Received: 31 August 2021 | Accepted: 22 February 2022

DOI: 10.1111/poms.13706

ORIGINAL ARTICLE

# Inventory control in production–inventory systems with random yield and rework: The unit-tracking approach

**Peter Berling**[1,2] | **Danja R. Sonntag**[3]

[1]Department of Industrial Economics and Technology Management, Norwegian University of Science and Technology, Trondheim, Norway

[2]Department of Industrial Management and Logistics, Lund University, Lund, Sweden

[3]Business School, University of Mannheim, Mannheim, Germany

**Correspondence**
Peter Berling, Department of Industrial Economics and Technology Management, Norwegian University of Science and Technology, Postboks 8900, Trondheim, Torgarden NO-7491, Norway.
Email: lars.p.berling@ntnu.no

**Handling editor:** Panos Kouvelis

**Abstract**

This paper considers a single-stage make-to-stock production–inventory system under random demand and random yield, where defective units are reworked. We examine how to set cost-minimizing production/order quantities in such imperfect systems, which is challenging because a random yield implies an uncertain arrival time of outstanding units and the possibility of them crossing each other in the pipeline. To determine the order/production quantity in each period, we extend the unit-tracking/decomposition approach, taking into account the possibility of order-crossing, which is new to the literature and relevant to other planning problems. The extended unit-tracking/decomposition approach allows us to determine the optimal base-stock level and to formulate the exact and an approximate expression of the per-period cost of a base-stock policy. The same approach is also used to develop a state-dependent ordering policy. The numerical study reveals that our state-dependent policy can reduce inventory-related costs compared to the base-stock policy by up to 6% and compared to an existing approach from the literature by up to 4.5%. From a managerial perspective, the most interesting finding is that a high mean production yield does not necessarily lead to lower expected inventory-related costs. This counterintuitive finding, which can be observed for the most commonly used yield model, is driven by an increased probability that all the units in a batch are either of good or unacceptable quality.

**KEYWORDS**
base-stock policy, inventory control, random yield, rework, unit-tracking approach

## 1 | INTRODUCTION

We consider a single-stage make-to-stock production–inventory system under random demand and random yield, where defective units are reworked. Random yield refers to the number of items meeting the desired quality requirements, set either by a company itself or externally, for example, by the US Food and Drug Administration (FDA) for pharmaceutical products or the Federal Communications Commission (FCC) for electronic products. Our goal is to determine the ordering policy that minimizes the total average inventory-related cost per period, comprised of holding costs for all units on stock and backorder costs for all units that cannot be satisfied immediately from stock on hand.

In particular, the high-tech industry has to deal with surprisingly low yield rates given the level of automation and sophisticated production equipment commonly used. This leads to many products that cannot be sold directly to customers but still incorporate substantial value. Gurnani et al. (2000) and Chen and Yang (2014) report that in the liquid crystal display (LCD) manufacturing industry, production yield rates of less than 50% are common. In the semiconductor industry, yield rates are usually between 50% and 70% (Gavirneni, 2004) but can vary from 0% to 100% between production runs (Leachman & Hodges, 1996). The reasons for producing defective items are manifold and range from imperfect or inconsistent raw material quality and the limited skill level of workers to limited machine capabilities. Although the high-tech industry is commonly used in the scientific literature as an example of an industry with random

Accepted by Panos Kouvelis, after one revision.

yields, other industries also have to deal with yield rates that are significantly below 100%. For example, yield rates for olive oil production can be as low as 30% or 40% (Kazaz, 2004), and for vaccines they are about 75% (White III & Cheong, 2012).

Products that do not meet quality requirements are commonly either scrapped (e.g., in the food industry, Öztürk, 2017), sold as lower quality products for a lower price (e.g., in the apparel industry, Moussawi-Haidar et al., 2016), or reworked (e.g., in the automobile industry, Sarker et al., 2008). Especially in industries where defective units incorporate substantial value, an incentive exists to rework these units. Rework activities are common in, for example, the semiconductor, glass, metal processing, chemical, pharmaceutical, and automobile industries (Buscher & Lindner, 2007; Chiu et al., 2007; Sarker et al., 2008; Widyadana & Wee, 2012). The following two examples from industry emphasize the considerable role of rework in the manufacturing process: Tesla, Inc. had to rework 84% of its Model 3 vehicles due to a low first-pass yield of only 16%. First-pass yields correspond to the percentage of units that leave the manufacturing process without requiring any rework, which in the automobile industry are commonly around 80% (Lopez, 2018). The Boeing Company announced as a result of its Boeing 787 Dreamliner quality issues that "the low production rates and rework are expected to result in approximately $1 billion of abnormal costs" (Gates, 2021), which emphasizes the necessity to incorporate random yields and rework.

