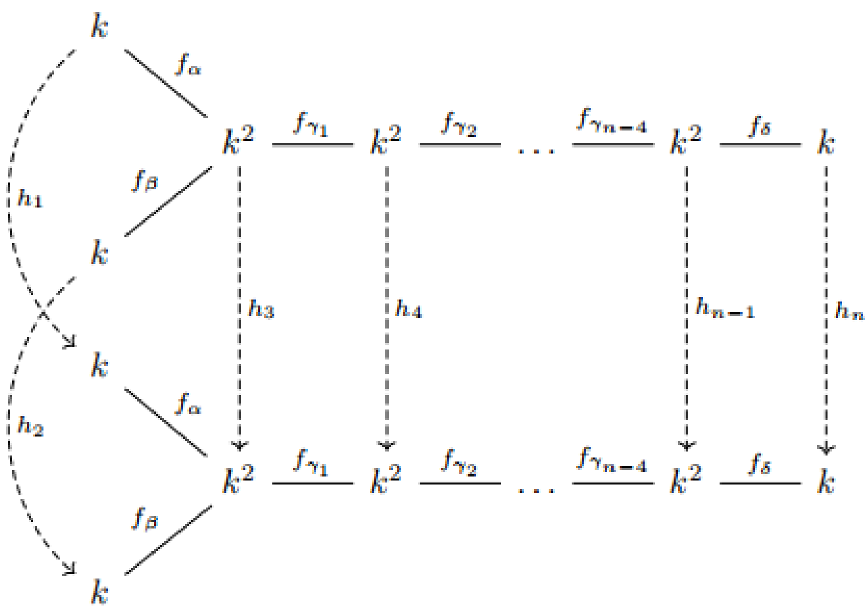
Bachelor's thesis

NTNU
Norwegian University of Science and Technology

Arthur Lund

Tree Modules for Quiver Representations

Bachelor's thesis in BMAT
Supervisor: Øyvind Solberg
August 2022



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Abstract

Given a quiver Γ over k of Dynkin D_n type, the amount of different expressions of tree modules with 0-1-matrices - or coefficient quivers (CQs) - is studied. After introducing the theory, we know that given quivers of finite representation type, the Dynkin types, the indecomposable representations of said quiver will always be tree modules. The method used is to draw the different possible (up to isomorphism) coefficient quivers that are trees, work back to make the representation belonging to it with the standard basis, and then check its indecomposability. Several reductions are made to the amount of cases we need to go through: from the Auslander-Reiten-quiver (AR-quiver) we learn that there is only one dimension vector of interest for each D_n; the duals of representations already dealt with are proven unnecessary to study; and the symmetry of the "arms" of the D_n graph is considered. It is found that given a representation over D_n when $n > 4$, what determines the amount of different CQs that produce tree modules is whether the arms have the same orientation or not. As a main result for $n > 5$, the CQs we are looking for will just be those for D_5 with $n - 5$ added identities.

1 Introduction

A tree module is a certain type of indecomposable quiver representation. Before defining this and other key terms we need to go through some notation and preliminaries from the representation theory of quivers.

1.1 Represantions and their bases

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver over the field k, where the set of vertices, Γ_0 (denoted by positive integers), and arrows, Γ_1 (denoted by greek letters), are both assumed to be finite sets. We denote the path algebra of Γ over k as $k\Gamma$. A representation $V = (V, f)$ of Γ over k is a collection of finite-dimensional k-vector spaces (or k-spaces), V, and the linear transformations between them, f. For every vertex i we have a k-space $V(i)$, and for every arrow $\alpha\colon i \to j$, we have a linear transformation $f_\alpha\colon V(i) \to V(j)$. A representation V of Γ over k can be thought of as a (finite-dimensional) $k\Gamma$-module (the module we get by taking the direct sum of the vector spaces: $\bigoplus_i V(i)$).

Frequently used in this text is the notion of an endomorphism of a quiver representation. This is a homomorphism from a representation to itself.

Definition 1.1. Let (V, f) and (V', f') be two representations of Γ over k. A *homomorphism* $h\colon (V, f) \to (V', f')$ is a collection of linear maps $h(i)\colon V(i) \to V'(i)$ for all $i \in \Gamma_0$, such that for all $\alpha\colon i \to j \in \Gamma_1$ the homomorphism commutes with the linear transformation f_α, i.e.

$$f'_\alpha h(i) = h(j) f_\alpha.$$

h is called an *isomorphism* if all the maps $h(i)$ are isomorphisms.

Let d_i be the dimension of $V(i)$ over k, and $d = \sum d_i$ be the *dimension* of V. We define a basis $\mathcal{B}$ of V as a subset of the disjoint union of the k-spaces $V(i)$ such that for any vertex i the set $\mathcal{B}_i = \mathcal{B} \cap V(i)$ is a basis for $V(i)$. The basis $\mathcal{B}_i$ is really a set of tuples of length $|\Gamma_0|$, where all the entries are zero except for the i'th spot, where we have a basis element of $V(i)$. As an example, take the following quiver Γ and representation V:

$$\begin{tikzcd}[row sep=small] & & 3 \\ 1 \arrow[r, "\alpha"] & 2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & \\ & & 4 \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & & k \\ k \arrow[r, "\begin{pmatrix}1\\1\end{pmatrix}"] & k^2 \arrow[ur, "\begin{pmatrix}1 & 0\end{pmatrix}"] \arrow[dr, "\begin{pmatrix}0 & 1\end{pmatrix}"] & \\ & & k \end{tikzcd}$$

Going for the standard basis in each of the vector spaces, we choose $\{1\}$ for the vector spaces k, and for k^2 we choose $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$. This gives us a basis for the representation V:

$$\mathcal{B} = \left\{(1,0,0,0), \left(0, \begin{pmatrix}1\\0\end{pmatrix}, 0, 0\right), \left(0, \begin{pmatrix}0\\1\end{pmatrix}, 0, 0\right), (0,0,1,0), (0,0,0,1)\right\}.$$

1.2 Coefficient quivers

For any arrow $\alpha\colon i \to j$ we can write f_α as a $(d_j \times d_i)$-matrix $M_{\alpha,\mathcal{B}}$. The rows are indexed by $\mathcal{B}_j$ and the columns by $\mathcal{B}_i$. The corresponding matrix coefficients are denoted by $M_{\alpha,\mathcal{B}}(b', b)$, where $b \in \mathcal{B}_i$ and $b' \in \mathcal{B}_j$ and they are defined by $f_\alpha(b) = \sum_{b' \in \mathcal{B}_j} M_{\alpha,\mathcal{B}}(b', b) b'$.

Example. Let the quiver $1 \xrightarrow{\alpha} 2$ over k have representation $k^2 \xrightarrow{f_\alpha} k^2$ with the standard bases and let f_α be defined by

$$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\0\end{pmatrix}.$$

The two equations

$$\begin{pmatrix}1\\1\end{pmatrix} = f_\alpha(b_1) = \sum_{b' \in \mathcal{B}_2} M_{\alpha,\mathcal{B}}(b', b_1) b' = M_{\alpha,\mathcal{B}}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)\begin{pmatrix}1\\0\end{pmatrix} + M_{\alpha,\mathcal{B}}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)\begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix} = f_\alpha(b_2) = \sum_{b' \in \mathcal{B}_2} M_{\alpha,\mathcal{B}}(b', b_2) b' = M_{\alpha,\mathcal{B}}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)\begin{pmatrix}1\\0\end{pmatrix} + M_{\alpha,\mathcal{B}}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)\begin{pmatrix}0\\1\end{pmatrix}$$

imply that

$$M_\alpha = \begin{pmatrix} M_{\alpha,\mathcal{B}}(b_1', b_1) & M_{\alpha,\mathcal{B}}(b_1', b_2) \\ M_{\alpha,\mathcal{B}}(b_2', b_1) & M_{\alpha,\mathcal{B}}(b_2', b_2) \end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}.$$

The *coefficient quiver* $\mathrm{CQ}(V, \mathcal{B})$ of the representation V under the basis $\mathcal{B}$ has the set $\mathcal{B}$ as the set of vertices, and there is an arrow from b to b' if and only if $M_{\alpha,\mathcal{B}}(b', b) \neq 0$.

Example. With the quiver and representation of the previous example, we see that there is an arrow from b_1 to both b_1' and b_2' and an arrow from b_2 to b_1', so the coefficient quiver of the representation is

$$\begin{tikzcd} \begin{pmatrix}1\\0\end{pmatrix}_1 \arrow[r] \arrow[dr] & \begin{pmatrix}1\\0\end{pmatrix}_2 \\ \begin{pmatrix}0\\1\end{pmatrix}_1 \arrow[ur] & \begin{pmatrix}0\\1\end{pmatrix}_2 \end{tikzcd},$$

where the subscripts indicate what basis - or which vertex in the original quiver - the basis elements belong to.

We will usually replace the basis elements in the CQ with circles $\circ$ when we know from context what they represent.

1.3 Tree modules

Now we have explained enough concepts to define tree modules.

Definition 1.2. An indecomposable representation V of a quiver over k is a *tree module* if there is a basis $\mathcal{B}$ of V such that the coefficient quiver $\mathrm{CQ}(V, \mathcal{B})$ is a tree.

