an image of mathematics that provides a consistent classroom culture. In the narrative interview with Alf related to this paper, I was energised by other awarenesses about my research process provoked by the discussion. Here are three quotations, one on interviewing, one on observing and one on purposes:

> LB Them telling their stories and the talk is stories of [*pause*] Interviewing as people telling their stories.

> LB I think that's where my conviction came that you don't do interpretation in the classroom, you collect the data.

> LB When I was visiting prospective teachers to observe them teach, I would ask them what they wanted my focus of observation to be, what they were working on at the time, looking for a purpose, I suppose.

PAPER 3: THE STORY OF MATHEMATICS

When *The story of mathematics* (Brown, 2001) was published, I had been working at the University of Bristol, SoE for about ten years, taking part in several research projects. The project reported on in this paper arose from the conclusions of a research report (Winter, Brown, & Sutherland, 1997) looking at *Curriculum materials to support courses bridging the gap between GCSE and A-level mathematics*:

> These comments encapsulate the importance of seeing the "story" of mathematics so that it has a coherence both in its teaching and in the experience of students. […] This finding is closely related to the mathematical competence and vision of individual, effective teachers. (p. 23)

The issue seemed to be that, due to teachers limiting their teaching to syllabus coverage for examinations at 16, students starting advanced level courses at 16+ had problems.

I am going to look at the research design of the story of mathematics project which led to a similar design on a successful major project in 1999, by which time my way of designing research feels stable. The findings of this project were first reported in a symposium at the American Educational Research Association (AERA) conference (Montreal, 1999), an invitation arriving after the conference for the set of papers to be published in a new educational policy, research and practice journal. The research questions for this project were:

- What are the stories of their mathematics for individual, effective teachers? How do their strategies and purposes in the teaching of mathematics support the doing of mathematics by their students?
- What stories are there within mathematics itself that can give an holistic sense of connections? What are the "big ideas" for mathematics?

Collaborative group

Three teachers were paired with three researchers forming a collaborative group. We met three times over about six months. At the first meeting we began the process of developing a common language through discussion of the research questions. My experience in the Stenhouse-influenced group (Paper 1) is apparent here. Between the first and second meetings of the group the researchers observed their teacher pair and each of the teachers was interviewed by me. The second meeting of the group focused on the interviews and observational data to consider sameness and differences in the teacher's practices. Common threads were identified through the mutual recognition amongst the teachers of similar practices and described through the developing common language. As researchers we were part of the conversation, feeding back anecdotes of related practice from our observations. In the last meeting, the teachers wrote lesson descriptions to illustrate the samenesses in practice which had been identified. The teachers' voices are given the space here, leading the conversations, with the labels, such as, same/different, being a "pedagogical tool through which mathematical ideas of order, inverse, pattern and structure can be explored" (p. 192) emerging from them, like "stoppers" in Paper 1.

Interviews

The interviews focused on the two research questions but for each "big idea" or teaching strategy mentioned there was an invitation to give an anecdote from a recent lesson to illustrate what was meant by the language used. Interviewing as people telling stories again.

Observations

Each researcher then used the transcript of the interview of their teacher to inform their observations of the teacher's lessons. The researchers looked for evidence from practice to illustrate what the teacher had described during the interview. Here the practice I recognise is what I have come to call "staying with the detail". No interpretation in the observational record beyond the researcher identifying practice related to descriptions from the interview. At the point of recognition, the researcher aims to capture what is said and what is done, a process supported by video data in later projects.

Findings

In the design of the project, we had overlooked a central emergent theme from the data analysis: These teachers are able to make their teaching contingent upon the story of mathematics of their students. I return to Teacher C, his pleasure in his students building their own frameworks. Staying with the detail of these three teachers' practices led to a new awareness, showing a pattern rather than the isolated case of Teacher C. Transferable strategies are at the wrong level. What was central was the motivating labels, basic-level categories that could be linked to a complex set of developing mathematics and teaching strategies used by an individual teacher. These can look different whilst the same label is used. All three teachers work on complex connections within mathematics and mathematics teaching for themselves. Their stories of mathematics are not fragmentary. The gap between the students' experiences of mathematics between stages is bridged because the teachers' actions are the same and because the story of mathematics is the same for the teacher and the students. At a meta-level, it does not seem to matter what the story is, as long as there is one, and it exists in a process of learning contingent upon the voices of the students, who themselves know how to act in doing mathematics.

DISTILLATIONS FROM RECENT PAPERS: PATTERNS OVER TIME

A doctoral student, who was interested to read some of the writing of her supervisors, myself and Alf Coles, reported that she had found our recent work hard to access but had found the earlier papers inspiring, leading to me to discussing the first three papers at length. For the final three papers, I am going to focus on the distilled principles that inform our research practices, always focused on mathematics, as communicated in our latest writing. I have chosen three papers that each seem to focus centrally on one of the three aspects of this talk to share what they say about current practice. These distillations are patterns over time: learning through finding new basic-level categories from staying with the detail of experience; establishing a classroom or research group culture through metacommenting and attending to the voices of the teachers and students when researching in collaborative groups. I will then look to the future.

Interviewing

The writing, a book chapter, chosen for this section is entitled *Mapping the territory: Using second-person interviewing techniques to narratively explore the lived experience of becoming a mathematics teacher educator* (Bissell, Brown, Helliwell, & Rome, 2021). In writing this chapter with colleagues, I had been enactivist for 25 years. Its theoretical underpinnings are therefore enactivist. The focus of the paper is on the articulations of an expert teacher, moving from being in the classroom to supporting teachers in a national setting:

> In the moment there is no time for reflecting. In moving to a new job, therefore, we act using what we have done previously. […] Using what we have done previously in a new environment will be followed by adapting when what happens is not effective. (p. 207)

Three interviews seek to support the interviewee in identifying this process of adapting. I have a well-developed protocol for the interviewing process, including how the interviewer works contingently with the first-person accounts of the interviewee. The first three items in the protocol are based on Claire Petitmengin's (2006) work:

- Stabilising attention: A regular reformulation by the interviewer of what the interviewee has said, asking for a recheck of accuracy (often in response to a digression or judgement). Asking a question that brings the attention back to the experience.
- Turning the attention from "what" to "how" (never "why").
- Moving from a general representation to a singular experience. This is what we term "story", a re-enactment, reliving the past as if it were present. Talking out of experience, not from their beliefs or judgements of what happened, often involves the teachers in a move to the present tense. Staying with the detail is important, a maximal exhaustivity of description that allows access to the implicit. (p. 209)

These three items are extended into a fourth by Brown & Coles (2019). The fourth fundamental way of acting encapsulates our enactivist take on learning through adapting basic-level categories, what were originally described in the first three papers as "purposes", "motivating statements" or "questions" that accrue a range of behaviours:

- Getting to new basic-category labels: After dwelling in the detail, telling stories and exploring without judgement or digressions, the invitation is to elicit statements of what is being worked on. [...] In this way, new basic-level categories might be identified, such as the straplines (a word used in editing newspapers, memorable, usually less than five-word, phrases) from this research of "listening for" or "setting up the culture". These awarenesses, triggering and being triggered by the environment can allow adapted and new behaviours to emerge. (p. 209)

The density of this writing compared to the earlier papers is clear and gives some insight into our learning over more than 20 years. Working with prospective teachers, I had developed a story of how they learnt to teach and here I am applying those ideas to my learning in moving from being a teacher to being a teacher educator. For Alf and me, in our work together, learning is through changing or extending basic-level categories. The process goes back to those interviews in the MEd paper, of inviting the other to tell stories from experience, supporting them to stay with the detail of what happened, focusing on resonance and dissonance, without judgements, opening up the possibility of acting differently. New basic-level categories emerging seems to be energising for the learner, who can accrue a range of behaviours to use flexibly. Alf and I, in talking after sessions when we taught together on the course for prospective teachers, used this process, which we call reciprocal narrative interviewing, on each other, turn taking as interviewer and interviewee to develop insights and awarenesses, to ourselves learn. With teachers new to teaching, looking for patterns across a sequence of interviews staying with the detail of first lessons of a new year with groups new to them that year, I would be looking for when the basic-level categories had stabilised, when the teacher literally knew what they were doing in acquiring their story of mathematics (Brown & Coles, 2008).

Observing

Another book chapter, *Learning to teach mathematics: The lesson de-brief conversation* (Brown, Brown, Coles, & Helliwell, 2020) focuses on the observation of prospective teachers followed by discussions with them and their mentor after the lesson. It is important to stay with what the prospective teacher is able to notice, what we call staying with the detail, and this is linked to asking them what their focus of observation is before the lesson. A prospective teacher is not sure what they want the culture of their classroom to be but, in getting to a label such as "being expressionless" for your own behaviour, we have come to recognise what we call "metacommenting" by experienced teachers, especially in the early days of working with a new class. What is seen by the teacher as behaviours by students, such as, "getting organised" or "looking for a counterexample" are shared so that the students know what to do to do mathematics. Over time some of these behaviours are favoured by the students and the teacher does not then need to metacomment, since a culture of doing mathematics in that classroom has been established.

It is striking to me, in re-reading this chapter, how we, the mathematics teacher educators and authors of the writing, use fictionalised accounts of de-brief conversations. This seems a long way from being asked not to use "story" in the title of a paper because readers might think it was fiction. Markku Hannula (2003) had a paper published in FLM when I was editor, where he used fictionalised experiences and, in this chapter, we do the same, citing Hannula. As well as participants being anonymised, it is possible to bring together parts from several stories that focus on stressing important principles and practice.

Researching in collaborative groups

In 1995 I was funded to develop a Master's module based on the participants working in a collaborative group to develop their teaching. By 2011, the principles for running such a group were established. In the academic journal paper, *Differentiation from an advanced standpoint*: *Outcomes of mathematics teachers' action research studies aimed at raising attainment* (Coles & Brown, 2021, pp. 169-170), these were stated as:

- the group size should be less than or equal to 10,
- meetings should be spread out over an extended period of time,
- the teachers should come from a range of schools and be volunteers rather than conscripts,
- the leader of the group sets up a loose structure for meetings and time is given to each participant to discuss their emerging thoughts about their issue,
- the leader of the group gives individual readings in between meetings to support participants thinking about their issue, or there could be tutorials for participants between meetings linked to their Master's study,
- the leader(s) of the group will make one or more visits to each teacher's school to further support thinking about the issue and/or data collection, and, in some cases, the teachers visit each other's classrooms. (Brown & Coles, 2011, p. 865)

The paper focuses on a subset of teachers researching in a collaborative group with a focus on using higher-level content with groups of low attainers. The research was to be written up as part of a Master's module. One teacher gives the following reasons, grounded in her practice, for extending the syllabus for these children:

> [a student] was able to tell me that the square root of 49 is 7 two weeks after having studied Pythagoras' Theorem. […] Even though the student probably hasn't remembered how to use Pythagoras' Theorem, they have taken away how to square root. This was one of the factors that led me to think I should try teaching these students more of the difficult topics. Not just for them to try and grasp these harder topics, but because they might take something else away from it. (p. 175)

LOOKING TO THE FUTURE

In 2018 I travelled to Indonesia to give a keynote entitled, *Global needs: Rethinking teaching and learning mathematics for future changes*. I ended that talk saying:

> We need citizens with higher-order thinking skills to provide creative suggestions for whatever happens in an uncertain future. It just might be that cross-curricular teams of teachers working with students on real, complex problems will become the norm for schooling.

Teachers being led by their students who are building frameworks within real contexts that are not their frameworks makes me optimistic for the future. Here is one example of what is happening right now. Each individual student graduates from the Green School, Bali with a presentation. Watch Bronson Parish's (2018) presentation on ocean flow, https://youtu.be/0ITm7v1yLGo. Many villages on his home island, Sumba, have no power or running water even though it is surrounded by water, continually moving. The presentation is his story of his practical ideas that he wants to be available to and affordable by anybody, harnessing the ocean's energy to desalinate water and support the growing of food in coastal communities affected by climate change, including building a working prototype with his father.

This is one student's journey solving a real problem in his community. Researching in collaborative groups with teachers gives me a vision of future schooling in which teachers are supporting children researching in collaborative groups to solve real problems in their communities.

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EXPLORATIONS ON VISUAL ATTENTION DURING COLLABORATIVE PROBLEM SOLVING

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In this plenary, I reflect on the MathTrack reseach project that examines the role of visual attention in the multimodal social interaction in the classroom contexts of collaborative non-routine problem solving. The project is using multiple mobile eye-tracking devices to record teacher and student visual attention when students work in groups solving a non-routine geometry problem. Project outcomes include methodological innovations for working with eye movement data, findings about joint representational attention, and the importance of eye contact in teacher-student interaction. Our experience suggests that eye movement research in classrooms should focus on analysing visual processes and making within-person analyses.

INTRODUCTION

In my plenary, I reflect on a research project, MathTrack (Mobile gaze tracking for the study of attention and emotion in collaborative mathematical problem solving). The pilot phase of the project begun in 2014, a grant from the Academy of Finland funded the main project 2016-2020, and the analysis of the collected data is still ongoing. In MathTrack we used multiple mobile eye-trackers in Finnish grade 9 mathematics classrooms to study teachers' and students' visual attention. The project extended eye-movement research into new areas and required methodological innovation.

Representations are essential in mathematics education (e.g. Kaput, 1987). Multiple representations may exist, for example, on board, in textbook, on computer screens, and as gestures and spoken words. In addition to these representations, the classroom is a social setting, where both teachers and students continuously observe and react to what others are doing. Such richness of information raises a question regarding how teachers and students navigate between the multiple channels.

Until recent decades, research on social interaction had focused on verbal communication. More resent research has started to acknowledge that communication is not just words; gestures, glances, body movement, and prosody are also important aspects of it (Radford, 2008). For example, signalling one's own attention and reading the attention of others is essential for fast-paced interactions when people collaborate (Clark & Schaefer, 1987). The facial region is especially important, serving to regulate the flow of conversation, to provide feedback on the reaction of others, to communicate emotions, and to communicate the nature of relationships (Argyle & Cook, 1976). In mathematics education, research on multimodal communication between teachers and their students is relatively recent (Radford, 2008; Arzarello et al., 2009).

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Recent reviews have summarized the research on eye movement related to mathematics education. A review on eye tracking research on learning (Lai et al., 2013) identified seven main areas of research: patterns of information processing, effects of instructional design, re-examination of existing theories, individual differences, effects of learning strategies, patterns of decision making, and conceptual development. From their review within mathematics education, Strohmaier et al., (2020) concluded that eye tracking seemed particularly beneficial for studying processes rather than outcomes, for revealing mental representations, and for assessing subconscious aspects of mathematical thinking. Furthermore, in a review of PME proceedings until 2018, Lilienthal and Schindler (2019) found altogether 33 papers using eye tracking, the earliest appearing in 2013. Six of these papers came from the MathTrack project. Other active PME researchers in this area are Lilienthal and Schindler themselves (6 papers) and Shvarts (e.g. 2018) with her collaborators (10 papers). The trends indicate an increasing interest in eye tracking, especially on dual and mobile eye tracking.

I will next do a brief review of the research on visual attention, focusing on problem solving and social interaction. Then, I will give an overview of the MathTrack project that examines the role of visual attention in the multimodal social interaction in the classroom contexts of collaborative non-routine problem solving. Finally, I will make recommendations for future research on visual attention in mathematics classrooms.

VISUAL ATTENTION AND VISUAL INTERACTION

The study of eye movements is based on the premise that gaze and thinking are related (eye-mind hypothesis, Just & Carpenter, 1980). Studies on human perception show that we can identify finer structures such as letters and the fine articulation of gestures only in the fovea of the eye, which spans less than two degrees of our perceptual field (Gullberg and Holmqvist, 1999). From a distance of 5 meters, that is about 17 cm in diameter, and for normal reading distance (40 cm) the diameter is about 1.4 cm. This means that we cannot recognize symbols, facial expressions, or finer gestures unless we look at the target. Outside the foveal area, light and motion recognition is good, (Gullberg and Holmqvist, 1999), allowing us to direct our gaze at interesting targets originally observed in our peripheral vision.

The nature of visual attention

Most important elements of eye movement are fixations (brief pauses when the eye is immobile) and very fast transitions called saccades to next fixation (Gullberg and Holmqvist, 1999). Perception takes place during fixations, which vary from some tens of milliseconds to a few seconds in duration. Fixations are typically around 250 milliseconds when reading, while in a natural situation (making tea) they vary more and are longer, being on average about 400 milliseconds (Land et al., 1999).

Different methods have been developed to analyse eye movement behaviour. As in all areas of research, a phase of qualitative research has been necessary to get a basic understanding of the eye movements in a specific task (e.g. reading, social interaction).

On this foundation, research has developed into a more systematic quantitative research with mostly experimental designs in laboratory settings. Some eye-movement analyses require pre-defining areas of interest (AOI) while some allow the areas to emerge trough the analysis. The data may include number and durations of fixations on AOI, sequence of transitions between different AOI, distance and speed of saccades, but also blinks and pupil dilation. Typical analysis is based on the knowledge that higher fixation frequency or longer duration of fixation often mean either greater interest in the target or that the target is complex and difficult to encode.

While eye-movement research is based on the assumption that what we look at tells something about our thinking, the relation between eye movement and cognition is not straightforward. One key issue is to make a distinction between more automatic (involuntary or bottom-up) attention regulation and more conscious (voluntary or top-down) attention regulation (Noudoost et al., 2010). This is well illustrated in the seminal experiment by the psychologist Afred Yarbus, where people were looking at the same painting with different instructions, each instruction leading to different viewing pattern (Tatler, et al., 2010). Moreover, in a social setting, the attention is often directed through the interaction with others. For example, pointing gestures (McNeill 1992) and gaze (e.g. Gullberg & Holmqvist, 1999) are important for directing attention, to the extent that they are typical means that magicians use to misdirect audience attention, when performing magic tricks (Kuhn et al., 2014).

Even when one fixates on a target, it is possible that one's attention is on something else, leading to inattentional blindness (Memmert, 2006). A classical example of this is when research participants watch a video and try to count how many times the players with white shirts pass the ball, they don't notice that a research assistant in a gorilla suit walks amongst the group of players, pounds her chest, and walks away (Simons and Chabris, 1999). Eye tracking (Memmert, 2006) confirmed that those who noticed and those who failed to notice the gorilla had equal amount of fixations on the gorilla. Hence, the blindness was not due to not seeing the gorilla, but because of not attending to the gorilla. Moreover, Memmert (2006) found out that expertise with basketball increased the likelihood to notice the gorilla. This suggests that those who have more automatized processes for the task can better notice unexpected stimuli.

Eye-movement research in natural contexts is more difficult and has evolved more slowly than research in laboratory settings (Tatler et al., 2019). Important areas of more recent advance in eye movement research have been in studying social interaction and perception in action (Foulsham, 2015), both relevant for classroom research.

Visual attention in problem solving

Regarding insight problems, experts (i.e., high-performers) are known to find the task-relevant features of the visual information faster than novices and their visual attention is focused more on the relevant areas of the visual stimulus (Gegenfurtner et al., 2011). Novices display more attentional transitions than experts while they also use longer gaze sequences for each task, compared with more expert counterparts (Kim et al., 2014). Knoblich et al. (2001) add that even novices are more likely to solve a task requiring insight successfully if they attend to the relevant areas.

The value of attentional transitions (i.e., switches between regions of interest) has been highlighted for geometry (Kim et al., 2014). A scanpath analysis (i.e. sequence of fixation targets) indicated that successful and unsuccessful solvers have mutually inverse direction of fixation targets. Specifically, unsuccessful solvers struggle both with decoding the problem and in locating relevant information (Tsai et al., 2012). Even teachers demonstrate the same expert—novice contrast in a concept-mapping task measuring subject-knowledge (Dogusoy-Taylan & Cagiltay, 2014). Though both teacher groups followed the same overall strategy in solving the problem, expert gaze focused more on relevant regions than novices.

Eye movement behaviour in a social context

Social interaction includes many processes related to eye movement. Skarratt et al. (2012) summarize earlier research to show that humans “prioritize other humans, their faces and, in particular, their eyes when viewing natural scenes” (p. 3). They conclude that eye movement behaviour is different when there is an actual person to look at compared to watching a video of a person. The potential for interaction seems relevant to eye movements. In social interaction, gaze can be used to make or avoid eye contact (Laidlaw et al., 2011), to communicate the direction of attention (Gullberg & Holmqvist, 1999; Skarratt et al., 2012), and more specifically to build joint attention (Pfeiffer et al., 2013).

The teacher has an important role in the classroom social interactions. There are two main functions for teacher gaze. One is the attentional (information seeking) function and the other is the communicative (information giving) function (McIntyre et al., 2017). An important foundation for successful collaborative work is socially shared regulation of learning (joint regulation of cognition, metacognition, motivation, emotion, and behaviour; Panadero & Järvelä, 2015). As gaze is important in social interaction, it is quite likely also involved in socially shared regulation of learning.

MATHTRACK RESEARCH PROJECT

The motivation for MathTrack was threefold. First, we wanted to learn about student visual attention when they solve mathematics problems as a group. After all, few studies have examined the visual attention during collaborative processes. Second, we wanted to learn about teacher visual attention when they observe and facilitate such problem solving activity. The classroom is rich in rapidly changing visual information, requiring efficient navigation across potential targets while teacher’s gaze is also an important communicative tool, having potential to direct student attention. Third, the short history of earlier work doing multiple person mobile eye tracking in natural context challenged us to develop new methodological solutions.

Methods

The MathTrack project used mobile eye-tracking devices and the algorithms and software developed in the Finnish Institute of Occupational Health (Toivanen et al., 2017). Toivanen worked for the MathTrack project and manufactured the eye-trackers. The method utilizes a 3D model of the eye, making the trackers robust to motion. The accuracy of the device was approximately 1.5 degrees of the visual angle, which is comparable to or better than commercial alternatives. The device consists of a glasses-like frame equipped with some electronics and three mini-cameras connected to a computer that was carried in a backpack (see Figure 1), allowing the participants' freedom to move. The software on the computer recorded the video frames and produced a video of the scene camera, superimposed with a gaze point. The frame rate of the video varied according to the amount of light; optimally, it was 30 fps.

For the main study, we collected data from seven ninth-grade mathematics classrooms. When recruiting participating students among volunteers we including both male and female students with both positive and negative affect towards mathematics. Moreover, in three of the classes students used GeoGebra for solving the problem, while in the other four they worked with pen and paper.

Three stationary video cameras and several microphones recorded the actions and conversations of the students and the teachers. Smartpens recorded students' writing and screen capture videos recorded students' work on computers. Most importantly, five sets of wearable eye-tracking devices recorded the eye movements of the teacher and the four focus students. We synchronized the camera clocks before each recording, but also used a physical clapperboard to signal the beginning of the recording to be able to synchronize the multiple channels of data that we collected.

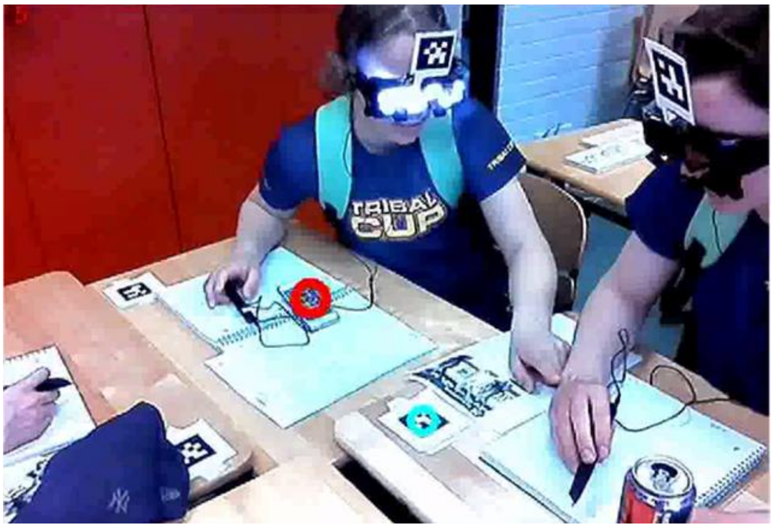

Figure 1. A frame from eye-tracking video showing students wearing the trackers. The red circle indicates the computed location of teacher gaze and the blue circle indicates the visual marker that is closest to gaze target. The bright light around student eyes is infrared, which is invisible to the naked eye.

For each class we recorded two mathematics lessons. During the first lesson, we calibrated the devices and let the teacher and the students get used to the equipment and researchers. The actual research data was recorded from the second lesson. For this lesson, the researchers gave a non-routine task (a four point Steiner tree problem) to the teacher in advance and instructed them to organize the lesson in a certain way. The students first worked on the task individually, then in pairs, then in groups of four, and finally there was a whole class discussion. The researchers instructed the teacher to engage in activating guidance, using questions and not revealing the key idea of the problem (Hähkiöniemi & Leppäaho, 2012).

After the second lesson, we individually interviewed the teacher and the four focus students with the lesson video as a stimulus. Specifically, we asked the students about the moments when they experienced curiosity, frustration, flow, anxiety, or boredom. The focus on teacher interviews was their observation of the focus students' progress and their decisions regarding when and how to intervene with different groups. We also collected questionnaire data from target students and teachers.

RESULTS

Some of our early observations related simply to the overall nature of teacher and student eye movements in mathematics lesson. One thing that we observed, but have not reported until now, is that teacher eye movement is much more volatile than student eye movement. Teachers really seem to pay attention to everything. We also noticed that both students and teachers pay a lot of attention to others' faces (e.g. Haataja et al., 2019). These early observations paved way for more focused research questions.

Our first analyses of data were qualitative case studies. This was a way to start making sense of the complex data we had generated. We learnt things that in retrospect seem expected and even trivial. For example, we witnessed a student to observe teacher's gestures and gaze cues quite closely to follow and even predict where the teacher wanted her students to focus next (Garcia Moreno-Esteva & Hannula, 2015), and how a silently gazing student's eye movements indicated interesting cognitive activity (Hannula & Williams, 2016). As we progressed, we begun to apply mixed methods, using quantitative analysis to report the patterns of eye movement and qualitative analysis to give meaning to it (Haataja et al., 2019, 2021; Määttä et al., 2021).

Because earlier eye movement research in natural social settings was quite limited, we had to look at some fundamental methodological issues. One of these was how students and teachers experienced the data collection with all the extraordinary equipment around. Fortunately, none of them reported that the equipment or the presence of the researchers affected their behaviour or learning significantly. Some compared the experience to watching a 3D-movie: as soon as the action began, they forgot the goggles. However, we noticed that the device was sometimes inconvenient, leading to some students repeatedly adjusting them. We also noticed students paying special attention toward the devices, illustrated as jokes about the video recording of peer's behaviour or examining where the smartpen's camera is.

One hindrance for making progress with eye movement research in natural environments has been the slow manual annotation of fixations. When we started our project, it took us two hours to code one minute of eye movement data. While we got twice as fast with more experience, it was still slow and prone to errors. During the project, we developed ways to use visual markers (see Figure 1) in the learning environment to identify fixation targets automatically (Hannula et al., 2019).

We also had to resolve what measures to use for eye-movement data. Depending on the research question, we reported, for example, number and average duration of dwells (Haataja et al., 2019) or the distribution of fixation durations (Hannula et al., 2019). We also developed our own methods for analysing eye movement data.

For eye movement behaviour, the sequence of fixation targets is important. To compare long sequences of eye movements, we developed a method to synthesize the information from hundreds of fixations as a *scanning signature* (Garcia Moreno-Esteva et al., 2020). A scanning signature gives a visual representation of a person's eye-movement behaviour across targets. Figure 2 shows an example how these can illustrate the different eye movement patterns, in this case just before the key insight (Garcia Moreno-Esteva & Hannula, 2021). This method even computes values for the number of fixations on different targets and transitions between them, as well as their temporal average occurrences all of which can be used for quantitative analyses.

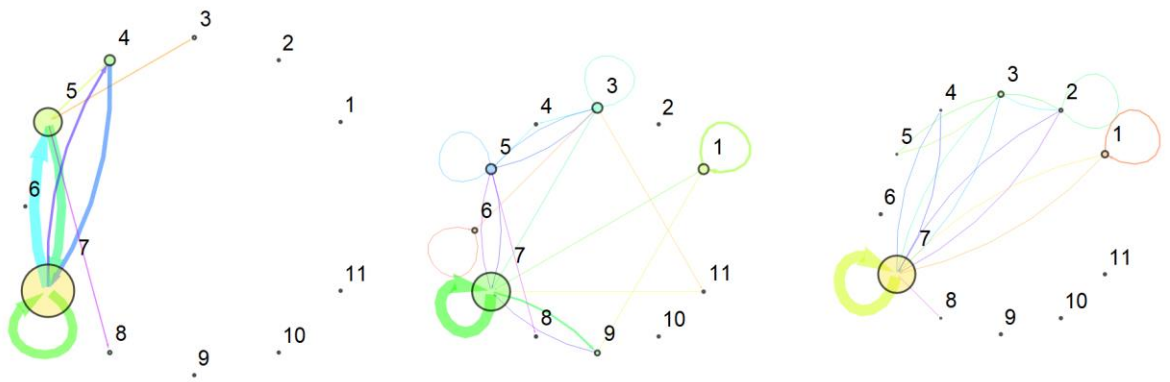

Figure 2. Scanning signatures for three students before an insight leading to the optimal solution (Garcia Moreno-Esteva & Hannula, 2021). The node 1 is the problem, nodes 2 to 10 are different solutions, and the node 11 is the area with computations. Circle diameter reflects the number of fixations on that target. Arrow thickness reflects the number of respective transitions. Temporal sequences are colour coded (red-yellow-blue-purple).

For our study on student visual attention during group problem solving (Salminen-Saari et al., 2021), we developed a novel method to measure the level of synchrony between two or more eye-movement patterns. Moreover, we extended the idea of joint attention (e.g. Tomasello 1995) to cover also episodes when students were looking at different representations of the same idea. Such *joint representational attention* was common, and it happened, for example, when students were watching similar solutions each on their own notebook. We found joint attention to be most frequent when students were verifying a solution and the moments of joint attention usually led to making progress in solving the problem. While our study was limited in scope, it suggests that joint attention is beneficial for collaborative problem solving process and that the moments of verification have strong potential to bring students attention together. Further work on joint visual attention, especially in mathematics, should pay attention to joint representational attention.

Our research on teachers' eye movement behaviour was the first one recording how teacher visual attention related to their scaffolding intentions and their interpersonal behaviour when facilitating group work. We found out that during cognitive scaffolding, teacher's visual attention was mostly on student written products, even though the student was then often trying to make eye contact with the teacher, while during affective scaffolding the teacher was more frequently watching the students' faces (Haataja et al., 2019). For the interpersonal behaviour, we used Kiesler's (1983) interpersonal theory, where communication is identified along two dimensions: communion (warmth) and agency. We found out that moments of higher teacher communion were often characterized by teacher-initiated eye contact, and related to more and longer student fixations on the teacher (Haataja et al., 2021). Our results highlight the importance of eye movement behaviour in teacher-student interaction. Specifically, moments of making and avoiding eye contact seem important communicative acts in this interaction.

LESSONS LEARNED IN THE PROJECT

Reflecting back our research so far, I will now summarize some lessons learned. I will first discuss the methodology of eye movement research in classrooms. Then I will reflect the research designs that are suitable for this approach.

Methods

There is extensive research done on mathematical thinking of students based on observational data and self-reporting. However, clinical interviews or think-aloud protocols distort the nature of social interaction and thus lack ecological validity. Interviews done afterwards – including stimulated recall – only have access to the student's post hoc reconstructions. Hence, these approaches have a limited possibility to access the automatic level of cognitive processes, which include both navigating in social interaction as well as interacting with the physical world. The automatic processes are typically very fast, fleeting, and inaccessible to introspection. Eye tracking data opens a new window to explore them and to contrast with earlier findings.

The eye movement research has developed into a paradigm that forefronts experimental research designs with carefully controlled stimulus and environment, typically in a research laboratory, watching a computer screen. Until recent decades, a major challenge for studying eye movements in natural settings was the lack of affordable and reliable equipment. While we now have the technology, the methods for data pre-processing and analysis developed for laboratory settings often do not work in natural settings. For example, in order to measure pupil dilation, it is important that the lighting is controlled, which is not possible in classrooms. Moreover, study designs in natural context are often explorative in the beginning, and as such, not always considered relevant by those entrenched in the experimental paradigm. This may lead to difficulties in publishing research.

With a novel and highly technical research method, we encountered several issues relating to research ethics (for more thorough reflection, see Hannula et al., 2022). The first issue concerned the nature of video research, which, according to Everri et al. (2020) has still surprisingly little ethical guidance. As people are recognizable from the video, the data becomes effectively a personality register, setting constraints for storing and sharing video data for research purposes. Moreover, video data may reveal 'special categories of personal data', such as racial or ethnic origin or religious beliefs (Finnish Social Sciences Data Archive, n.a.), making the personal data sensitive. As video data can be used to address questions not foreseen at the time of data collection, it is challenging to describe the intended research to participants, their guardians, and ethics review boards in a way that is at the same time informative and not unnecessarily restrictive. This applies especially to eye tracking in naturalistic settings, where participant gaze may reveal more than they expect. This has made us aware of our responsibility for being sensitive to what we analyse and report.

Another ethical issue for the project was the definition of the physical integrity when wearing the eye tracker (Hannula et al., 2022). When new technology is used for social and behavioural research, it requires revisiting old definitions as some may be more invasive in terms of privacy or bodily experience than others (e.g., Duru, 2018). In Finland, only studies involving deviation from informed consent or risks—such as a violation of physical integrity—must be reviewed by the ethics review board. It was not clear whether the wearable trackers would be considered to fall into this category. Our study actually became a precedent for re-defining 'intervention in physical integrity'. When the Finnish National Board on Research Integrity (2019) revised the national guidelines for ethics review for non-medical research involving human participants, they added a new definition for the intervention in the physical integrity to happen if participants cannot free themselves from the devices within a reasonable time.

There were several lessons related to management of staff and data. Having a novel, highly technical setting requires specialized technical staff. MathTrack would not have been possible without Miika Toivanen. He had been one of the developers of the mobile eye-tracking glasses and the related software before working in the MathTrack project. Collecting and post-processing the eye-tracking data was his special expertise. The necessity for highly specialized technical expertise, embodied in a single person, makes such research vulnerable. Therefore, one of the priorities in his work was to document the method and to teach it to other researchers in the project

Data management is a particularly complex issue in a project like ours. Because of the large number of video files, such projects need more data storage space than usual in educational sciences. In MathTrack, we stored for each lesson altogether 28 video files, including stimulated recall videos, screen captures, and the raw data from ten cameras recording eye movements. During the project, we realized how important it is to have a good metadata for each file and a clear structure in the data archives. This was highlighted because of several researchers and research assistants doing different analyses on different parts of the data. While we had a master document for each file in a joint network folder, doing the actual analysis required downloading the files on one's own computer. Keeping track of these processed files and deciding when and which ones to upload in the network file became a non-trivial task.

Some wearable eye-tracking devices are sensitive to movement. In natural settings with longer data collection (e.g. full lesson) it is likely that the device will be touched. Even a slight movement (called "slippage") will reduce the accuracy of data. On the other hand, our device was robust against movement to the extent that the participant could remove the device and put it back on later without a need to calibrate again. This allowed us to calibrate the equipment one day and to focus on uninterrupted collection of data on another day. When doing eye tracking in natural settings, the robustness against slippage is an essential feature.

To avoid laborious and error-prone human annotation of fixations, I warmly recommend using visual markers in the environment. In our context, we were able to identify the location of student gaze automatically for 74% of their fixations. Moreover, this allows also generating heat maps, a useful way to illustrate how visual attention is distributed. While automatic object recognition is rapidly advancing and is successful in some contexts (e.g. Jongerius et al., 2021), its performance seems not yet sufficiently reliable for eye movement research in real classrooms.

For data analysis, we point out the potential of methods based on graph theory, which underlie both our method to recognize moments of joint attention (Salminen-Saari et al., 2021) and the method to synthesize fixations and saccades as a scanning signature (Garcia Moreno-Esteva et al., 2020).

The basic assumption of eye movement research is that the fixation on a target is informative about our visual attention. It seems that peripheral vision is sufficient for some elements, such as large gestures. We also believe that in a natural context teacher and students may sometimes rely on their memory of something they have looked at. Such peripheral vision and memory may be sufficient in some situations. Yet, based on our experiences, the basic premise of attention correlating with fixations seems justified for research even in natural contexts. However, making conclusions based on fixation duration is more difficult. When reading, longer fixations typically indicate more demanding cognitive processing. However, in our study the long fixations when working with GeoGebra were more commonly related to the difficulty of moving the cursor to an intended location on the screen (Hannula et al., 2019). The use of gaze to define a target of action is something that seems to include long fixations (see also Land et al., 1999).

Research design

Eye movement data is continuous, providing hundreds of data points each minute. While eye movement data can be used to analyse how much different targets receive attention overall, this type of dense data has a specific potential for analysing processes. Combining it with other continuous data seems especially fruitful (e.g. Haataja et al., 2021). Transitions between targets inform how things are connected. The sequence of events can be used to examine which ideas were attended to before coming up with a new idea (e.g. Garcia Moreno-Esteva & Hannula, 2021).

Eye-movement data collected in natural contexts is not well suited for between-person comparisons. While eye movements have some universal characteristics, there is also significant idiosyncracy, i.e. each person has their personal pattern of fixations and saccades (Poynter et al., 2013). Our data showed significant variation in the distribution of fixation durations, specifically between teacher and the students but also between different students. The individual idiosyncrasies led us to leave out several students from our analysis comparing eye movements while using GeoGebra vs. using pen and paper (Hannula et al, 2019). In principle, this could be handled statistically if samples were sufficiently large, but that would significantly extend the data collection in scope. On the other hand, eye-movement data suits very well within-person analyses, providing convincing evidence even with small samples (e.g. Määttä et al. 2022).

One of the benefits for research in natural classrooms is the ecological validity of the data. However, we found it useful to control the learning situation by asking the teachers to conduct a lesson around the same task and using the same instructional approach. This way, we could pool data from different lessons for a meaningful analysis. Yet, we also made systematic variation to the context by asking some of the teachers to have their students solve the task using GeoGebra. As it is likely that the amount of data researchers can collect in natural contexts will be quite limited also in the future, we recommend reducing the variation of contextual variables. You should be clear to identify what you wish to vary and try to limit the variation of other features.

CONCLUSIONS

One of the key things to know about eye movement research "in the wild" is that it is not easy. The eye-tracking methodology requires investing in devices and technical expertise. Hence, eye-movement research in classrooms should focus on such questions that are difficult or impossible to study with other approaches.

One obvious area to continue exploring is teacher eye-movement behaviour in the classroom. While we have used the eye-movement data to examine teacher-student interaction (e.g. Haataja et al., 2019, 2021), such data could inform also research on teacher's professional vision and decision-making (Stahnke et al., 2016). We know that eye movement behaviour watching live people is different from eye movements when watching a video. Therefore, mobile eye tracking can access such aspects of teacher visual attention that other methods cannot.

Because of the importance of gaze in social interaction, collaborative work is another area where eye movement research has potential. We have examined joint visual attention during problem solving (Salminen-Saari et al., 2021), which is definitely an area worth further examination. Moreover, studying the student eye movements in their multimodal communication would inform about when and what visual information (diagrams, gestures, facial expressions) students attend to in such communication.

Another promising avenue for research on visual attention in classrooms is to examine eye movements of those students whose attention deviates from average. In a meta-analysis Armstrong and Olatunji (2012) synthesized research on how affective disorders bias attention towards emotional stimuli. As mathematics anxiety is an enduring problem in mathematics education (e.g. Hannula, 2018), studying anxious students' eye movements in classrooms is another valid venue for mobile eye tracking. Furthermore, eye tracking has been used to study the attentional processes of people with attention disorder (Maron et al., 2021) or those on autism spectrum (Laskowitz et al., 2022). So far, this has been done almost exclusively in laboratory settings. As the trend in education is to integrate students with special needs into ordinary classrooms, it is of utmost importance to study their attention in these natural contexts.

As an overall conclusion, it is clear that mobile eye tracking in real classrooms is a viable research approach. It provides a unique approach to studying the visual attention of teacher and students. Specifically, it captures the visual processes as they unfold during the lesson – rather than studying them in retrospect. Moreover, while eye tracking is not mind reading, it provides new information about the automatic and non-conscious processes in mathematics teaching and learning. Hence, it should be an essential part of the research agenda on mathematical thinking and interaction.

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INTERTWINING CUMULATED AND CURRENT MATHEMATICAL PROBLEM-SOLVING DEVELOPMENTS TO FRAME AND SUPPORT TEACHING PRACTICES

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The COVID-19 pandemic has modified social and school activities worldwide. The irruption of digital technologies to organize and carry out educational tasks not only shaped and marked the implementation of school activities during the critical period of the pandemic, but also, contributed to identify resources and new ways to interact with students to foster, monitor, and assess their learning. How could cumulated and current results and advances in problem-solving research be interpreted to frame teaching and learning scenarios to coordinate teachers and students' online and face-to face work? To address this question, a synthesis of problem-solving themes and research results is reviewed to sketch a route to frame a teaching/learning scenario for students to develop problem-solving competencies.

INTRODUCTION AND BACKGROUND

Human beings constantly face a variety of questions or problem situations. That is, posing questions and looking for different ways to solve them are activities that distinguish human behaviours from other living species performances. How many matches will be played in a tennis tournament? How much water does a family consume in one month? How much land is needed to plant 100 apples trees? How can a quadratic equation be solved and what does that solution mean? What should a lesson on the concept of derivative involve? How to frame problem-solving learning scenarios that consider teachers and students' remote and face-to-face work? Some problems might require the direct application of rules and procedures to solve them (e.g., how to prepare a meal or buy a cell phone), while others demand that individuals access specific resources and strategies as part of a systematic plan to approach those problems (e.g., engineering, science, or mathematical problems). In general, problem-solving processes involve the identification and use of knowledge resources, problem representations, and strategies to reason, explore, and solve those problems (Schoenfeld, 2022; Santos-Trigo, 2019).

In this context, how people develop knowledge and strategies to formulate and solve disciplinary problems have been themes of the academic agenda across different areas, or disciplines including mathematics education. Thus, research programs that aim to understand problem-solving approaches to learn mathematics have provided significant information and frameworks to implement learning scenarios that foster students' construction of robust mathematical knowledge and problem-solving performances (Foster, Burkhardt, and Schoenfeld, 2022).

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Furthermore, the development of disciplinary knowledge depends on available artifacts or tools that an individual or group of people activate during the process of understanding concepts or posing and solving problems. Tools amplify human cognition in areas that involve memory, computation, representation of ideas, use of symbols, etc. (Arcavi, 2020). For instance, Babylonian mathematicians (1830-1531 BC) used clay tablets as a tool to register problems, methods, and results in areas such as arithmetic, geometry, and equations. Tools like the straightedge and compass inspired the ancient Greek mathematicians to work on geometry problems. Euclid (325 BC) introduced the axiomatic method to support and validate mathematical relations and results. Descartes (1596-1650) developed the coordinate system that originated an algebraic approach to the study of geometry (analytic geometry). In general, an artifact or tool is a material object such as a straightedge or a compass, an abstract object (Cartesian system or an algorithm to solve quadratic equations), or a digital application such as GeoGebra that offers a set of affordances to represent, explore, and solve mathematical problems.

Hence, mathematical developments can be traced and explained in terms of what problems were posed at different times, the tools and methods used to approach and solve those problems, and the types of results or achieved solutions. Features of mathematical developments or advancements in different time periods reveal what were the prominent tools and ways of reasoning used to present and support mathematical results. Problem approaches and attempts to find solutions and results provide relevant information regarding themes, contents, research directions, and scope of the discipline. Indeed, Halmos (1975) pointed out that "what mathematics is really all about is solving concrete problems" (p. 467). How are mathematical problems formulated and what the process of approaching and solving problems involve? How do students develop problem-solving competencies? These questions have inspired mathematicians and math educators to investigate what the process of formulating and solving mathematical problems entails and how students understand mathematical concepts and solve problems. Although research programs are designed and pursued within regional contexts or educational traditions, it is common that research communities get shaped and influenced by global research developments (Liljedahl & Santos-Trigo, 2019).

The COVID-19 pandemic challenged all educational systems and institutions moved to remote or hybrid teaching scenarios in which teachers and students rely on digital apps to work and monitor the school tasks. Foster, Burkhardt & Schoenfeld (2022) recognized that school systems were not ready to face the needs and challenges required to quickly answer the problems that emerged during the pandemic. Institutions relied on the use of digital apps to continue and follow up school tasks and instructors and students encountered obstacles such as uneven internet access to activate the apps, the absence of proper materials to support instruction, and a lack of tools to monitor and assess students' learning and problem-solving competencies. Schools also faced internal problems to provide teachers and students essential resources and constant support to carry out their tasks and assignments. It became evident that this new learning space mediated by digital tools opened new themes to discuss, such as curriculum adjustments, tool appropriation by teachers and students, supporting materials, access to online platforms, interactions between teachers and students, and the development of appropriate assessments to register student's achievements. To this end, teachers and students' problem solving and learning experiences developed during their remote work not only expanded the ways to deal with mathematical tasks, but also, they need to be integrated in current teaching practices. Thus, it becomes relevant to analyse and discuss how extant research results and developments in mathematical problem solving could be interpreted and extended to support and frame teaching practices in current learning spaces.

CONCEPTUAL FRAMEWORKS IN MATHEMATICAL PROBLEM SOLVING

Mathematics and mathematics education communities have been interested in understanding and characterizing what the process of formulating and solving mathematical problems entails. A robust characterization of what problem-solving processes involve provides important information to support mathematics curriculum proposals and ways to frame learning scenarios. Polya (1945), based on his own experience as a mathematician, proposed a framework that identifies four intertwined problem-solving phases: understanding the problem, devising a solution plan, carrying out the plan, and looking back. In terms of teaching, Polya suggested that teachers should foster their students' inquiring approach to work on all problem-solving phases. Thus, questions are a medium for students to understand and make sense of problem statements, to propose, implement, and monitor a solution plan, to reflect on problem-solving solutions, and to extend problems initial domains, methods, and mathematical results. In addition, Polya illustrated how the use of heuristic strategies could help students work on problems or overcome difficulties that might appear during their solution process.

Thus, looking for particular or simpler cases, exploring symmetry, searching for patterns, or working backwards, etc. are relevant strategies for students to consider and implement during their problem-solving approaches. Polya's work has provided foundations to support research programs in mathematics education. For example, Schoenfeld (1985) implemented a mathematical problem-solving course for university students that enhanced the participants' use of heuristics during the process of dealing with nonroutine problems. This course was part of a research program to characterize what mathematical thinking entails and the extent to which the students' systematic use of heuristics in approaching mathematical problems contributes to the students' learning and to their development of mathematical problem-solving competencies.

Based on empirical results, Schoenfeld (1985) proposed a conceptual framework that characterizes and explains students' problem-solving behaviours in approaching mathematical tasks. He identified four interrelated dimensions that influence and shape students' problem-solving performances: (1) the students' identification and access of resources or knowledge base to understand and work on the problems; (b) the use of cognitive or heuristic strategies to make sense of problem statements and to represent and explore ways of solving those problems; (3) the use of metacognitive or self-monitoring and control strategies to make decisions regarding what path solutions to pursue to approach mathematical tasks; and (4) students' conceptions and beliefs about mathematics and solving problems. That is, students' beliefs systems shape and permeate how they behave and engage while working on mathematical tasks. These categories have oriented the design of curriculum proposals and helped teachers to structure and implement problem-solving activities in learning environments. Indeed, Schoenfeld's framework became a seminal contribution that continues to be a significant referent in problem-solving research. Koehler and Mishra (2011) proposed a framework that integrates content with pedagogical and technological knowledge to implement a technology-enhanced learning environment. Specifically, they recognize the importance for teachers to discuss and analyse what changes the systematic use of digital technologies bring to mathematical contents and pedagogy to learn the discipline.

Curriculum proposals recognize that problem-solving activities are central to structure mathematical contents and support teaching scenarios worldwide. The Common Core State Mathematics Standards curriculum proposal (CCSMS) (2010) identifies problem solving as a process standard supporting core mathematical practices that involve reasoning and proof, communication, representation, and connections. Curriculum proposals, in general, promote the use of digital technologies throughout the study of the discipline; however, the irruption of digital apps in pandemic times makes it necessary to analyse what the student's tools appropriation entails and what ways of mathematical reasoning students develop with the consistent use of those tools. Conceptual frameworks are key referents to support research and to orient problem-solving instruction, and current events related to the COVID-19 pandemic require to update and adjust such problem-solving frameworks and curricula.

THE IMPLEMENTATION OF PROBLEM-SOLVING APPROACHES

In terms of implementing problem-solving instruction in classrooms, Schoenfeld (2022) recognized the importance for students to study and approach mathematics as a sensemaking discipline. Thus, students themselves have an opportunity to develop mathematical connections and insights, and consequently, they tend to conceptualize their learning as a problem-solving activity that they can understand, participate, and make it their own. To this end, there are salient principles that support the construction of a learning scenario that fosters problem-solving activities:

(a) An inquisitive or inquiring strategy to delve into concepts and to understand and work on mathematical problems. That is, learners are encouraged to constantly pose and pursue questions to examine and understand definitions and concepts, to make sense of problem statements, and to solve, extend or formulate new problems. The

Nobel laureate I. I. Rabi mentioned that, when he came home from school, “while other mothers asked their kids ‘*Did you learn anything today*?’ [my mother] would say, ‘*Izzy, did you ask a good question today*?’” (Berger, 2014, p.67).

(b) Tasks, problems, or mathematical situations (learning a concept) are the vehicle and departure point for students to participate and engage in mathematical discussions and reflections. In this context, problem-posing activities emerge and are part of what learning mathematics involves. The idea is that students share their task’s approaches with peers and discuss the extent to which their solution methods could be applied to solve others’ family of problems. In this process, they look for problem extensions and pose and explore new related problems (Schoenfeld, 2022).

(c) Looking for multiple ways or methods to solve a problem is an essential activity for students to discuss concepts, resources and strategies that appear in each way to solve the problem. Thus, students should have an opportunity to always look for several ways to represent, explore, and solve mathematical problems. In Chinese classrooms, it is common that teachers implement a problem-solving approach around three intertwined activities: *one problem multiple solutions*, *multiple problems one solution*, and *one problem multiple changes* (Cai & Nie, 2007).

(d) Learning mathematics and solving problems involve a continuous process in which students openly discuss and refine their ideas within a community that values and fosters individual and collective participation and contributions. That is, learning a concept or solving a mathematical problem involve achieving partial goals that eventually get extended to connect and delve into other concepts and new problems.

The implementation of a problem-solving approach to learn mathematics might take different paths and is often influenced by countries’ educational systems and traditions. For example, in the Netherlands, problem-solving activities are structured and fostered within the Realistic Mathematics Education perspective, that privileges situating mathematical tasks in realistic and familiar contexts (Doorman, et al. 2007). Thus, students have an opportunity to access their mathematical knowledge and strategies to approach those tasks. The goal is for students to work on tasks that are embedded in real world, fiction, or mathematics contexts which are familiar to them, so that they can model and solve those problems through mathematical concepts and resources.

THE ROLE AND IMPORTANCE OF PROBLEMS AND TASKS IN MATHEMATICS LEARNING

To support and foster a problem-solving approach implies providing opportunities for learners to develop a way of thinking that is consistent and compatible with mathematics practices. What are the relevant features that distinguish a students’ problem-solving thinking to learn mathematics? A salient feature is that students conceptualize and think of mathematical tasks as a set of dilemmas that they need to elucidate and solve in terms of mathematical resources and strategies. Thus, they need to problematize what they learn (mathematical concepts) and their way to solve problems.

Problematizing means that students pose questions to examine or delve into concept meaning and to engage themselves into the problem-solving process. Thus, questions are a means for students to comprehend concepts and to solve mathematical dilemmas. Accordingly, the task of learning concepts such as the concept of function or derivative can be framed as a problem-solving endeavour in which students engage in learning activities that help them represent, explore, and delve into the concept meaning and ways to apply it to solve diverse problems.

It is recognized that for teachers and students to understand, learn and construct mathematics knowledge, they need to solve different types of problems. In general, problems or exercises that appear in mathematics textbooks are routine tasks in which students activate contents and algorithms previously studied or they follow procedures and rules taken from worked examples that are applied to solve them. However, for students to develop a deep understanding of concepts and a robust mathematics thinking, they need to work and discuss nonroutine problems. That is, problems or tasks in which they need to identify problems' deep structure to access, activate and coordinate different resources, concepts, and strategies that are relevant to solve and extend those problems. For example, Selden et al. (2000) asked university students who had taken a calculus course to solve five nonroutine problems. An example of these problems is:

Find at least one solution to the equation $4x^3 - x^4$ *= 30 or explain why no such solution exists. (p. 138).*

It is observed that the terms involved in the statement do not explicitly refer to calculus concepts (derivative, maximum, minimum, etc.); and the students experienced serious difficulties to solve it, since they tried to use algebraic methods to find possible roots.

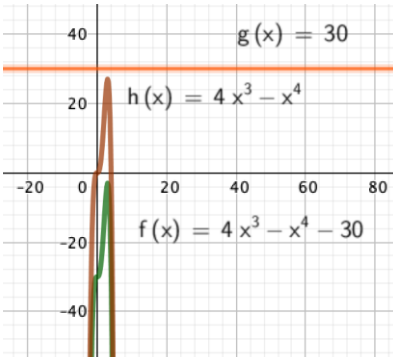

Figure 1: Graphic representation of the equation elements and the associated function

They failed to apply calculus concepts to determine and analyse the behaviour of the associated function (increasing, decreasing, maximum/minimum points, etc.) to find that the function does not intersect the x-axis and therefore, it does not have real roots. In this case, the deep structure of this example involves recognizing that calculus concepts and resources are important to solve the task. Figure 1 shows a graphic representation of this problem.

How can teachers think of nonroutine problems and incorporate in their teaching practices? To address this question, it is argued that even routine problems that appear in textbooks can be transformed by students into a set of activities or new extended problems that require more than the application of certain rules or procedures to solve them (Santos-Trigo, 2019; Santos-Trigo & Reyes-Martínez, 2019). To this end, it is important that learners always think of multiple ways to approach and solve the problems. These solution paths might involve representing or modelling the problem algebraically, quantifying objects attributes to find and explore patterns, or constructing a dynamic model to find mathematical relations among objects to solve the problem geometrically (Santos-Trigo, 2019, 2020).

DIGITAL TECHNOLOGIES AND MATHEMATICAL PROBLEM SOLVING

Digital technologies and online developments are transforming ways in which people communicate, interact, exchange information, and solve daily problems. In education, it is common that teachers and students rely on diverse digital technologies and online developments or platforms to deal with teaching and learning tasks. Thus, they look for information to solve problems, consult short videos to understand concepts or they check solved examples to identify strategies and possible ways to solve other problems. In this context, curriculum proposals and teaching practices or learning scenarios need to be revised to explicitly incorporate and value the students' coordinated use of digital technologies during the process of learning concepts and solving mathematical problems.