This paper shows how to manage production–inventory systems under random demand, random yield, and rework, focusing on minimizing inventory-related costs. Because of the complexity of this planning problem, optimal policies are either unknown or very complex and practically irrelevant, wherefore heuristic solutions are commonly used. In particular, when considering rework, all existing approaches rely on approximations. We follow this line of literature and focus on identifying heuristic solution approaches to this rarely studied problem.

We develop a new extension of the unit-tracking/decomposition approach—originally developed by Axsäter (1990)—to derive an exact expression for the expected holding and backorder costs per period under a base-stock policy and an improved state-dependent ordering policy. The idea behind the unit-tracking/decomposition approach is to decompose the problem into a series of single-unit problems and minimize the expected cost per unit by comparing the cost of ordering the unit now with the cost of ordering it in a later period. We adjust the unit-tracking/decomposition approach by taking order-crossing into account, which occurs when units ordered in later periods that pass quality control enter the warehouse before units ordered earlier that require rework.

In summary, the contributions of this paper are as follows: (1) Problem-wise, the contribution of this paper is that it analyzes a rarely discussed but practically highly relevant make-to-stock production–inventory system under random demand, random yield, and rework. We especially consider that items might have to undergo rework multiple times until they satisfy the quality requirements, which has not been studied in make-to-stock production–inventory systems under random demand and yield before. (2) Methodology-wise, the contribution is threefold. First, we extend the unit-tracking/decomposition approach to formulate the exact expression of the per-period cost under a base-stock policy and a simple approximation of this cost. Second, we use the same approach to develop a state-dependent ordering policy. state-dependent policy, take order-crossing into account, which is prohibited by assumption in existing literature on the unit-tracking/decomposition approach (see, e.g., Berling & Martínez-de-Albéniz, 2016a; Muharremoglu & Tsitsiklis, 2008). This emphasizes the paper's contribution also for other planning problems. Third, our solution approaches are, unlike existing approaches in the literature, not limited to a specific demand or yield model and, therefore, applicable to a variety of different systems. (3) Numerical results reveal that especially for low yields of 50%, as observed in the high-tech industry, our state-dependent approach leads to cost reductions of up to 6% compared to the base-stock policy and 4.5% compared to an existing approach from the literature. (4) Managerial-wise, we provide simple decision rules that help decision makers to decide when to use a base-stock policy or a state-dependent ordering policy. Finally, we show that for the most commonly used yield model, namely, stochastic proportional yield, high production yields do not necessarily reduce the inventory-related cost of the system. This finding is counterintuitive but can be explained by an increased probability of the extreme outcomes, that is, all units in a batch being good or needing rework, which has a negative effect on the overall cost.

The remainder of this paper is organized as follows. We review the relevant literature in Section 2 and describe our problem in detail in Section 3. In Section 4, we derive the base-stock policy using the unit-tracking/decomposition approach. Based on this, we develop a heuristic for a state-dependent ordering policy in Section 5. In Section 6, we analyze the performance of the state-dependent ordering policy compared to the optimal base-stock policy and an existing policy from the literature. In Section 7, we discuss the assumptions made in Section 3 regarding their effect on the solution procedure and the applicability of the model to solve practically relevant problems. We conclude the paper with a summary and an outlook on future research opportunities in Section 8.

## 2 | LITERATURE REVIEW

In Section 2.1, we discuss papers that consider make-to-stock production systems under random demand and yield with a focus on papers discussing similar planning problems and methodological aspects as we do in this paper. We review the literature on the unit-tracking approach in Section 2.2.

### 2.1 | Make-to-stock production–inventory systems under random yield

Research into make-to-stock production systems under random demand and yield began in the late 1980s with a study by Ehrhardt and Taube (1987). They analyzed a single-period inventory model, in which ordered units were subject to random yield and defective items were disposed of. The work by Ehrhardt and Taube (1987) has been extended by Henig and Gerchak (1990) and Wang and Gerchak (1996) to handle multiple periods and capacity restrictions on the production quantity. Bollapragada and Morton (1999) and Huh and Nagarajan (2010) analyze stationary multiperiod inventory systems under random yield and present different heuristic solution approaches, mainly within the class of linear inflation policies, because the optimal policy is very complex (Henig & Gerchak, 1990). Under a linear inflation policy, the order quantity calculated based on a classical base-stock policy is inflated depending on the yield rate. Linear inflation policies show excellent performance and are therefore commonly used for such systems; this is also one of the reasons why our analysis relies on (adjusted) base-stock policies.