With directed graphs, there can be some ambiguity in defining trees. Here, we define directed graphs as trees when their underlying undirected graph is a tree, meaning it is connected with no cycles. From the following result from graph theory, we know that a tree module V of dimension d admits a basis $\mathcal{B}$ such that exactly $d-1$ matrix coefficients are non-zero, because there are $d-1$ arrows in the coefficient quiver $\mathrm{CQ}(V, \mathcal{B})$.

Theorem 1.1. *A tree with n vertices has $n-1$ edges.*

Proof. Corollary 2.6 in [Bol79]. □

To simplify matters further, it will now be shown that tree modules can be expressed by 0-1-matrices, as bases can be changed to make all the non-zero matrix coefficients 1. The proof follows that of [Rin98].

Definition 1.3. Let V be a representation of a quiver over k. The bases $\mathcal{B}$ and $\mathcal{B}'$ of V are *proportional* if for every element $b \in \mathcal{B}$, there is a scalar $\lambda(b) \in k$ such that $\lambda(b)b \in \mathcal{B}'$.

Proposition 1.1. *Let $\mathcal{B}$ be a basis of V such that $CQ(V, \mathcal{B})$ is a tree. Then there is a basis $\mathcal{B}'$ of V proportional to $\mathcal{B}$ such that all the non-zero coefficients $M_{\alpha,\mathcal{B}'}(b', b)$ are equal to 1.*

Proof. Fix one of the elements b_0 of $\mathcal{B}$, let $\lambda(b_0) = 1$. Assume that b and b' are neighbours in $\mathrm{CQ}(V, \mathcal{B})$ and that $\lambda(b)$ is already defined. If there is an arrow α from b to b', let $\lambda(b') = \lambda(b) \cdot M_{\alpha,\mathcal{B}}(b', b)$; otherwise, the arrow goes the other way and we define $\lambda(b') = \lambda(b) \cdot M_{\alpha,\mathcal{B}}(b, b')^{-1}$. Since the coefficient quiver is a tree, this procedure defines non-zero scalars $\lambda(b)$ for all $b \in \mathcal{B}$ in a unique way. Setting $\mathcal{B}'$ as the set of elements $\lambda(b)b$ with $b \in \mathcal{B}$, it can be checked that $M_{\alpha,\mathcal{B}'}(\lambda(b')b', \lambda(b)b) = 1$ when $M_{\alpha,\mathcal{B}}(b', b) \neq 0$, and $M_{\alpha,\mathcal{B}'}(\lambda(b')b', \lambda(b)b) = 0$ otherwise. □

1.4 Quivers of finite representation type

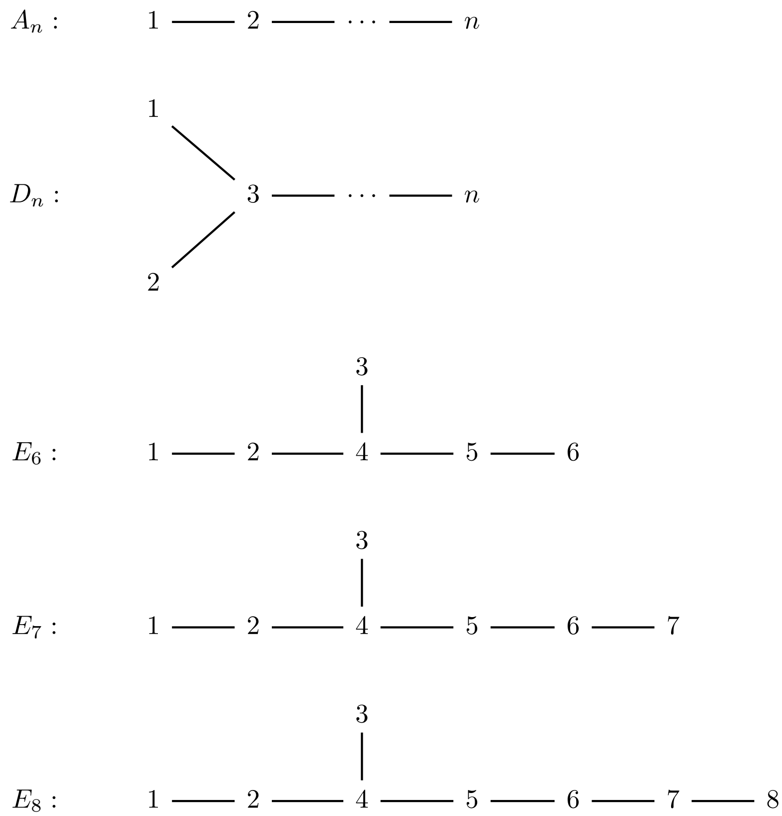
Definition 1.4. A quiver is of *finite representation type* if it has only finitely many isomorphism classes of indecomposable representations.

Theorem 1.2 (Gabriel)**.** *An acyclic quiver is of finite representation type if and only if the quiver is a (directed) Dynkin graph.*

Proof. Theorem 5.5 in section VIII of [ARS95]. □

If Γ is of finite representation type, then all indecomposable representations V of Γ over k are tree modules, because $\mathrm{Ext}^1_{k\Gamma}(V, V) = 0$, i.e. V is exceptional, and, as proven in [Rin98], exceptional modules are tree modules.

The Dynkin Diagrams are given below. A quiver (representation) will be said to be of a certain type if its underlying graph is of that type. This text will study how many different coefficient quivers we can get for representations of type D_n quivers, where $n \geq 4$. The same question for the A_n quivers is trivial, as it can be shown that each vector space for any representation in that case is one-dimensional, giving only one trivial CQ for each representation.



2 D_4

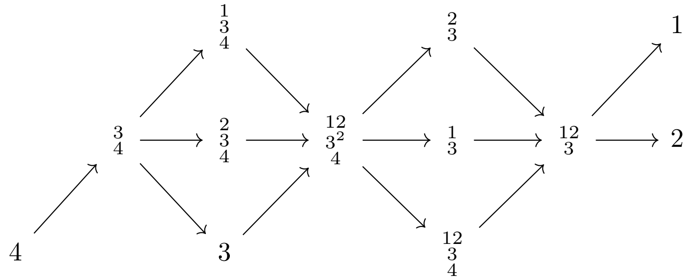
We begin with the lowest amount of vertices, 4. This chapter will introduce the methodology of finding coefficient quivers that give tree modules. A handful of results will be shown or referenced to, so that we do not study any more cases of dimension vectors, arrow configurations, and coefficient quivers than what is strictly necessary. The approach and results here will be used in the subsequent chapters.

2.1 D_4^1

We name the quiver of type D_4 with all of the lines directed rightward D_4^1:

$$\begin{tikzcd}
1 \arrow[dr, "\alpha"] & & \\
& 3 \arrow[r, "\gamma"] & 4 \\
2 \arrow[ur, "\beta"] & &
\end{tikzcd}$$

The Auslander-Reiten-quiver (AR-quiver) for D_4^1 is given below:



This shows the dimensions of the vector spaces in the type D_4 representations that are indecomposable for some collection of linear maps, as mentioned on page 277 of [ARS95].

Definition 2.1. The *dimension vector* of a representation V of a quiver Γ over k is a tuple of length $|\Gamma_0|$ such that the i'th element is the dimension of $V(i)$ over k.

For instance the top left entry in our AR-quiver is $\substack{1\\3\\4}$, which refers to the dimension vector $(1,0,1,1)$ and the set of representations

$$\begin{tikzcd}
k \arrow[dr, "f_\alpha"] & & \\
& k \arrow[r, "f_\gamma"] & k \\
0 \arrow[ur, "f_\beta"] & &
\end{tikzcd}$$

The map f_β must be zero, and the maps f_α and f_γ will be scalar maps, making any representation of this type and dimension vector isomorphic to

$$\begin{tikzcd}
k \arrow[dr, "1"] & & \\
& k \arrow[r, "1"] & k \\
0 \arrow[ur, "0"] & &
\end{tikzcd}$$

The accompanying coefficient quiver is simply $\circ \longrightarrow \circ \longrightarrow \circ$. This is a tree, so the representation counts as a tree module (since it is also indecomposable), but it is uninteresting in the sense that it can be inferred unambiguously from the AR-quiver. This will be the case for any representation where none of the k-spaces have dimension greater than one, since any linear map then is determined to be either 0 or isomorphic to 1. For two more examples of this case, the entries i for $i \in \Gamma_0$ give coefficient quiver $\circ$ and $\substack{12\\3}$ gives coefficient quiver

$$\begin{tikzcd}
\circ \arrow[dr] & \\
& \circ \\
\circ \arrow[ur] &
\end{tikzcd}$$

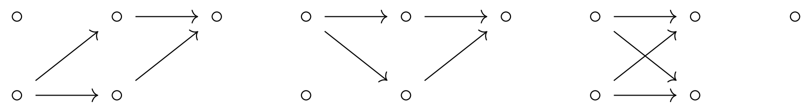
To get more than just the trivial coefficient quiver for a tree module, we therefore need at least one k-space to be of dimension greater than one, which is the case for only one of the entries in the AR-quiver, $\substack{12\\3^2\\4}$, referring to the set of representations

$$\begin{tikzcd}
k \arrow[dr, "f_\alpha"] & & \\
& k^2 \arrow[r, "f_\gamma"] & k \\
k \arrow[ur, "f_\beta"] & &
\end{tikzcd}$$

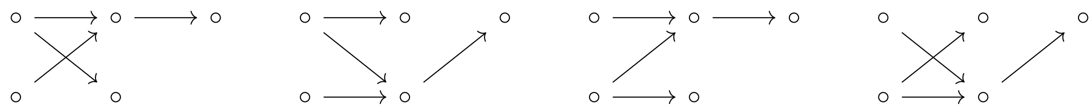
The dimension of this representation is 5, so the number of arrows in the CQ is 4. We use the standard basis:

$$\begin{array}{ccc}
1_1 & \begin{pmatrix}1\\0\end{pmatrix} & 1_4 \\
& & \\
1_2 & \begin{pmatrix}0\\1\end{pmatrix} &
\end{array}$$

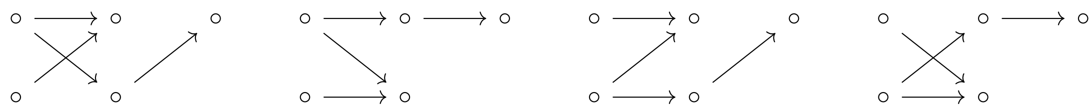
When making the $\binom{6}{4} = 15$ different coefficient quivers we can have with four arrows in six spots we get five different cases:
Case 0. The coefficient quiver is not a tree.



