Optimal Transmission-Constrained Scheduling of Spatio-temporally Dependent Observations using Age-of-Information

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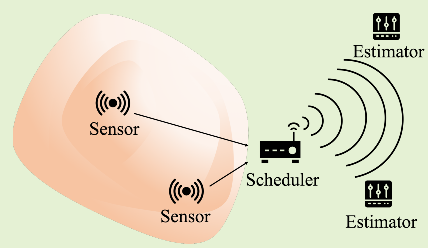
***Abstract*—This paper proposes an optimal scheduling policy for broadcasting spatio-temporally dependent observations to two remote estimators over a finite time horizon. The system comprises a scheduler that can broadcast one observation from one out of two spatio-temporally dependent processes at each time instant. Since the number of broadcasting instants for each sensor is constrained, the scheduler must plan the broadcasting that minimizes the time-averaged estimation error. As the scheduler cannot observe the measurements, it determines the expected estimation error based on the age-of information (AoI). Using AoI as a state variable, we derive a set of optimal scheduling policies that minimizes the average mean squared error (MSE) for any given time horizon. The policies provide the optimal number of transmission instances for each sensor and time-varying AoI thresholds for when to be scheduled. By studying how the MSE evolves with respect to the AoI generated by a given scheduling sequence, we can obtain an optimal policy using a low-complexity numerical method. Numerical results validate the theory and demonstrate how utilizing spatio-temporal dependencies together with AoI can enhance the estimation accuracy in a communication-constrained sensor network.**



***Index Terms*—Age-of-information, Scheduling, Spatio-temporal correlation, Wireless sensor networks**

I. Introduction

ADVANCEMENTS in wireless communication have enhanced the ability to monitor and control systems remotely. Wireless sensor networks (WSNs) and networked controlled systems rely upon sensors observing physical processes and communicating their measurements to remote estimators that track process parameters. However, the information-update rate is limited by both the number of communication channels and the sensor nodes' energy storage capabilities. In many WSNs, sensors transmit their measurements over parallel channels, where each channel can be occupied by one sensor at a given time. If the number of channels is less than the number of sensors, all sensors transmissions must be coordinated. For this reason, a common task is to design scheduling schemes that assign available time slots for each sensor [1], [2].

It is important to design scheduling policies that maximize the utility of the WSN. In the context of remote estimation, the focus is on designing scheduling policies that achieve high estimation accuracy. A common objective is to derive optimal scheduling policies that minimize the time-averaged estimation error. The structure of each optimal scheduling policy depends on the system settings and the nature of the measurement process distribution. Optimal scheduling schemes have been studied under different system scenarios such as limited packet sizes for different source processes [3], limited battery [4], single or multiple communication channels [1], [5], and in the presence of eavesdroppers [6].

Finding optimal scheduling policies involves solving sequential decision-making problems, mainly through dynamic programming, as processes are generally dynamic. As the time horizon grows, the number of potential scheduling trajectories increases, and deriving optimal policies for large time-horizons becomes prohibitive due to high computational complexity. In order to reduce the computational complexity, one approach is to employ a greedy method and find scheduling sequences that minimize the estimation error over shorter time horizons [7], [8]. The disadvantage is that it can lead to sub-optimal performance for extended time horizons.

For large time horizons, a practical approach is to consider an infinite time horizon [1], [5], [9]. This assumption can simplify the mathematical framework when deriving optimal scheduling policies. However, when dealing with non-stationary processes [10] or with stochastic energy supply, e.g., energy harvesting [11], it is beneficial to be able to quickly derive optimal scheduling policies for finite horizons. The performance discrepancy of finite and infinite approaches depends on the time horizon they converge.

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TABLE I: Table of symbols

Symbol	Definition
$\theta_i[k]$	Value of the ith process
σ_i	Standard deviation of the ith process
ρ_{ij}	Spatial correlation between the ith and the jth process
$\rho_t(\cdot)$	Temporal correlation given the time difference
$x_i[k]$	Measurement of the ith sensor
$w_i[k]$	Measurement noise at the ith sensor
ξ_i	Standard deviation of the measurement noise at the ith sensor
$\boldsymbol{\pi}[k]$	Index of the sensor scheduled (broadcasted)
$\Delta_i[k]$	AoI of the ith sensor
$\Delta_{ij}[k]$	AoI differences between the ith and the jth process
$\gamma_k(\cdot)$	Scheduling strategy mapping $\boldsymbol{\Delta}[k-1]$ to $\pi[k]$
$y_i[k]$	Most recently broadcasted measurement from the ith sensor
T	Time horizon
γ	Scheduling policy, i.e., $\gamma = (\gamma_1, \gamma_2, \dots \gamma_T)$
$\hat{\theta}_i[k]$	Estimated value of the ith process
$\Delta_i^\gamma[k]$	AoI of the ith sensor given policy γ
n_i^γ	Number of scheduling instances for the ith sensor given policy γ
$\bar{n}_i$	Transmission constraint for the ith sensor
$J(\gamma)$	Time averaged MSE given policy γ
γ^*	Optimal scheduling policy minimizing $J(\gamma)$
n_i^*	Optimal number of scheduling instances for the ith sensor
$E(\boldsymbol{\Delta}[k])$	MSE at instant k given the AoI
$h_i^\gamma[k]$	*AoI peak* of the ith sensor at instant k, i.e., the immediate AoI before the ith sensor is scheduled or before the complete time period has elapsed
$g_i^\gamma(l)$	Number of instances the ith sensor reaches AoI $l \in \mathbb{N}_+$ given policy $\gamma \in \Gamma$
$\mathbf{m}$	Vector of time-varying AoI thresholds
$\gamma^{\mathbf{m}}$	Policy defined by threshold vector $\mathbf{m}$
$\bar{E}(m)$	Time averaged MSE for scheduling segment of length m where the AoI of Sensor 1 reaches $m-1$ before Sensor 2 is scheduled
$\hat{m}$	Value m minimizing $\bar{E}(m)$
$\delta(\cdot)$	Dirac delta function
$\mathbb{1}(\cdot)$	Indicator function having value 1 if the condition in the argument is true and 0 otherwise

Although sensor observations are correlated in practice, only a few works investigate scheduling policies that account for dependency among the observations. In resource-constrained WSNs, sensor dependencies have been exploited to achieve energy-efficient routing [12], [13], missing data inference in environmental crowd sensing [14], optimal sensor location selection [15], and reducing traffic load [16]. For a remote estimation WSN where sensors transmit noisy measurements of a Gaussian source process over multiple access channels (MAC), the estimation accuracy is normally restricted by the transmission powers of the sensors and the number of antennas at the remote estimator [17], [18]. However, the results in [18] demonstrate once the transmission power allocation is optimized to maximize the overall estimation accuracy, a higher degree of dependence among measurements significantly reduces the minimum MSE. In [11], [19], an optimal scheduling policy is obtained, assuming a scheduler that observes measurements from all the sensors before selecting a finite number of measurements to communicate to the remote estimator. However, such a solution may not always be feasible due to privacy and latency constraints.

For an observation-driven scheduling scheme, all sensors consistently measure and transmit measurements to the centralized network scheduler. This approach can result in additional delays due to the communication bottleneck at the scheduler. In addition, it makes the scheduler a vulnerable security point to be targeted by an eavesdropper or malicious attacker [20]. Finally, it presumes all sensors continually measure and transmit measurements to the scheduler, spending excessive energy.

In this paper, we present an optimal scheduling policy for a finite time horizon for two sensors observing spatio-temporally observations. At each time instant, an observation from one of the processes is broadcast through a network scheduler to two remote estimators with each tracking one process. We consider a system model similar to [19], but with two differences. Firstly, we assume a transmission constraint on each sensor to account for limited energy resources at the sensor nodes. Secondly, the scheduler cannot view the measurements. Instead, the scheduling policies can be derived based on the timeliness of the information received at the estimators, i.e., the age-of-information (AoI) [21]–[23].