Based on the early works on random yield systems with disposal of defective items, various extensions have been analyzed, focusing either on the effect of different yield models, for example, binomial, stochastic proportional, or all-or-nothing yield (see Yano & Lee, 1995), on the system costs (see Erdem & Özekici, 2002; Inderfurth & Vogelgesang, 2013; Kutzner & Kiesmüller, 2013; Sonntag & Kiesmüller, 2016), on the extension of the system to nonzero production times (see Inderfurth & Kiesmüller, 2015; Kiesmüller & Inderfurth, 2018), or multistage production processes (see Choi et al., 2008; Dettenbach & Thonemann, 2015; Sonntag & Kiesmüller, 2017). Voelkel et al. (2020) complement this work by analyzing the impact different costly tracking possibilities—always, never, or dynamic—have on ordering decisions.

None of these papers model rework activities. Rework processes in make-to-stock production systems with stochastic demand and yield have, to the best of our knowledge, only been considered by Gotzel and Inderfurth (2005) and Sonntag and Kiesmüller (2018). Gotzel and Inderfurth (2005) analyze a different system to this paper, considering an additional stock point before the rework process, which is perfect. They use a two-parameter policy with both a produce-up-to level and a rework-up-to level to determine the production and rework quantities in each period.

Sonntag and Kiesmüller (2018) analyze the same production–inventory system as considered in this paper but only consider a perfect rework process. They modify an approach by Inderfurth and Kiesmüller (2015) under disposal to incorporate the possibility of reworking defective items. Sonntag and Kiesmüller (2018) approximately determine the expected per-period cost and the optimal base-stock level under a traditional base-stock policy and then present an alternative base-stock policy, which we describe in Section 6. Unlike Sonntag and Kiesmüller (2018), we derive the true expected cost and the optimal base-stock level using an exact approach. Furthermore, we compare the performance of the heuristics proposed in this paper with the performance of the alternative base-stock policy proposed in Sonntag and Kiesmüller (2018) and show that our methods outperform the existing approach.

Although Sonntag and Kiesmüller (2018) discuss the same system under perfect rework as considered in this paper, nonperfect rework processes have not been considered so far. Furthermore, their model is limited to normally distributed demand and stochastic proportional yield. In contrast, our proposed methods are able to handle various demand and yield distributions.

### 2.2 | Unit-tracking approach

The unit-tracking or unit decomposition approach by Axsäter (1990) determines the total cost for the system by tracking the cost for each item ordered. This cost is made up of the holding cost while the item is in storage or the backorder cost for the time a customer has to wait for the unit to be available. The method differs from traditional approaches in inventory management that determine the cost by tracking the inventory level at the different stocking points. Axsäter (1990) introduces the unit-tracking approach for a divergent two-echelon inventory system under Poisson demand and a one-for-one replenishment policy under continuous review. Axsäter (1993a) extends the approach to batch ordering, and Axsäter (1993b) considers periodic instead of continuous review.

Whereas Axsäter (1990) introduces the unit-tracking methodology for the class of one-for-one replenishment policies, Muharremoglu and Tsitsiklis (2008) and Janakiraman and Muckstadt (2009) use the approach to derive structural insights regarding the optimal solution. Muharremoglu and Tsitsiklis (2008) consider a serial multiechelon inventory system under stochastic lead times but without order-crossing. They show that a state-dependent echelon base-stock policy is optimal and present an efficient algorithm to calculate the optimal base-stock levels. Janakiraman and Muckstadt (2009) consider a similar system but add capacity limits to the production stages. Berling and Martínez-de-Albéniz (2016a) extend the analysis by Muharremoglu and Tsitsiklis (2008) and allow for expediting but again do not allow order-crossing. Berling and Martínez-de-Albéniz (2016b) apply the results of Berling and Martínez-de-Albéniz (2016a) in the context of transportation within a supply chain.