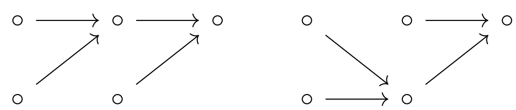
Case 1.



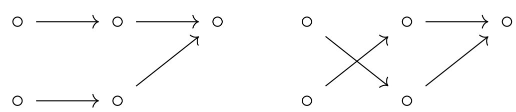
Case 2.



Case 3.



Case 4.



Cases 1-4 are isomorphism classes for coefficient quivers - we see that a tree is isomorphic to another if and only if they belong to the same case above.

Up to isomorphism we have four coefficient quivers for the representations at hand that are trees. But we have not checked for the pre-condition to be a tree module: whether or not the representation is indecomposable. We only need to check that one CQ from each case corresponds to an indecomposable representation, as the following proposition states that indecomposability status does not change within a CQ isomorphism class.

Proposition 2.1. *Let V and V' be representations of the same D_4 type quiver. Let V be indecomposable and $\mathcal{B}$ be a basis of V such that $CQ(V, \mathcal{B})$ is a tree. If $CQ(V, \mathcal{B}) \simeq CQ(V', \mathcal{B})$, then V' is also indecomposable.*

Proof. Since V and V' are of the same quiver and their CQs are isomorphic, the amount of arrows and their direction between two columns in the CQ will be the same, so the CQs can be made equal to each other by transposing the circles within the columns. The transposition of the two first basis elements, which are not from the same basis, corresponds to interchanging the arrows α and β in the original quiver representation, which will not change indecomposability. Otherwise, the transpositions correspond to changing the basis, which will not change the representation at all. Therefore, V' will also be indecomposable. $\square$

The leftmost in each case will be chosen as the representative of the isomorphism class, and we check for indecomposability of the representation we get working back from the CQ with the standard basis.

Theorem 2.1. *A representation is indecomposable if and only if its endomorphism ring is one-dimensional.*

Proof. Corollary 5.14 on page 294 of [ASS06]. $\square$

Now we can calculate which of the four CQ's give a tree module. Case 1 is represented by the mappings

$$f_\alpha = \begin{pmatrix}1\\1\end{pmatrix}, \quad f_\beta = \begin{pmatrix}1\\0\end{pmatrix}, \quad f_\gamma = \begin{pmatrix}1 & 0\end{pmatrix}:$$

and we calculate whether the endomorphism ring of the quiver representation, which we call $D_4^1(1)$, is one-dimensional:

$$\begin{tikzcd}
k \arrow[dr, "\begin{pmatrix}1\\1\end{pmatrix}"] \arrow[ddd, dashed, bend right, "a_1"'] & & \\
& k^2 \arrow[r, "\begin{pmatrix}1 & 0\end{pmatrix}"] \arrow[ddd, dashed, "\begin{pmatrix}a_{31} & a_{32}\\ a_{33} & a_{34}\end{pmatrix}"] & k \arrow[ddd, dashed, bend left, "a_4"] \\
k \arrow[ur, "\begin{pmatrix}1\\0\end{pmatrix}"] \arrow[ddd, dashed, bend right, "a_2"'] & & \\
k \arrow[dr, "\begin{pmatrix}1\\1\end{pmatrix}"] & & \\
& k^2 \arrow[r, "\begin{pmatrix}1 & 0\end{pmatrix}"] & k \\
k \arrow[ur, "\begin{pmatrix}1\\0\end{pmatrix}"] & &
\end{tikzcd}$$

$$\begin{aligned}
\begin{pmatrix}1\\1\end{pmatrix} h_1 = \begin{pmatrix}a_1\\a_1\end{pmatrix} = h_3 \begin{pmatrix}1\\1\end{pmatrix} &= \begin{pmatrix}a_{31}+a_{32}\\a_{33}+a_{34}\end{pmatrix}, \\
\begin{pmatrix}1\\0\end{pmatrix} h_2 = \begin{pmatrix}a_2\\0\end{pmatrix} = h_3 \begin{pmatrix}1\\0\end{pmatrix} &= \begin{pmatrix}a_{31}\\a_{33}\end{pmatrix} \\
\implies h_3 &= \begin{pmatrix}a_2 & a_1-a_2\\ 0 & a_1\end{pmatrix}. \\
\begin{pmatrix}1 & 0\end{pmatrix} h_3 = \begin{pmatrix}a_2 & a_1-a_2\end{pmatrix} &= h_4 \begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}a_4 & 0\end{pmatrix} \\
&\implies a_4 = a_2 = a_1 \\
&\implies \dim(\mathrm{End}(D_4^1(1))) = 1,
\end{aligned}$$

so the quiver representation $D_4^1(1)$ is indecomposable and therefore a tree module.

We see that $D_4^1(2)$ has the same beginning of the coefficient quiver, so the h_3 map will be the same, but the f_γ makes a difference:

$$\begin{tikzcd}
k \arrow[dr, "\begin{pmatrix}1\\1\end{pmatrix}"] & & \\
& k^2 \arrow[r, "\begin{pmatrix}0 & 1\end{pmatrix}"] & k \\
k \arrow[ur, "\begin{pmatrix}1\\0\end{pmatrix}"] & &
\end{tikzcd}$$

$$\begin{aligned}
\begin{pmatrix}0 & 1\end{pmatrix} h_3 = \begin{pmatrix}0 & a_1\end{pmatrix} &= h_4 \begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & a_4\end{pmatrix} \\
&\implies \dim(\mathrm{End}(D_4^1(2))) = 2,
\end{aligned}$$

since a_1 and a_2 are free variables. Hence, $D_4^1(2)$ decomposes and is not a tree module.

Further computation shows that $D_4^1(3)$ decomposes (endomorphism dimension 2), but $D_4^1(4)$ does not and is a tree module.

2.2 D_4^2

Looking at other orientations of the arrows in the quiver representation, it is useful to know that we can use the same AR-quiver to find the dimension vectors, as long as the underlying graph is the same.

Proposition 2.2. *Indecomposable representations of Dynkin quivers that have the same underlying graph have the same dimension vectors.*

Proof. Page 265 in [ASS06]. ☐

Knowing that the dimension vector is (1,1,2,1) no matter the orientations, we may treat vertices 1, 2 and 4 interchangeably, leaving only three more orientations to consider. However, for our purposes this can be reduced even further.

$$\begin{tikzcd}
1 \arrow[dr, "\alpha"] & & \\
& 3 & 4 \arrow[l, "\gamma"] \\
2 \arrow[ur, "\beta"] & & 2
\end{tikzcd}
\qquad
\begin{tikzcd}
1 & & \\
& 3 \arrow[ul, "\alpha"] \arrow[dl, "\beta"] & 4 \arrow[l, "\gamma"] \\
2 & & 1'
\end{tikzcd}
\qquad
\begin{tikzcd}
1 & & \\
& 3 \arrow[ul, "\alpha"] \arrow[dl, "\beta"] \arrow[r, "\gamma"] & 4 \\
2 & & 2'
\end{tikzcd}$$

Definition 2.2. The *dual of a directed graph/quiver* has the same vertex set, but all the arrows are reversed.

The quivers $1'$ and $2'$ are the duals of 1 and 2. It will now be shown that the CQs of the tree modules over those quivers are simply the duals of the CQs that make tree modules over 1 and 2, which means we do not need to study $1'$ and $2'$ if we have studied 1 and 2. We can just reverse the arrows.

Definition 2.3. The *dual of a quiver representation* $V = (V, f)$ of the quiver Γ over k is defined as the representation of the dual quiver Γ^t over k, $V^t = (V^t, f^t)$, where the k-spaces are the same, i.e. $V^t(i) = V(i)$ for all $i \in \Gamma_0$, but for any arrow $\alpha\colon i \to j \in \Gamma_1$ we have $\alpha^t\colon j \to i \in \Gamma_1^t$, and for any $f_\alpha\colon V(i) \to V(j) \in f$ we have $f^t_{\alpha^t}\colon j \to i \in f^t$, where $f^t_{\alpha^t} = f_\alpha^T$, the transpose of f_α.

Note that the dual of the dual of a directed graph/quiver/representation is the original directed graph/quiver/representation. A one-to-one correspondence between the CQs of a representation and those of its dual is now demonstrated.