The AoI is defined as the freshness of information, i.e., the elapsed time between the received measurement at the estimator and its generation at the source. In remote estimation systems, the real-time estimation accuracy depends on the AoI at the estimator. However, the relationship is commonly nonlinear and, minimizing the AoI does not consistently lead to optimal performance [22]. Instead, the AoI can be utilized as a state variable to design optimal scheduling policies in remote estimation tasks. Most previous works related to AoI in the context of remote estimation and scheduling assume independent sensor observations [7], [24]–[29].

Recent works have allowed for dependent observations when utilizing AoI to find optimal scheduling policies. In [30], the authors propose a policy that minimizes the average AoI for a WSN, where observations from multiple sensors need to be collected to reconstruct one of many source processes. In [31], the authors consider the problem of sensors with overlapping observation areas, monitoring processes that generate updates at a random frequency. Our work is mostly related to [4], [32], [33], which study the transmission frequency of spatio-temporally correlated sensor measurements, also modeled by a random field. In contrast, our policy provides a scheduling decision based on the AoI for each available discrete time slot instead of deciding transmission rates for each sensor, which allows us to exploit full channel capacity. Additionally, we assume observations being corrupted by measurement noise and allow for processes with different marginal variances.

To clarify the difference to the related work [19] regarding scheduling observations of two correlated Gaussian processes for a remote estimation WSN, we assume that: i) the two processes are spatio-temporally correlated and derive an optimal policy over a finite time horizon; ii) the scheduler cannot observe the measurements and determines the scheduling decision based on the AoI, and; iii) observations are corrupted by measurement noise.

The main contributions of this paper are as follows. We first prove the existence of an optimal scheduling policy for a general set of spatio-temporal correlation functions with sensor transmission constraints. After that, we show how to derive an optimal scheduling sequence for a given horizon using a low-complexity method. Although dependency adds additional complexity to scheduling problems, our method does not rely on dynamic programming, and the computational complexity does not become prohibitive beyond a given time horizon. The results corroborate the theory and demonstrate the performance enhancement of exploiting spatio-temporal dependencies in communication-constrained systems. Furthermore, we can utilize the results to derive necessary energy resources to achieve optimal performance in similar systems.

The remainder of the paper is organized as follows. Section II presents the system model and the scheduling problem. Next, in Section III, we demonstrate the structure of an optimal policy and how to derive an optimal policy. Section IV presents numerical results to validate the theory, and compare the performance improvement of exploiting spatio-temporal correlation. Section V concludes the paper.

II. Background and Problem Formulation

As illustrated in Fig. 1, we consider a WSN consisting of one scheduler, two sensors, and two remote estimators. The ith sensor observes the stochastic process $\theta_i[k] \in \mathbb{R}$, with $\theta_i[k] \sim \mathcal{N}(0, \sigma_i^2)$, at time instant $k \in \mathbb{N}_+$ and $i = 1, 2$. The distributions of the two processes are not assumed to be homogenous, where without loss of generality $\sigma_1 \geq \sigma_2$. Furthermore, they are correlated over space and time with the cross-covariance given by a positive-definite function [34]

$$\mathbb{E}[\theta_i[k]\theta_j[l]] = \sigma_i \sigma_j \rho_{ij} \rho_t(|k-l|), \quad i, j \in \{1, 2\}, \tag{1}$$

where $\rho_{ij} \in [-1, 1]$ denotes the spatial correlation and $\rho_t : \mathbb{R}_+ \to (0, 1]$ represents the temporal correlation, which is a strictly decreasing function with $\rho_t(0) = 1$ and $\lim_{n\to\infty} \rho_t(n) = 0$. At time instant k, the ith sensor acquires measurement $x_i[k] \in \mathbb{R}$, which is modeled as

$$x_i[k] = \theta_i[k] + w_i[k], \quad k \in \mathbb{N}_+, \quad i = 1, 2, \tag{2}$$

where $w_i[k] \in \mathbb{R}$ is independent measurement noise with distribution $w_i[k] \sim \mathcal{N}(0, \xi_i^2)$. For each process $\theta_i[k]$, there exists a corresponding remote estimator that tracks the stochastic process and computes an estimate $\hat{\theta}_i[k]$ based on sensor measurements transmitted by the network scheduler, as can be seen in Fig. 1.

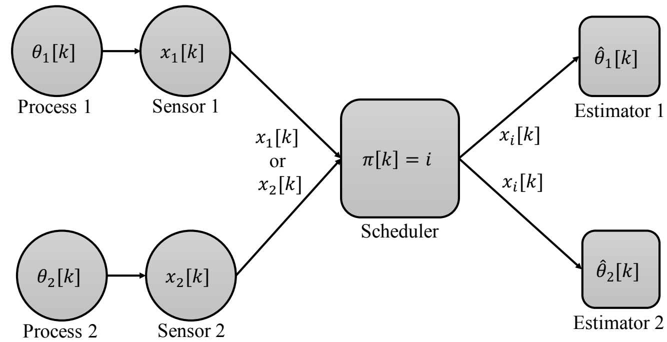

Fig. 1: Scheduling problem for remote estimation in a WSN consisting of two sensors and two remote estimators.

A. Scheduler

At every time instant, the scheduler can broadcast only one sensor observation to the remote estimators. Since the processes are spatio-temporally correlated, the estimators can use every measurement to improve the local estimation accuracy.

Let $\pi[k] \in \{1, 2\}$ denote the index of the sensor that is scheduled at time instant k. The AoI of the ith sensor is denoted by $\Delta_i[k] \in \mathbb{N}_+$, $i = 1, 2$, and defined as the time elapsed between two successive measurement transmissions from the ith sensor [32], i.e.,

$$\Delta_i[k] = \begin{cases} 0, & \text{if } i \in \boldsymbol{\pi}[k], \\ \Delta_i[k-1] + 1, & \text{if } i \notin \boldsymbol{\pi}[k]. \end{cases} \tag{3}$$

The scheduler is not allowed to observe the measurements, $\boldsymbol{x}[k] = [x_1[k], x_2[k]]^{\mathrm{T}}$; however, it can keep track of the AoI for each sensor using $\boldsymbol{\Delta}[k] = [\Delta_1[k], \Delta_2[k]]^{\mathrm{T}}$. Let γ_k denote the *scheduling strategy* at time k, i.e.,

$$\boldsymbol{\pi}[k] = \gamma_k(\boldsymbol{\Delta}[k-1]), \tag{4}$$

which provides a mapping from $\boldsymbol{\Delta}[k-1]$ to the scheduling decision at instant k.

B. Remote Estimators

The data available at Estimator i, at time instant k, consists of AoI $\boldsymbol{\Delta}[k]$ and $\boldsymbol{y}[k] = [y_1[k], y_2[k]]^{\mathrm{T}}$, representing the most recently broadcasted measurement from each sensor, i.e.,

$$y_i[k] = x_i[k - \Delta_i[k]], \quad i = 1, 2. \tag{5}$$

The minimum mean square error (MMSE) estimate of $\theta_i[k]$ given the information $\{\boldsymbol{\Delta}[k], \boldsymbol{y}[k]\}$ is computed as

$$\hat{\theta}_i[k] = \mathbb{E}[\theta_i[k] | \boldsymbol{\Delta}[k], \boldsymbol{y}[k]], \quad i = 1, 2. \tag{6}$$

C. Scheduling Policy

The *scheduling policy* γ is defined as the collection $\gamma = (\gamma_1, \gamma_2, \ldots \gamma_T)$, where T indicates the time horizon. The performance measure, or cost, is taken as the average mean squared error (MSE) of the estimate (6) over T time slots and is given by

$$J(\gamma) = \frac{1}{T} \sum_{k=1}^{T} \sum_{i=1}^{2} \mathbb{E}\Big[(\theta_i[k] - \hat{\theta}_i[k])^2 \Big| \boldsymbol{\Delta}^{\gamma}[k] \Big], \tag{7}$$

where $\boldsymbol{\Delta}^\gamma[k] = [\Delta_1^\gamma[k], \Delta_2^\gamma[k]]^\mathrm{T}$ is the AoI at time k generated by γ and $\boldsymbol{\Delta}[0] = [1, 0]^\mathrm{T}$ when initializing the system.