Unlike previous works focusing on either divergent or serial multiechelon inventory systems, Yu and Benjaafar (2008) consider a single-stage periodic review system with correlated, nonstationary demands. In line with the findings of Muharremoglu and Tsitsiklis (2008), they show that the optimal policy is a state-dependent base-stock policy. Berling and Martínez-de-Albéniz (2011) extend the analysis of single-echelon continuous-review systems by considering variable purchase prices for the considered product, for

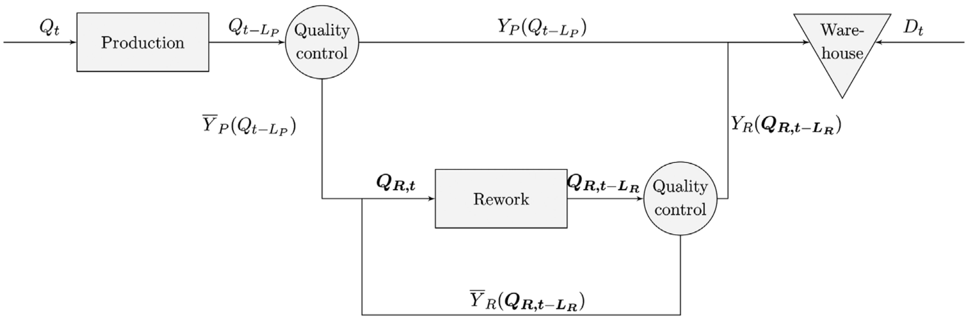


**FIGURE 1** Periodic review production–inventory system with random yield and nonperfect rework in period $t$

example, on the commodity market. They show how to determine the optimal price-dependent policy with regard to stochastic demand and stochastic prices.

In summary, the unit-tracking approach has shown broad applicability to various inventory-related problems. However, the problem in the present paper requires a significant adjustment of the method in order to take order-crossing into account. This has not been done so far due to its complexity. The present paper thus contributes significantly to other inventory-related problems where order-crossing is present (e.g., under stochastic lead times as discussed in Muharremoglu & Tsitsiklis, 2008, under the assumption of no order-crossing).

## 3 | MODEL FORMULATION

We consider a periodic review production–inventory system with random demand and random yield where defective items are reworked. In the following, we will first present the sequence of events and then explain the yield processes in more detail. The practical implications of the assumptions made in the model are discussed in detail in Section 7. The sequence of events in each period $t$ is as follows:

1. The period demand $D_t$, which is stochastic and independent and identically distributed (i.i.d.) across periods with a known distribution, is observed and satisfied from available stock-on-hand together with the backordered demand from previous periods. All demand that cannot be met is backordered at a cost of $b$ per unit, and there is a holding cost of $h$ per unit that is in stock at this point. We assume that demand occurs at the beginning rather than at the end of a period in order to simplify the notation.
2. As visualized in Figure 1, the warehouse is restocked with all items leaving the production/rework process with sufficient quality, that is, $Y_P(Q_{t-L_P})$ and $Y_R(\boldsymbol{Q_{R,t-L_R}})$, respectively. A detailed explanation of the notation is given below.
3. An order of a batch of $Q_t \geq 0$ units is placed, where the order quantity $Q_t$ depends on the chosen ordering policy (see Sections 4 and 5).

The number of units leaving the production process with sufficient quality in period $t$, $Y_P(Q_{t-L_P})$, depends on the number of items entering the production process in period $t - L_P$, $Q_{t-L_P}$, where $L_P$ denotes the constant production lead-time. The production yield is stochastic with a known distribution which is i.i.d. across production runs of the same size $Q$. The $\overline{Y}_P(Q_{t-L_P}) = Q_{t-L_P} - Y_P(Q_{t-L_P})$ units that do not pass the quality inspection enter the rework process directly.

Regarding the rework process, we consider two variants: perfect and nonperfect rework, where perfect rework is a special case of the more general nonperfect rework process. Under perfect rework, the items entering the rework process in period $t$, $\boldsymbol{Q_{R,t}}$, equal the nonconforming items from the production process, $\overline{Y}_P(Q_{t-L_P})$. After a constant rework lead time $L_R$, the number of items leaving the rework process in period $t$, $Y_R(\boldsymbol{Q_{R,t-L_R}})$, equals $\overline{Y}_P(Q_{t-L_P-L_R})$.