Online platforms and encyclopedias (Khanacademy, Wikipedia, YouTube) offer activities and information that students can consult to review or extend their understanding of mathematical concepts, to contextualize themes involved in problem statements, to check videos that explain concepts or to analyze solved problems. It is common that students turn to available mathematical learning resources before or during the development of classes. Here, teachers should guide their students on how they can use platforms content to work on mathematical tasks (Santos-Trigo, 2020a). Similarly, mathematical action technologies (GeoGebra, Wolframalpha, etc.) provide a set of affordances for students to model and explore problems dynamically (Santos-Trigo, 2020a). As a result, they can identify and analyse mathematical behaviours of objects and possible relationships that might appear from moving objects within the problem model. For instance, dragging objects such as points, lines, or segments within the problem representation becomes a powerful strategy for students to look for properties and patterns associated with objects' attributes behaviors. The use of digital technologies might also support students' collaborative work. "Online resources that are now at the disposal of our students enable them to direct their own learning, collaborating with teachers and fellow students all over the world" (Engelbrecht & Oates, 2020, p. 39).

In this perspective, technology affordances offer students an opportunity to dynamically model or represent concepts and problems and to extend the use of heuristics, and resources to identify objects' relations that are important to solve problems. The way in which problem statements are stated influences what resources, concepts, and strategies that students think of and activate to solve them. The idea is that students should work on a variety of tasks or problems situated in different contexts (Santos-Trigo, 2019, 2020a). For example, in a mathematical class, what do students see in Figure 2 and what mathematical questions could they formulate?

Figure 2: What mathematical questions could you formulate?

Terms and concepts that might appear in the students' questions include water, volume, time, taps, etc. and a possible problem statement can be posed:

A tank or container of water is filled with one tap in four hrs and another tap fills the same tank in 6 h, how much time is needed to fill the same tank when both taps are open at the same time?

This is a common word problem that students solve in secondary school level and a standard approach is to model algebraically the problem to solve it. The use of a Dynamic Geometry System (GeoGebra) provides affordances to represent and explore it geometrically (Santos-Trigo, 2020a).

a. The use of a Cartesian system. An approach to address this task might involve representing the given information in a Cartesian system in which it is important to decide what units to consider for the axes. For instance, coordinates for X and Y axis might include time units (hr) and volume of the tank (liters) respectively (Figure 3).

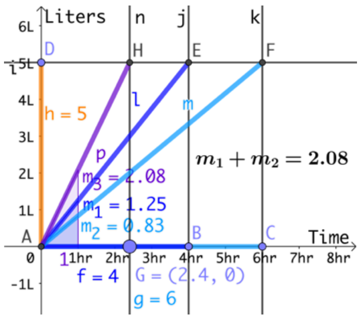

Figure 3: Using a Cartesian system to represent key information of the problem

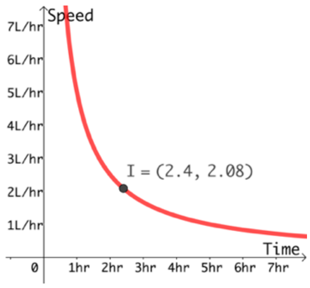

Figure 4: Exploring the behaviour of the slopes of a family of segments AH that appears when point G is moved along x-axis

Further, points B and C are situated on the time coordinates at 4 and 6 hr and represent the time in which taps 1 and 2 fill the tank respectively; point D at the value of 5 litres on the y-axis represented the volume of the tank. The tank volume is an arbitrary quantity since it does not appear in the problem statement. What do the coordinates of points E and F mean? The first coordinate represents the time in which the taps fill the volume of the tank, and the second coordinate is the total volume of the tank (five litres).

b. Connecting slopes of segments AE, AF with AH. Situate any point G on the x-axis (time coordinate) and draw a perpendicular to the x-axis that passes through point G. This perpendicular intersects the perpendicular to the y-axis that passes through D at point H. It is observed that the slope m_3 of segment AH changes when point G is moved along the x-axis. Figure 3 shows the position for point G in which the slope of segment AH is the sum of the slopes of AE and AF (the solution of the problem). That is, when both taps are open the tank is filled in 2.4 h (GeoGebra interactive model).

c. Graphic representation of the variation of slope of AH. How does the slope of AH change when point G is moved along time axis? Figure 4 shows a Cartesian system where the x-axis represented the time and as y-axis the slope values of segment AH. Thus, point I is defined as $I = (x(G), m_3)$ where $x(G)$ is the x-coordinate of point G and m_3 is the slope of AH. What is the locus of point I when point G is moved along the x-axis? Figure 4 shows the locus of point I when point G is moved along the x-axis, and when the coordinates of point I become (2.4, 2.08), it means that the two taps fill the tank in 2.4 hr.

Similarly, another approach might involve connecting the position of point F with the amount of litres that both taps fill at that time (Figure 5). In this case, the volume of the tank is 10 litres. That is, segment FN represents the sum of FI and FE. Segment FN represents the sum of litres that fill the tank when both taps are open at the same time. Then, when point N reaches the value of (2.4, 10) then the x-coordinate of point F is the time taken by the two taps to fill the tank (Figure 6), GeoGebra interactive model).

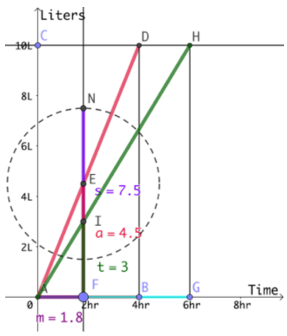

Figure 5: Representing the filling time and tank volume for each tap and both taps filling the tank

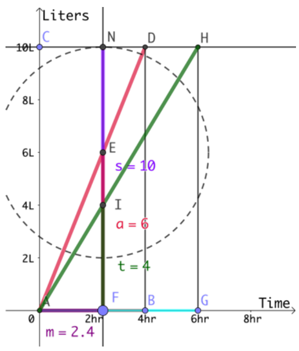

Figure 6: Moving point F to solve the task

d. Task parametrization. The idea is to express slope m_3 as the sum of slopes m_1 and m_2. That is, if AD represents the tank total volume l and x is the length of segment AG (x -coordinate of G) (Figure 3), then, the condition to solve the task is when $m_3 = m_1 + m_2$, that is, $\frac{l}{x} = \frac{l}{4} + \frac{l}{6} = \frac{3l+2l}{12}$, which leads to $x = \frac{12}{5} = 2.4$

CURRENT CONTEXT AND POST-CONFINEMENT LEARNING SPACES

The social confinement that was imposed worldwide to control the spread of the COVID-19 pandemic has altered and transformed not only the ways people communicate, interact, and carry out daily activities, but has also produced significant changes in the educational arena. Suddenly, teachers shifted from face-to-face teaching scenarios to remote or online models based on the use of digital technologies and students needed to adjust their ways to work on mathematical tasks. Thus, school settings and the structure and organization of learning environments were transformed worldwide to cope and deal with the effects and consequences of the prolonged COVID-19 pandemic confinement. Thus, each country, depending on its available infrastructure, resources, and traditions responded to the challenge of organizing and implementing online activities to continue school tasks. As a result, there appeared a variety of proposals to structure learning spaces and to carry out learning activities based on the use of digital technologies. A remote or online teaching/learning scenario might transcend the place and time where learning activities happen (Edwards, et, al., 2021).

A hybrid model that combines remote and face-to-face activities seems to emerge for structuring and organizing current learning spaces. Thus, the challenge is to provide teachers a continuous support and assistance to integrate available technologies into their teaching practices. Kearney, Burden and Schuck (2020) pointed out that "…given the tools that students currently have at their disposal, we might need to consider afresh both the curricula we offer and our lesson designs so that we can support students to learn in ways that are relevant and meaningful" (p. 13).

In a post confinement problem-solving approach to learn mathematics, both strategic and tactical plans become relevant for learners to engage in mathematical tasks. That is, a strategic plan helps them identify what resources or online developments to consult, what digital tools are important to use; how and with whom they should interact to understand concepts and to solve problems; what material needs to be revised, etc. While a tactic plan involves the actual actions that students take and perform to understand concepts and to develop problem-solving competencies. It involves the activation of technology affordances to model, explore, solve problems and to communicate results. It also includes the use of technology apps to discuss ideas and to share mathematical results. "Identifying problems, defining and implementing effective solutions, even adopting innovative strategies, is the core process of problem-solving" (Rovida & Zafferri, 2022, p. 106). Thus, during the process of understanding concepts and solving mathematical tasks, students pose and discuss questions, develop a mathematical language to express and share their ideas, and monitor and register their own learning. Thus, in a post-pandemic learning space, teachers need to orient their students on ways to work on remote mode that involve consulting online materials, analyzing selected videos about concept explanations, and working on online assignments. This students' remote work then is reviewed and discussed in a face-to face environment that fosters the individual and group students' participation.

THE CONSTRUCTION OF DYNAMIC MODELS, HEURISTICS AND PROBLEM SOLVING

It is argued that the construction of dynamic models of concepts and problems opens novel ways for students to explore concept properties and to solve mathematical problems. For instance, the study of conic sections that appears in an analytic geometry course privileges the process of finding the equation or algebraic model of loci of points that hold or fulfill certain properties. What about tasks for modelling and exploring dynamically conic sections? What routes might students take to grasp fundamental concepts of analytic geometry?

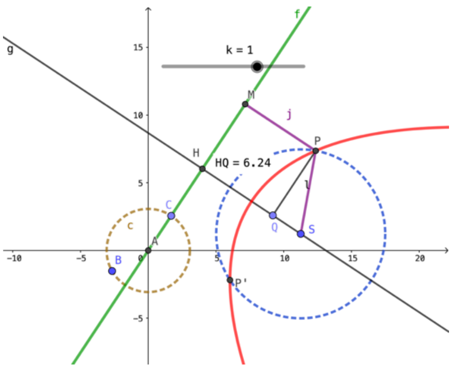

Figure 7: A dynamic model to study the conic sections
https://www.geogebra.org/m/ffzbgjwk

With the use of a Dynamic Geometry System such as GeoGebra, a dynamic model that involves a fixed line (directrix), a fixed point (the focus) and a point that moves in such a way that its distance from the focus bears the same ratio to its distance from the directrix leads to analyse the conic sections in terms of the ratio value (Drew, 1869).

Figure 7 shows a fixed line AC (directrix) and a fixed-point S, line HS is perpendicular to line AC, point Q is movable point on line HS, k is a given constant that takes values between 0 and 2 and the radius SP of the circle with centre at S is *k* times the length of segment HQ ($\boldsymbol{r = SP = k \times HQ}$). QP is perpendicular to HS, P is the intersection of perpendicular line QP and the circle with centre S, and PM is perpendicular to line AC, thus, the ratio $\boldsymbol{\frac{SP}{PM} = k}$, $(\boldsymbol{HQ = PM})$.

It is observed that points P and P' generate a parabola when point Q is moved along line HS, in this case $\boldsymbol{\frac{SP}{PM} = k = 1}$ which means that SP is always equal to PM (definition of parabola). What about if the value of k changes? What curve is generated by point P when point Q is moved along line HS? Figures 8 and 9 show the conic sections that appear when k takes values less and more than 1 correspond to an ellipse and a hyperbola.

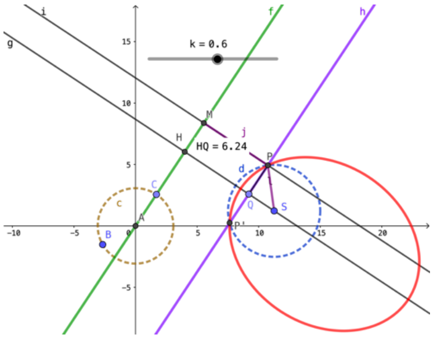

Figure 9: when the ratio $\frac{\mathrm{SP}}{\mathrm{PM}} = \mathrm{k} < 1$, the conic is an ellipse

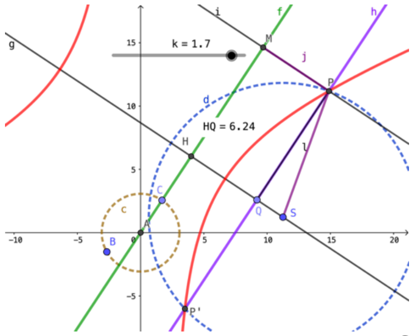

Figure 10: when the ratio $\frac{\mathbf{SP}}{\mathbf{PM}} = \mathbf{k} > \mathbf{1}$, the conic is a hyperbola

The two examples, the word problem and the conic tasks, show that the construction of the tasks' dynamic model allows teachers and students not only to extend traditional ways to deal with these types of tasks; but also, to activate a set of new problem-solving heuristics to represent, explore, and solve the problems. The use of the tool (GeoGebra) enhances the application of canonical heuristics such as examining particular or simpler cases or looking for patterns through the activation of tool's proper affordances. These include dragging objects, tracing loci, measuring object attributes, using sliders, etc. which become relevant to identify and explore mathematical relations to solve the tasks.

LOOKING BACK AND CONCLUDING REMARKS

Educational institutions life and activities were significantly altered due to the COVID-19 pandemic long social confinement. The irruption of digital technologies to organize and carry out educational tasks not only shaped and marked the implementation of school activities during the critical period of the pandemic, but also, contributed to identify resources and new ways to interact with students to foster, monitor, and assess their learning. What have we learned from the ways in which institutions faced the pandemic crisis? What changes are important to analyse and incorporate in school settings or learning environments and in mathematical curriculum proposals and their implementation to consider pandemic experiences? How to interpret cumulated research results and update conceptual frameworks to support post-pandemic learning spaces? These types of questions are relevant and part of the current research or academic agendas in mathematics education. To advance the discussion, it might be important to identify and frame conceptual issues to update and support a possible research agenda to address preservice mathematics teachers' education and teachers' professional development programs. Throughout this paper, four interrelated elements have been outlined to structure a learning space to guide and promote students' learning and development of mathematical knowledge and problem-solving competencies:

1. A problem-solving approach that privileges the students' development of an inquiring or inquisitive behaviour or method to delve into concepts and to solve problems. Questions are the medium for students to grasp concepts, to understand problem statements, to solve and extend initial tasks and to formulate new problems. To this end, students always look for different ways to represent, explore and solve mathematical problems and share and discuss the ideas, concepts, strategies, and solutions they used to work and solve the problems.

2. The systematic and consistent use of digital technologies. Researchers, teachers, and students relied on the use of different apps and online developments to work on school task and activities. Thus, it is common that teachers and students use a tablet, smart phone, or a laptop computer to activate communication apps (Zoom, Google Meet, or Teams) to implement and work on mathematical tasks. Similarly, mathematics action apps such as **GeoGebra** or **Wolframalpha** and online developments (**Wikipedia**, **KhanAcademy**, etc.) are important resources for students to use or consult to contextualize problem statements, represent, and explore tasks, or to review or extend their understating of involved concepts. In general, the use of these types of technologies and online developments not only provide students/teachers with an opportunity to extend their ways to work on mathematical tasks; but it also demands that teachers orient and guide their students on ways to select and use those resources.

3. A supporting system. During the pandemic crisis, teachers and students experienced different types of obstacles to implement remote activities that include robust access to internet, lack of proper materials to work on mathematical tasks, technical problems to use technologies, etc. Students' difficulties also include ways to receive feedback or answer to their questions or doubts that emerge while consulting online materials or working on mathematical tasks. In this context, a system to continuously support students is needed, and it might include a synchronous teacher-students interaction or the use of discussion forums including the use of chats or social networks (Engelbrecht & Oates, 2021).

4. Course materials and problem-solving assessment. Mathematical tasks or problems are essential for students to understand concepts and to develop problem-solving competencies. It was clear during the pandemic, that students are the centre for designing and implementing learning activities. From this perspective, all possible resources that students could rely on to understand mathematical concepts and to solve problems should be available and ready to be used during the students work. The material could include short videos in which an expert or teacher explains concepts and poses questions for students to follow and discuss with their pairs through communication apps. In this material, there will be examples of solved problems to illustrate the importance for learners to always look for different ways or methods to solve a problem. Here, students are encouraged to discuss concepts and resources used to reach the solution. It could also include links to platforms in which students can consult, review, and extend their understanding of involved concepts (Santos-Trigo, Barrera-Mora & Camacho-Machín, 2021). Throughout the course materials, students need to complete quizzes, and solve proposed problems to monitor their concepts understanding and problem-solving competencies. In terms of problem-solving assessment, the use of technology can also provide some tools for students to register and monitor their work and learning experiences. A digital wall or a problem-solving digital notebook is introduced for students to register and monitor their learning experiences (Santos-Trigo, et al, in press). Students are asked to record on weekly basis their work, questions, comments, and ideas, and experiences of their attempts to solve a task, their work, and problem solutions that include:

a. Questions they pose to understand concepts and problem statements

b. Online resources and platforms they consult to contextualize problems and review and extend their understanding of involved concepts

c. Different ways to solve a mathematical problem. The type of problems for students to work include problems like those discussed during the class, those that can be solve by the same methods but differ from those solved in instruction and new problems that were not addressed in class sessions.

d. Concepts and strategies used to solve the problem

e. Identification of other problems that can be solved with the methods that were used to solve the initial problem

f. Digital technologies and online resources used to solve the problem

g. Dynamic models used to solve the problem and strategies used to identify and explore mathematical relations (dragging objects, measuring object attributes, tracing loci, using sliders, etc.)

h. Formulation of new related problems including possible extensions for the initial problem

i. Discussion of solutions of some new problems.

j. Short recorded video presentation of their work and problem solutions.

k. Reflection on how their problem solutions relied on other peers' ideas and the extent to which their own work influenced and shaped the group work.

Finally, what we lived during the pandemic crisis not only has altered the way we move, interact, and carry out daily and our professional activities; but also, has brought an opportunity to rethink and update our academic agenda to incorporate experiences cumulated during pandemic confinement in current educational practices. The consistent use of digital technologies not only helped teachers and students to work on mathematical tasks, but also, provided affordances to extend mathematical activities beyond face-to face activities. We argue that the students' systematic and coordinated use of digital technologies extend their way of reasoning to solve mathematical problems. Then, mathematics teachers should experience the changes in mathematical contents, class dynamics, and assessment that are needed to fully incorporate the use of technologies in their teaching practices.

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CONTRASTING PERSPECTIVES BETWEEN THE TEACHER AND STUDENTS: A REFLECTION ON THE LEARNER'S PERSPECTIVE STUDY

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The Learner's Perspective Study (LPS) has provided a vehicle for the work of an international community of classroom researchers. The distinguishing characteristic of the research design for the LPS is the inclusion of complementary accounts of classroom events. Among the different levels of complementarity, the current paper focuses on the various accounts of classroom participants to discusses how we take the teacher and students perspectives together by contrasting and juxtaposing them to explore the co-constructed nature of mathematics classroom. By referring to previous studies in the LPS as well as to a "spin-off" study, the author argues that examining participant perspectives on the same classroom event provides better understanding of and insights into classroom practice with expanding the researcher's perspective.

INTRODUCTION

The ultimate goal of any research on classroom practice is to improve teaching with the enhancement of students' learning. For this goal, various approaches and methodologies have been adapted in the international comparative studies on classrooms. In the earlier years, among others, international comparative studies of mathematics classrooms were conducted based on the classroom observations and interviews with the teachers and students to reveal similarities and differences found in the classroom in different cultural/social background (e.g. Becker et al., 1990; Stevenson & Stigler, 1992). Since mid-1990s, with the advancement of digital technology and facilities available for collecting and analysing classroom data including videos, large-scale international studies of classroom practice have been conducted. In particular, the video component of the Third International Mathematics and Science Study (TIMSS1995 Video Study) was the first attempt ever made to collect and analyse videotapes from the classrooms of national probability samples of teacher at work (Stigler et al., 1999). The study was a breakthrough as a scientific exploration into the classroom practice, showing the feasibility and the potential of applying videotape methodology in wide-scale national and international survey of classroom instructional practice, and it was followed by an extension of the study with the same research design (e.g. Hiebert, et al., 2003).

These international studies have tried to identify coherent sets of actions, and associated attitudes, beliefs and knowledge, that appear to constitute culturally-specific *teacher practices* (Stigler & Hiebert, 1999). On the other hand, we certainly need to hypothesize that there is also a set of actions and associated attitudes, beliefs, and

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knowledge of students that constitute a culturally-specific coherent body of *learner practices*. Research on classroom practice needs to focus on both teacher practices and learner practices as produced by co-construction of those practice through the activities of all the participants.

The Learner's Perspective Study (LPS) was designed to examine the practices of eighth grade mathematics classrooms in a more integrated and comprehensive fashion than had been attempted in previous international studies such as TIMSS Video Study. An essential thesis of the LPS is that international comparative research offers unique opportunities to interrogate established practice, existing theories, and entrenched assumptions (Clarke, 2017; Clarke et al., 2006; Clarke, Keitel, & Shimizu, 2006). The findings of the study included rich descriptions of the practices of participants in eighth grade mathematics classrooms in the participating countries, predominantly from the perspective of the learner, supplemented by the perspectives of the teacher and the researcher (Clarke, Keitel, & Shimizu, 2006; Clarke et al., 2006; Shimizu et al., 2010; Kaur et al., 2013; Leung et al., 2014).

With a reflection of the earlier studies in the LPS, this paper reconsiders how we can take the teacher and students perspectives together in classroom research for understanding complex phenomena of practice in the mathematics classroom. The findings from the research in the LPS and its "spin-off" project are briefly reviewed with an intention of examining the significance of contrasting and juxtaposing the teacher' and students' perspectives. The author argues that contrasting those different perspectives provides us an opportunity to explore the co-constructed nature of quality instruction in mathematics and that an integration of those different perspectives can offer better understanding of complex phenomena in the mathematics classroom as well as implications for improving classroom instruction.

LEARNING FROM THE FINDINGS OF TIMSS VIDEO STUDY

The TIMSS1995 Video Study collected and analysed videotapes from the classrooms of national probability samples of teacher at work (Stigler & Hiebert, 1999; Stigler et al., 1999). Focusing on the actions of teachers, it has provided a rich source of information regarding what went on inside eighth-grade mathematics classes in Germany, Japan and the U.S. with certain contrasts among three countries. In addition, objective observational measures of classroom instruction were developed to serve as valid quantitative indicators, at a national level, of teaching practices in the three countries. It was interesting to learn that the findings of the study included aspects of mathematics lessons as identified with a strong resemblance between Germany and the U.S. with Japan looking differently (Shimizu, 1999). Before looking back the findings from the study, an episode is provided.

The Didactic Triangle Revisited

Stigler and Hiebert (1999) reported on a meeting in which 'distinguished researchers and educators from Germany, Japan, and the United States' (p. 25) were invited to review and discuss the classroom recordings made for the TIMSS video study. I had an opportunity to participate in this particular meeting as one of the Japanese consultants. Stigler and Hiebert (1999) noted that one participant shared his reflections after viewing video recordings made in Japanese, German and US mathematics classes as follows:

> In the Japanese lessons, there is the mathematics on one hand, and the students on the other. The students engage with the mathematics, and the teacher mediates the relationship between the two. In Germany, there is the mathematics as well, but the teacher owns the mathematics and parcels it out to students as he sees fit, giving facts and explanations at just the right time. In the U.S. lessons, there are the students and there is the teacher. I have trouble finding the mathematics; I just see interactions between students and teachers. (pp. 25-26)

The reader may recognize the didactic triangle in which student, teacher, and content form the vertices (or nodes) of a triangle that is the classical trivium used to conceptualize teaching and learning in mathematics classrooms in three countries. By quoting the episode above, Goodchild and Sriraman (2012) suggest that the observations made over a decade ago are still relevant today.

There also appears to be some differences among the lessons coded in the study. For example, a Japanese classroom would never have someone come into the lesson while it was under way. It would also be very rare for a lesson to be interrupted by a public announcement in the school. These events certainly took place in the lessons coded in the U.S. classrooms and to some extent in German classrooms. As a natural consequence of such events in the classroom, students' views on a lesson should be quite different among three countries. Stigler and Hiebert (1999) provide another episode from the meeting of TIMSS Video Study where a Japanese researcher incredulously asking about the moment of sudden interruption by the public announcement in the middle of the lesson (Stigler and Hiebert, 1999, pp. 55–56).

Although we could reconceptualize teaching and learning in mathematics classrooms by extending the model of triangle to a model of tetrahedron by adding one more vertex or node (e.g. technology or artifact (Goodchild & Sriraman, 2012), I reflect on the episode of watching the videos from mathematics classrooms with the recognition that the typical framing of the didactic triangle is narrower than it should be and that it should be broadened to view classroom activities from a more social/cultural perspective (Schoenfeld, 2012).

A Japanese Perspective on the Findings from the TIMSS Video Study

When I watched the videos from three countries, lessons from each country seemed quite differently. German teachers, for example, seemed to teach mathematics in a "one-to one" question and answer mode with an authority of mathematics behind them. On the other hand, Japanese teachers seemed to behave keeping their position at somewhere in between mathematics and students without authority of mathematics behind them. Part of my early impressions seemed to be confirmed by the objective observational measures of classroom instruction developed by the TIMSS Video. I thought that we need to have a framework with which we describe co-constructed nature of classroom activities by integrating the teacher's and the learner's perspective.

It should be noted that the focus of the TIMSS Video Study was basically on the actions of teachers. A more complete description of the practice of mathematics classrooms would be obtained if learner's perspective was incorporated into the research design. Student interpretations of mathematics classrooms are necessary for a detailed account of classroom practice.

The analysis of video data collected in the TIMSS Video Study, as reported by Stigler and Hiebert (1999), centred on the proposition that the teaching practice of a nation could be explained to a significant extent by the teacher's adherence to a culturally-based "teacher script" at least in the case of mathematics. Central to the identification of these cultural scripts for teaching were the Lesson Patterns reported by Stigler and Hiebert (1999) for Germany, Japan and the USA. Their contention was that at the level of the lesson, teaching in each of the three countries could be described by a "simple, common pattern" (Stigler & Hiebert, 1999, p. 82).

In the TIMSS Video Study, using the sub-sample of 90 lessons coded by the Math Content Group, explicit linking was coded that the teacher wants students to understand in relation to each other (Stigler et al., 1999, p.117). Two kinds of linking were coded. Linking across lessons and linking within a single lesson. In the study, linking was defined as an explicit verbal reference by the teacher to ideas or events from another lesson or part of the lesson. The reference had to be concrete (i.e., referring to a particular time, not to some general idea); and the reference had to be related to the current activity. The results of this coding show a resemblance between Germany and the U.S. with Japan looking differently. The highest incidence of both kinds of linking, across lessons and within lessons, was found in Japan. Indeed, Japanese teachers linked across lessons significantly more than teachers of German lessons, and made significantly more linkings within lessons than teachers of both German and U.S. lessons.

Table 1 shows percentages of lessons that include explicit linking by the teacher to ideas or events in a different lesson, and to ideas or events in the current lesson (Stigler et al., 1999, p.118).

	Germany	United States	Japan
In a different lesson	55	70	91
In a current lesson	41	40	96

Table 1: Linking to ideas or events in a different/current lesson (%)

Japanese teachers usually plan a lesson as a part of a unit, a sequence of several lessons. This means that each lesson in a unit has a different purpose for attaining the goals of the unit, depending on the phase in the unit. At an introductory phase in a unit, for example, the lesson will be a concept-oriented for introducing a new idea or concept. On the other hand, at the summative phase, the lesson may be more practice or skill oriented. Thus, lesson script for lessons at each phase can be slightly different from each other even in the same country. Collecting and analysing data over a lesson sequence are needed for clarifying a diversity of lessons at different phases in the unit.

THE LEANER'S PERSPECTIVE STUDY: AN INCLUSIVE METHODOLOGY

The methodology of survey-style TIMSS Video Study can be complemented by research techniques intended to give prominence to the perspective of the learner (Clarke, 2001). Among the methodologically most interesting aspects of the LPS has been the collaborative negotiation of the study design, the method of data generation, the general and local analyses, and the process whereby the various complementary accounts can be integrated into a rich and useful portrayal of mathematics classrooms internationally. Inclusivity as a methodological principle was as pervasive in the LPS research design as complementarity (Clarke et al., 2006; Williams, 2022). The inclination to integrate rather than segregate is at the heart of the LPS, since it was intended from the project's inception that any documented differences in classroom practice be interpreted as local solutions to classroom situations and, as such, be viewed as complementary rather than necessarily oppositional alternatives, within a broadly international pedagogy, from which teachers in different countries might choose to draw in light of local contingencies. International comparative classroom research is viewed as the exploration of similarity and difference in order that our understanding of what is possible in mathematics classrooms can be expanded by consideration of what constitutes good practice in culturally diverse settings.

Data Generation in the LPS

The TIMSS Video Study and the LPS differ in their breadth of data capture, and thus in the number and nature of research questions they support. Data generation in the LPS used a three-camera approach (Teacher camera, Student camera, Whole Class camera) that included the onsite mixing of the Teacher and Student camera images into a picture-in-picture video record that was then used in post-lesson interviews to stimulate participant reconstructive accounts of classroom events. These video records were supplemented by student written material, and by test and questionnaire data from students and the teacher. These data were collected for sequences of at least ten consecutive lessons occurring in the "well-taught" eighth grade mathematics classrooms of teachers in participating countries and regions. Each participating country used the same research design to collect videotaped classroom data for at least ten consecutive mathematics lessons and post-lesson video-stimulated interviews with at least twenty students in each of three participating 8th grade classrooms. The three mathematics teachers in each country were identified for their locally-defined 'teaching competence'. In the key element of the post-lesson student interviews, in which a picture-in-picture video record was used as stimulus for student reconstructions of classroom events, students were given control of the video replay and asked to identify and comment upon classroom events of personal importance. Teachers were also interviewed using a similar protocol.

Contrasting Perceptions of Lesson Events between the Teacher and Students

In this section, the findings from previous study (Shimizu, 2006a) is reconsidered. The methodology employed in the LPS allows participants to identify those events that were significant to them. Namely, in the post-lesson video-stimulated interviews, which occurred on the same day as the relevant lesson, the teacher and the students were asked to identify and comment upon classroom events of personal importance (Clarke, 2006, See Table 2 for the examples of prompt).

Prompt Four: Here is the remote controller for the video-player. Do you understand how it works? (Allow time for a short familiarization with the control). I would like you to comment on the videotape for me. You do not need to comment on all of the lessons. Fast-forward the videotape until you find sections of the lesson that you think were important. Play these sections at normal speed and describe for me what you were doing, thinking and feeling during each of these videotape sequences. You can comment while the videotape is playing, but pause the tape if there is something that you want to talk about in detail.
Prompt Seven: Would you describe that lesson as a good one for you? What has to happen for you to feel that a lesson was a “good” lesson? Did you achieve your goals? What are the important things you should learn in a mathematics lesson?

Table 2: Selected Prompts in Post-lesson Video-stimulated Interviews

The analysis of LPS data has revealed both patterns and variations in the ways in which the teacher and students perceived the lesson. The LPS design provides the researchers with the opportunity to explore the commonalities and differences in perceptions of mathematics lessons by teachers and students by means of juxtaposing their reconstructive accounts of the classroom. Although video-stimulated interviews have been used in other studies to examine teachers’ and students’ ideas and beliefs, earlier studies have not focused on contrasting teacher and student perceptions of the same lesson they have just experienced.

Perceptions of lesson events

A key result of analyses conducted on the LPS data has been the suggestion that “the lesson” is unsuitable as a unit of comparative analysis of classroom practice and that the “lesson event” (those recurrent activities from which lessons are constructed) is more suitable for the purposes of comparative analysis, corresponding more closely to the decisions made by each teacher regarding the structure of any particular lesson, and being more effective in distinguishing between the practices of particular classrooms (Clarke et al., 2008). In the classroom, teacher and student practices can be conceived as being in a mutually supportive relationship. This is not to presume that the teacher and students have the same goals or even that they perceive lesson events in the same way. By contrasting their perceptions of particular lesson events, it is possible to identify the discrepancies between the teacher and the students in their perceptions of classroom practice. These discrepancies can help us to understand each participant's contribution to the activities of the mathematics classroom. Also, through the analysis of their perceptions of lesson events and the associated values held by the teacher and the students, the influence of both perceptions and values on learning mathematics can be explored.

Valuing Matome

The LPS approach of juxtaposing teacher and student perceptions of mathematics lessons has the capacity to bring out the symbiotic nature of the teacher's and the students' contributions to the teaching and learning process. Japanese teachers, for instance, often try to organize an entire lesson around a few problems with a focus on the students' alternative solutions to them. In this "structured problem solving" approach, the "summing up" (Matome) phase is indispensable to a successful lesson (Shimizu, 2006b). Students' solutions are shared and pulled together during the phase in light of the goals of the lesson. While Japanese teachers may devote considerable effort to planning and structuring their lessons around the "climax" of the lesson, the structure may not be perceived by the students or may be perceived differently.

Teacher	Student 1	Student 2
3:50		
		6:00
9:29	9:23	
14:00	14:25	14:22
	16:00	
16:50		
		17:30
		24:44
	27:09	
28:00		
29:30		
32:26		
33:30		
	34:30	
	37:45	37:26
		40:00
42:50	42:55	43:02
	45:20	

Table 3. Elements in the lessons felt to be significant: J1-L5

The comparison displayed in Table 3 of perceived events felt to be significant by the teacher and the students clearly shows that there are agreements and discrepancies between them on what is important in mathematics lessons and what is not. This analysis raises the issue of the possible influences of such agreements and discrepancies in perceptions of lesson events between teacher and students on students learning and teacher planning. The result illustrated in Table 3 also raises the need to attend to the meanings constructed by the teacher and the students as the participants in the same lesson. By examining the post-lesson interview data closely, any differences can be understood as discrepancies between the teacher's and the students' perceptions of classroom events.

The purpose of this section has been to highlight the differences between the perceptions of what constitutes a significant lesson event, as held by the teacher, the student(s), and (implicitly) the researcher. In the next section, we see leaners' perspective on what constitute a good lesson in mathematics compared with the teacher's perspective.

"Good" Mathematics Lessons from the Learner's Perspective

The Japanese term for teacher's behaviour in the classroom related teaching is "Gakushushido" that literally means "Guidance of Learning". It should be noted that the term "Gakushushido" is used as one word. Here we see a tradition of recognizing that teaching and learning are interdependent activities within a classroom setting and that classroom practices should be studies as such. If we focus on the teaching and learning as interdependent activities, we need to look into what participants, both the teacher and students, value in the classroom and how they perceive the lesson with associated values embedded in activities in classroom. In the following part of this paper, associated values attached to a "good" lesson are explored from the learner's perspective with a reference to the findings of Shimizu (2009).

Data collection for the study was conducted at three public junior high schools in Tokyo. All of the three mathematics teachers had experience of teaching mathematics more than twenty years. Two of them were writers of mathematics textbooks widely and commercially available in Japan. The criteria for identifying them reflected a locally-defined "teaching competence". Namely, the three mathematics teachers were identified for their active roles in the study groups of teachers in Tokyo, and the recognition in the community of mathematics teachers as a teacher who teach mathematics in excellent ways.

During the data collection that followed the LPS methodology, semi-structured video-stimulated interviews with the students occurred on the same day as the relevant lesson. In each lesson two students sitting next to each other were selected as "focus students" for that particular lesson. These students were interviewed individually after the lesson. Among three of them, two teachers were interviewed three times, roughly once a week, during the period of videotaping and one teacher was interviewed twice. The post-lesson interviews with sixty students, twenty students from each of three schools, were transcribed and subjected to the analysis. For the analysis of the interview data, a coding system was developed (Shimizu, 2009). Table 4 shows the description of each coding category with an illuminating example of students' response to Prompt Seven on good lessons.

Code	Description	Example
Understanding/ Thinking	Those responses that refer to their understanding and thinking in the classroom	*I can understand the topics to be learned*. (J2-03M)
Presentation	Those responses that refer to presenting their ideas in the classroom	*I can present my solution on the blackboard*. (J3-07I)
Classmates	Those responses that refer to other students' presentations and explanations	*There is an opportunity of listening to classmates.* (J1-09S)
Whole class discussion	Those responses that refer to the whole class discussion	*We all in the classroom exchange ideas actively*. (J1-06U)
Teacher	Those responses that refer to teacher's explanation	*I listen to teacher's final talk. I always take a note and check a point*. (J3-06S)
Other	Other responses	*By preview the topic at home, I attend the lesson with a preparation*. (J3-09K)

Table 4: The Description and Examples of Categories for Coding

The first five categories and one additional category, "other", had appeared from the initial analysis of transcriptions from one of the three schools (labelled as "J1"). Then all the students' response to the prompt were classified into six categories for coding by the author and research assistant. When discrepancies in coding between coders appeared, they were resolved by discussions. It should be noted that these codes do not constitute a mutually exclusive coding system.

Codes	Responses
Understanding/Thinking	27 (45.0) *
Presentation	10(16.7)
Classmates	4(6.7)
Whole class discussion	16(26.7)
Teacher	10(16.7)
Other	10(16.7)

Table 5: Students' Response to the Prompt Seven in Video-stimulated Interview * Numbers in parentheses denote the percentage of each category.

Table 5 shows the result of the analysis as a whole of students' response to the prompt seven in video-stimulated interview. It is noted that the percentages do not add up to 100, because the coding system is not mutually exclusive.

As Table 5 shows, nearly half of the students interviewed (45.0%) described "understanding" or "thinking" to be happened in a "good" lesson. As the example in the Table 3, "I can understand the topics to be learned", illustrates, the students in this category regarded a lesson as "good" one if he can have a clear understanding of mathematical topic taught. Those students who mentioned to "understanding/thinking" seemed to attach values directly to the importance of their own understanding and thinking in the lesson. Some students in this category also referred to other activities in the classroom. MANA, a student from J2, for example, mentioned to teacher's explanation as the object of understanding: "*Even if your answer is wrong…to be able to understand what the teacher explained. If that happens, I think//that it was a good lesson*." Roughly a quarter of the students (26.7%) identified "whole class discussion" as the "component" of a "good" lesson. Then, two categories "presentation" and "teacher" follow the "whole class discussion". Only four students (6.7%) explicitly described the activities related to their "classmates" in mathematics classroom.

There is a difference between the first four categories and "teacher" category in terms of types of the activities referred by the students. That is, first four categories are directly related to students' own learning activity, while "teacher" category is related to both students' learning and teacher's instructional activities. The example of "teacher" category in Table 4, for instance, is the one that referred to "teacher's final talk" (highlighting and summarizing the main point), taking a note, and checking the key point of lesson. This example illustrates that teacher's instructional activities can also be a component of a "good" lesson.

Relating teacher and learners perspectives

To understand the characteristics of a "good" mathematics lesson, a detailed analysis was also conducted with an eye of relating students' responses to Prompt Seven to those by the teacher who taught each classroom.

Suzu, a student from the school J3, for example, responded to the questions, "When you think it's a good class?" and "What should happen in the class?", as follows.

01. INT: When you think it's a good class,

02. SUZU: Yes.

03. INT: What should happen in the class?

….

04. INT: Do you have anything that you think is a good class?

05. SUZU: I can present my answer, and then listen to my friend's way as well,

06. INT: Yeah?

07. SUZU: The teacher's final comment, or answer,

08. INT: Yeah?

09. SUZU: Listen to it carefully, and to make a good note from it.

The student clearly mentioned to the importance of presenting his answer to the problem to the class and of listing to his classmates' method to solve the same problem. He also referred to listening to "*The teacher's final comment, or answer*" carefully and of "*making a good note from it*". These comments suggest that, students' views on a "good" lesson are shaped through the classroom practices co-constructed by the teacher and the students. If the teacher keeps summarizing and highlighting the main points of the lesson as a daily routine, the students may become aware of the importance of the particular lesson event which tends to come on the final phase of lesson in the form of teacher's public talk together with time for note-taking. Then he or she will "listen to it carefully" and try to "make a good note from it". The teacher's summarizing and highlighting, in turn, have to rely upon students' understanding of the mathematical topic taught to be summarized and highlighted.

Teachers' comments on what constitutes a "good" lesson also suggest the co-constructed nature of a "good" mathematics lesson. Mr. K, the teacher of JP3, in the second interview, for example, mentioned to the importance of students thinking on alternative solutions and their understanding in a "good" lesson as follows: "*practically, what I think is that the students think in many ways...and they understand it well ...The students can ask me or each other where they can't understand*." Here, Mr. K expressed that he valued to have his students think in many ways and understand the topic well through the interaction with him and classmates.

The comment suggests that in a "good" lesson teacher and student practices can be conceived as being in a mutually supportive relationship. This is not to presume that teacher and students have the same goals or values, or even that they perceive the importance of particular classroom activities in the same way. The analysis described above suggests that a "good" lesson is a co-constructed classroom practices by the teacher and the students.

Unpacking the Technical Vocabulary of Japanese Mathematics Teachers

In this section, the findings of recent study on Japanese lexicon (Shimizu, et al., 2021) is shared and discussed. The study is as a part of the International Lexicon Project which is a "spin-off" project of the LPS to document and compare the naming systems of mathematics teachers on phenomena related to teaching in ten countries: Australia, Chile, China, the Czech Republic, Finland, France, Germany, Japan, South Korea, and the USA (Mesiti, et al., 2021). A study of teachers' naming systems is seemingly related to teachers' perspective but I argue that Japanese lexicon includes terms and phrases based on the teacher's reflections on learners' perspective.

The Japanese tradition of Lesson Study has created a teaching community in which observation and discussion of teaching are integral parts of professional practice with particular lexical terms of specific significance. The study aims to explore the constituent elements and structural characteristics of the Japanese lexicon. Documenting and comparing teaching vocabulary will provide an opportunity for us to understand the nature of teaching and facilitates international comparative research (Mesiti, et al., 2021).

In the Lexicon Project, local teams of researchers and experienced teachers in each country, classify a common set of video records of mathematics lessons, drawn from all participating countries, in order to identify those terms in their local language that in combination constitute the national pedagogical lexicon, by which we discuss, analyse, reflect upon and theorize about the mathematics classroom.

Data generation was undertaken simultaneously in Japan and other participating countries. Each participating team contributed a videotaped lesson which was included in a stimulus package. This stimulus material was viewed by team members in each country to identify the well-established pedagogical terms or phrases of used in the communities. These terms are supplemented with the clearest possible operational definitions in English, describing both the form and function of each named term. The combined classroom video material from the participating countries then becomes a source of video exemplars of each of the named terms.

The Japanese Lexicon team members took part in the video viewing process to identify the terms and phrases used by Japanese mathematics teachers. The Japanese team consisted of two researchers, two experienced teachers and two doctoral students. The Japanese Lexicon was constructed by watching video material, time-stamped transcripts and classroom supporting material for one lesson of mathematics at an eighth grade classroom in a public school in Ibaraki prefecture.

An electronic survey was conducted for a national validation to examine how familiar the terms were for the mathematics teacher in Japan. Overall, the terms were very familiar to the respondents, although some terms were somewhat less frequently in use. A total of 70 terms or phrases are identified as the Japanese Lexicon (Shimizu, et al., 2021). Characteristic of the relationships among terms within the Japanese Lexicon were the multi-layered intentions of Japanese teachers as these were inferred by the Japanese team and represented in the constituent elements of the Lexicon.

For example, the term “hatsumon”, asking a key thought-provoking question, has a specific meaning within the system of terms and phrases related to teaching through problem solving. Teacher’s question with a particular intention should be effective in relation to the goal of today’s lesson and to the status of students’ understanding. It is considered as embedded in the system of particular style of teaching mathematics. In planning a lesson, for instance, the teacher anticipates alternative solutions to the problem and identifies possible students’ misunderstanding and mistakes. Thus, an enactment of “hatsumon” cannot be planned without the teacher’s views on and thinking about what students think and how they solve the problem.

Another example is “kikan-shido” which literally means “instruction between desks”. This particular term refers to the teacher’s behaviour during students’ problem solving by their own. However, an important aspect of “kikan-shido” is in that the teacher scans students’ work purposefully and uses the knowledge gained to select students to present solutions to the classmates (Hino, 2006). Also, the term has “evolved” to similar but different term “kikan-shien” that means “supports between desks” (Shimizu, et al., 2021). The terms “kikan-shido” and “kikan-shien” are constituted so as to incorporate what students expected to do and what the teacher anticipate and expect in students learning. In this sense, the Japanese Lexicon includes terms and phrases in which learners’ perspective is “amalgamated” and integrated into the teacher’s views on their teaching.

This study provides insights into the naming system employed by mathematics teachers in relation to their classroom practice by documenting and interpreting the constructs that are well-known and used among them. Documenting the Japanese Lexicon reveals that teachers’ use of terms and phrases related teaching is profoundly influenced by the learner’s perspective amalgamated with teacher’s perspective.

DISCUSSION

Complementary Accounts of Classroom Events

The distinguishing characteristic of the research design for the LPS is the inclusion of complementary accounts of classroom events (Clarke, 2001; Ellerton, 2008; Williams, 2022). In the research in the LPS, four levels of complementary accounts can be identified (Clarke, et al., 2006): (a) at the level of data, the accounts of the various classroom participants are juxtaposed; (b) at the level of primary interpretation, complementary interpretations are developed by the research team from the various data sources related to particular incidents, settings, or individuals; (c) at the level of theoretical framework, complementary analyses are generated from a common data set through the application by different members of the research team of distinct analytical frameworks; and (d) at the level of culture, complementary characterizations of practice and meaning are constructed for the classrooms in each culture by the researchers from each culture and these characterizations can then be compared and any similarities or differences identified for further analysis.

Given the complementarity in the studies discussed in this paper is directly related to the level (a) and (b), the discussion can be examined and expanded further at the level of both theoretical framework and culture. Williams (2022) draws attention to the potential for the Complementary Accounts Methodology to contribute to support for researchers employing diverse theoretical frameworks. The “rich” data sets offer an opportunity for researchers not only to examine the nature of the classroom practice in details but also to expand researchers’ perspective by interrogating their own implicit assumptions about classroom practices.

Significance of Generating Integrated Data Sets

With the reflection on the previous studies conducted in the LPS, it is safe to say that the strength of the project resides in the generation of integrated data sets of classroom practices. The collection of integrated data sets in the LPS enables the researchers to examine the questions related to the teacher and students practices and interrelatedness or mutually supportive relationship between them. As the name of the project suggests, the inclusion of students' interviews with a direct connection to their specific classroom activity they just experienced is crucial to the methodology. Furthermore, connecting the learner's perspective with the teacher's perspective facilitates our understanding of the complexity of classroom practices.

Focusing on the frame for comparisons in the LPS, Schoenfeld (2022) points out as "when people think about teaching, they think about the teacher; it's easy for the learner to get lost". The original idea of having the focus on "the learner's perspective" derived from the accumulation of David Clark' works on classroom practice and his ideas (Clarke, 2001) as well as our reflections on the significance and limitations of earlier studies of classroom practice such as TIMSS Video Study.

The characteristic of the research design of the LPS can be found in its focus on the practice of competent teachers. Instead of gathering statistically representative sample for finding an "average" lesson, the LPS methodology generates integrated data sets from the classroom taught by experience teachers. This research design raises researchers an opportunity to examine the cultural specificity of "good" classroom practice and opens the door to thinking about improvements of teaching.

CONCLUDING REMARKS

Accumulated international comparisons under the umbrella of the LPS have made clear how culturally-situated are the practices of classrooms around the world, and the extent to which students are collaborators with the teacher, complicit in the development and enactment of patterns of participation that reflect individual, societal and cultural priorities and associated value systems.

I have discussed how we take the teacher and students' perspectives together for understanding the classroom practice in mathematics by referring to the previous studies in the LPS and a related project. Contrasting and juxtaposing the teacher and students' perspectives provide us an opportunity to explore the nature of quality instruction in mathematics. The complex phenomena in the mathematics classroom can be understood better by taking both the teacher and students perspectives into considerations in research on classroom practice that describes both similarities and differences in participants' perspectives on the same classroom event.

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PLENARY PANEL

MATHEMATICS TEACHER EDUCATION SHOULD BE RESPONSIVE TO A RAPIDLY CHANGING WORLD

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INTRODUCTION (OLIVE CHAPMAN)

The theme of the PME-45 conference, *Mathematics education research supporting practice: Empowering the future,* not only highlights the importance of research to improving the quality of mathematics teaching and learning but also the importance of teacher education to empower teachers to impact practice in a timely, relevant, and meaningful way. Hence, this plenary panel considers the latter by debating the motion: *Mathematics teacher education should be responsive to a rapidly changing world.*

Rationale

In response to perceived needs of the 21st-century, knowledge-based, digital society, global education policies, research, and professional organizations advocated reforms to mathematics education that included teaching to support conceptual understanding of mathematics and mathematical thinking. Such reforms could be linked to the focus on developing teachers' *mathematics knowledge for teaching,* needed to support deep mathematical understanding, with an orientation towards content knowledge and pedagogical content knowledge. However, the 21st century has presented us with a rapidly changing world with implications for mathematics education and consequently mathematics teacher education (MTE) that are more complex and challenging than initially perceived; implications that could require (re)consideration and new/renewed understanding of the nature of the mathematics classroom, the mathematics teacher, and MTE to enable them to live well in this changed and changing world. Thus, focusing on MTE (including preservice and inservice teachers), this plenary panel aims to draw attention to, and provoke discussion of, what the rapidly changing world means to MTE and whether MTE should be responsive to the rapid changes.

Teacher education in a changing world

Darling-Hammond and Bransford's (2007) edited book, *Preparing teachers for a changing world,* recommends that all new teachers must have strong disciplinary knowledge and basic understanding of how people learn and develop. It suggests that teachers must be able to apply that knowledge in developing curriculum that attends to students' needs, the demands of the content, and the social purposes of education that include teaching content to diverse students, managing the classroom, assessing student performance, and using technology in the classroom. These suggestions of

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factors to be considered in preparing teachers for a changing world seems to align with those that have been addressed in MTE based on research in the field. But they do not fully or appropriately represent concerns of the rapidly changing world we have been experiencing and will continue to live in with seemingly unpredictable effects of some changes. However, recent publications on the changing context of mathematics education and implications for MTE have further addressed technology and equity.

Most recent publications on mathematics education have focused on the changing world in terms of the *digital age* and *equity*. Regarding the digital age, Clark-Wilson et al.'s (2021) edited book, *Mathematics education in the digital age*, addresses the impacts the digital age has, and will continue to have, on the learning and teaching of mathematics within formal education systems/settings. It suggests that it is important for the design and evaluation of MTE and professional development programs to embed the knowledge, skills, and attributes to teach mathematics with digital resources. Niess et al. (2016) also focused on the digital age in the *Handbook of research on transforming mathematics teacher education in the digital age*. The book addresses the development of teachers' knowledge for the integration of technologies to improve classroom instruction. Based on research on emerging pedagogies for preservice and inservice teachers, it suggests how mathematics teacher educators must and can think beyond their own backgrounds to incorporate current and emerging technologies into their efforts to prepare students to teach mathematics.

Regarding equity, Xenofontos's (2019) edited book, *Equity in* m*athematics education: Addressing a changing world*, reconsiders the concept and/or practice of equity, and its related concept, social justice, and the role of mathematics education research in addressing and promoting a fairer world. It offers "practical suggestions" on how equitable teaching practice could be included in MTE and suggestions for inservice teachers to implement in their classrooms. Other publications have focused on the impact of the Covid-19 pandemic on mathematics education with consideration of both technology and equity; for example, the Educational Studies in Mathematics special issues on *Mathematics education in a time of crisis—a viral pandemic* (vol. 108, issues 1-2, 2021). Some studies have taken a sociopolitical turn that "seek not just to better understand mathematics education in all of its social forms but to transform mathematics education in ways that privilege more socially just practices" (Gutiérrez, 2013, p. 40), with different implications for MTE.

While this emerging body of literature addresses issues related to mathematics education in a changing world with MTE being a way to react to proposed changes in the classroom, the plenary panel intends to put MTE at the forefront regarding its role to being proactive and responsive to a rapidly changing world. This positioning of MTE links it to factors, such as technological, sociocultural, sociopolitical, geopolitical, biopolitical; and socioeconomical global situations that underlie our rapidly changing world. Specific situations include artificial intelligence; epidemics; mental health; environmental/climate change; war; displaced populations (migratory crisis); diversity, equity, and inclusion; and decolonization. There is no question that such situations have implications for mathematics education based on the global needs of the students in our classrooms. The effects of these situations are present in the learners who arrive in our mathematics classrooms every day. Thus, it is important for teachers and teacher educators to become knowledgeable of the interplay between school mathematics and mathematics education and the issues arising from these situations that impact students and their communities. This is necessary for mathematics education to realize its full potential for the 21st century. However, achieving this potential is likely to be challenging for the field of mathematics education given the significant shifts in conceptualizing mathematics and the learning and teaching of mathematics that will be needed to address the changing world, the lack of a global theory/perspective of mathematics pedagogy, and educational policies outside of the control of the field. These challenges are also applicable for MTE.

For MTE to realize its full potential to support initial teacher education and teacher professional development, it should be responsive to the changing world. But to what extent is this achievable or even possible? MTE was adapted or changed to address the requirements of the reform movement in mathematics education, but after about three decades we are still far from achieving reform on a significant level across the globe. So, would MTE be more successful in being responsive to a rapidly changing world?

One concern for MTE is teacher knowledge. Research on *mathematics teacher knowledge for teaching* suggests different but related models of what this knowledge should be. Would some models be more appropriate now or would new models need to be developed? Would focusing on identity and power (Gutiérrez, 2013), for example, be of more importance than current models without this focus? As Gutiérrez states,

> we are also at a time when not attending to identity and power means we are at best fooling ourselves about future prospects and at worst likely to ensure that mathematics education will be unable to realize its full potential for the 21st century. (p. 38)

Then, what would such a shift in teacher knowledge or characteristics mean for mathematics teacher educators' knowledge? Mathematics teacher educators' knowledge for teaching teachers is not well established since research on mathematics teacher educator knowledge is fairly new. In order to be responsive to a rapidly changing world, what knowledge will teacher educators need? This knowledge will depend on what ought to be the purpose of MTE in a rapidly changing world. In addition to teacher educators' knowledge and ability to respond to all types of change, there are several issues/questions that must to be addressed to understand the practicality of MTE being responsive to changes in the world. For example, there are issues associated with: autonomy to decide on what changes are relevant; cost of implementing change; teachers of different education levels and contexts; diversity in meaning of being responsive; and diversity in mathematics pedagogy and teacher education around the world. The plenary panel addresses these and other issues.

Structure of plenary panel and presentation

The presentation adopts the Oxford debate structure consisting of a panel of debaters and a moderator. Based on this structure, the 90-minute live session, moderated by Olive Chapman, includes opening remarks by each panelist, followed by an intra-panel discussion, then a question-and-answer period involving the audience, and finally brief closing remarks by each panelist. The panel consists of two teams of two members on opposite sides of the motion being debated. Paola Sztajn and Nancy Chitera make the case ***for*** MTE being responsive to a rapidly changing world. Nuria Climent and Jaguthsing Dindyal make the case ***against*** MTE being responsive to a rapidly changing world. In the following sections of this paper, the panelists provide summaries of their main arguments to support or oppose the motion. Each section was written solely by the panelist, as indicated in the heading of the section, and thus, reflects the thinking of that panelist. We hope the preceding discussion in the introduction section and the ideas in sections that follow will inspire further discussion of the motion.

"WE" IN MATHEMATICS TEACHER EDUCATION: BEING RESPONSIVE TO A RAPIDLY CHANGING WORLD (PAOLA SZTAJN)

"I can't do that at my school. That is not what my principal wants." "I am seeing practices in the classroom that I can't reconcile with what I am learning in my teacher education courses." "My mentor teacher told me to teach more like them. So I did, because it works in the classroom."