Let n_1^γ and n_2^γ, respectively, denote the number of instants Sensor 1 and Sensor 2 are scheduled using policy γ. The values of n_1^γ and n_2^γ can be computed as

$$n_1^\gamma = \sum_{i=1}^{T} \mathbb{1}(\Delta_1^\gamma[i] = 0), \quad n_2^\gamma = \sum_{i=1}^{T} \mathbb{1}(\Delta_2^\gamma[i] = 0), \tag{8}$$

where n_1^γ, $n_2^\gamma \in \{0, 1, ..., T\}$, $n_1^\gamma + n_2^\gamma = T$ and $\mathbb{1}(\cdot)$ is an indicator function which equals 1 if the condition in the argument is true, 0 otherwise.

Without loss of generality, let us assume each transmission costs unit energy. Also, let $\bar{n}_1, \bar{n}_2 \in \mathbb{N}_{++}$, be the energy resources available for transmitting data, i.e., Sensor i can transmit $\bar{n}_i$ measurements. We assume that the total energy available at the sensors satisfies $\bar{n}_1 + \bar{n}_2 \geq T$.

Our objective is to find an *optimal scheduling policy* γ^* that minimizes the average cost in (7) over any time-horizon

$$\begin{aligned} \min_{\gamma \in \Gamma} \quad & J(\gamma), \\ \text{s. t.} \quad & n_i^\gamma \leq \bar{n}_i, \quad \bar{n}_i \in \mathbb{N}_{++}, i = 1, 2, \\ & n_1^\gamma + n_2^\gamma = T, \end{aligned} \tag{9}$$

where Γ is the set of all feasible policies, and $\bar{n}_i$, $i = 1, 2$, is a transmission constraint on each sensor.

III. Optimal Scheduling Policy

In this section, we will derive an optimal scheduling policy in two steps. We begin by deriving an optimal scheduling policy for the specific case; having no transmission constrains, i.e., $\bar{n}_i > T$, $i = 1, 2$, and the number of transmission instances for the ith sensor must equal $n_i \in \mathbb{N}_+$, $i = 1, 2$. This results in the following optimization problem

$$\begin{aligned} \min_{\gamma \in \Gamma} \quad & J(\gamma), \\ \text{s. t.} \quad & n_i^\gamma = n_i, \quad i = 1, 2, \quad n_1 + n_2 = T. \end{aligned} \tag{10}$$

To solve (10), we first derive an expression for how the MSE at instant k depends on $\boldsymbol{\Delta}[k]$ and, then, analyze how the MSE evolves for any given process $\boldsymbol{\Delta}^\gamma[k]$. Thereafter, we present a method to attain the optimal number of scheduling instances for Sensor 1, n_1^*, and Sensor 2, n_2^*, for different transmission constrains $\bar{n}_i$, $i = 1, 2$. Finally, once n_1^* and n_2^* are known, we use the solution for (10) to derive an optimal policy γ^*.

In order to find γ^*, we need to calculate the cost in (7) for any given policy $\gamma \in \Gamma$. The MSE at time k depends on $\boldsymbol{\Delta}^\gamma[k]$, which is perfectly known using (3). Let $E(\cdot)$ denote the MSE at instant k given the AoI $\boldsymbol{\Delta}[k]$ and be defined as

$$E(\Delta_1[k], \Delta_2[k]) = \sum_{i=1}^{2} \mathbb{E}\Big[(\theta_i[k] - \hat{\theta}_i[k])^2 \Big| \boldsymbol{\Delta}[k]\Big]. \tag{11}$$

As shown in [9], we can derive a closed-form expression for the MSE (11) as

$$\begin{aligned} & E(\Delta_1[k], \Delta_2[k]) = (\sigma_1^2 + \sigma_2^2) \\ & + \beta[k]\Big(2(\sigma_1\sigma_2\rho_{12})^2 \rho_t(\Delta_1[k])\rho_t(\Delta_2[k])\rho_t(\Delta_{12}[k])(\sigma_1^2 + \sigma_2^2) \\ & - \rho_t^2(\Delta_1[k])(\sigma_2^2 + \xi_2^2)((\sigma_1\sigma_2\rho_{12})^2 + \sigma_1^4) \\ & - \rho_t^2(\Delta_2[k])(\sigma_1^2 + \xi_1^2)((\sigma_1\sigma_2\rho_{12})^2 + \sigma_2^4)\Big), \end{aligned} \tag{12}$$

where $\beta[k] = \big((\sigma_1^2 + \xi_1^2)(\sigma_2^2 + \xi_2^2) - (\sigma_1\sigma_2\rho_{12})^2\rho_t^2(\Delta_{12}[k])\big)^{-1}$ and $\Delta_{ij}[k] = |\Delta_i[k] - \Delta_j[k]| \in \mathbb{N}_+$ is the AoI-difference between the two sensors. By analyzing the properties of $E(\Delta_1[k], \Delta_2[k])$ in (12), we can better understand how the MSE evolves for any process $\boldsymbol{\Delta}^\gamma[k]$, which can help in the derivation of an optimal policy γ^*.

From (3), we know that one sensor is scheduled at each time instant k and the process $\boldsymbol{\Delta}^\gamma[k]$ evolves as

$$[\Delta_1^\gamma[k], \Delta_2^\gamma[k]]^\mathrm{T} = \begin{cases} [0, \Delta_2^\gamma[k-1] + 1]^\mathrm{T}, & \text{if } \pi[k] = 1, \\ [\Delta_1^\gamma[k-1] + 1, 0]^\mathrm{T}, & \text{if } \pi[k] = 2. \end{cases} \tag{13}$$

For $E(1, 0) \geq E(0, 1)$, we see that the following properties hold for the function in (12):

$$E(0, \Delta_2[k]) \leq E(\Delta_1[k], 0), \quad \forall \Delta_2[k] \leq \Delta_1[k], \tag{14a}$$

$$E(0, \Delta_2[k]) \leq E(0, \Delta_2[k] + \epsilon) \leq E_2^\infty, \quad \epsilon \in \mathbb{N}_+, \tag{14b}$$

$$E(\Delta_1[k], 0) \leq E(\Delta_1[k] + \epsilon, 0) \leq E_1^\infty, \quad \epsilon \in \mathbb{N}_+, \tag{14c}$$

where the upper bounds in (14b) and (14c) are given by

$$\begin{aligned} E_1^\infty &= \lim_{\Delta_1[k] \to \infty} E(\Delta_1[k], 0) \leq \sigma_1^2 + \sigma_2^2, \\ E_2^\infty &= \lim_{\Delta_2[k] \to \infty} E(0, \Delta_2[k]) \leq \sigma_1^2 + \sigma_2^2. \end{aligned} \tag{15}$$

Equations (14) imply that if $E(1, 0) \geq E(0, 1)$ holds, any given AoI value, $\Delta \in \mathbb{N}_+$, generated at Sensor 1 results in a higher MSE than if generated at Sensor 2, i.e., $E(\Delta, 0) \geq E(0, \Delta)$, $\forall \Delta \in \mathbb{N}_+$. As seen in (12), the inequality, $E(1, 0) \geq E(0, 1)$, depends on the statistical properties $\sigma_1, \sigma_2, \xi_1$, and ξ_2, where σ_1 and σ_2 are the dominent factors. For our system, we assume that $E(1, 0) \geq E(0, 1)$ holds based on the following reasons. Firstly, we know that the marginal variance of Process 1, σ_1, is greater or equal to the marginal variance of Process 2, σ_2, i.e., $\sigma_1 \geq \sigma_2$. Secondly, we assume that the variances of the measuement noises ξ_1 and ξ_2 have similar magnitude, i.e., $\xi_1 \approx \xi_2$, and are not greater than the variances of the process $\sigma_i \geq \xi_j$, $i, j = 1, 2$. For this scenario, based on (12), we state the following assumption.

Assumption 1: The inequality $E(1, 0) \geq E(0, 1)$ holds, yielding that the properties in (14) also hold.

