Journal of Mathematical Behavior 67 (2022) 100997

Contents lists available at ScienceDirect

Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Teaching practices promoting meta-level learning in work on exploration-requiring proving tasks

Anita Valenta *, Ole Enge

Department of Teacher Education, Norwegian University of Science and Technology (NTNU), 7491 Trondheim, Norway

ARTICLE INFO

Keywords:
Meta-level learning
Proving
Teaching practices
Commognition

ABSTRACT

In this study, we explore teaching practices promoting students' learning to prove in a context where students work on exploration-requiring proving tasks. In these tasks, the goal is to provide an argument for the validity of a mathematical statement, and students are not given a particular procedure to follow. We propose a theoretical framing describing the students' and teacher's actions. Learning to prove in mathematics involves endorsing some meta-level rules that have been established historically and is an example of meta-level learning. We analyze two classroom episodes to show how the theoretical framing can be used in analysis and what the teaching practices can be.

1. Introduction

Mathematics is different from students' everyday experiences regarding the validation of general statements, that is, statements concerning infinitely many cases, as, for example, "a sum of two even numbers is even" (Jahnke, 2007; see also Fishbein, 1982). To validate general statements in everyday life, for example, "summer temperatures in Trondheim are about 18 degrees," there are different routines than in mathematics. In everyday life we do not take the statement so strictly ("well, it does not mean that it cannot happen a day with 5 degrees"), the notions we use are not well defined (*about* 18 degrees, at what time of the day), and we validate the statement by using empirical data. Proving, the way general statements are validated in mathematics, is governed by certain rules that are rather different from everyday discourse and are historically established.

In commognition theory, Sfard (2008) describes mathematics as a particular form of discourse about mathematical objects. Learning is then described as an extension of the discourse on the object- or meta-level. Object-level learning is about extending the discourse on already familiar mathematical objects with some new properties of the objects and new ways to communicate about or act on them. On the other hand, meta-level learning is about changing the meta-rules of the discourse, that is rules about the discourse that are historically established. Learning to prove—how to proceed, what are mathematically valid forms of reasoning in proving, what can be taken as known without further explanation, how to present the proof, and so forth—is meta-level learning.

Nachlieli and Elbaum-Cohen (2021) call attention to the need for several empirical studies on meta-level learning, particularly on teaching practices that can promote it. They point out that the theoretical considerations on meta-level learning and teaching practices supporting it have been exemplified and discussed only in a few classroom episodes: in the studies of Ben-Zvi and Sfard (2007) and Sfard (2007). Nachlieli and Elbaum-Cohen (2021) contribute with another empirical study—they identify and discuss the teaching practices that can promote meta-level learning in the transition from real to complex numbers. In their study, the teacher introduces

* Correspondence to: Institutt for lærerutdanning, NTNU, 7491 Trondheim, Norway.
E-mail address: anita.valenta@ntnu.no (A. Valenta).

https://doi.org/10.1016/j.jmathb.2022.100997
Received 27 August 2021; Received in revised form 27 June 2022; Accepted 27 July 2022
Available online 1 August 2022

complex numbers through direct instruction and explicitly emphasizes the similarities and differences between the current and new discourse on numbers. On the other hand, Cooper and Lavie (2021) suggest that transitions to new discourses can be initiated by designing some exploration-requiring task situations that can provide opportunities for students to engage in new routines in an explorative way.

Teaching mathematical proving can be done as direct instruction, for example, in proof courses for undergraduate students (e.g., Selden & Selden, 2008). The teacher can introduce a particular procedure for proving (as proof by induction), discuss explicitly its reasonableness and effectiveness, and provide tasks for practice. However, teaching proving is often similar to the approach Cooper and Lavie (2021) describe—it is initiated by some tasks that give opportunities to students to engage in proving without the particular procedures being given. This is the case described in several studies of teaching and learning proving (e.g., Zack, 1997; Yackel, 2002; Stylianides & Ball, 2008; Ellis et al., 2022). In addition to designing appropriate tasks, the teacher would have a vital role in helping students during the work on the task, trying to introduce the new meta-rules for the students. Our aim is to theoretically and empirically shed light on the teaching practices that can promote students' learning to prove. Based on the notion of exploration-requiring task situation (Cooper & Lavie, 2021), we introduce the notion of exploration-requiring *proving* tasks as proving tasks where the goal is to provide an argument for the validity of a mathematical statement and no particular procedure is given to follow. Our research question is as follows:

> What can be teaching practices that aim to promote students' learning to prove in work on exploration-requiring proving tasks?

By building on the works of Cooper and Lavie (2021) and Nachlieli and Elbaum-Cohen (2021), we develop a theoretical framing adopted for mathematical proving. In the framing, we describe students' and teacher's participation in work on exploration-requiring proving tasks through a commognition lens, with an emphasis on the notion of routine. To illustrate how the theoretical framing can be used to identify teaching practices in classroom episodes and what the teaching practices can be, we analyze two episodes where students work on an exploration-requiring proving task. The study contributes to research on meta-level learning and the teaching practices supporting it, but it also contributes to research on work on proving in school, as we discuss below.

There are several processes related to the validation of mathematical statements in mathematics: justifying, proving, and formal proving, and their aim is to change the epistemic value (e.g., true or false) of a given statement (Jeannotte & Kieran, 2017). Justifying, proving and formal proving are defined inclusively, with an increasing degree of deductive structure and stringency. Justifying is a process that "allows for modifying the epistemic value" (p.12), but the modification can also be from "likely to more likely". In other words, justifying does not need to be mathematically valid. But proving does need to be mathematically valid, as it is a process that "by searching for data, warrant, and backing, modifies the epistemic value of a narrative from likely to true" (p.12). In the current study, we consider all work on the validation of mathematical statements in classroom situations as *work on proving* because even though the question for students can be to justify and/or students' arguments are more informal justifications than proofs, the teacher can use the situation to support students towards proving.Definition of proving in the framework of Jeannotte and Kieran (2017) is in accordance with the definition of proof developed by A. Stylianides (2007). Proving, as described by Jeannotte and Kieran (2017), can be seen as a process of development of a proof, as defined by Stylianides (2007). Stylianides developed the definition of proof particularly for use in (primary) school and mathematics education. He defines proof as a mathematical argument that fulfils three criteria:

i. it uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
ii. it employs forms of reasoning (*modes of argumentation*) that are valid and known to, or are within the conceptual reach of, the classroom community; and
iii. it is communicated with forms of expression (*modes of argument representation*) that community (*Stylianides, 2007, p. 291; emphasis in original*).

The definition respects mathematics as a discipline, and the terms "valid" and "true" are to be understood in the context of what is typically agreed upon in mathematics. At the same time, the definition emphasizes that modes of argumentation and representation, and the set of accepted statements, depend on the community.

We argue that seeing work on proving through the commognition lens can give new insights into learning and teaching to prove. Proving is highly dependent on historical development and what is accepted in mathematicians' community, but also dependent on a community engaged in proving. As pointed out by Stylianides (2007), a proof can differ in definitions used, modes of representations and arguments, depending on what is accessible in the community. At the same time, teaching can be seen within commognition as an activity aiming to bring students' discourse close to mathematicians' discourse (Tabach & Nachlieli, 2016, p. 299). Therefore, seeing work on proving through commognition can provide an opportunity to analyze the complexity of work on proving in classrooms because it both takes account of a given community and the canonical discourse. The way we frame work on proving as meta-level learning within commognition captures both the students' and teacher's actions, an aspect Shinno and Fujito (2021) call for. Stylianides, Stylianides, and Weber (2017) point out the need for more studies on the social aspects of proving, and the present study is a contribution in this area. We show how the framing can be used for the identification of teaching practices in two episodes, and we suggest that the approach we use in the study can be used in the development of a framework of teaching practices that can support students' learning to prove.

2. Theoretical framework

Below, we present how teaching and learning mathematics is conceptualized in the commognitive framework, and we present studies on teaching practices that can promote meta-level learning. Further, in Section 2.3, we present our theoretical framing on teaching and learning to prove in work on exploration-requiring proving tasks in light of the commognitive framework. In Section 2.4 we take a closer look at previous research on teaching and learning proving, which we need to better understand students' and teacher's participation in work on proving.