As a scholar and practitioner in MTE, I have heard statements such as the ones above too many times. They stem from a tension that continues to exist between what is taught in post-secondary mathematics teacher preparation programs and what is experienced in P-12 school mathematics teaching. This tension positions the two groups as "us" versus "them". It is my position that until mathematics teacher preparation and mathematics classroom teaching partner in ways that honor both practices and their different knowledge, we will continue to experience this tension in the education of mathematics teachers. Because I consider such tension unproductive, it is important to bridge the divide between MTE and mathematics classroom teaching to create a space in which the preparation of mathematics teachers is examined as the work "we" (classroom teachers and teacher educators) do. I contend that making MTE more responsive to a rapidly changing world, in ways that are more similar to what teachers are often asked to do, is one step toward bridging the divide.

Mathematics teacher education as boundary encounters

In Sztajn et al. (2014), we proposed that the work of MTE happens at the boundary that separates and connects different communities. At that time, we were interested in the question of how research-based and practice-based knowledge interact in MTE. We suggested it was myopic to place the knowledge needed to improve teaching (and I add, to improve MTE) either within the research domain or classroom-practice domain. Instead, we proposed that research and practice needed to come together as partners in

MTE, and we conceptualized the partnership that happens between researchers and practitioners within MTE as a boundary encounter (Wenger, 1998).

We noted two important premises for the partnership between research and practice. First, the research community has knowledge about students' mathematics and mathematics learning that has the potential to be useful to teaching. Second, the teaching community has knowledge about students' mathematics and mathematics learning in context that is of utmost importance for mathematics education researchers. From these premises, we suggested that bringing together those who work in the preparation of mathematics teachers and those who teach P-12 mathematics created opportunities for knowledge exchange among these communities—opportunities that I now claim are key to strengthening MTE. Further, we noted in the paper that it was the difficulty of creating and exchanging knowledge at the boundary that made those encounters transformative.

I believe the transformative processes happening at the border can unify different stakeholders working in MTE and create a space in which those working in post-secondary mathematics teacher preparation and in P-12 school mathematics teaching come together to design MTE programs that are shared in true partnership across the two communities. It is this work in partnership, which is responsive in nature, that can avoid statements like the ones listed in the opening paragraph, creating a "we" in MTE.

Mathematics teacher education as responsive work

Brokers who participate in boundary encounters translate, coordinate, and create alignment across groups (Wenger, 1998). They facilitate transactions across communities and introduce practices of one group into the other. Because by their design and due to their direct interactions with communities, schools are often expected and asked to be responsive to a rapidly changing world. I suggest that brokers working in MTE can bring issues that require responsive approaches into the post-secondary preparation of mathematics teachers. In what follows, I suggest a few areas of change that are impacting P-12 mathematics teaching and for which post-secondary educators' responsiveness seems important. In particular, I focus on three areas that are pertinent in the U.S. context: new technologies, societal changes, and students' wellbeing.

New Technologies

Technologies for classroom use are changing quickly and impacting P-12 classrooms, particularly in more urban and suburban affluent schools. In the United States, for example, one-to-one computer initiatives were implemented in school districts before they became common in post-secondary education. Due to the presence of vendors from various companies who are trying to sell "solutions" to school districts, it is not unusual for teachers to be piloting new technologies before they are considered in teacher education. The presence of tools that use artificial intelligence and approaches that collect and display data for teachers and other school leaders are also growing in schools, probably before they grow in many teacher education programs.

In such context, what is the role of those working in post-secondary mathematics teacher training? Whereas one can argue that higher education mathematics teacher educators can support analyses of the pros and cons of different technologies in light of different learning goals and theories, I would argue that a more responsive approach would include active participation in discussions and implementations related to new technologies, particularly in the decision-making processes in which school mathematics leaders are engaged when they have to make choices about product purchase. An interesting approach, for example, would be for innovative technologies to be tried simultaneously in the classroom and in teacher preparation. How would that transform practices at all levels of mathematics teaching? Spaces where all those involved in MTE are collaboratively looking at new existing technologies and websites for mathematics teaching and learning could promote the selection and use of resources that extend from teacher preparation to classroom practice.

Of course, there are also new technologies designed for teacher learning and not necessarily designed to promote learning of K-12 students. In these cases, responsive approaches to these technologies could also include partnership with P-12 teachers as they would be able to examine these technologies to consider their own learning. Thus, even in these cases, responsive approaches in which new technologies for mathematics teaching are examined as boundary objects within boundary encounters would be productive. These encounters would need to be ongoing and dynamic and operate at a pace that, sometimes, is faster than the usual pace in which higher education responds to technological changes.

Societal Changes

Changes in social contexts impact students at all levels. In P-12 schools, the more direct contact with families and communities can sometimes bring such changes faster into the classroom than in higher education. Again, as those working in mathematics teacher preparation and mathematics classroom teaching come together into boundary encounters, how can a shared examination of societal changes and emerging issues impact MTE, making it more responsive? In the case of societal changes, this responsiveness is key for engagement with learners.

In the United States, the past couple of years, beyond COVID, have been ones of social unrest. Different perspectives about the past, the ways in which different groups have been treated and the resulting existing inequalities have divided the nation creating a polarization that was not as evident in prior years. This divide has made the discussion of important issues and the education of all students (meaning really each and every student) more complex. This divisiveness impacts mathematics teaching and learning. Socially shared perceptions about who can and cannot learn mathematics, as well as discussions about the role of mathematics in helping students make sense of the (different) worlds in which they live matter.

Being responsive to societal changes and allowing discussions and issues that are part of students' lives into their mathematics education and the preparation of mathematics teachers is a key component for moving toward more equitable instruction. Because equity (or lack of thereof) continues to be a predominant educational problem in the United States, particularly in mathematics and other gate-keeping disciplines, MTE needs to respond to issues that are impacting the lives of learners. This response is not something that once done is complete; it needs to be systemic and systematic to matter.

Students' Wellbeing

Youth mental health has become a silent pandemic in our time. This is a national problem of such degree that the United States Surgeon General recognized the problem was widespread: students' feelings of helplessness, depression and suicidal thoughts are on the rise, and 20% of the young population is experiencing mental, emotional or behavioral disorders (Murthy, 2021). One more time, in this context, we see schools moving faster than higher education in recognizing and addressing the issues of student wellbeing, probably in part because there is a more consistent group of adults in K-12 schools that sees students on a regular, daily basis and can detect indicators of changes faster than more isolated adults in the teacher preparation context. But how will mathematics teacher educators take these new data and information about students' struggles with wellbeing into account? This is not an issue that can be ignored and on where, again, I argue mathematics teacher educators need to be responsive.

We in mathematics teacher education

Being responsive to the issues of our time, from my perspective, is not an option. Rather, it is part of the work we need to do in MTE, particularly if we conceive of such work as work that happens at the border between higher education and P-12 classrooms. I suggest this responsiveness can, in fact, be part of what brings MTE together as a single community that integrates those working in mathematics education in higher education institutions and those teaching mathematics in P-12 classrooms. Perhaps it is this coming together, around current issues in which both communities are responsive, that can bring groups together and improve MTE.

SHOULD MATHEMATICS TEACHER EDUCATION BE RESPONSIVE TO A RAPIDLY CHANGING WORLD? (NANCY CHITERA)

Does MTE get affected or influenced by global challenges? If so, are these influences across countries similar challenges in a rapidly changing world? How do countries respond to such influences? Are mathematics teacher educators taught on how to be responsive in the face of global challenges? This section of the paper discusses whether MTE should be responsive to global issues. The world is faced with many challenges that affect the education system. These global challenges include epidemics (e.g., Covid-19); climate change; displaced populations; and equity among others. Little work has been undertaken to understand the impact of global challenges on MTE and how responsive MTE has been. Most of the work published or documented focuses on impact of these global challenges on the education system and how the education system responds. I argue that the global challenges have had a big impact on MTE and therefore raises the need for MTE globally to be responsive to our rapidly changing world despite the nature of existence of such challenges.

The context of Malawi teacher education

Malawi is a country in the southern part of Africa. English is its official language and Chichewa, which is spoken by about fifty per cent of the population (Baldauf & Kaplan, 2004), is its national language. The primary teacher education program runs for two years in teacher training colleges while secondary teacher education runs for four years. Both teacher education systems have two phases: the residential training when student teachers stay in college/university, attend lectures and complete projects and assignments, and the school-based education, which consists of teaching practice in various primary and secondary schools. Student teachers are expected to become conversant with content related to pedagogical knowledge such as the technical skills of lesson planning and teaching methods, as well as the specific learning areas that are offered in primary or secondary education, which includes numeracy and mathematics among others, consisting of subject content and methodology.

Purpose of mathematics teacher education

In teacher education institutions, the main objective is to help student teachers learn how to teach mathematics. In the face of global challenges, what would be the purpose of MTE and how would the teacher educators make their adjustments? Who will decide on what kind of adjustments are to be made and how should they change? For whose impact and for what purpose?

Teaching and learning mathematics, whether at primary, secondary, or tertiary level, aims at encouraging and enabling learners to appreciate and recognize that mathematics saturates the communities around us in addition of teaching them to become mathematicians. Learners will need to appreciate its worthiness and application in solving real-world problems. This is critical because mathematics provides learners with opportunities to develop their intellectual skills in solving real-world problems, development of deductive as well as inductive reasoning, coupled with creative thinking. Thus, MTE should aim at helping student teachers learn how to bring out the mentioned critical elements in an individual, which will involve consideration of the global challenges that the world is facing, such as COVID-19 and climate change.

The challenge that exists is that in most cases, preparing a teacher to teach mathematics has been taken for granted. So, does this mean that the purpose has to respond to global changes? Mathematics also helps us to understand the world and is a tool for developing our mental abilities. For example, logical and critical thinking, having creative ideas, abstract or spatial intelligence, development of problem-solving capabilities, as well as good business communication skills are all encouraged in mathematics. This shows that there is an implied great responsibility on MTE to model the teaching of mathematics accordingly. The assumption is that MTE should be modelled in such a way that student teachers should be able to draw on even in the face of global challenges. This implies that the type of MTE discourse and how it is presented to the student teachers is very crucial. Student teachers learn and develop familiarity and confidence with such kind of discourse that is required in the face of global calamity.

Mathematics teacher education responsiveness to global challenges

Drawing on Wenger's (1998) notion of shared repertoire as the resources and tools used for knowledge (re)creation, it can be argued that for any development as a response to global challenges will need to be undertaken first to gain awareness of the complexity of MTE under such diverse global challenges. For example, what practices need to be addressed/challenged/included within the MTE to better prepare teachers to teach mathematics under the global challenges? There is also need to reflect on how classroom practices can contribute to the development of these transformative practices that will embrace new ways of thinking and acting in the face of a global challenge that are different than those that have guided MTE in the past. Guided MTE therefore calls for a proactive approach of doing things rather than being reactive, implying that MTE takes responsibility for the outcomes of its system, which will have to be accompanied by careful planning, being relevant and resourceful. This requires not only mathematics teachers, but all stakeholders charged with such responsibility at all levels.

Applying the notion of Community of Practice (CoP) developed by Wenger (1998) the implementation of the transformative response to global challenges as explained above requires to be seen within an emergent relationship between different stakeholders who can come together around a joint enterprise (a common area of interest), characterised by the existence of mutual engagement in the social practices such as the process of developing common understandings, routines, activities, stories, and ways of speaking and acting. Taking MTE as professional work, any transformative practices would require sharing experiences of norms and practices, together with sharing the ways of using certain tools. With the nature of global challenges and in the spirit of being pro-active, MTE needs to identify knowledge resources oriented to constructing responsive repertoires in the education system. This process includes how and what exactly should occur to implement meaningful responses to global challenges Figure 1 offers a model of changes that should be considered for MTE to be responsive to global challenges.

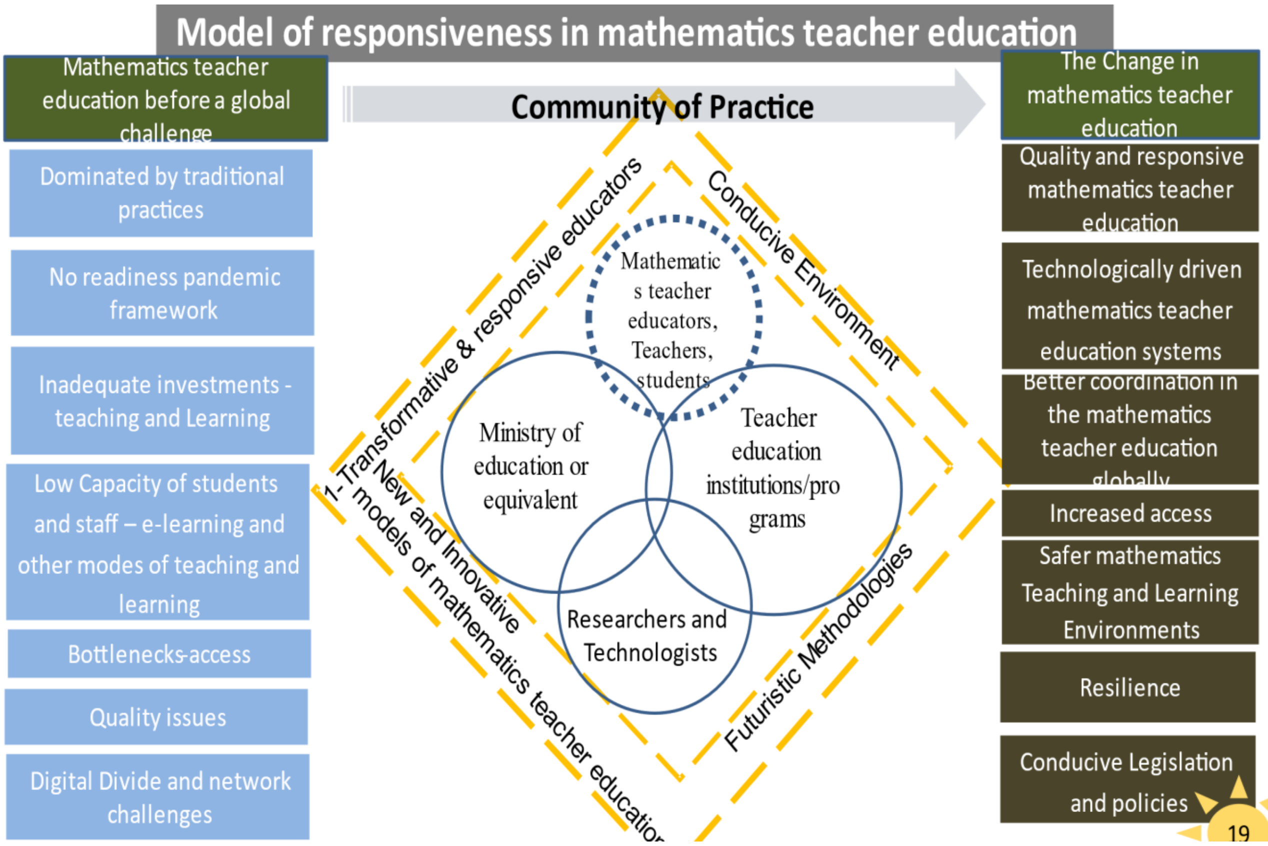

Figure 1: Model of responsiveness in MTE

MATHEMATICS TEACHER EDUCATION CANNOT BE CONSTANTLY RESPONDING TO A CHANGING WORLD (NURIA CLIMENT)

This section of the paper argues against the claim: *Mathematics teacher education should be responsive to a rapidly changing world*. The arguments are based on the notions of immediacy and "world" understood as a global homogenous world.

Immediacy versus responsibility

MTE cannot be drastically changing at every turn. Significant changes in MTE demand resources, political decisions, and social agreement. Moreover, a teacher education program needs coherence and a clear and solid orientation. All of these factors are incompatible with a continuous shifting from one position to another. This is not to say that MTE should not take into account socially and culturally important features and changes. While we cannot attend to the particularities of each new situation, we can attend to structuring elements within them. One of these is inequity. Many of the global and local crises are based on inequality and, as a result, inequality increases (Chan et al., 2021). Moreover, the capacity to react to rapid change is itself unevenly distributed (and heavily dependent on countries' richness, size, and cultural constrains, among others). As Chan et al. noted:

> In our research, we may find it important to study the local, immediate needs but also look

> at the big questions and examine the structure beneath the crisis. What is invariant? And where that structure is unjust, how can it be changed? (p. 6)

Working on inequality in MTE is to take responsibility for what happens from a deep and thoughtful response but not based on immediacy. Preparing teachers to be sensitive to inequality and difference will promote a kind of mathematics teaching oriented towards developing future citizens who are more respectful and sensitive to others.

From adapting teacher education to preparing flexible teachers

If MTE were orientated towards a flexible, critical teacher, adaptable to different situations, MTE would not need to be constantly changing. Over the last two decades, research into mathematics teacher professional development has built up the profile of the adaptive, critical and reflective teacher. In this perspective, the teacher makes his or her own decisions taking into consideration the context in which the teaching takes place. Both awareness of the context and teacher autonomy play an important role. Within this paradigm, the teacher regards him or herself as an authority, with the capacity "to evaluate different perspectives in terms of what he or she values and considers to be empirical evidence" (Cooney et al., 1998, p. 312).

Some training experiences have empowered teachers to adapt to crises such as COVID-19. Ramploud et al. (2021) show the effects of a formative strategy based on lesson study framed in a perspective of cultural transposition. According to them, this perspective "is aimed at giving teachers, who have come into contact with teaching practices from different countries, the opportunity to become aware of their own unthoughts" (p. 4) (including cultural beliefs about teaching and learning). The goal is the emancipation of teachers to make educational decisions based on their intentions. This training develops resilience skills ("such as finding one's educational intentionality and flexibly trying to find ways to design one's educational proposals corresponding to such an intentionality, even in unexpected situations" (p. 5)). Ramploud et al. present the case of a teacher and the dilemmas she faced in her experience during the COVID-19 outbreak, in which flexibility became central. This goal of emancipation in this study closely parallels that of autonomy in Cooney et al.'s (1998) analysis. Both of these studies draw attention to the importance of (future) teachers reflecting on their assumptions if the goal of becoming an adaptive teacher is to be achieved. These assumptions include both how they see themselves as teachers, and how they understand the social and cultural parameters within which they must act. The aim is to make teachers as aware as possible of these constraints, and to understand when it may be necessary to change them.

Some of the voices advocating that both mathematics teaching and MTE must be adapted to emerging critical problems, such as the COVID 19 pandemic, call for teacher training to foreground certain mathematical content (such as graphics, statistics and modelling). Recognizing the importance of these contents to understand the current world, if teachers reflect on the possible role of mathematics not only to interpret and respond to crises, but to create and shape them as well (Skovsmose, 2021), they can contribute to producing educated citizens capable of understanding and engaging more humanely in problematic situations.

Given that teachers will have to deal with different situations of change, it is essential that they become aware of their training as a continuous process, which begins in initial training and lasts throughout their professional life. This reinforces the desirability of training models for both preservice and inservice teachers that foster shared learning based around practice (Carrillo et al., 2020). Such models emphasize for both preservice and inservice teachers that foster shared learning based around.

A global world

When we refer to a rapidly changing world, we think of a global world where problems are evenly perceived and given equal importance across the planet. To what extent is this an unquestionable and positive assumption? Not all problems are equal; some are deemed to be crises while others are considered local (Skovsmose, 2021). However, in a particular place, a local problem may be more critical to its population than a global one, and so-called crises can be identified as such because they affect richer and more powerful countries. The global perspective may imply, in addition, a standardization of MTE without considering the local needs of mathematics teaching (Gellert, 2021). MTE, then, should not only be guided by global problems; rather MTE needs to make (prospective) teachers aware of the importance of local issues relating to mathematics.

Teacher education can only provide local answers

While MTE needs to be adapted to social and cultural changes, some of which may be common in different places, the answer must consider the particularities of where such education takes place. There is no global consensus of what is effective mathematics teaching, and perhaps such an aim is even undesirable. While western countries are generally agreed on approaches that emphasize student centeredness and inclusiveness, "how relevant for other settings are our western understandings of effective mathematics pedagogy?" (Walshaw, 2014, p. 299). Global approaches to mathematics education may involve a new type of colonialism, imposing certain perspectives on national educational policies (Schubring, 2021). These perspectives are often a reflection of the hegemony of Western states. Therefore, a uniform global response in terms of MTE is not desirable. Research has revealed important differences in MTE even in geographically, historical and political close countries, as the Czech Republic and Hungary (Novotná et al., 2021). These authors explain that similar questions are not always answered similarly.

Conclusion

Refuting the claim that MTE should be responsive to a rapidly changing world is not to say that education programs should insulate themselves from what is happening in the world at large. Teacher training, and mathematics teaching itself, must evolve and adapt to changing circumstances. But we must also be aware of two considerations in doing so. First, given the difficulties and differences involved in orchestrating a rapid response, would it not be better to focus our attention on the core features of crisis situations, and on a stable teacher profile, such that MTE remains consistent over the medium term, rather than undergoing constant change? Such an orientation is achievable if our teacher profile is that of a flexible professional, who recognizes that the underlying cause of the majority of crisis situations is inequality. Second, the world is variegated, and mathematics teaching and MTE are highly context dependent activities. Both the identification of the issues meriting a response, and the capacity and means to do so, should be considered from a situated perspective. We must be alert to the dangers of inadvertently furthering the interests of the most powerful, and establishing a standardization of MTE and hence of mathematics teaching.

MATHEMATICS TEACHER EDUCATION IN AN EVER-CHANGING WORLD (JAGUTHSING DINDYAL)

This section of the paper argues against the motion: *Mathematics teacher education should be responsive to a rapidly changing world*. In this section, teacher education refers to the structures, institutions, and processes by means of which people are prepared for work in elementary and secondary schools, including preschool and kindergarten (Taylor, 2016) and the education of both preservice and inservice teachers is included. Teaching is viewed as "a complex practice and hence not reducible to recipes or prescriptions" (National Council of Teachers of Mathematics [NCTM], 1991, p. 22) and "The quality of an educational system cannot exceed the quality of its teachers" (Barber & Mourshed, 2007, p. 16). In addition, MTE, in any country, is considered as being embedded in specific socio-cultural, economical, political, and ideological contexts that limit rapid and frequent changes to its nature and content.

How do we see MTE?

Does MTE have an independent existence? MTE is not a separate entity. It is not tangible but is an underlying process that can be perceived or sensed in a timeframe in any country. MTE is identified with the school mathematics curriculum. Accordingly, it is contextual within each specific socio-cultural and socio-economic environment. as advocated by Apple (2001) who claimed "teacher education does not stand alone. It is deeply connected to more general tendencies in educational politics" (p. 183). Most importantly, both mathematics education and MTE in any country are in the political domain where important decisions about mathematics curriculum and MTE are made by the policy makers. Apple (1992) added that "mathematics education exists as part of the larger curriculum" (p. 429) and as such, policy decisions about general education impact mathematics education and certainly impact MTE as well. Very often financial priorities affect the quantum of the GDP allocated to education and as such to MTE. In view of the current pandemic, Schleicher (2020) has reiterated that our capacity to react effectively and efficiently in the future will hinge on governments' foresight, readiness, and preparedness, which connects clearly with policy decisions about MTE.

The decision about what to change and what not to change following changes in the world resides with the policy makers in each country. We know that policies take time to change. As such, we cannot change the mathematics curriculum and the concomitant MTE curriculum too rapidly. Do countries have the ability and resources to respond to MTE following rapid changes in the world? Schleicher (2020) reports that funding in many countries have been diverted in the health sector and the economy. How fast can universities and other similar institutions responsible for MTE bring about rapid changes in the MTE curricula?

Stigler and Hiebert (1999) have highlighted that teaching is a cultural activity. There is a uniqueness about culture in all countries that hinders it from being modified or changed rapidly. An important aspect is the value system that is so unique to a country. It takes time for values to change or be modified in some ways. Even within the same country, educational initiatives from the federal government and other national bodies are viewed with a lot of suspicion by state authorities. For example, in the U.S., proposals for reform by the NCTM led to the so-called *Math Wars* between the *Mathematically Sane* and the *Mathematically Correct* groups. So, imagine what will be the issues if individual countries started to respond to all types of proposals for change in their MTE, based on "rapid" changes in the world.

Rapidly changing world: Myth or reality?

The notion of rapidly changing world is not an issue in the sense that the world has always been a changing world as exemplified by this quote from the Greek philosopher Heraclitus who has proposed: *"There is nothing permanent except change" or the equivalent idea that "We never cross the same river twice". This implies that the notion of a changing world, however significant, is not new as we have always dealt with this idea in one way or the other. Examples pertaining to the mathematics curriculum and the corresponding effect on mathematics education worldwide include:*

- *There were reforms embodied in the New Math movement of the 1960s, the back-to-basics in the 1970s, the problem-solving movement in the 1980s and the standards movement in late 1980s and 1990s. Although these happened mainly in the U.S., their repercussions reached all parts of the world.*
- *Influential reports such as the Cockcroft Report (1982) of the UK also impacted MTE worldwide, given the considerable number of past British colonies.*
- *Theories about how students learn mathematics which started mainly in Western countries: behaviourism, guided discovery, and constructivism have influenced mathematics education and certainly impacted MTE worldwide. There were even proposals for using teacher-proof approaches in schools.*
- *Data from the international comparative studies such as TIMSS and PISA, unexpectedly, showed much higher levels of performance from countries in South East Asia as compared to Western countries and debunked some myths about class size, and teaching approaches and led to several countries adopting curricula published elsewhere.*

- *Influential books such as The teaching gap (Stigler & Hiebert, 1999) emphasized the idea that teaching is a cultural activity and highlighted lesson study approaches used in Japan, especially, to the Western world.*
- *Advances in technology which led to the use of calculators, graphing calculators, CAS and DGS also influenced the mathematics curriculum and de facto, the MTE programs worldwide.*
- *Recent initiatives such as STEM education are already affecting mathematics education in many parts of the world, with several issues (Bakker et al., 2021).*

So, there have always been changes in mathematics education in parts of the world to which the rest of the world has been responding for good or bad based on their own local policies and available resources. So, what are these rapid changes that we should be reacting to? What guarantee do we have that these so-called rapid changes are not market driven to exploit countries worldwide? We have to be extremely careful, as Apple (2001) cautioned, that as "market-based approaches are growing internationally, there are concomitant moves to create uniformity and a system of more centralized authority over what counts as important teacher skills and knowledge" (p. 183).

What does "should be responsive" mean?

There is a lot of vagueness associated with the term "responsive". Are we talking about a significant review of the MTE curriculum? Are we suggesting massive retraining/re-education of teachers each time there is what we (or a group of people or institutions) consider a major change in the world? If so, where is the funding coming from? In many countries funding for education has been directed away from education to other so-called priority sectors by governments worldwide (see Schleicher, 2020). Recent experience with the pandemic shows that we are not prepared but the main issue with the pandemic affecting classroom practice was that schools were in a state of lockdown. There was no physical classroom and so the need was for a virtual mode of delivery through a teaching platform such as Zoom or Microsoft Teams for which teachers had not previously been prepared for in most countries. Inaccessibility of the classroom seems to be the main issue and not the CK, PCK, or MPCK of the teachers.

MTE has to be certainly responsive to changes in the local environment and that the way it has been in all countries, whether there is a single institution (as in small countries) or several institutions (as in larger countries) involved in the preparation of teachers. MTE is not seen as perfect in any situation. It is in a state of flux, always changing and adapting to local needs of the curriculum as and when there are changes in the curriculum. There is still much to be done locally in improving the relevance of MTE in any country. To what extent can countries deploy their limited resources to counter or adapt to so-called rapid changes in the world? Who decides what is a global change or a global issue? Are we not advocating some kind of uniformisation of teaching practices across different countries? (see Apple, 2001) Also, focusing only on MTE does not make sense without any concomitant focus on the broader mathematics curriculum, actual delivery of lessons and, perhaps most important, students' learning. So, why not focus on initial teacher preparation? As Barber and Mourshed (2007) stated, "Teachers develop the bulk of their instructional capability during their first years of training and practice" (p. 28). As such, having better preservice MTE can develop more versatile teachers who are more adaptable to new changes.

Other than in extreme cases such as epidemics, wars and environmental issues, which may affect the basic needs of human beings worldwide, MTE has always existed in a changing world. The notion of rapidly changing world is not clear and the idea of being responsive is vague. MTE has always tried to be relevant to local needs in all countries. Rather than continually changing MTE it is suggested that more resources be directed to get the right people to become teachers, develop them into effective educators, and ensure that each child gets the best possible instruction (see Barber & Mourshed, 2007).

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PME 45

Alicante · Spain · 2022

RESEARCH FORUMS

MATHEMATICS TEACHERS' INTERACTIONS WITH RESOURCES THROUGH A LANGUAGE LENS

Coordinators: Jill Adler, Núria Planas, Luc Trouche, Janine T. Remillard
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INTRODUCING THE TOPIC AND FOCUS OF THE RESEARCH FORUM

Research on resources in/for mathematics teachers' classroom work and/or professional development is a vibrant domain that has been addressed through a number of diverse approaches and emphases (e.g., Adler, 2012, 2021; Barwell, 2018; Gueudet, Pepin, & Trouche, 2012; Planas, 2018; Remillard, 2019; Ruthven, 2019). In this domain, the scope of a culturally-driven concept of resource has been expanded and used or contrasted to explain possibilities and challenges of mathematics teaching and professional development. Despite the ambitious agendas, the domain has not yet been placed in relation to advances in mathematics education research with a language lens, that is, field research that takes the study of mathematics education processes as integral to the study of the language underlying these processes.

We (a diverse group of domain researchers) believe this domain of research would benefit from discussion on the extent to which and how a language lens differently traverses and shapes or could shape different domain approaches across cultures and theoretical traditions. While much of the emphasis in the Resource Approach to Mathematics Education (e.g., Trouche, Gueudet & Pepin, 2019), for example, has been on material resources such as textbooks and digital technologies, the use of language in the interaction with these resources has hardly been explored. In a similar vein, much of the emphasis in the Language as a Resource Approach (e.g. Barwell, 2018; Planas, 2018) has been on symbolic resources such as discourse practices and linguistic moves, with scarce attention to their interaction with teaching and/or developmental material resources. Still drawing on one more example, the Mathematics Discourse in Instruction Framework (MDI, e.g., Adler, 2021) has placed emphasis on specific resources like examples and explaining, and their connections within language in use. In all these approaches, the concept of resource is a central focus for research and thus more interaction between them could have been expected.

Complementarily to the research forum in construction in the form of a 2023 ZDM Special Issue with L. Trouche, J. Adler and J. Remillard as guest editors, this Research Forum aims at putting forth a language-based discussion within the research domain on mathematics teachers' interactions with resources in school teaching and/or in professional development. We seek to provide newer understandings of and synergies around: i) how language is or can be a resource for mathematics teaching and developmental practice, and ii) how it interacts or can interact with other resources

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towards their realization for mathematics teaching and teacher developmental practice. We hope to give focus and direction to two questions, each of which is connected to the major goals of learning from and expanding the discussion amongst frameworks:

- RF-Q1. How do we (as a mathematics education research community) understand language as a resource in our studies with curriculum, mathematics teachers and teaching?
- RF-Q2. How do we understand teachers interacting with resources in crossing languages and contexts?

Rather than summarizing findings in the domain over the last years, we choose to challenge ourselves to think beyond the boundaries of our apparently disconnected frameworks by trying to discern what could be gained, refined, or added through the introduction of either a language lens and/or the study of effects of the interaction of language with other resources at play in our settings. In doing so, we hope promising directions for future research will emerge, as we build more interconnected frameworks. Importantly, the risks of building the research domain in parallel to advances of mathematics education research with a language lens will be reduced.

SESSION 1 – FOCUS ON RF-Q1

How do we (as a mathematics education research community) understand language as a resource in our studies with curriculum, mathematics teachers and teaching?

In Session 1, we will discuss some of the theoretical-analytical approaches to language as a resource in research work on curriculum and teaching, and on the ways in which mathematics teachers meet language in their interactions with other curricular and developmental resources. While the focus on language as a resource in mathematics education research has mostly been developed with respect to students' home languages and learning, this field-based focus was importantly prompted by the analyses of mathematics teaching in the seminal work of Adler (2001). A number of complementary or alternative theoretical-analytical frameworks have progressively emerged to capture different aspects of the complexity involved in language use, and to differently account for how language intersects other resources such as knowledge, but also textbooks, digital tools, lesson plans, classroom tasks or developmental sessions. Following the introduction to the RF and a brief explanation of how its two sessions relate to each other and to mathematics education literature on resources and language, this session will focus on the presentation of three particular approaches that allow us to see a dynamic scene of mathematics teacher education research in which a language lens is fundamental.

The first contribution (N. Planas, J. Adler, & L. Mwadzaangati) will present theoretical-analytical tools originated in sociocultural (MDI) and sociolinguistic (SFL - Systemic Functional Linguistics) frames. These are tools oriented to use language with mathematics teachers for the design and promotion of mathematical discourse practices in their content teaching. Situated insights within country contexts of Malawi and Catalonia-Spain will provide the basis for exploring challenges around the realization of language as a resource in the thinking, preparation and implementation of content mathematics teaching. The second contribution (H. Sabra & J. Alshwaikh) will present theoretical-analytical tools from the MDI and the Documentational Approach to Didactics (DAD, Trouche, Gueudet, & Pepin, 2020), approaches as applied in their study of the use of the mother tongue (Arabic) of mathematics teachers in Palestine and its realization as a resource to support their classroom teaching. Issues on how the teachers interact with their language in the use and interpretation of other teaching resources available (e.g., textbooks), as well as the eventual modification of the languages (e.g., naming of mathematical objects) in them, will be addressed. Adopting a language lens through a social semiotic framework, the third contribution (H. Van Steenbrugge & J. Remillard) will present theoretical-analytical tools designed to explore multimodal modes of communication put into use in the design of mathematics curriculum resources. Printed lesson guides and digital platforms for elementary mathematics education in Sweden, USA and Flanders-Belgium will illustrate the discussion of how images in/and written texts, in both printed and digital resources, communicate representational meanings about how the mathematical content is taught and learned, and how the relations between teacher and students evolve around encounters with curriculum resources.

A commentary and provocation by R. Barwell reflects on the approaches presented, pointing to possible directions of convergence and divergence and raises critical questions for engagement in this session in the forum.

SOCIOCULTURAL FRAMES FOR A FOCUS ON THE RESOURCE OF MATHEMATICS TEACHING TALK

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Introduction

In our studies with secondary school mathematics teachers in Spain, South Africa and Malawi, we share sociocultural frames in the approach to the mediational role of language and to its potential role as a resource in/for mathematics teaching and learning. At the intersection of Vygotskian sociocultural theory and Hallidayan functional linguistics, and with different emphases in our respective research contexts, we view language as an integral aspect of what makes teaching and learning possible, whose use in interaction with other resources (e.g., time, knowledge, curriculum texts) can be investigated and supported in developmental work with teachers. Related to our interest in the resource of language, we share the interest in the more particular resource of mathematics teaching talk as a means to enable and support learners' participation in the mathematical discourse.

We consider the focus on mathematics teaching talk very timely in the current moment of mathematics teacher education research and practice that is language related. Despite the value given to this talk, it is often subordinated to the mathematical discourse practices of reasoning and argumentation, and learners' productions of these, and remains under-researched as an object on its own. In our collaboration, we argue for mathematics teaching talk as an equally prominent focus of research about language-responsive mathematics classrooms, that does not compete with the focus on mathematical discourse practices and learner participation in mathematical discourse. It was a focus on mathematical teaching talk that led to substantial progress in the early years of mathematics education research on language (e.g., Pimm, 1987; Lampert, 1988). That said, the more recent developments of this research stimulated by the influential construct of the mathematical discourse practices (e.g., Moschkovich, 2007) requires renewing the debates around what kind of resource is mathematics teaching talk, or what is it for.

What mathematics teaching talk is for?

With respect to the question of what mathematics teaching talk is for, we find inspiration in the example of lesson work in Lampert (1998), with the blocks of a tangram and the task, "Can two of them be joined to make a hexagon?" (p. 1). In the middle of disagreement about whether the angles in one of the figures proposed should be measured with respect to the "inside" or the "outside" of the figure (p. 3), the teacher explicitly talked about these angle types and related this to whether it was or was not a hexagon. Mathematical discourse practices throughout the lesson with reasoning challenges such as "Does every figure that has six sides also have six angles?" (p. 4), developed with moment-to-moment teaching talk in which learner expressions such as "inside and outside angles", "equal sides" or "two of the same shape", were discussed. Importantly, the teacher inserted "relationship" in her talk ("So the fact that a hexagon has six sides that you started out saying there, and the relationship between these shapes…", p. 2), and by doing so she offered and connected vocabulary and reasoning. Mathematics teaching talk is here a resource that draws on word use and reasoning, and the relationship between these, capturing the learning challenges that evolve out of the participation in the task at play.

In our research strategy around a focus on a notion of mathematics teaching talk that aims at enabling and supporting learner participation in mathematical discourse, we build on two theoretical-analytical tools at the intersection of Systemic Functional Linguistics (SFL) (Halliday, 1978) and the Vygotskian-informed Mathematics Discourse in Instruction (MDI) (Adler, 2021). Both SFL and MDI refer to the meaning potential of linguistic interaction and word use in talk. Specifically, SFL argues the meaning potential of lexicon (e.g., "relationship" in Lampert's example) and grammar (e.g., "two figures of the same shape") in any language, and how it can be realised within concrete registers (e.g., school geometry) in communication. MDI considers the lexicogrammar level when examining word use in mathematics teaching and the role played by naming (how mathematical objects, processes and procedures are referred to e.g. a figure with six sides and six angles is a hexagon) and explaining (how these are reasoned about, or given legitimacy as to what counts as mathematical e.g. that 'joining' shapes to make a hexagon is 'relating' them) within mathematical discourse. We thus zoom in from language to mathematics teaching talk, and then to naming and explaining as resources in/for mathematics teaching.

Two sites of professional development practice

We need to state that whether in relation to SFL or MDI, our attention to teaching talk, as it focuses on word use, and brings the teacher into focus, is frequently interpreted as concerned only with vocabulary or technical language and/or promoting "teaching as telling" through what comes to count as an explanation. Lampert's example hopefully counters these interpretations. At the same time, we acknowledge that the study of naming, explaining and more generally word use is not new in the community of mathematics education research on language, indeed this goes back to Pimm (op. cit.) over 30 years ago now. However, we contend that more remains to be done in the theorization of word use that can inform work with teachers on language-responsive mathematics teaching (Prediger, 2019).

From the perspective of the interplay between theoretical and practical work (and of the scope of application of the theory in professional development practice), our conceptualizations of the tools or resources of lexicon and grammar, and naming and explaining continue to evolve, and remain challenged by numerous tensions. Some recent insights come from workshops conducted with secondary school mathematics teachers in Malawi and Spain, with a focus on word use in the teaching of angles.

In Spain, the tools of naming and explaining, in mathematics teaching talk, are being approached with respect to mathematical contents of the secondary school curriculum and content learning challenges faced by many learners as reported in the field literature. Naming and explaining are then operationally linked together and defined as: words and sentences with the potential to communicate meanings and induce reasoning or discourse practices to support the overcoming of learning challenges whose experience can easily evolve out of the learners' participation in a concrete mathematical task. Given the task of replacing the machine that rotates the pieces in an image, and the widely documented challenge of the static thinking of angles, naming and explaining the centre of the rotation angle, during the task resolution, are important resources in mathematics teaching that can, in turn, enable and support learners in moving out of methods of 'guessing' the place for the machine.

In Malawi, in a lower secondary school lesson focusing on the meaning of an 'exterior' angle of a triangle, and its relationship to interior opposite angles, the teacher asked learners what they thought an exterior angle of a triangle was. Using their knowledge of interior angle as angle inside a triangle, learners referred to exterior angle as angle outside the triangle. This latter informal naming was reflected in some learners pointing to the reflex angle outside the triangle, others drawing a line intersecting a vertex of a triangle and pointing to angle between the intersecting line and side of a triangle as exterior angle, and others extending a side of the triangle and marking the exterior angle formed without describing it. This unfolding was in the enactment of a first lesson plan in a lesson study, and as participating teachers reflected together on the lesson, they discussed the interaction of the informal talk with learners' thinking. They replanned the lesson so as to enable and support the naming and explaining of the exterior angle by linking learners' informal association of 'outside' with its specific mathematical meaning as the adjacent angle formed by extending a side of the triangle.

The numerous tensions at the research and theoretical levels of a focus on word use in mathematics teaching talk appear recreated at the practical level in the work with the teachers in Spain and Malawi. These are versions of well-known and ever-present tensions – but even more reason that they are included in teacher education practice on language-responsive mathematics teaching, and related research.

COMBINING MATHEMATICAL ARABIC AND THE TEXTBOOK FOR TEACHING THE SIGN OF QUADRATIC FUNCTIONS

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Introduction

In our study, we present the use of the mother tongue (Arabic) of mathematics teachers in Palestine and its realization as a resource to support their teaching. We focus on how Arabic language and the textbook interact for teaching "functions" in Grade 10. To better understand the current challenges of teaching mathematics in Arabic, a historical overview seems essential.

The movement to translate Greek mathematics into Arabic began towards the end of the 8th century. It was accompanied by the foundation of institutions to organize research activities and to produce new scientific knowledge; and led to developments in the Arabic language itself (Rashed, 2019). In the 19th century, another movement of translation happened to adopt the new science from Europe to different regions of the Arab world (Crozet, 1999). At this period, scientists in Egypt adapted the "new knowledge" by translating texts into Arabic. Educators sought to adapt new knowledge to develop course materials for higher schools. The products of these processes have mainly served as resources for teaching 'translated mathematics.'

In the 1950s and 1960s, there were attempts to look at mathematics curricula to promote "uniformity in Arab education systems" (Jurdak & Jacobsen, 1981). Common textbooks were designed and translated into Arabic, each country adapted it and modified it separately. In this period, education in Palestine was influenced by the different rulers, mainly Egyptian and Jordanian and the ongoing Israeli occupation. The Palestinian Ministry of Education became responsible for education in the West Bank and Gaza Strip in 1994. A unified curriculum of the two parts of the territory saw the light only in 2000. The textbooks are designed and delivered by the Ministry of Education; they are the main resource for mathematics teachers.

Thus, Arab teachers, including Palestinians, were provided textbooks containing translated mathematics. The development of a suitable language for teaching was left to them. The main question we address here is: how Palestinian teachers combine the use of the textbook and the Arabic language in their teaching?

Theoretical framework and methodology

To designate what the teachers have to develop as language, we rely on Halliday's (1978) concept of *mathematics register*. Our interest is in the way that teachers develop their own 'new mathematics register,' which we call 'mathematical Arabic.' Hence, following Halliday (1978), we define *mathematical Arabic* as the set of meanings that involve "the introduction of new thing-names, the ways of referring to new objects or new processes, properties, functions and relations." Therefore, we want to understand the way mathematics teachers develop their own mathematical Arabic by interacting with what is available to them in terms of resources, especially the textbook. Mathematical Arabic resources the expression of textbook content and facilitates student's learning of it.

Adopting Mathematical Discourse in Instruction - MDI (Adler & Ronda, 2015), we focus on explanatory communication as a tool of exploration to better understand how mathematical Arabic is used as a resource. In addition, MDI looks at the types of language used within a lesson, whether it is colloquial language or mathematical language. Even for the latter, MDI defines three types of mathematical language; school, semi-formal, and formal mathematical language. Similarly, and in order to evaluate how teachers justify mathematically, MDI suggests three categories: non-, partial and full mathematical justifications.

In addition, we draw on the Documentational Approach to Didactics - DAD (Trouche, et al., 2012) to characterize the way the textbook influences teaching, and the way in which teachers' dispositions guide their use of the textbook. DAD also helps us to study the combination that teachers create between the use of textbooks and deployment of their mathematical Arabic. For our study, we consider the teachers' schemes of use of the textbook and mathematical Arabic, and the way the teachers justify their choices when interacting with the resources.

We hypothesize that the DAD and MDI are complementary and come together to allow us to understand how the combination of mathematical Arabic and the textbook is constructed. While MDI enables us to understand what mathematical Arabic is used and the degree of formality of mathematics presented, DAD explores the scheme of use and the components of these schemes.

Our field of study is based on three Palestinian teachers (T1, T2, and T3); and a specific teaching aim of "how to determine the sign of quadratic function" for Grade 10. The data collected is related to the Grade 10 textbook, audio-records of classroom sessions, and interviews with teachers. We listened to all recorded sessions and selected one common episode among the three teachers, and we transcribed those episodes. We then interviewed each teacher to reflect on their mathematical Arabic and the way they used the textbook. The interviews contain two parts. The first was the teacher's profile and the characteristics of her own mathematical Arabic. The second part was related to the teaching aim; we asked questions that allowed us to refine our analysis of the scheme of use of the textbook and the mathematical Arabic to reach that aim.

Discussion and conclusion

It appears that the responsibility for developing a language for the dissemination of mathematics is beyond the responsibility of teachers. We noticed that teachers use formal mathematical terms mentioned in the textbook when they give formal justification through definitions, rules and laws. However, when teachers presented justifications for students during the lesson to explore the sign of the quadratic function, we observed more use of their own mathematical language. Some of those justifications were partially mathematical such as "it's a law, and I memorize it" (قانون أنا حافظاه) referring to the formula of the discriminant. Despite the fact that the three teachers mention three steps in teaching the sign of the quadratic function, we observed that each teacher has a different way of doing so.

We observed that Palestinian teachers develop their own mathematical Arabic, either by referring to their experience as students, or –if applicable– their teaching training. The teaching of mathematics in English at university seems to constitute a break in the process of maturing their own mathematical Arabic. The teaching experience is probably the main ground for shaping one's mathematical Arabic. In practice, teachers shape their mathematical Arabic according to the subject taught, the students' needs (e.g., possible difficulties), and the textbook.

Furthermore, we identified three different forms of combination of use of the textbook and the mathematical Arabic for each teacher: complementarity in the case of T1, tension in the case of T2, and pattern of equivalence in terms of "when the textbook is lacking, mathematical Arabic bridges the gap" in the case of T3. The degree of agency that teachers have toward the textbook seems to be correlated with the development of their own mathematical Arabic.

This study opens up avenues for investigating different issues in teaching mathematics in Arabic. For example, an area of research is curricular studies; investigating the sources of the choices made in the Palestinian curriculum seems to be crucial for defining an adapted language. Another issue is the need for further investigation for defining foundations to help teachers build their own mathematical Arabic.

THE MULTIMODALITY OF CURRICULUM RESOURCES AND THE COMMUNICATION OF SOCIAL RELATIONS

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Curriculum resources as multimodal

We understand language broadly and as a multimodal form of communication. Designers of contemporary curriculum resources use “an ensemble of semiotic features” or modes “that shape what learning is and how it may take place” (Bezemer & Kress, 2008, p. 168). Teachers are generally a primary audience for these features. Increasingly, curriculum resources are offered in digital formats, in addition to or in lieu of print resources. Digitalization extends the possibilities for how authors communicate with teachers and who else they communicate with (Pepin et al., 2017). Using a multimodal lens, we examine how designers of different types of mathematics curriculum resources from Sweden, USA, and Flanders, communicate with teachers, focusing on implicit messages about social relations between the teacher, students, artefacts, mathematics, and eventual other social actors.

Analyzing messages about social relations

Our understanding of the implicit messages about social relations in curriculum resources reflects nearly ten years of work analyzing how print and digital resources communicate with teachers. In the process of structuring mathematics learning opportunities, curriculum authors communicate implicit messages about the social relations at stake in teaching and learning mathematics, specifically with respect to positioning, authority, and agency. The move from print to digital resources expands the typical social relations at stake, requiring us to expand our analytical framework.

Social relations have received relatively limited attention in curriculum resource analysis, which tends to focus on mathematics content and explicit pedagogy (Fan et al., 2013). Our analysis builds on the work of Herbel-Eisenman (2007), which examined linguistic features in textbooks using discourse analysis to uncover messages about the relative positioning of the students, teacher, textbook, and mathematics. Many of these messages are implicit and subtle. By using a multimodal lens, we are able to uncover how these subtle messages are communicated, not just through written language, but also through modes such as image, layout, and connectivity.

Framework for analyzing print lesson guides

The multimodal framework we used to analyze print lesson guides focused on three modes: writing, images, and layout. Bezemer and Kress (2008) explain that modes of communication have different *modal resources,* which specify the possibilities for variation within each mode. Written communication, for example, has graphical resources, like font size, lexical resources, like content, and grammatical resources, that shape the style of communication. Images can vary in size, color, shape, and content. Layout involves the arrangement of these resources on the page. In this way, modes and their modal resources are not discrete, but intersect and work together in multimodal artefacts. Images represent a prominent mode in lesson guides, yet the size and placement of images can also be seen as modal resources in the mode of layout. The modal resources we analyzed for each mode are summarized in Table 1.

Mode	Modal Resources
Written communication	Quantity of writing Focus of writing How written messages were communicated
Layout	Placement of components on page Visual markers and text to signal guide navigation of page
Image	Content of images Size of images Location of images

Table 1: Modes and modal resources focal in our analysis of print lesson guides

We found that layout, images, and written communication work in combination to structure teachers' interactions with the guides and communicate messages about the process and source of mathematics learning. These modes a) structured teachers' reading path through the guide, b) signalled the locus of instruction, and c) configured the relationship between teacher, students, guide, and other instructional materials.

The mode of connectivity in digital resources

Digitalization allows curriculum resources to connect to other resources, people, and to make connections within the resource. Drawing on Akkerman and Bakker's (2011) depiction of boundary objects and boundary crossing, we conceptualized the mode of connectivity as having three modal resources, shown in Table 2. Applying the framework to two digital curriculum resources' (DCRs) adaptability and networking features, we surfaced implicit messages about a) the relations at stake in a typical classroom, b) those that expand beyond this typical web, to include outside actors, and c) the agency implied in these relations.

Mode	Modal Resources
Connectivity	People and objects being connected (e.g., teacher, students, parents, material and semiotic resources, content, units) Domains at stake (school, everyday, virtual, policy) Visibility of connections

Table 2: Modal resources of the mode of connectivity in digital curriculum resources

We found that the two DCRs differed in visibility of adaptability, which we related to messages about agency between the teacher, students, and the DCR. In one DCR, for instance, learning trajectories were made explicitly visible to both the students and the teacher. This visibility positions students and teachers as having control over learning and teaching, in relation to the DCR. We also noted how the visibility of the DCRs' networking between teacher, colleagues, principal, and parents communicated messages about teacher agency.

Conclusion

Our findings illustrate ways that, in their design, curriculum resources communicate subtle messages about social relations that can either reproduce or challenge typical lines of authority and visions of mathematics teaching and learning. We assert that multimodal frameworks and comparative analyses are especially adept at uncovering these messages. As such, these analyses might uncover some of the covert elements of culture in these resources.

SESSION 1. COMMENTARY AND PROVOCATION

UNDERSTANDING LANGUAGE AS A RESOURCE IN MATHEMATICS EDUCATION WITHIN A DIALECTIC-DIALOGIC TENSION

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In this commentary, I frame my remarks with a broad characterisation of how the role of language in mathematics classrooms is theorised in mathematics education. There is, I propose, a long-standing tension in this area of research between a modernist, dialectic orientation, and an alternative, dialogic orientation (Barwell, 2016). In the former, associated with Vygotsky, language, including classroom talk and written or printed texts, is a cultural tool through which learners are initiated into socially established scientific concepts and forms of reasoning. In this perspective, differences in interpretation that arise in mathematics classrooms are to be resolved or overcome so that successful learners internalise established forms of mathematics knowledge, including standard or formal mathematical discourse. In the dialogic orientation, associated with Bakhtin, language is a kind of living social relation in which learners and teachers participate and which mutually defines them. The pre-given nature of language means that meaning precedes participants' intentions, so that they must grapple with the entire history of language to make mathematical meaning. For Todorov (1984), understanding is ultimately, not only an interpersonal process, but also "a relation between two cultures" (p. 109). From a dialogic perspective, then, differences in interpretation are not seen as problems to be resolved, but as fundamental to mathematical meaning-making (Wegerif, 2008). Rather than a focus on a goal or endpoint, in which the teacher passes on approved knowledge to learners, a dialogic perspective acknowledges the mutually constitutive, relational nature of mathematical meaning-making.

The (dialogic) tension between these two orientations is not so much apparent in the theorists cited or positions explicitly adopted by researchers. A quick look at the research literature would reveal a much greater prevalence of references to Vygotsky or authors working in the sociocultural tradition. Rather, this tension links the most common goal of mathematics education to impart, transfer, or transmit an established body of mathematics knowledge on the one hand, and the progressive desire to give space to students' voices and involve them in charting their way in their mathematics learning on the other. This tension is often managed by incorporating dialogic ideas about voice, for example, into a broader dialectic, sociocultural framework (in an attempt, perhaps, to resolve or overcome the differences). In reality, the tension is irresolvable.

Language as a resource in mathematics education emerged as a response to deficit perspectives, such as the idea that bilingualism causes confusion or that language diversity creates obstacles to learning mathematics. In the literature, this metaphor is largely aligned with a sociocultural framework. The word resource evokes a substance or material to be used in order to complete a task. In mathematics education research, the idea of language as a resource generally means that learners use language to acquire the desired mathematical knowledge. In many studies, learners' multiple languages are indicated as the resource(s); in others, particular language practices are indicated, such as code-switching or use of narrative; in yet others, it is the various mathematical meanings or interpretations at play that are seen as the resource (see Barwell, 2018). The first three contributions to this research forum contribute to a recent development in work on language as a resource in mathematics classrooms by adopting ideas from Halliday's systemic functional linguistics or related work (although Halliday's work has informed mathematics education in other ways since the 1970s) (Planas, 2018). A key idea in this approach is that of 'meaning potential'. Language is understood as a semiotic system from which speakers choose among multiple alternatives to realise specific meanings.

The introduction of Halliday's notion of meaning potential is, I suggest, an extension of dialectic sociocultural perspectives on language as a resource. In effect, the material idea of resource has been interpreted as a form of potential (like oil reserves, perhaps). The task of learners is to learn how to exploit this potential (resource) to make prescribed mathematical meanings. In Planas, Adler and Mwadzaangati, the focus is on the role of the teacher in this process. The teacher's role is to use language as a 'mediational means' to guide learners to the correct mathematical meanings. In the illustrative example from a classroom in Malawi, we see the heteroglossia of students' everyday language and multiple meanings, and the teacher's efforts to mediate between their everyday language and meanings and the required mathematical meanings. In their contribution, Sabra and Alshwaikh provide a fascinating account of how mathematics textbooks play a mediating role between a formal Arabic mathematics register and the Palestinian teachers' (and presumably learners') individual versions of that register. Finally, Van Steenbrugge and Remillard examine how choices among "modal resources" in digital curriculum resources produce social relations in relation to mathematical meaning making. These texts are seen as mediating teachers' and learners' mathematical activity in terms of how they navigate the text and the authority relations such activity entails.