Under nonperfect rework, reworked items might fail the quality inspection and require an additional rework cycle. Let $L_R^m$ denote the constant rework time for the $m:th$ rework cycle, $Q^m_{R,t-L_R^m}$ equal the number of units that entered the $m:th$ rework cycle in period $t - L_R^m$, and $\boldsymbol{Q_{R,t-L_R}} = (Q^1_{R,t-L_R^1}, Q^2_{R,t-L_R^2}, ...)$. The number of reworked units that enter the warehouse in period $t$ after the $m:th$ rework cycle equals $Y_R^m(Q^m_{R,t-L_R^m})$, wherefore the total number of units entering the warehouse after a successful rework process in period $t$ equals $Y_R(\boldsymbol{Q_{R,t-L_R}}) = \sum_{m=1}^{\infty} Y_R^m(Q^m_{R,t-L_R^m})$. The yield of the rework process, $Y_R^m(Q^m_{R,t-L_R^m})$, is stochastic with a known distribution that is i.i.d. across periods but do depend on the batch-size $Q^m_{R,t-L_R^m}$ and may depend on the rework cycle $m$. All reworked items not satisfying the quality requirements, $\overline{Y}_R^m(Q^m_{R,t-L_R^m})$, have to undergo rework for at least one additional cycle. Therefore, $Q^{m+1}_{R,t} = \overline{Y}_R^m(Q^m_{R,t-L_R^m}) = Q^m_{R,t-L_R^m} - Y_R^m(Q^m_{R,t-L_R^m})$ for $m \geq 1$ and $Q^1_{R,t} = \overline{Y}_P(Q_{t-L_P})$.

The optimal decision in each period is to order the amount of goods, $Q_t^\star$, that minimizes the expected sum of all future holding and backorder costs. Determining $Q_t^\star$ is not a trivial task because the expected amount of inventory available at the end of period $t + L_P$ (and beyond) depends not only on the inventory position—the stock-on-hand minus backorders plus all outstanding orders in the production and rework process—after the order in period $t$ has been placed but on the exact location of the pipeline inventory because of possible order-crossing. We analyze the order-crossing phenomenon and the resulting challenges in detail in the next section.

To the best of our knowledge, the structure of the optimal policy for the described problem is unknown and a complete enumeration of all possible outcomes to find the optimal policy is computationally intractable. Rather than search for the truly optimal decision in each period, we focus on finding the optimal base-stock policy in Section 4 and then introduce a state-dependent ordering policy in Section 5.

## 4 | BASE-STOCK POLICY

We use a base-stock policy as starting point for our analysis on how to determine the production quantities in each period. Using an (adjusted) base-stock policy is reasonable for several reasons. First, base-stock policies are simple and commonly used in inventory management as a proxy for the optimal ordering policy. Second, the base-stock policy is known to be the optimal policy under perfect yield (Federgruen & Zipkin, 1986). Third, an adjusted base-stock policy—known as the "linear inflation policy" (Huh & Nagarajan, 2010)—has shown excellent performance in random yield systems with disposal of defective items, and is commonly used in the literature (see, e.g., Bollapragada & Morton, 1999; Huh & Nagarajan, 2010; Inderfurth & Kiesmüller, 2015). Finally, (adjusted) approximate base-stock policies are also commonly preferred within the existing literature on random yield systems with rework of defective items (Gotzel & Inderfurth, 2005; Sonntag & Kiesmüller, 2018).

The new solution procedure used to find the optimal base-stock policy, which has not been determined for the considered problem so far, is inspired by the unit-tracking/decomposition approach introduced by Axsäter (1990). This method aims to decompose the problem into a series of independent "order the next unit now or later" problems that are solved sequentially. Therefore, each unit is matched with a specific demand that it will satisfy, and the expected cost for this pair is then determined. The expected cost is composed of a backorder cost multiplied by the expected time the customer has to wait for the unit and a holding cost multiplied by the expected time the unit has to wait for the demand. Both expectations are calculated based on the lead-time until an item enters the warehouse and the distribution of the arrival time of customer demand.