2.1. Learning and teaching mathematics—commognitive lens

Sfard (2008) describes mathematics as a particular form of discourse about mathematical objects. Discourse includes both thinking and communication with others, as the term "commognition" indicates. Mathematics discourse is characterized by the use of some particular words (e.g., *sum* and *even numbers*) and visual mediators (e.g., symbols + and *x*) used to identify the objects of communication. Another characteristic of mathematical discourse is mathematical narratives, which are defined as "any text, spoken, or written, that is framed as a description of objects, or of relations between objects or activities with or by objects, and that is subject to endorsement or rejection, that is, to being labeled as true or false" (Sfard, 2007, p. 572). Hence, in a mathematical discourse, the narratives on properties of and relations between mathematical objects are constructed, and they are endorsed as true or rejected as false in the community. Definitions, theorems, proofs, and conventions are examples of mathematical narratives. A discourse is also characterized by the routines that participants regularly employ, such as calculating or deductive reasoning. These routines are regulated by some rules, either object-level rules related to the properties of mathematical objects (as what to do to find a sum of two numbers) or meta-level rules about the discourse that are historically established, such as how we proceed in mathematics to verify that something is true.

When mathematics is regarded as a discourse, learning mathematics can be seen as a change of a discourse, an expansion. (Sfard, 2007, 2008). Sfard makes a distinction between object-level learning and meta-level learning. In object-level learning, the discourse is expanded by new vocabulary, new routines based on object-level rules, or by constructing new endorsed narratives about mathematical objects that are already part of the discourse. On the other hand, meta-level learning involves new routines governed by some meta-level rules, such as changing routines for the identification of geometric forms from visual to the use of a definition, as suggested by Sfard (2007). Sfard (2007) points out that although object-level learning can take place through students' explorations without much involvement from the teacher, object-level learning has a different nature: it is based on meta-discursive rules that have been established historically and are difficult to discover by learners on their own. Therefore, the role of the teacher in supporting meta-level learning is even more crucial than for object-level learning.

Meta-level learning involves a change in routines based on new meta-rules. Lavie et al. (2019) build on commognition and further emphasize the role of routines when participating in a specific discourse; they suggest that learning can be seen as the routinization of actions in a given discourse. Furthermore, they point out that on their way to new routines, learners must pass, if only briefly, through the stage of ritualized performance or imitation (see also Sfard, 2008). Here, rituals are understood as socially oriented; they are the acts of solidarity with coperformers (often teachers), so the learner may not be aware of the mathematical purpose of the routine. Gradually, learners will become more internally motivated, and their goal will be producing and endorsing new narratives, which Lavie et al. (2019); Sfard and Lavie (2005) call explorative participation. However, as Cooper and Lavie (2021) point out, ritual and explorative participation are extremes, and there is a range of ways to participate.

Lavie et al. (2019) emphasize that analysis of routines is complex because a person's actions highly depend on their interpretation of a given task. Different people act differently in the same task because they have different interpretations of the task, depending on previous experienced situations. Lavie et al. (2019) suggest that the notion *task situation*, defined as "any setting in which a person considers herself bound to act—to do something" (p. 160), can be valuable in the operationalization of the notion of routine. In other words, to analyze the routines a participant employs in a discourse, one can analyze how the person views the task and what procedure she implements to complete that task.

As mentioned before, within commognition, teaching can be seen as "communicational activity the motive of which is to bring the learners' discourse closer to a canonical discourse" (Tabach & Nachlieli, 2016, p. 299). Both verbal and nonverbal actions are seen as communicational activities. In teaching mathematics, the teacher regularly employs some teaching practices, and the teaching practices can be seen as routines in mathematical classroom discourse (Nachlieli & Elbaum-Cohen, 2021). Building on Lavie et al. (2019), Nachlieli and Elbaum-Cohen (2021) suggest that we can see teaching practice as "the task as seen by the performing teacher together with the procedure she executed to perform that task" (p. 7). Here, the task can have both a mathematical and pedagogical nature. The teacher interprets the different situations that arise in a mathematical classroom as particular tasks and then acts to perform these tasks.

2.2. Earlier research on teacher practices promoting meta-level learning

In an episode concerning transition from positive numbers to multiplication of negative numbers, Sfard (2007) points out that students cannot discover new meta-level routines by themselves. The teacher must take the leading role and find a way to communicate the new meta-rule. Another example of meta-level learning is a case of learning to identify geometrical figures using properties and definitions (Sfard, 2007). In the initial discourse, students identify triangles directly, that is, visually, in the same way that they identify nonmathematical objects. The teacher wants the students to use the geometrical properties of the triangle in the identification process. To support students' learning of the new meta-rule, the teacher repeats the identification routine several times during the discussion, and the students gradually start to employ the same routine. As Sfard (2007) points out, the students do it for social reasons, that is, ritually, and will later gradually experience why the new routine is meaningful in mathematics. A similar case is described in Sfard and Lavie (2005) in an episode about young children and the counting routine. Ben-Zvi and Sfard (2007) point out that there must be agreement in the community about the roles of the participants, and the leadership of the teacher is necessary so that meta-learning can take place and the discourse can change in the direction of the canonical mathematical discourse. Nachlieli and Elbaum-Cohen (2021) focus on the transition from real to complex numbers and the teaching practices that can promote this transition. The teacher in the data they analyze emphasizes the similarities and differences between the current and new discourse, and she takes the leading role in the transition, as Ben-Zvi and Sfard (2007) recommend. Further on, the teacher provides the students with the opportunity to participate ritually in the new discourse by giving them tasks where they can practice the new procedures.

Based on the research of Nachlieli and Tabach (2012), Cooper and Lavie (2021) suggest that transitions to new discourses can be initiated by designing some task situations requiring exploration, which can provide opportunities for students to engage in new objects and routines in an explorative, not ritual, way. The task should have an interdiscursive potential, that is, it should be designed so that it can facilitate students' transition into the new discourse. The students should be able to use familiar routines from the initial discourse (along with known keywords, visual mediators, and narratives) while they are engaging with words, routines, narratives, and visual mediators that characterize the new discourse. Cooper and Lavie (2021) consider task situations to be exploration-requiring depending on the degree to which they fulfill the following conditions:

1. Initiation: The student's goal should be to achieve something, not to complete a procedure.
2. Procedure: Students should be able to choose a procedure from among some alternatives, making independent decisions, and not simply perform a procedure suggested by a teacher.
3. Closure 1: The task has been completed when students provide a narrative detailing the mathematical reasoning involved.
4. Closure 2: The student should be able to decide when the task has been successfully completed. (p. 4)

In an exploration-requiring task situation designed to support the transition to a new discourse, no procedures are presented and thus the students cannot participate ritually. Cooper and Lavie (2021) suggest that the students would participate in an explorative way in some "midway discourse" because they still will not be aware of the new meanings and purposes. The teacher's role in the transition is designing appropriate exploration-requiring task situations and supporting students' reinterpretation of the task situation in a way that is consistent with new meta-rules.

In their study of proving using commognition as a theoretical framework, Shinno and Fujita (2021) point out that work on proving involves learning on both the object- and meta-level. On the object-level, students develop a new narrative on properties of known mathematical objects; in the process, their use of words, visual mediators, and routines related to the given mathematical object can change. On the meta-level, the learning is about new routines for how to proceed in substantiating a mathematical narrative (Shinno & Fujita, 2021). In our study, we are only interested in learning to prove (i.e., in meta-level learning). Shinno and Fujita (2021) point out that their theoretical framing does not sufficiently consider the role of the teacher, and our aim is to capture the teacher's role in more detail.

2.3. Our framing of teaching and learning to prove in work on exploration-requiring proving tasks

We build on the work of Lavie et al. (2019) and Nachlieli and Elbaum-Cohen (2021) and their emphasis on routines as central for learning and teaching. We are interested in teaching practices that aim to promote meta-level learning, and such learning involves a change in students' routines. Therefore, we have chosen to focus particularly on routines within the commognition. We operationalize both the students' and teacher's routines as task–procedure pairs: how the person sees the task and what procedure she implements to complete that task. While students' task situations are within the current mathematical discourse, a teacher's task situations can also have a pedagogical nature, and their goal is to bring students' routines closer to what is acknowledged in mathematical discourse. Further, we build on the notion of exploration- requiring task situations and the teacher's role in supporting students' learning within the work in such situations, as suggested by Cooper and Lavie (2021).