The empirical examples given in the three contributions illustrate well the dialectic-dialogic tension. In particular, in each case, there is the strong goal of the mathematics that students are required to learn, leading teachers or textbook authors to search for ways to guide them. At the same time, we see the inevitable heteroglossia of learners' or teachers' diverse ways of participating in language, including mathematical language, and we see hints of how their participation in language mutually defines them. Van Steenbrugge and Remillard refer to Bezemer and Kress (2008) who, in other work have noted a change in the organisation of textbooks since the 1930s, and a shift from a standpoint of 'vertical' authoritative relations, to one of 'horizontal' participatory relations (Bezemer & Kress, 2010). Nevertheless, textbooks are, in some sense always authoritative and so we see again the dialectic-dialogic tension at work. If the empirical examples illustrate the dialectic-dialogic tension, the dialectic, sociocultural theorisation of language as a resource is based on or creates various orderings, often in the form of binaries: formal mathematical discourse–everyday language; teacher–learner; official curriculum– local variation; official language–heteroglossia; semiotic system–participants' utterances. From a dialogic perspective, such orderings serve to define the relation between participants as a form of alterity. The teacher exists in relation to the learner, mathematical discourse exists in relation to everyday language, not as abstract entities, but as entirely relational. One cannot exist without the other. The way in which we theorise these relations is, therefore, crucial since these theorisations serve to structure and organise them.

The etymological roots of the word 'potential' are derived from power (as in 'potent') and resources are often implicated in power relations: humans fight to control resources or to have access to them. In language policy research, Ricento (2005) has argued that the discourse of language as resource often constructs marginalised languages as subservient to dominant ones. We can ask 'whose resource? Who controls this resource? Who consumes it? For what purpose? What kinds of relations are produced?' In mathematics education, research on language as a resource is mostly conducted in socially stratified contexts: children in Malawi or South Africa from African language backgrounds learning mathematics in English; Latinx students in the United States; or students from immigrant backgrounds in several parts of the world. Does the idea of language as a resource mean that learners' diverse language repertoires are to be harnessed just until they can do mathematics in the official language of instruction? Does the idea of language as a resource in relation to textbooks or curriculum materials mean that learners' own diverse ways of talking about mathematics should be harnessed just until they have mastered the desired form of mathematical discourse? Does this theoretical approach not risk reproducing in a more subtle way the social stratification that the more progressive goals of mathematics teaching might hope to dismantle? What alternative theorisations might we consider? There is no tidy answer to these questions but the tensions I have discussed are necessary if we are to think in new ways.

SESSION 2 – FOCUS ON RF-Q2

How do we understand teachers interacting with resources in crossing languages and contexts?

In Session 2, we will focus on the *words* used by teachers as well by researchers when describing/analyzing these interactions with resources, and the correspondence between these words when moving from one language to another one. It constitutes a zoom in perspective towards words, their structure in a language, and their association across languages. Considering words as "saturated with sense" (Vygotsky), we hope that this perspective will allow us to deepen our understanding of mathematics teachers' interactions with resources.

The first contribution (M. Artigue, C. Knipping, J. Novotná, & B. Specht) comes from the International Classroom Lexicon Project which set out to document the terms and the professional vocabulary that teachers use for describing the phenomena of middle school mathematics classrooms around the world. The study of this vocabulary leads to evidence of different *naming systems* on which teachers' discourse is based in different cultures. These naming systems constitute then windows into teachers' resource systems, revealing part of their content and structure. The second contribution (M. Shao, I. Kayali, I. Osta, G. Gueudet, B. Pepin, & L. Trouche), using again the DAD, analyses in Arabic, Chinese and French the same episode of an English mathematics teacher interacting with resources. It leads the authors to think about the instantiation of the same concepts in three languages, and then to rethink the concepts themselves. The third contribution (C. Wang, Y. Shinno, B. Xu, & T. Miyakawa) questions also the process of translating a theoretical framework dedicated to analyze teachers' interactions with resources, here from an anthropological point of view. It identifies different factors (linguistic, cultural, and social) that cause the difficulties or confusions encountered during the translation work: e.g the different status of teachers' collective work in the West and the East. The issues of translations are explored from two levels of the cross-cultural perspective: between the West and the East, and between China and Japan. The provocation, by L. Radford, who with others, has analyzed resource-use crossing theoretical views, and how these refract cultural aspects of teaching and learning, will offer commentary and through this open up discussion.

NAMING SYSTEMS AS A WINDOW INTO TEACHERS' RESOURCE SYSTEMS

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Documenting and understanding naming systems

The International Classroom Lexicon Project set out to document the terms and the professional vocabulary that teachers use for describing the phenomena of middle school mathematics classrooms around the world (Mesiti et al., 2021, Mesiti et al., 2022). When identifying teachers' naming systems for classroom phenomena in the Lexicon Project, important differences appeared (ibid.). The term 'milieu' from the French lexicon, for example (Artigue et al., 2021, p. 195), has no English translation and no obvious equivalent in other lexicons. French teachers use this professional term to describe: "The element of a learning situation with which the students interact, and provide them objective feedback." (Artigue et al. 2021, p. 208). This French term has its origin in mathematics education research and is adopted by teachers to describe classroom situations. The generated lexicons, like this French one, indicate forms of conventionalized classroom practice in ten different countries (Mesiti & Clarke, 2018; Artigue et al., 2017). But it turned out that the naming systems in the lexicons per se were too limited to understand how they orient teachers' visions, analyses of mathematics classrooms, and their complex interactions with resources in the class.

For capturing these multifaceted visions, we created narratives, methodological artefacts, based on the lexicons and classroom lessons of the Czech Republic, France and Germany. These narratives and their comparisons turned out to be promising for understanding how the lexicons, seen as cultural objects, shaped visions of the classroom and of associated phenomena (Artigue et al., 2017). Approaching lexicons in this way, understanding them as cultural artefacts and using them as windows into teachers' resource systems (Trouche, Gueudet, & Pepin, 2020) is an enlargement of our lexicon based comparative perspectives so far.

Teachers' resource systems

Nevertheless, also this approach does not excavate the full range of teachers' resources related to classroom situations. The videos used as spurs in the Lexicon Project show that the teachers involved use a diversity of material resources and tasks in their lesson. All of them make extensive use of the board, and often more or less sophisticated projection devices. They use calculators, mathematics software and physical models, together with diverse tasks from textbooks, school platforms and created by themselves. However, the lexicons mirror this diversity in a very limited way. Some reasons may explain this situation: the fact that the lexicons document key pedagogical and didactical terms used to denote classroom phenomena, not classroom tools; the choice not to include names denoting mathematical software or tools such as rulers and protractors in the lexicons.

The focus in our comparative analyses is on narratives about classroom situations and the lexicon terms used in these. This approach by its own is already promising and reveals the perspectives concealed in the lexicon terms on classrooms, teaching and learning. So, language as a resource for teachers is discussed in this contribution primarily in respect to teachers' conceptualizing of classroom practice, instruction and learning, across different cultural contexts and linguistic traditions. Talking about classroom practice and describing teaching and learning, is a common resource that mathematics teachers and researchers use to get a better and shared understanding of classroom instruction and learning. Documenting and comparing naming systems that teachers use, reveals how the different cultural naming systems capture disparate views on classroom practice, teaching and learning (see Mesiti et al., 2022).

Comparing lexicons and narratives - First results

Despite differences observed, comparing lexicons and narratives also revealed undeniable common linguistic resources for describing different types of tasks proposed to students as learning progresses, between different pedagogical methods, to describe the pedagogical and didactic management of the class by the teacher and to evaluate learning. However, there are also striking differences. The French lexicon and narratives differ from the other two in its mathematical and didactic focus. The Czech lexicon, which is clearly more pedagogical and its vocabulary is much less technical, is more descriptive and closer to ordinary language. The German lexicon and narratives occupy an intermediate position and exploit the possibilities by the German language to create compound words that are richer in meaning than their constituents, and may combine the concrete and the abstract, which results in a technical language that is both concise and accessible. The French narratives describe the mathematical-didactic management of the classroom by the teacher, going into great detail about the mathematical activity. The mathematical contents at stake are also well present in the German narratives, combined with a marked attention to the structure of the lessons, the teaching methods and the way in which teachers exploit learning opportunities. The Czech lexicon and narratives draw our attention to several possible forms of explanations. They also demonstrate that the pedagogical interactions described are closely bonded to the mathematical content at stake.

Even if limited to three lexicons and nine narratives produced by the same teams that produced the lexicons, our research tends to show that the cultural comparative approach of the lexicons and narratives can help to understand language as a resource for teachers describing mathematics classrooms teaching and learning, through the regularities and differences it reveals. However, this approach also has some limits. In approaching teachers' linguistic resources, the lexicons were subjected to a rigorous selection process, privileging terms reasonably shared by teachers. Finally, the nine videos that served as stimuli may also have more or less limited the repertoire of terms considered by the different teams.

DEEPENING THE CONCEPTUALIZATION OF TEACHERS INTERACTIONS WITH RESOURCES BY TRANSLATING A CASE ANALYSIS IN DIFFERENT LANGUAGES

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Introduction

The Documentational Approach to Didactics (DAD) is a theoretical framework that investigates mathematics teachers' professional development through the lens of their interactions with resources (Trouche, Gueudet, & Pepin, 2020). The theory was originally conceptualized in French and developed mainly in English. How can we properly translate DAD into other languages? How could the translation processes help to deepen the approach itself?

Methods

In this contribution, we try to respond to these questions by translating DAD in 'situ' (meaning across a case analysis). Our contribution is based on the DAD-Multilingual project (Trouche, 2020; Shao et al., submitted) and we focus on two languages involved in it: Chinese, and Arabic. They are far from English and both have ancient and rich cultural and curricular traditions related to mathematics teaching resources.

In terms of the DAD concepts to be translated, we focus on two dualities: *resource – document* and *instrumentation – instrumentalisation*. A *resource* is defined as everything that can 're-source' (Adler, 2000) the teacher's activity; a *document* is a hybrid entity composed of a set of resources and a scheme of usage of these resources for facing a given situation. *Instrumentalisation* is the process in which a teacher adapts the resources to his/her didactical goals; *instrumentation* is the process in which the resources, with their affordances and constraints, influence the teacher's activities.

In terms of the 'situ', we select one case of an experienced mathematics teacher (named George). The case consists of three lessons he taught to grade 13 students (aged 17-18 years) in England on the topic of 'volume of revolution' (Kayali, 2020). George expressed confidence and willingness to use a 3D visual software – Autograph – and a variety of other resources in the lessons. We identify, in this case analysis, the instantiations of the DAD concepts, and translate both the concepts and their instantiations from English into the two target languages. We discuss issues arising from this process, taking into account whether the instantiations would also emerge in a similar 'situ' in other educational contexts corresponding to the target languages.

Reflection on the translation process

Due to space limitations, we will not elaborate on the case, but directly present the instantiations of DAD concepts and discuss the related translation issues.

Duality resource-document

The *resources* used by George in the case include: textbooks, past-examination questions and grading standards, formulae cards, students' discourses in class, George's mathematics knowledge about volume of revolution and so on. All the resources, together with a global *scheme of use* attached to them, are considered as a document. The scheme corresponds to a class of situations involving two main goals: introduce the volume of revolution and prepare students for the exam.

The Arabic translation for the global notion of *resources* – مَوارِد (mawaarid) – holds a historic connection with the vital need for water in the desert; the singular form of the term – مَورِد (mawrid) – originally means water spring (source of water) and is now extended to mean 'the place to go for getting informed or inspired'. With this term, the idea of 're-source' in the concept of *resource* is more evident. For the notion of *scheme*, the two potential Arabic terms reveal two different interpretations of this concept: مُخَطَّط (moukhattat) (a plan with a static and linear structure) versus صيغة (sighah) (an intertwining and flexible structure open to redesign and modifications). Obviously, the latter term is closer to the notion of scheme used in DAD.

In Chinese, *resource(s)* is widely translated as 资源 (zī yuán), and here we mainly discuss the translation of some concrete resource – *textbook* – 课 本 (kè běn). The 课 and 本 respectively indicate *lesson* and *book*, suggesting that mathematics teachers in China need to closely comply with the textbook and preview all its content carefully to teach a similar lesson, but it is not the case of George, who only noticed an illustrative figure in the textbook towards the end of the first lesson.

<u>Duality instrumentation-instrumentalisation</u>

We also notice many instantiations of *instrumentation* – *instrumentalisation* in the case. George instrumentalised Autograph by creating a solid of revolution of graph $y=x(3-x)$ in it (Fig. 1). He was *instrumented* by the textbook which provided an illustrative image (Fig. 2) and stated that the students must be able to use definite integration to find the area under a curve. George *instrumentalised* the textbook by: deciding to use its image to explain the formula of volume of revolution, selecting the textbook questions for student practice and connecting them with the exam standards.

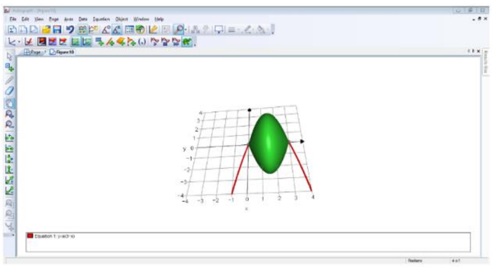

Figure 1: Rotating shaded area around x-axis

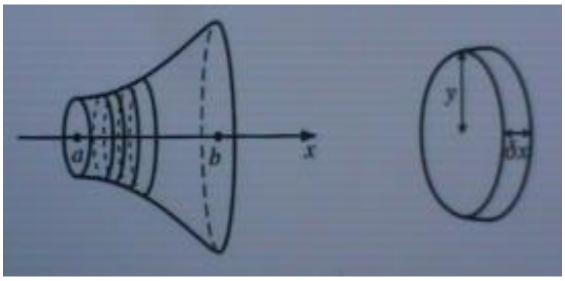

Figure 2: Illustrative figure in the textbook

In Arabic, *instrumentation* and *instrumentalisation* correspond to two nouns: إمْداد (imdaad), which means *supply*, and تَسخير (taskhiir), which means adapt according to one's needs and practices. The most significant example in George's case showing the differences between the two terms is related to the use of the textbook figure (Fig. 2):

لم يبحث جورج في البداية عن الصورة التوضيحية. إنَّما <u>أَمَدَّ</u> الكتابُ عَمَله بالصورة وبفكرة استخدامِها.

ثم <u>سَخَّر</u> جورج تلك الصورة لتقديم إيضاحٍ تمثيليّ للقاعدة

which respectively mean: The book <u>instrumented</u> his work with the figure and the idea of using it; then, he <u>instrumentalised</u> that figure to illustrate the idea of the formula.

In Chinese, we focus on the translation of *instrumentation*. The corresponding Chinese term is 工具化, which has a connection with the philosophical field implying the deviated usage of an object leading to its impoverishment and/or enrichment; inspired by this we could consider that George's *instrumentalisation* of Autograph involves an impoverishment as he only mobilized a limited set of functions of the software (constructing functional graphs and creating their solids of revolution).

For a similar 'situ' in China, teachers' *instrumentalisation* of a software similar to Autograph (e.g., GeoGebra) could also happen, but Chinese teachers seldom instrumentalise a textbook by connecting textbook questions with exam standards as the former questions are often basic and much easier than the standards of the Chinese college entrance exam (高考， gāo kǎo).

Conclusions

As can be seen, the translation 'in situ' affords a bridge to communicate instantiations of the DAD concepts in the English educational context and those 'potentially' existing in the educational context corresponding to a target language. Some instantiations in the English educational context are not totally equivalent to their linguistic counterparts in another context, like textbook versus 课本. Even if we can properly express an instantiation in a target language, the instantiation itself (e.g., the instrumentalisation with respect to a textbook) may not exist in the corresponding educational context. These linguistic non-equivalences open up a perspective for contrasting teachers' interactions with resources in the crossing educational contexts and cultures.

In terms of deepening DAD concepts, the translation process in section 3 shows that we can draw inspiration from the educational, cultural, theoretical traditions in other cultural spheres to enrich the connotation of the theoretical concepts. Also, contrasting the different possibilities of translation for the same concept, in connection with their use 'in situ', can help clarify critical aspects of the concept.

Above all, the more one goes into the translation of details of DAD, the more one can deepen the comparison of teachers' interaction with resources across cultures. The theory itself will also be 'enlightened' by the different cultural analyses and nuances in the target language that needs to be considered during the translation process.

TRANSLATION WORK FROM AN ANTHROPOLOGICAL PERSPECTIVE

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Translation of a theoretical framework

The DAD Multilingual Project (Trouche, 2021) was launched in 2020 to gather translations of the English article that introduces the theoretical framework, *the Documentational Approach to Didactics* (DAD). We were involved in this project and worked on Chinese and Japanese translations. Our translation work entailed various difficulties and issues due to the linguistic and cultural distance between the West and East. We explored these difficulties and issues and attempted to reveal cultural specificities of teachers' and researchers' work related to the resources.

Our study adopts a perspective of *the Anthropological Theory of the Didactic* (ATD), specifically the concepts of *praxeology* and *transposition* (Chevallard, 2019). In general, praxeology is a tool to model the knowledge and practice related to teaching and learning. This notion may be applied to different practices. Regarding our study, teachers' work with resources can be modelled in terms of the *didactic* or *paradidactic praxeology*, and DAD developed to investigate and understand this praxeology can be considered as an element that constitutes a *research praxeology*, which models researchers' practice and knowledge (Artigue et al., 2011). Furthermore, the translation work of a theoretical framework is considered as a process of *transposition* of a part of research praxeology (*logos* block) from a research *institution* (e.g., French or English) to another (e.g., Chinese and Japanese). Taken together, the overall structure of our translation work can be outlined like Figure 1 in terms of ATD.

Another critical hypothesis of ATD is that any praxeology cannot survive in an empty society but in an *institution*. In the institution, a praxeology is always subject to *conditions* supporting it and *constraints* hindering it. In our case, the transposition process or translation work is exposed to the conditions and constraints entailed in the target institution, which presumably produce the difficulties of translation. Thinking upside down, we investigate the difficulties and issues of translation to identify linguistic and cultural elements that constitute the system of conditions and constraints, which is called *ecology* in ATD.

Some linguistic and cultural issues of translations

We faced several challenges in the translation work, which appeared at least at two levels, linguistic and cultural, which were sometimes intertwined.

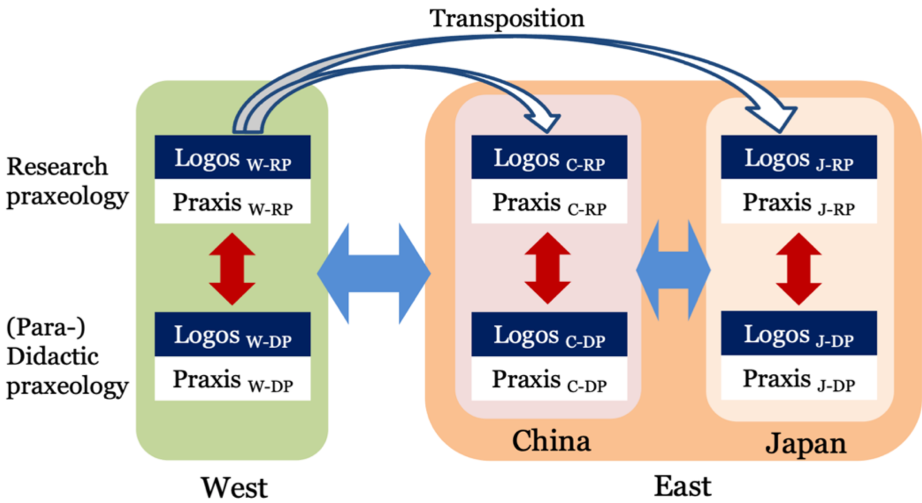

Figure 1: The overall structure of our work in terms of ATD

The distance in terms of the language between the West and East is not limited. This can be firstly attributed to the non-use of alphabets, but the use of Chinese characters and/or other letters. One important issue for us was to create new terms for technical terminologies, due to the linguistic distance such as the inexistence of one-to-one correspondence between Western and Eastern terms, the grammatical difference of the rules of derived words, and so forth. For example, we struggled with the translation of the term *documentation*, which seems easy to be translated within Romance languages. We had multiple candidates for translating the term *document*, and an additional term was necessary to express the meaning given by the suffix "-ation". Considering the literal or contextual meaning of the original term, we finally arrived at the Chinese and Japanese translations, 文献纪录 (wén-xiàn jì-lù) and 文書活動 (bunsho katsudō), as a new term with two terms that already exist in the respective language.

Even if we found a word (at literal or contextual level) of a given English (or French) word, we sometimes faced other issues owing to cultural differences, such as teachers' terminologies, educational researchers' terminologies, and researchers' perspective. For example, our translation of the key word *resource* was 资源 (zī yuán) in Chinese and リソース (risōsu) in Japanese, by taking into account the definition given within DAD. We were wondering to what extent these terms are appropriate and will be used for studying Chinese and Japanese teachers' work. In the educational context, most teachers in both countries are not very familiar with these terms in their professional communities. Instead, teachers use certain specific terminologies for describing and discussing mathematics teaching including resources. In China and Japan, the teachers use the terms 教材 (teaching material; kyōzai in Japanese and jiào-cái in Chinese) and 教具 (teaching instrument: kyōgu in Japanese and jiào-jù in Chinese), which are similar but not identical to the term *resource* in DAD. From a scientific perspective, the evolution of science requires the development of new theoretical concepts and terminologies to better understand the object of study. However, in the Chinese and Japanese communities of mathematics education, teachers and researchers often share terminologies for their collaborative work. The terms 教材 and 教具 mentioned above are used not only by the teachers but also by the researchers. Thus, the choice of translation was made according to teachers' and researchers' terminologies as well as researchers' interpretations of the theoretical framework and expectations of how the theoretical term could be received in their local contexts.

Conditions and constraints in transpositions

Language was one of the biggest constraints that causes difficulties of translation and shapes the theoretical concepts of DAD. The linguistic distance between West and East was greater than between China and Japan. Exploring translation issues leads us to determine cultural similarities between our two countries rather than the differences, such as teachers' terminologies related to the resources, the critical role of textbooks, teachers' individual/collective work like Lesson Study and Teacher Research Group (TRG); and researchers' work in mathematics education.

A critical aspect highlighted in our study was the nature of research praxeology in the East, specifically the close relationship between research praxeology and (para-) didactic praxeology. Teachers and researchers often share terminology for their collaborative work as mentioned above. Related to this, the multiple roles played by researchers were also highlighted: conducting scientific research; working with teachers and playing the role of a "knowledgeable other" in Lesson Study or TRG; and the transposition of research praxeology from the West to the East.

Further, the discussion on the translation in terms of the transposition also questions the viability of the Western theoretical framework (or research praxeology) in the East. A "theoretical framework" is often received in the East as prescriptive or normative, which can be used for developing, designing, and improving educational practices. This would be a crucial question for the further development of DAD.

SESSION 2. COMMENTARY AND PROVOCATION

THE CHALLENGES OF TRANSLATING AS A CULTURAL ENCOUNTER

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In dealing with RF-Q2, one of the central themes that arises is the one of translating teacher resource use from one cultural context to another. In fact, two of the three papers of this RF session deal with the difficulties that researchers have found in translating the ideas of the DAD into other languages. Before I comment on the challenges of translation, and using the DAD as an example, I present a short overview of what we may term the epistemological apparatus of the DAD.

The epistemological apparatus of the DAD

At the epistemological core of the DAD lies the concept of resource—a concept that appears to have different meanings in current mathematics education literature. An encompassing meaning was suggested by J. Adler: everything that can re-source the teacher activity (see Shao et al.). In the DAD, however, for something to be a resource, a *schema of usage* is required. The alluded schema must be understood in the sense of Vergnaud's reformulation of Kant/Piaget's idea, namely in terms of *psychological operative invariants* that organize human behaviour for a given class of situations. As we can see, there is, in the concept of schema, a removal of materiality—an abstraction in Aristotle's sense—that gives the schema its invariance and readiness to be used in front of similar situations. And this is how the schema appears in the DAD (see, e.g., Trouche, 2004).

The ergonomic approach that runs underneath the DAD brings to the fore the need to somehow reverse the Aristotelian abstraction in the didactic cogitations and to return to the materiality of the world. This materiality did not seem to have been relevant to Kant who found in the faculties of the mind (e.g., the faculties of understanding and imagination) enough ingredients to account for its functioning, or to Piaget for whom the objects of his experiments were instrumental means to elicit the logical-mathematical children's schemes. Materiality is the substance of the ergonomic approach, which is a response to late modernity: precisely, a response about our dealings with concrete objects; it is about the interface between body and matter—matter seen Piagetianly; that is, as we accommodate it to our ends (instrumentalization), and, following a Vygotskian thread, matter as it affects us cognitively (instrumentation). Thus, in Shao et al.'s paper we see how the teacher "was instrumented [affected] by the textbook" but also how he "instrumentalised the textbook"; that is, how he accommodates the book to his thinking and needs.

Translating the DAD as a cultural encounter

In the human sciences, a theory is a complex cultural artefact that intends to explain something while at the same time bearing and conveying a specific outlook of reality. There is no exception when the something is mathematics education. The DAD, as any other Western educational theory, has been shaped by a series of conceptions about learning, knowing, knowledge, the teacher, the student, etc. Its main concepts arose in specific historical conditions and have been refined, modified, and adjusted, as new circumstances have required. This is why the DAD, as any other theory, cannot be neutral. It makes assumptions about the very fabric of the educational world. In other terms, the DAD and any other educational theory is unavoidably ideological (it unavoidably conveys a specific cultural system of ideas). Thus, drawing on its assumptions, the DAD sees things as occurring in certain ways: George, the teacher, is instrumented by cultural objects; he acts following some Piagetian schemas, etc.

The fact that theories are ideological invites us to consider translation as a delicate process. For one thing, it would be perilous to consider translation as ideologically free.

To do so would amount to adhere to the view that the earth's various cultural forms of life are in the end all the same—even worse that they are the same as *ours*, which is nothing less than adopting an ethnocentric view of humans and, in the case of educational theories, of how humans learn.

It is precisely the dissonances between various forms of life that surface in the process of translation. Confronted by these dissonances, Shao et al. remark that "Even if we can properly express [a DAD's concept-word, e.g., 'resource'] in a target language, the [concept word] . . . may not exist in the corresponding educational context." The same remark is made in Knipping et al.'s and Wang et al.'s papers. The target language, indeed, responds and co-responds to an altogether different cultural view with its own history and its own political, economic, and social conceptions of the school and learning. The conceptual Kantian schema and the material resources the DAD brings to the fore are foreigners to the Asian cultural views where translation tries to find its niche. The DAD's concepts of schema, resource, language, etc. are part of an ideological apparatus of the Western world through which such a world intended to respond to its own culturally situated needs. There were, in particular, the need to shape a new Western conception of the modern subject (Radford, 2021), the need for a rationality understood instrumentally (Bohy-Bunel, 2022), and the need to come to grips with the question of materiality in face of the Western world's understanding of progress as a technological event (Radford, 2004). These three needs find an answer in the Kant-Piaget-Vergnaud lineage of ideas as challenged by the conception of matter of late modernity.

We see hence that a great deal of the difficulties of the process of translation rests on translating a cultural form of life into a different one. These difficulties do not prevent one from translating one cultural theory into the language of another culture. The problem is not (or not only) a question of language. The problem is to find one's way into the practice of what I want to term a post-colonial, culturally responsible translating practice; that is, one that emphasizes the aesthetics of cultural pluralism; one that places the translated ideas in the web of metaphors and cultural significations of the target culture; one which, for example, makes room to understand the Chinese textbook not as a mere technical tool but as an artefact imbued with the meanings of its own culture and ways of conceiving of the teacher and teaching and learning (see Shao et al.). A post-colonial, responsible translating practice should also be one that is not unidirectional, but *dialogical*. Shao et al. contend that the impossible matching of the DAD terms in the language of the target culture "open[s] up a perspective for contrasting teachers' interactions with resources in the crossing educational contexts and cultures." They go on to say, "We can draw inspiration from the educational, cultural, theoretical traditions in other cultural spheres to enrich the connotation of the theoretical concepts." Whose concepts? DAD's? What about the other cultures and their indigenous ways of conceiving of learning, knowledge, the teacher, and the student? What about their *influence* on the DAD's theoretical assumptions? How do the indigenous philosophers and educators challenge Kant, Piaget, Vergnaud, and all those that inform the DAD?

RF-Q2 points to a profound problem that is always present in the encounter of cultures, namely, that the theories to which we resort in our work are carriers of historically produced ideological stances. These stances surface when we encounter the Other. Taking into account these ideological stances, it seems to me, is a prerequisite to the practice of genuine translating. RF-Q2 moves us beyond the possibilities of language and brings us into the domain of culture, power, and ethics.

CONCLUSION AND INVITATION TO PARTICIPANTS

To summarise, this Research Forum aims to place the longstanding domain of research on resources in mathematics teaching in conversation with the longstanding domain of research on language as an inherent shaping process in the mathematics classroom. These domains have a number of natural intersections, including shared commitments to understanding mathematics teaching and learning within and across differing cultural contexts. At the same time, neither domain embodies a singular theoretical perspective and builds on differing epistemological underpinnings. Moreover, within each domain, digital evolutions and cultural boundary crossing have necessitated expansion and re-conceptualisation of key constructs and processes. Bringing the two bodies of work together invites additional complexities and debates. Still, we find doing so to be crucial and productive if we seek to understand the work of mathematics teaching in the current digital, connected, and translanguaging world.

We have organized the forum to open discussion on two focal questions repeated here:

- RF-Q1. How do we (as a mathematics education research community) understand language as a resource in our studies with curriculum, mathematics teachers and teaching?
- RF-Q2. How do we understand teachers interacting with resources in crossing languages and contexts?

To explore these questions, we have brought together six contributions from mathematics education researchers around the globe, seeking to intertwine research on language and teachers' interactions with resources. These contributions offer and make use of differing lenses and empirical approaches for uncovering and interpreting the use of language in mathematics classrooms and resources and their consequences for how mathematics learning is framed and understood. The presentations in Part 1 offer theoretical-analytical frameworks for examining the intersection of language and resources, particularly in teaching and teachers' curriculum work. The presentations in Part 2 focus specifically on language in the form of words, their translations, and their related meanings across cultural and linguistic boundaries. In addition, two well-versed scholars offer commentary and provocations from differing theoretical perspectives, inviting contributors and members of the audience to explore critical questions. Richard Barwell, asks us to consider the potential consequences of framing language as a resource when used in relation to mastery of "mathematics in the official language of instruction." Luis Radford raises questions about what he calls "post-colonial, culturally responsible translating practice", and specifically the cross-cultural applicability of a framework like DAD informed by Western-world philosophies.

Crucially, this research forum invites participants to bring their perspectives, questions and critique to the above issues. Additional questions readers might consider include:

How might we make use of the issues emerging as we put these two domains in conversation with one another to examine our own assumptions?

How do we navigate the tension between merging/building new frameworks and the need to dismantle existing structures in order to take seriously issues of power and reproduction in cross-cultural, cross-linguistic research?

What might we look for in our initiated collaboration for continuing the investigation of how a language lens can support our understanding of mathematics teachers' interactions with resources?

The co-ordinators and all authors look forward to what we hope will be fruitful engagement with these questions and issues over the course of the conference sessions for this Research Forum.

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MATHEMATICAL PROBLEM POSING: TASK VARIABLES, PROCESSES, AND PRODUCTS

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Mathematical problem posing (MPP) has been at the forefront of discussion for the past few decades, and a wide range of problem-posing topics have been studied. However, problem posing is still not a widespread activity in mathematics classrooms, and there is not yet a general problem-posing analogue to well-established frameworks for problem solving. This paper presents the state of the art on the effort to understand the cognitive and affective processes of problem posing as well as task variables of problem posing at the individual, group, and classroom levels. We end this paper by proposing a number of research questions for future studies related to task variables and processes of problem posing.

POSING A PROBLEM ABOUT PROBLEM POSING – PROMPT DESIGN

To open the floor for a discussion about problem posing, we invite readers to engage with a problem-posing activity. Consider the initial Situation A and several related prompts for problem posing below. How would the different prompts impact your problem posing based on Situation A?

Situation A: ABC is an equilateral triangle. D, E, and F are midpoints of the sides of ΔABC. Show that the area of ΔDEF is ¼ the area of ΔABC.

Prompt 1A: Based on the above problem, use the "what if not" strategy to pose two mathematical problems.

Prompt 2A: Based on the above problem, use the "what if not" strategy to pose as many mathematical problems as you can.

Prompt 3A: Based on the above problem, use the "what if not" strategy to pose two "easy" mathematical problems and two "difficult" mathematical problems, where the relative difficulty takes into account the levels of students.

Five of us independently responded to the question (How would the different prompts impact your problem posing based on Situation A?). A clear difference between the prompts is in the request for the number of posed problems: two in Prompt 1A, two easy and two difficult in Prompt 3A, and "as many as you can" in Prompt 2A. Further, the addition of "relative difficulty" and "levels of students" in Prompt 3A is appropriate for a problem-posing activity with teachers and can be omitted in work with students. However, the reference to difficulty may entice problem posers to consider a greater

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variety of problems and attend to what can make a problem easy or difficult. Moreover, problem posers' interpretation of "difficulty" can be a fruitful venue for investigation.

Common to Prompts 1A, 2A, and 3A is the reference to the "what if not" strategy. As such, the expected variations in problem posing can attend to any of the problem attributes:

V1: What if ABC is a not-equilateral (right angle, isosceles, scalene) triangle? What then is the ratio of the areas of ΔABC and ΔDEF?
V2: What if D, E, and F are not midpoints but divide the sides in some common ratio. What then is the ratio of the areas of ΔABC and ΔDEF?
V3: What if the ratio of the areas of ΔABC and ΔDEF is a given R. How then should we place points D, E, and F on the sides of ABC to obtain the given ratio of the triangle areas?
V4: What if we are not considering ΔABC and ΔDEF? What other triangles are determined in Situation A? What is the relationship between their areas?
V5: What if the starting figure is not a triangle but a quadrilateral (or a special quadrilateral, like a square) and the "inner" quadrilateral is constructed by connecting mid points (or points placed on the sides of that quadrilateral) using a given ratio. What then is the relationship between the starting areas and the inner quadrilaterals? What if it is not a quadrilateral but any polygon?
V6: What if we aren't looking for areas? Can you determine any relationship between the attributes of ΔABC and ΔDEF?

Situation A mentions the relationship between areas. As such, five of the six examples above explicitly mention areas of triangles. But a particular focus can be on the prompt rather than on the situation. Consider Situation B and several related prompts below.

Situation B: D, E, and F are midpoints of the sides of equilateral triangle ΔABC.

Prompt 1B: Consider the (ratio of) areas of ΔABC and ΔDEF. Use the "what if not" strategy to pose two mathematical problems.
Prompt 2B: Consider the (ratio of) areas of ΔABC and ΔDEF. Use the "what if not" strategy to pose as many mathematical problems as you can.
Prompt 3B: Consider the (ratio of) areas of ΔABC and ΔDEF. Use the "what if not" strategy to pose two "easy" mathematical problems and two "difficult" mathematical problems, where the relative difficulty takes into account the levels of students.

The focus on areas appears in the theme itself in the case of Situation A and in the prompts in the case of Situation B. This is the main difference between the two situations so far. The problem-posing variations V1 to V6 responding to prompts 1B, 2B, and 3B are not expected to be different from those resulting from Prompts 1A, 2A, and 3A. However, Situation B is more open and can be followed up with more open-ended prompts:

Prompt 4B: Based on the described Situation B, pose two mathematical problems related to ratios of measures of the attributes in the problem.
Prompt 5B: Based on the described Situation B, pose two mathematical problems related to ratios of measures (e.g., area, lengths, perimeter) of the attributes (e.g., segments, areas) in the problem.
Prompt 6B: What can you say about the described Situation B? Formulate this as questions about the different attributes and the relationships among them.

Prompts 4B and 5B both specify the number of problems as well as the focus on ratios of measures of the attributes. However, Prompt 5B explicitly suggests what measures and what attributes are to be considered. We consider Prompt 6B to be very open in terms of attributes in the focus and the number of problems to be considered. The choice to use a more open or a more specific prompt can depend on the population of problem posers and on their previous experience. Furthermore, the last three prompts (4B, 5B, and 6B) do not mention the "what if not" or any other particular strategy. Although the "what if not" strategy is a good tool for starting a problem-posing activity, other formulations can open the task for creative adventures.
For example, Prompt 6B can be modified to appeal to the affective domain of problem posing.

Prompt 7B: What can you say about the described Situation B? Formulate this as questions about the different attributes and the relationships among them that for YOU would be interesting to answer.

Prompt 7B can be used with either teachers or students. Here are several examples of what was "prompted" by Prompt 7B for us.

V7: A turtle walks along the sides of an outer ΔABC and the inner ΔDEF, beginning at point A and finishing at the same point. Can it walk so that every segment would be walked only once? If yes, suggest as many as possible trails for the turtle. If not, why not?
V8: ΔABC is an equilateral triangle. D, E, and F are points of the sides ofΔABC that divide the sides in the same ratio. That is, $AD:DB = BE:EC = CF:FA = x:y$. What should the ratio $x:y$ be so that ΔADF, ΔBDE, and ΔCEF would become: (1) an acute angle; (2) a right angle; and (3) obtuse?
V9: ΔABC is an equilateral triangle. D, E, and F are points of the sides of ΔABC that divide the sides in ratios X, Y, and Z. Suppose $AD:DB = X$; $BE:EC = Y$; and $CF:FA = Z$. Is there a relationship between the ratios X, Y, and Z and the ratio of the areas of ΔABC and ΔDEF (where $X = Y = Z$ it is a variation of V8)?

V10: ΔABC is an equilateral triangle. D, E, and F are midpoints of the sides of triangle ABC.

1. Show that the area of ΔDEF is ¼ the area of ΔABC and the perimeter of DEF is 1/3 the perimeter of ΔABC.
2. Consider the following process: The middle triangle DEF is removed, midpoints of the sides of three remaining triangles (ΔAFE, ΔFBD, and ΔEDC) are drawn, and each of these three triangles is split into four triangles as has been done for the initial triangle ABC. Then, again, the middle triangle in each of the three triangles is removed. What would be the area and the perimeter of the figure resulting from all the remaining triangles?
3. Imagine that the above process is repeated many times. Approximate the area and the perimeter of the figure consisting of all the remaining triangles after 100 iterations.
4. What would be the area and the perimeter when the number of iterations approaches infinity?

V11: What transformation(s) can map ΔABC to ΔDEF?
V12: Reverse construction: Given ΔDEF, which is the "inner" triangle? Construct ΔABC such that points D, E, and F are midpoints of AB, BC, and CA. Easy: Start with equilateral ΔDEF. Harder: Start with scalene ΔDEF. Very hard: Construct ΔABC such that points D, E, and F divide the sides of ΔABC in the given ratio.

We invite readers to examine the suggested prompts and consider which ones, if any, they will choose when working with students or teachers in their respective environments. What considerations determine your preference? What task variables are featured? Further, will the choice of a prompt be different if it is intended to be used for research data collection? What additional or different considerations will determine your choice? We also invite readers to engage in prompt design, considering Situation B as a prelude to the forthcoming discussion of processes and variables of problem posing at individual, group, and classroom levels. In the following sections, we discuss problem-posing research with regard to processes and task variables.

PROBLEM-POSING PROCESSES: PROGRESS

Mathematical problem posing (MPP) has been at the forefront of discussion for the past few decades (Brown & Walter, 1983; Cai, 1998; Ellerton, 1986; English, 1998; Kilpatrick, 1987; Silver, 1994; Silver & Cai, 1996). Recent years have seen increased research activity in the domain of problem posing as reflected in journal special issues (Cai & Hwang, 2020; Cai & Leikin, 2020; Singer, Ellerton, & Cai, 2013), books (e.g., Felmer, Pehkonen, & Kilpatrick, 2016; Singer, Ellerton, & Cai, 2015), and conferences (e.g., ICME-14: TSG 17). This increased research on problem posing has also been reflected in the wide range of problem-posing topics studied (see Cai, Hwang, Jiang,

& Silber, 2015, and Singer et al., 2013, for examples of such topics) and review papers (e.g., Baumanns & Rott, 2021; Cai & Leikin, 2020; Cai et al., 2015).

One of the important topics studied is the *processes* of problem posing as experienced by students and teachers. Although we know that students and teachers are capable of posing mathematical problems, we have a considerably less fine-grained understanding of how they go about posing those mathematical problems in any given situation. Some researchers have identified general strategies students may use to pose problems (e.g., Brown & Walter, 1983; Cai & Cifarelli, 2005; Christou, Mousoulides, Pittalis, & Pitta-Pantazi, 2005; Cifarelli & Cai, 2005; English, 1998; Koichu, 2020; Koichu & Kontorovich, 2013; Pittalis, Christou, Mousoulides, & Pitta-Pantazi, 2004; Rott, Specht, & Knipping, 2021; Silver & Cai, 1996). Others have explored some of the variables that may influence students' problem posing (e.g., Kontorovich, Koichu, Leikin, & Berman, 2012; Leung & Silver, 1997; Silber & Cai, 2017). Still others have explored the affective processes of mathematical problem posing (e.g., Schindler & Bakker, 2020).

However, there is not yet a general problem-posing analogue to well-established frameworks for problem solving such as Pólya's (1945) four phases of problem solving, Garofalo and Lester's (1985) cognitive-metacognitive processes of problem solving, and Schoenfeld's (1985) problem-solving attributes. More research is needed to develop a broadly applicable understanding of the fundamental processes and strategies of mathematical problem posing. For now, we remain in the beginning stages of understanding the cognitive and affective processes of problem posing, and this is one of the reasons for which this activity is implemented in mathematics instruction in a rather cursory way (Cai & Hwang, 2020; Cai & Leikin, 2020).

Even though the products of problem posing (i.e., new problems) are important as they constitute the heart of mathematical activities, problem-posing processes are equally important because it is in the processes that problem posers come up with ideas for new problems, evaluate those ideas, and develop or reject them (Baumanns, in press).

Earlier attempts at understanding problem-posing processes

In several earlier studies (e.g., Cai & Hwang, 2002; English, 1998; Silver & Cai, 1996), researchers have tried to use students' posed problems as a base for examining problem-posing processes. For example, Cai and Hwang (2002) used pattern situations to examine students' problem posing and problem solving. They observed that the sequence of pattern-based problems posed by students appeared to reflect a common sequence of thought when solving pattern problems (gathering data, analyzing the data for trends, making predictions). Silver and Cai (1996) found that students tend to pose related and parallel problems when they were asked to pose three problems. They observed a clear tendency of students to pose later problems by varying a single element in earlier problems, which is known as the "what if not" strategy (Brown & Walter, 1983) referred to in several of the prompts considered in the previous section.

Earlier studies have also tried to identify problem-posing strategies as a way to understand problem-posing processes. There are consistent findings about the use of the "what if not" strategy in problem posing (Cai & Ciffarelli, 2005; Cifarelli & Cai, 2005; Lavy & Bershadsky, 2003; Song, Yim, Shin, & Lee, 2007). For example, Lavy and Bershadsky (2003) identified two stages to pose problems. In the first stage, all the attributes included in the statement of the original problem are listed. In the second stage, each of the listed attributes is negated by asking "what if not attribute k?" and alternatives are proposed. Each of the alternatives could yield a new problem.

Phases of the problem-posing process

For problem solving, several models of the problem-solving process have been developed, initiated by reflections on their processes by mathematicians, most notably Poincaré (1908) and Pólya (1945). Later, researchers from mathematics education picked up this topic; important representatives of such research are Mason, Burton, and Stacey (1982), Fernandez, Hadaway, and Wilson (1994), and Schoenfeld (1984; see Rott et al., 2021, for an overview). For problem posing, on the other hand, as stated above, there is no well-known and generally accepted phase model (cf. Cai et al., 2015, p. 14). Some researchers argue that both problem solving and problem posing are strongly related and that there might be no need for a specific problem-posing-process model; however, we argue that cognitive processes in both kinds of processes are different enough to warrant individual models (cf. Baumanns & Rott, 2022; Pelczer & Gamboa, 2009).

Before going into detail regarding research on problem-posing models, we ponder the question of why such models are important. Process models can be used for normative and descriptive purposes (Rott et al., 2021). On the one hand, *normative* models sketch a (more or less) ideal process, stripped of unnecessary detours, that can be used in teaching and instruction. For example, Pólya's four-step problem-solving model is a rather simple model that in its sequence of steps does not account for errors, being stuck, or realizing that the problem formulation needs to be read again. However, it was never intended to map real processes in their "non-smooth" nature but to instruct problem solvers in what steps to do and how to become a better problem solver or poser, respectively. On the other hand, *descriptive* models are designed to account for non-ideal sequences of steps in processes. Such models are used by researchers (or educators) to interpret processes they have observed, make sense of their observations, look for patterns, compare processes by experts and novices, and so on. Reviews of the literature reveal that for problem solving, mostly normative and only very few descriptive models have been developed (Rott et al., 2021) and, for problem posing, only a handful of models has been developed at all (Baumanns & Rott, 2022). In their review, Baumanns and Rott (2022) identified three models of the problem-posing process and added their own—all of which are descriptive phase models. These five models will now be described briefly.

The first model identified by Baumanns and Rott is that of Cruz (2006), who described the process of problem posing in teaching-learning situations and, thus, included educational needs and goals (see Figure 1). After setting a *goal*, a teacher *formulates a problem* and *tries to solve it*, which might fail or lead to regressions. After the problem has been solved, the problem is reflected upon, possibly *improved* to meet the goals, and then selected or rejected. This is a normative model of the problem-posing process intended to guide teachers; actually, it is based on a professional development program for teachers.

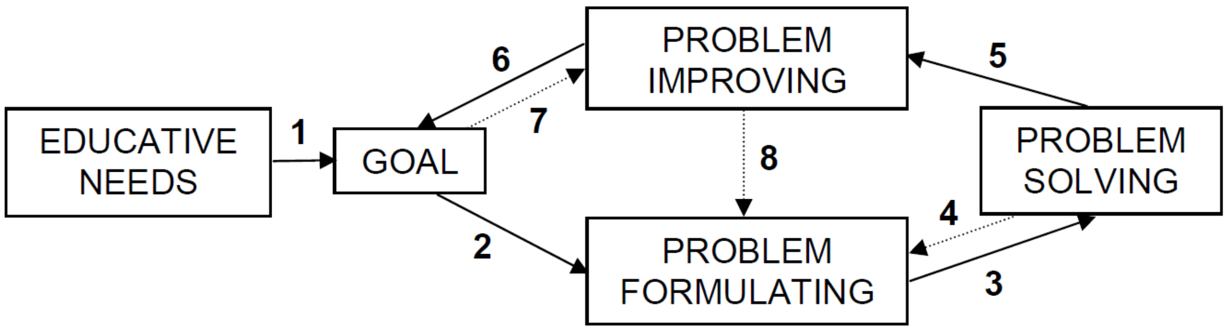

Figure 1. Problem-posing phase model by Cruz (2006)

The second model, based on an analysis of problem-posing processes, is that by Pelczer and Gamboa (2009), who developed a descriptive phase model with five phases, namely *setup*, *transformation*, *formulation*, *evaluation*, and *final assessment*. The setup phase is the starting point, including a reflection about the context of a given situation and the required knowledge. In the transformation phase, the given situation is analyzed and possible modifications are reflected upon and then executed. During the formulation phase, problem formulations and possible alterations are explored. In the next phase, the posed problem is evaluated to see whether it satisfies the initial conditions. In the final phase, much like Pólya's looking-back phase, the whole process is reflected upon.

Koichu and Kontorovich (2013) also developed a descriptive model. Based on two activities by prospective mathematics teachers called "success stories," they identified four phases of problem posing. The first phase is called *warming-up*, in which spontaneous ideas and typical problems regarding a given situation are posed. The next phase is called *searching for an interesting mathematical phenomenon*, in which the initially posed problems are critically considered and modified. In the next phase, problem posers are *hiding the problem-posing process in the problem formulation*, which was a behavior that had not been observed before (Koichu & Kontorovich, 2013, p. 82). In the final *reviewing* phase, the posed problems are evaluated and possibly tested with peers.

Zhang et al. (2022) described the problem-posing process as comprised of the following three major steps: (a) understanding the task (i.e., the context of the problem-posing task); (b) constructing the problem involving selecting and determining which elements to be used and recognizing the relationships among them to construct a new problem space; and (c) expressing the problem which involves organizing the language to express the problem space obtained in the previous stage.

Baumanns and Rott (2022) then developed their own descriptive phase model, the development of which was based on the problem-posing processes of 64 preservice mathematics teachers (see Figure 2). After an initial *situation analysis*, the model allows for differentiation between activities of *variation*, in which a given problem is altered, and *generation*, in which a new problem is generated—a differentiation that had been proposed by Silver (1994) but that had not been made in an operationalized way with empirical data. The duplication of Figure 2 aims to denote that after one problem has been posed, the process can be repeated for posing the second problem, third problem, and so on.

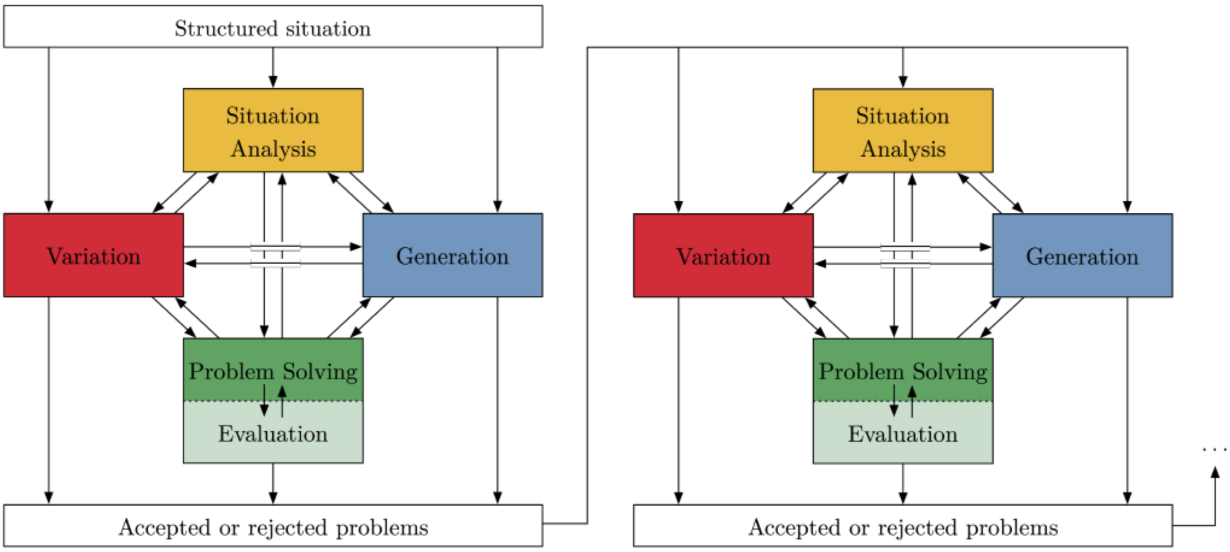

Figure 2. Problem-posing phase model by Baumanns and Rott (2022)

As is the case for different problem-solving models, different problem-posing models serve different goals. For example, Koichu and Kontorovich described a problem-posing process in which one high-quality problem gradually emerges from the pool of initial problem-posing ideas, whereas Baumanns and Rott's (2022) model attends to problem posing as a sequence of repeated problem-posing cycles where each problem posed is considered to be a separate product.

Affective processes of problem posing

Regarding research on problem solving, the whole affective dimension, ranging from emotions to attitudes to beliefs (Philipp, 2007), with a focus on beliefs, has proven very useful and important (Schoenfeld, 1992). Regarding research on problem posing, however, the affective dimension has only recently been systematically addressed by means of a special issue in *Educational Studies in Mathematics* (Cai & Leikin, 2020). This special issue, encompassing for example studies dealing with teachers' beliefs (Li, Song, Hwang, & Cai, 2020) or students' motivation and self-efficacy (Voica, Singer, & Stan, 2020), can only be the starting point of systematic research on affect in mathematical problem posing. In our initial example, Prompt 7B capitalizes on affect in problem posing.

TASK VARIABLES IN STUDYING PROBLEM POSING

Focusing on task variables

There are many ways in which mathematics education research might investigate the cognitive and affective processes of problem posing in an effort to better incorporate problem posing in the teaching and learning of mathematics (Cai et al., 2015). In this paper, we focus on task variables to explore the affective and cognitive processes of problem posing as has been successfully done in research on problem solving. There are two main reasons for such a focus. The first is that we have prior research to draw from on task variables in problem solving (Goldin & McClintock, 1984). The second reason is that we have prior research to draw from on how specific characteristics of mathematical tasks of different natures can affect teachers' and students' responses, in terms of both thinking and instruction (e.g., Koichu & Zazkis, 2021; Liljedahl, Chernoff, & Zazkis, 2007; Zazkis & Mamolo, 2018).

In mathematical problem-solving research conducted over the past several decades, researchers have explored the effects of various task variables on students' problem solving. For example, several classifications of task variables related to problem solving are considered in Goldin and McClintock (1984): syntax variables, content and context variables, structure variables, and heuristic behavior variables. Syntax variables are factors dealing with how problem statements are written. These factors, such as problem length as well as numerical and symbolic forms within the problem, may contribute to ease or difficulty in reading comprehension. Content variables refer to the semantic elements of the problem, such as the mathematical topic or the field of application, whereas context variables refer to the problem representation and the format of information in the problem. Structure variables refer to factors involved in the solution process, such as problem complexity and factors related to specific algorithms or solution strategies. Finally, heuristic process variables refer to the interactions between the mental operations of the problem solver and the task. Considering heuristic variables separately from subject variables (factors that differ between the individuals solving the problem) is difficult because heuristic processes involve the problem solver's interactions with the task. However, the interaction between heuristic processes and the other task variables can have a significant impact on problem-solving ability.

Problem-posing tasks

Just as there are many types of problems and problem-solving tasks, there are many types of problem-posing tasks. Although researchers have proposed categorization schemes for problem posing (e.g., Baumanns & Rott, 2021; Stoyanova & Ellerton, 1996), in this paper, we adopt the idea of a problem-posing task as consisting of two parts: situations and prompts (Cai & Hwang, in press), as exemplified in the first section by means of an example in the context of geometry. The problem situation is what provides the context and data that the students may draw on (in addition to their own life experiences and knowledge) to craft problems. Figure 3 shows the various types of problem situations (Cai & Hwang, in press).

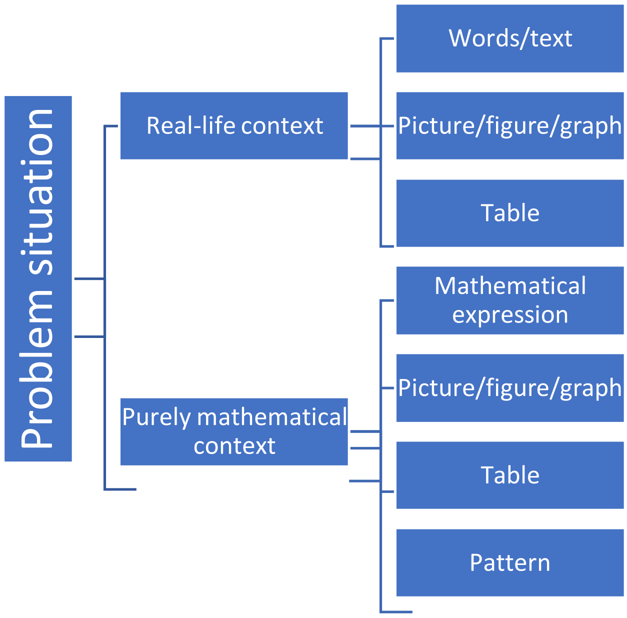

Figure 3. Types of problem situations in problem-posing tasks (Cai & Hwang, in press)

In addition to a problem situation that provides context and data for students to use in their posed problems, a problem-posing task must include a prompt that lets posers know what they are expected to do (Cai & Hwang, in press). Depending on the goal of the task, for the same problem-posing situation, there can be many kinds of prompts. Some possible prompts include:

- Pose as many mathematical problems as possible
- Pose problems of different levels of difficulty (e.g., “Pose one easy problem, one moderately difficult problem, and one difficult problem.”)
- Given a sample problem, pose similar problems (or problems that are structurally different)

The choice of prompt can influence both the mathematical focus for the students and the level of challenge or affective engagement that the posing task presents. Indeed, from a research perspective, it is not yet well understood what prompts are best to pair with a given problem situation or what prompts are most suited to achieving a desired degree of challenge or to address particular learning goals. That is, research has not yet illuminated the connections between different kinds of problem-posing prompts and different cognitive processes in problem posers.