Proposition 2.3. *Let V be a representation. The CQ of V under the standard basis is the dual of the CQ of V^t under the standard basis.*

Proof. The parts of the CQ correspond to the linear maps of the representation, so we need only consider each linear map of V, i.e. the representations $k^i \xrightarrow{f_\alpha} k^j$ and their CQs. Now suppose M is the matrix of f_α with respect to the standard basis. Then $M_{ij} \neq 0$ if and only if $M^T_{ji} \neq 0$. There is an arrow $i \to j$ in the CQ of V if and only if $M_{ij} \neq 0$. Similarly, there is an arrow $j \to i$ in the CQ of V^t if and only if $M^T_{ji} \neq 0$. Hence, there is an arrow $i \to j$ in the CQ of V if and only if there is an arrow $j \to i$ in the CQ of V^t. The result follows from this local understanding. ☐

Proposition 2.4. *The dual of an indecomposable representation is indecomposable.*

Proof. The proof is for modules, which correspond to representations. Denote the dual morphism by D. Then, given modules A and B, we have

$$D(A \oplus B) \simeq D(A) \oplus D(B)$$
$$D^2 = \mathrm{id}$$

Let X be indecomposable and assume $D(X)$ decomposes, i.e. $D(X) \simeq A \oplus B$ for some nonzero modules A and B. We get

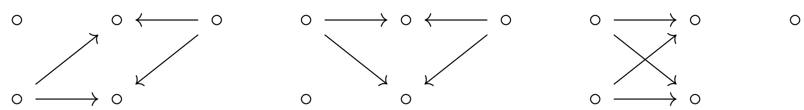
$$X = D(D(X)) \simeq D(A \oplus B) \simeq D(A) \oplus D(B),$$

but since the duals of nonzero modules are nonzero, we have shown that X decomposes, which is a contradiction, so $D(X)$ must be indecomposable as well. ☐

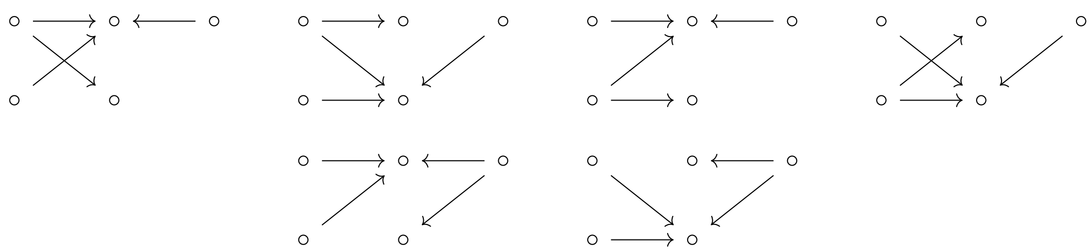
Now we consider the remaining quiver representations D_4^2:

$$\begin{tikzcd}
k \arrow[dr, "f_\alpha"] & & \\
& k^2 & k \arrow[l, "f_\gamma"] \\
k \arrow[ur, "f_\beta"] & &
\end{tikzcd}$$

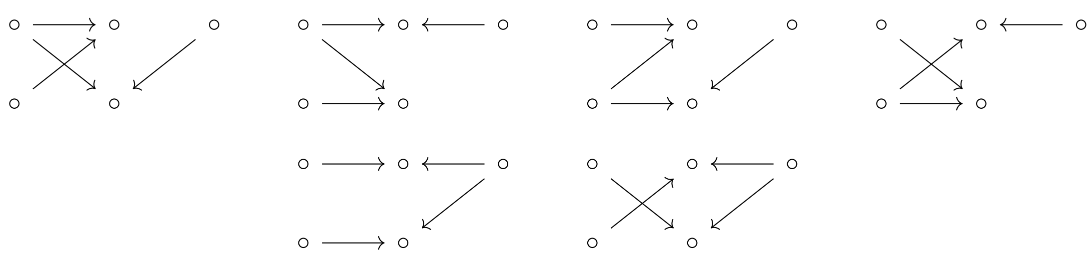
Case 0. The coefficient quiver is not a tree.



Case 1.



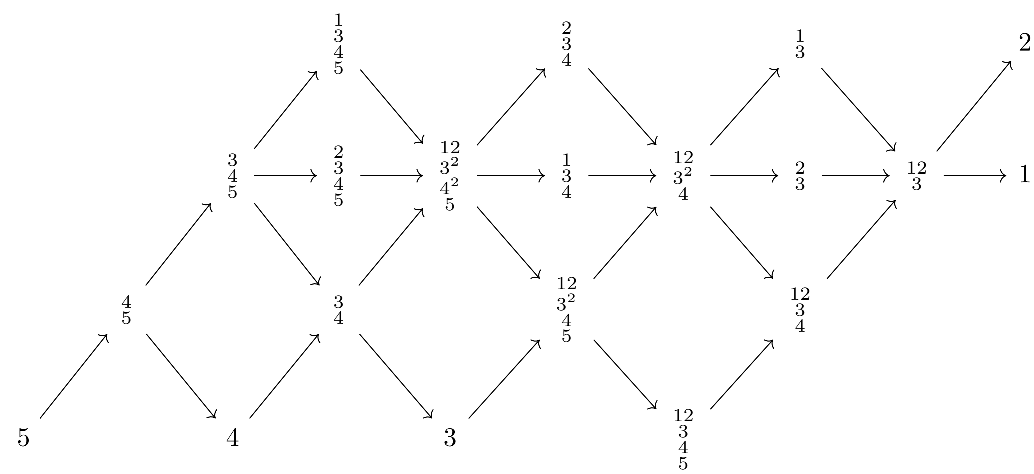
Case 2.



Leaving out the details, we find that the case 1 CQs give decomposable representations, while case 2 CQs give indecomposable representations, i.e. tree modules.

3 D_5

The AR-quiver for the quiver D_5^1 $\begin{tikzcd} 1 \arrow[dr, "\alpha"] & & & \\ & 3 \arrow[r, "\gamma"] & 4 \arrow[r, "\delta"] & 5 \\ 2 \arrow[ur, "\beta"] & & & \end{tikzcd}$ is given below:



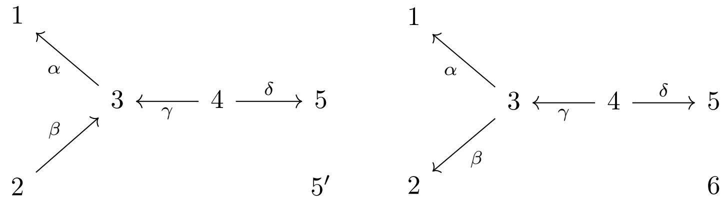
The nontrivial cases are of dimension vector $(1,1,2,2,1)$, $(1,1,2,1,1)$ and $(1,1,2,1,0)$, however the latter two have been studied in D_4 already; there will just be an added mapping $f_\delta = 1$ or $f_\delta = 0$ at the end. Hence we only have to find the different coefficient quivers of the representations

$$\begin{tikzcd} k \arrow[dr, "f_\alpha"] & & & \\ & k^2 \arrow[r, "f_\gamma"] & k^2 \arrow[r, "f_\delta"] & k \\ k \arrow[ur, "f_\beta"] & & & \end{tikzcd}$$

in the D_5^1 case. There are $2^4 = 16$ arrow configurations, but we know that the dual of a representation already studied will have the dual CQ, so we do not need to look at more than half of that.

$$\begin{tikzcd} 1 \arrow[dr, "\alpha"] & & & \\ & 3 \arrow[r, "\gamma"] & 4 \arrow[r, "\delta"] & 5 \\ 2 \arrow[ur, "\beta"] & & & 1 \end{tikzcd}
\qquad
\begin{tikzcd} 1 \arrow[dr, "\alpha"] & & & \\ & 3 \arrow[dl, "\beta"] \arrow[r, "\gamma"] & 4 \arrow[r, "\delta"] & 5 \\ 2 & & & 2 \end{tikzcd}
\qquad
\begin{tikzcd} 1 & & & \\ & 3 \arrow[ul, "\alpha"'] \arrow[r, "\gamma"] & 4 \arrow[r, "\delta"] & 5 \\ 2 \arrow[ur, "\beta"] & & & 2' \end{tikzcd}$$