The properties in (14) show that the MSE increases with the AoI but is bounded as MSE $\leq \sigma_1^2 + \sigma_2^2$. Furthermore, as seen in inequality (14a), a given AoI for Sensor 1 has a larger MSE than for Sensor 2. Thus, we make the following two conclusions of an optimal policy; firstly, given the optimal number of transmission instances for each sensor, n_i^*, $i = 1, 2$, an optimal policy should result in a scheduling sequence that minimizes the maximum AoI for each sensor. Secondly, if $\bar{n}_1 \geq \bar{n}_2$, then Sensor 2 is not scheduled more times than Sensor 1. Based on these two properties, we will in the following theorem present a solution to (10) and a reduced feasible set for n_1^* and n_2^*. To do so, we first introduce some necessary mathematical notations.

Let $h_i^\gamma[k] \in \mathbb{N}_+$, $i = 1, 2$, represent the *AoI peaks*, i.e., the immediate AoI before Sensor i is scheduled or before the complete time period has elapsed, and be defined as

$$h_i^\gamma[k] = \begin{cases} \Delta_i^\gamma[k], & \text{if } k = T \text{ or } \Delta_i^\gamma[k+1] = 0, \\ 0, & \text{else .} \end{cases} \tag{16}$$

From (13), together with initial AoI $\mathbf{\Delta}[0] = [1, 0]^{\mathrm{T}}$, the cumulative sum of the AoI peaks satisfy

$$\sum_{k=1}^{T} h_i^\gamma[k] = \begin{cases} n_2^\gamma + \mathbb{1}(\gamma_1(\mathbf{\Delta}[0]) = 2), & \text{if } i = 1, \\ n_1^\gamma, & \text{if } i = 2. \end{cases} \tag{17}$$

Let $\bar{\Delta}_i : \mathbb{N}_+ \to \{0, 1, ..., T+1\}$, represent the lowest maximum AoI (minmax) for Sensor i, given that Sensor i is scheduled l instances during interval $k \in [1, T]$, and be defined as

$$\bar{\Delta}_i(l) = \min_{\gamma \in \Gamma} \sup_{k=1,...,T} \left\{ h_i^\gamma[k] \middle| n_i^\gamma = l \right\}, \quad i = 1, 2. \tag{18}$$

Since $\mathbf{\Delta}[0] = [1, 0]^{\mathrm{T}}$, to achieve (18), given pair $n_i^\gamma = l$ and $n_j^\gamma = T - l$, $i \neq j$, the scheduling sequence of Sensor 1 and 2 must be as evenly distributed as possible. For the ith sensor, such a sequence will result in either l, or $l+1$, non-zero AoI peaks, i.e., $h_i^\gamma[k] > 0$, having either one, or two, distinct values, i.e.,

$$h_i^\gamma[k] \in \left\{0, \bar{\Delta}_i(n_i^\gamma) - \mathbb{1}(n_i^\gamma \neq T), \bar{\Delta}_i(n_i^\gamma)\right\}, \ i = 1, 2. \tag{19}$$

The value $\bar{\Delta}_i(l)$ is approximately the average of all the non-zero AoI peaks, rounded-up, to adjust for $\bar{\Delta}_i(l) \in \mathbb{N}_+$. From (17), we derive a closed form expression of $\bar{\Delta}_i(l)$ as

$$\begin{aligned} &\bar{\Delta}_1(l) = \left\lceil \frac{T+1-l}{l+1} \right\rceil, \ \bar{\Delta}_2(l) = \left\lceil \frac{T-l}{l+1} \right\rceil, \ l = 0, ..., T-1 \\ &\bar{\Delta}_1(T) = \bar{\Delta}_2(T) = 0, \end{aligned} \tag{20}$$

where $\lceil \cdot \rceil$ is the ceil operator.

Finally, let $g_i^\gamma : \mathbb{N}_+ \to \mathbb{N}_+, i = 1, 2$, represent the number of instances the ith sensor reaches AoI $l \in \mathbb{N}_+$ given policy $\gamma \in \Gamma$, and be defined as

$$g_i^\gamma(l) = \sum_{k=1}^{T} \mathbb{1}(\Delta_i^\gamma[k] = l). \tag{21}$$

Theorem 1: $i)$ Under Assumption 1, a policy $\gamma \in \Gamma$ is a solution to (10), if it minimizes the maximum AoI as

$$\Delta_i^\gamma[k] \leq \bar{\Delta}_i(n_i), \quad k = 1, ..., T, \quad i = 1, 2, \tag{22}$$

and the number of instances the AoI equals $\bar{\Delta}_i(n_i)$ and $\bar{\Delta}_i(n_i) - 1$, $i =, 1, 2$, given in (20), can be computed as; if $n_i = T$ and $n_j = 0$, $i \neq j$, then $\bar{\Delta}_i(n_i) = 0$, and

$$g_i^\gamma(0) = n_i, \quad g_j^\gamma\left(\bar{\Delta}_j(0)\right) = 1, \tag{23}$$

else, if $1 \leq n_j \leq n_i$, then $\bar{\Delta}_i(n_i) = 1$, and

$$\begin{aligned} g_i^\gamma(1) = n_j&, \quad g_i^\gamma(0) = n_i \\ g_j^\gamma\left(\bar{\Delta}_j(n_j)\right) &= n_i - (n_j+1)\left(\bar{\Delta}_j(n_j) - 1\right) + \mathbb{1}(j = 1), \\ g_j^\gamma\left(\bar{\Delta}_j(n_j) - 1\right) &= n_j + 1 - \mathbb{1}(j = 1)\mathbb{1}\left(\bar{\Delta}_j(n_j) = 2\right) \\ &\quad - \mathbb{1}\left(\bar{\Delta}_j(n_j) = 1\right). \end{aligned} \tag{24}$$

$ii)$ Under Assumption 1, for problem (9), the optimal number of scheduling instances satisfies

$$\begin{aligned} n_1^* &= \bar{n}_1, & &\text{if} \quad \bar{n}_1 \leq T/2 \\ n_1^* &\geq T/2, & &\text{if} \quad \bar{n}_1 > T/2, \end{aligned} \tag{25}$$

and $n_2^* = T - n_1^*$.

Proof: To prove Theorem 1, we first, using (12), define the two functions $\tilde{E}_i(k) : \mathbb{N}_+ \to \mathbb{N}_+$, $i = 1, 2$, which gives the cumulative MSE of scheduling Sensor $i = 1, 2$, for k consecutive time instances, where $\tilde{E}_i(0) = 0$ and

$$\tilde{E}_1(k) = \sum_{i=1}^{k} E(0, i), \quad \tilde{E}_2(k) = \sum_{i=1}^{k} E(i, 0), \quad k \geq 1. \tag{26}$$

From (26), we find that $\tilde{E}_i(k)$ can be expressed as

$$\begin{aligned} \tilde{E}_1(k+\epsilon) &= \tilde{E}_1(k) + \sum_{l=k+1}^{k+\epsilon} E(0, l), \quad \epsilon \in \mathbb{N}_{++}, \\ \tilde{E}_2(k+\epsilon) &= \tilde{E}_2(k) + \sum_{l=k+1}^{k+\epsilon} E(l, 0), \quad \epsilon \in \mathbb{N}_{++}. \end{aligned} \tag{27}$$

Thus, from (14) and (27), the following inequality holds

$$\begin{aligned} \tilde{E}_i(k) + \tilde{E}_i(\epsilon) &\leq \tilde{E}_i(k+\epsilon), & &\epsilon \in \mathbb{N}_+, \, i = 1, 2, \\ \tilde{E}_1(k) &\leq \tilde{E}_2(k), & &\forall k \in \mathbb{N}_+. \end{aligned} \tag{28}$$