Admittedly, there are many different levels with which to approach research related to task variables and their associated processes in problem posing. In this paper, we describe three such levels: the individual level, group level, and classroom level.

Problem-posing prompts at the individual level

The first level with which we approach problem-posing research described in this paper is the individual level. Research on problem-solving tasks has established that different prompts can elicit different cognitive processes and impact students' problem-solving performance (Goldin & McClintock, 1984). Thus, it is reasonable to expect that the prompt in a problem-posing task also shapes students' engagement with the task. A few studies have investigated how different prompts in problem-posing tasks impact students' or teachers' problem-posing performance and processes (e.g., Silber & Cai, 2017). Silber and Cai (2017) compared preservice teachers' problem posing using structured prompts and free prompts, finding that the preservice teachers in the structured-posing condition more closely attended to the mathematical concepts in each task. Moreover, the effect of the prompt depends, in part, on the setup of the task. For example, in their review of problem posing in textbooks, Cai and Jiang (2017) identified four common types of problem-posing tasks: posing a problem that matches the given/specific kinds of arithmetic operations, posing variations of a question with the same mathematical relationship or structure, posing additional questions based on the given information and a sample question, and posing questions based on given information. A similar prompt (e.g., "Pose a mathematical problem.") could be used with many of these types of tasks, but its meaning to the student could be different for each type.

Leung and Silver (1997) developed and analyzed a Test of Arithmetic Problem Posing (TAPP) which they then used to examine how the presence of numerical information affected preservice teachers' problem-posing abilities. The instructions, which are the prompts we are focusing on, include: "(1) Consider possible combinations of the pieces of information given and pose mathematical problems related to the contexts; (2) Do not ask questions that are not mathematical problems; (3) Set up as many problems as you can think of; (4) Think of problems with a variety of difficulty levels. Do not solve them; (5) Set up a variety of problems rather than many problems of the same kind; (6) Include unusual problems that your peers might not be able to create; (7) You can change the given information and/or supply more information" (Leung & Silver, 1997, p. 8). The first prompt seems to be advice for the participants on how to pose problems. The second prompt emphasizes that the problem posed should be accepted by the community of mathematicians. The third through sixth prompts are related to the "many," "different kinds" or unusual, and "different difficulty levels" mentioned earlier. The last prompt tells the participants what they can do with the data (either change the given information or add more information). Responses were analyzed along two dimensions: quality and complexity. With respect to quality, the responses were classified as mathematical or nonmathematical, as plausible or implausible, and as containing sufficient or insufficient information. With respect to complexity, the responses were classified according to the arithmetic complexity of the solution of the posed problem (i.e., the number of steps to answer the question). Results from the TAPP indicated that the teachers performed better on tasks that included specific numerical information than on tasks without specific numerical information. This might "be due to their being able to 'use the numbers' in the given information rather than having to supply their own numbers or rather than engaging in the generation of qualitative reasoning problems which would not need to contain numerical information" (Leung & Silver, 1997, p. 20). This result provides some insight into how task variables can impact problem posing. Adapting the TAPP to examine how different characteristics of problem situations affect subjects' problem posing could offer a way to study the effect of other task variables on problem posing.

Zhang et al. (2022) replicated and extended the study by Leung and Silver (1997), focusing on elementary school students' problem posing. They examined the cognitive process of mathematical problem posing in three stages: a) input—understanding the task, b) processing—constructing the problem, and c) output—expressing the problem. They also found that the provision of specific numerical information in the problem-posing situation was associated with better problem-posing performance but only in the stages of understanding the task and constructing the problem stages. Students' performance in the stage of expressing posed problems did not show a significant difference with respect to provision of specific numerical information. A similar pattern was revealed for the problem-posing situations with or without contexts, favoring the task format with contexts. Students performed better in all three problem-posing stages on the problem-posing situations with contexts.

English (1998) compared third-grade children's problem-posing performance in formal and informal contexts. In the formal context, children were first asked to make up a story problem to given number sentences like 12 – 8 = 4 and were then asked to think of a completely different problem that could also be solved by the number sentences. Three kinds of informal contexts were presented to the children. The first informal context was a real-life situation presented in pictures. A photograph of children playing with sets of colored items was shown to the participants, then they were asked to make up story problems about something that could be seen in the photographs. The second informal text was a real-life situation presented in words—for example, a card with a statement like "Sarah has five dolls on one shelf and four toy cars on another." Then, the participants were asked to make the statement into a problem they could solve. The third informal context was a piece of literature supported with a list of numbers of native animals. English found that all children offered a significantly greater number of basic change/part-part-whole problems for their first attempt in the formal context, but many of them had difficulty creating a second problem for the given number sentence. Comparatively, they generated more compare problems in the informal context. Encouragingly, several participants even posed multistep problems in the informal context.

Silber and Cai (2021) presented two kinds of problem-posing tasks to undergraduate students taking a noncredited developmental mathematics course so they could be ready to take the foundational mathematics courses required for their major. One kind of problem-posing task consisted of a purely mathematical context presented in a linear graph (i.e., the Graph of a Line posing task). The other was a real-life context described in words only (i.e., Handshakes and Making Change) or in words and pie charts (i.e., Food Drive). Students were required to pose three problems for each context. The problems posed were categorized as mathematical questions, mathematical statements, or nonmathematical responses. The mathematical questions were further analyzed based on their solvability. Among the three real-life contexts, the Food Drive context seemed to be the most familiar context for the participants because they possibly had experienced it when they learned percentages and pie charts. The Making Change context seemed to be the second most familiar because it was often used as a model for addition and subtraction (cost + change = pay) and the model for the system of linear equations (e.g., ten coins [dimes and half-dollars] to pay $2.20). The Handshakes problem, which involves modelling (using points or circles to represent people and the line between any two points as a handshake), is usually used for patterns in algebra. The results obtained in Silber and Cai's (2021) study revealed that the percentages of problems that were solvable mathematical problems for Food Drive, Making Change, Handshakes, and Graph of a Line were 98%, 90%, 88%, and 52%, respectively. Thus, the familiarity level of the contexts might need to be taken into consideration in future studies.

Effect of problem-posing prompts at the group level

The second level with which to approach research related to task variables in problem posing is the group level, that is, how a small group poses mathematical problems and how the task variables affect group problem posing (e.g., Kontorovich et al., 2012).

As early as 1987, Kilpatrick pointed out that group work can provide a fruitful setting for mathematical problem posing because the dialogue between problem posers may have a synergetic effect. In his words,

> When students work together, they often identify problems that would be missed if they were working alone. A poorly formulated idea brought up by one student can be tossed around the group and reformulated to yield a fruitful problem. Students participate in a dialogue with others that mirrors the kind of internal dialogue that good problem formulators appear to have with themselves. (Kilpatrick, 1987, pp. 141-142)

Despite the broad attention that this seminal article has attracted in the mathematics education research community, research on problem posing in groups is still relatively rare. A Google Scholar search using the key words "group problem posing" + "mathematics," "collaborative problem posing" + "mathematics," and "collective problem posing" + "mathematics" returns dozens of results (50, 137, and 95, respectively) as compared to the thousands of results returned by a parallel search in which “problem posing” is replaced with “problem solving.” Furthermore, in many of the studies identified in the search, “group,” “collaborative,” or “collective” problem posing are mentioned merely as potential counterparts of “group,” “collaborative,” or “collective” problem solving, with the main attention given to the latter rather than to the former activity.

Armstrong (2014) alluded to collective problem posing as an emergent phenomenon in school discourse. She argued that the insufficient attention paid to group problem posing thus far could partially be explained by specific features of the mainstream line of research on problem posing as it had been developed since the 1990s. Namely, many of the problem-posing studies operate with written products of problem posing as a focus of analysis and value large-size pools of participants and large collections of problems posed that can be categorized in a variety of ways. Arguably, this focus, as useful as it is, leaves aside problem-posing processes and in turn leaves aside phenomena related to the dynamics of group work on problem-posing tasks, as has been suggested by Kilpatrick (1987). Indeed, Kilpatrick’s provisional argument was about group processes that can lead inexperienced problem posers to formulating fruitful ideas rather than about the quantity of the resulting problems posed.

However, it is safe to say that research on problem posing at the group level is gradually growing. While recognizing that the critical mass of studies that would enable us to clearly identify trends has not yet accumulated, we can (tentatively) identify four different approaches to treating group work as a variable in problem-posing research.

In the first approach, the fact that students work on problem posing in small groups is provided as contextual information, but the findings are reported in an aggregated manner that hides within-group processes. A study by English and Watson (2015) serves as a characteristic example. The study explores the problem-posing products of 20 groups of fourth-grade students working in groups of four in the context of statistical literacy. The main results are reported per group, as the following quotation shows: “Of the 20 groups, 9 posed three or four different types of questions, 10 created two types, and 1 group, just one type” (English & Watson, 2015, p. 11). The between-group differences in this study are attributed to individual differences between students’ pre-existing knowledge and preferences but not to the dynamic processes in the groups. Another example comes from Leung and Wu (1999), who first reflected on two problem-posing lessons as if each group was an individual student (e.g., “the six groups changed the problem in three ways”; p. 113) and then stopped on ideas of a particular student expressed in front of the class (of note is that this study can also be considered in the section on problem posing at the classroom level).

The second approach focuses on individual students in the context of small-group problem posing. For instance, one of the results of the study by Headrick et al. (2020) is that even when students are organized in small groups, they tend to individually pose problems to the teacher as opposed to their groupmates. Another study, by Koichu

(2020), showed that students in small groups who face a multistage task including problem solving, exploration, and problem posing tend to distribute the load and work separately on different parts of the task so that problem posing essentially turns into an individual enterprise. These results do not contradict but rather complement the findings by Schindler and Bakker (2020), who found that a group setting can play a positive role in shaping an affective field of individual problem posers. In their case study of one student working in a small group on a series of problem-posing and problem-solving tasks, the student overcame the initial anxiety rooted in her prior experiences, increased her interest in problem posing, and became an open-minded and active participant in the project due to the group collaboration that provided her with the feeling of safety and appreciation. Furthermore, Ellerton (2015) pointed out that working in groups may either support or hinder the problem-posing progress of individuals. Her study suggests the importance of keeping a delicate balance between the collective and the individual in problem posing as well as the importance of learning how to give and take feedback on the problem-posing ideas of others in productive ways.

The third approach attends to the richness of problem-posing performance in small groups working on the same task while featuring summative rather than dynamic descriptions. Armstrong (2014) developed an original methodology (called "tapestries") that blurs the data but provides visual representations of collective patterns of problems posed. This methodology was used in a study with four groups of 12-year-old students to compare the across-group problem-posing products as related to the group problem-posing strategies and tactics. Armstrong introduced the term "group's personality" (p. 62) and compared the groups in the following manner: For example, a group that tended to deeply explore concepts and connect participants' ideas posed more problems than another group that tended to argue about every problem's formulation, aiming at reaching a consensus. In contrast, Cai (2012) compared two groups of preservice teachers working on a task in the context of numerical sequences by summarizing the main mathematical ideas developed in each group. Despite methodological differences, both studies converge to conclusions about the opportunities embedded in well-chosen problem-posing tasks that trigger rich mathematics discussions and learning.

Finally, the fourth approach is heavily informed by sociocultural perspectives on teaching and learning mathematics and therefore considers within-group problem-posing interactions as the main data to be analyzed as opposed to the written problems as the main data. For example, English, Fox, and Watters (2005) argued for the potential of problem posing and solving with mathematical modelling while systematically demonstrating how problem-solving and problem-posing ideas emerge and evolve in small-group discussions. In this study, the argument for the usefulness of combining problem posing, problem solving, and modelling relies not only on the demonstrated benefits of the chosen types of tasks for student learning of mathematics but also for the development of their collaborative learning skills. Meanwhile, an in-depth analysis of student interactions of low-track eighth-grade students who were engaged in small-group work on a problem-posing task in the context of geometry is the focus of a study by Agarwal (2020). The analysis of six groups revealed how the students shifted their actions and restructured their activity towards organizing for collective agency in mathematical problem solving while balancing risk-taking behaviors (e.g., there is a risk to be misunderstood or mocked) and agency-driven behaviors in favor of emotional courage and productive participation.

We conclude this section by reviewing a study by Kontorovich, Koichu, Leikin, and Berman (2012) that has an explicit focus on handling the complexity of problem posing in small groups. These authors present a confluence exploratory framework that aims to explain the emergence of problem-posing products from problem-posing processes as shaped by five facets: task organization, students' knowledge base, problem-posing heuristics and schemes, group dynamics and interactions, and individual considerations of aptness. The framework is presented in Figure 4.

This framework was used to make sense of the work of two groups of tenth-grade students who were given the Billiard Ball Mathematics Task (adapted from Silver, Mamona-Downs, Leung, & Kenney, 1996, and used in several additional studies). The analysis attempted to explain the quantity and quality of the resulting problems posed in each group by systematically attending to all the facets included in the framework. In particular, the analysis showed the role of group dynamics captured in terms of normalization, conformity, and innovation—three social processes well-known from the literature on group interaction and development in the context of coping with challenging (though not necessarily mathematical) tasks (e.g., Wit, 2007). The analysis also shed light on the importance of functional roles that group members assumed in a small-group discussion.

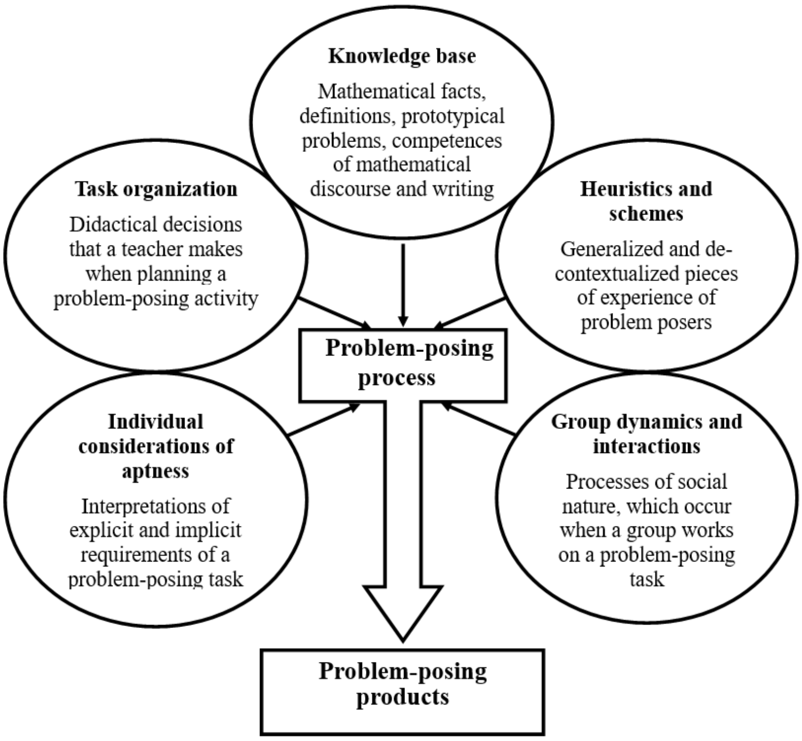

Figure 4. A confluence framework of problem posing in small groups (Kontorovich et al., 2012)

Along with the aforementioned studies by English et al. (2005) and Agarwal (2020), a study by Kontorovich et al. (2012) supports and empirically substantiates Kilpatrick's (1987) vision of group work as a fruitful but immensely complex pedagogical setting for further promoting problem posing in school. Needless to say, more research on problem posing at the group level is needed.

Effect of problem-posing prompts at the classroom level

Finally, the third level with which to approach research related to task variables in problem posing is the classroom level. Mathematics can be taught through engaging in problem posing, and researchers have begun to explore what teaching mathematics through problem posing looks like and to develop problem-posing cases to illustrate problem-posing instruction (Cai & Hwang, 2020; Ellerton, 2015; Zhang & Cai, 2021). However, it is not yet clear how we should design the problem-posing tasks used in such instruction so as to create greater learning opportunities for students. For example, for a given situation in the classroom, students could be asked to pose three mathematical problems or to pose three problems with different difficulty levels such as easy, moderately difficult, and difficult (Cai & Hwang, 2002). How would such different prompts impact classroom instruction and students' learning?

The past two decades and especially recent years have seen increased research on implementing problem posing into classrooms (Cai & Hwang, 2020; Cai et al., 2015). Researchers have begun to explore what teaching mathematics through problem posing looks like (Çakır & Akkoç, 2020; Cai, 2022; Chen & Cai, 2020; Crespo & Sinclair, 2008; Ellerton, 2013; Christou et al., 2005; Klaassen & Doorman, 2015; Stoyanova & Ellerton, 1996; Zhang & Cai, 2021), develop problem-posing teaching cases to illustrate problem-posing instruction (e.g., Chen & Cai, 2020; Zhang & Cai, 2021), and examine the impact of problem-posing instruction on students and teachers (Akben, 2020; Bevan & Capraro, 2021; English, 1997, 1998; Li et al., 2020; Klaassen & Doorman, 2015; Kopparla et al., 2019; Suarsana, Lestari, & Mertasari, 2019; Yang & Xin, 2021).

In teaching through problem posing, students are encouraged to pose problems that may be meaningful to them personally or socially. Thus, classroom activity around problem posing involves the negotiation of socio-mathematical norms, such as in determining criteria for what counts as a mathematically interesting problem (Çakır & Akkoç, 2020; Crespo & Sinclair, 2008).

Cai (2022) proposed a problem-posing task-based instructional model (see Figure 5). In a lesson, there might be more than one problem-posing task or a combination of problem-solving and problem-posing tasks. This model describes using one problem-posing task to teach mathematics. The first step is to present a problem-posing situation (see specifics in Figure 3), the second step is to provide a problem-posing prompt, and the third step is for students to pose problems either individually or in group. The later steps in Figure 5 show how teachers can handle the posed problems based on the learning goals.

Zhang and Cai (2021) analyzed 22 problem-posing teaching cases based on the work of Merseth (2016) and Stein, Henningsen, Smith, and Silver (2009). They described a teaching case as the following:

> A teaching case includes major elements of a lesson and related analysis, but it is not a transcribed lesson. Teaching cases include narratives describing instructional tasks and related instructional moves for the tasks. Cases also include information about the underlying thinking of major instructional decisions as well as reflections on and discussions of those decisions. The development of teaching cases is based on real lessons and typical instructional events from the lessons. (Zhang & Cai, 2021, p. 962)

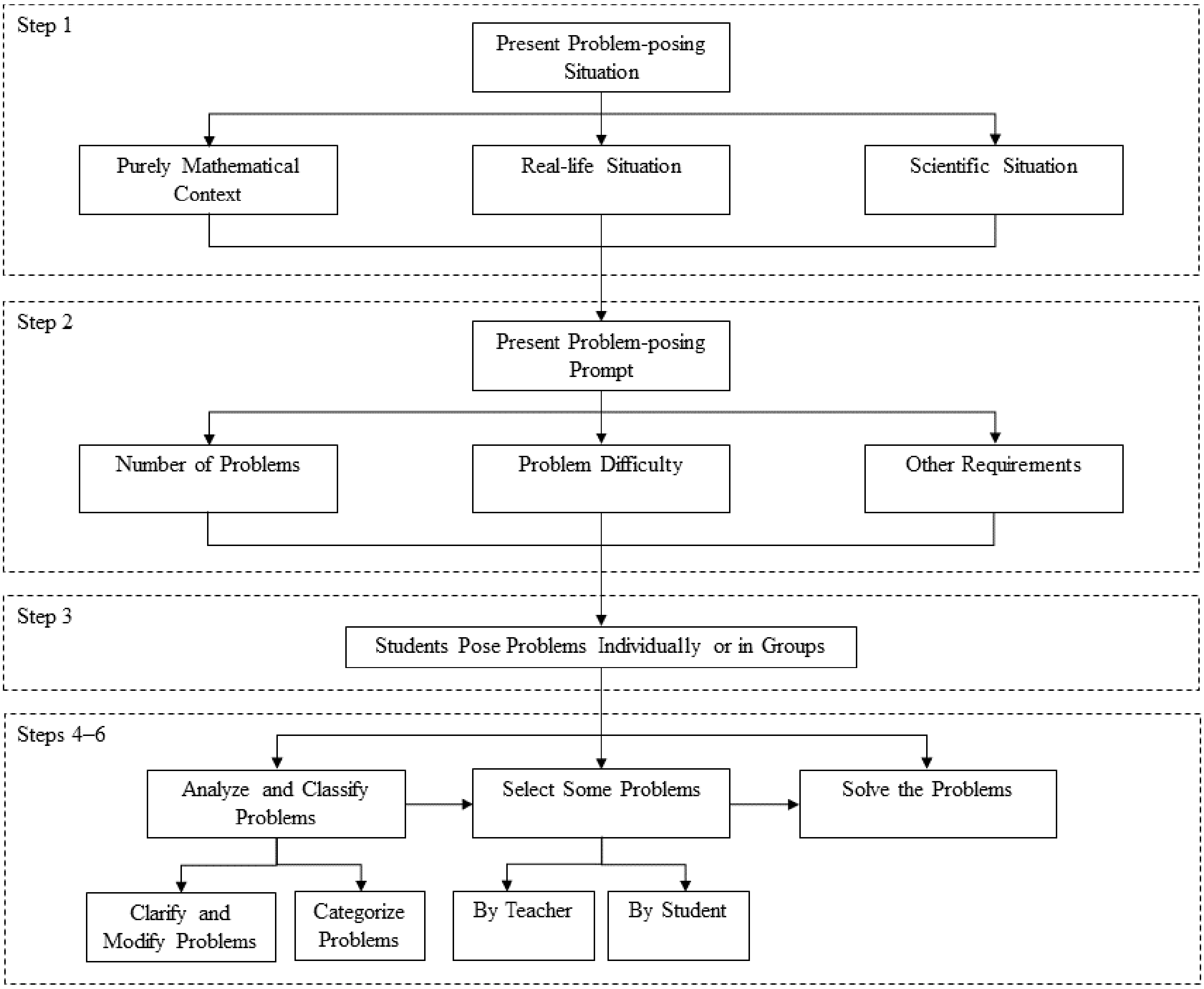

Figure 5. A problem-posing task-based instructional model (Cai, 2022)

In their analysis of problem-posing teaching cases, Zhang and Cai (2021) found that teachers used different prompts in their problem-posing tasks, such as posing a problem that matches the given or specific kinds of arithmetic operations, posing problems based on given information, and posing variations on a question with the same mathematical relationship or structure. Their analysis found no teaching case that explored the effect of different prompts for the same problem-posing situation. In fact, thus far, there are no studies that have studied the effect of different prompts on students' problem posing at the classroom level.

Using problem posing as an instructional intervention, researchers have found positive effects of problem posing not only on teachers' own development (Li et al., 2020) but also on students' learning along both cognitive and noncognitive measures (e.g., Akben, 2020; Bevan & Capraro, 2021). Although such positive effects of problem posing on both students and teachers are encouraging, none of these studies include information about the effects of problem-posing tasks with different prompts. In fact, as Klaassen and Doorman (2015) summarized, there has been no specific investigation of the effects of the variety of prompts researchers have used in classrooms, prompts such as:

- Write a problem to the story so that the answer to the problem is a specific one
- Write an appropriate problem for the specific information, such as the expression or picture
- Ask as many questions as you can, and try to put them in a suitable order
- Write a problem that you would find difficult to solve

As part of a larger research project, Cai, Muirhead, Cirillo, and Hwang (2021) began to explore how teachers view the impact of different prompts on students' problem posing. Each teacher was presented with three pairs of tasks, each of which uses the same problem situation but includes different prompts (Prompt A: Pose three different mathematical problems that can be solved using this information; Prompt B: Pose one easy mathematical problem, one moderately difficult mathematical problem, and one difficult mathematical problem that can be solved using this information). Each teacher was asked to anticipate their students' responses to Prompt A compared to Prompt B and to describe how these variations in the wording of the prompts might affect their students' responses. According to one sixth-grade teacher, Prompt A is less wordy and more accessible for students. However, the teacher thought that Prompt B engaged students more in their thinking because they must think about posing problems with different difficulty levels. Thus, Prompt B "forces" or "encourages" students to think more. The teacher also thought that in implementing problem posing in the classroom, teachers can scaffold problem-posing tasks with Prompt A to problem-posing tasks with Prompt B.

In practice, it does seem that encouraging students to pose different difficulty levels of problems has some advantages for eliciting deeper student thinking about certain kinds of problems (Cai & Hwang, 2002) and adjusting the level of challenge of the task relative to each student. For example, the prompt, "Create a problem that would be difficult for you to solve," can challenge each student to stretch toward the edge of their own ability. Although each student may still engage in the problem-posing task at a level that is appropriate for their existing mathematical understanding, such a prompt could result in the overall level of challenge increasing. Ultimately, we believe that the choice of problem-posing prompt has the potential to make a difference in how students engage with problem-posing tasks in the classroom.

CONCLUSION

The purpose of this paper is to summarize some progress in problem-posing research related to processes and task variables. We end by presenting some research questions for the field of mathematics education.

As mentioned, problem-solving variables include syntax variables, content and context variables, structure variables, and heuristic behavior variables. Can all these types of variables be adapted to problem posing? Should the additional variables be considered to pinpoint not only similarity of problem solving and problem posing but also characteristic differences between these activities? How can systematic variation of problem-posing situations and prompts inform our understanding of the relationship between problem-posing processes and products and between problem posing and problem solving? How do student-related variables (e.g., knowledge, affect, experiences) interact with task-related problem-posing variables?

In addition, we not only need to continue the effort to examine the cognitive processes of problem posing related to task variables but also affective processes of problem posing. For example, how do beliefs influence problem-posing processes and posed problems? How do problem-posing activities influence students' (epistemological) beliefs regarding mathematics? Do problem-posing activities—used in teaching settings—impact students' sense-making and motivation regarding mathematics? Cai and Leikin (2020) provided additional research questions about affect in problem posing.

The literature offers characterizations of teacher knowledge needed to incorporate problem solving in teaching (e.g., Chapman, 2015). Similarly, we can ask: What teacher knowledge is needed for successful integration of problem posing in the classroom? (In other words, we can think of task-related variables, student-related variables, and teacher-related variables.)

In this paper, we have discussed the impact of task variables (specifically problem-posing prompts) on problem posing at the individual, group, and classroom levels. In addition to the discussion of the impact at different levels, there is also a need to understand how teachers handle posed problems (Cai, 2022; Zhang & Cai, 2021), because of its importance in integrating problem posing in mathematics learning and instruction. In problem solving, we have more or less clear criteria to measure success (i.e., successfully solved problems). In problem posing, it is much harder to determine whether a problem-posing process was successful or not (given that posed problems could be nonmathematical, repetitive, boring, not challenging, etc.). How do we measure "success" in problem posing? Compared to problem solving (where you can easily identify whether a problem has been solved), it is often unclear when a problem-posing process is finished or whether the posed problems are "good" or not. To effectively teach mathematics through problem posing, we have to address these questions in general and to develop strategies to deal with students' posed problems in particular.

Using a clinical interview methodology or a large-sample-size survey, we could examine how different types of problem-posing tasks with different situations and prompts influence students' problem-posing processes. Such research requires coordination and international collaboration, and the ideas presented in this paper will be a step towards establishing it.

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INNOVATIVE PERSPECTIVES IN RESEARCH IN MATHEMATICAL MODELLING EDUCATION

Organisers: Gabriele Kaiser[1], Stanislaw Schukajlow[2]

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The research forum shares and discusses innovative perspectives in research on mathematical modelling education. Specifically, the proposed research forum intends to give an overview of current perspectives from different research strands (amongst others, psychologically and pedagogically oriented research) and from different social-cultural contexts (including Eastern and Western contexts). Finally, the research forum aims to develop prospects for further developments in modelling education research.

AN OUTLINE OF THE THEORETICAL BACKGROUND OF THE RESEARCH TOPIC

Research on mathematical modelling education, as well as its orientation and relationship to various neighbouring disciplines, has become a prolific and productive field that is growing with enormous speed especially in the last decade. Various special issues on the field have been published by high-ranking mathematics educational journals (see, for example, the recent special issue on mathematical modelling competences in *Educational Studies in Mathematics* (Kaiser & Schukajlow, 2022) and the special issue on psychologically influenced approaches in *Mathematical Thinking and Learning* (Kaiser et al., 2022). In addition, rigorously peer-reviewed proceedings on the topic have been continuously published for several decades.

For decades, research on mathematical modelling education has had pedagogical goals; in other words, it has aimed to improve mathematics education with empirically developed and evaluated examples of mathematical modelling (Kaiser & Brand, 2015). The design of innovative teaching methods and use of technology are two central areas of research with high relevance to the learning of mathematical modelling. Psychological topics, such as affect, intuition and creativity, have been recently introduced to the research discourse on mathematical modelling education (Schukajlow et al., 2018). Affect and intuition were demonstrated to be critical for learning in earlier research, whereas creativity when solving modelling problems is an emerging topic of research. The relevance of socio-cultural perspectives on modelling research has been emphasised more strongly in the last few years, especially by performing East–West comparisons and ethno-mathematical studies. Overall, the current field of mathematical modelling education research can be described as experiencing a diversification of dominant approaches and the introduction of new

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research perspectives, such as new media/technology and its usage in education, especially during the COVID-19 pandemic.

In the research forum, these perspectives will be presented in more detail, focusing on their novelty and potential to advance the current research discourse on mathematical modelling and, more generally, mathematics education. In his commentary, Wim Van Dooren addresses new developments in research on modelling that arose after the 2014 PME research forum in Vancouver (Cai et al., 2014).

GOALS AND KEY QUESTIONS OF THE PROPOSED RESEARCH FORUM

This research forum addresses three strands of research on mathematical modelling education:

Pedagogically oriented research perspectives

- Innovative research approach used to explore teaching approaches and their role in the promotion of modelling competences: Werner Blum, Berta Barquero, Rina Durandt
- New media and technologies and their role in modelling education research: Stefan Siller, Mustafa Cevikbas, Vince Geiger, Gilbert Greefrath

Socio-culturally oriented research perspectives

- Cultural and socio-cultural influences on the implementation of mathematical modelling education and consequences for mathematical modelling education research including the ethno-mathematical perspective: Xinrong Yang, Björn Schwarz, Milton Rosa

Psychologically oriented research perspectives

- The influence and role of affective aspects within mathematical modelling activities: Stanislaw Schukajlow, Janina Krawitz, Susana Carreira
- The influence of creativity on mathematical modelling and its role within mathematical modelling activities: Xiaoli Lu, Gabriele Kaiser, Roza Leikin
- The role of intuition within mathematical modelling: Rita Borromeo Ferri, Corey Brady

Discussion of the perspectives: Wim Van Dooren

FORMAT OF THE RESEARCH FORUM

The format of the research forum will integrate brief formal presentations, small group discussions, pre-prepared commentary and coordinated Q&A sections. Each of the two 90-minute sessions of the forum will start with formal presentations that introduce the research topic by sharing existing research and perspectives on mathematical modelling education. The participants will then be invited to join small group discussions, which provide a good opportunity to ask questions and learn more about research approaches in mathematical modelling education. During these discussions, participants may be invited to share what they know about these approaches and/or further perspectives. A summary of the information shared during the discussions and further explanations will be prepared and used as a commentary to finish the first 90-minute session. In the second 90-minute sessions, the contributors will present innovative psychologically oriented perspectives to research on mathematical modelling education. These presentations will be followed by a commentary led by the coordinators of the research forum. The session will end with a Q&A involving the audience, contributors and discussants.

INNOVATIVE RESEARCH APPROACHES FOR EXPLORING TEACHING ENVIRONMENTS DESIGNED TO PROMOTE MATHEMATICAL MODELLING COMPETENCY

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The paper addresses the question of which teaching designs can advance students' modelling competency. After some general results of empirical investigations, five examples of research studies on the secondary and tertiary level are described in which teaching environments for modelling have been constructed and investigated which proved to be effective.

MATHEMATICAL MODELLING COMPETENCY

There is a broad consensus in the educational debate that mathematical modelling has to be an integral part of mathematics teaching on all educational levels. One essential aim of teaching modelling is to advance the *competency of mathematical modelling*, that is the ability to deal with extra-mathematical problems by using or creating suitable mathematical models, working within these models, and interpreting the obtained results for the solution of the problems.

We know from research that mathematical modelling is cognitively demanding because it usually requires several skills and abilities as well as mathematical and real-world knowledge. Each step in the modelling process may be a cognitive hurdle for learners (Blum, 2015). So, an important question is: How can students' mathematical modelling competency be advanced? In this paper, we focus on the following, slightly more specialised question: Which teaching designs promise or have proven to enhance progress in school or university students' ability to solve modelling tasks?

ADVANCING MODELLING COMPETENCY

To advance modelling competency requires well-aimed teaching methods which are designed so as to fulfil criteria of quality teaching. Several studies report on encouraging results which show that it is indeed possible to advance the ability to solve modelling tasks by means of suitable learning environments (for an overview see Niss & Blum, 2020, chapter 6; Cevikbas et al., 2022). An analysis of these studies reveals that, globally speaking, those teaching designs are particularly effective which contain, on the one hand, *instructional* elements (well-designed teaching material, with appropriate tasks, and adaptive teacher guidance) and, on the other hand, *constructional* elements (students' self-directed activities in solving modelling tasks and students' use of suitable strategies). Crucial seems to be a permanent balance between these elements, that means students ought to work as independently as possible, supported by minimal teacher interventions when necessary.

RESEARCH INTO ADVANCING MODELLING COMPETENCY

In the following, we refer to five studies which can be regarded as examples of innovative research into possibilities of advancing the competency to solve modelling tasks. In all these studies, teaching environments have been constructed which constitute a certain blend of constructional and instructional elements as outlined above.

Example 1: In the German interdisciplinary research project DISUM (see Blum & Schukajlow, 2018), the effects of a more independence-oriented teaching style, called "operative-strategic", was compared in a ten-lesson mathematical modelling unit in altogether 26 grade 9 classes (14-15-year olds) with the effects of a more teacher-guided style, called "directive", and with an improved version of the operative-strategic style, called "method-integrative". The same 14 modelling tasks were treated in the same order in all designs. In both the operative-strategic and the method-integrative design, the major aim was maintaining a permanent balance between teacher's guidance and students' independence, and encouraging individual solutions, whereas in the directive design, the teacher guided the students and developed common solution patterns. In the method-integrative design, a meta-cognitive aid called "solution plan" (essentially a four-step modelling cycle) was in students' hands, and the teacher introduced its use by demonstrating in the fourth lesson how modelling tasks may be solved. It turned out in a pre-/post-test research design that all teaching styles had significant and similar effects on students' technical mathematical skills, but only the two independence-oriented styles had significant effects on students' modelling competency. Moreover, the method-integrative classes outperformed the operative-strategic classes in mathematical modelling.

Example 2: In an Australian study (see Galbraith, 2018), the effects of a systematic teaching program in grade 8 (12-year olds) over a year was investigated in which the development of concepts and skills as well as the ability to apply mathematics, was pursued by means of mathematical modelling focused around a sequence of carefully selected problems. The grade 8 medium ability group of a school was chosen as the trial group whereas the high ability group followed a conventional teaching programme involving exposition of ideas, techniques, and worked examples by the teacher followed by consolidation exercises. In the trial group, mathematical concepts and skills were introduced and developed through application contexts, while simultaneously the process of modelling itself was practised, but also traditional homework exercises were set for concept reinforcement and skill practice. A six-step modelling cycle served as a set of meta-cognitive prompts for the students. Remarkably, the trial group outperformed the conventional group in a standard grade 8 mathematics test at the end of the school year. In addition, the trial group showed substantial progress also in modelling competency, assessed by qualitative data, taken from students' journals and from oral interviews.

Example 3: There is broad empirical evidence that the use of meta-cognitive strategies can be a substantial support in solution processes. This holds also for modelling processes. In the LIMo project, a five-step “Solution Plan” with strategical hints was used as a meta-cognitive tool, and its effects were controlled in a comparative study in 29 grade 9 classes with a pre-/post-test design (see Beckschulte, 2020). Both the experimental group and the control group were exposed to a four-lesson mathematical modelling unit with the same four tasks. In the experimental group, the Solution Plan was introduced at the end of the first lesson. The tests revealed significant progress in students’ modelling competency in both groups, with a significant advantage of the experimental group in the sub-competencies of Interpreting and Simplifying, especially strong in the follow-up test. So, the researchers conclude that working with a meta-cognitive instrument has advantages particularly in the long run.

Example 4: In line with the research approach of the study and research paths (SRP) for the teaching of modelling, Barquero et al. (2011) describe the design and analysis of an SRP about population dynamics which was tested with first-year engineering students over six consecutive years. The implementation took place in a “mathematical modelling workshop”, in parallel with regular lecture sessions, facilitating a mixture of more constructional with more instructional teaching elements. Throughout the entire year, students received different sets of population data and were asked to develop models to forecast the size of the population. To align the workshop to the standard syllabus of a first-year mathematics module, the SRP was divided into three branches: considering time as discrete and a population with independent generations; forecasting in discrete time while distinguishing groups in a generation; considering time as continuous with one or more generations distinguished. Despite the expected variation among the implementations, students’ submissions (weekly teams’ reports, individual reports at the end of each branch, and three individual tests) provided substantial empirical evidence about their modelling competency progress. This research also provided robust designs which have been later transferred to teacher education and to secondary education.

Example 5: In a study also at the tertiary level, two teaching designs similar to those used in the DISUM study were implemented in a five-lesson mathematical modelling unit with three groups of first-year engineering students in South Africa (see Durandt et al., 2022). Two groups were instructed according to the teacher-directive design, and one group followed the method-integrative teaching design with its characteristic mixture of constructional and instructional teaching elements. The same ten modelling tasks were treated in the same order in all three groups. The students' progress in mathematical modelling and in mathematical topics underlying the modelling tasks was measured by a pre-/post-test design. It turned out that, like in the DISUM project, the group taught according to the method-integrative design had the biggest competency gains in modelling, while the progress in mathematics was the same for all groups. There were also differences between the two directive groups, presumably due to the fact that two different lecturers taught these groups, thus pointing to the high importance of the teacher variable in such investigations.

Several more such studies can be found in recent special modelling issues of journals: Carreira & Blum (2021a,b); Kaiser & Schukajlow (2022); Kaiser, Schukajlow, & Stillman (2022).

THE ROLE OF DIGITAL RESOURCES IN MATHEMATICAL MODELLING RESEARCH

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The integration of mathematical modelling into instruction can promote learners' understanding of mathematical content, ideas, and concepts as well as offering an approach to solving real world problems. Resources such as digital tools, media, and simulations hold great potential for the implementation of mathematical modelling in school classrooms. The use of digital resources can be used to support the generation of solutions to real world problem. In this paper, we present a concise synthesis of research regarding the role of digital resources and their potential for promoting learners' capability with mathematical modelling. We conclude the paper by identifying future directions for research into digital resources enhanced mathematical modelling instruction.

MATHEMATICAL MODELLING VIA EMERGING DIGITAL RESOURCES

Emerging digital resources are becoming increasingly important in the context of mathematical modelling instruction in school classrooms. These emerging digital resources open new possibilities for exploring mathematical situations. Research on mathematical modelling has been consistently extended to include the use of digital tools (e.g., Geiger, 2011; Greefrath et al., 2018), although its integration has attracted discussion within academic discourse (e. g., Doerr et al., 2017; Monaghan et al., 2016). Research into digital resources enhanced mathematical modelling has focused on its affordances when solving with real world problems and how it can best be used to enhance and support classroom instruction. Most of this research, however, has focused on the use of the digital tool itself rather than how digital resources can be integrated into thinking about the strategies needed to solve a problem in a real-world context. Considering digital resources as thinking tools when dealing with real world problems, however, has been largely theorized rather than the focus of empirical research. Further, there are a limited number of studies that have investigated the effective integration of digital resources into learning environments, for example, the use of simulations of real-world scenarios based on mathematical models as in the case of computer apps or virtual reality technologies. With this paper, we make it clear that despite the limited availability of findings, there is clear research potential inherent in the underlying approach. To explore this potential, we first present a perspective on the role of digital tools in mathematical modelling, and then present open questions based on (current) empirical studies on modelling with digital resources.

PERSPECTIVE OF THE ROLE OF DIGITAL TOOLS IN MATHEMATICAL MODELLING

A number of studies have pointed to difficulties learners may experience during the modelling process and how the use of digital tools can act as a bridge between the real model and the mathematical results (e.g., Galbraith & Stillman, 2006). Doerr and Pratt (2008) point to the potential of digital resources for providing different representations of a real-world problem adding that the technology-based model represents a new field of knowledge with its own learning opportunities. Consistent with this perspective, Confrey and Maloney (2007) describe a holistic approach to modelling with digital tools by positioning relevant digital resources as mediators of meaning through the different representation that can be generated. Other studies have complemented this perspective by providing evidence that digital resources can provide affordances that can support mathematical modelling throughout the process (e.g., Geiger, 2011; Siller & Greefrath, 2010). Greefrath et al., 2018, however, note that a holistic view on the use of digital tools during the modelling process, rather than focusing on specific sub-competencies, better describes learners' actual approach to modelling when using digital resources.

RECENT EMPIRICAL STUDIES ON MODELLING WITH DIGITAL RESOURCES

To date, there have been only a limited number of systematic reviews of research into mathematical modelling (see for example, Cevikbas et al., 2022). In Cevikbas et al.'s (2022) review, the literature on conceptualizing mathematical modelling competencies and their measurement and fostering is described based on the analysis of the papers published between 2003-2021. However, none of the studies focused specifically on the role of digital resources in teaching and learning mathematical modelling. For this paper, we conducted a new systematic literature search by using three well-known electronic databases (Web of Science, ERIC, and EBSCO Teacher Reference Center). In this search, the Boolean string "(mathematical model*) AND (technology* OR digital)" was employed with a focus on titles and abstracts of the peer-reviewed journal articles and book chapters written in English. Our review encompassed studies published between 2012-2021. As the International Community of Teachers of Mathematical Modelling and Applications (ICTMA) conferences have been influential, we also conducted a manual search of ICTMA book chapters (1984-2021) and identified 30 eligible publications in total.

The results of this search reveal that a variety of digital resources can be used to promote the teaching and learning of mathematical modelling. Many studies focused on the role of digital resources in thinking in modelling, not simply as a means to complete computational tasks. The digital resources that were most often investigated across these studies was Dynamic Geometry Systems (DGSs) (37%), followed by Internet (33%), spreadsheets (27%), Computer Algebra Systems (CASs) (17%), mobile devices (20%), computers (17%), graphic calculators (17%), simulations (computer-generated representations of real world situations) (13%), specialized software such as 3D design software and Game Maker Studio (13%), videos and videogames (10%), motion detectors (7%), apps (7%), applets (7%), sensors (7%), smartboards (3%), programming languages (3%), 3D printers (3%), simulators (3%) and electric circuits (3%) and animations (3%). No studies were identified that related specifically to new pedagogical approaches (e.g., flipped classroom) or innovative technologies (e.g., augmented and virtual reality, artificial intelligence). An examination of the identified studies indicated that the above-mentioned digital resources were used for various purposes in the modeling process including: (a) finding information or data; (b) enhancing to explore possible solution pathways; (c) formulating problems, equations, schemas, or diagrams; (d) visualization; (e) calculation; (f) interpreting results; and (g) validating solution. Results suggest that the use of digital resources can be beneficial at different points in the modelling cycle, consistent with Geiger (2011) and Siller and Greefrath (2010). From a different perspective, a number of studies were concerned with the notion of the "black box" approach to the use of technology – this term refers to any complicated device whose inputs and outputs we know, however whose inner workings we do not know (O'Byrne, 2018). Concerning this issue, studies reported that the problems encountered in the solution processes of modelling, which are seen in digital group work, seem to be a direct result of the automatic calculation provided by the technology. In addition, there were a small number of studies that noted that an impediment to the use of digital resources by teachers and learners in the practice of mathematical modelling was a lack of experience.

THE NEED FOR RESEARCH ON MODELLING WITH DIGITAL RESOURCES AND OPEN QUESTIONS

Our review of the literature identified only 30 relevant publications in total, indicating that digital resources enhanced mathematical modelling remains under researched, even though it is no longer a new area of interest to scholars in the field. Further, the number of publications in high-ranking journals, such as those indexed in Social Science Citation Index is especially limited, with chapters from ICTMA books making up the bulk of the identified literature. Our review also indicates that most studies were based on what might now be considered conventional digital tools that have an established role in teaching and learning modelling (e.g., computers), rather than new and emerging technologies (e.g., augmented and virtual reality) that may have great potential for instruction in responding to real-world problems. Overall, our review indicates that areas that require further attention in research include: How to improve the experience and knowledge of educators and students on the use of digital resources in modeling? What innovative technology active teaching approaches may be effective in supporting student learning in modelling? How can digital resources be used in the modeling process while avoiding black-box related issues? Ultimately, many interesting questions remain open for current research.

CULTURAL AND SOCIO-CULTURAL INFLUENCES ON THE IMPLEMENTATION OF MATHEMATICAL MODELLING EDUCATION AND CONSEQUENCES FOR MATHEMATICAL MODELLING EDUCATION

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In the paper, we first discuss main social cultural factors which influence the implementation of mathematical modelling education such as differences of theoretical perspectives of modelling or ways of teaching. We then review the differences of mathematical modelling competences between students from Western and Eastern contexts identified in available comparative studies in this field. We also discuss the approach of ethnomodelling to expand the understanding of social and cultural influence on mathematical modelling education. We close the paper with a few recommendations.

CULTURAL AND SOCIO-CULTURAL INFLUENCES ON THE IMPLEMENTATION OF MATHEMATICAL MODELLING EDUCATION

Mathematical modelling is a central part of mathematical education, which for example becomes obvious by its embeddedness into theoretical frameworks of studies on both,

students' competences (e.g. the concept of mathematical literacy in PISA, OECD, 2003) as well as (future) mathematics teachers' competences (e.g. TEDS-M, Blömeke et al., 2014). However, there is no joint understanding of mathematical modelling or mathematical modelling competences and instead, various approaches can be identified (Cevikbas et al., 2022). Kaiser and Sriraman (2006) distinguished various perspectives on modelling such as epistemological and realistic modelling.
Moreover, the teaching and learning of mathematical modelling in mathematics classroom is of course embedded into approaches of teaching and learning of mathematics in general. Thus, cultural and socio-cultural differences concerning the teaching and learning of mathematics in general also influence the teaching and learning of mathematical modelling in particular, becoming manifest for example in different accentuations in the curricula as well as different ways of teaching. A prominent distinction with regard to different approaches in East Asian and Western countries was formulated by Leung (2001). He formulates various dichotomies according to which the teaching and learning of mathematics differs between the two regions for example referring to rote versus meaningful learning or whole class teaching versus individualised learning. It is obvious, that respective differences can have a strong influence on how mathematical modelling is taught. However, also the analysis of processes of the teaching and learning of mathematics in research has to take social-cultural aspects into account (Lerman, 2001) as well as teacher education (Presmeg, 1998).

CONSEQUENCES FOR MATHEMATICAL MODELLING EDUCATION

In the past years, researchers have started to compare students' and teachers' mathematical modelling competencies between different educational environments. For example, Ludwig and Xu (2010) compared the overall mathematical modelling competence levels of 1108 secondary school students (Grade 9 to Grade 11) from Germany and Mainland China and mainly found that the general performance of the participants was nearly the same, except that students from Mainland China were found to progressively improve their competencies from Grade 9 to 11.

Recently, Chang, Krawitz, Schukajlow and Yang (2020) compared a specific sub-competence of mathematical modelling, namely making non-numerical and numerical assumptions, between secondary school students from Germany and Taiwan. They found that the German participants performed slightly better for the making assumption tasks than their counterparts from Taiwan. Furthermore, it was found that if students in the two educational systems were on the same level of mathematical knowledge, the German participants were found to have higher modelling performance compared to the participants from Taiwan in solving the same modelling tasks. Similarly, Hankeln (2020) compared the mathematical modelling processes between 18 French secondary school students (from Grade 10 to 12) and 12 German students with the use of think-aloud methods. It was also found that even though none of the participants was familiar with open modelling problems, the French participants were hindered more by the underdetermination of the task, false assumptions and wrong representation of the situation. By contrast, the German participants were found to reflect upon the real-world situation rather superficially and to be hindered by difficulties in the calculation.

Quite recently, Yang, Schwarz and Leung (2022) compared pre-service mathematics teachers' professional modelling competencies between Germany, Mainland China and Hong Kong. It was found that pre-service teachers from Germany demonstrated the strongest MCK and MPCK of mathematical modelling, while those from Hong Kong demonstrated the weakest professional competencies, with pre-service teachers from Mainland China falling in between. Specifically, Bonferroni-adjusted post hoc tests showed that significantly more participants from Hong Kong and Mainland China displayed low or very low levels of MCK and MPCK of mathematical modelling, and by contrast, more participants from Germany were found to possess high or very high levels of MCK and MPCK of mathematical modelling.

Overall, such differences identified between different educational systems and countries may be explained by differences of tradition of mathematical modelling in mathematics curricula, mathematics textbooks, teacher education, and teaching culture in these systems.

ETHNOMATHEMATICS AND THE SOCIOCULTURAL PERSPECTIVE OF MATHEMATICAL MODELLING

Historical evolution enabled the development of alternative mathematical knowledge systems that provide explanations of daily problems and phenomena, which leads to the elaboration of ethnomodels as representations of facts present in our own reality. Ethnomathematics helps members of distinct cultural groups to draw information about their own realities through the elaboration of representations that generate mathematical knowledge that deals with creativity and invention. According to D'Ambrosio (2006), ethnomathematics is a way in which people from particular cultures use their own mathematical ideas, procedures, and practices for dealing with quantitative, qualitative, spatial, and relational daily phenomena. This process legitimates and validates their own mathematical experience that is inherent to their lives. Similarly, it is important to argue that, in an ethnomathematical perspective, mathematical thinking is developed in different cultures in accordance with the common problems that are encountered within the sociocultural context of their members.

In this regard, D'Ambrosio (2006) has affirmed that in order to solve specific problems, members of distinct cultural groups develop non-generalizable solutions that cannot be adapted to other purposes. These members also create methods that are generalized to solve similar situations in their own contexts, and then theories that are developed from these generalizations so that they are able to understand these phenomena through the development of ethnomodels. In the ethnomathematics context, these members come to develop mathematical representations in ways that are quite different from school/academic/Western mathematics as taught in schools, which can be represented through the elaboration of ethnomodels.

OUTLOOK

During the past years, there has been an increasing interest to explore and investigate the social and cultural aspect of mathematical modelling, however, firstly, more empirical studies are needed, especially more cross-cultural comparative studies are needed which compare students' and teachers' modelling competencies between Western and Eastern contexts and which involve more countries and regions. At the moment, almost all the comparative studies mainly involve Germany as the typical Western representative. In addition, it will be also necessary to conduct more comparative studies within Western or Eastern context as well to understand more deeply about how a specific social and cultural context influences the development of mathematical modelling competencies.

Secondly, it is needed to develop more cross-culturally reliable and valid instruments in the field. At the moment, only one or two modelling tasks were employed in most of the available comparative studies to measure participants' modelling competences, therefore, it is very possible that their modelling competences were not fully measured. In addition, from the statistical point of view, it is impossible to make more advanced statics analysis such as causal inference analysis with the involvement of a wide range of other variables such as knowledge and affective factors.

THE INFLUENCE AND ROLE OF AFFECTIVE ASPECTS IN MATHEMATICAL MODELLING

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Affective aspects, such as motivation and emotions, are essential for the teaching and learning of mathematical modelling. However, research on students' affect in modelling is just beginning. In this contribution, we summarise the knowledge acquired in recent years about students' affect with respect to modelling problems, how instruction in mathematical modelling influences students' affective outcomes, and which affective constructs were found to be important for students' progress in mathematical modelling.

INTRODUCTION

Affective aspects of students' learning are essential for their life-long learning, career choices, and future lives. For example, while choosing classes in high school, college, or university, students greatly rely on what they are interested in, what they like, and whether they consider themselves able to successfully face the demands of their mathematics classes. For a long time, research on modelling – similar to research on other competencies and content areas in mathematics education – was focussed on cognitive outcomes, whereas noncognitive outcomes were largely ignored. Broadly speaking, affect includes all noncognitive variables, such as motivation, emotions, attitudes, and beliefs. Important characteristics of affective outcomes are their valence (positive, negative, or neutral), temporal stability (stable traits vs. unstable states), and objects (e.g., learning, mathematics, strategies, or competencies) (Schukajlow et al., 2017). To adhere to the space restrictions, we focus on the roles that affect plays for school and university students but not for teachers. We review studies on students' perceptions of affect regarding modelling problems and the relationship between affect and performance, summarise findings on the effects of teaching methods on affective outcomes, and analyse research on affective variables as predictors of performance in modelling.

STUDENTS' AFFECT REGARDING MODELLING PROBLEMS

Mathematics problems with a relationship to the real world are expected to be motivating for students and enhance their positive beliefs, attitudes, and emotions. These considerations are in line with theories in the area of affect (e.g., expectancy-value theory of motivation, interest theory or control-value theory of achievement emotions). In the expectancy-value and control-value theories, research has suggested that task (utility) value might be higher if a problem's solution is useful in real life. Theories of interest suggest that connections to reality might be an additional source of students' interest that adds to their interest in the underlying mathematical problem. On the basis of these theories, one would expect higher interest, enjoyment, and value for modelling problems compared with intramathematical problems (i.e., problems that are not related to reality). However, prior research has not supported these expectations and has indicated similar or even lower motivation and positive emotions while solving modelling problems than intramathematical problems in school students. One explanation for this result is that not all problems that are anchored in the real world are relevant for students. Moreover, as school tests rarely include modelling problems, their relevance for students whose goals are to improve their grades and pass exams might be low. Summarizing this line of research, we ask teachers to choose contexts that have relevance for students and encourage teachers to emphasize the relevance of the specific modelling problem while presenting it in the classroom. Ways to increase the relevance of problems include developing problems that refer to the local context or personalizing tasks with digital tools so that the context captures students' interests.