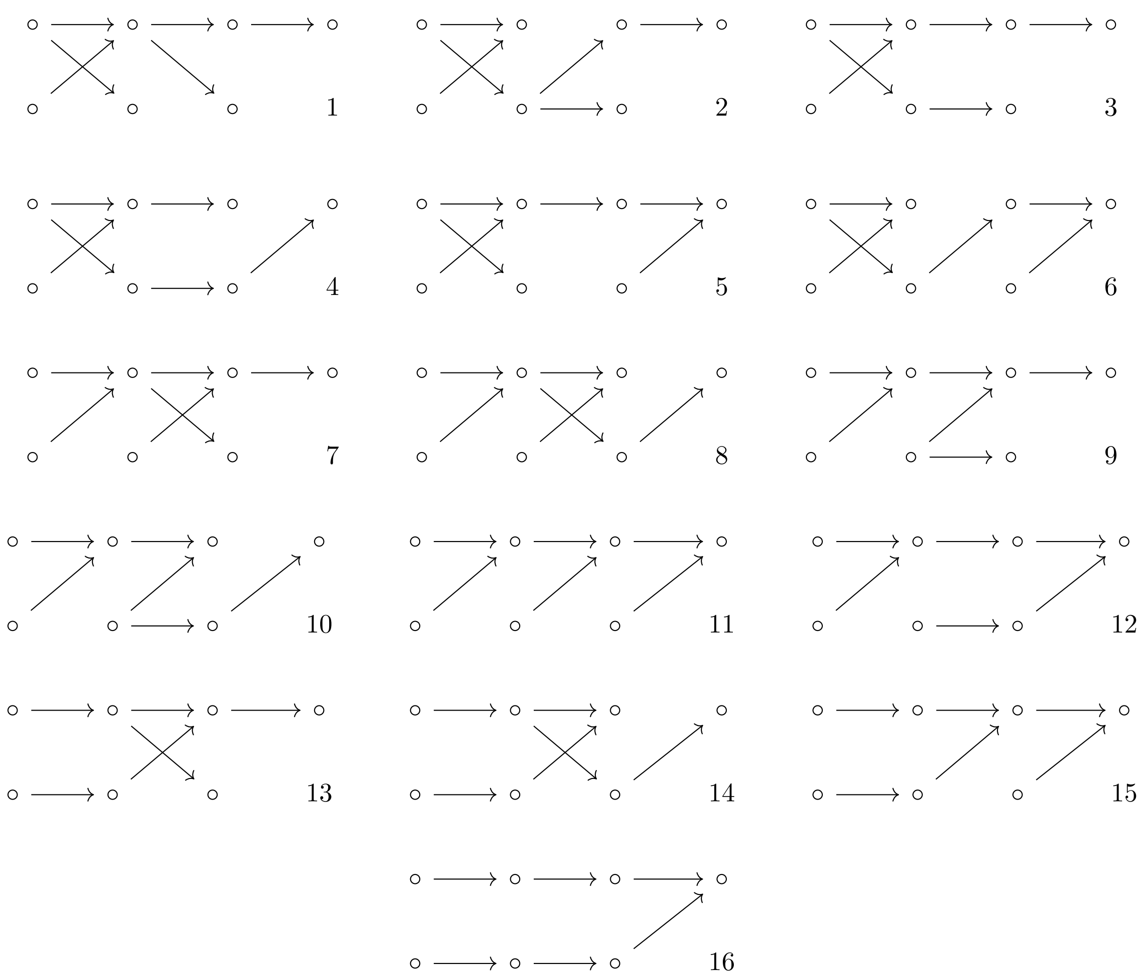
$$\begin{tikzcd} 1 & & & \\ & 3 \arrow[ul, "\alpha"'] \arrow[dl, "\beta"] \arrow[r, "\gamma"] & 4 \arrow[r, "\delta"] & 5 \\ 2 & & & 3 \end{tikzcd}
\qquad
\begin{tikzcd} 1 \arrow[dr, "\alpha"] & & & \\ & 3 & 4 \arrow[l, "\gamma"] \arrow[r, "\delta"] & 5 \\ 2 \arrow[ur, "\beta"] & & & 4 \end{tikzcd}
\qquad
\begin{tikzcd} 1 \arrow[dr, "\alpha"] & & & \\ & 3 \arrow[dl, "\beta"] & 4 \arrow[l, "\gamma"] \arrow[r, "\delta"] & 5 \\ 2 & & & 5 \end{tikzcd}$$



Furthermore, there is a symmetry on the first two vertices: we can interchange them to get another CQ which will be isomorphic to the one we had. If their arrows have the same orientation, this "flip" gives the same representation and CQ, so we cannot make any reductions in those cases. So we remove $2'$ and $5'$ as they are similar cases to 2 and 5.

3.1 D_5^1

The different tree coefficient quivers that we can draw with six arrows have been carefully drawn below so that only one representative of each isomorphism class is included. The number n in the bottom right will give the name of the CQ and representation $D_5^1(n)$.



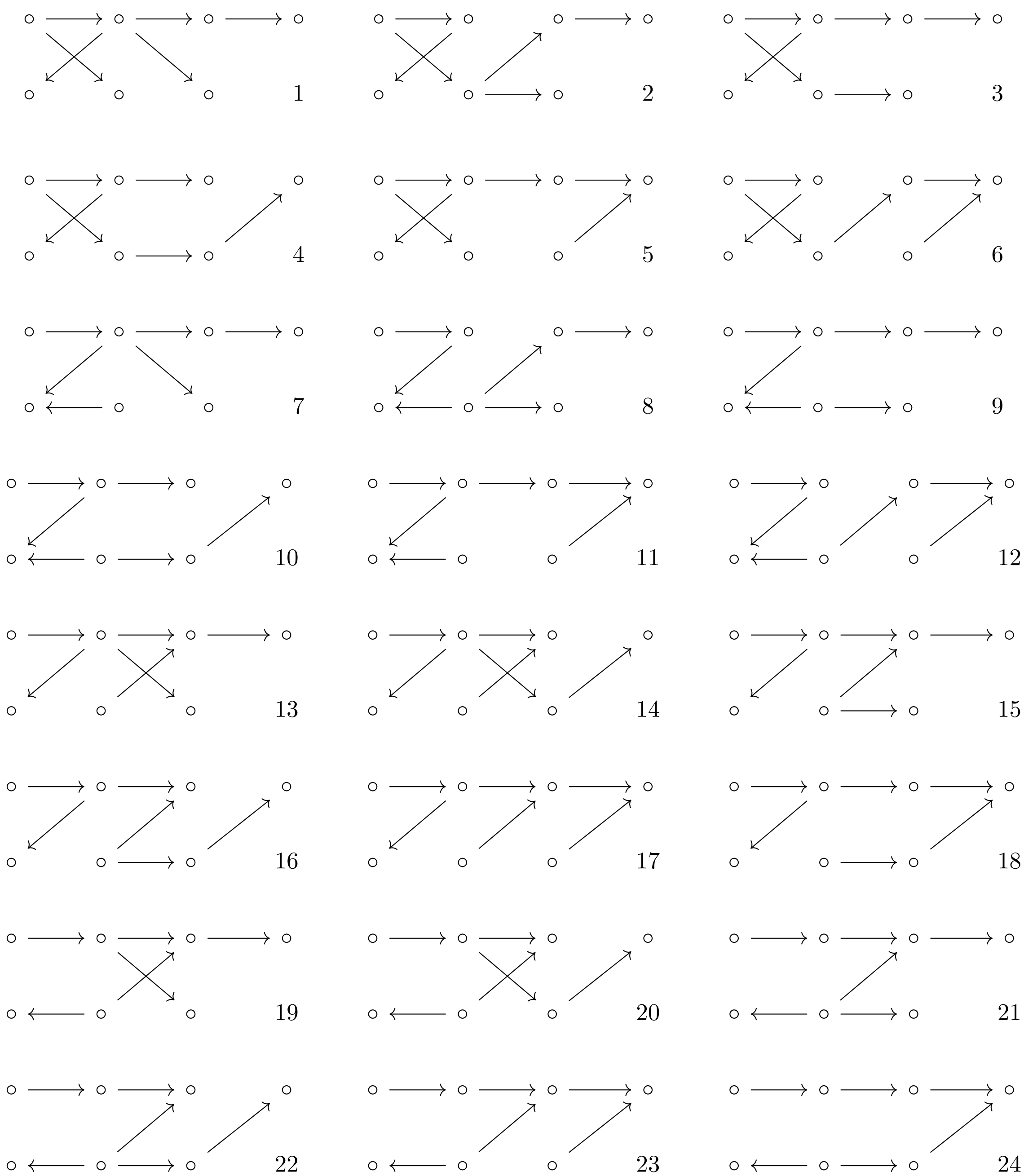
Of these we calculate by the same method as with D_4 that the CQs number 3, 13 and 16 give indecomposable representations/tree modules.

3.2 D_5^2

The representations $D_5^2(n)$ are of the form

$$\begin{tikzcd}
k \arrow[dr, "f_\alpha"] & & & \\
& k^2 \arrow[r, "f_\gamma"] \arrow[dl, "f_\beta"] & k^2 \arrow[r, "f_\delta"] & k \\
k & & &
\end{tikzcd}$$

with the following CQ's:



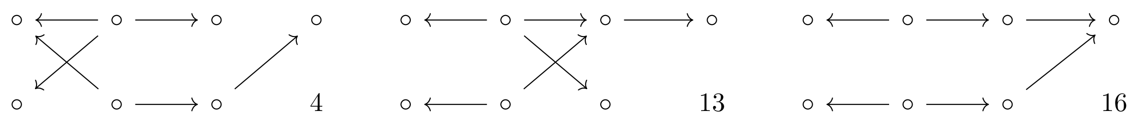
It is calculated that only CQs with number 4, 9, 13, 15 and 18 induce tree modules.

3.3 D_5^3

The representations $D_5^3(n)$ are

$$\begin{tikzcd}
k & & & \\
& k^2 \arrow[ul, "f_\alpha"'] \arrow[dl, "f_\beta"] \arrow[r, "f_\gamma"] & k^2 \arrow[r, "f_\delta"] & k \\
k & & &
\end{tikzcd}$$

We get a one-to-one correspondence between the CQ $D_5^1(n)$ and $D_5^3(n)$ for each n by turning the arrows between circle columns 1 and 2. We only draw those that produce tree modules here, and they are $D_5^3(4)$, $D_5^3(13)$ and $D_5^3(16)$.