There can be situations where a procedure (e.g., mathematical induction) is given to students to follow in work on proving tasks. Here, the students' goal can be mainly to complete the procedure, not to substantiate a given narrative (the substantiation would be a biproduct). In the current study, we are interested in situations where no procedure is given, and the emphasis is on the construction and substantiation of the narrative (as e.g., in Zack, 1997; Yackel, 2002; Stylianides & Ball, 2008; Ellis et al., 2022; Knox & Kontorovich, 2022). We suggest that teaching proving in such situations is often similar to the approach Cooper and Lavie (2021) describe—it is initiated by some tasks that give opportunities to use some familiar routines from the initial discourse and also engage in work on proving and developing new routines in a proving discourse.

In her work, Knox (2021) argues that the task she uses with students to promote learning to prove can be considered as having the similar characteristics as exploration-requiring task situation described by Cooper and Lavie (2021). She points out that the students, while working on the task, can use known words, visual mediators, narratives, and routines to explore the conjecture. Concerning interdiscursive potential, she particularly emphasizes the role of paper strips, designed within the task to afford students presentation of the general number, and also the role of the question stressing the distinction between always, sometimes, and never. Knox (2021) points out that the role of the teacher would be more crucial in case of proving than in examples of exploration-requiring task situations proposed by Cooper and Lavie (2021), since students will need more support in deciding whether the task is completed. We suggest that the task Knox (2021) discusses has similar features as tasks that are usually used in promoting meta-level learning to prove, and we elaborate on this below.

We suggest that in any proving task, there are two narratives to be constructed[1] and substantiated:

1. A *concluding narrative* on some properties and/or relations between mathematical objects (e.g., “sum of two even numbers is even”) that can be substantiated by mathematical deduction.
2. An *argument narrative* aiming to show the validity of the concluding narrative (e.g., “it is true in all examples I have tried: 6 +8 =14, 22 +40 =62, 42 +12 =56. So, it is always true.”), which is substantiated based on sociohistorical meta-rules and can be a mathematically valid (that is, a proof) or not.

Therefore, in all tasks where students are asked to prove or justify something, an argument for why a concluding narrative is true or not is a crucial part of the answer. Learning to prove within what we call exploration-requiring proving tasks is the focus of our study, and we will now define more clearly what we mean by this term.

We will call a task situation an *explorative-requiring proving task* if it is designed to promote students’ learning to prove and:

1. The students’ goal should be to construct and substantiate a mathematical narrative (*concluding narrative*), not to complete some procedure.
2. Students should be able to choose a procedure, not simply perform a procedure suggested by a teacher.
3. The task is completed when students provide both a concluding narrative and an argument narrative (showing the validity of the concluding narrative).
4. The students should be able, together with the teacher, to decide when the task has been completed successfully.

The conditions 1–3 above draw directly on the conditions for exploration-requiring task situations, described in the previous section (Cooper & Lavie, 2021), and are shaped for proving tasks. When it comes to condition 4 given by Cooper and Lavie (2021), that students should be able to decide when the task has been completed successfully, we suggest that a modification “together with the teacher” is necessary in the case of proving. If the task is designed to promote meta-level learning (which implies that students are not very familiar with the given meta-level rules), the students cannot be sure on their own whether the proving task has been completed successfully. The sociohistorical meta-rules, that are the bases for the substantiation of the argument as valid, are still not part of the students’ discourse, they are the learning goal. Therefore, the teacher must have a central role in promoting the meta-rules and their reasonableness is the substantiation of the argument-narrative. So, in an exploration-requiring proving task, we suggest that the students, *together with the teacher*, should be able to decide when the task has been completed successfully. After a while, the students will become more able to participate in the substantiation of arguments on their own.

As discussed by Cooper and Lavie (2021), the potential for interdiscursivity is important if the task is to be used to support students’ meta-level learning. We suggest that the proving tasks, those that are designed for learning to prove, are often designed so that words, narratives, visual mediators, and routines from the initial discourse can be used. Some tasks will have more potential to help students into the new, proving discourse (e.g., by providing particular visual mediators or emphasizing the need to convince someone), some will have less potential, maybe just simply asking the students to justify a particular statement. In the latter case, the teacher’s role will be even more prominent so that condition 4 above is fulfilled and the students can actively participate in deciding when the task the solution of the task is good enough.

As mentioned before, Cooper and Lavie (2021) suggest that students who are working with exploration-requiring tasks can participate exploratively in a midway discourse because their goal can be producing a mathematical narrative and they have no procedure they can try to imitate to participate ritually; however, they are still not aware of the significance of and reasoning behind the new objects and/or routines. We suggest that this can be the case in work on exploration-requiring proving tasks, too. Supported by the teacher, students can develop more and more mathematically advanced and valid arguments without being aware of why it is necessary.

Cooper and Lavie (2021) point out that in addition to designing appropriate task situations, the teacher would have a vital role in helping students reinterpret the task situation in a way that is consistent with new meta-rules. When starting to work on an exploration-requiring proving task, students will interpret the task and how to act according to their previous experiences. Through interactions with students, the teacher will try to support their reinterpretation of the task situation toward canonical discourse and new meta-rules, as illustrated in Fig. 1. The teaching practices can also be seen as task–procedure pairs, but, as discussed before, the teacher’s task and procedures can have both a mathematical and pedagogical nature.

To show how our theoretical framing of teaching practices that aims to promote learning to prove can be used in analysis and what the teaching practices can be, we analyze two episodes where students work on an exploration-requiring proving task. Before that, we take in the next section a closer look at earlier studies on learning and teaching proving that can shed more light on the three components of the figure: (1) what characterizes students’ initial routines in validation of general statements when no procedure is given, (2) what characterizes mathematicians’ routines in similar situations, and (3) how teachers can support students in the transition.

[1] In some proving tasks, those given in the form of “prove that.”, the concluding narrative is given, and need not be constructed.

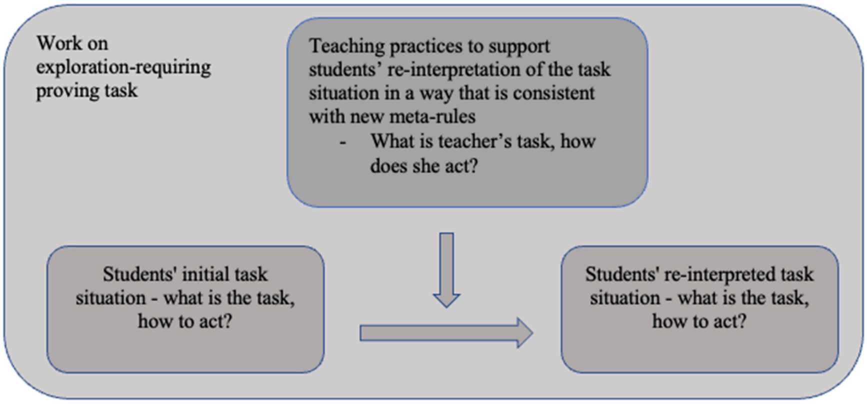


Fig. 1. Teaching and learning proving in work on explorative proving tasks—students' and teacher's routines as seen as task situations (task–-procedure pairs).

2.4. Earlier research on work on proving

The studies we discuss here are not originally conducted within commognition, but we are interpreting the results using the commognitive framework. We start the review by discussing aspects of typical students' approaches work on validation of general statements, and mathematicians' approaches. Then we present some studies that shed light on teachers support in the process of learning to prove.

To validate a general mathematical statement, students often proceed by using examples and providing an empirical argument (see Reid & Knipping, 2010 for an overview of the studies concerning the topic). Empirical arguments are typical in discourses other than mathematics, and it is not surprising that students try to use the same routine in mathematics. One can say that students' initial task situation is typically to consider the task as "check the statement on some examples to see whether it holds", and the procedure is to choose some examples and check. In some cases, students chose some special example (an example with big numbers, and such) and use it to provide a crucial experiment (Balacheff, 1988), thinking that "if the statement is valid for such unusual, random example, then it must be valid for all examples". Psychologically, crucial experiment can be more convincing than an empirical argument, but it is still mathematically invalid form of argument. Several studies show that learners use empirical arguments even in cases when they know that they are not valid mathematically because they struggle to find another way to verify general statements (Stylianides A.J, 2009; Weber, 2010). They struggle to find a way to proceed and to identify properties, structures and known results that can be useful for proving a given statement, the challenge Weber (2001) relate to students' lack of strategic knowledge. Within commognition, one can say that students have limited experience with proving tasks and a way to proceed is not an established routine for them yet.