RELATIONSHIP BETWEEN AFFECT AND MODELLING PERFORMANCE

Theories of affect assume a bidirectional relationship between affect and achievement, including modelling performance. Students with high initial motivation and positive emotions are expected to engage more deeply in solving modelling problems and to demonstrate better modelling performance. Students with high prior performance experience higher situational interest, enjoyment, autonomy, and competence while solving modelling problems and see increases in their self-efficacy expectations and positive emotions regarding modelling. On the basis of these considerations, researchers hypothesized the existence of a feedback loop between affective and cognitive variables. Some empirical studies have confirmed these expectations for some affective constructs. For example, enjoyment in solving modelling problems in mathematics classes was positively related to modelling performance assessed after mathematics classes. Self-efficacy in modelling was found to be positively related to modelling performance in university students and in school students. In one study, researchers asked students to report their interest and enjoyment in solving modelling problems prior to solving the problems. Higher prior interest and enjoyment in solving modelling problems was positively related to students' modelling performance. In a study with engineering students (Gjesteland & Vos, 2019), students also reported high flow (i.e., they forgot about time and experienced happiness) while solving modelling problems. The authors attributed these findings to task characteristics, such as the openness and accessibility of the task.

INFLUENCE OF INSTRUCTIONS IN MODELLING ON AFFECT

In the last decade, an increasing number of studies have evaluated the effects of teaching methods for modelling problems on affect with mixed results. Studies that compared student-centred and teacher-directed teaching methods for modelling in ninth and tenth graders revealed positive effects on students' enjoyment, interest, and self-efficacy, whereas no differences were found for students' value of modelling and attitude towards mathematics, even though qualitative analyses of students' responses indicated that students preferred the student-centred teaching method and more specifically cooperative group work. In a study with engineering students in South Africa, a student-centred teaching method that was enriched with some directive elements showed greater development in students' interest, effort, and value than a teacher-directed teaching method, but the effects just missed significance (Durandt et al., 2022). In the framework of a mathematical modelling competition, solving modelling problems was demonstrated to improve self-efficacy in mathematics.

No differences in students' interest, enjoyment, or boredom were found between German school students who solved modelling problems in the classroom by paper and pencil and outside the classroom by using MathCityMap. Therefore, both teaching methods can be beneficial for students' affect. The authenticity of the problem seems to play a more important role than where the students are when solving the problems.

In order to uncover possible mechanisms behind how learning environments affect students' learning in the classroom and how learning in the classroom in turn affects modelling performance, several studies have addressed the specific characteristics of modelling problems in teaching interventions. Providing students with reading comprehension prompts (i.e., presenting questions about the situation described in the task in this study) improved students' situational interest in solving modelling problems in Germany and Taiwan (Krawitz et al., 2021). The authors attributed these positive effects to an increase in students' reading comprehension and their greater involvement in problem solving resulting from engaging in the processing of reading comprehension prompts. Further, a series of studies compared the effects of prompting students to develop multiple solutions for modelling problems with the effects of prompting students to find one solution on students' affect. Prompting students to develop multiple solutions for modelling problems that required them to make assumptions increased students' experiences of competence, autonomy, enjoyment, and interest and decreased boredom while solving modelling problems. Positive effects of prompting students to apply two different mathematical procedures while solving modelling problems were found on students' experiences of competence. Further indirect effects from this teaching method were found on students' self-efficacy in mathematics via experiences of competence and enjoyment as intervening variables. Consequently, affective aspects can explain how an intervention influences modelling and which affective aspects teachers should focus on in mathematics classrooms.

AFFECTIVE ASPECTS AS PREDICTORS OF THE DEVELOPMENT OF MATHEMATICAL MODELLING

Another important line of research involves affective outcomes as predictors of students' learning. In a study on teaching modelling with digital tools, self-efficacy in using software but not attitude towards software predicted school students' development of mathematizing. In a study on students' drawing strategies, researchers assessed students' enjoyment of and anxiety towards drawings before problem solving. Students who enjoyed making drawings used the drawing strategy more often and more often solved the modelling problems; students who were anxious about using this strategy rarely made drawings and rarely solved the modelling problems. In an intervention study on knowledge about drawings, strategy-based motivation (self-efficacy and cost) at pretest were found to predict the quality of drawings and modelling performance at posttest (Schukajlow et al., 2021).

SUMMARY AND FUTURE DIRECTIONS

Our analysis indicates that it is not always easy to improve students' affect. Increasing the relevance of the context might be a promising way to foster students' interest, motivation, and positive emotions regarding modelling. Future research should clarify which characteristics of modelling problems contribute to higher affect. Initial studies confirmed that some affective constructs are related to performance in modelling, and we call for more research to collect indications of the relationships between affect (e.g., self-efficacy, values, emotions, identities) and engagement, performance, and other achievement outcomes in the short and long terms. Intervention studies have indicated that student-centred teaching methods and specific teaching approaches, such as prompting students to develop multiple solutions or offering solution plans in the classroom, improved some students' affective outcomes. More research is essential to clarify which teaching methods are beneficial for which students' learning outcomes. Further, studies have revealed the importance of strategy-based motivation and emotions or self-efficacy regarding software as predictors of students' progress in modelling. We suggest that researchers target different objects of affect in the context of modelling. Teachers' beliefs about modelling and their judgements of students' affect are other important areas of research, even though we did not address them in this review due to the space limits.

THE INFLUENCE OF CREATIVITY ON MATHEMATICAL MODELLING AND ITS ROLE WITHIN MATHEMATICAL MODELLING ACTIVITIES

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Although mathematical modelling is playing an increasing role in mathematics education, only recently was this approach connected with creativity and its development in mathematics education. In this contribution we describe recent empirical studies connecting mathematical modelling and creativity, referring to long-standing approaches on the conceptualization and measurement of mathematical creativity. The empirical studies point out that mathematical modelling requires creativity at each step of the modelling process. However, the studies present somewhat contradictory descriptions of the relations between the components of creativity and the adequacy of mathematical modelling approaches.

CREATIVITY IN MATHEMATICS EDUCATION

Developing creativity is one of the major goals of mathematics education. Its importance is rooted in two main observations. First, the activity of professional mathematicians is directed at mathematical invention and leads to the development of mathematics as science. Thus, this activity is inherently creative. Second, following Vygotsky's approach, creative ability is one of the foundations of knowledge development, and knowledge development and creativity have a mutually supportive relationship (see Leikin & Sriraman, 2022).

Interest in creativity in the field of mathematics dates back to the mathematicians Poincaré and Hadamard, who analyzed creative processing among professional mathematicians. Poincare stressed the importance of intuition and a feeling that mathematics is beautiful for mathematical creation. Hadamard identified four stages of the creative process: preparation, incubation, illumination and verification. Later, it was argued that creativity is a critical component of advanced mathematical thinking related to mathematicians' ability to perceive original and insight-based solutions to complex mathematical problems. Connections between mathematical creativity and mathematical giftedness were pointed out.

At the school level, mathematical creativity in mathematics (education) was overlooked for several decades. From 1960 to 1970, scholars developed connections between mathematical creativity and psychological theories. Of specific importance to the current discourse is the model of creativity proposed by Torrance (1974), which posited that creativity is composed of fluency, flexibility, originality and elaboration. It was suggested that open-ended problems requiring divergent production are effective for the development and evaluation of creativity. Referring to Torrance’s model, Silver (1997) proposed that creativity could be fostered by instruction rich in mathematical problem-solving and problem-posing. Later, Leikin (2009) suggested a model for the evaluation of mathematical creativity using multiple-solution tasks.

In their survey paper, Leikin and Sriraman (2022) reported that, during the past decade, there has been a meaningful growth of interest in research on creativity in mathematics education. Based on a systematic literature survey of mathematics education and creativity from 2010 to 2021, the authors identified three major lines of research: research examining the relationships between creativity in mathematics and other characteristics, research analyzing instructional practices and mathematical tasks, and research focused on teachers’ creativity-related conceptions and competencies. Referring to the instructional practices and tasks used in these studies, Leikin and Sriraman (2022) report that there are hardly any studies examining creativity related to mathematical modelling. In this paper, we attempt to describe the relevance of the relationship between creativity and modelling in mathematics education.

CREATIVITY IN MATHEMATICAL MODELLING EDUCATION

Earlier research in mathematical modelling education

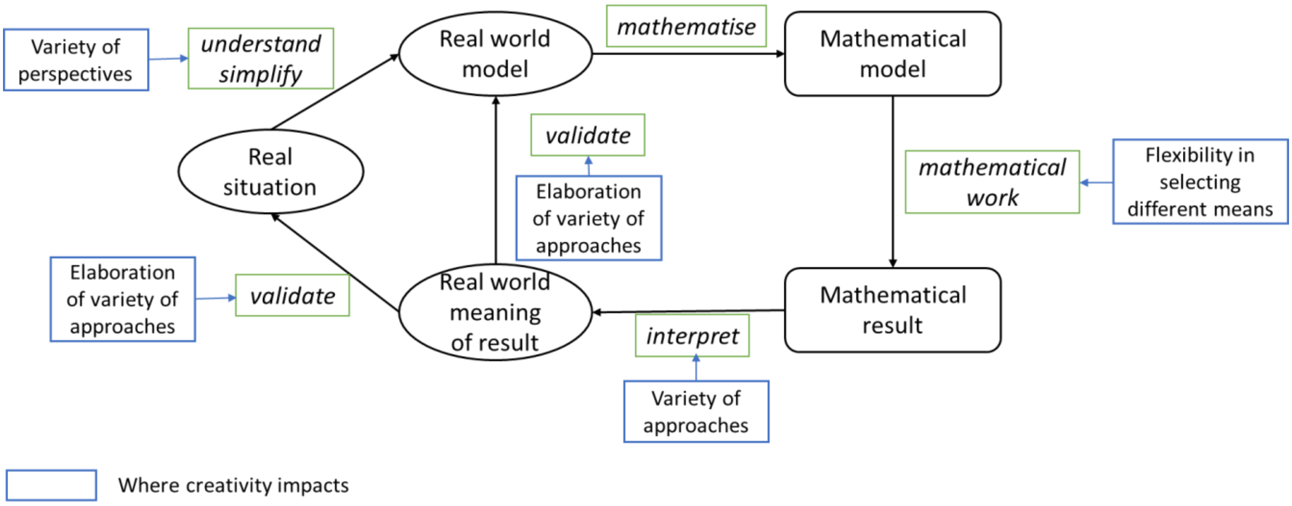
Mathematical modelling has gained increasing importance across the world in the last few decades, bringing real-life contexts to mathematics classes. Modelling practices in school have the potential to motivate students, help them develop appropriate views on mathematics, foster mathematical and extra-mathematical literacy, promote in-depth understandings of mathematical content, and, as a result, promote civic competences for which creativity is a crucial component (Maaß et al., 2019).

Until now, only a few empirical studies on modelling education have involved creativity. Dan and Xie (2011) measured university students’ modelling skills and levels of creative thinking and found a strong positive correlation between modelling and creative competence. Chamberlin and Moon (2005) pointed out that complex, open, non-routine model-eliciting tasks could motivate learners to develop models and elicit creative applied mathematical knowledge. Based on this approach and Torrance’s model of creativity, Wessels (2014) defined creativity in modelling as comprised of four components: fluency, flexibility, novelty (originality) and usefulness. Of these, usefulness is of specific importance for modelling, since modelling is characterized as applicable mathematics, unlike mathematics in general.

Recent research in mathematical modelling education

In their studies, Lu and Kaiser (2022a, b) pointed out the necessity of creativity in all phases of modelling process. They argued that creativity allows for a rich understanding of real-world situations through their analysis. This is particularly important when developing mathematical solutions that reflect the value of varied mathematical content and when elaborating ideas for interpreting and validating mathematical results that link the results with new understandings of real-world situations. Overall, they proposed that creativity should be incorporated into the construct of modelling competences and the modelling cycle be enriched by creativity.

Fig. 1: Enriched modelling cycle



Based on Wessels' (2014) work and studies on creativity in problem-posing and problem-solving (e.g., Leikin, 2009), Lu and Kaiser (2022 a, b) further differentiated creative components in the modelling process through two empirical studies that focused on upper secondary school students and pre- and in-service mathematics teachers in China. These components are as follows:

- Usefulness, which describes the efficiency of a modelling approach for solving a task. Higher levels of usefulness are assigned when an approach has the potential to be applied to other situations.
- Fluency, which refers to the application of various solutions to the task.
- Originality, which describes the relative rarity of the modelling approach.

Lu and Kaiser developed a framework for measuring creativity in modelling that includes these components. This framework includes an independent measurement of modelling competencies based on an analysis of the adequacy of participants' modelling approaches, and it is enriched by evaluation of the three creativity-related components of participants' modelling approaches.

With this framework, Lu and Kaiser (2022a) evaluated the modelling approaches used by upper secondary school students and pre-service and in-service teachers from China. They found (1) a significant positive correlation between adequacy of the modelling approach and usefulness and fluency, (2) a negative correlation between usefulness and originality, and (3) dependency of the chosen modelling approach on the mathematical knowledge of the participants, although the influence was less strong than expected. In their second study on upper secondary school students, who had more experience in tackling modelling tasks than their peers, Lu and Kaiser (2022b) also identified significant positive correlations between fluency and originality, but inconsistent correlations between usefulness and fluency or originality.

Overall, the results of these studies indicate the importance of including creativity in mathematical modelling, as well as the relation of usefulness as part and characteristic of modelling problems. The components of creativity remain ambiguous.

OUTLOOK

Given that existing studies have produced ambiguous results, further empirical work is needed. It is especially important to investigate the role of the adequacy of the modelling approach and the relation of adequacy to the components of creativity. In addition, the role of culture in mathematics education must be considered. Thus, studies should be conducted in other parts of the world. Furthermore, the characteristics and complexity of the modelling task strongly influence the originality of the modelling solution and chosen approach; future studies must examine the influence of the kind of modelling tasks and its complexity. Overall, it seems necessary to develop refined, and partly standardized, measurement instruments.

INTUITION AND INNOVATION IN MODELLING

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In this paper, we aim to motivate the study of intuition in the field of mathematical modelling. Experiences of intuition and of "a-ha" moments can be significant episodes in the development of mathematical dispositions and identities. We thus argue that research on intuition serves an important equity goal: grounding an approach to mathematics education that assumes students are capable of innovations in mathematics—of creating mathematics that is new to them. In such an approach, intuition plays a primary, active role, along with other, less well-studied "ways of knowing." We close with a call to study intuition alongside these other facets of mathematical knowledge, in a shared to construct a modelling education that more adequately engages the full range of student experience.

FRAMING THE STUDY OF INNOVATION

Researchers have tended to characterize intuition in contradistinction to processes of ratiocination, and to contrast intuitive knowledge with knowledge that the knower can articulate explicitly. Building on this tradition, Sinclair (2009) groups intuition with aesthetics, gesture, and embodied cognition under the heading of "covert" ways of knowing mathematically, as opposed to the "propositional" forms that dominate externalized manifestations of mathematical thinking and knowing.

The challenges of characterizing and studying intuition appear concretely in the context of modelling. Observing individuals solving modelling tasks, for example, one asks the questions, whether the solution already exist in the unconscious mind and merely took time to rise to conscious awareness? Or, alternatively, is there a parallel and unconscious problem-solving process taking place, whose results then enter consciousness in the moment when intuition is experienced? For example, Davis et al. (1998) argue that unconscious perception and action can be recognized by individuals, but that it is difficult for those individuals to describe such states or connect them with intuition.

INTUITION'S ROLE IN MATHEMATICAL INNOVATION

At the same time, researchers have recognized that intuition and intuitive knowledge do in fact play vital roles in modelling. Borromeo Ferri and Lesh (2013) distinguish between implicit (intuitive) and explicit worlds of modelling. Moreover, they hypothesize that when modellers are provoked into conscious reflection on their interpretation systems, they may begin to articulate explicit models that are based in or concordant with their intuitive models. Otherwise, their externalized work may represent only the 'tip of the iceberg' of their implicit models.

Intuition thus plays a role that should not be underestimated in modelling and in other creative mathematical work. An ample literature (e.g. Fischbein, 2002), affirms that intuition can be central as a trigger or even a driving force in mathematical learning processes. Moreover, it plays a prominent role in famous 'a-ha moments' of mathematical discovery (Liljedahl, 2005).

WHO CAN INNOVATE?

Thus, one may ask: How central should intuition and the "a-ha" experience be in our designs of learning environments for mathematics education, and in our expectations about what students are capable of? Liljedahl (2005) found that the a-ha experiences that pre-service teachers recalled were disappointingly shallow, but that they were nevertheless very significant moments for developing mathematical identities and dispositions. These findings are ambiguous: one's interpretation of it depends on one's beliefs and values. We argue that the positive impact of a-ha moments urges us to identify opportunities for our students to develop and use intuition, and to create occasions for them to see themselves as innovative makers of mathematics.

The importance of intuition and innovation implicates our beliefs and values, since it causes us to ask, what proportion of the population do we expect are capable of producing and experiencing innovation in mathematics? If we believe this is a *small* proportion of the population, then we are likely to view the support and study of intuition and innovation as a subfield of 'gifted and talented' education. If, on the other hand, we believe that *every* student has the capacity to make original mathematics, we will feel obligated to provide all students with opportunities to engage in mathematical innovation, throughout their educational lifespan.

INTUITION AND INNOVATION IN THE PHILOSOPHY OF SCIENCE

The question of whether intuition and innovation are focused in an elite few or are resources for all people extends well beyond Mathematics Education. It has been central as well in debates on the philosophy of science. In particular, in his account of the nature of science, Kuhn (2012) depicts genius and originality erupting in discontinuous innovations that transform fundamental paradigms. Kuhn's division between 'normal science' and 'revolutionary science' treats creativity and intuition as rare and mysterious phenomena. Under this perspective, the study of scientific discovery is the province of exceptional psychology.

In contrast to this 'irrationalist' view, Lakatos (1976) argues for studies of the "*logic* of discovery" that emphasize collective and discursive interactions as a source of innovation and creative power. Lakatos paints a picture of mathematical work that features bold conjectures made by ordinary participants, along with a collective discursive process that struggles to 'prove' and foregrounds collective efforts to 'improve' these fallible conjectures.

Mathematicians as a group are not unified in endorsing either view. On one hand, Hadamard's (1945) study of "the psychology of invention in the mathematical field" can be seen as contributing to a Kuhnian perspective, as it focuses on exceptional names in history; on the other hand, the account of innovation humanizes these historic figures. Thom (1971), the famous topologist, wrote about the importance of cultivating "intuition" in *all* mathematics students; but Dieudonné opposed this perspective, arguing that only "four or five men in the eighteenth century, about thirty in the nineteenth, and not more than a hundred" in the twentieth had mental faculties that he would describe as valuable "intuition." In particular, Dieudonné argued that *teachers* of mathematics should instead focus on becoming "adequately educated" in correct formalism and use that as their guide.

Henderson and Taimiņa (2005) argued that viewing intuition as the property of the few can be damaging. A heavy focus on formalism can be alienating and can separate mathematics from learners' lived experience. Instead, they described geometry courses with pre-service teachers, in which embodied, intuitive, and aesthetic ways of knowing contribute to 'alive mathematical reasoning' being described by Hilbert described as 'intuitive understanding', which is offering a more immediate grasp of the objects one studies, a live *rapport* with them, which stresses the concrete meaning of their relations.

INTUITION IN THE MODELLING CURRICULUM

The success of the international research agenda in modelling makes it vital to consider the role of intuition and innovation. The discussion above underlines the ethical and philosophical stakes, but there are also exciting opportunities for cooperative research. Studies on the relationship between creativity and modelling have already made theoretical and practical progress (see this Research Forum and Lu & Kaiser, 2022a. b). Moreover, research suggests that intuition lies at the intersection of consciousness and creativity, and the interaction between these cognitive functions can have a strong influence on modelling. Creativity and intuition can therefore also be mutually dependent, and at the places where Lu & Kaiser (2022a, b) locate creativity in the modelling cycle, intuition may also play a role. More generally, with a greater appreciation of the value and interconnections among what Sinclair (2009) has described as ‘covert’ ways of knowing and their role in modelling, we will be better able to position *all* students to develop these facets of mathematical identity.

DISCUSSION: MATHEMATICAL MODELLING AS EMBLEMATIC FOR RESEARCH IN THE PSYCHOLOGY OF MATHEMATICS EDUCATION

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The Research Forum discusses important developments in research on mathematical modelling along three different strands: (1) teaching approaches, (2) socio-cultural approaches and (3) psychological aspects. In this discussion, I point at several links between these strands, indicating how more insight is needed into the implications of research on socio-cultural and psychological aspects for the design of mathematical modelling tasks and learning environments in which they are used.

Mathematical modelling is not only being more and more acknowledged as a major and essential part of the mathematical curriculum (thereby also providing opportunities for STEM teaching in which other scientific disciplines, technology, and engineering approaches are integrated); it also has become a mature field of research in its own right. The current Research Forum brings together some recent lines of research on mathematical modelling, organised in three different strands: (1) teaching approaches for enhancing a mathematical modelling competency (including the use of technological tools) and (2) socio-cultural issues and (3) psychological aspects of what it implies to be or to become competent in mathematical modelling. I will discuss what I see as major remaining challenges in these areas, and I will try to show that these challenges might be met by looking into the insights gained in the other strands.

Blum, Barquero, and Durandt rightfully describe mathematical modelling as a very demanding competency, in which many skills and abilities come together (and – as we see from the other contributions in the Research Forum – also many affective aspects play a role). Blum et al. review studies that show which teaching designs seem promising for enhancing students' ability to solve modelling tasks. It is very nice to see that in recent years, such teaching designs have been studied empirically, partly also by means of experimental studies with a systematic measurement of learning gains. And even better: Findings seem to converge to the importance of a balance between instructional elements such as teacher/material guidance on the one hand and constructional elements including student's self-directed activity on the other hand. Still, I missed a theoretical elaboration as to why this importance of a balance between instructional and constructional elements would be specifically important for the teaching of mathematical modelling. What is specific in a mathematical modelling competency that necessitates the teaching of it to have such a balance? And importantly: what is the optimal blend? For what aspects of mathematical modelling and at which moments in the sequence of teaching and learning activities is it important to be more teacher directed and when is it crucial to be more student initiated? And is the answer to these questions the same for students of all expertise levels? Is it similar across cultures, given the insights of crosscultural studies reported by Yang et al.? Is the balance essential to elicit some of the desirable affective processes and outcomes (as discussed by Schukajlow et al.), and what do the insights in the teaching and learning of creativity and the stimulation of intuition tell about the importance of a balance between instructional and constructional elements?

Also, Siller, Cevikbas, Geiger and Greefrath consider the ways in which teaching of mathematical modelling can be done, with a specific focus on the potential of digital resources. They report a systematic review revealing that there is research on a wide range of digital resources, ranging from tools such as dynamic geometry systems, computer algebra systems and spreadsheets to technology that allows to bring complex reality to the classroom, such as simulations, videos and video games. Importantly, the review indicates that these technologies can serve different purposes in the modelling process and thus have great potential, but it also points at potential fallacies, for instance in using technology that automatically provides the result of calculations. It seems that besides the open questions that Siller et al. raise after their review, future research may also benefit from focusing on the psychological aspects of the acquisition of a modelling competency and the (theoretical) affordances of specific types of technology: As Schukajlow et al. suggest, authentic tasks may be more motivating, and AR/VR technology may be used in increasing the authenticity of modelling tasks. Still, research may need to show whether students indeed consider tasks offered in such technology as being authentic and sufficiently competitive with "real" problems. Technology may also be used to act on the self-efficacy of students: Certain tools like dynamic geometry systems, excel sheets and computer algebra systems may take away the burden to work through formal mathematics, and to focus on the mathematising and interpretation phase of modelling. As such, this kind of tools may allow students to try out many solutions to a problem, to finetune them, to make predictions and check conjectures without the burden of calculating, manipulating expressions and drawing geometric constructions. This may make room in students' minds for creativity to occur (see Lu et al.), or for some intuitions to arise (Borromeo Ferri and Brady). But such effects cannot be taken for granted. The modelling tasks and the learning environment may need to be designed to facilitate that, and there may be an important role for the teacher to direct students to these processes that are deemed important in modelling, which otherwise may still not occur.

Yang, Schwarz and Rosa convincingly show that mathematical modelling is a socio-cultural construct, and that various approaches can be identified. Cultural and socio-cultural differences regarding mathematical modelling can be related to – but certainly do not completely coincide with – cultural and socio-cultural differences in the teaching and learning of mathematics more generally. Yang et al. refer to a number of studies that typically compare students' and teachers modelling competencies in two countries, often a European and an Asian country. And differences are indeed found. One can argue that mathematical modelling may be more susceptible to cultural and socio-cultural differences than other aspects of mathematics education. Modelling tasks are often quite complex and open tasks, susceptible to multiple solution approaches, the adequacy of which need to be considered, discussed and negotiated. Classroom norms – which often remain implicit – play an important role in such situations: Students and teachers need to negotiate and come to an agreement about what constitutes as a good solution to a complex and open modelling tasks, what can be considered as valid arguments and considerations in proposing a certain solution, how a solution needs to be communicated, and so on. While the research reviewed by Yang et al. convincingly shows the cultural and socio-cultural embeddedness of mathematical modelling, it remains far from clear what the implications for teaching mathematical modelling are. Should learning environments and modelling tasks be designed very differently, taking into account the socio-cultural context? And if so, does that imply that the final goal of such learning environment, i.e. the mathematical modelling competency that one tries to establish in learners, is also different across contexts?

Given that mathematical modelling is an activity that has a strong socio-cultural embedding, it is not surprising that affect plays an important role. Schukajlow, Krawitz and Carreira provide an overview of the main affective constructs that play a role in the acquisition of a mathematical modelling competency, and also clearly argue why affective outcomes are also part of the learning outcomes of teaching and learning activities around modelling. The main kinds of affect that they address in their review relate to motivation (including interest and enjoyment) and self-efficacy. Importantly, they do not only show the (recursive) correlation between such constructs and modelling achievement; they also show that affect can be influenced, although – as explained above – this line of research be deepened specifically in relation to the insights of the need for a balance in instructional and constructional teaching approaches. An affective aspect that may also deserve some attention in this respect relates to learners' goal orientation, which can be performance oriented or learning oriented (Dweck, 1990): Modelling tasks are often complex and open, and the assessment is not straightforward. Unlike for many other mathematical tasks, there often is no simple distinction between a correct and an incorrect answer. Learners who have a strong performance-oriented goal orientation may feel insecure when involved in assessments with such open tasks, as opposed to learners with a more learning-oriented goal orientation.

Lu, Kaiser, and Leikin elaborate on the construct of creativity, how it is essential in the activity of mathematics, and particularly in mathematical modelling. While being overlooked for a long time, creativity has taken an important place in research on mathematics education. The theoretical construct has been operationalized, and in their contribution, Lu et al. propose an enriched modelling cycle in which it is shown that in the various stages of the modelling cycle, creativity plays a role. They also review the first studies that link creativity to performance on modelling tasks. However, they also clearly indicate that much further work int his field is needed, for instance in theorizing and in developing adequate measurement instruments. I wish to add that there is a need to come to a deeper understanding of whether and how creativity in mathematical modelling activities can be enhanced in instruction. If teaching is conceived as the systematic, methodical design and organization of certain teaching activities in order to elicit specific learning activities in learners for them to achieve pre-specified learning goals, teaching for creativity seems almost a *contradiction in termini*. Still, if we acknowledge the important role of creativity, this will be the challenge: How can we design tasks and organize tasks in a learning environment so that learners can experience the importance of being creative, and their creativity is stimulated.

The contribution of Borromeo Ferri and Brady on the role of intuition is somewhat on the same line as that of Lu et al. on creativity. Creativity and intuition share some important characteristics, be it that intuition has a more controversial history in mathematics education. Mathematics is often seen as a purely rational, deductive activity in which reasoning relies on consciousness and logic. However, mathematical problem solving – including mathematical modelling – often is not, and intuition seems a particularly fruitful pathway. Once more the challenge is how one can make room for intuition – and even stimulate it – in teaching/learning environments that are focused on mathematical modelling, and what role a teacher can play in it. Making the link to the contribution of Blum et al. showing the importance of a balance between instructional and constructional approaches in education, teachers may at some occasions act as a "model" in solving mathematical modelling problems, verbalize their thinking processes, thoughts, heuristics, considerations, etcetera, but also make explicit their intuitions, their search and hope for an "aha" experience, thereby revealing that also expert mathematical modellers do rely on it. They may show that the use of certain technological tools, as described by Siller et al., may shed a different light on a modelling problem, thereby potentially facilitating – but not guaranteeing – such an "aha" experience.

Based on the above considerations, a tentative conclusion that I want to draw is that the perspectives across three major strands of the Research Forum need to be integrated. I want to argue that exactly this is the essence of what research in the Psychology of Mathematics Education should do: Based on research that investigates the psychological processes of what it implies to learn a mathematical skill and/or to acquire a specific mathematical competency (in this case: modelling) and the understanding of its socio-cultural embeddedness, the field should aim to unravel the principles that would guide the teaching of these competencies, and thus the design of learning environments (including the technology used in them). As I see it now, the contributions that focus on the teaching approaches aimed at enhancing a mathematical modelling competency would benefit from a close consideration of the socio-cultural and psychological issues that are involved in acquiring a modelling competency, while the contributions that review the insights from research on socio-cultural and psychological aspects may need to go more deeply into the implications for the teaching of mathematical modelling and into the principles underlying the design of learning environments.

OVERALL SUMMARY AND CONCLUSION

The present research forum demonstrates impressively that mathematical modelling is a dynamic research field with diverse research topics, theories, methodologies, and practical implications. While preparing this research forum, we build upon the previous research forum (Cai, 2014), recent special issues in research on modelling in in *Education Studies in Mathematics* (2022), in *Mathematical Thinking and Learning* (2022) and in *ZDM – Mathematics Education* (2018), as well es overviews about empirical research on teaching and learning of mathematical modelling (Schukajlow, Kaiser, & Stillman 2018), about the current discussion on mathematical modelling competencies (Cevikbas et al., 2022), and about research on modelling from a cognitive perspective (Schukajlow et al., 2021). The research forum provides a unique opportunity in presenting and discussion of innovating perspectives in research on mathematical modelling education. As a result of the synthesis of the contribution of the research forum we call for (1) integration of various theoretical perspectives such as social-cultural approaches into research in mathematical modelling, (2) taking into account theoretical foundations from other research areas such as teacher education, intuition, creativity or technology for the development of teaching methods for improving modelling competencies, and (3) considering social, cognitive and affective process and outcomes in research on modelling.

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LAYERING METHODOLOGICAL TOOLS TO REPRESENT CLASSROOM COLLECTIVITY

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Our research is guided by the question: "How might we observe, document, display, and analyze data from a collective systems perspective?" In this research forum, we share new research tools for studying mathematics classrooms, highlight opportunities for observation and analysis by layering these tools, and then illustrate how the layering of tools allows for visual distinctions across lessons and classrooms.

INTRODUCTION

For nearly 30 years, the researchers in this forum have worked individually, in subgroups, and as a collective to explore, analyze, and report on data related to collective action in mathematics classrooms (e.g., Davis & Simmt, 2016; Martin & Towers, 2015; Thom & Glanfield, 2018). For the past eight years, we have engaged in a methodological research project with the goal of exploring how we might observe, document, display, and analyze classroom data from the perspective of collective systems. That is, we intentionally shift the unit of analysis from individual students to the classroom as one 'body.'

This research forum builds on our previous PMENA working group (McGarvey, et al., 2015), NCTM research symposium (McGarvey et al., 2017), PME-42 research forum (McGarvey, et al., 2018), and PMENA colloquium (Thom, et al., 2021). These forums have been essential for engaging with the research community. Through the comments, criticisms, and suggestions received, we have taken up, expanded, extended, and revised our work. Here, we review our work to date, share new methodological tools, and examine, discuss and debate these tools as well as the insights gained about mathematics classrooms and lessons when these tools are layered.

BACKGROUND AND THEORETICAL FRAMING

This project is rooted in complex systems research—an approach to inquiry that investigates how relationships between parts of a system can give rise to collective behaviours. Examples of complex systems include birds flocking, ants foraging for food, weather systems, the Internet, and many others. Each complex system arises from the layering of biological, social, societal and environmental subsystems (Davis & Simmt, 2016). Complex systems are challenging to model and difficult to predict, but

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 177-201). PME.

are often understandable in retrospect. A key aspect of complex systems is the dialectical entanglement of the system and its environment (Varela et al., 2017).

Our overriding project goal is to develop methodological tools to better understand the dynamics of the classroom as a collective whole, rather than continuing to treat classroom interactions as a series of distinct individual contributions. While we do not discount the value of research that explores individual understanding, teacher actions and decision making within classroom contexts are often based on the teacher's sense of the class as a whole. We choose to understand the whole by developing tools where the unit of analysis is the whole class, rather than individuals within it. Because of this, we need different analytical tools.

Grounded in diverse yet complementary frameworks that include complexity science, network theory, embodied cognition, and enactivism, our work attempts to conceptualize the entire classroom as one collective agent. In our work, we explore and generate new techniques for representing, analyzing, and interpreting group activity by making use of modelling techniques to represent classroom lessons as a visual whole. In doing so, we highlight one or more features of classroom collective action all-at-once without attributing actions or utterances to specific individuals. This approach is useful for observing particular aspects of a system that may contribute to its global traits.

At PME-42 in Sweden, we presented four methodological tools under the metaphor of "vital signs" including utterance distribution, actions on the board, a mathematical ideas network, and the dynamics of ideas based on the Pirie-Kieren (P-K) model (Pirie & Kieren, 1994). We found the use of "vital signs" to be useful way to foreground a particular feature of classroom activity, while recognizing that such tools must be layered in order to provide a more robust indicator of the health of the body. Utilizing feedback, suggestions, and criticism received at PME-42 and at other forums, we developed two additional tools including Lesson Activity Mapping and Bodymarking, and revised the Dynamics of Ideas into two related tools: Persistence and Movement of Ideas. We focus on these tools in this forum.

Lesson Activity Mapping visually represent the collective actions and interactions in the classroom, such as whole class lecture, small group discussion, individual seatwork, along with the focus of interaction, such as problem solving, sharing solutions, providing explanations, and so on. The second tool, Bodymarking, emphasizes the collective gestures and gaze orientation of the class. Third, we have substantially revised the dynamics of ideas, so that it can be visually interpreted in the same way as the other tools. The other advancement of our work has been to establish a visual standard for all of the tools so that they can be layered, so that we may, for example, examine (Non)Actions on the Board, Bodymarking, and Lesson Activity Mapping for one lesson simultaneously. We can also compare this set of tools across different lessons to illustrate holistic differences in patterns, and point to key moments in a lesson.

Our project is intended to be exploratory as we continue to examine the potential for conceiving of classroom collectives as adaptive and self-maintaining complex systems. As such, we use complex systems as an interpretive frame for observing, analyzing, and comparing mathematics lessons and classrooms.

COMMON DATA

As in other presentations and publications, we use the TIMSS videos (timssvideo.com) as a common source of data in which to explore, develop, and illustrate the methodological tools (see McGarvey et al., 2018). The advantage of using the TIMSS videos is that they are publicly available and capture classroom activity in a way that is common in mathematics education research. That is, there is a video of the full lesson based on a single camera following the teacher and a set of transcripts. In addition, the TIMSS resources provide “lesson graphs” that outline the lesson activities and timing (see Figure 3).

For this forum, the tool developers were asked to analyze several TIMSS videos. In this paper researchers describe their methodological took using one or both of the following two lessons: (1) Solving Inequalities (JP4) in Japan, and (2) Exponents (US3) in the United States. The two lessons are approximately 50 minutes in length and there is a similar number of students in each class (i.e., 35 and 36 respectively). The lessons offer contrasting features including the physical arrangement of desks (i.e., rows and grouped desks), and style of teaching (i.e., whole class and small group). Figures 1 and 2 offer a storyboard for each of the two lessons.

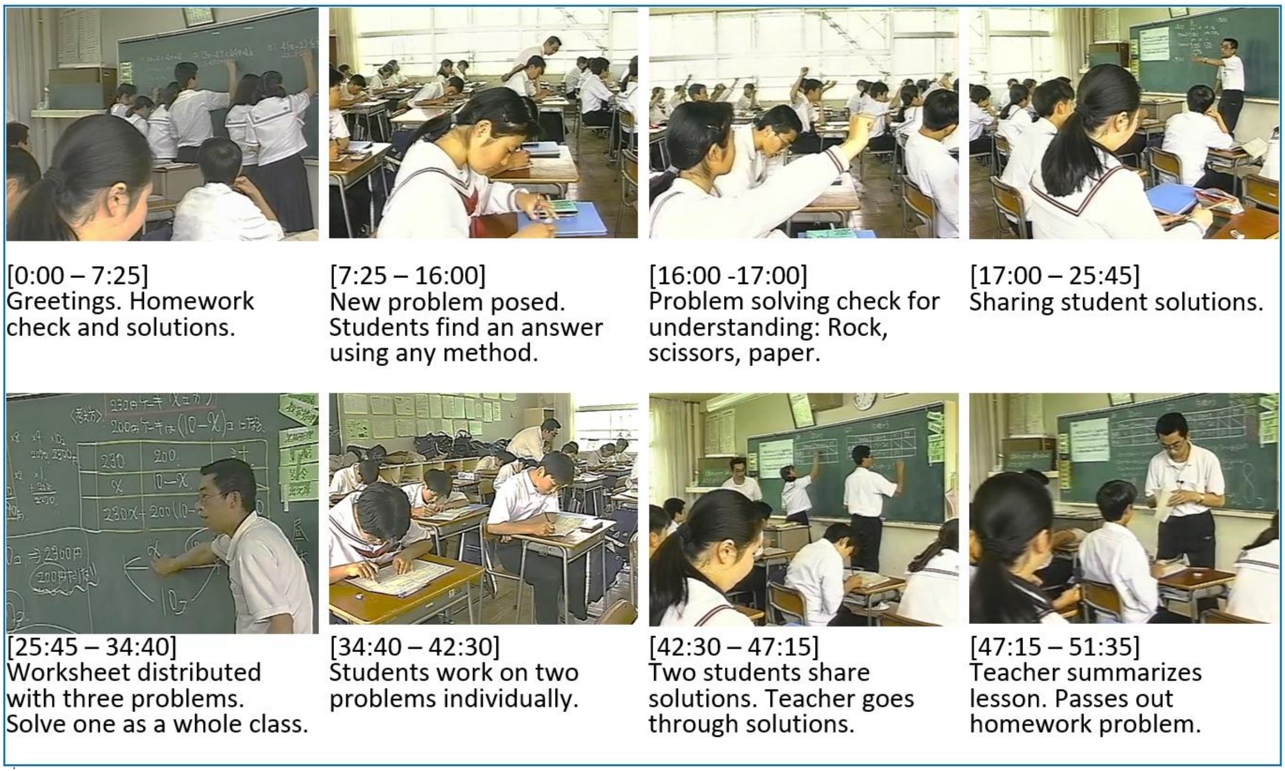

Figure 1: Japan lesson storyboard

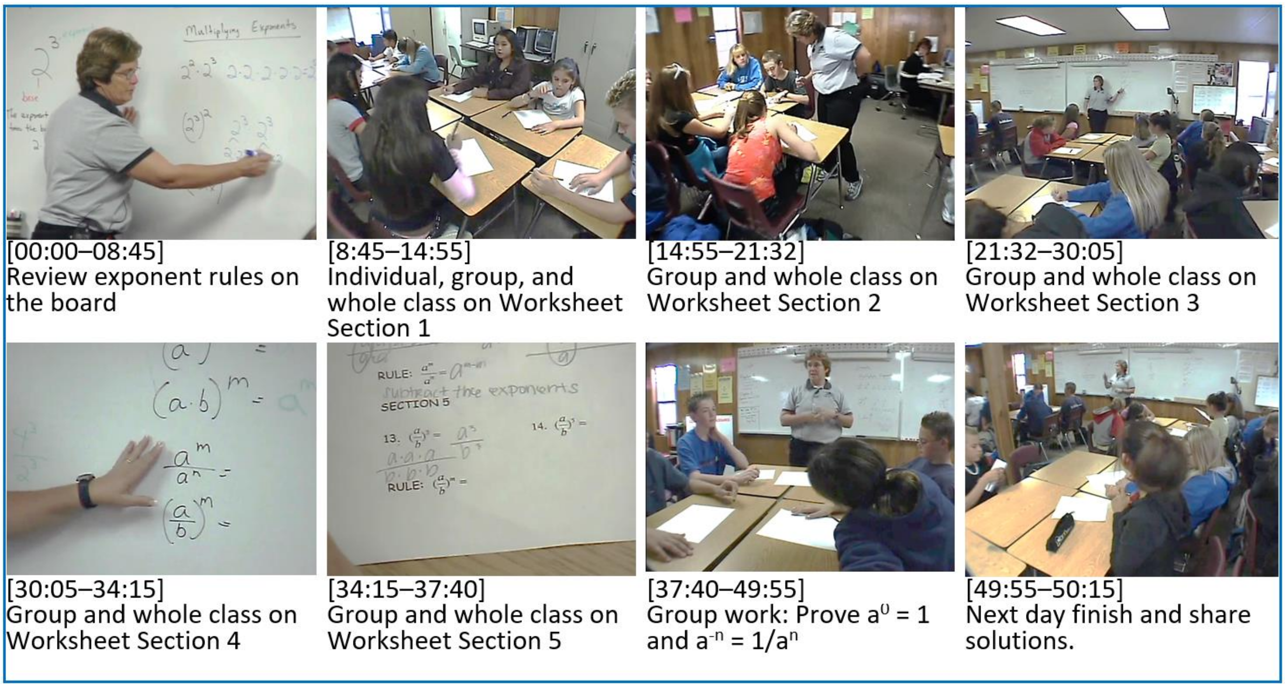

Figure 2: US lesson storyboard

Analysis of the two lessons for each of the three tools presented in this forum are illustrated below. They include (1) Lesson Activity Mapping; (2) Bodymarking; and (3) Persistence and Movement of Ideas.

LESSON ACTIVITY MAPPING

Elaine Simmt and Lynn McGarvey

University of Alberta

INTRODUCTION

The general impetus for our research team is to explore and develop methodological tools that model some aspect of classroom collectivity as a visual whole. The tools developed to date range from modelling relatively simple and specific aspects of the classroom to more complex features. When contemplating what aspect might have value when making comparisons across lessons with different content, tasks, and discourse styles, we landed on some common lesson structure questions: Are students engaged in whole class, small group, or individual activity? Where is the locus of control for engaging in the task? That is, are students generating solutions or applying learned processes? And what is the source or central focus of the activity? Are they preparing to engage in a task, engaged in a task, sharing solutions, and so on? By layering these forms of engagement, we could model general lesson structures.

The structure of mathematics lessons is an ongoing area of interest in mathematics education. In fact, the 1995 and 1999 TIMSS Video Studies brought considerable attention to the variability in lesson structures worldwide (e.g., Hiebert et al., 2003). Attention to a number of features, such as public and private work, mathematical and non-mathematical engagement, the types of mathematical activities, and so on. For the most part, an aggregate of lesson features for each country were described using descriptive statistics, and resulted in lesson patterns or "scripts" for each country. For example, the script ascribed to Japanese lessons included: presenting a problem, having students working individually or in groups, discussing various methods for solution, and summarizing key conclusions. In comparison, U.S. lessons were describe by an acquisition/application script based on the pattern of reviewing material, teacher demonstration, practice, and seatwork (Hiebert et al., 1996; Stigler & Hiebert, 1999).

The contrast in scripts spawned new reform-based lesson structures, such as Launch-Explore-Discuss (Stein et al., 2008) that emphasized "Task-First" rather than "Teach-First" approaches (Russo & Hopkins, 2017). However, as we might expect, the lesson patterns ascribed to entire countries are much more varied when not reduced to a general form, and that it may be more useful to make comparisons at the level of the "lesson event" (Clarke, et al., 2007). We considered several aspects of the lesson structures valuable to our work in modelling classrooms as collectives, and explored how to visualize these features under the vital signs metaphor.

BACKGROUND

We developed the Lesson Activity Mapping tool using the video, transcripts, and "lesson graphs" provided as resources for the TIMSS videos (see Figure 3 for a portion of the US3 lesson graph). Lesson graphs are one-page summaries of the activities in each lesson. As seen in Figure 3, the lesson graph chunks the class period into timed segments (left column) making distinctions between public and private class work (right column). Information about the mathematics content, and teacher and student actions are also provided. In most instances, the segments are described as either "Public Class Work" or as "Private Class Work." Public class work includes such activities as reviewing homework, sharing solutions, posing problems, class discussions, teacher demonstrations, and so on. Private class work typically referred to individual seatwork where students were working independently on problems from the textbook or worksheets provided.

Under each public or private block, the lesson graph provides a brief description of the activities or tasks in that timeframe, and general descriptions of the teacher and student actions. As we can see in the US3 lesson graph (Figure 3), in the first 9.5 minutes of the lesson the teacher announces the topic of the day, comments on a teaching aid she is using to represent exponents, and then demonstrates how to multiply exponents using three examples.

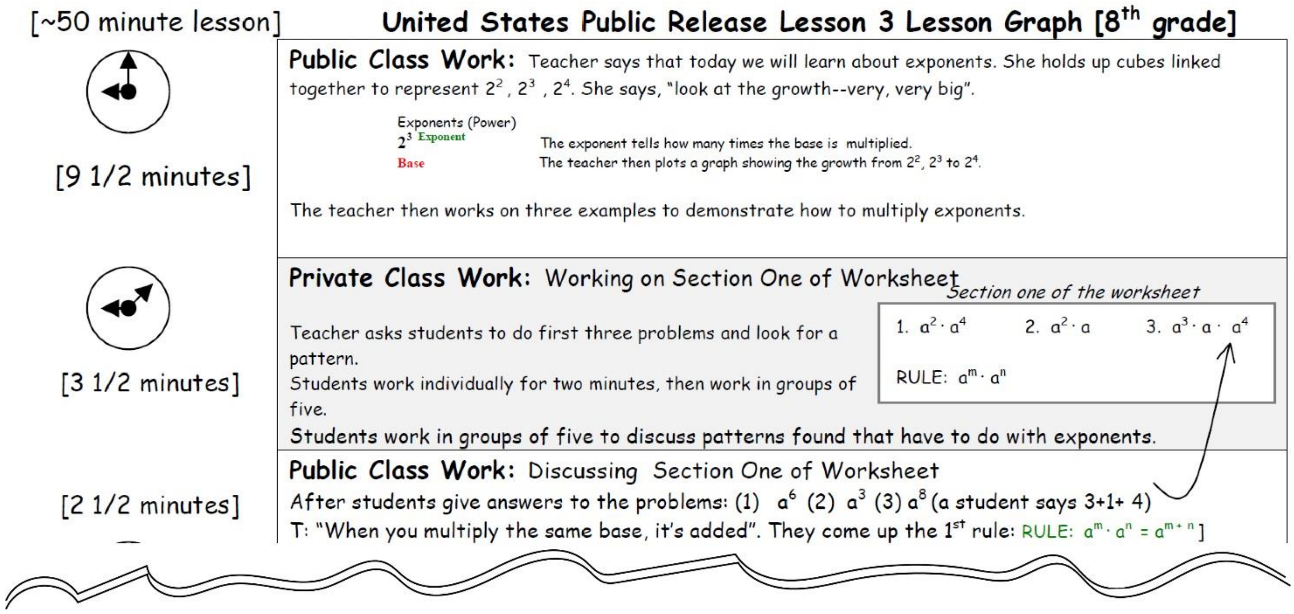

Figure 3: US3 Exponents lesson graph

We found the lesson graphs to be valuable summaries of the lesson activities. In particular, the distinction between public and private work contributed to our view of collective activity in which two or more actors offered actions and utterances to others, in contrast to when students acted (primarily) on their own.

The other component we believed would contribute well to collective modelling of lessons are broad-based activity segments. Activity segments, described by Stodolsky (1988) are the "instructional or managerial" aspects of a lesson that have an intended purpose and with relatively clear starts and stops (p. 11). Different activity types or variations of activity segments have been explored including "setting up," "working on," "sharing," and "demonstration (Stigler et al., 1999); reviewing, demonstrating, practicing, correcting/assigning (Clarke et al., 2007); and "development,", "student work,", and "review of student work" (Kaur, 2021). The lesson graphs also showed the general activity segments for each lesson.

METHOD AND CODING

For the purposes of generating a new tool or vital sign, we chose to visually represent lesson structures based on public/private indicators, as well as demarcated activity segments. Because we are specifically interested in collective activity, we coded for two categories of public class work, whole class and small group engagement, as well as private work according to the following descriptors:

- *Public-Whole* (dark green) involves periods of time where information and ideas were at the level of the whole class.
- *Public-Group* (light green) includes periods of time where information and ideas were discussed at the level of a sub-group of the class in which two or more students were involved. The public-group periods were coded when students were directed to work with a group. That is, the grouping was an intentional

aspect of the lesson. It did not include periods in which students turned to one another for brief moments.

- *Private* (yellow) refers to time periods where individuals worked predominantly independently. Again, this was an intentional aspect of the lesson where the teacher instructed students to work on their own

For activity segments we selected four categories of actions: non-mathematical, presenting, engaging, and sharing.

- *Non-mathematical* (grey) refers to segments of the lesson that involve activities such as greetings, announcements, moving into groups, and other forms of housekeeping.
- *Presenting* (pink) involves segments where information is offered in preparation to engage in work or reviewing and summarizing completed work. These segments included the presentation of tasks, procedures, instructions, worked examples, explanations, and question-answer interchanges.
- *Engaging* (blue) includes periods where most students in the collective were actively involved in a task.
- *Sharing* (chartreuse) segments are predominantly a reflection on completed tasks by providing solutions.

Figure 4 provides the colour coding used for the two sets of codes. We placed the activity segments at the top and the public-private segments at the bottom.

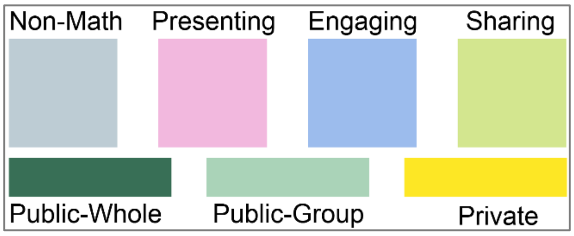

Figure 4: Colour codes used for activity segments (top) and public-private (bottom)

Although other researchers have examined activity segments using additional codes we felt that visually it was important to limit the number of segments. Our goal is to seek out lesson patterns broadly, rather than detailed descriptions of the lessons.

As with the other tools, we examined the lesson in 15-second segments and coded for public-private and for the activity segment. Rather than overlap codes within a 15-segment time period we chose to code to the nearest 15-second mark. This was intended to provide a clearer picture of the overall lesson pattern. Creating a standardized visualization allows us to layer the Lesson Activity Mapping tool with other vital sign visualization tools from the same lesson, enabling the researcher to look for critical points in the lesson.

RESULTS

Using the public-private and activity segment codes above we coded the United States (US3) and Japan (JP4) lessons (see Figure 5).

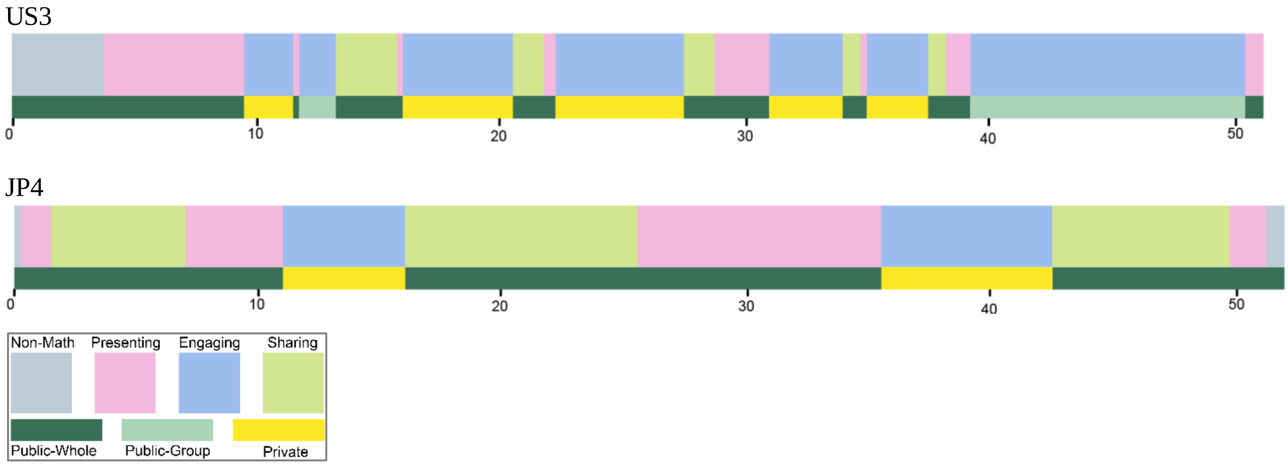

Figure 5: US3 (top) and JP4 (bottom) Lesson Activity Mapping based on public-private and activity segments

The visualization tool shows a number of common features. For these two lessons, presenting and sharing activity segments always occur with public-whole class engagement. Engaging is most often paired with private work; however, there are two instances in the US lesson in which the teacher explicitly tells the class to work with their group. In these instances, we see engaging with public-group work.

When the two lesson structures are viewed as a whole, we see a number of visual differences. In the lesson graph provided for the US3 lesson, it shows the lesson in twelve segments. However, when coding at 15-second intervals we see twenty-one different segments. In this particular lesson the majority of the lesson was based on students completing sections of a worksheet, so we see the teacher present information to prepare students to answer the question in a section; the students complete the section individually or in small groups, and the solutions to those sections are shared. This results in a frequent cycling through of presenting-engaging-sharing.

The JP4 lesson shows eleven segments. The number of mathematical segments (nine) matches with the number of segments shown on the lesson graph provided. (The additional two segments are non-mathematical occurring at the beginning and end of the lesson.) We see much longer segments, for all three types of lesson activities.

Another contrast between the two lessons are how the lessons begin and end. While the US lesson begins with presenting information to prepare students to complete the worksheet, the Japan lesson begins with a sharing of solutions to the homework from the previous night. The US lesson ends with a lengthy period of engagement while the Japan lesson ends with sharing solutions and a brief period of presenting in which the teacher summarizes the key insights of the day.

We also notice the difference in the pattern for engaging in public and private work. The US lesson shows frequent cycling between public and private while the Japan lesson is predominantly public, except for two periods where students work privately and independently on a problem.

CONCLUDING REMARKS

The visualization offered through the Lesson Activity Mapping tool provokes questions for us: What is happening when the activity moves from public to private? How do the patterns of shifting back and forth impact the development of mathematics in the lesson? What are the patterns that exist for other lessons? Is the pattern an artifact of the content of the lesson or pedagogical distinctions of the teacher or culture? Do different patterns of activity segments and public-private actions point to different lesson structures already known in the mathematics education community?

We acknowledge that examining only two lessons does not provide warrants for generalization. Rather, presenting these two lessons illustrates that the visualization of a lesson can stimulate questions for the mathematics education researcher.

We believe that the visualization offered by the Lesson Activity Mapping tool offers possibilities for analysis of multiple lessons by the by the same or different teachers, topics, grade levels, and cultures. By making such comparisons, the visualizations can help us identify overarching patterns representing the dynamics of the system. There is a trivialization of the situation with this tool; however, using it across multiple examples as a way to make visual comparisons offers an opportunity to identify new insights and questions.