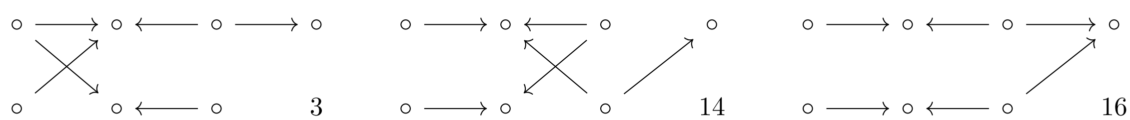


3.4 D_5^4

The representations $D_5^4(n)$ are

$$\begin{tikzcd}
k \arrow[dr, "f_\alpha"] & & & \\
& k^2 & k^2 \arrow[l, "f_\gamma"] \arrow[r, "f_\delta"] & k \\
k \arrow[ur, "f_\beta"] & & &
\end{tikzcd}$$

and the CQs correspond to those of D_5^1 with the arrows between columns 2 and 3 pointing left. The tree modules are $D_5^4(3)$, $D_5^4(14)$ and $D_5^4(16)$.

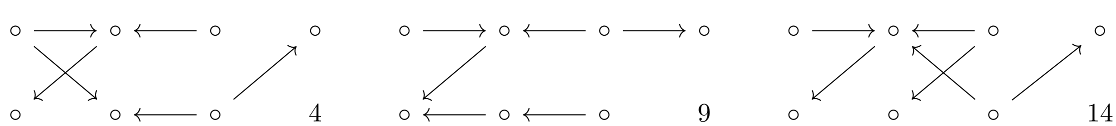


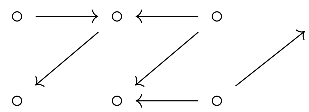
3.5 D_5^5

The representations $D_5^5(n)$ are

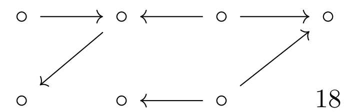
$$\begin{tikzcd}
k \arrow[dr, "f_\alpha"] & & & \\
& k^2 \arrow[dl, "f_\beta"] & k^2 \arrow[l, "f_\gamma"] \arrow[r, "f_\delta"] & k \\
k & & &
\end{tikzcd}$$

and the CQs correspond to those of D_5^2 with the arrows between columns 2 and 3 pointing left. The tree modules are $D_5^5(4)$, $D_5^5(9)$, $D_5^5(14)$, $D_5^5(16)$ and $D_5^5(18)$.





16



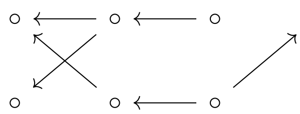
18

3.6 D_5^6

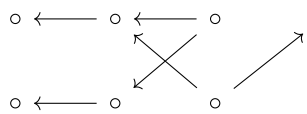
The representations $D_5^6(n)$ are

$$\begin{tikzcd}[row sep=small]
k & & & \\
& k^2 \arrow[ul, "f_\alpha"'] \arrow[dl, "f_\beta"] & k^2 \arrow[l, "f_\gamma"] \arrow[r, "f_\delta"] & k \\
k & & &
\end{tikzcd}$$

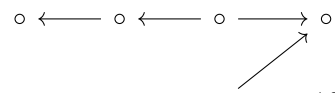
and the CQs correspond to those of D_5^1 with the arrows between columns 1 and 2 and columns 2 and 3 pointing left. The tree modules are $D_5^6(4)$, $D_5^6(14)$ and $D_5^6(16)$.



4



14



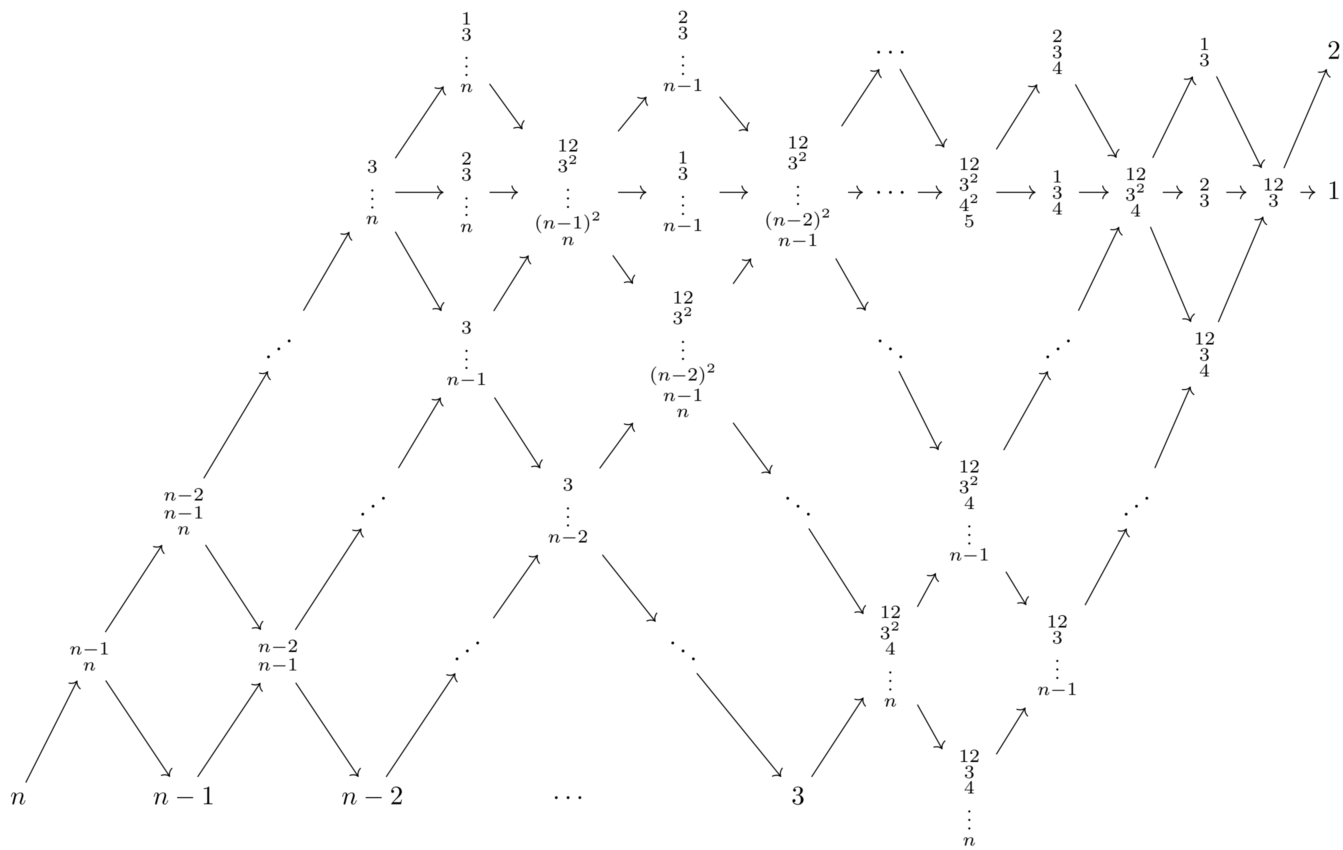
16

4 D_n

Now we turn to the general quiver D_n, where we assume that $n > 4$:

$$\begin{tikzcd}[row sep=small]
1 \arrow[dr, dash, "\alpha"] & & & & & & \\
& 3 \arrow[r, dash, "\gamma_1"] & 4 \arrow[r, dash, "\gamma_2"] & \dots \arrow[r, dash, "\gamma_{n-4}"] & n-1 \arrow[r, dash, "\delta"] & n \\
2 \arrow[ur, dash, "\beta"] & & & & &
\end{tikzcd}$$

From the AR-quiver below for D_n with all arrows turning right, we see that the only novel case to study also for the general case D_n is of dimension vector (1,1,2,2,2,...,2,2,1).



So we are studying representations of the following type:

$$\begin{tikzcd}
k \arrow[dr, no head, "f_\alpha"] & & & & & & \\
& k^2 \arrow[r, no head, "f_{\gamma_1}"] & k^2 \arrow[r, no head, "f_{\gamma_2}"] & \dots \arrow[r, no head, "f_{\gamma_{n-4}}"] & k^2 \arrow[r, no head, "f_\delta"] & k \\
k \arrow[ur, no head, "f_\beta"] & & & & &
\end{tikzcd}$$

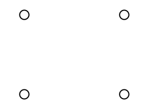
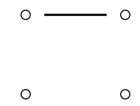
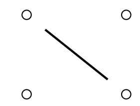
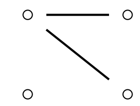
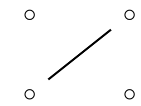
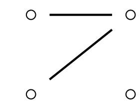
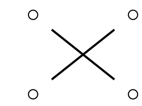
Since we will not take interest in the dual of an already discovered CQ, we can turn the last arrow δ to the right and then consider all configurations of the other arrows in the representation. The approach taken in the general case, however, will not be to study the $3 \cdot 2^{n-4}$ arrow configurations individually. We will look at the "shapes" in the coefficient quivers we have seen, and their properties.