Incorporating random yields and (nonperfect) rework increases the complexity substantially because units can cross each other in the pipeline. Order-crossing implies that it is uncertain which demand will be matched with which unit because this will depend on past and future ordering decisions. For example, let us consider a situation under perfect rework with three units in stock and an outstanding order of one unit placed in the last period. With perfect yield, these units will be used to satisfy four consecutive demands, and the "next unit" to be produced will be used to fulfill a fifth demanded item. Now consider a system under random yield where the unit ordered in the last period needs to be reworked for two (or more) periods. That unit will then be available at the end of period $t + L_P + 1$ (or later), whereas the "next unit" will be available at the end of period $t + L_P$ if it is of good quality. Thus, these units will cross each other in the pipeline, so the "next unit" will be used to satisfy the fourth rather than the fifth demand, and the unit ordered in the last period will be used to satisfy the fifth demand (or possibly an even later demand if $L_R > 2$ and it is crossed by more units). An illustration of this behavior with $L_P = 4$ and $L_R = 2$ and a perfect rework process is shown in Figure 2.

In Section 4.1, by extending the unit-tracking approach, we derive an exact expression of the expected cost per period under a base-stock policy and perfect rework. The structure of this cost is discussed in Section 4.2, along with its implications for the search procedure for the optimal base-stock level. In Section 4.3, an accurate approximation of the marginal cost of increasing the base-stock level is derived. This approximate expression facilitates the search for the optimal base-stock level and provided the correct value in all instances calculated in Section 6. In Section 4.4, we show how this analysis can be extended to systems with a nonperfect rework process.

### 4.1 | Base-stock policy: Exact cost expression under a perfect rework process

The derivation of the expected cost per period is inspired by the unit-tracking/decomposition approach and is carried out in three steps. The notation used is summarized in Table 1. Based on the number of units, $Q_0$, ordered in an arbitrary period 0, we first match each of the $Q_0 - i$ good units, that is, units that pass the quality inspection, and the $i$ reworked units with a demand. We then formulate the expected backorder and holding costs for each such unit-demand pair. Finally, based on the probability of each demand–yield scenario, we summarize the single-unit costs into an expected cost per period. In this subsection, we take a closer look at all three steps. Note again, that we assume a perfect rework process and discuss adjustments under nonperfect rework in Section 4.4.

#### Step 1: Match each produced unit with a demand

A base-stock policy implies that the order placed at the end of period $t$ resets the inventory position to the desired base-stock level $S$. In a traditional model with perfect yield, the $Q_0$ units ordered at the end of period 0 would thus be used to satisfy the $S - Q_0 + 1 : th$, $S - Q_0 + 2 : th$, …, $S : th$ demand occurring "after" period 0. However, this is not the case with nonperfect yield due to the possibility of order-crossing. The $Q_0 - i$ good units will cross the $j$ units in $\mathbf{Q_B} = [Q_{-L_R}, \dots, Q_{-1}]$ that need rework and will therefore be used to satisfy the $S - Q_0 + 1 - j : th$, $S - Q_0 + 2 - j :$

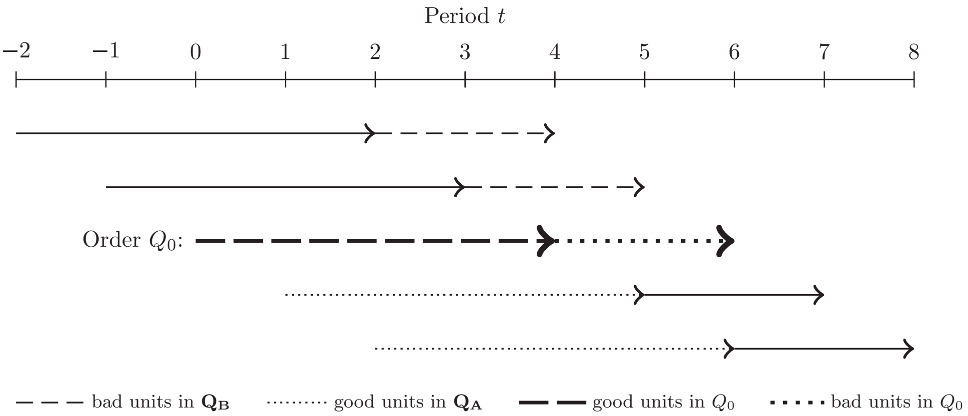


**FIGURE 2** Timeline showing when orders are placed and when units arrive at the warehouse for $L_P = 4$, $L_R = 2$, and perfect rework