To conclude we would like to reiterate that the purpose of the Lesson Activity Mapping tool is to observe lessons in ways we have not before and to see things that may have gone unnoticed. Finally, when using this tool in conjunction with others, we can identify what might be interesting moments in the lesson, and as we look across content and contexts we may be able to identify dynamics of lessons that help us better understand learning systems.

BODYMARKING

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THEORETICAL CONSIDERATIONS

Bodymarking is a methodology and tool for understanding collective action through the ways bodies gesture in the classroom. Observing, tracking, and analyzing this kind of movement is prominently featured in the theories and approaches collected under the rubric of body pedagogics (Shilling, 2017), an embodied approach to the study of various cultural practices, including in formal teaching/learning contexts. In such educational contexts, researchers have tracked specific teacher movements and the paths teachers have taken as they move around classrooms (Andersson & Risborg, 2018; 2019), as well as other embodied phenomena, including gaze and gesture (Kaanta, 2012). Spurred by the availability and affordability of eye-tracking technology, gaze, in particular, has become a well-studied phenomenon, especially in the mathematics classroom, and is frequently used as a means of measuring student engagement and the pedagogical priorities of teachers (e.g., Abrahamson et al., 2015; McIntyre et al., 2019; Seidel et al., 2021). Building on this work we have chosen to focus on the kinds of actions and gazes that we believe might best illuminate collectivity in mathematics classrooms and to attempt to capture, with a digital tool, these everyday aspects of classroom life.

BODYMARKING TOOL DESCRIPTION

The Bodymarking tool focuses on six everyday classroom actions, which we denote as strands, including intentional movements of the hand or body, gaze, writing, and other kinds of tool use. We argue that each of the strands, which we discuss in detail below, captures a distinct gestural expression in the classroom.

Following a taxonomy used in other fields, such as neuroscience and neuropsychology, gestures can be characterized as either transitive (i.e., involving tool use) or intransitive (i.e., not requiring tool use). We have chosen to observe and record both types of gesture through Bodymarking. Strands associated with transitive gestures involve tool-oriented actions in the classroom. These include writing in public spaces (Boardwork), writing in private spaces, (Writing), and the use of other tools, such as mathematics manipulatives (Manipulating Tools). We argue these are three of the most prominent means of interacting with the material world in the mathematics classroom.

As described in Mgombelo et al.'s (2018) "Vital sign 2: (Non)actions on the board"—whiteboard, chalkboard, computer screen, etc.—often orients classroom action. In the Japanese lesson, for example, we see the board used by the teacher to convey information and by students to engage in problem solving. In contrast to Mgombelo et al.'s vital sign, we only remark on engagement with the board through the addition or subtraction of material, whether it be by a student or teacher. Our interest lies more in the distinct cadences of public work and the complex ways it couples with other classroom phenomena, not the nature of any one particular engagement. We are similarly interested in the ways the classroom works privately, which we capture through the Writing strand, and how it engages other materials in the environment, either publicly or privately.

We have also chosen to observe three distinct gestures that do not require the use of tools—Pointing, Hand and Body Movement, and Shared Gaze—to focus on as aspects of collective action in the classroom. These kinds of gestures have been frequent objects of study in mathematics education (e.g., Alibali et al., 2014), and more generally, have been shown to play a fundamental role in learning (Novack & Goldin-

Meadow, 2015). We focus on intentional movements of the hand or body, such as when a student raises a hand to ask a question, counts out a sum on their fingers, or measures a length with outstretched arms. By intentional, we mean gestures that we interpret as emerging out of the interactional domain of the classroom. This includes gestures that intimate actions, describe abstract ideas, and orient the gesturer or others; they can be deliberate, communicative gestures, or the kind of unconscious gesturing that often accompanies speech in conversation. Though these gestural movements may be spontaneous, they are not random. In the Bodymarking process, we do not record movements we interpret to be random or reflexive, such as when a student taps their foot.

Though the Hand and Body Movement strand could be considered inclusive of actions such as pointing and gazing, we conceive of pointing and gazing as specific kinds of gesture worthy of closer scrutiny and have therefore separated these from the other hand and body movements we track. Though pointing is clearly a particular kind of hand gesture, we believe it often functions in ways other hand and body movements do not. Our emphasis on pointing speaks to our interest in understanding how actors in the classroom are oriented by and toward each other and their environment at the collective level. In studying interaction in the context of virtual spaces for collaboration, Luff et al. (2013) noted that if "there is one collaborative activity that exemplifies the embedded character of practical action then it is reference, and in particular, pointing" (p. 2). Moreover, as Cooperrider (2021) noted, pointing often goes beyond the directing of attention to include a host of iconic and communicative phenomena. Finally, pointing is unique with respect to the other two strands denoted as intransitive in that it can be incorporated with tool use. By focusing on the phenomenon of pointing, not just its physical instantiation, Bodymarking can capture the complex ways we use the material environment to orient ourselves.

How we conceive of gaze in Bodymarking is similarly nuanced. Gaze is an increasingly studied phenomenon in the context of mathematics education (see Strohmaier et al., 2020) and is frequently associated with quantifying measures of visual attention. In work more closely aligned to our use of gaze, Abrahamson et al. (2015) sought to identify emergent patterns of sensorimotor activity, including gaze, and mathematical discourse. We are particularly interested in studying collective action and so in the Bodymarking tool we capture instances of shared gazing, those that involve all or most of the class and those that may only involve small groups.

THE CODING PROCESS

Using TIMSS video as source material, we recorded observations for each of the six strands at 15-second intervals for the duration of each lesson. We coded entirely without sound or subtitles, an approach also adopted by Wilmes and Siry (2021) in their study of multimodal interaction in the science classroom. Although the object of their study is a better understanding of how students enact science, and so is explicitly focused on a lesson's content, they note that viewing video with no sound allows the researcher to “draw analytical focus…to the embodied aspects of interaction” (Wilmes & Siry, 2021, p. 79). In this sense, their approach is consistent with ours: we are not so much interested in the mathematical content intimated by an individual’s iconic gesture, for example, but rather what the cadences of actions at the classroom scale can tell us about the states and dynamics of collective knowing.

For each of the strands except Shared Gaze, we recorded only the occurrence of a gesture, not its frequency. For example, there is no distinction made between a 15-second interval in which only one instance of pointing occurs and a 15-second interval in which there are many. If a gesture is observed, we assign a colour-code to the relevant strand for that 15-second interval (Figure 6). For the Shared Gaze strand, we code a 15-second interval as one of two colours (see Figure 6) if we determine that the shared gaze occupies at least half of the interval.

Bodymarking Strands	15 Sec. Intervals	Description of Strands
Pointing		Using fingers or objects, such as a pencil, to focus attention on aspects of written work, identify key ideas or missing steps, etc.
Hands/Body		Gestures involving the hand while not engaged in pointing; bodily movement, such as modeling distance with outstretched arms.
Shared Gaze (Focused)		Majority of individuals in frame share a common object of gaze
Shared Gaze (Diffused)		Common objects of shared gaze are multiple or only orient a minority of individuals in frame
Boardwork		Involves addition and/or removal of work by an individual in public view, such as on a chalkboard, whiteboard, or overhead projector
Writing		Addition and/or removal of work in private view, such as on a student’s worksheet or notebook
Manipulating Tools		Using any sort of tool, including a calculator, working with manipulatives, etc.

Figure 6: Bodymarking Strand Colour-Coding and Descriptions

As described above, only one strand, Shared Gaze, requires further distinction. For that strand, we distinguish between two types of gazes, those shared by most or all of the individuals during an interval (dark brown) and those shared by only some individuals during an interval (light brown). It is worth noting an important limitation of the TIMSS video data: we are constrained by the view of the camera. To address this limitation when coding for this strand, we only consider individuals shown in the camera’s view in discerning gazes shared by most and some. If an interval depicted four individuals all gazing at a single object, it would be coded as Shared Gaze (Focused); if only two of the four individuals were gazing at the object, but the other two were each looking at something else, it would be coded as Shared Gaze (Diffused). The purpose of the Shared Gaze strand is to tell us something about how the collective gaze of the classroom is oriented: is the whole classroom oriented by a common project? Are small groups of individuals focussed on a multiplicity of objects? To best capture this ebb and flow, we colour code the distinct gazes along a single strand, Shared Gaze.

Applying the Bodymarking tool to both of the Japanese and US lessons yielded 332 and 357 unique observations, respectively, across the six strands (Figure 7).

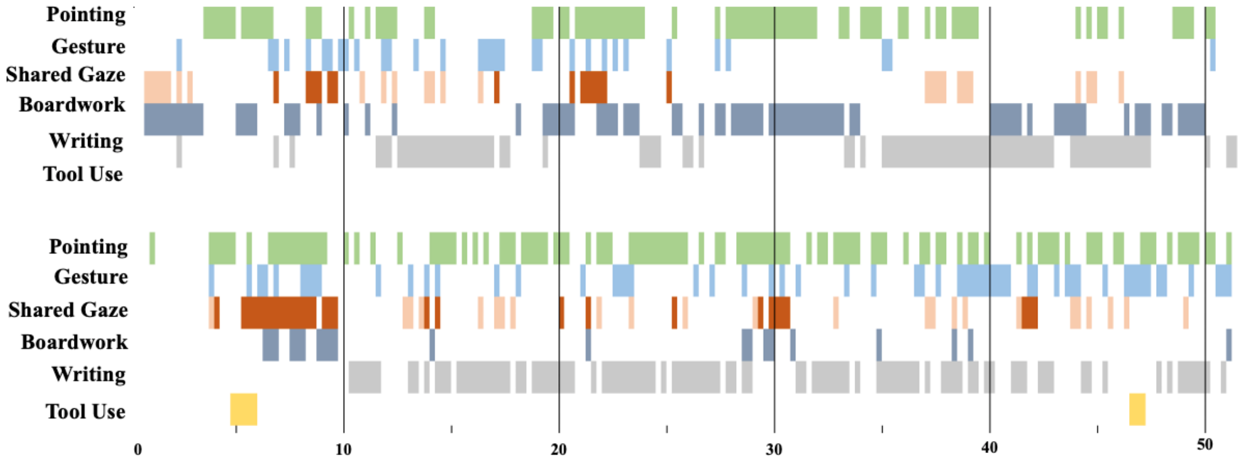

Figure 7: Bodymarking Visualizations for Japanese (Top) and US (Bottom) Lessons

DISCUSSION

The lesson storyboards shown in Figures 1 and 2 reveal the ubiquity of everyday actions, such as pointing and gazing in the classroom. Although we observe and record these individual gestures through the Bodymarking process, the name we have chosen also points to our interest in marking out the ephemeral body of the collective as it emerges through classroom action. For the two lessons in this analysis, we found our attention drawn to two phenomena. The first concerns the ways in which Shared Gaze couples with other actions in the classroom. We would expect Shared Gaze (Focused), shown in dark brown, to occur naturally alongside other strands, such as Boardwork, and this is evident in Figure 7. What stands out to us is the observation that Shared Gaze (Focused) is the only strand that did not frequently occur in the absence of the others. That is, Shared Gaze (Focused) is almost always coupled with at least one other strand. In fact, there is only one 15-second interval, early in the Japanese lesson, in which Shared Gaze (Focused) occurs in the absence of other strands. This leads us to question how moments in which there is a focused gaze shared by the classroom differ from those in which there are multiple objects of shared gaze or none at all. Moreover, we are interested in what those moments might tell us about how the actions captured by the other strands couple with each other and with gaze.

A second phenomenon of interest involves the potential for observing cadences of classroom action over the course of a lesson. Figure 8 highlights three intervals in the Japanese lesson in which Boardwork is prominently featured. Intervals A and B show Boardwork coupling with intransitive gestures, such as Gaze and Pointing, while interval C shows it coupling with a transitive gesture, Writing. The intervening periods in which Boardwork is absent depict unbroken blocks of writing.

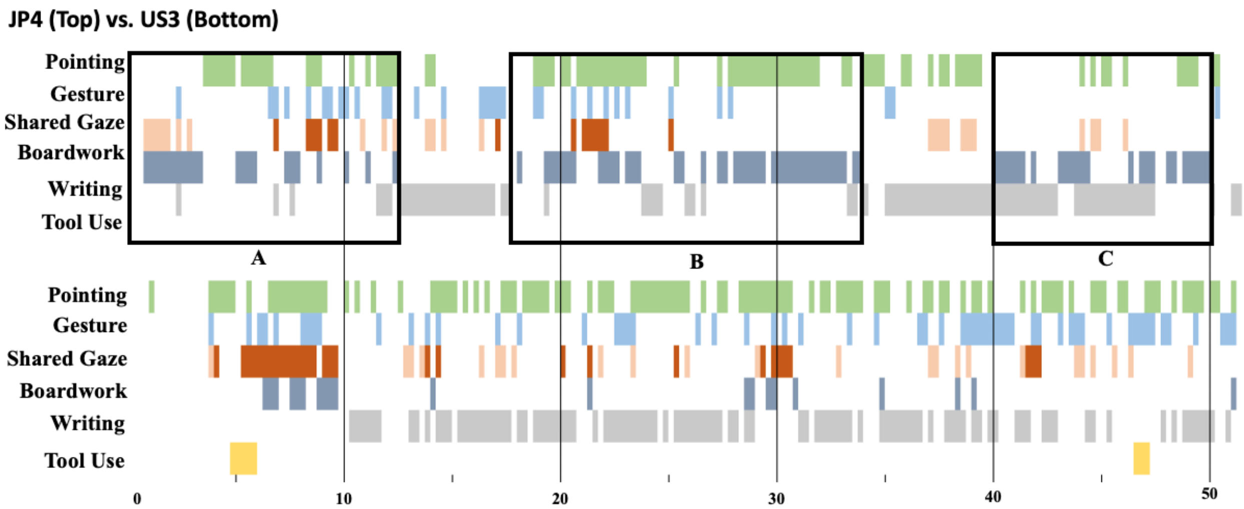

Figure 8: Intervals of Boardwork and Other Gestures

The cadence of the Japanese lesson contrasts with the US lesson, which depicts no discernable rhythm. To be sure, this is a function of classroom pedagogy—the Japanese lesson, for example, alternates introducing new content with practice, while the US lesson involves small group work on a problem set for most of the period—but we argue the Bodymarking visualizations may yield additional insight when applied to a larger set of classroom data. How might similar pedagogies manifest in different settings? What could variations within a lesson, as depicted in intervals A, B, and C, tell us about the way collective action emerges in the classroom? And how are they reflected in other vital signs?

CONCLUDING REMARKS

Bodymarking offers a means of visualizing everyday classroom action. By focusing on both transitive and intransitive gestures, such as writing and pointing, respectively, we argue it has the potential to provide insight into the collective engagement of the material world. Moreover, in attending to how these gestures couple with one another, and how those couplings are reflected in other vital signs, Bodymarking may provide insight into how collective knowing and doing emerges in the mathematics classroom.

PERSISTENCE AND MOVEMENT OF IDEAS

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CONTEXT

In 1989 Pirie and Kieren introduced a model (Figure 9) of a dynamical theory for the growth of mathematical understanding. The authors characterized mathematical understanding as an embodied process that was inherently dynamic, levelled but non-linear and recursive (Pirie & Kieren, 1994). The model featured eight nested yet unbounded levels: Primitive Knowing as "the starting place for the growth of any particular mathematical understanding" (p. 170); Image Making as the activity by which to "make distinctions in previous knowing and use it in new ways" (p. 170); Image Having as the "use [of] a mental construct about a topic without having to do the particular activities which brought it about" (p. 170); Property Noticing as the action by which to "manipulate or combine aspects of images to construct context specific relevant properties" (p. 170); Formalising as activity which "abstracts a method or common quality from the previous image dependent know how which characterised noticed properties" (p. 170); Observing as "reflect[ing] on and coordinat[ing] formal activity and express[ing] coordinations as theorems" (p. 171); Structuring as involving "formal observations as a theory" (p. 171); and Inventising which entails the "break[ing] away from preconceptions ... and creat[ing] new questions [that] might grow into a totally new concept" (p. 171).

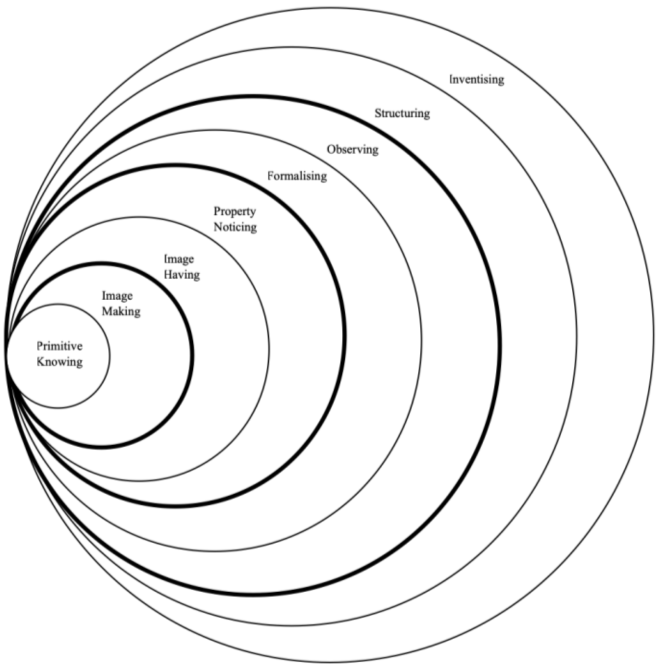


Figure 9: Pirie-Kieren Model

The nested structure of the model shown in Figure 9 reflects each level as including all inner levels as well as being integral to all outer levels.

To date, the Pirie-Kieren model/theory has predominantly been used to illuminate the understanding of individual students. In contrast, we use Pirie and Kieren’s model/theory to attend to the emergence and dynamics of ideas at the collective level, in mathematics classes, as suggested by Thom and Glanfield (2018); Kieren and Simmt (2002); Martin and Towers (2003; 2015); Davis and Simmt (2003); and Pirie and Kieren (1994).

Using the JP 4 TIMMS lesson we first identified concepts and ideas within the lessons. We identified the level at which the ideas emerged, monitored the ideas as they were (re)iterated or elaborated upon, and tracked during each lesson as they moved back and forth across the different levels of the model. There were two ideas in the Japan lesson around the concept of inequality. Idea 1 (I1[JP]), the first idea to emerge, involved the procedure(s) used to solve an inequality. Idea 2 (I2[JP]), the second idea to emerge, related to how an inequality expression could be used to model a specific context.

DYNAMICS OF IDEAS TOOL

In the first iteration of the Dynamics of Ideas Tool we used the Pirie-Kieren theory to code the mathematical ideas within the lessons and map the emergence, (re)iteration(s), elaboration(s), and the dynamics, or movement back and forth, of those ideas onto the Pirie-Kieren model, according to the eight levels (as seen in Figure 10 which shows the mapping of the first 17 minutes of the Japan lesson). Five minutes 33 seconds of this period were not coded. Two minutes 57 seconds consisted of going over homework related to I1[JP]. I1[JP] emerged, and for the most part, stayed at the *Formalising* level. The balance of time, 8 minutes 30 seconds, was spent on I2[JP]. Interestingly, I2[JP] arose in manners that were not specific to any one level in the model but indeed, clearly beyond *Image Having*. To distinguish these events, we mapped the moments in which I2[JP] occurred *Beyond Image Having* as dotted spheres on the boundaries between levels. In addition to this, and unlike I1[JP], I2[JP] moved across levels, back and forth, from *Primitive Knowing* through to *Formalising*, and *Beyond Image Having*.

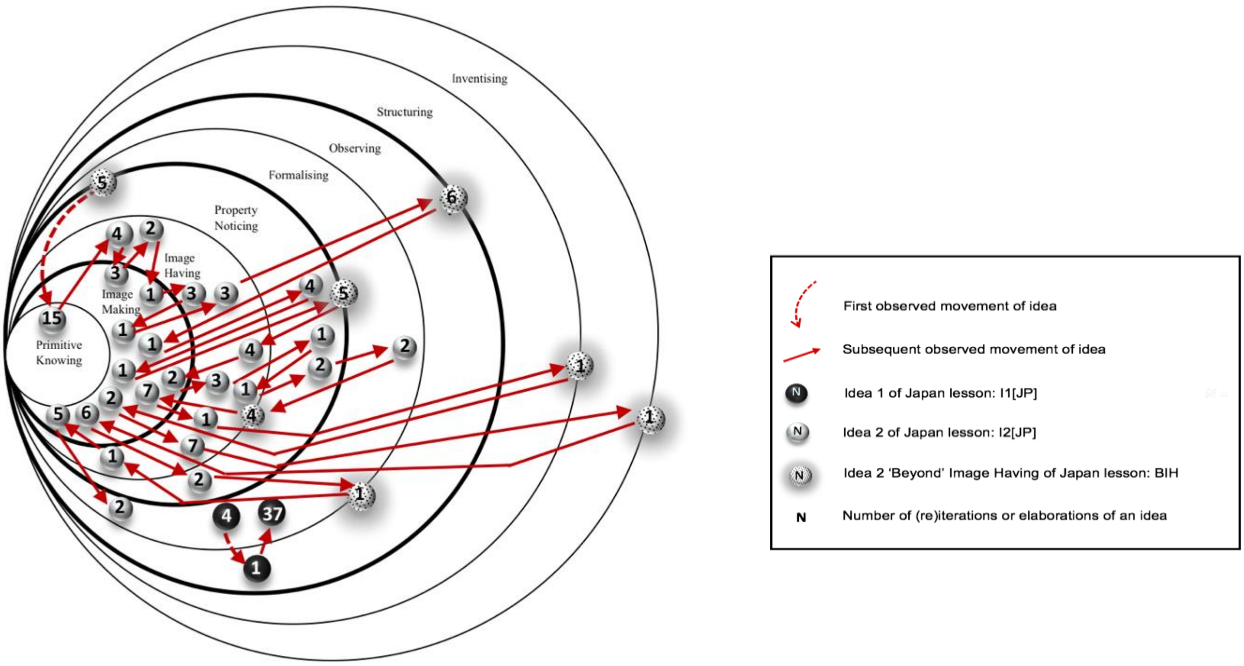


Figure 10: 0:00-16:59 of the Japan lesson

We encountered challenges in using this tool. First, the tool was not clear due to the sheer density of the ideas; that is, as they emerged and underwent (re)iterations, elaborations, and moved across the levels. Second, while the nested model allowed for chronological sequencing, it did not allow for mapping along linear time which meant that specific moments in time within any one lesson could not be compared with other tools being developed.

FROM ONE TO TWO TOOLS

In designing the second iteration of the tool, we separated the two dynamics: the persistence (i.e., the (re)iterations and elaborations) of the ideas and the movement (or lack thereof) of the ideas across the Pirie-Kieren levels in order to address the first challenge. For each of the dynamics, we then mapped them in 15-second increments to address the second challenge. The addition of the 15-second increments as a standard timeline allowed for identifying a specific moment in time and as well, comparison of any moment across tools.

PERSISTENCE OF IDEAS TOOL

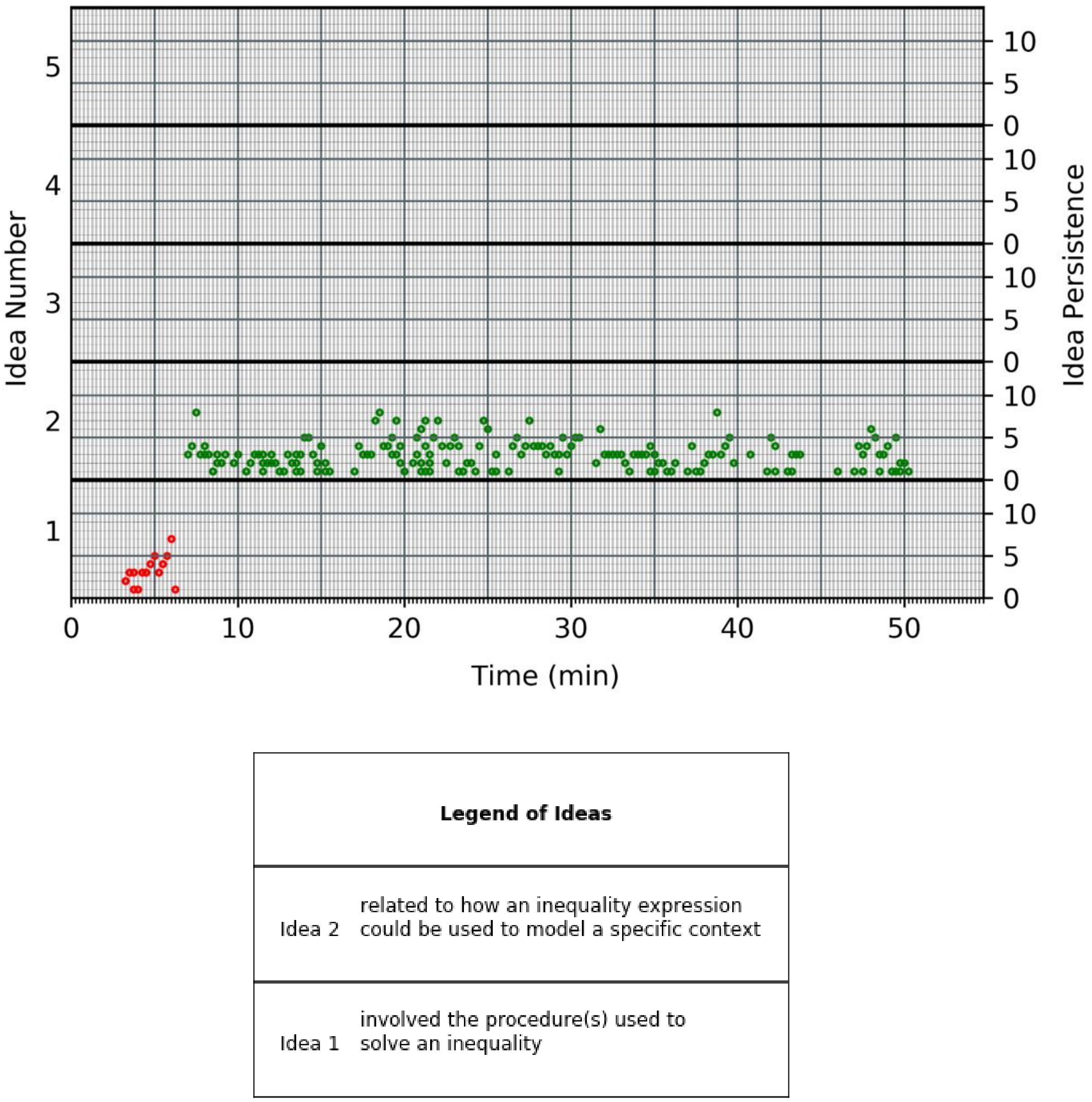
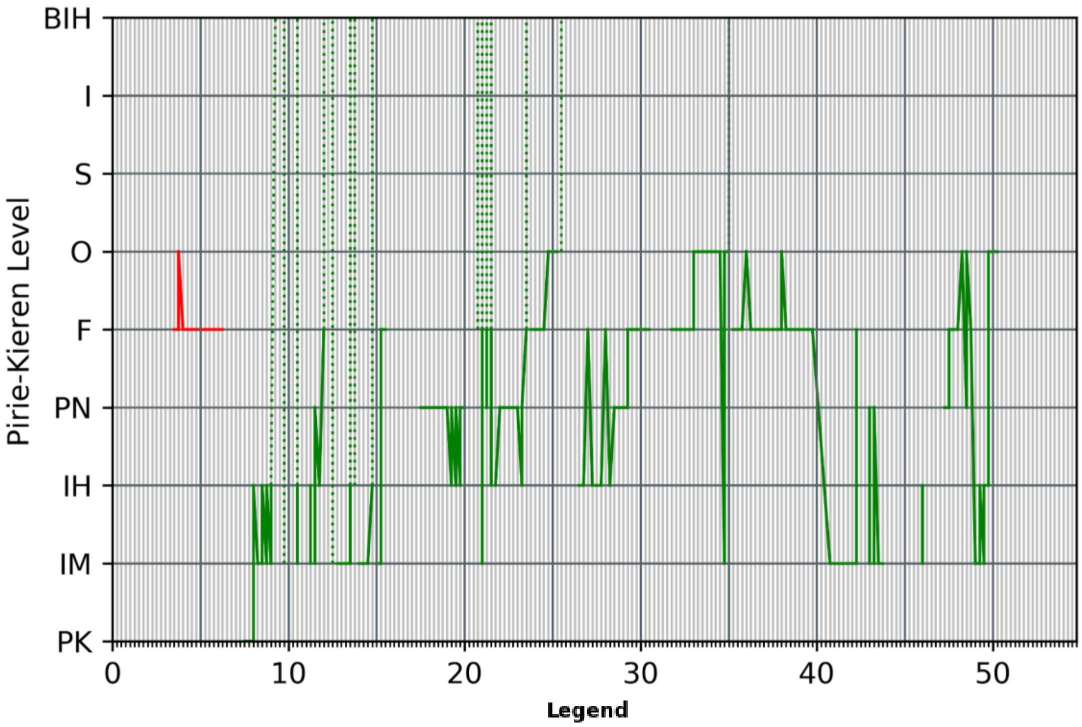


Figure 11: Persistence of Ideas for JP 4

Graphically, we can clearly see the persistence of Idea 1 and Idea 2 as well the difference of persistence between the two ideas across the whole period of time (see Figure 11). The ways in which the ideas persisted across the whole lesson could not be seen in the Dynamics of Ideas Tool as mapped on the Pirie-Kieren model. This new tool monitors the observed ideas as they emerge, are elaborate upon, and reiterated within the classroom as a collective.

MOVEMENT OF IDEAS TOOL



BIH	Beyond Image Having
Pirie Kieren Levels	
I	Inventising
S	Structuring
O	Observing
F	Formalising
PN	Property Noticing
IH	Image Having
IM	Image Making
PK	Primitive Knowing

Figure 12: Movement of Ideas for JP4

The movement (or lack thereof) of ideas can be observed across the Pirie-Kieren levels and across the whole period of lesson time (see Figure 12). This could not be seen in the Dynamics of Ideas Tool as mapped on the Pirie-Kieren model. Neither were instances observed as not mathematical located on the model. The breaks in the graph are periods of time that could not be coded for a variety of reasons (e.g., no mathematical ideas were expressed for approximately the first 3 minutes of the lesson, and no mathematical ideas emerged at approximately the 7 minute and 15-17 minutes marks of the lesson). I1[JP] emerged, and for the most part, stayed at the Formalising level. Interestingly, I2[JP] arose in manners that were not specific to any one level in the model but indeed, clearly beyond Image Having. To distinguish these events, we mapped the moments in which I2[JP] occurred Beyond Image Having using dotted lines. Unlike I1[JP], I2[JP] moved across levels, moved back and forth, from Primitive Knowing through to Formalising and Observing, and Beyond Image Having throughout the lesson. This tool monitors how the ideas moved within the Pirie-Kieren levels as they are taken up within the classroom as a collective.

VERTICALLY ALIGNING THE TWO TOOLS

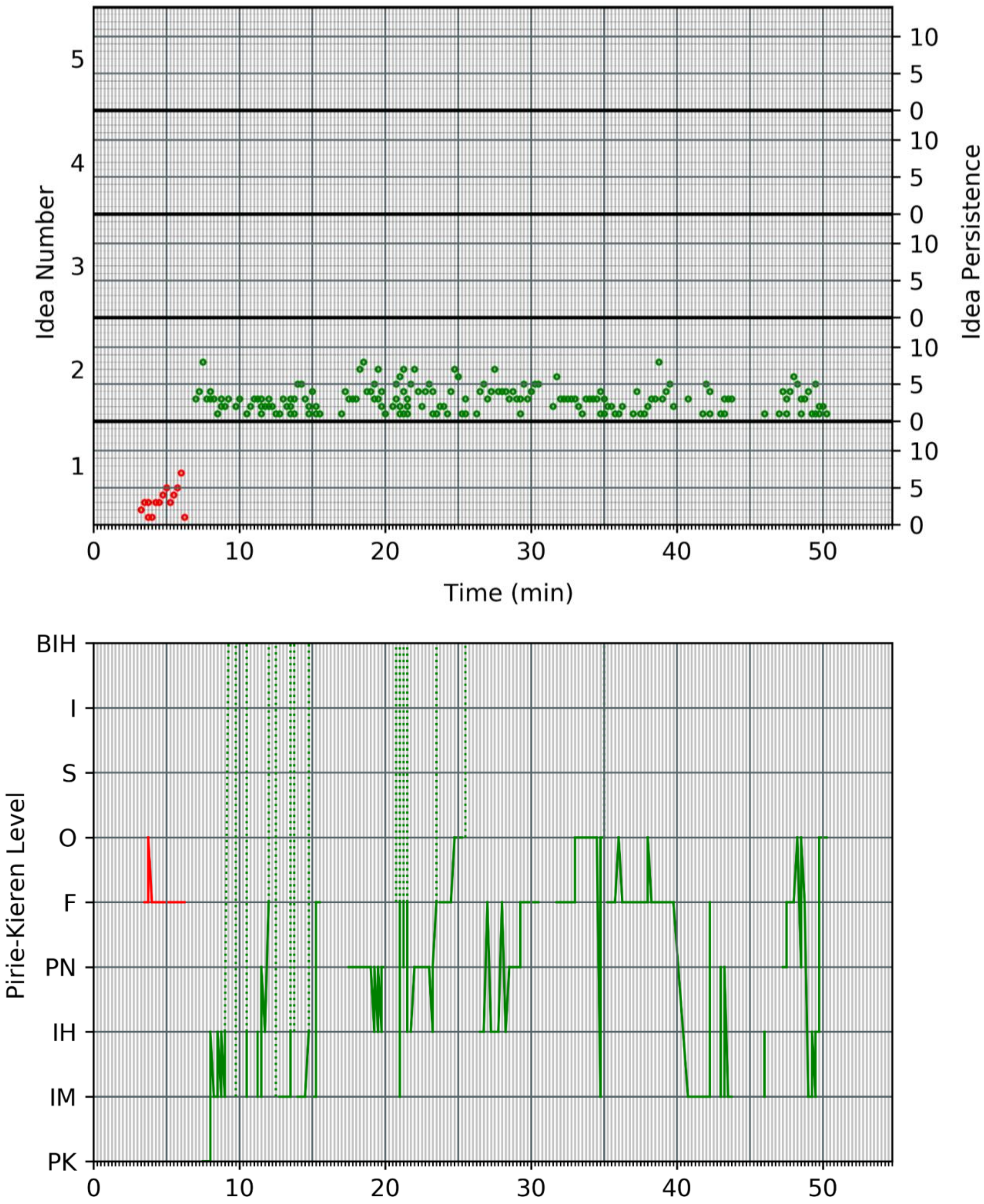


Figure 13: Aligning the Two Graphs

When the two tools are aligned vertically, we can see at any moment in time, which idea is being taken up in the collective, the persistence of that idea, and the Pirie-Kieren level at which the ‘taken up’ occurs (see Figure 13). So, for example, between the 17 and 20-minute period of the lesson, Idea 2 was elaborated upon or reiterated 1 to 8 times at any given moment within the collective. Within this time period, the idea moved back and forth between Property Noticing and Image Having. In contrast, during the 40- to 45-minute time period in the lesson, Idea 2 can be observed as persisting between 1 to 5 times while moving back and forth from Formalising to Image Making to Property Noticing then back to Image Making. These two examples could not be seen in the Dynamics of Ideas Tool. Further still, at approximately the 13 to 15-minute time period, the persistence of Idea 2 also occurs 1 to 5 times, however, the idea moves between Property Noticing, Image Having, and Beyond Image Having.

CONCLUDING THOUGHTS

The first iteration of the Dynamics of Ideas Tool attempted to observe two dynamics at the same time. The second iteration involved the separation of the two dynamics into two distinct Tools. The two tools offer a clearer way to monitor the persistence and movement of mathematical ideas within the classroom as a collective.

LAYERING THE TOOLS

We offer Figure 14 as an initial layering of three tools for the Japan lesson (JP4): Lesson Activity Mapping, Bodymarking, and Movement of Ideas. By aligning the tools vertically we may begin to see visual patterns across the three visualizations. As we might expect, the lesson pattern has many features in common with the Bodymarking. For example, boardwork and pointing occur predominantly in the public activities of presenting and sharing, while the writing occurs during private engagement times. These are also the time segments when we see more movement in ideas to the different levels of the Pirie-Kieren model.

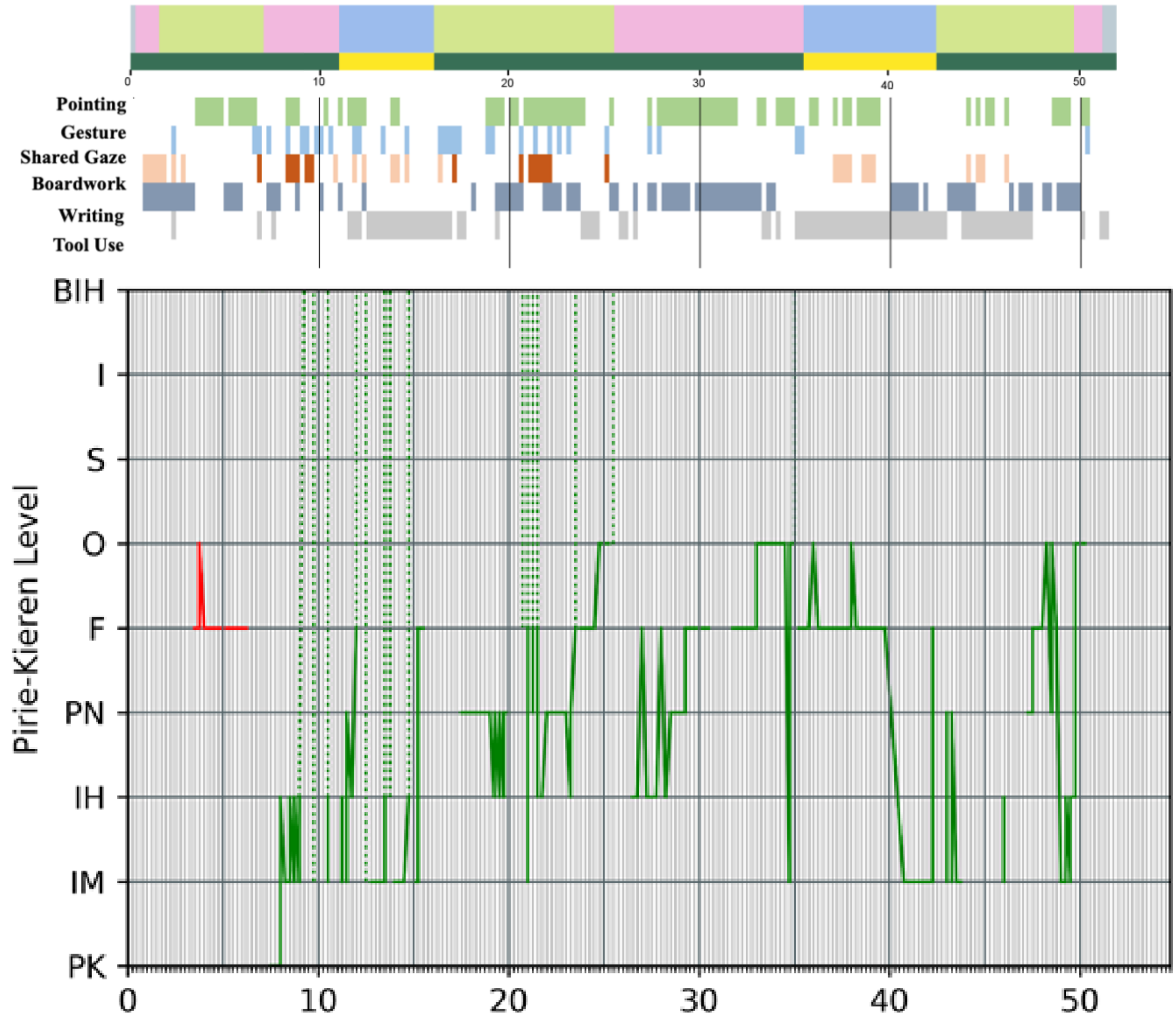


Figure 14: Layering Tools for Collective Activity

We believe that it is by layering multiple tools that we may be able to notice possible moments of interest, emergence, activity, and inactivity within a classroom. It is by exploring different modelling techniques of different aspects of collective activity that we can gain insight into global traits and group activity of collective systems.

Acknowledgement

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WORKING GROUPS

THE CHALLENGE OF TEACHING MATHEMATICS "AT THE FRONTIER"

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The metaphor of frontier inspires this WG's topic. Over the past 30 years, research considered at the "frontiers" of mathematics education included a focus on minority/underrepresented populations (e.g., Butler-Barnes, Cheeks, Barnes, & Ibrahim, 2021); technology and digital environments, especially in the context of the developing world (e.g., Srinivas, Bose & Kumar, 2019); and the theories, methodologies, and tools that drive and support such research (Sriraman & English, 2010). More recent "frontiers" include teaching at distance, especially in the context of the Covid-19 pandemic (e.g., Brunetto, Bernardi, Andrà & Liljedahl, 2021), and a general focus on meeting the mathematics and socioemotional needs of every student, teacher, and family (Andrà & Bernardi, 2020; Courtney, Austin & Glasener, in press). Moreover, the metaphor is not new to the PME community, as the theme of PME38 in Vancouver was: "Mathematics Education at the Edge", relating to cutting-edge research as well as to issues with groups that are often positioned at the edge or periphery of educational research such as social justice, peace education, equity, and Indigenous education. In this WG, we aim at reflecting on this multi-faced understanding of "frontiers" and to offer new theoretical and operational ways of dealing with frontiers, from a mathematics teacher perspective.

The goals of this working group will be to: 1) build a shared definition of what it means to be "at the frontier" in mathematics education and identify several current and emerging frontiers ripe for examination; 2) discuss the theoretical/methodological frameworks used by contributing researchers and other working group members to examine teachers' challenges at these frontiers; and 3) establish a network of researchers interested in doing research in and developing/adapting existing frameworks for such contexts.

The first 90-minute slot (Slot 1) will be dedicated to examining how research about "frontiers" emerge in literature, as well as to extend it through three episodes taken from WG leaders' previous research (Andrà & Bernardi, 2020; Brunetto et al., 2021; Courtney et al., in press). Slot 1 unfolds as follows: i) [10 mins] Present ways in which previous PME WGs and RRs addressed the metaphor of frontiers; ii) [15 mins] Introduce three examples of possible "frontiers"; iii) [30 mins] Small group discussions to identify other potential examples taken from WG participants' research, with the specific task of accounting for why each example can be considered as being at the frontier; iv) [30 mins] Share out with the entire WG, each small groups' discussions,

2022. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 205-206). PME.

examples and characterizations of "frontiers in Mathematics Education"; and v) [5 min] Conclude with a tentative definition of frontiers in Mathematics Education.
Slot 2 is dedicated to theoretical approaches that support examination of the frontiers in mathematics education, in particular for mathematics teachers who live with them. Slot 2 unfolds as follows: i) [10 mins] Summarise the discussions from Slot 1, as well as provide a definition of frontiers constructed by the WG leaders from the work of Slot 1; ii) [10 mins] Recall the three examples presented in Slot 1 with a focus on theoretical approaches used to analyse them; iii) [30 mins] Invite WG participants to share, in small groups, the theoretical approaches used to analysed the examples provided by them in Slot 1; iv) [30 mins] Share with the entire WG, the theoretical approaches used by participants in each small group, and discuss potential integration of theories; v) [10 mins] Propose to establish a network of researchers willing to continue the work of the WG over the next year.

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(RE)CONCEPTUALISING THE EXPERTISE OF THE MATHEMATICS TEACHER EDUCATOR

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Furthering discussions emergent from working groups of the same topic at both PME43 and PME44 (Helliwell & Chorney, 2019; 2021) and building on PME working groups of the past (e.g., Goos et al., 2011), we continue to explore (re)conceptualisations of expertise of the mathematics teacher educator (MTE) that look beyond the boundaries of the individual to material and social elements of constitution and constraint. Currently, several descriptions of MTE expertise exist that make use of and extend descriptions of mathematics teacher knowledge. For instance, Chick and Beswick (2018, pp. 479-482) present a framework of 22 categories of Pedagogical Content Knowledge (PCK) for school mathematics teachers (which they label SMTPCK), each mapping to a corresponding category of PCK for mathematics teacher educators (which they label MTEPCK). In fact, category-based descriptions of mathematics teacher and mathematics teacher educator knowledge proliferate the literature on the subject. Chapman (2021), however, suggests that category-based perspectives on MTE knowledge can provide a simplistic view of what it is and that "research needs to give attention to other ways of representing it as a complex system or way of thinking" (p. 412). The aim of the present working group is to generate alternatives to category-based perspectives of MTE expertise that capture its complex nature. One suggestion is to frame MTE expertise by turning our gaze outward, by drawing on Hutchins' (1995) model of "distributed cognition" as a balance between knowledge and external agencies. Of particular interest is to explore and develop potential methodologies and methods that support these distributed frameworks.

At both PME43 and PME44, we established a foundation of inquiries and themes towards perspectives of non-centralisation that drew on notions of distributed cognition (Hutchins, 1995). From PME44, emergent issues included: 1) Ways of differentiating who and where MTEs are (e.g., university-based MTEs or facilitators of professional development); 2) What MTEs attend to in the moment of teaching mathematics teachers; and 3) The meaning of content in mathematics teacher education (e.g., ways of describing mathematics education). In terms of the present working group, we intend the subgroups formed to continue their conversations and develop ideas further as well as welcoming new participants.

AIMS OF WORKING GROUP

- To summarise some of the interests and questions from the participating community from the two previous working groups to lay foundations for further refinement and development in thinking about and researching MTE expertise.

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- To explore and develop research questions and potential methodologies that support researching these various interests and questions.

OUTLINE OF SESSIONS

Session 1

- Introductions and summary of previous discussions on MTE expertise. The presenters will share some personal experience of expertise that emerged from distributed activity. The presenters will engage in a method of reading each other's experience of expertise through a distributed lens as a potential model for group activities in session 1.
- Participants share in groups their experiences of MTE expertise discuss with others possible interpretations of these experiences.
- Whole group discussion with a focus on interpretations and what forms of distribution emerge. Themes will be noted for session 2.

Session 2

- Building off session 1, groups will be organised by interest, according to discussions in session 1. Groups will develop their own questions, but the leaders will provide prompts to support engaging with questions from a distributed approach.
- Each group will share responses and then discuss on next steps for future collaborations, including consideration of a joint output for participants such as a special issue for the *Journal of Mathematics Teacher Education*.

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AN EMBODIED PERSPECTIVE ON DIVERSITY IN MATHEMATICS EDUCATION

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The topic of diversity and the need to account for various developmental and cultural paths increasingly gains attention within mathematics education. This growing concern in different learning conditions arising from aspects like language, culture, (dis)ability has been reflected in the recent past in their repeated centralization in conference panels (e.g., Prediger on multilingualism at CERME-12; a panel discussion on the consequences of the Covid pandemic for equity in classrooms at ICME-14; Wagner's call for diversification of mathematics education at PME-44). An ESM special issue in progress has been dedicated to the role of racism in math education, following an editorial in response to the Black Lives Matters protests in 2020 (Wagner et al., 2020). At the same time, embodiment as concerning the role of lived bodily experiences and embodied interactions, including gestures, motor coordinations, eye movements, full body actions, for understanding mathematics has been acknowledged with growing interest within mathematics education (e.g., Abrahamson et al., 2015; Núñez et al., 1999; Shvarts et al., 2021). Following this, it does indeed matter how learners' bodies occupy and act in space we live in (Sinclair & de Freitas, 2019), with our physical and cultural profiles influencing our mathematical thinking and learning. The embodiment lens hence allows for a perspective on diversity that emanates from these conditions as central to knowing and understanding and hence, to mathematical education.

The proposed working group seeks to extend and widen the exploration of the relationships between embodiment and diversity to understand better the challenges and opportunities of diverse populations through the lens of embodiment to be better able to respond to them. It builds on, synthesizes and extends the work of past PME discussion and working groups on embodiment (e.g., 2012, 2017, 2020), inclusion (e.g., 2018), and marginalization (e.g., 2015) in mathematics thinking and learning.

Main topics, guiding questions and objectives of the WG

We propose two main topics that will guide the work in this group: ***The first one*** concerns the *diversity of bodies* and the influence of learners' *sensory-motor profiles* on learning mathematics. Questions of interest are, for example, related to mathematical epistemology, that is, in how far mathematical cognition and grounded mathematical concepts might differ for people with different bodily configurations and sensory profiles (Krause, 2017). Related to this, we might explore how these differences can shape our approaches for designing the variety of bodies and lived experiences. ***The second topic*** concerns what we call the *diversity of voices* (e.g., genders, ethnicities), captured by the notion of *social-cultural profiles*. Here we

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wonder, beside others, how similar embodied experiences are expressed differently depending on belonging to minority/majority groups and what kind of instructional support might enable learners from various populations to express their experiences in a mathematical conversation. With this, the ***main objectives*** of this working group are (i) to engage the discourse about the role of the body in diversity and disability with respect to mathematics thinking, learning, and instruction, (ii) to raise key questions for future research and praxis, and (iii) to preparing the basis for a colloquium for the next PME conference.

Activities and structure

Session 1: The first session starts with a short introduction of the organizers, the participants, and the objectives of the WG (10 min), followed by a brief kick-off presentation on general ideas of embodiment and diversity (5 min). We will then explore the diversity of the participants' perspective, experiences, and interests in the topic (20 min) to work in small groups on different aspects of diversity and their relationships to embodiment to gather research questions (30 min). The first session will close with a plenary discussion on the results of the small groups' work (25 min).

Session 2: After a first revision of the first session (7 min), we will give short presentations to introduce the ideas of social-cultural and sensory-motor profiles with respect to embodiment in mathematical thinking and learning (20 min). This is followed by a video-based group work on the influence of social-cultural profiles and sensory-motor profiles on mathematics teaching and learning (33 min). We will then wrap up by discussing and summarizing key topics and questions evolved (10 min) and conclude on next steps and potential future collaborations (10 min).

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INTERNATIONAL PERSPECTIVES ON PROOF AND PROVING: RECENT RESULTS AND FUTURE DIRECTIONS

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This working group is a continuation of the working group on international perspectives on proof and proving at PME 43 (Reid et al., 2019a, 2019b). The group was not able to meet in 2020 and 2021 due to the pandemic. The aim is to foster research on proof and proving from an international perspective by bringing together research on proof and proving and international comparison. The long-term goal is to present the results of a number of comparative international researches on proof and proving in a PME Research Forum.

The past two decades have seen a strong increase in research into proof and proving in mathematics education. Much of this has been conducted in single national and cultural contexts, although there have been, and continue to be, a few comparisons that have compared proof in a few contexts. For example, since the studies referenced by Reid et al. (2019a), there has been a comparison of word use in curricula and standards in Norway, the USA and Germany (Reid, 2022), a proposed framework from a cultural perspective (Miyakawa & Shinno, 2021), a study of proof-related reasoning in upper secondary school textbooks in Sweden and Finland (Bergwall, 2021), and a study of Estonian and Finnish students' views about proof (Viholainen et al., 2018). This slowly growing research base on proof and proving from an international perspective is much needed as it remains unclear whether existing research results from single national and cultural contexts are transferable, or, indeed, if the assumptions on which the studies are based are valid elsewhere. Notwithstanding the small amount of existing comparative research on different aspects of proof and proving, comparatively little information exists about the role of proof and proving in educational contexts from an international perspective. Additional international comparisons involving a wider range of countries could shed light on the *teaching and learning of proof and proving* in areas such as curriculum (including textbooks and other teaching and learning materials); student learning and achievements; teaching (including teaching practices, teachers' knowledge, and teacher education or professional development of teachers); and assessment.

At PME 43 subgroups were formed. These were selected based on the interests of those present, and include some overlap. Three groups formed around issues specific to education levels, and three others around topics across education levels. The topics chosen were:

- Pre-Primary and Primary Argumentation and Proof

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- Secondary Level Argumentation and Proof
- University Level Proof Teaching and Learning
- How are Argumentation and Proof conceptualised internationally?
- Proof in the Primary & Secondary school Curriculum
- Visualisation and Proving

Each group identified research questions and continued sharing information over the year following. As part of the PME 45 Working group sessions there will be brief reports from some of these subgroups. Some groups will not be present to report on their progress but the organisers will be able to summarise their activities. All groups are very much welcoming new participants.

STRUCTURE OF THE SESSIONS

Involvement of participants from different countries is essential to the functioning of the group. Over the two sessions the following activities are planned:

- Introduction to the working group, it aims, goal, and history.
- Reports from participants in the subgroups formed at PME 43, and the questions they have considered and steps they have taken.
- Proposals of new themes based on the interests of new participants in the working group.
- Discussion in subgroups, including possible theoretical and methodological approaches.
- Whole group discussion of ways to expand collaborations among researchers.

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CONSTRUCTS AND METHODS FOR IDENTIFYING PATTERNS OF INTERACTION IN MATHEMATICS CLASSROOMS

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In this working group, we use data from our investigation of storylines available to minoritised mathematics students. We invite participants to reflect on their own research projects that investigate patterns of interaction. Together we consider the constructs of storylines and other alternatives, the challenges of identifying them, and the insights that become available due to our methodological choices.

MOTIVATING CONTEXT

The context that motivated the questions we wish to bring to this working group is a participatory research project (MIM: Mathematics Education in Indigenous and Migrational contexts) that seeks to identify storylines impacting the experiences of minoritised mathematics students. Using the storylines these youths and the people around them use to make sense of their interactions, we will work together to develop new strengths-based pedagogies.

Our research raises many questions about storylines—particularly related to how to identify them. An underlying question is *why storylines*? Researchers use other concepts to describe patterns of interaction in mathematics classrooms. Thus we will use some stories and data from our project and also invite participants to draw on their own research experiences. Together, we will work at questions such as: (1) What is a storyline? How is it different from other similar constructs? (2) How do/might we identify storylines in a text/interaction? (3) What are the implications of our choice of construct and our way of identifying? What do our methodological choices foreground and how do the emergent insights inform mathematics classroom interaction?

STRUCTURE OF THE WORKING GROUP

We will begin the working group with MIM data: in particular, interviews with school leaders from Northern Norway where Indigenous groups (including Sami and Kven) have long histories and where many new migrants have arrived recently. Though the project focus is on the experiences of the minoritized youth, we note that the leaders of the schools in which these youth learn mathematics impact the students' experiences. To analyse these interviews we look at theorizations of positioning in which storylines are prominent (Davies & Harré, 1990), briefly described below. We will ask what storylines could be and we will compare to other constructs for describing interactions that people draw upon to make sense of action and speech. In particular, we know that

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some of these identified storylines could be otherwise described as discourses (e.g., Foucault, 1982), figured worlds (e.g., Holland et al., 1998), or themes.

STORYLINES

Storylines mediate and structure interactions. These phenomena that shape the way discourse happens are approached in various ways in scholarship. We choose the construct of storylines because it focuses on action and story and because the theorization honours negotiability. In other words, people in interaction can choose storylines that serve them and the others well in the interaction. Storylines are part of a core triad in the theorization of positioning theory (Davies & Harré, 1990). Scholarship in positioning theory usually focuses on positioning, with insufficient attention to storylines. This raises methodological questions about necessary and key elements of storylines and how they manifest in interaction. In our efforts to identify storylines, we have tried focusing on verbs and the way subjects are positioned with the verbs, focusing on personal pronouns, focusing on words that imply force and influence, focusing on words that identify emotions, and intuitive approaches to coding. We note that it is very different to identify storylines in a mathematics classroom compared to identifying classroom storylines in interviews talking about the classrooms (e.g. students experiencing storylines vs. teachers or principals talking about the storylines available to students for interpreting their classroom experiences).