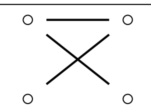
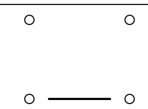
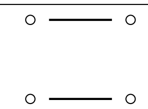
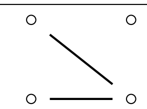
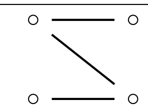
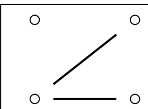
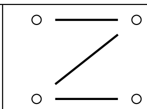
4.1 Shapes

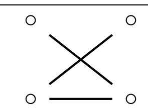
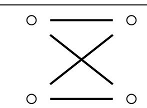
Looking at which of the previous coefficient quivers that induce indecomposable representations, one can notice some combinations of arrows repeat themselves between each of the bases, while others never make the cut. We will call a combination of arrows between two bases a *shape*, and studying these individually gives insights and clarity to give an overview of all the CQs of tree modules of type D_n, $n > 4$.

Representations of this type have three parts, which we name the arms $\begin{tikzcd}[row sep=small] k \arrow[dr, no head, "f_\alpha"] & \\ & k^2 \\ k \arrow[ur, no head, "f_\beta"] & \end{tikzcd}$, the body

$k^2 \overset{f_{\gamma_1}}{\overline{\qquad}} k^2 \overset{f_{\gamma_2}}{\overline{\qquad}} \dots \overset{f_{\gamma_{n-4}}}{\overline{\qquad}} k^2$, and the tail $k^2 \overset{f_{\delta}}{\overline{\qquad}} k$. The 16 possible line configurations between two sets of two basis elements are listed and named below. Transposition of the two basis elements on the right induces our equivalence classes here, as we work from left to right in the CQ regardless of arrow direction and hence regard the two elements on the left as fixed.

CQ							
Name	O	F	$\simeq F$	T	H	S	$\simeq I$

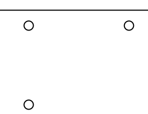
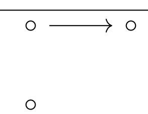
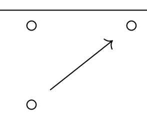
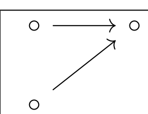
CQ							
Name	X	$\simeq H$	I	$\simeq S$	$\simeq X$	B	Z

CQ		
Name	$\simeq Z$	A

The names are based on the associations we get from the shapes; for example the F stands for flat, the H for hill, the S for seven, the I for identity, the X looks like a cross, and the Z looks like a Z. Adding an r as a subscript for any of the shapes would indicate arrows pointing right, and l means left. For the arms we may also have m to indicate the arrows move right from the top left circle and left towards the bottom left circle. The letter names made up here can also refer to the matrix corresponding to the shape. For instance, X_r can be the CQ above with the arrows pointing right, or the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and H_l can mean $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

The arms are two vector transformations instead of a single matrix transformation, so this works differently. Another index, which is either 1 or 2, will denote whether the map goes to/from the vertex 1 or 2. Together they make the shape, and are denoted the same as with the body, unless we must make clear that the shape is for arms, in which case 1,2 is put as an index to refer to the collection of maps to/from vertices 1 and 2. For instance, $I_{r_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $I_{r_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $I_{r_{1,2}} = \{I_{r_1}, I_{r_2}\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, and $S_{m_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $S_{m_2} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ make up $S_m = S_{m_{1,2}} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix} \right\}$.

The shapes for the tail are easier to list, and we have decided the direction is to the right.

CQ				
Name	O	F	H	S

Again, the letter may also refer to the map, and in this case subscripts are not necessary.

With the shape letters we can more succinctly write down the CQs of tree modules that we have already seen, as is done in the table below. The letter sequence goes from left to right in the quiver representation, starting with the arms and ending with the tail.

D_4^1	X_rF_r	I_rS_r			
D_4^2	I_rS_l				
D_5^1	X_rI_rF	I_rX_rF	I_rI_rS		
D_5^2	X_mI_rH	Z_mI_rF	S_mX_rF	S_mZ_rF	S_mI_rS
D_5^3	X_lI_rH	I_lX_rF	I_lI_rS		
D_5^4	X_rI_lF	I_rX_lH	I_rI_lS		
D_5^5	X_mI_lH	Z_mI_lF	S_mX_lH	S_mZ_lH	S_mI_lS
D_5^6	X_lI_lH	I_lX_lH	I_lI_lS		

4.2 The Reduction Number

When looking at the endomorphisms of D_n type representations

$$\begin{tikzcd}
k \arrow[dr, dash, "f_\alpha"] \arrow[ddd, dashed, bend right=60, "h_1"'] & & & & & \\
 & k^2 \arrow[r, dash, "f_{\gamma_1}"] \arrow[ddd, dashed, "h_3"] & k^2 \arrow[r, dash, "f_{\gamma_2}"] \arrow[ddd, dashed, "h_4"] & \dots \arrow[r, dash, "f_{\gamma_{n-4}}"] & k^2 \arrow[r, dash, "f_\delta"] \arrow[ddd, dashed, "h_{n-1}"] & k \arrow[ddd, dashed, "h_n"] \\
k \arrow[ur, dash, "f_\beta"] \arrow[ddd, dashed, bend right=60, "h_2"'] & & & & & \\
k \arrow[dr, dash, "f_\alpha"] & & & & & \\
 & k^2 \arrow[r, dash, "f_{\gamma_1}"] & k^2 \arrow[r, dash, "f_{\gamma_2}"] & \dots \arrow[r, dash, "f_{\gamma_{n-4}}"] & k^2 \arrow[r, dash, "f_\delta"] & k \\
k \arrow[ur, dash, "f_\beta"] & & & & &
\end{tikzcd}$$

we can see that there is 1 variable for each of h_1, h_2 and h_n, and 4 variables for each of the $n-3$ homomorphisms on the body, for a total of $4n-9$ variables. These must reduce to one to have a one-dimensional endomorphism ring, which makes the representation indecomposable. The amount of linear equations we can make from exploiting the commutativity of the homomorphisms with the different linear transformations is 2 for each of the arms, 4 for each of the $n-4$ squares inside the body, and 2 for the tail. This adds up to $4n-10$, one less than the amount of variables. This means that the represenation is indecomposable if and only if every equation reduces the number of free variables by one. We now have a condition for each transformation and first make a definition before stating it.

Definition 4.1. Let M be an $i \times j$-matrix over k, and V_M be the representation $k^j \xrightarrow{M} k^i$. The *reduction number of* M is defined as $\rho(M) = i^2 + j^2 - \dim(\mathrm{End}(V_M))$.

The reduction number of a matrix M tells us how many variables have been reduced in the endomorphism of the quiver representation V_M in the definition above. We need the reduction number to equal the amount of equations we get with each transformation, which will be $i \times j$, in which case we call it *permitted*. The 2×2-matrices f_{γ_m} in the body must have reduction number 4, as we get 4 linear equations in the expression

$$h_{m+3}f_{\gamma_m} = f_{\gamma_m}h_{m+2}$$

for rightward γ_m and

$$f_{\gamma_m}h_{m+3} = h_{m+2}f_{\gamma_m}$$

for leftward.

Here are some examples of matrices for the body.

Example. The matrix $X_r = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$:

$$\begin{tikzcd} k^2 \arrow[r, "X_r"] \arrow[d, dashed, "\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}"] & k^2 \arrow[d, dashed, "\begin{pmatrix} a_5 & a_6 \\ a_7 & a_8 \end{pmatrix}"] \\ k^2 \arrow[r, "X_r"'] & k^2 \end{tikzcd}$$

$$h_2 X_r = X_r h_1$$
$$\begin{pmatrix} a_5 + a_6 & a_5 \\ a_7 + a_8 & a_7 \end{pmatrix} = \begin{pmatrix} a_1 + a_3 & a_2 + a_4 \\ a_1 & a_2 \end{pmatrix}$$
$$\implies a_7 = a_2, \quad a_8 = a_1 - a_7 = a_1 - a_2, \quad a_5 = a_2 + a_4, \quad a_6 = a_1 + a_3 - a_2 - a_4$$
$$\implies \rho(X_r) = 4.$$

Example. The matrix $H_l = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$:

$$\begin{tikzcd} k^2 \arrow[d, dashed, "\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}"] & k^2 \arrow[l, "H_l"'] \arrow[d, dashed, "\begin{pmatrix} a_5 & a_6 \\ a_7 & a_8 \end{pmatrix}"] \\ k^2 & k^2 \arrow[l, "H_l"] \end{tikzcd}$$

$$h_1 H_l = H_l h_2$$
$$\begin{pmatrix} a_2 & 0 \\ a_4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a_5 & a_6 \end{pmatrix}$$
$$\implies \rho(H_r) = 3.$$

Example. $\rho(I_r) = \rho(I_l) = 4$, since $h_1 = h_2$ in that case.

With more computations we see that only X, Z and I in either direction have reduction number 4, making them the only shapes permitted in the body.

The 1×2- and 2×1-matrices of the arms and tail must have reduction number 2, as we get 2 equations. For the arms, a shape is permitted if and only if its parts are permitted and we can say for instance that $\rho(I_{r_{1,2}}) = \rho(I_{r_1}) + \rho(I_{r_2}) = 2 + 2 = 4$ even though $I_{r_{1,2}}$ is not a matrix. In this fashion we define the *reduction number of a shape*: for arms we have $\rho(M_{1,2}) = \rho(M_1) + \rho(M_2)$ and otherwise the number is the same as that of its matrix, $\rho(M)$. There are four equations in the arms, so we are looking for shapes that give reduction number 4.