Using the definitions in (26) and (16)-(17), we reformulate the optimization problem in (10), where the number of scheduling instances must equal n_i, i.e., $n_i^\gamma = n_i$, $i = 1, 2$, as

$$\begin{aligned} \min_{\gamma \in \Gamma} \quad & \frac{1}{T}\Big(\sum_{k=1}^{T} \tilde{E}_1(h_2^\gamma[k]) + \sum_{k=1}^{T} \tilde{E}_2(h_1^\gamma[k]\Big) \\ & - T^{-1}\mathbb{1}(\gamma_1(\mathbf{\Delta}[0]) = 2)E(1, 0) \\ \text{s. t.} \quad & \sum_{k=1}^{T} h_1^\gamma[k] = n_2 + \mathbb{1}(\gamma_1(\mathbf{\Delta}[0]) = 2), \quad \sum_{k=1}^{T} h_2^\gamma[k] = n_1. \end{aligned} \tag{29}$$

As seen in the constraints of (29), the cumulative sum of the AoI peaks is constant for Sensor 2, and can only take two possible values for Sensor 1. Thus, we infer from (28) that to solve (29) the AoI peaks $h_i^\gamma[k]$, $i = 1, 2$, should be minimized. To achieve this, a solution to (29) must result in an evenly distributed scheduling sequence that, firstly, minimizes the maximum AoI, which using (19), gives

$$h_i^\gamma[k] \in \{0, \bar{\Delta}_i(n_i) - \mathbb{1}(n_i \neq T), \bar{\Delta}_i(n_i)\}, \ i = 1, 2. \tag{30}$$

Secondly, a solution that maximizes the number of instances the AoI equals $\bar{\Delta}_i(n_i) - \mathbb{1}(n_i \neq T), i = 1, 2$. Hence, a solution to (10), is obtained by solving the optimization problem

$$\begin{aligned} \max_{\gamma \in \Gamma} \quad & g_i^\gamma\left(\bar{\Delta}_i(n_i) - \mathbb{1}(n_i \neq T)\right), \quad i = 1, 2 \\ \text{s. t.} \quad & \Delta_j^\gamma[k] \leq \bar{\Delta}_j(n_j), \quad j = 1, 2, \, k = 1, ..., T, \end{aligned} \tag{31}$$

where the constraints in (31) proves (22).

Proving (23) is straightforward; if only Sensor i is scheduled, i.e., $n_i^\gamma = T$ and $n_j^\gamma = 0$, $i \neq j$, the AoI will never be greater than zero, $\Delta_i[k] = 0$, $k = 1, ..., T$, while for Sensor j the maximum AoI will be reached only once at $k = T$.

To prove (24), where $n_i \geq 1$, $i = 1, 2$, we form a linear expression given the criteria in (30) as

$$\sum_{k=1}^{T} h_i^{\gamma}[k] = \alpha_i \bar{\Delta}_i(n_i) + \beta_i \left(\bar{\Delta}_i(n_i) - 1 \right), \tag{32}$$

where $\alpha_i, \beta_i \in \mathbb{N}_+$, represent the number of AoI peaks $h_i^{\gamma}[k]$, $i = 1, 2$, that equal, respectively, $\bar{\Delta}_i(n_i)$ and $\bar{\Delta}_i(n_i) - 1$. Based on (29)-(32), we formulate (21) in terms of α_i and β_i as

$$g_i^{\gamma}\left(\bar{\Delta}_i(n_i)\right) = \alpha_i, \tag{33}$$

$$g_i^{\gamma}\left(\bar{\Delta}_i(n_i) - 1\right) = \begin{cases} n_i, & \text{if } \bar{\Delta}_i(n_i) = 1, \\ \alpha_i + \beta_i - 1, & \text{if } i = 1,\ \bar{\Delta}_i(n_i) = 2, \\ \alpha_i + \beta_i, & \text{else.} \end{cases}$$

From (31) we see that β_i, $i = 1, 2$, should be maximized and we must derive expressions for α_i and β_i, given n_1 and n_2.

For $n_1 < n_2$, we have that $\bar{\Delta}_2(n_2) = 1$, $\gamma_1(\mathbf{\Delta}[0]) = 2$, $\sum_{k=1}^{T} h_1^{\gamma}[k] = n_2 + 1$, and $\alpha_1 + \beta_1 = n_1 + 1$, with (32), gives

$$\alpha_1 = (n_2 + 1) - (n_1 + 1)\left(\bar{\Delta}_1(n_1) - 1\right),\ \beta_1 = n_1 + 1 - \alpha_1,$$
$$\alpha_2 = n_1, \quad \beta_2 = 0. \tag{34}$$

For $n_1 = n_2$, we have that $\bar{\Delta}_i(n_i) = 1$, $i = 1, 2$, and

$$\alpha_1 = \alpha_2 = n_2, \quad \beta_1 = \beta_2 = 0. \tag{35}$$

For $n_1 > n_2$, we have that $\bar{\Delta}_1(n_1) = 1$, $\gamma_1(\mathbf{\Delta}[0]) = 1$, $\sum_{k=1}^{T} h_1^{\gamma}[k] = n_1$ and $\alpha_2 + \beta_2 = n_2 + 1$, with (32), gives

$$\alpha_1 = n_2, \quad \beta_1 = 0, \tag{36}$$
$$\alpha_2 = n_1 - (n_2 + 1)\left(\bar{\Delta}_2(n_2) - 1\right),\ \beta_2 = n_2 + 1 - \alpha_2.$$

Given the values of n_1 and n_2, $n_i \geq 1$, $i = 1, 2$, we substitute the corresponding expression (34)-(36) into (33), proving (24).

To prove (25), we see from (28) that for γ^* to satisfy (29), Sensor 1 should be scheduled an equal number of instants, or more, than Sensor 2. This gives, that if $\bar{n}_1 \geq T/2$, Sensor 1 should be scheduled $n_1^* \geq n_2^*$; otherwise, if $\bar{n}_1 < T/2$, then $n_1^* = \bar{n}_1$, which proves (25). ∎

Theorem 1 implies that if the number of scheduling instances n_1^* and n_2^* are known, we can use expressions (22) - (24) to derive an optimal policy γ^*. The criteria for an optimal policy is that; for Sensor i, $i = 1, 2$, the maximum AoI must equal $\bar{\Delta}_i(n_i^*)$ in (22), and the number of instants the AoI reaches $\bar{\Delta}_i(n_i^*)$ and $\bar{\Delta}_i(n_i^*) - 1$ during time interval $[1, T]$ must satisfy (23) and (24). Hence, we conclude, that there can be one or more optimal policies as long as they satisfy (22) to (24).

From (25), we know that if $\bar{n}_1 \leq T/2$, then $n_1^* = \bar{n}_1 = T - n_2^*$ and we can again use (22)–(24), to obtain γ^*. In the next section, we show how to derive n_1^* and n_2^* for the case $\bar{n}_1 > T/2$ in (25), to obtain γ^*.

A. Optimal Scheduling Policy for $\bar{n}_1 > T/2$

In Theorem 1, we presented the structure of an optimal policy γ^* if either; n_1^* or n_2^* is known. To find n_i^*, $i = 1, 2$, one approach is to construct an optimal policy that satisfies Theorem 1, for all feasible pairs n_1^{γ} and n_2^{γ} in (9), and compare the performance using (7) and (12). However, this becomes cumbersome if the number of feasible pairs are large.

In this section, we will present a low-complexity method to reduce the number of feasible values of n_2^*, and subsequently derive γ^*. Based on the results in Theorem 1, we begin formulating the structure of an optimal policy for the case $\bar{n}_1 > T/2$. The structure is based on defining AoI thresholds, indicating when Sensor 1 is to be scheduled. We then derive optimal thresholds, which implicitly gives us n_2^*.

In Theorem 1, we see in (25) that if $\bar{n}_1 > T/2$ then

$$\max\{T - \bar{n}_1, 0\} \leq n_2^* \leq \lfloor T/2 \rfloor \leq n_1^* \leq \bar{n}_1, \tag{37}$$

and, from (24), that $\bar{\Delta}_1(n_1^*) \leq 1$. This implies that γ^* always results in Sensor 1 being scheduled immediately after Sensor 2, i.e., if $\Delta_2^{\gamma^*}[k] = 0$, $1 \leq k \leq T - 1$ then $\Delta_1^{\gamma^*}[k+1] = 0$. Based on this property, we define a policy that satisfies the conditions of an optimal policy γ^*. We begin by defining the scheduling strategies in (4) that the policy will consist of.