When we begin to recognize storylines, we ask what characteristics are necessary in naming/identifying them. Should a storyline be a complete sentence with a subject and action/positioning, or is a theme word enough? (Some storylines in the literature have very brief descriptions.) Most importantly, for any of our work with storylines we notice ourselves feeling compelled to evaluate them. Which ones could potentially strengthen mintoritised students, and thus warrant promotion? Which ones should be resisted? With these questions, we notice the intersectionality of storylines as they conflict with and/or support each other. And we ask who has the right or responsibility to decide which storylines are most appropriate in mathematics classrooms with minoritised students. Finally, we ask how to promote, resist or develop new storylines.We will be interested to discuss with working group participants how they have analysed their data to characterise mathematics classroom patterns of interaction and the relational structures that emerge from and drive those interactions. We see that our questions about storylines are important for any of these similar constructs.

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CONNECTING EARLY MATHEMATICAL MODELING WITH CULTURALLY RESPONSIVE MATH TEACHING

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This working group will focus on exchanging research around early mathematical modeling with attention to culturally responsive mathematics teaching with pedagogical practices that support optimal participation for diverse student groups. At each session, we will have participants explore ways mathematical modeling affords opportunities to develop three domains: students' knowledge and identities, rigor and support, and power and participation through modeling experiences situated in real world issues in students' local communities.

The goal of the working group is to exchange research ideas and examine ways mathematical modeling (MM) can promote student engagement across multiple cultural and community contexts. MM is an iterative process of making assumptions, identifying variables, formulating a solution, interpreting the result, and validating the usefulness of the solution (Blum & Ferri, 2009). Different theoretical perspectives around the world focused on teaching and learning of MM at secondary and tertiary level have been well documented (Stillman, Blum, & Kaiser 2017). More recently, a handbook on early mathematical modeling (Suh et al., 2021) detailed the nature of MM in the early grades with tasks situated in local contexts and illustrated the emergent modeling competencies of elementary students. Our working group frames MM as a humanizing endeavor that authentically connects mathematics to the real world, starting with ill-defined, often messy community-based problems and providing opportunities for students to develop empathy and compassion toward other people, living things and our planet (Aguirre et al, 2019; Gutiérrez, 2018; Lee et al., 2021; Turner et al, 2022). Successful community-based MM requires a teaching approach that centers children's cultural funds of knowledge, honors diverse ways of thinking, empowers decision-making, addresses power dynamics and offers opportunities for children to take action that will help their communities. We will introduce a Culturally Responsive Mathematics Teaching framework that supports community-based math modeling (del Rosario Zavala & Aguirre, 2021).

Through this working group, we invite the international PME community to collaborate and build global perspectives on community-based MM and CRMT in the early grades (English, 2006; Kaiser & Sriraman, 2006). In the first WG session, participants will engage in MM with community-based tasks examining fairness, access, representation, and community uplift (i.e., water crisis, community gardens, access to diverse books) and examine how CRMT can be used as a transformative and

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humanizing experience for students through classroom videos and artifacts. In the second session, we will facilitate exploration on ways MM and CRMT enhances equity and empathy and will provide time for participants to get into smaller research groups so that they find synergistic research interests with others. Our working group brings an urgent perspective of bridging culturally responsive mathematics teaching with community-based mathematical modeling to foster innovative scholarship and meaningful learning in elementary-aged children across the globe.

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SEMINAR

CARTOONS IN MATHEMATICS EDUCATION RESEARCH, TEACHER PROFESSIONAL DEVELOPMENT, AND IN THE MATHEMATICS CLASSROOM

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Representations of profession-related requirement contexts (e.g. Buchbinder & Kuntze, 2018), such as classroom situations, have a great potential: they can function, for instance, as a starting point for pre-service teachers to analyse and reflect on particular classroom situations, or to develop ideas on how to provide help to students in a specific situation. Profession-related requirement contexts can be represented by means of different formats; frequently used formats are texts, videos, and cartoons. As discussed elsewhere (e.g. Friesen & Kuntze, 2016), different formats have individual advantages and disadvantages. The cartoon format, however, combines advantages and avoids many disadvantages of other formats: cartoons, for instance, can relatively easily be varied in systematic ways. Further, they allow to reduce the complexity of profession-related situations and to focus on particular aspects, while still providing a high level of authenticity (Friesen & Kuntze, 2016). Cartoons therefore open up a variety of possibilities for research in mathematics education as well as for teacher professional development: for instance, cartoons representing actions of teachers in specific situations (e.g. when a student makes a mistake) can be used to elicit and challenge beliefs of (pre-service) teachers (e.g. Skilling et al., 2021); a stimulus to reflect on alternatives; or to imagine responding to potential teaching situations.

Beyond this, cartoons can also provide various opportunities for students' learning in the mathematics classroom and for corresponding research: when students reflect on situations in which other learners are represented, for instance, cartoons can stimulate metacognitive reasoning (Mevarech, Verschaffel, & de Corte, 2018), e.g. with a focus on strategies for problem solving or on students' mathematical argumentation.

Goals and activities of the seminar

This seminar aims at providing practical insight into the various possibilities of the use of cartoons in learning opportunities and in mathematics education research, with a particular focus on new researchers and teacher educators. A key part of the seminar is the joint development and reflection of cartoons in small groups with the aim of learning how cartoons can easily be designed with the help of a digital tool (without specific prior knowledge requirements), how to use cartoons for a variety of purposes in different contexts, and how to deal with practical challenges. The activities of the

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seminar will, further, provide opportunities for exchange and networking between the participants regarding cartoon-based research and teaching related to different topics.

In particular, the activities of the seminar are structured as follows:

Session 1:

- Short introduction of a theoretical framework for the development and use of cartoons and concept cartoons in research, teacher professional development, and in the mathematics classroom (20min)
- Introduction of a digital tool for creating cartoons and the methodology of concept cartoons (20min)
- Development of cartoons and concept cartoons with the digital tool in small groups with a focus on multiple purposes, addressees, and topics; each group is individually supported by a member of the seminar team (40min)
- First reflection and exchange between the groups (10min)

Session 2:

- Continuing and completing collaborative group work on cartoons and concept cartoons, individually supported by the seminar team (45min)
- Presentation, review, and discussion of products developed in the small groups; outlook on further innovative cartoon-based approaches (45min)

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NATIONAL PRESENTATION

SPANISH RESEARCH ON MATHEMATICS EDUCATION

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INTRODUCTION

Didactics of mathematics as a research area

Mathematics education is a dynamic area of research in Spain. The Spanish Society of Research in Mathematics Education (SEIEM) now has around 240 members almost covering all Spanish universities. They work in different groups and use a variety of approaches to focus on the teaching and learning of mathematics in various contexts and at all educational levels, from pre-school to university. The development of this research community has taken place during the past 40 years, particularly during the last two decades.

An important date to trace this evolution is the Spanish law "for the university reform" of 1982, established during the first years of democracy after Franco's dictatorship (1939-1975). This law led to the recognition of Didáctica de las Matemáticas in 1984 as one of the "areas of knowledge" that structured university departments, which was a key factor in its consolidation as a scientific and academic discipline (Rico et al., 2002). Moreover, the law consolidated the integration of teacher education into universities, which provided institutional support to research in this area. The fact that teacher educators became members of university departments encouraged the production of doctoral theses and the constitution of research groups. Some educators already had a PhD in mathematics and reoriented their research in mathematics education. Others started their PhD abroad, mainly in France, but also in Italy, the UK or the US. In the late 1980s, PhD programmes in Didactics of Mathematics were initiated in the universities of Granada, Valencia and Autonomous of Barcelona, followed by other universities some years later. To run the programmes, universities established fruitful relationships with recognised international researchers and research institutions to offer doctoral courses and guidance with thesis supervision. The PhD candidates for these programmes were teacher educators, secondary school teachers, some recent graduate students in mathematics, and numerous students from Latin America who did not have the opportunity to carry out a PhD in mathematics education in their countries. The number of doctoral theses defended is a clear illustration of this short and intense evolution (Figure 1).

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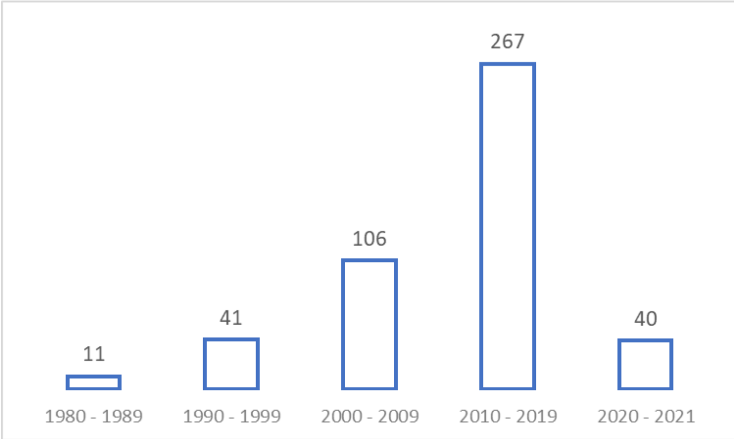

Figure 1: Number of doctoral theses in Didactics of Mathematics defended in Spain. (Source: our elaboration from https://dialnet.unirioja.es)

A terminological discussion about the denomination of the area also took place at that time: should it be Didáctica de las Matemáticas or Educación Matemática? The influence of the English-speaking communities enhanced the use of the latter, while the name traditionally used in teacher education and the continental European communities (France, Italy, Germany) pushed to maintain the former. Both expressions are used today in a quasi-interchangeable way, even though the Spanish society of researchers adopted “mathematics education” in contrast to the official denomination of the university knowledge area, which maintains the denomination of “didactics of mathematics”.

The Spanish Society of Research in Mathematics Education

Born in 1996, the Spanish Society of Research in Mathematics Education (SEIEM, www.seiem.es) reached almost 100 members from 31 different universities only one year later, and, by 2021, it had more than doubled its membership. Among its objectives, it aims at maintaining a space for conceptual and methodological debate about research on mathematics education, encouraging the constitution of research groups and their collaboration, promoting mathematics education in research institutions and educational agencies, helping disseminate research outputs, and fostering the cooperation and exchange between research and practice throughout the educational system and other societal contexts. The SEIEM integrates ten thematic groups on “Learning geometry”, “Teacher knowledge and professional development”, “Didactic of statistics, probability and combinatorics”, “Didactic of mathematics as a scientific discipline”, “Digital environments”, “Didactics of analysis”, “History of mathematics education”, “Early childhood education”, and “Numerical and algebraic thinking”.

During its 26 years of existence, SEIEM has been celebrating an annual symposium in different universities the only interruption being in 2020 due to the pandemic lockdown. The next one, the 25th symposium, will be held in Santiago de Compostela in September 2022. SEIEM symposia always include plenary sessions on specific research topics, the presentation of communications and posters, and time slots devoted to the thematic groups with more informal presentations and discussions. Together with the sister Portuguese Society, SPIEM, they organise a Summer School to contribute to young researchers' training, strengthen links between expert and junior researchers and foster cooperation between researchers from different universities in Spain and Portugal.

Finally, to "contribute to the advancement of knowledge of the processes involved in mathematics education and mathematics education research", the Society created a research journal in 2012, *Avances de Investigación en Educación Matemática* (AIEM, www.aiem.es) publishing two issues per year. It accepts papers in Spanish, English, Portuguese and French, and appears indexed in SJR and ESCI databases.

Since 2004, SEIEM is part of the Spanish Committee for Mathematics (CEMat), a re-structuration and extension of the Spanish IMU Committee, which integrates the different societies related to mathematics: the Royal Spanish Society for Mathematics, the Catalan Mathematical Society, the Spanish Society for Applied Mathematics, the Spanish Federation of Associations of Mathematics Teachers, the Spanish Society of History of Science and Technology, and the SEIEM. At the SEIEM's creation in 1996, the mathematician M. de Guzmán, who was the President of the International Commission on Mathematical instruction (ICMI) at that time, emphasised the need for the Society to channel its activity with a vision of integration. CEMat appears from this perspective as the common forum for all societies dealing with mathematics. It enables the collaboration between researchers in mathematics, researchers in mathematics education, teacher educators and mathematics teachers of all educational levels. The time has maybe come to open up to other Spanish educational societies and to establish links with other analogous associations in Europe and beyond.

International collaborations

As mentioned above, many of the research groups that started developing investigations in didactics of mathematics counted on the cooperation of international researchers who came to Spain to give research courses and seminars, or to host novice researchers in their universities. During this same period, many students from Latin America came to Spain for their doctoral studies in the newly created programmes. This situation helped maintain international collaborations and affiliations. Therefore, it is not unusual to find Spanish researchers in almost all international European and Latin American research organisations, and elsewhere in the world.

Spanish researchers have been involved in the European Society for Research in Mathematics Education (ERME) since its constitution in 1998, and regularly participate in its congresses (CERMEs). The Spanish contribution to CERMEs is notable, considering there has almost always been a Spanish researcher in the IPCs. Furthermore, the fourth congress was held in Girona, Spain, in 2005, chaired by M. Bosch, many Thematic Working Groups have had Spanish leaders and co-leaders

(Table 1), and there has been some participation in plenary activities, such as C. Batanero's plenary talk in CERME9.

Adult mathematics education
Affect and mathematical thinking
Algebraic thinking
Applications and modelling
Different theoretical perspectives in research in mathematics education
Geometrical thinking
History in mathematics education
Mathematical curriculum and practice
Mathematics and language
Mathematics education in multicultural settings
Teacher education
Stochastic thinking
University mathematics education

Table 1. CERME Thematic Working Groups with the participation of Spanish researchers as leaders or co-leaders

Since 2016, ERME has been promoting Topic Conferences (ETCs) organised on specific research themes by working groups associated with CERME conferences. Spanish contributions have been key in three ETCs. The first one is related to the International Conferences on the Anthropological Theory of the Didactic (CITAD), which has been held alternatively in Spain and France since 2005. The second ETC in which Spanish inputs are worthy of note is the International Network for Didactic Research in University Mathematics (INDRUM). Spanish researchers participated in its creation in 2015, collaborated in the organisation of the first INDRUM conference in 2016, co-chaired the third (2020) and fourth (2022) conferences, and possibly will organise the fifth one in 2024. The third ETC with Spanish researchers in its IPC corresponds to the ERME group about Language in the Mathematics Classroom, whose first conference was held in 2018.

Outside Europe, the most important Spanish collaborations are with Latin American research communities and organisations. Participation in the RELME conferences, CIAEM-IACME conferences (Inter-American Conference of Mathematics Education affiliated at ICMI) and engagement in the Latin American Committee of Educational Mathematics (CLAME) should be stressed.

Finally, broader conferences and organisations like PME, CIEAEM, or the ICMI itself, have always counted on Spanish participants assuming different responsibilities. Many are worth mentioning. First, the celebration of ICME8 (1996) in Seville and PME20 in Valencia the same year and the participation of M. de Guzmán (1991-2002), C. Batanero (2003-2006) and N. Planas (2021-2024) in the ICMI Executive Committee. Regarding ICMI thematic studies, C. Batanero, N. Planas and M. Bosch were IPC members of the 18th (2008), 21st (2014) and 24th (2018) ICMI Studies on Statistics, on Language Diversity, and on Curriculum Reform respectively. More recently, A. Gutiérrez has been designated as co-leader of ICMI Study 16 devoted to the learning and teaching of Geometry. Finally, T. Recio (2008-2016) was involved in The ICMI Klein Project and M. Bosch in the starting period of the ICMI AMOR project.

Determining the research areas

So far, we have discussed the emergence and development of research in didactics of mathematics in Spain from the perspective of researchers: their training, organisation and collaborations inside and outside the country. To present an overview of their productions, perspectives and work areas, we searched publications in some of the most relevant research journals released during the past 15 years, namely *Educational Studies in Mathematics*, *Enseñanza de las Ciencias*, *International Journal of Science and Mathematics Education*, and *Journal of Mathematics Teacher Education*. Contributions to past PME proceedings (2010-2021) were also included. This search led to identifying 13 work areas that were grouped into four sections, developed below.

The first and longest section presents research focusing on the learning of mathematical topics such as calculus, numerical and algebraic thinking, stochastic thinking, and geometry. The second section approaches language, discourse, and interaction, while the third one is related to teacher knowledge and professional development. The last and fourth area includes two theoretical approaches—the ATD and the OSA—that bring together researchers working on different problems from a common methodological perspective. Unfortunately, some topics remain outside the classification, like studies relating history to mathematics education, those approaching mathematics for students with special needs, the ones about attitudes and beliefs, or more global analyses about curricula and international comparisons. This a posteriori classification seeks to account for the diversity—and local coherence—of the investigations carried out by a community that integrates more than 200 researchers working from different theoretical perspectives and using different methodologies.

RESEARCH ON THE LEARNING OF MATHEMATICS

This section presents the Spanish research on teaching and learning mathematics that focuses on students' learning processes. It is divided into four parts, corresponding to the four mathematical areas whose learning processes have been most investigated in Spain over the past decades.

Research on learning calculus

Since 1997, when the research group for mathematical analysis education (GIDAM) of the SEIEM was created (www.seiem.es/grp/gidam.shtml), its members have been conducting research on the teaching and learning of calculus, in particular with ICT tools. Said research was carried out in secondary school and the first years of university education. Azcárate et al. (2015) present a review of the research produced by the various teams integrating the GIDAM. It involved different theoretical and conceptual frameworks, like Actions-Processes-Objects-Schemas (APOS), the onto-semiotic approach (OSA), and Advanced Mathematical Thinking (AMT), amongst others. The present summary provides a detailed review of the research performed by this research team. It is organised around the different topics approached: sequences and series, functions, limits, continuity, derivatives, integrals, ordinary differential equations (ODEs) and modelling.

Researchers from the University of Valladolid have conducted studies focusing on identifying and interpreting students' strategies to convert representations when working with real intervals (Pecharromán et al., 2019). Licera et al. (2019) used the framework of the Anthropological Theory of the Didactic (ATD) to study the problem of the uncertain status of real numbers in secondary education caused by the absence of an explicit approach for measuring magnitudes. Claros et al. (2016) used the phenomenology of Freudenthal, representation systems, and AMT to analyse the cognitive structure secondary school students develop when studying number sequences. Codes et al. (2013) studied university students' understanding of infinite series. As for functions, Ortega and Pecharromán (2010) used an instructional design to work on the properties of functions based on graphical representations, while Berciano et al. (2015) identified a significant learning improvement when working on the interpolation and extrapolation of functions graphically instead of algebraically.

The research group at the University of Granada has focused on aspects of the meaning of the concept of limit, considering its semantic and personal meaning, and as an element closely related to understanding. Fernández-Plaza et al. (2016), for instance, tried to characterise the meanings manifested at different levels of cognitive complexity, which allows integrating meanings expressed in definitions using the analysis of arguments.

Other researchers have stressed the importance of coordination in the processes in terms of the APOS theory, used to provide an understanding of limits (Valls et al. 2011), and have paid close attention to students' understanding of the derivative of a function (García et al., 2011; Orts et al., 2018; Fuentealba et al., 2019). The research groups at the universities of Alicante, Autonomous of Barcelona, and Seville analysed this topic in great depth, using the APOS theory. They tried to characterise different underlying structures of the derivative schema in terms of student ability to explicitly transfer the relationship between a function and its first derivative to the derivative function and the second derivative. This detailed analysis was performed both in secondary education and the first years of university education. Concerning differential calculus, Lucas et al. (2017) proposed a broader perspective to connect it with algebraic-functional modelling, using the ATD.

Ariza et al. (2015) characterised the development of engineering students' definition of the integral schema using fuzzy metrics to establish the level of development on intra-, inter- and trans- levels. Other topics that received attention involve the learning difficulties associated with integrals. Camacho et al. (2010) conducted a study to determine the difficulties students face in understanding the concept of the definite integral based on the study of areas of plane figures using symbolic calculus software. Boigues et al. (2010) presented an extensive study using fuzzy theory to analyse the validity of a scheme for the definite integral. González-Martín (2005) identified an epistemological obstacle combining two obstacles originated by the students' conviction that a finite (infinite) 3D figure has a finite (infinite) area or volume when learning the generalisation of the concept of integral (improper integral).

Researchers at the University of La Laguna have studied the derivative and its applications but in terms of the functions that are involved in solving ordinary differential equations and the modelling phenomena they organise. Camacho et al. (2012) studied the conceptual and cognitive processes that arise when introducing ODEs via problem-solving in a university freshman chemistry course, and Guerrero-Ortiz et al. (2016) analysed the difficulties students have interpreting ODE models that result when analysing growth phenomena using specific software designed for this purpose. From the ATD perspective, Barquero et al. (2019) focused on the modelling activity and the proposal of study and research paths considering an entire course of mathematics for engineers.

Research on numerical and algebraic thinking

In this summary, we address numerical thinking and algebraic thinking together, as they have elements and relationships in common. We highlight the main lines in which Spanish researchers have participated and which have given rise to relevant results and publications.

One of the research groups of the SEIEM is called Numerical and Algebraic Thinking. Its members' productions can be found in the proceedings of their annual meetings since 1997 (https://seiem.es/pub/actas).

A research group managed for decades by Luis Rico from the University of Granada has addressed topics in this research agenda (https://fqm193.ugr.es/). Different approaches (e.g. González-Marí et al., 2009) have been used in his publications. Enrique Castro led a research line on calculation and estimation (Segovia & Castro, 2009) and arithmetic problems (Castro & Frías, 2013). Other groups working on arithmetic problems are located at the universities of Extremadura (Gil et al., 2006) and Valencia (Gómez & Puig, 2014).

After Castro (1995), a research line emerged focusing on patterns and generalisation. It was developed through several research projects until 2013 (Cañadas et al., 2009; Molina et al., 2008). Since 2014, there have been three projects on algebraic thinking involving children aged 3-12 years (https://pensamientoalgebraico.es/en). These projects focus on different approaches to algebraic thinking (Ayala-Altamirano & Molina, 2020; Pinto et al., 2021; Ramírez et al., 2022; Torres et al., 2021), problem-posing in algebra (Cañadas et al., 2018; Fernández-Millán & Molina, 2016), and secondary school students' errors and difficulties with algebra (Castro et al., 2022; Molina et al., 2017). It is worth mentioning De Castro (2018) and Alsina (2016) with regard to patterns and algebraic aspects in early childhood education, involving children aged 3-6 years. Other results by these authors can be found on their ORCID pages (orcid.org/0000-0002-2246-5402; orcid.org/0000-0001-8506-1838). Moreover, at the University of La Laguna, there has been interesting research on students' errors and difficulties in algebra (Socas, 2007), and number sense (Almeida et al., 2014).

At the University of Valencia, there has been a tradition of research on historical studies, analysis of textbooks, and problem-solving, as, for example, Puig (2018) (orcid.org/0000-0002-7074-6110), or the books and book chapters by Gómez (e.g., Gómez, 2013) (www.uv.es/gomezb/Mispublicacionesdedominio publico.html).

As part of this Spanish agenda, some groups have focused on the learning processes of children with special needs, such as Down syndrome or autism, and the learning of arithmetic (Bruno & Noda, 2019; Polo-Blanco et al., 2021). They have also studied the learning processes of mathematically gifted students regarding pre-algebra (Gutiérrez et al., 2018). More publications about these topics can be found in: orcid.org/ 0000-0002-0154-8073, orcid.org/0000-0001-6425-6337, orcid.org/0000-0001-7187- 6788.

Research on students' stochastic thinking

Research on students' stochastic thinking was initiated in Spain in the 1980s at the University of Granada by a specific research group. It consists of seventeen researchers, and they have been involved in it all this time. They were pioneers in the field at a time when stochastics was not taught in primary school. Related research focused on the cognitive development of children and the reasoning biases in decision-making was carried out in the field of psychology. Other researchers became progressively involved in stochastic education at the universities of Cádiz, Jaén, La Laguna, Lleida, Valencia, and, more recently, Girona, Oviedo, the Basque Country, and Zaragoza. These teams have cooperated through the SEIEM and activities related to the International Association for Statistical Education, such as the IASE/ICMI Study (Batanero et al., 2011), in which an important group of Spanish participants took part. Their work on students' stochastic thinking is based on previous epistemological studies (e.g., Batanero, 2000) and textbook studies (e.g., Gea et al., 2015; Lonjedo et al., 2015; Serradó et al., 2005). Said studies identified new variables that had not been considered in psychological research and are relevant to the students' learning. For example, the results pointed to the multiple meanings of probability, which, until that time, had been limited to its traditional meaning in teaching and research, and opened up an unexplored research field related to the understanding of the frequentist and subjective meanings of probability. They also revealed the existence of semiotic conflicts in the interpretation of fundamental stochastic ideas. For instance, both textbooks and students mixed up the frequentist and Bayesian meanings of hypothesis testing, which is often interpreted deterministically (Batanero, 2000).

Part of the studies on reasoning and learning focus on university students, including several doctoral theses developed by statistics lecturers. This research helped to identify the main difficulties and errors related to topics such as association and correlation, hypothesis testing, normal distribution, confidence intervals, analysis of variance, random variables, and the central limit theorem. Teaching experiments based on the use of technology have also been designed and tested. For instance, Batanero et al. (1998) described the local, algebraic, unidirectional, and causal conceptions of statistical association, and showed the persistence of the latter after instruction. In another study, Batanero et al. (2004) reported the evolution of the personal meaning of normal distribution a group of students progressively acquired over several weeks of work using ICT tools.

Research on primary and secondary school students has considered the understanding of statistical graphs and tables, central position measures, probability, combinatorics, and sampling. Cañizares (1997) performed an extensive study with children aged 10-14 years, identifying the influence of age, combinatorial reasoning, and language comprehension in solving probability problems. Regarding combinatorics, the implicit combinatorial model was found to be a variable affecting strategies and errors in solving combinatorial problems (Batanero et al., 1997). Pallauta et al. (2021) pointed out the major difficulties of secondary school students in converting graphs into tables and interpreting tables. Batanero et al. (2020) demonstrated that, contrary to what was supposed in previous studies, students in compulsory secondary education understood the variability of small samples better than that of large samples. More recently, interest has grown in describing the emergence of intuitive stochastic ideas in early childhood and in offering activities to encourage children's statistical thinking (Rodríguez-Muñiz et al., 2021). One of the conclusions is that the first ideas about chance and probability occur much earlier than assumed in previous psychological research (Vásquez & Alsina, 2019).

Stochastics has also been reflected in teacher education research. For example, Alonso-Castaño et al. (2021) used problem-posing and problem-solving to analyse teachers' probabilistic knowledge; Berciano et al. (2021) connected Science, Technology, Engineering, Arts and Mathematics (STEAM) activities and early childhood prospective teachers' stochastic reasoning; González et al. (2011) summarised existing research on teachers' knowledge of statistical graphs, and Martins et al. (2012) and Serradó et al. (2006) analysed teachers' attitudes towards stochastics and its teaching.

In summary, Spanish research on stochastic thinking has been performed by an increasingly cohesive group of researchers for over 40 years across all educational levels and subjects.

Research on learning geometry

Research on teaching and learning geometry in Spain started with the study of the Van Hiele levels, characterisation of visualisation, and learning of proof. The Van Hiele model (Gutiérrez et al., 1991) has been the basis for several studies characterising students' reasoning in plane isometries, similarity, and organisation of 2D and 3D geometric objects. Guillén (2004) adapted the Van Hiele model to 3D geometry and presented learning activities that highlight the need to extend the characteristics of the Van Hiele levels to space geometry. More recent research extended the field of interest towards contexts of inclusion and mathematical talent (Aravena et al., 2016).

The processes students go through to articulate visualisation and geometric reasoning have been studied from primary school to university. Learning Trajectories for early childhood education on 2D and 3D geometry, patterns, and representations of itineraries have recently been studied. Researchers at the University of Alicante used configural reasoning to theoretically support that articulation (Clemente et al., 2015; Llinares et al., 2014, 2019) and Duval's theories to analyse grade 3 students' reasoning using polygons (Bernabeu et al., 2021). Both show the importance of visualisation for the development of students' classification and deduction processes in geometry. Results showed the influence of the figures provided in the problems and the subsequent modifications of those figures by the students for the development of their understanding of geometric concepts. This articulation has also been considered by the OSA (Godino et al., 2012) to advance in the establishment of skill levels in tasks requiring visual reasoning. It shows that there may be different cognitive configurations at each level, the levels depending on both conditions of the task and the visualisation skills required (Blanco et al., 2019). At the universities of Valencia and Granada, Gutiérrez et al. (2018) and Ramírez and Flores (2017) showed that visualisation is a cognitive process characterising mathematically gifted students. It is advanced for determining categories of the students' justification and proof abilities. Other publications about these topics can be found in: orcid.org/0000-0002-3292-6639, orcid.org/0000-0001-7187-6788, orcid.org/0000- 0002-8462-5897.

Technological environments and resources are facilitators for teaching and learning geometry. In the past decades, GeoGebra has been identified as a useful tool for proving propositions and exploring the understanding of geometric concepts and properties, such as properties of triangles, loci, and symmetry. In recent years, the development of students' argumentative skills in a GeoGebraTUTOR intelligent tutoring system environment has been explored. Paneque et al. (2017) showed that tutor-teacher-student interactions produced learning opportunities, inducing conjectures, and promoting the transition from empirical to deductive arguments. Automated reasoning tools of GeoGebra have also been explored from an educational perspective to contribute to increasing students' curiosity and critical spirit. Gómez-Chacón and Kuzniak (2015) studied how GeoGebra can influence students' geometric work. Their results showed a wide diversity of students' approaches because of variations in their interactions both with software and geometry. Other teaching experiments have been carried out with robots, 3D printers, and modelling, obtaining statistically significant improvements in computational thinking, as shown in Diago et al. (2021). The study of shapes in 3D geometry and their relationship with 2D geometry has been a major objective at the University of Almería, where the dynamic 3D geometry software NeoTrie VR of immersive virtual reality was developed (Rodríguez et al., 2021). Their results, together with those of researchers exploring augmented reality (Sua et al., 2021), seem to indicate substantial benefits compared to traditional methods.

Some researchers have focused on obstacles and difficulties encountered by students in learning geometry. González-Regaña et al. (2021) showed the conflicts pre-service teachers experienced in the transition from descriptions to formal definitions of polyhedra. Advanced aspects in vector geometry and its connection with algebra have been investigated (Borji et al., 2020), and teaching sequences have been proposed for the transition from the Euclidean to the Cartesian conceptions of a tangent. They aim to promote changes in the register of different coordinate systems and to encourage a dynamic and global conception of geometric loci (Gaita & Ortega, 2014). At lower educational levels, that connection has been investigated for the generalisation in geometric pattern sequences, including some results indicating that the difficulties in modifying the different interpretations and the lack of coordination between geometric and arithmetical structures could explain the difficulties students have with algebraic generalisation (Callejo et al., 2019).

Spanish researchers have also examined the importance of textbooks as a classroom resource, presenting a view of plane and space geometry, paying particular attention to compound proportion, lines and notable points of a triangle, and solids of revolution. Their results showed that there are few activities in textbooks aimed at explorations and formulation of conjectures and relationships. Recent research focuses on children's spatial orientation, measurement and visualisation in the study of areas of plane figures in primary school, translations at secondary school, and the transition from natural to axiomatic geometries at the university level. They also show that textbooks give preference to certain types of visualisations in area problems, thus preventing other ways of seeing (Marmolejo & González Astudillo, 2015).

RESEARCH ON LANGUAGE, DISCOURSE AND INTERACTION

In this section, we provide a brief overview of three lines of mathematics education research with emphasis on the following concepts: 1) language, 2) discourse, and 3) interaction. We argue that considered together, these three lines constitute a strong empirical context for the mathematics education research developed in Spain during the past two decades. This research views educational and professional practices as mediated by social aspects of teaching, learning, and thinking. However, they are differently rooted in sociolinguistics, cultural studies or psychology in conjunction with theories of mathematics education. A primary quality of mathematics education research and practice is thus *the social* (social mind/ practice/ context/ development/ work, in Planas & Valero, 2016), which cannot be dismissed without an important loss of meaning and coherence in knowledge construction. Another basic assumption is that language, discourse, and interaction are fluid, interrelated concepts. On the one hand, language cannot be studied in depth without looking into discourse, and, on the other, language itself is a process and product of social interaction. These three lines build up a common yet plural research agenda in which contexts of mathematics education are represented as linguistic, discursive and interactional social contexts.

Mathematics and language

Since the early work on classroom norms and language diversity in Gorgorió and Planas (2001), the line of classroom research on language and mathematics has documented challenges: for primary and secondary school learners of mathematics with migrant backgrounds and home languages other than the language of instruction; and for their teachers in trying to understand learners' thinking processes in the absence of full communication. Drawing on the integrated nature of the social, cultural, political and linguistic aspects of mathematics teaching and learning, language was initially characterised as a social tool in the processes of sharing meanings and values within the mathematics classroom. Since then, languages of doing mathematics within the classroom and deficit approaches in mathematics education research and practice have been questioned in ongoing projects.

Several empirical studies using bilingual and multilingual school lessons have reported how mathematical discourse practices of reasoning and explanation of content meaning are hindered by frequent shifts from one language to another to focus on linguistic accuracy in the language of instruction (Planas & Setati, 2009). These shifts in language are prompted by both teachers in whole-class discussions and peers in small group interactions. Findings have shown that access to and development of opportunities for mathematical learning in classrooms are mediated by distinct valorisations of the languages and mathematical meanings of some groups of students and the languages and mathematical meanings represented as appropriate in the school culture. Paying attention to lessons and their implications for pedagogies of mathematics instruction has motivated moves towards the study of mathematics teaching practices and the content of teacher talk in the classroom (Planas et al., 2018). A recent study identifies and interprets some of the language-based professional challenges of student teachers who are expected to support their pupils in processes of mathematics learning by exposing them to the teacher talk for understanding (Caro & Planas, 2021).

The development of this line of research has started to produce findings on teaching and learning useful mathematics in developmental work with secondary school teachers around mathematical-pedagogical aspects of the languages used when teaching content in the classroom. The study of the connection between content talk and mathematics teaching in the communication of mathematical meaning at the word and sentence levels, and in the enhancement of mathematical discourse practices have hence started to inform mathematics teacher education research and practice.

Mathematics and discourse

In this line of research, researchers have adopted the commognitive framework developed by Anna Sfard. It considers mathematics as a type of discourse including some special characteristics: word use, visual mediators, endorsed narratives, and routines. According to this framework, learning means changing one’s discourse to become a participant in another discourse (for instance, the discourse of mathematicians, or the discourse of mathematics teachers).

Several studies have employed this framework to better understand how students perform mathematical tasks related to the mathematical process of defining. Gavilán Izquierdo et al. (2014) studied the changes in students’ mathematical discourse when they described and defined mathematical objects (2D figures). They identified several types of situations depending on what changed (or did not change) in the students’ discourse. Sánchez and García (2014) showed that two types of discourses coexist in the colloquial mathematical discourse of pre-service primary teachers when they describe and define 2D objects: socio-mathematical discourse and mathematical discourse. Those two types of discourses have their own features and norms. Their coexistence leads to the appearance of commognitive conflicts, which are encounters between participants in the discourse who use mathematical words in different ways. The resolution of these conflicts is one of the main sources of learning.

More recently, there have been studies on how pre-service primary teachers describe and define 3D geometric solids. Fernández-León et al. (2021) studied the routines the participants performed, which served to shed light on how undergraduate students define. They found that some students did not have a clear idea of what a definition is and that they sometimes mixed describing and defining. They also reported differences between the discourse of students when defining 2D figures described in Gavilán Izquierdo et al. (2014) and the discourse of students when defining 3D figures. The study of the routines of the students allowed the authors to infer the existence of commognitive conflicts between the discourse of pre-service primary teachers and the discourse of mathematicians. One of the conflicts was similar to a commognitive conflict identified in Sánchez and García (2014) that appeared due to a conflict between socio-mathematical and mathematical norms. Since the participants in the studies of Fernández-León et al. (2021) and Sánchez and García (2014) were pre-service primary teachers, their results may have implications for teaching and learning 3D geometry in primary schools.

Toscano et al. (2019) investigated the discourse of pre-service primary teachers when they solved didactic-mathematical tasks instead of mathematical tasks. By didactic-mathematical tasks, they meant real activities proposed by teacher educators with the aim of “bringing future teachers closer to the reality of the professional activities of a primary teacher" (Toscano et al., 2019, p. 3). The authors identified two different discourses: one in which pre-service teachers act as students solving a classroom task, and another one that resembles the primary teachers' discourse. This distinction is important because, for students to become teachers, they should become participants of the community of practice of primary teachers and adopt their distinctive discourse. However, Toscano et al. (2019) reported that few pre-service primary teachers showed signs of adopting a discourse resembling the discourse of teachers, which "could be due to the difficulty that the pre-service teachers have in assuming the role of a teacher, despite this being what the tasks demanded" (p. 12).

Alongside studies focused on pre-service teachers, there have been some analyses of the discursive activity of in-service mathematics teachers. Gavilán-Izquierdo and Gallego-Sánchez (2021) investigated the discursive activity of an upper secondary school teacher when she introduced the derivative. They identified several types of visual mediators and routines, which served them to deduce what resources secondary school students will have in their transition to university discourse.

The study of discourse conducted in local research has opened a door to a better understanding of what happens in classroom situations of teaching and learning mathematics and of mathematics education in university training programmes.

Mathematics and interaction

This third line of research recognises that students' mathematical learning is in part developed in the classroom interaction between students and teacher, and amongst students in communicative practice throughout involvement in mathematical tasks. The social quality of mathematics teaching and learning is thus basically examined from the perspective of the interactional processes in the classroom.

Two main research directions have been developed to study student-teacher and student-student interaction. In the case of the interaction between the teacher and the students, research has been focused on joint problem-solving in the classroom. An analysis system was designed, based on a detailed view of the interaction, which takes the cycle as a measure unit, each of which is categorised according to the processes promoted and the participation of the students and the teacher. Processes can be cognitive depending on the level of reasoning, or metacognitive depending on the reflective and regulatory capacity that creates awareness of one's own cognition. Participation of students and teachers is understood as the participation that each of them has in the construction of learning and knowledge (Sánchez-Barbero et al., 2019). This system of tools allows analysing if a teacher solving problems together with students in the mathematics classroom does so in a rather superficial way or in a genuine way. A superficial way emphasises the mathematical aspects of the problem to obtain the solution without forming a model or considering situational information. A genuine way implies a focus on the mathematical aspects of the problem and other aspects such as the intentional, temporal and causal structure of the situation, while also making sense of the students' mathematical reasoning (Rosales et al., 2012).

Moreover, the analytical system developed allows examining whether the type of problem influences the nature of the interaction between students and teachers for its resolution or not. Some studies showed there is little reasoning and low participation of students when solving routine problems (Rosales et al., 2012). The few studies including non-routine problems showed that student reasoning and participation, as well as metacognitive processes, increased (Sánchez-Barbero et al., 2019). Possible future research directions could aim at analysing further incidental specificities in the development of interaction, the nature of the student or teacher's questions, the forms of help, or the cycles started by students. Other research possibilities could be to compare the development of the interaction in relation with the knowledge of students, across different educational levels and subjects.

A study carried out using similar analytical methods and theoretical stances was also carried out at the university. Here, written productions of pre-service primary school teachers about their perceptions of their training process allowed interaction with the teacher educator about the development of the subject and the progress in pre-service teacher learning (Chamoso et al., 2012).

Concerning the interaction between students, Juárez Ramírez et al. (2020) analysed the influence of virtual forums on the modifications made by engineering students during a mathematical modelling project. The results showed that the level of interaction depended on how the experience was carried out, and the highest levels of interaction corresponded to the greatest improvements in the work of the students. In another study, the interaction between pre-service teachers enabled them to improve the mathematical tasks they had created (Cáceres et al., 2015).

All these different analyses and findings regarding interactional processes have contributed to a better understanding of how mathematics learning takes place. Proposals for learning improvement can thus be adopted through the planning, facilitation and enhancement of particular interactional processes in formal settings.

RESEARCH ON TEACHER KNOWLEDGE AND PROFESSIONAL DEVELOPMENT

In recent years, the interest in research into mathematics teachers' knowledge and professional development has increased in response to a perceived social need to improve the quality of teaching. A range of national and international research projects have brought researchers together in collaborative networks, such as RED 8 (Network 8: Mathematics Education and Teacher Training, (web.ua.es/es/red8educacion-matematica/red-8-educacion-matematica-y-formacion-del-profesorado.html), comprising researchers from eight different Spanish universities (Badillo et al., 2019), and RED MTSK (the Mathematics Teacher's Specialised Knowledge Network, https://redmtsk.net), which allows researchers from universities in Spain and Latin

America to cooperate. Broadly speaking, there are two major concerns framing the majority of studies in this area: (i) the identification, characterisation, conceptualisation and evaluation of the specific knowledge required for teaching, and (ii) the characterisation of teacher training programmes and the development of prospective teachers' competencies.

The range of different research aims requires the selection of specific conceptual frameworks within which to interpret results and make appropriate inferences from the conclusions, find answers to research questions, and propose new ones. Researchers more aligned with the concerns of (i) have tended to ground their work on models such as Didactic-Mathematical Knowledge and Competencies (DMKC), based on an onto-semiotic approach, or on models such as the Mathematics Teachers' Specialised Knowledge (MTSK) model. Researchers more in line with (ii) the activity of teaching itself and how appropriate competencies are developed have often based their work on the construct of professional noticing (Fernández et al., 2018). Framing one's work within theoretical models is not unique to mathematicians. Research studies are often situated within the scope of wider-reaching theories such as instrumental genesis, discourse analysis, or the ATD (Barquero et al., 2022). The section below describes various developments in research that are currently taking place in Spain and the contributions of theoretical frameworks to the advancement of research.

Research into the characterisation, conceptualisation and evaluation of teacher knowledge

The Mathematics Teaching Research Group (SIDM) at the University of Huelva has carried out research aimed at identifying the knowledge brought into play by teachers in the course of working through mathematical content in the classroom. The group's research involves the comprehensive analysis of classroom practice in collaborative research projects aimed at improving teacher education. The MTSK model (Carrillo et al., 2018) provides a holistic approach to the notion of specialised knowledge and comprises a system of categories which maximise its analytical potential.

The MTSK model enables the mathematics teacher's specialised knowledge to be interpreted at any educational level, or about any mathematical content. The group is currently carrying out research into the application of MTSK to teacher training (in terms of task design), and the work of mathematics teacher educators themselves, among other areas. The studies follow an interpretative paradigm and are based on case studies and teaching experimentation. Recent work has sought to connect studies and teacher knowledge from the perspective of MTSK with other theoretical frameworks for mathematics education, such as Mathematical Working Spaces (MWS), ethnomathematics, and didactic analysis.

The research group at the University of Barcelona place their work in the theoretical framework of the DMKC model for mathematics teachers. It is based on notions of the OSA to cognition and mathematical instruction (Godino et al., 2019). One key notion in this model is didactic suitability (DS), a checklist of six educational facets against which the suitability of a teaching procedure can be evaluated (by oneself or by someone else) to provide quality instruction and improve future implementations. The orientation of the group's research methodology is primarily qualitative. It is aimed at describing the practical argumentation and development of competencies of prospective and in-service primary and secondary school teachers, through the design and implementation of training cycles called experiments in the development of teachers' competencies and knowledge.

The GIPEAM research group has developed a line of research focused on characterising the mathematical and pedagogical knowledge shown by prospective primary teachers and in-service secondary teachers in training sessions by means of reflection in, for and about video excerpts of classroom practice. These qualitative studies are grounded in professional noticing and MTSK (de Gamboa et al., 2021).

Another line of research considers the role of knowledge in posing and solving problems in teacher education courses (Perdomo-Díaz et al., 2019; Piñeiro et al., 2021). This focus is currently being widened and applied to contexts of modelling and the integration of technology (Hernández et al., 2020).

Research has also been carried out on the affective dimension of prospective and in-service teachers concerning mathematics and its teaching and learning (Fernández-Cezar et al., 2020). Likewise, the relationship between mathematical and pedagogical knowledge and the affective domain has been studied to promote inclusive practices (Blanco et al., 2021).

Research linked to the analysis of classroom practice, the acquisition of teaching competency and professional development

Over the past few years, the GIDIMAT-UA group has developed training strategies for promoting professional noticing in mathematics teaching-learning contexts for primary and secondary teacher education programmes (Fernández et al., 2018; Fernández & Choy, 2020; Llinares, 2019). Some studies centre on defining learning trajectories as a means of developing prospective teachers' professional noticing of students' mathematical thinking (Ivars et al., 2020; Moreno et al., 2021; Sánchez-Matamoros et al., 2019). The methodological approach used in these studies is that of research cycles, in which learning environments are designed to integrate teaching records (diary-type notes, short video recordings, etc.), theoretical inputs taken from the research literature, selected to support the reasoning of the prospective teachers, for example, learning trajectories of specific concepts (Ivars et al., 2020), and guiding questions, to develop prospective teachers' professional noticing. The learning environments are considered as spaces for promoting teachers' reflection and broadening their understanding of their practice. The conceptual tools are theoretical knowledge. Some results have demonstrated that limited knowledge of mathematical content is likely to handicap the development of teaching competencies concerning organising mathematical content for teaching, and the interpretation of how students learn mathematics (Buforn et al., 2022). The use of teaching records allows prospective teachers to familiarise themselves with situations approximating classroom reality, and to develop the teaching competency of professional noticing, while also generating a professional discourse to describe classroom situations.

The theoretical models underlying the research into the affective domain in teacher education include both professional noticing and MTSK, models concerned with diversity training, and models seeking to develop functional thinking as a route into early algebraic reasoning (Oliveira et al., 2021). For those models based on MTSK, the studies aim to establish indicators for the knowledge implemented in teaching and learning mathematics with students that have special educational needs.

The use of technological tools (BlocksCAD) in teacher training has also been studied, and training strategies have been used following the Study and Research Path (SRP) model within the framework of the ATD (Florensa et al., 2021).

Another area which has recently made strides is research into teacher education through the development and implementation of STEAM activities. It seeks to analyse the practice of teachers using a methodology based on projects combining mathematics with other disciplines (Diego-Mantecón et al., 2021). The findings indicate a positive progression among teachers towards implementing integrated approaches.

End note

Collaboration through networks and research teams affects the results of the two main lines of research considered here. It combines the comprehensive analysis of the knowledge evidenced in in-service teachers' practice with the design of (pre-service or in-service) training tasks. In both cases—the analysis of classroom practice and task design—the researchers draw on theoretical models that interact with one another. Both domains follow a predominantly qualitative research paradigm in the form of case studies and teaching experiments. One implication of these studies is to transfer results to the design and evaluation of the effectiveness of teacher education programmes.

DEVELOPMENT AND ARTICULATION OF MATHEMATICS EDUCATION THEORIES IN SPAIN

In this paper, we synthesise the evolution of two research approaches in didactics of mathematics in Spain. They emerged intending to contribute to the development of didactics in the direction proposed by Guy Brousseau in the 1980s with the founding principles of the Theory of Didactic Situations in Mathematics (TDSM). They aspire to build a didactic science to deepen the study and understanding of didactic phenomena, i.e., those phenomena related to the production and dissemination of mathematical knowledge, and whose essential principle is the problematisation of mathematical knowledge. It abandons the assumption that didactics is exclusively concerned with the selection, sequence, and distribution over time of "given" mathematical content. It now postulates that the primary object of study is the mathematical activity itself. An epistemological approach is thus constituted, broadening the object of study of didactics, and hence extending the pedagogical-cognitive approach. The "Brousseaunian revolution" (Gascón, 2013) can be considered the *raison d'être* of the DMSC group (Didactics of Mathematics as a Scientific Discipline), set up in the SEIEM in 1998. It derives from the regular seminars organised since 1991 within the Inter-University Research Seminar.

Within the DMSC group, the Onto-Semiotic Approach to Mathematical Knowledge and Instruction subgroup emerged (Godino & Batanero, 1994; Godino et al., 2007, 2019), initially promoted by the FQM126 Group at the University of Granada. In parallel, the ATD (Chevallard, 1992, 2015) appeared, from the research group led by Josep Gascón at the Autonomous University of Barcelona, and Marianna Bosch at the University of Barcelona (BAHUJAMA Group).

This paper aims to present the most salient characteristics of the research developed in both theoretical approaches and their developments in recent decades. Their dialogue is still alive today, beyond their origins and the internal evolution in the DMDC group, as shown by the works published in *For the Learning of Mathematics* (Gascón & Nicolás, 2017; Godino et al., 2019) or in the course about dialogue between theories in the *Intensive Research Programme on the ATD* held at the Centre of Mathematical Research (Barcelona) in 2019.

Emergence and development of the OSA

Faced with the diverse theories in mathematics education, each addressing partial questions about teaching and learning, using different languages and theoretical tools, the Onto-semiotic Approach (OSA) aims to construct a theoretical system that allows addressing the epistemological, ontological, semiotic, cognitive and instructional dimensions involved in mathematics education in a unified manner.

Thus, in the OSA seminal article (Godino et al., 1994), an explanation of the meaning of mathematical objects and its relationship with other notions, such as concept, conception, representation, schema or understanding, is provided. An anthropological and pragmatist view of mathematics is assumed (Font et al., 2013). Therefore, the activity of people when solving problems is considered the central element in the generation of mathematical knowledge, as well as in its teaching and learning. It is further assumed that mathematics is not only a human activity but also an organised system of culturally shared objects. This is the reason why we need to analyse the various objects and processes emerging from the types of practices, as well as their structure. It is necessary to deepen our understanding of the problems of learning and teaching. Consequently, it was natural to develop an ontology and semiotics for the description of mathematical activity, as well as the processes of communication and production inherent to this activity. The pragmatist theory of the meanings of mathematical objects, as well as the typology of objects and processes developed, have served as the basis for the elaboration of a theory of mathematical instruction that considers the triple dialectic between content, teacher and students.

In addition, a theoretical DMKC model has been developed within the OSA (Breda et al., 2017). The global competency of analysis and didactic intervention of the mathematics teacher is believed to consist of five sub-competencies, which are associated with five conceptual and methodological OSA tools: *analysis of global meanings* (based on the identification of problem-situations and operative, discursive and normative practices involved in their resolution);, *onto-semiotic analysis of practices* (identification of the network of objects and processes involved in the practices);, *management of didactic configurations and trajectories* (identification of the sequence of interaction patterns between teacher, student, content, and resources);, *normative analysis* (recognition of the web of norms and meta-norms that condition and support the instructional process);, and *analysis of didactical suitability* (assessment of the instructional process and identification of potential improvements).

The problem of articulating theoretical frameworks is one of the central problems that gave rise to the OSA: the attempt to understand, compare, coordinate, and integrate theories used in mathematics didactics, such as the Theory of Didactic Situations (TDS) in mathematics (Brousseau), the Theory of Conceptual Fields (TCF) (Vergnaud), the Theory of Didactic Transposition (TDT) (Chevallard), the Theory of Semiotic Representation Registers (TRSR) (Duval), amongst others. Several articles in which these articulations are addressed (Godino et al., 2006), as well as a large number of articles with applications of OSA tools to various mathematical contents, have been published. These publications show the impact and dissemination of the OSA in different parts of the world. They are available at http://enfoqueontosemiotico.ugr.es.

Emergence and development of the ATD

The Anthropological Theory of the Didactic (ATD) originated as a research programme in the 1980s with the developments of the theory of didactic transposition and has evolved over the past 30 years (Bosch & Gascón, 2006). Today, approximately one hundred researchers collaborate in the development of the ATD, mostly in Europe, Latin America, Canada, and Japan. It is worth highlighting that the international congress of the ATD (CITAD), held since 2005, facilitates the discussion of the research advances. The dialogue between theories, curriculum issues and teacher education are recurrent discussion topics.

Since its beginnings, the ATD has adopted a broad institutional perspective. Creating, teaching, learning, and disseminating mathematics are human activities that take place in different institutional settings and through complex transposition processes. A general model of human activity is proposed using the key notion of *praxeology* (Chevallard, 1992). It constitutes the minimal unit of analysis of human activities and the knowledge used and generated. In the same direction as the TDS, when addressing a didactic problem related to certain knowledge (numeracy, algebra, negative numbers, calculus, among others), a fundamental step is to question the dominant epistemological models in the institutions involved, which provide a particular vision of the knowledge at stake. For this purpose, all the steps in the didactic transposition process are considered, and the so-called reference epistemological models (REMs) (Bosch & Gascón, 2006) are proposed as key methodological tools for providing didactics with an emancipatory point of view.

Closely linked to the previous developments, more recent work focuses on the transition between pedagogical paradigms. Chevallard (2015) characterises the current situation considering the dominant paradigm of “visiting works”, in contrast with a broader paradigm of “questioning the world” in which enquiring into problematic questions plays a crucial role. To investigate the conditions that allow the advancement from one paradigm to the other, some reference didactic models have been developed through the proposal of study and research paths (SRPs) (Bosch, 2018). Several SRPs have been designed and implemented at different educational levels, as well as in teacher education, identifying new needs in teaching devices and epistemological infrastructures (Barquero et al., 2021; Ruiz-Olarría et al., 2019; Sierra et al., 2012).

The ecological approach is a central research methodology in the ATD. It focuses on examining the conditions that encourage certain school activities to exist and grow, and the constraints that hinder them from existing and growing in certain institutional settings. In this ecological analysis, it is crucial to locate at which (mathematical, didactic, pedagogical, school, social) levels these conditions and constraints appear. The main aim is to study the range of possibilities offered by educational institutions and to anticipate difficulties when changes are introduced. Previous research (Barquero et al., 2019) has highlighted the usefulness of combining epistemological and ecological dimensions to compare different theoretical approaches, and to understand their impact on the formulation of research problems and the delimitation of the corresponding unit of analysis.

DISCUSSION

In 2008, S. Llinares, one of the former SEIEM presidents, explored what he called the “agendas of research in Mathematics Education in Spain” and found “a variety of perspectives and theoretical approaches used by researchers to try to answer the wide range of questions raised” (Llinares, 2008, p. 25). He also pointed out the need for strong theories from the point of view of their ability to explain the phenomena under study, putting the spotlight on the impact of research on the educational system and marking it as a task to be carried out.

Almost 15 years later, the variety of research questions and approaches remains, with the logical evolution that permeates any scientific field, partially as a consequence of the new social demands. Despite several local initiatives carried out by the SEIEM and the CEMat, the issue of the impact of research is still a pending task. Mathematics education in Spain has acquired a level of maturity, productivity and visibility that puts it on a par with neighbouring countries. However, the SEIEM status and the kind of influence it can provide to current teaching practice and institutions are sources of open questions that demand a lot of effort from the entire community, and certainly the development of new methods and theoretical perspectives.

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PME 45

Alicante · Spain · 2022

INDEX AUTHORS AND CO-AUTHORS

INDEX OF AUTHORS AND CO-AUTHORS (ALL VOLUMES)

C

D

E

F

G

H

I

J

K

L

M

N

O

P

Q

R

S

V

W

X

Y

Z

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