Example. $S_r = S_{r_{1,2}} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$:

$$
\begin{tikzcd}
k \arrow[dr, "S_{r_1}"] \arrow[ddd, dashed, bend right, "a_1"'] & \\
& k^2 \arrow[ddd, dashed, "\begin{pmatrix} a_3 & a_4 \\ a_5 & a_6 \end{pmatrix}"] \\
k \arrow[ur, "S_{r_2}"] \arrow[ddd, dashed, bend right, "a_2"'] & \\
k \arrow[dr, "S_{r_1}"] & \\
& k^2 \\
k \arrow[ur, "S_{r_2}"] &
\end{tikzcd}
$$

$$
\begin{gathered}
S_{r_1} h_1 = \begin{pmatrix} a_1 \\ 0 \end{pmatrix} = h_3 S_{r_1} = \begin{pmatrix} a_3 \\ a_5 \end{pmatrix} \\
S_{r_2} h_2 = \begin{pmatrix} a_2 \\ 0 \end{pmatrix} = h_3 S_{r_2} = h_3 S_{r_1} \\
\implies a_2 = a_1, \quad a_3 = a_1, \quad a_5 = 0 \\
\implies \rho(S_r) = 3.
\end{gathered}
$$

Example. $S_m = S_{m_{1,2}} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix} \right\}$:

$$
\begin{tikzcd}
k \arrow[dr, "S_{m_1}"] \arrow[ddd, dashed, bend right, "a_1"'] & \\
& k^2 \arrow[dl, "S_{m_2}"] \arrow[ddd, dashed, "\begin{pmatrix} a_3 & a_4 \\ a_5 & a_6 \end{pmatrix}"] \\
k \arrow[ddd, dashed, bend right, "a_2"'] & \\
k \arrow[dr, "S_{m_1}"] & \\
& k^2 \arrow[dl, "S_{m_2}"] \\
k &
\end{tikzcd}
$$

$$
\begin{gathered}
S_{m_1} h_1 = \begin{pmatrix} a_1 \\ 0 \end{pmatrix} = h_3 S_{m_1} = \begin{pmatrix} a_3 \\ a_5 \end{pmatrix} \\
h_2 S_{m_2} = \begin{pmatrix} a_2 & 0 \end{pmatrix} = S_{m_2} h_3 = \begin{pmatrix} a_3 & a_4 \end{pmatrix} \\
\implies a_3 = a_1, \quad a_5 = 0, \quad a_2 = a_3 = a_1, \quad a_4 = 0 \\
\implies \rho(S_m) = 4.
\end{gathered}
$$

After calculations the conclusion is that X_r, X_l, I_r, I_l, X_m, Z_m and S_m are the only permitted arm shapes.

We see that all the tail shapes (except of course O) appear in the D_5 case and are thus permitted.

4.3 The Cost Number

The number of arrows we can have puts another condition on which shapes that can be together in a CQ. As a shorthand for the amount of arrows in a shape, we use the following definition.

Definition 4.2. The *cost of a shape/matrix* M is the amount of arrows/non-zero matrix elements in the shape/matrix, denoted $\kappa(M)$.

Below is a summary of the permitted shapes with the cost number for each given.

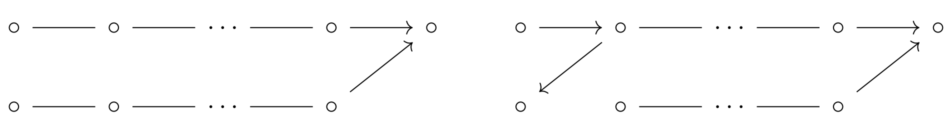
Arm Shape	$X_{r_{1,2}}$	$I_{r_{1,2}}$	$X_{m_{1,2}}$	$Z_{m_{1,2}}$	$S_{m_{1,2}}$	$X_{l_{1,2}}$	$I_{l_{1,2}}$
κ	3	2	3	3	2	3	2

Body Shape	X_r	Z_r	I_r	X_l	Z_l	I_l
κ	3	3	2	3	3	2

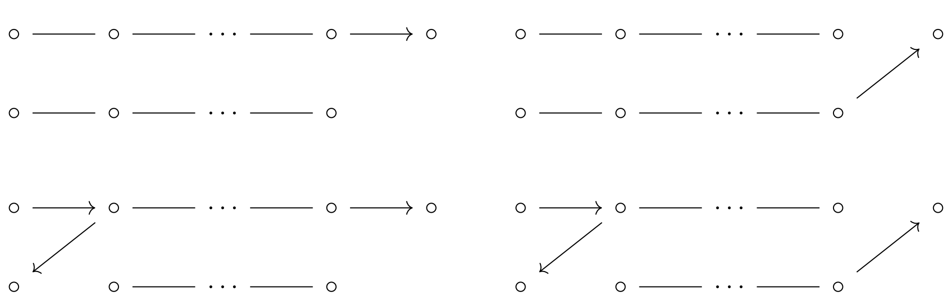
Tail Shape	S	F	H
κ	2	1	1

We know a tree module V can be expressed with 1-0-matrices with the amount of 1's being one less than the dimension of V, so we have a total cost $\sum_{M \in V} \kappa(M) = \dim(V) - 1 = 2n - 3 - 1 = 2n - 4$ in our case, D_n. This cost is to be distributed on $n-2$ shapes, making the average shape cost 2 exactly. Each shape will have at least cost 1. The tail is the only place where the cost can be 1, which would make the cost of exactly one other shape 3.

If the cost of the tail shape is 2, then the entire CQ is determined, given arrow directions; the only arm shape of cost 2 is I for uni-directed arms, and S_m for bi-directed arms (directed from vertex 1 and towards vertex 2). Let (I) stand for an arbitrary (possibly zero) amount - although the total amount of shapes in a D_n shape sequence is supposed to be $n-2$ - of identity shapes, each of direction r or l. We see that the representations induced by CQs $(I)S$ and $S_m(I)S$ are indecomposable and therefore valid tree modules from the fact that the identities will make the homomorphisms in the body all equal and that $I_{r/l}IS$ and S_mIS are valid in the D_5 case.



Now we can assume that the cost of the tail shape is 1, so there will be a shape of cost 3 elsewhere. Any further valid sequence of shapes for D_n would have to be made by adding a shape of cost 3 to the front or middle of the type D_{n-1} shape sequences $(I)^{n-4}F$ or $(I)^{n-4}H$, or in the middle of $S_m(I)^{n-5}F$ or $S_m(I)^{n-5}H$.



Uni-directed arms. The body identities can be omitted in this investigation, as the next homomorphism will then equal the previous. Hence, for instance, $I(I)X_r(I)F$ is valid if and only if IX_rF

is valid. We know from D_5 that it is $\left(\text{whether we have } I_{r_{1,2}} \text{ or } I_{l_{1,2}},\ h_3 \text{ will be } \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\right)$. Furthermore, we find that we can have $I(I)X_l(I)H$, but neither $I(I)X_r(I)H$ nor $I(I)X_l(I)F$. Note that $I(I)Z(I)F/H \simeq I(I)X(I)F/H$ as graphs, so the shape Z is only for the bi-directed arms case. Since we know X_rIF and X_lIH are valid, while X_rIH and X_lIF are not, we may sum up the valid cases found here as $(I)X_r(I)F$ and $(I)X_l(I)H$ (where $X_{r/l}$ is possibly an arm shape).

Bi-directed arms. By the same argument, adding $n-5$ identities to valid sequences in D_5^2 and D_5^5, we find that $X_m(I)H$, $Z_m(I)F$, $S_m(I)X_r(I)F$, $S_m(I)X_l(I)H$, $S_m(I)Z_r(I)F$ and $S_m(I)Z_l(I)H$ complete our list of valid cases for D_n, $n > 4$.

4.4 Conclusion

Below is a table showing all the different kinds of coefficient quivers for tree modules of type D_n, $n > 4$. We see that they are really just the same as those for D_5 with $n-5$ added identity shapes. How many CQ isomorphism classes each kind contains is also given. For example, $(I)X_r(I)F$ and $(I)X_l(I)H$ each represent $(n-3)2^{n-4}$ different CQs up to isomorphism, because there are $n-3$ choices for where to put the X, and 2^{n-4} choices for the orientations of the I's.

CQ type	$(I)S$	$(I)X_r(I)F$	$(I)X_l(I)H$	$S_m(I)S$	$X_m(I)H$	$Z_m(I)F$
# isoclasses	2^{n-3}	$(n-3)2^{n-4}$	$(n-3)2^{n-4}$	2^{n-4}	2^{n-4}	2^{n-4}

CQ type	$S_m(I)X_r(I)F$	$S_m(I)X_l(I)H$	$S_m(I)Z_r(I)F$	$S_m(I)Z_l(I)H$
# isoclasses	$(n-4)2^{n-5}$	$(n-4)2^{n-5}$	$(n-4)2^{n-5}$	$(n-4)2^{n-5}$

Adding it up, the total amount of tree module coefficient quivers we can have for D_n, $n > 4$, up to isomorphism and duality is

$$2^{n-3} + 2(n-3)2^{n-4} + 3 \cdot 2^{n-4} + 4(n-4)2^{n-5} = (2 + 2n - 6 + 3 + 2n - 8)2^{n-4} = (4n-9)2^{n-4}.$$

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