The transition from informal argument to proof is another challenge that has been identified by several studies on proof (see, e.g., Stylianides et al., 2017 for an overview of studies concerning the topic). In commognition-terms, we can say that it is related to both interpretation of what the task is (what is expected as a solution of the task and how it is to be presented) and also how to proceed (what gaps are to be filled, how to fill them). One can say that in all mathematics tasks the answer consists of some result and the reasoning leading to the result. In proving tasks, the emphasis is on this reasoning (argument narrative), the result (concluding narrative) is almost just a biproduct, and there is an expectation about how the reasoning is to be presented. Students are not used to these requirements from non-proving discourses, and the transition can be challenging.

Mathematicians know that they must infer the concluding narrative by deductive reasoning, and that the argument narrative must be presented clearly, justifying all the steps. They consider identifying the key idea as crucial in the proving process when there is no obvious procedure to choose (Raman, 2003). A key idea is an (informal) understanding that shows why a particular statement is true and can be developed into a formal proof. One can say that identifying a key idea can be considered an overall routine employed by

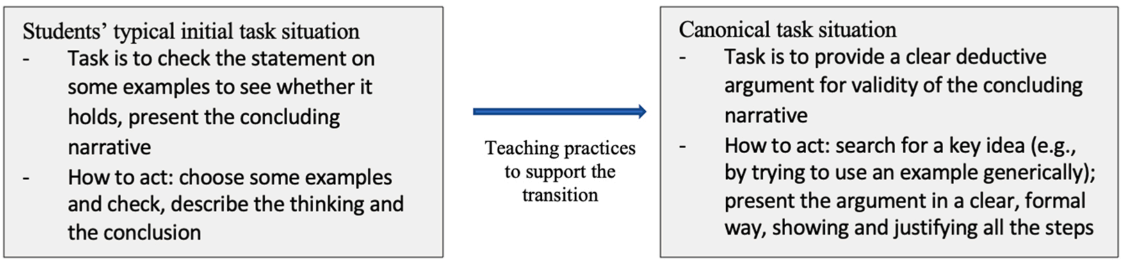


Fig. 2. Description of students' task situation and canonical task situation within work on validation of a general statement, based on the previous studies.

mathematicians in proving. Often, they approach search for a key idea by using examples, but not for providing empirical arguments as students. Mathematicians use examples for experimenting and obtaining insights into the conjecture and structures involved (Epstein & Levy, 1995; Lynch & Lockwood, 2019), and for developing arguments by generic examples (Lockwood et al., 2016). An argument by generic example uses a particular example as a representative of a whole class and is presented such that its role is the "carrier of the general" (Mason & Pimm, 1984, p. 287; see also Rowland, 1998; Garuti et al., 1998). Balacheff (1988) points out that argument by generic example is significantly different from empiricism: "The generic example involves making explicit the reasons for the truth of an assertion with operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class's" (p. 219). In other words, once an argument by generic example is constructed, the key idea of the proof is revealed. For a mathematician, the way further to a more formal proof is short. Mathematical proving is characterized by logic and rigor, and in presentation of a proof mathematicians aim to clearly express and justify the inferences from statement to statement (Sfard, 2000).

Summing up, Fig. 2 shows what we can call students' typical initial task situation, early in their process of learning to prove, and the canonical task situation when the mathematician does not see some obvious way to proceed in validation of a general statement.

We will now review some studies pointing to teaching practices that can support the transition between the two discourses. Several designed experiments have aimed to develop the learners' skepticism toward empirical arguments (e.g., Stylianides G.J, 2009; Brown, 2014). Viewed through commognition, one can see the goal of such experiments as changing students' view on what the task is. The way one interprets the task and the way one acts are, of course, related. However, even though students' interpretation of the task can be changed through the teaching experiment and students know that what is expected is something else than checking the statement on examples, studies show that it is difficult for them to find another way to act than empirically (Stylianides G.J, 2009; Weber, 2010).

Examples can be useful in proving, but Alcock and Weber (2010) point out that it is not enough to simply suggest using examples for students: they need more support in learning how to do so in a productive way (see also Iannone et al., 2011). (Mason, 2019; see also Mason & Pimm, 1984) suggests that students' attention somehow needs to be directed to see structural relationships in the examples; he points out that once the student can "see the general in the particular," in other words, see the example as generic and recognize a general structural relation in one particular example, one barrier toward proving has been overcome. There is a discussion in the mathematics education research community on whether an argument by generic example can be considered mathematically valid or not (see Leron & Zaslavsky, 2013; Yopp et al., 2015; Reid & Vergas, 2018). Nevertheless, there is agreement in that generic examples play an important role in proving because they engage learners in the main ideas of a proof in an intuitive and familiar context, and many researchers have promoted the role of generic examples as a transition from empirical to deductive argument (Rowland, 1998, 2001; Alcock & Inglis, 2008; Leron & Zaslavsky, 2013). In other words, use of examples can have an interdiscursive potential bringing students into proving discourse. Recently, Ellis et al. (2022) have identified several types of instructional support that can help learners move from empirical arguments to more structural use of examples: experiencing the need for verification, fostering contextual interpretation, fostering reflection and justification, and fostering pattern exploration. The search for patterns is also emphasized in the study of Brown and Coles (2000): looking for what is the same and what is different among examples was promoted by the teacher, and gradually, it became a part of the students' actions and was shown to be powerful in helping the students in proving process. Both searching for patterns and trying to use examples generically can be seen as routines that can help students in searching for a key idea.

However, even when students have identified a key idea, expressing it in form of proof is not easy. As Sfard (2000) points out, the logic and rigor needed in a proof in not something students can discover on their own, they must have teacher's support. The teacher's role is to challenge students to be clearer on what is known, what the rationales are, how statements follow from one to the other, and how the argument can be expressed (Stylianides & Ball, 2008; Yackel, 2002; Zack, 1997). In other words, the teacher can lead students in the direction of more mathematically valid justifications and proof through challenging their informal arguments, emphasizing the key idea, and helping students to clearly express and justify the inferences in the argument.

3. Episodes from the classroom

In the current study, we are interested in a detailed understanding of a specific phenomenon and participant actions (see, e.g., Cohen et al., 2011, pp. 219–223), that is, the phenomena of teaching and learning proving within work on explorative proving tasks. Our sampling is theory based because we use the data to elaborate on our theoretical framing (Mertens, 2005, p. 242). In particular, we use the data to illustrate how the framework can be used in analysis and what the teaching practices supporting students' reinterpretation of task situations may be.

3.1. Context

The episodes we analyze are collected as part of a larger research project[2] aiming to develop resources that can support teaching and learning to reason and prove in primary school.[3] As part of the project, the researchers, in collaboration with three primary school teachers, designed lessons and tasks to be tried out in the teachers' classes. The episodes are from the fourth lesson, which was designed and tried out in the seventh grade.

The class consisted of 20 students who were 12–13 years old. Before the project started, the class had not worked explicitly on

[2] ProPrimEd - Reasoning and Proving in Primary Education. The project is a collaboration between the Norwegian University of Science and Technology (NTNU) and Trondheim municipality. For more information about the research project: https://www.ntnu.edu/ilu/proprimed.

[3] Primary school in Norway is grades 1–7.

justification and proving. Researchers from the project were present in the class for two weeks in usual mathematics lessons before the designed experiment and then later in lessons designed in collaboration between the researchers and teachers. In the designed lessons, the whole-class discussions were led by the teacher, but the researchers participated as additional teachers in interactions with the students when they were working in groups. As part of the lesson planning, the researchers and teachers discussed (in rather general terms) what could be difficult for the students and how they could be supported. In each lesson, whole-group discussions, and group work in two to three groups were video-recorded.

In the lesson we analyze here, two researchers were present and supported the students during their group work. One of the researchers is one of the authors of the paper. We are mainly interested in the process of learning to participate in a mathematical discourse of proving and the teaching practices that can support that learning, who the knowledgeable individual is not important for the study. Therefore, we denote both the teacher and researchers as "the teacher" in our data, making no distinction between persons.

The data we analyze are the transcriptions and written works of two groups, each consisting of four students. The groups worked on the following task:

Sum of consecutive numbers

Consecutive numbers are those coming after each other, for example:

13, 14 are two consecutive numbers

20, 21, 22, 23 are four consecutive numbers

3, 4, 5, 6, 7, 8, 9 are seven consecutive numbers

We study the sum of the consecutive numbers. The question for you to examine, make conjecture about, and justify is as follows:

Does it happen always, never, or sometimes that the sum of three consecutive numbers is divisible by 3?