**TABLE 1** Notation

| | |
|---|---|
| $S$ | Base-stock level |
| $D_t$ | Demand in period $t$ |
| $D_{min}$ | Minimum demand in a period |
| $D_{max}$ | Maximum demand in a period |
| $Prob(D_t)$ | Probability that the demand in period $t$ is $D_t$ |
| $\mathbf{D_B} = [D_{-L_R}, \ldots, D_{-1}]$ | Demand in the $L_R$ periods before period 0 |
| $\mathbf{D_A} = [D_1, \ldots, D_{L_R}]$ | Demand in the $L_R$ periods after period 0 |
| $\mathbf{D_{L_R}}$ | Set encompassing all demand combinations over $L_R$ periods |
| $Prob(\mathbf{D_N})$ | Probability that the demand over $L_R$ periods is $\mathbf{D_N}$ |
| $\Sigma D_A$ | Total number of unit demanded in the $L_R$ periods after period 0 |
| $Q_t$ | Order placed in period $t$ |
| $Prob(x\|Q_t)$ | Probability that $\overline{Y}_P(Q_t) = x$, i.e., that $x$ of the $Q_t$ units ordered in period $t$ need rework |
| $\mathbf{Q_B} = [Q_{-L_R}, \ldots, Q_{-1}]$ | Orders placed in the $L_R$ periods before period 0 |
| $\Sigma Q_B$ | Total number of units ordered the $L_R$ periods before period 0 |
| $\mathbf{Q_A} = [Q_1, \ldots, Q_{L_R}]$ | Orders placed in the $L_R$ periods after period 0 |
| $\Sigma Q_A$ | Total number of units ordered the $L_R$ periods after period 0 |
| $\mathbf{Q_N}$ | Set encompassing all order quantity combinations over $L_R$ periods |
| $Prob(x\|\mathbf{Q_N})$ | Probability that $x$ of the units in $\mathbf{Q_N}$ need rework |
| $p_{n,t}(x)$ | Probability that the $x:th$ demand after period $n$ occurs in period $n+t$ |

$th$, …, $S-i-j:th$ demand occurring "after" period 0. The $i$ units ordered in period 0 that need rework will, in contrast, be crossed by the good units ordered in period 1 to $L_R$ (see Figure 2). Because the order placed in period $L_R$ resets the inventory position to $S$, these units will be used to satisfy the $S-i+1-k:th$ to $S-k:th$ demand occurring "after" period $L_R$, where $k$ is the number of units in $\mathbf{Q_A} = [Q_1, \ldots, Q_{L_R}]$ that do not pass the quality inspection. To simplify the notation, we assume that good units ordered in period $t$ are sequenced before reworked units ordered in period $t-L_R$. As these units arrive at the same time, the sequencing rule can be changed without affecting the cost.

## Step 2: Determine per unit holding and backorder cost

The expected cost for a good unit used to satisfy the $x:th$ demand occurring "after" period 0, given $D_0$ and $\mathbf{D_B} = [D_{-L_R}, \ldots, D_{-1}]$, is equal to

$$C_G(x, D_0, \mathbf{D_B}) = \begin{cases} b(L_P+1+t), & -\sum_{\tau=0}^{t} D_{-\tau} < x \le -\sum_{\tau=0}^{t-1} D_{-\tau},\ t \in [0, L_R] \\ b\sum_{t=1}^{L_P}(L_P+1-t)p_{0,t}(x) + h\sum_{t=L_P+2}^{\infty}(t-(L_P+1))p_{0,t}(x), & 0 < x. \end{cases} \tag{1}$$

The first case in Equation (1), that is, when $x \le 0$, represents a scenario where the unit is used to satisfy a demand that has already occurred. Hence the quotation marks around the word *after* used earlier. The cost for the unit is thus the backorder cost paid from when the demand occurred until it is satisfied in period $L_P+1$. If $-D_0 < x \le 0$, the unit is matched with a demand that occurred in the current period. Similarly, it is matched with a demand in period $-1$ if $-(D_0+D_{-1}) < x \le -D_0$, period $-2$ if $-(D_0+D_{-1}+D_{-2}) < x \le -(D_0+D_{-1})$, and so forth. The corresponding customer waiting time is $L_P+1$, $L_P+2$, … and backorder costs are accumulated for all these periods, resulting in the first case in Equation (1). If $x > 0$, then the unit will be used to satisfy a demand that truly occurs after period 0. If this demand occurs in period $t < L_P+1$, then the unit is not available in stock when the demand occurs and the customer has to wait $L_P+1-t$ period(s). If, on the other hand, the demand occurs after period $L_P + 1$, then the unit will have to be kept in stock for $t - (L_P + 1)$ period(s). The second case in Equation (1) provides the resulting expected value of the backorder and holding costs with the different scenarios weighted by their respective probabilities, $p_{0,t}(x)$.