Let γ_k^m, $k \in \mathbb{N}_+$ be a scheduling strategy defined as

$$\gamma_k^m(\mathbf{\Delta}[k-1]) = \begin{cases} 1, & \text{if } \Delta_2[k-1] + 1 < m, \\ 2, & \text{if } \Delta_2[k-1] + 1 \geq m, \end{cases} \tag{38}$$

where $m \in \mathbb{N}_+$ and $m \geq 2$. The value m of γ_k^m is referred to as a *threshold* of γ_k^m, which implies that Sensor 2 will be scheduled at instant k if the AoI of Sensor 2, otherwise, becomes greater than or equal m, i.e., $\Delta_2[k] \geq m$.

Let $\gamma^{\mathbf{m}}$ be a policy consisting of scheduling strategies as in (38) expressed as

$$\gamma^{\mathbf{m}} = (\underbrace{\gamma_1^{m_1}, \ldots, \gamma_{m_1}^{m_1}}_{m_1}, \underbrace{\gamma_{m_1+1}^{m_2}, \ldots, \gamma_{m_1+m_2}^{m_2}}_{m_2}, \ldots, \gamma_T^{m_N}), \tag{39}$$

where threshold vector $\mathbf{m} = [m_1, m_2, \ldots, m_N]^{\mathrm{T}}$, $m_i \in \{2, 3, ..., T+1\}$, $i = 1, \ldots, N$, and $\mathbf{m} \in \mathbb{R}_+^N$, $N \leq T$. The policy $\gamma^{\mathbf{m}}$ is defined by $\mathbf{m}$, which will be referred to as the *threshold vector*. Let $c(m, \mathbf{m})$ be a function that counts the number of threshold elements in $\mathbf{m}$ that equals m, i.e.,

$$c(m, \mathbf{m}) = \sum_{i=1}^{N} \mathbb{1}(m_i = m), \quad m \in \{2, 3..., T+1\}, \tag{40}$$

which satisfies

$$N = \sum_{m=2}^{T+1} c(m, \mathbf{m}). \tag{41}$$

The sum of all the thresholds satisfy

$$T \leq \sum_{i=1}^{N} m_i \leq T + 1. \tag{42}$$

From (42), the scheduling decision at instant T for $\gamma^{\mathbf{m}}$ is

$$\pi[T] = \begin{cases} 1, & \text{if } \sum_{i=1}^{N} m_i = T + 1, \\ 2, & \text{if } \sum_{i=1}^{N} m_i = T. \end{cases} \tag{43}$$

For example, if $N = 2$, $\mathbf{m} = [m_1, m_2]^{\mathrm{T}}$ and $\sum_{i=1}^{N} m_i = T$, the policy would generate a scheduling sequence as

$$(\pi[1], \pi[2], \ldots, \pi[T]) = (\underbrace{1, 1, \ldots, 1, 2}_{m_1}, \underbrace{1, 1, \ldots, 1, 2}_{m_2}). \tag{44}$$

As seen in (44), a policy $\gamma^{\mathbf{m}}$ results in N scheduling segments, where threshold m_i corresponds to segment i, in which, Sensor

1 is scheduled $m_i - 1$ consecutive time instances. From (8) and (43), the number of instances Sensor 1 and Sensor 2 is scheduled during $k \in [1, T]$ is

$$n_1^{\gamma^{\mathbf{m}}} = \sum_{i=1}^{N} (m_i - 1) \tag{45}$$

$$n_2^{\gamma^{\mathbf{m}}} = \begin{cases} N, & \text{if } \sum_{i=1}^{N} m_i = T, \\ N-1, & \text{if } \sum_{i=1}^{N} m_i = T+1. \end{cases}$$

The policy $\gamma^{\mathbf{m}}$ satisfies criteria (25) in Theorem 1. Thus, by finding an optimal threshold vector $\mathbf{m}^*$ that satifies

$$\min_{\mathbf{m}} \quad J(\gamma^{\mathbf{m}}), \tag{46}$$

where $\mathbf{m}^* = [m_1^*, m_2^*, ..., m_{N^*}^*]^T$ and $N^* \leq T$, we can derive an optimal policy γ^* as $\gamma^{\mathbf{m}^*}$, which minimizes (9).

In the following proposition, we present conditions that an optimal threshold vector $\mathbf{m}^*$ must satisfy.

Proposition 1: Under Assumption 1, if $\bar{n}_1 > T/2$, for any optimal threshold vector $\mathbf{m}^*$, there exist an optimal threshold $m^* \in \{2, 3, ..., T+1\}$ that satisfies

$$m^* - 1 \leq m_i^* \leq m^*, \quad \forall i = 1, ..., N^*,$$

where $N^* = c(m^*, \mathbf{m}^*) + c(m^* - 1, \mathbf{m}^*)$.

If $n_2^* = T/2$, then

$$m^* = 2, \quad c(m^*, \mathbf{m}^*) = \frac{T}{2}, \quad c(m^* - 1, \mathbf{m}^*) = 0, \tag{47}$$

else,

$$\begin{aligned} m^* = \left\lceil \frac{T+1}{n_2^* + 1} \right\rceil, \quad c(m^*, \mathbf{m}^*) + c(m^* - 1, \mathbf{m}^*) = n_2^* + 1, \\ c(m^*, \mathbf{m}^*) = T + 1 - (n_2^* + 1)(m^* - 1), \\ c(m^* - 1, \mathbf{m}^*) = n_2^* + 1 - c(m^*, \mathbf{m}^*). \end{aligned} \tag{48}$$

Proof: Equations (47) and (48) follow from Theorem 1, definitions (39) and (40), and equations (41)-(45). ■

For $\bar{n}_1 > T/2$, Proposition 1 shows that an optimal threshold vector $\mathbf{m}^*$ consists of either one or two threshold values, i.e., $m_i^* \in \{m^* - 1, m^*\}$, $i = 1, ..., N^*$. From (48), we see that the optimal number of scheduling instances for Sensor 2, n_2^*, implies the optimal threshold m^*. Hence, if n_2^* is known, we can derive an optimal policy γ^* using the results in Proposition 1.

In the following, we will reduce the feasible set of n_2^* to simplify the derivation of n_2^*, by analyzing how the average error depends on the scheduling segment length m. Let $\bar{E} : \{2, 3, ..., \infty\} \to \mathbb{R}_+$, represent the average error over a scheduling segment of length m, and be defined as

$$\bar{E}(m) = \frac{\sum_{i=1}^{m-1} E(0, i) + E(1, 0)}{m}, \tag{49}$$

which converges to the upper MSE boundary in (15), i.e., $\lim_{m \to \infty} \bar{E}(m) = E_2^{\infty}$.

If n_2^*, and thus, m^*, is known, we use (49) to express the cost in (7) as a combination of $\bar{E}(m^*)$ and $\bar{E}(m^* - 1)$, as

$$J(\gamma^*) = \bar{E}(2), \quad \text{for} \quad m^* = 2, \tag{50}$$

else, for $m^* > 2$, i.e.,

$$\begin{aligned} J(\gamma^*) = \frac{1}{T} \Big(& m^* \bar{E}(m^*) c(m^*, \mathbf{m}^*) \\ & + (m^* - 1) \bar{E}(m^* - 1) c(m^* - 1, \mathbf{m}^*) - E(1, 0) \Big) \end{aligned} \tag{51}$$

For $\bar{n}_1 > T/2$, using (37), let $\mathcal{L}$ be the set of feasible values of n_2^*, $n_2^* \in \mathcal{L}$, given by

$$\mathcal{L} = \big\{ n \in \mathbb{N}_+ \big| \max\{T - \bar{n}_1, 0\} \leq n \leq \min\{\bar{n}_2, \lfloor T/2 \rfloor\} \big\}.$$

From (50) and (51), we find that n_2^* should be equal to the value n, $n \in \mathcal{L}$, satisfying

$$\min_{n \in \mathcal{L}} f(n) = J(\gamma^*), \tag{52}$$

with $f : \mathcal{L} \to \mathbb{R}_+$ defined as

$$f(n) = \begin{cases} \bar{E}(2), & \text{if } n = T/2, \\ \nu(n) \bar{E}(m) + \omega(n) \bar{E}(m-1) + w_0, & \text{else}, \end{cases} \tag{53}$$

where $\nu(n), \omega(n)$ and m are derived by substituting (48) in (51) as

$$\begin{aligned} m &= \left\lceil \frac{T+1}{n+1} \right\rceil, \quad w_0 = -\frac{E(1, 0)}{T}, \\ \nu(n) &= \frac{m}{T} \big(T + 1 - (n+1)(m-1) \big), \\ \omega(n) &= \frac{m-1}{T} \left(n + 1 - \nu(n) \frac{T}{m} \right), \end{aligned}$$

and $\nu(n) + \omega(n) = (T+1)/T$ for all $n \in \mathcal{L}$.