The task was designed to help students in the transition from the initial toward the proving discourse and to satisfy conditions of an exploration-requiring proving task. A similar task concerning the sum of consecutive numbers has been used in several other studies on proving (see, e.g., Lynch & Lockwood, 2019; Ozgur et al., 2019). In our case, the task was designed to support students learning to prove. The students had very limited experience with proving tasks, and no procedure was given to follow. As an solution of the task, the students were expected to construct and substantiate a concluding narrative and provide an argument narrative for its validity. Our thinking was that the students, together with the teacher, could decide when the task is completed successfully. Furthermore, the task was situated in a numerical discourse that was familiar for the students: about consecutive numbers, sum and divisibility; and they could use routines from that discourse in their work. However, the words "conjecture" and "justify" in the task belong to another discourse, the proving one. These words were emphasized in previous lessons as a "conjecture is a statement we need to investigate more to determine its truth value" and "to justify is to explain *why* a mathematical statement is true or not." We chose to approach justification and proving through an emphasis on "explaining why" because the explanatory role of a proof could be an effective way to introduce the notion of proof to students (see, e.g., Knuth, 2002). One can say that putting emphasis on explaining why could have an interdiscursive potential and help students in interpreting the task situation more in accordance with the proving discourse. Furthermore, as Buchbinder and Zaslavsky (2019) point out, stressing the distinction between always, sometimes, and never, as it was done in the design of the task, could give the opportunity to students to experience the roles examples could play in proving situations. In other words, this distinction might also support the students in the transition between their initial discourse and the proving discourse.

3.1.1. Method of analysis

Our research question focuses on teaching practices that can promote learning to prove. In line with Nachlieli and Elbaum-Cohen (2021), we consider teaching practice as a task–procedure pair, and our method of analysis is inspired by the method they use in their study on teaching practices in the transition from real to complex numbers. The approach builds on a commognitive framework, particularly the operationalization of the routine concept in Lavie et al. (2019).

We are interested in the teaching practices related to interactions between students and the teacher during the students' work on the given task. Hence, we start our analysis by identifying sequences in the data where the teacher intervenes in the students' work. Further, we analyze teaching practices in these sequences as follows: starting with empirical observation of the teacher's action, we describe the action more generally as a procedure. Then, we try to identify what task the teacher is trying to accomplish by performing the action. Our analysis was performed first independently by each of us researchers before we compared and agreed on the results. We present the analysis of each teaching practice in the form of a table giving an overview of all three elements: empirical observation, procedure, and task. In the name of each teaching practice (which is also the name of the table), we try to capture both the teacher's task and the procedure. Further on in the analysis, we point out how the teaching practice seems to affect the students' task situation, that is, their view on what their task is and how to proceed.

3.2. Teaching practices in the episodes

In the following, we present the two episodes, alternating between sequences of data where the teacher interacts with students and our analysis.

3.2.1. *Episode 1*

There are four students in the group: Marie, Aida, Ozra, and Loran.[4] The students start the work by giving examples of numbers that are divisible or not by 3, discussing what “consecutive” and “sum of consecutive” numbers are and checking the statement on the examples $1 + 2 + 3$, $2 + 3 + 4$ and $3 + 4 + 5$. They conclude that “It is true” and start to write this on the task sheet. However, they notice that they need to justify and stop the writing for a moment. Marie asks the others to suggest another “three consecutive numbers, completely random.” The examples $4 + 5 + 6$ and $9 + 10 + 11$ come up, and the students calculate and find out that the sums are divisible by 3. Then, they conclude that the statement is always true, write their conclusion and examples on the sheet, and start to work on another task. The teacher comes by, and the students tell her that the sum of three consecutive numbers is always divisible by 3 and that they have checked on many examples.

122 Teacher: Then, you only know that it is true sometimes.

123 Marie: Yeah, we have just tried some examples. How can we show that it is always true?

The students first interpret the task situation as the “need to find out whether the statement is true always, sometimes, or never and can do it by trying out on some examples.” When they notice that they need to justify, they reinterpret the task situation to include not just a concluding narrative (whether the statement is true always, sometimes, or never), but also an argument narrative. However, the argument the students develop is still invalid, it is a crucial experiment now (using “random” examples). After the teachers’ intervention in line 122, they realize that their argument is not good enough and that the task is about more than concluding based on checking examples. Our analysis of the task and the procedure related to this teacher’s utterance is given in Table 1.

After realizing that argument based on checking some examples is not good enough, the students struggle to find a way to proceed. The following is an excerpt from the discussion while the teacher is talking to the group:

124 Marie:We cannot show for all the numbers. [Teacher: No] But if we show the number up to 10?

128 Teacher: Why up to 10?

129 Loran: Because. if it goes up to 10, then it always goes.

130 Teacher: Oh, why is that?

131 Loran: Because everybody has said so.

132 Marie: That was everybody have said, yeah (laughs)? [Loran: Last time, yes!]

133 Teacher: I am skeptical! (laughs).

134 Marie:Yeah, but how can we show that it works for absolutely all numbers?

135 Aida:Maybe we can check all numbers?

136 Marie: We don’t have enough space for all numbers here (points to the sheet). Are you aware of how many numbers there are … or?

137 Loran:OK. Try one million and one, one million and two …

138 Marie: We cannot sit here with that the whole time. It does not help at all.

In the previous lesson, the students worked on a task where the construction of numbers up to 10 (using the numbers 3 and 5 only, with addition and subtraction) was used to conclude that all natural numbers can be constructed this way (which was the question). They suggest now that it is enough to “check up to 10.” Our analysis of the teacher practice appearing in the sequence is presented in Table 2.

The two teaching practices identified above seem to be both about changing students’ views of what the task is and how to act. Concerning the task – it involves more than checking examples and concluding, it requires that one reasons for all the steps, not just use some procedures blindly. Concerning how to act – checking random examples or examples up to 10 is not good enough procedure. After the teacher’s contributions, the students give up the idea of justifying the statement by checking examples, but they still do not know how to proceed. The discussion continues as follows:

139 Teacher: What do you know about three consecutive numbers?

140 Marie: Since they are three.

141 Loran: Since you are adding three consecutive numbers, it will be a number in a 3-times table.

142 Teacher: If I have a slip of paper and some number is written on it. What is the next number?

143 Aida: One more.

[4] The names are pseudonyms.

Table 1
Teaching practice: Communicating to students that empirical argument is not valid as a proof of general statements in mathematics.

Task	Procedure	Empirical observation
Leading students to realize that empirical arguments are not good enough to justify a general statement	The teacher informs the students that checking some examples only shows that the statement is true sometimes	Teacher: Then, you only know that it is true sometimes. (line 122)

Table 2
Teaching practice: Emphasizing to students the need to be skeptical and defend reasonableness of the procedures to be used.

Task	Procedure	Empirical observation
Making students aware of the need to be skeptical and reason on whether some previously used procedure makes sense	Teacher is asking why it is enough to check up to 10, and she is expressing skepticism	Teacher: Why is that [enough to check up to 10] (128, 130). I am skeptical (133)

144 Teacher: Yes, the number plus 1. And the next one?

145 Loran and Ozra:2.

146 Teacher: Yes, right. So, we have the number, and the number plus 1 and the number plus 2. And we add them. Why does it have to be divisible by 3, then?

147 Marie:I don’t really get it (laughs).

The teacher suggests that the students discuss it more and goes to another group.

In this excerpt, the teacher seems to be trying to help students find a more general approach to the justification of the statement than checking examples, but it leads only to students’ rewording of the statement in line 141. The teacher tries further (142, 144, 146), but it does not seem to help. Our analysis of the teacher’s intervention in the sequence is given in Table 3.

After the except above, the students continue to discuss how to proceed, and Loran suggests again to check examples, specifically some special examples that are “most difficult to find out.” However, Marie says the following:

200 Marie: Yes, but that doesn’t explain at all why it works for all numbers! It is just an example! All we have written are examples. We don’t explain how it is.

The students’ work is not progressing, and they ask the teacher for more help.

219 Teacher: Can you try to think very cleverly when I say, let’s say 99 plus 100 plus 101? (Loran calculates, not audible) Because?

220 Loran: Because 99 plus 101 is equal to two hundred and plus 100, it is 300.

221 Marie: You can move the 1 from 101 to 99; then, we have 100, 100, 100. [Teacher: Yeah!]

224. Marie: So actually, you add all the numbers with each other. For example, 6, 7,

8. You move from 8 to 6, so it becomes 7, 7, 7.

In line 219, the teacher suggests a concrete example for students to consider and suggests for them to try to think “very cleverly.” This suggestion leads the students to recognize a general structure in a sum of three consecutive numbers. Our analysis of the teacher’s intervention in the sequence is presented in Table 4.

“Thinking cleverly” seems to be an entry into looking for a structure in the sum of three consecutive numbers. Finding a clever way to calculate involves seeing the sum differently than just the result of the adding process—the same calculation process, no matter the numbers. In addition, the teacher suggests a concrete example for the students to consider, so her intervention is rather explicit. After the intervention, the students recognize the structure in the examples and proceed on a justification by using a generic example in line 224.

The first two teaching practices in the episode are mainly about what the task is. One can say that the students’ view on what the task is developed from “the task is to check on some examples/examples up to 10 and conclude based on that” to “the task is *more* than

Table 3
Teaching practice: Trying to highlight for the students a general structure across the examples.

Task	Procedure	Empirical observation
Helping students find the general structure across the examples	The teacher is asking questions, leading students toward a general structure of three consecutive numbers	Teacher: What do you know about three consecutive numbers? (139) Here is the number, the number plus 1, and the number plus 2; why must the sum be divisible by 3? (142, 144, 146)

Table 4
Teaching practice: Suggesting for students how to proceed in the proving process to recognize a structure in examples.

Task	Procedure	Empirical observation
Helping students find the structure by suggesting an appropriate example that makes the structure visible	1. The teacher invites the students to find another way to calculate the sum where the property of being three consecutive numbers is used 2. The teacher suggests an example where the structure of three consecutive numbers is prominent and the use of it makes the calculation trivial	Teacher: Can you try to think very cleverly when I say, let's say 99 plus 100 plus 101. (line 219)

checking on examples, we need to argue for *all* examples". However, even though they see the task differently than in the initial discourse and their view is more in accordance with a canonical discourse, they do not know how to proceed. Through the third teaching practice (Table 3), the teacher tries to help the students in searching for a general structure across the examples, but with no success. The procedure she suggests seem to be too far away from the students' initial discourse. In the fourth teaching practice (Table 4), the teacher suggests for the students a particular example, and she asks them to "think cleverly" in calculating the sum. Working on examples and calculating sums is known to students in their initial discourse, they participate exploratively and discover the structure in the given example. Supported by the teacher, they realize that other examples have the same structure. Probably they do not see yet the full strength of the approach and their participation is motivated socially, but still – they make their first steps into a proving discourse. We can say that they participate exploratively in some midway discourse. Now that they have experienced how one can proceed, we suggest that their view on the task has developed further, from only knowing what is not good enough after the first two teacher interventions, to having at least one experience of what else the task asks for (more than checking examples). Through the teacher's intervention in the episode, the students' task situation changes, both considering how they view what the task takes and how to proceed to complete it.

3.2.2. *Episode 2*

There are four students in the group: Laura, Oskar, Rasmus, and Erik. Much like group 1, they start by using different examples to understand what the statement is about and then to test it. The students pick up several examples of three consecutive numbers, all in the interval between 15 and 25, add them, divide by 3, and conclude that "it works". In some cases, they make mistakes in their calculations and try again. All the work is only oral. The teacher comes by and observes students' work for one to two minutes.

204 Teacher:You can write the sums (points to the paper in front of Oskar). [Oskar: Of those numbers?] Yes. [Oskar: 63] And then, you can write what you get when you divide by 3. [Oskar: Divided by 3, it is 21.] Yes. So, if you take 20, 21, and 22, then you get … [Oskar: 63] 63. And if you divide by 3, you get the answer 21.

205 Oskar: So, the answer must be one of the numbers?

206 Teacher: Mhm. It is in this case, at least. You can try other numbers.

207 Oskar: Mmm…

208 Teacher: (to Erik) You said, what was that you said? 15, 16, 17. What is the sum?

209 Rasmus: 15, 16, 17. It is 48.

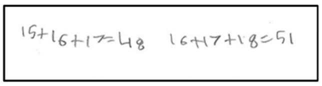
210 Oskar: And 48 was 16.

…

227 Rasmus: I think that it must always happen. Because if you take 16, 17, 18. It is 48. That was what he said, right (points to Erik).

228 Teacher: Write the numbers you have talked about, so you know what you.

229 Rasmus: No matter if you take one more, I mean 17, 18, 19 [Teacher: OK.]. It becomes three more. (Teacher nods and goes to another group of students.)



In line 204, the teacher suggests that the students write down their example work. The aim of the utterance seems to be to suggest

Table 5
Teaching practice: Directing students' attention to pattern searches in their work on examples.

Task	Procedure	Empirical observation
Helping students to find a pattern across the examples they work on	The teacher suggests for the student to write down their work using examples	Teacher: You can write the sums (points to the paper in front of Oskar). And then you can write what you get when you divide by 3 … (line 204), also the utterances in lines 206, 208, 228

that the students should look for patterns in examples. The students perceive the teacher's suggestion that way and start to search for patterns. The teacher promotes writing further in the sequence, too. The teacher's intervention changes the way the students act—from orally checking whether the statement is true on examples to searching for the patterns supported by written calculations of examples. Our analysis of the teacher's action in the sequence is given in Table 5.

After the teacher's contribution above, the students' work moves forward in the proving process, and they use examples generically and build an argument using informal induction, that is, looking for a relation between the sum of three consecutive numbers and the sum of the next three consecutive numbers. The teacher comes by again after some time, and the following discussion takes place:

250 Teacher:How is it going with you?

251 Rasmus:It happens always. Because 15, 16, 17, it is 48. And then 16, 17, 18, it is 51. [Teacher: Yes] So if we take one more, no matter what we had before, it will be 3 more, plus 3 all the time. [Teacher: Yeah] It is …

252 Teacher:Why is it important that it is plus *3*?

253 Rasmus: It is plus 3 all the time if we take one more, so.

254 Teacher:Yes, but why is it important that it is plus 3?

255 Erik: Because it is divisible by 3.

256 Rasmus: Because it can be divided by 3, no matter then.

257 Teacher: Yes, so if the first one is divisible by 3, then the next one must be divisible by 3. [Erik: Ah, yeah!]

258 Rasmus: Yes, right. 1 plus 2 plus 3. [Laura: Yeah!] 1 plus 2 plus 3; that is 6. That is divisible by 3.

The students present both the concluding narrative and argument-narrative in line 251, showing that they see both parts as necessary to answer the task. The teacher contributes in utterances 252, 254 and 257. Our analysis of the teacher's action in the sequence is presented in Table 6.

Here, the students have a key idea, talk in general terms (last part in line 251), have developed a justification, and are now communicating their argument narrative to the teacher. By asking questions in lines 252 and 254, the teacher communicates the need to make the rationale on the inductive step (for a sum starting in k to the sum starting in $k + 1$) more explicit, and in line 257, the teacher helps students express this clearly. This leads students to realize the need to be sure that the base case ($1+2+3$) satisfies the statement, and they check it in line 257.

So, in Episode 2 we have identified two teaching practices supporting students' reinterpretation of the task situation. The first one, when the teacher directs students' attention to search for a pattern by suggesting writing examples, is about how to act. The second teaching practice is mainly about view on what the task is: through the utterances in line 252, 254 and 257, the teacher communicates to the students what is expected as an answer to the task, that all the steps in the argument need to be clear and explicitly justified. Even though the teaching practice is mainly about what the task is, what is expected, it also indicates for the students how to proceed: you need to fill the gaps, say more, be explicit. In lines 252 and 254 the teacher asks the questions pointing to an unclear inference in the students' argument, and by answering the students make the step clearer. In line 257 the teacher models how to proceed to clearly express the inference.

In this episode, the students' initial discourse is not characterized by empirical arguments as in Episode 1. Here, students' view of the task and how to proceed is closer to the canonical one, and the question is mostly on how to present the argument narrative, what is expected. As in Episode 1, the students participate exploratively (with focus on development of narrative) in something (filling the gaps) that is different than in their initial discourse The teacher must ask the same question several times, and the students answer the way they do mainly because of the social reasons, probably without seeing the mathematical reasons of need to clarify and justify so explicitly. So, as in Episode 1, also here we can say that the students participate exploratively in a midway discourse.