One way to find n_2^* is to evaluate every element in $\mathcal{L}$ and see which minimizes (52). However, as mentioned earlier, if the cardinality of $\mathcal{L}$ is large, this becomes cumbersome. Instead, we can reduce the set of feasible values for n_2^* based on mathematical properties of $\bar{E}(m)$ presented in [9].

For $E(1, 0) \geq E(0, 1)$, the value $\bar{E}(\hat{m})$, $\hat{m} \geq 2$, is a minimum point, satisfying

$$\begin{aligned} \bar{E}(\hat{m}) \leq \bar{E}(\hat{m} - l) \leq \cdots \leq \bar{E}(2), \\ \bar{E}(\hat{m}) \leq \bar{E}(\hat{m} + l) \leq \cdots \leq E_2^{\infty}, \end{aligned} \tag{54}$$

i.e., $\bar{E}(\hat{m}) \leq \bar{E}(\hat{m} + l)$, for $\hat{m} \geq 2$, $l \in \mathbb{N}$ and $2 - \hat{m} \leq l < \infty$. Furthermore, for $\hat{m} \geq 2$, the value $\hat{m}$ is $\hat{m} = \infty$ if

$$\sum_{i=1}^{\infty} \Big(E_2^{\infty} - E(0, i) \Big) \leq E(1, 0) - E_2^{\infty}, \tag{55}$$

else, if (55) does not hold, the value $\hat{m}$ is finite and given by

$$\hat{m} = \inf \left\{ m \geq 2 \Big| \frac{\sum_{i=1}^{m-1} E(0, i) + E(1, 0)}{m} \leq E(0, m) \right\}. \tag{56}$$

From definition (56), we formulate the following Theorem.

Theorem 2: Under Assumption 1, for $\bar{n}_1 > T/2$, the optimal number of scheduling instances for Sensor 2, n_2^*, satisfies (52) and belongs to set $n_2^* \in \{n^-, n^+\}$, where

$$\begin{aligned} n^- &= \sup \left\{ n \in \mathcal{L} \Big| \left\lceil \frac{T+1}{n+1} \right\rceil > \hat{m} \right\}, \\ n^+ &= \inf \left\{ n \in \mathcal{L} \Big| \left\lceil \frac{T+1}{n+1} \right\rceil \leq \hat{m} \right\}. \end{aligned} \tag{57}$$

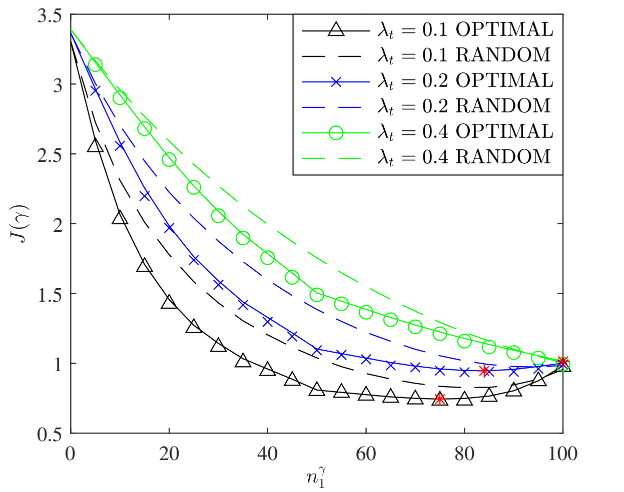

Fig. 2: Average cost $J(\gamma)$ versus n_1^γ for system parameters $\sigma_1 = 2, \sigma_2 = 1, \rho_{12} = -0.5$, $T = 100$, $\bar{n}_1 = \bar{n}_2 = 100$, and $\xi_1 = \xi_2 = 0.5$. Solid lines show results derived from theory and markers show simulation results. Red asterix show optimal performance at n_1^*.

Proof: Let $\mathcal{L}^m \subseteq \mathcal{L}$, represent the number of scheduling instances of Sensor 2 that correspond to threshold m as

$$\mathcal{L}^m = \left\{ n \in \mathcal{L} \middle| m = \left\lceil \frac{T+1}{n+1} \right\rceil \right\}. \tag{58}$$

If the optimal threshold m^* is known, the optimal number of scheduling instances for Sensor 2, n_2^*, satisfies (52) as

$$n_2^* = \arg\min_{n \in \mathcal{L}^{m^*}} f(n).$$

The inequalities in (54) imply; that if $\bar{E}(m) \leq \bar{E}(m-1)$, the following inequality holds for (53)

$$\min_{n \in \mathcal{L}^m} f(n) \leq \min_{n \in \mathcal{L}^{m-l}} f(n), \quad l \in \mathbb{N}_+, \tag{59}$$

where $2 \leq m - l \leq m$. Also, if $\bar{E}(m) \leq \bar{E}(m+1)$ then

$$\min_{n \in \mathcal{L}^{m+1}} f(n) \leq \min_{n \in \mathcal{L}^{m+1+l}} f(n), \quad l \in \mathbb{N}_+. \tag{60}$$

From (59) and (60), we conclude that if either $\hat{m}$, or $\hat{m}+1$, are feasible thresholds, one of these two threshold values equals m^*. Otherwise, m^* is either of the feasible thresholds lying closest to $\hat{m}$, and $\hat{m}+1$. Hence, an optimal threshold m^* lies in $m^* \in \{m^-, m^+\}$,

$$\begin{aligned} m^- &= \sup_{n \in \mathcal{L}} \left\{ \left\lceil \frac{T+1}{n+1} \right\rceil \middle| \left\lceil \frac{T+1}{n+1} \right\rceil \leq \hat{m} \right\}, \\ m^+ &= \inf_{n \in \mathcal{L}} \left\{ \left\lceil \frac{T+1}{n+1} \right\rceil \middle| \left\lceil \frac{T+1}{n+1} \right\rceil > \hat{m} \right\}. \end{aligned} \tag{61}$$

For any $m > 2$, the sum, $\nu(n) + \omega(n)$, in (53), is constant for all $n \in \mathcal{L}^m$ and depend on n as

$$\nu(n+k) \leq \nu(n), \quad \omega(n) \leq \omega(n+k), \quad \forall n, n+k \in \mathcal{L}^m.$$

Thus, if m^* is known, then n_2^* is found by maximizing the weight $\nu(n)$ or $\omega(n)$, $n \in \mathcal{L}^{m^*}$, which is multiplied with the smallest of the two elements; $\bar{E}(m^*)$ and $\bar{E}(m^*-1)$. Together

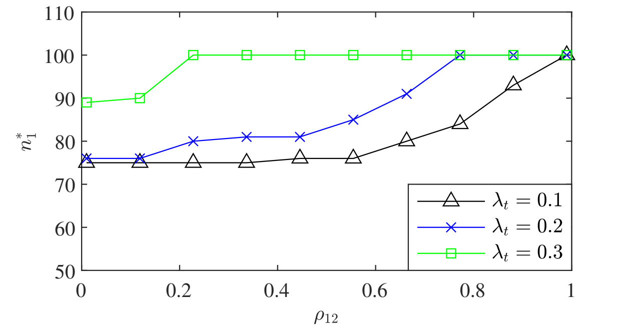

Fig. 3: Optimal number of scheduling instances of Sensor 1, n_1^*, versus spatial correlation ρ_{12} for system parameters $\sigma_1 = 2, \sigma_2 = 1$, $T = 100$, $\bar{n}_1 = \bar{n}_2 = 100$, and $\xi_1 = \xi_2 = 0.5$.