3.3. Discussion

Our research question asks what the teaching practices promoting students' learning to prove in work on exploration-requiring proving tasks can be. Using the theoretical framing we have developed (summed up in Fig. 1), we analyzed two classroom episodes and identified six such practices:

Table 6
Teaching practice: Challenging students to develop and express clear rationales in each step of an argument.

Task	Procedure	Empirical observation
Help students to identify and express the parts of the argument that are not made explicit enough	1. The teacher presses students to make explicit parts of the argument 2. The teacher is helping students to express the missing rationale	Teacher: Why is it important that it is plus 3? (line 252); Yes, but why is it important that it is plus 3? (line 254); Yes, so if the first one is divisible by 3, then the next one has must be divisible by 3. (line 257)

A. *Communicating the limitations of empirical arguments in proving general statements*
B. *Emphasizing to students the need to be skeptical and defend the reasonableness of procedures to be used*
C. *Trying to highlight for the students a general structure across the examples*
D. *Suggesting for students how to proceed to recognize a structure in the examples*
E. *Directing students' attention to pattern searches in their work on examples*
F. *Challenging students to develop and express clear rationales for each step of an argument*

We suggested in our framing that the teaching practices in the work on explorative proving tasks would support students' reinterpretation of the task situation toward a more canonical discourse. Below, we discuss the six practices we have identified (A–F above) in relation to earlier research on learning and teaching proving. We emphasize how the teaching practices can be seen as supporting students' reinterpretation of the task situation and how the approach we suggest can contribute to research on proving. Further, we discuss the results of the study in relation to other studies on teaching practices that can promote meta-level learning.

We suggest that through teaching practices A, B, and F, the teacher is supporting students' reinterpretation of what the task is, what the expected product is when we ask in mathematics whether a general statement is true always, sometimes, or never, and why this is the case. In line with Buchbinder and Zaslavsky (2018), we see that the design of the task plays an important role in students' transition to a proving discourse. As we can see in Episode 1, the students continue their work after checking some examples because they read in the task that they need to justify their work. They realize then that an argument narrative is necessary, in addition to the concluding narrative. Later, the wording "always, sometimes, and never" in the task is central in the teacher's action in practice A, pointing to a clear distinction in mathematics between knowing that something is valid sometimes and valid always. Making students aware of limitations of empirical arguments is pointed out as important step in learning to prove also in the study of Ellis et al. (2022), and Stylianides G.J (2009). In our study, we see that the task design is supporting the teachers' intervention on this point. Through practice B, the teacher communicates that all choices and procedures need to be justified (not just used blindly), and in practice F she emphasizes the need to develop and express clear rationales in all steps of the argument. Similar teacher's actions, where the teacher is helping students to fill the gaps in the arguments, have been discussed earlier, for example in the studies of Zack (1997), Stylianides and Ball (2008) and Ellis et al. (2022). In our study we suggest that these practices are supporting students' reinterpretation of what the proving task is about, what is expected as an answer to a proving task. At the same time, as discussed in the analysis of Episode 2, the practice F is indicating for the students how to proceed: by expressing clear rationales for each step. Also through practices A and B the teacher affects the students' procedures—by communicating that the procedures checking examples in general and checking examples up to 10 are *not* good enough. However, as discussed in the analysis of Episode 1, the students then only know how *not* to proceed, not how to proceed, as is the case of the episode 2 and teaching practice F.

Teaching practices C, D, and E are more explicitly related to how to proceed and aim to help students find a mathematically valid approach to prove a general mathematical statement. Through practice C, the teacher is promoting looking for properties and structure that can play the role for the given relation, a practice that can support students' development of strategic knowledge, as discussed by Weber (2001). Looking for a general structure in an example, as in practice D, is a routine Mason and Pimm (1984) emphasize in the development of an argument by generic example. Practice E is about looking for patterns among examples, a routine Brown and Coles (2000) point out as useful in proving. One can say that through practices C, D and E, the teacher communicates some routines for finding a key idea (Raman, 2003). The practice C, pointing to a structure in a general way, did not help the students in our data to proceed further in the proving process. On the other hand, interventions D and E, which made use of examples, did support students in developing an argument based on generic examples. This can indicate the importance of the use of examples in proving as tools for teachers in supporting students in their work on proving, as suggested by Mason and Pimm (1984), also Rowland (1998). Examples are close to students' initial discourse, they can participate exploratively in a midway discourse, as in Episode 1: while work on an example and calculating "cleverly", the students generalize the structure in the example, even though not fully aware why it is important. So, we see in our data that the teaching practices involving use of examples have interdiscursive potential. Even though some teaching practices (as C, D and E) are mostly related to how to proceed, we suggest that they also implicitly affect students' view on the task. For example, in Episode 1 we suggest that the students further develop their view on what the task is and what is expected, when they, supported by the teaching practice D, find a procedure that the teacher approve. At this moment it can still be difficult for students to say in general what the procedure is (*think through one example, see whether it happens something there which will happen in other examples*?). However, they have experienced what it takes to validate a general statement and it is reasonable to think that they view the task differently–from "task is about more than checking examples" to "task is to point to something that is similar across all the examples".

As discussed above, the teaching practices we have identified in the data are to a large extent reported earlier in other studies on proving. However, here we put forward how they are arising from the teacher's goal to bring the students' discourse close to mathematicians' discourse and how the actions are aiming to support students' reinterpretations of what the task is and/or how to proceed to complete the task. Through framing the work on exploration-requiring proving tasks within commognition and seeing both students' and teacher's actions as routines operationalized as a task–procedure pair, our approach (Fig. 1) can help analyze the complexity of work on proving in classrooms; specifically, it can help by taking account both the given community and canonical discourse. Further on, we suggest that our distinction between exploration-requiring proving tasks and proving tasks where a particular procedure is given and students are expected to practice it can be valuable for elucidating the kind of participation we can expect from students, either ritualized or explorative, and how students can be supported in each case. In the case of work on exploration-requiring proving tasks, our analysis shows what the teaching practices can be. The approach can be used in developing a more complete framework for teaching practices based on more extensive data.

Meta-level learning and the teaching practices aiming to support it can take different forms depending on the mathematical content and teaching approach. As we presupposed, the teaching practices we have identified in work on explorative proving tasks are less explicit than those in the case of transition from real to complex numbers described by Nachlieli and Elbaum-Cohen (2021). The teacher in our episodes is not making explicit what the routine was before and what has changed now (e.g., that before or in other tasks, we could accept more "sloppiness" in saying that something is true, but now we will do it more mathematically proper). Instead, the change in the discourse is initiated by the task the students are working on, and the teaching practices that can promote meta-level learning are communicated implicitly to the students while they work on the task. Also, in the geometry case presented by Sfard (2007) and the counting case described by Sfard and Lavie (2005), we see that the new routines are communicated by the teacher implicitly, that is, through repeated questions, suggestions, and confirmation of wanted answers in a given task situation. However, there are differences between these situations and our episodes. One difference is that the wording in the task situation is different in our data, indicating that something has changed. While the questions in a geometry situation are the same as in present discourse (e.g., which of these figures are triangles?), there are words in the task situation in our case that indicate a new discourse (conjecture, justify, always, sometimes, or never). Another difference is in the complexity: while it is the single routine the teacher communicates to the students in the geometry-case, the teaching practices concern several routines in the case of proving—the always–sometimes distinction, how to proceed, how to make rationales explicit. This complexity is something that makes both teaching and learning to prove challenging.

The teaching practices we have identified in the episodes are specific to the given task and the context, and the data we analyze is rather limited. However, our goal was not to make a survey, but rather to give an example of what teaching practices of this type can look like, how they can be analyzed, and how they are similar or different from those teaching practices related to other types of meta-level learning.

The current study contributes to research by framing work on proving as meta-level learning within commognition, here in a way that captures both the students' and teacher's actions, as Shinno and Fujito (2021) call for. We show how the framing can be used for the identification of teaching practices, and we suggest that the approach can be used in further development of framework describing teaching practices in work on explorative proving tasks. Another contribution is providing an empirical example of meta-level learning and the teaching practices that can support it. As Nachlieli and Elbaum-Cohen (2021) point out, there are few such examples, and our study gives an example that differs from others, as discussed above. The teaching approach we study is based on the use of exploration-requiring proving tasks, similar to those Cooper and Lavie (2021) suggest in their theoretical study, and we show here how these tasks can play out in the classroom in case of learning to prove.