with expressions (59) and (60), this gives that n_2^* belongs to the set $n_2^* \in \{n^-, n^+\}$, where

$$\begin{aligned} n^- &= \arg\min_{n \in \mathcal{L}^{m^+}} f(n) = \text{maximum}\{\mathcal{L}^{m^+}\}, \\ n^+ &= \arg\min_{n \in \mathcal{L}^{m^-}} f(n) = \text{minimum}\{\mathcal{L}^{m^-}\}. \end{aligned} \tag{62}$$

By substituting (61) for $\mathcal{L}^{m^-}$ and $\mathcal{L}^{m^+}$ in (62), we derive (57). ■

For $\bar{n}_1 > T/2$, Theorem 2 implies that γ^* can be derived by, firstly, calculating $\hat{m}$ from expressions (55) and (56). This can be done in a straightforward way using the algorithm in [9, Alg. 1]. Secondly, by deriving n^- and n^+ in (57) and evaluating the value that minimizes (52) as

$$n_2^* = \arg\min_{n \in \{n^-, n^+\}} f(n).$$

Finally, deriving γ^* by applying n_2^* to expressions (47) and (48) in Proposition 1.

To summarize the theoretical results presented in Section III, we have presented two important properties that an optimal scheduling policy must satisfy. First, Theorems 1 and 2 show that an optimal scheduling policy corresponds to an optimal number of transmission instances for each sensor, n_i^*, $i = 1, 2$. Theorem 1 in (25) specifically states that if the transmission constraint for Sensor 1, $\bar{n}_1$, is more than half the number of total scheduling instances, i.e., $\bar{n}_1 \geq T/2$, the optimal number of scheduling instances for Sensor 1 should be more than, or equal to, the optimal number of scheduling instances for Sensor 2, i.e., $n_1^* \geq n_2^*$. In Theorem 2, we presented a simple numerical method to reduce the number of possible values of n_2^* to evaluate, to a set of only two values, $\{n^-, n^+\}$. For $\bar{n}_1 \leq T/2$, the value n_1^* should simply be maximized, i.e., $n_1^* = \bar{n}_1$.

The results in Theorem 1 in (25) demonstrate that in order to maximize the overall MSE for a WSN where two sensors observe Gaussian spatio-temporally correlated processes, the sensor with the highest MSE for a given AoI should be transmitted more frequently. Thus, it must be allocated more transmission power resources. These results can be compared to [17], where, in order to maximize the SNR, sensors are allocated different resources for transmission depending on their measurement noise.

TABLE II: Simulation Parameters

Parameter	Values
Standard deviation of Process 1, σ_1	2
Standard deviation of Process 2, σ_2	1
Standard deviation of measurement noises, ξ_1 and ξ_2	0.5
Spatial correlation, ρ_{12}	-0.5
Temporal correlation decay factor, λ_t	$0.1, 0.2, 0.3, 0.4$
Time horizon, T	100
Transmission constraint, $\bar{n}_i, i = 1, 2$	100

Secondly, as shown in Theorem 1, if n_1^* and n_2^* are known, an optimal scheduling policy should result in a scheduling sequence that minimizes the maximum AoI for each sensor. In other words, an optimal scheduling policy can be derived by finding a scheduling policy that minimizes the maximum AoI for each sensor while considering the constraints n_1^* and n_2^*. The structure of an optimal scheduling sequence in this paper is similar to the form of an optimal scheduling sequence for a system where sensors observe independent linear time-invariant processes, sharing a single communication channel [1]. Although this paper has focused on minimizing the overall MSE, an optimal scheduling sequence that minimizes the AoI interconnects with the large body of work regarding the AoI that has considered minimizing the AoI under different system set-ups, e.g., [21], [31], [35]

IV. Numerical Examples

We assume a system with statistical parameters $\sigma_1 = 2$, $\sigma_2 = 1$, $\rho_{12} = -0.5$, and $\xi_1 = \xi_2 = 0.5$. For the temporal correlation ρ_t in (1), we use $\rho_t(x) = e^{-\lambda_t x}$, $x \in \mathbb{N}_+$ [34], where $\lambda_t \in \mathbb{R}, \lambda_t > 0$, indicates the temporal correlation decays with respect to the AoI. Thus, a larger value of λ_t corresponds to a weaker temporal correlation. The simulation parameters for our numerical examples can be found in Table II.

Fig. 2 shows the average cost $J(\gamma)$ versus n_1^γ for $\lambda_t = (0.1, 0.2, 0.4)$ with time horizon $T = 100$ and $\bar{n}_i = T$, $i = 1, 2$. Given the value n_1^γ, an optimal policy is derived from the results in Theorems 1 and 2, and Proposition 1, referred to as OPTIMAL. Solid lines depict theoretical values obtained from (7)-(12) and markers show Monte Carlo simulations of 1000 sequences with $T = 100$, which matches the theory. The red markers show the performance given the optimal number of transmission instances $n_1^* = (76, 84, 100)$. The respective values for n_1^* can be compared to the respective optimal threshold values in (54), $\hat{m} = (4, 6, \infty)$, which indicate that Sensor 2 is scheduled around every $\hat{m}$th time instance. The results show that for $\lambda_t \leq 0.2$, an optimal performance is achieved for $n_1^* < \bar{n}_1 = T$.

In Fig. 2, the optimal performance is compared to a policy where Sensor 1 is scheduled n_1^γ instances in a random order during interval $[1, T]$, referred to as RANDOM. The performance is calculated as the average MSE after simulating 1000 scheduling sequences for each value n_1^γ. We see that an optimal scheduling order outperforms a random scheduling policy for every value of n_1^γ. Furthermore, we see that as the temporal correlation increases, i.e., $\lambda = 0.1$, the best performance for RANDOM is at $n_1^\gamma < T$.

Fig. 2 demonstrates that as the temporal correlation increases, i.e., $\lambda_t \to 0$, both optimal and RANDOM policy performance increases. The reason for this can be found in (12), where an increase in the temporal correlation, i.e., $\rho_t \to 1$, results in a reduction of the MSE given the AoI. Thus, we can conclude that if the degree of temporal correlation increases, the performance of any given scheduling policy will improve or remain the same.

Fig. 3 shows the optimal number of scheduling instances for Sensor 1, n_1^*, versus the spatial correlation ρ_{12} for $\lambda_t = (0.01, 0.05, 0.1, 0.14)$. The results show that n_1^* is lower bounded at $\rho_{12} = 0$ with values, respectively $n_1^* = (75, 75, 88)$, and then increases with ρ_{12}. We see that as the temporal correlation increases, i.e., $\lambda_t \to 0$, n_1^* decreases.

V. Conclusion

This paper studied a finite-horizon optimal scheduling policy for broadcasting observations from two spatio-temporally processes. At each time instant, a network scheduler can broadcast a measurement from one of the two processes to two remote estimators, each tracking one process. The number of scheduling instances for each sensor is limited by each sensor's energy supply. The scheduler cannot observe the measurements and decides the scheduling policy based on the AoI. We derived an optimal scheduling policy using the AoI as a state variable. The policy can be attained for any time horizon using a low-complexity numerical method. The numerical results matched the theory. The numerical results in this paper support earlier findings that exploit spatio-temporal correlation in scheduling tasks to improve the estimation accuracy in resource-constrained WSNs in various system settings [18], [32], [36].

Future work includes exploring the possibility of deriving an optimal finite time horizon scheduling policy using a different dynamic spatio-temporally process, e.g., a Gauss-Markov model [1], [37]. Another extension of our work would be to consider using a sequential estimator that incorporates past measurements and derive an optimal scheduling policy for such a system to further improve the estimation accuracy at the remote estimators.

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