We want to remark that even though the support given by the teachers in our episodes helps the students move forward in proving, it does not mean that the students will use the routines in new task situations. In Episode 1, for instance, the teacher's contribution helps students realize the limitation of the empirical argument in the validation of general statements, but this does not mean that the routine is individualized by the students. Already in the next task, the students can go back to empirical arguments. The individualization of routines takes time and experiencing different task situations, as pointed out by Sfard (2007) and Lavie et al. (2019). In particular, students' inclination to accept and offer empirical argumentation has been well documented in research as difficult to change (Brown, 2014; Stylianides G.J, 2009; Weber, 2010).

Funding

This work is a part of a bigger research project, ProPrimEd - Reasoning and Proving in Primary Education, funded by the Norwegian Research Council and Norwegian University of Science and Technology (NTNU).

Declarations of interest

None.

References

Alcock, L. J., & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics, 69*, 111–129.
Alcock, L., & Weber, K. (2010). Undergraduates' example use in proof construction: Purposes and effectiveness. *Investigations in Mathematics Learning, 3*(1), 1–22.
Balacheff, N. (1988). Aspects of proof in pupil's practice of school mathematics. In D. Pimm (Ed.), *Mathematics, Teachers and Children* (pp. 216–235). Hodder & Stoughton.
Ben-Zvi, D., & Sfard, A. (2007). Ariadne's thread, Daedalus' wings and the learners' autonomy. *Education et didactique, 1*(3), 117–134.
Brown, L., & Coles, A. (2000). Same/different: A 'natural' way of learning mathematics. In T. Nakahara, & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 113–120). Hiroshima: Hiroshima University.
Brown, S. A. (2014). On skepticism and its role in the development of proof in the classroom. *Education Studies in Mathematics, 86*(3), 311–335.
Buchbinder, O., & Zaslavsky, O. (2019). Strengths and inconsistencies in students' understanding of the roles of examples in proving. *The Journal of Mathematical Behaviour, 53*, 129–147.
Cohen, L., Manion, L., & Morrison, K. (2011). Research methods in education (7th ed.). Routledge.
Cooper, J., & Lavie, I. (2021). Bridging incommensurable discourses–A commognitive look at instructional design in the zone of proximal development. *The Journal of Mathematical Behaviour, 61*, Article 100822.
Ellis, A., Lockwood, E., & Ozaltun-Celik, A. (2022). Empirical re-conceptualization: From empirical generalizations to insight and understanding. *The Journal of Mathematical Behavior, 65*, Article 100928.
Epstein, D., & Levy, S. (1995). Experimentation and proof in mathematics. *Notice of the AMS, 42*(6), 670–674.
Fishbein, E. (1982). Intuition and proof. *For the Learning of Mathematics, 3*(2), 9–24.

Garuti, R., Boero, P., & Lemut, E. (1998). Cognitive units of theorems and difficulty of proof. In A. Olivier, & K. Newstead (Eds.), *Proceedings of the 22nd International Conference of the International Group for the Psychology of Mathematical Education* (Vol. 2, pp. 345–352). University of Stellenbosch.
Iannone, P., Inglis, M., Mejía-Ramos, J. P., Simpson, A., & Weber, K. (2011). Does generating examples aid proof production? *Educational Studies in Mathematics, 77*(1), 1–14.
Jahnke, H. N. (2007). Proofs and hypotheses. *ZDM, 39*(1), 79–86.
Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics, 91*(6), 1–16.
Knox, J.I., 2021, How primary school students' arguments develop: Taking initial steps in a deductive discourse. [Unpublished PhD thesis]. University of Auckland. ⟨https://hdl.handle.net/2292/56621⟩.
Knox, J. & Kontorovich, I., 2022, Capitalizing on interdiscursivity to support elementary students to bridge the empirical-deductive gap: The case of parity of numbers. Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education. Advance online publication: ⟨https://storage.ibrida.io/public/cerme12/TWG01/paper/Knox_TWG1_final.pdf⟩.
Knuth, E. J. (2002). Proof as a tool for learning mathematics. *The Mathematics Teacher, 95*(7), 486–490.
Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics, 101*(2), 153–176.
Leron, U., & Zaslavsky, O. (2013). Generic proving: reflections on scope and method. *For the Learning of Mathematics, 33*(3), 24–30.
Lockwood, E., Ellis, A. B., & Lynch, A. G. (2016). Mathematicians' example-related activity when exploring and proving conjectures. *International Journal of Research in Undergraduate Mathematics Education, 2*(2), 165–196.
Lynch, A. G., & Lockwood, E. (2019). A comparison between mathematicians' and students' use of examples for conjecturing and proving. *The Journal of Mathematical Behavior, 53*, 323–338.
Mason, J. (2019). Relationships between proof and examples: Comments arising from the papers in this issue. *The Journal of Mathematical Behaviour, 53*, 339–347.
Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the. *Educational Studies in Mathematics, 15*(2), 277–289.
Mertens, D.M. (2005). Research and evaluation in education and psychology: Integrating diversity with quantitative, qualitative and mixed methods (2nd ed.). Sage.
Nachlieli, T., & Elbaum-Cohen, A. (2021). Teaching practices aimed at promoting meta-level learning: The case of complex numbers. *The Journal of Mathematical Behaviour, 62*, Article 100872.
Nachlieli, T., & Tabach, M. (2012). Growing mathematical objects in the classroom–The case of function. *International Journal of Educational Research, 51*, 10–27.
Ozgur, Z., Ellis, A. B., Vinsonhaler, R., Dogan, M. F., & Knuth, E. (2019). From examples to proof: Purposes, strategies, and affordances of example use. *The. Journal of Mathematical Behaviour, 53*, 284–303.
Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics, 52*(3), 319–325.
Reid, D., & Knipping, C. (2010). Proof in mathematics education: Research, learning and teaching. Sense Publishers.
Reid, D., & Vergas, E. V. (2018). When is a generic argument a proof? In A. J. Stylianides, & G. Harel (Eds.), *Advances in Mathematics Education Research on Proof and Proving. An International Perspective* (pp. 239–252). Springer International.
Rowland, T. (1998). Conviction, explanation and generic examples. In O. Alwyn, & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Physiology of Mathematics Education* (pp. 65–72). University of Stellenbosch.
Selden, A., & Selden, J. (2008). Overcoming students' difficulties in learning to understand and construct proofs. In M. Carlsen, & C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics* (pp. 95–110). Mathematical Association of America.
Sfard, A. (2000). On reform movement and the limits of mathematical discourse. *Mathematical Thinking and Learning, 2*(3), 157–189.
Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *Journal of the Learning Sciences, 16*(4), 565–613.
Sfard, A. (2008). *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*. Cambridge University Press,.
Sfard, A., & Lavie, I. (2005). Why cannot children see as the same what grown-ups cannot see as different? Early numerical thinking revisited. *Cognition and Instruction, 23*(2), 237–309.
Shinno, Y., & Fujita, T. (2021). Characterizing how and when a way of proving develops in a primary mathematics classroom: A commognitive approach. *International Journal of Mathematical Education in Science and Technology*, 1–26.
Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289–321.
Stylianides, A. J., & Ball, D. L. (2008). Understanding and describing mathematical knowledge for teaching: Knowledge about proof for engaging students in the activity of proving. *Journal of Mathematics Teacher Education, 11*(4), 307–332.
Stylianides, A. J., & Stylianides, G. J. (2009a). Proof constructions and evaluations. *Educational Studies of Mathematics, 72*(2), 237–253.
Stylianides, G. J., & Stylianides, A. J. (2009b). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education, 40*(3), 314–352.
Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 237–266). National Council of Teachers in Mathematics.
Tabach, M., & Nachlieli, T. (2016). Communicational perspectives on learning and teaching mathematics: Prologue. *Educational Studies in Mathematics, 91*(3), 299–306.
Weber, K. (2001). Student difficulty in construction proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*(1), 101–119.
Weber, K. (2010). Mathematics majors' perceptions of conviction, validity, and proof. *Mathematical Thinking and Learning, 12*(4), 306–336.
Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *Journal of Mathematical Behaviour, 21*, 423–440.
Yopp, D. A., Ely, R., & Johnson-Leung, J. (2015). Generic example proving criteria for all. *For the Learning of Mathematics, 35*(3), 8–13.
Zack, V. (1997). You have to prove us wrong': proof at the elementary school level. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the International on the Psychology of Mathematics Education* (Vol. 4, pp. 291–312). University of Finland.