The expected cost for a reworked unit ordered in period 0 that is used to satisfy the $x:th$ demand occurring "after" period $L_R$, given $D_0$ and $\mathbf{D_A} = [D_1, \ldots, D_{L_R}]$, is equal to

$$C_{RW}(x, D_0, \mathbf{D_A}) = \begin{cases} b(L_P + 1 + t), -\sum_{\tau=0}^{t} D_{L_R-\tau} < x \leq -\sum_{\tau=0}^{t-1} D_{L_R-\tau},\ t \in [0, L_R] \\ b\sum_{t=1}^{L_P}(L_P + 1 - t)p_{L_R,t}(x) + h\sum_{t=L_P+2}^{\infty}(t - (L_P + 1))p_{L_R,t}(x),\ 0 < x. \end{cases} \tag{2}$$

The formula is much in line with the formula for the expected cost for a good unit in Equation (1). The difference between the two expressions occurs in the first case where $x \leq 0$. In this case, the period of the demand matched with the reworked unit is $L_R$ if $-D_{L_R} < x \leq 0$, and $L_R - 1$ if $-(D_{L_R} + D_{L_R-1}) < x \leq -D_{L_R}$, … . This results in a backorder cost of $b(L_P + L_R + 1 - L_R)$, $b(L_P + L_R + 1 - (L_R - 1))$, … , or equivalently $b(L_P + 1 + t)$ for $t = 0, 1, \ldots$ .

## Step 3: Determine the cost per period

In a stationary state, that is, when the inventory position at the end of the last $L_R$ periods or more has been the base-stock level $S$, the order quantity at the end of period $t$ equals the demand occurring at the beginning of period $t$, that is, $Q_t = D_t$. This is in order to replace the inventory used to fulfill the demand and reset the inventory position to the base-stock level $S$. Using the unit-demand matching procedure described above, the expected cost per period under perfect rework as a function of $S$ can be formulated as

$$\begin{aligned} EC_P(S) = \sum_{D_0=0}^{D_{\max}} Prob(D_0) \sum_{i=0}^{Q_0=D_0} Prob\,(i\,|Q_0\,) \\ \cdot \Bigg[ \sum_{IP=S-Q_0+1}^{S-i} \sum_{\mathbf{D_B} \in \mathbf{D_{L_R}}} Prob(\mathbf{D_B}) \\ \cdot \sum_{j=0}^{\Sigma Q_B} Prob(j|\mathbf{Q_B} = \mathbf{D_B}) C_G(IP - j, D_0, \mathbf{D_B}) \\ + \sum_{IP=S-i+1}^{S} \sum_{\mathbf{D_A} \in \mathbf{D_{L_R}}} Prob(\mathbf{D_A}) \\ \cdot \sum_{k=0}^{\Sigma Q_A} Prob\,(k\,|\mathbf{Q_A} = \mathbf{D_A}\,)\, C_{RW}(IP - k, D_0, \mathbf{D_A}) \Bigg]. \end{aligned} \tag{3}$$

The first two sums in Equation (3), that is, those with $Prob(D_0)$ and $Prob(i|Q_0)$, allow us to consider each possible combination of $Q_0$ and $i$—the number of units ordered and the number of units requiring rework—with its probability of occurring. For each such combination, the first triple sum in the parentheses provides the expected cost for the $Q_0 - i$ good units, and the second triple sum provides the expected cost for the $i$ units that need rework.

Equation (3) can be simplified further by using the i.i.d. property of the demand so that it reads:

$$\begin{aligned} EC_P(S) = \sum_{D_0=0}^{D_{\max}} Prob(D_0) \sum_{i=0}^{Q_0=D_0} Prob\,(i\,|Q_0\,) \\ \cdot \sum_{\mathbf{D_N} \in \mathbf{D_{L_R}}} Prob(\mathbf{D_N}) \sum_{n=0}^{\Sigma Q_N} Prob\,\left(n\,|\mathbf{Q_N} = \mathbf{D_N}\,\right) \\ \cdot \Bigg( \sum_{IP=S-Q_0+1}^{S-i} C_G(IP - n, D_0, \mathbf{D_N}) \\ + \sum_{IP=S-i+1}^{S} C_{RW}(IP - n, D_0, \mathbf{D_N}) \Bigg), \end{aligned} \tag{4}$$

where $\mathbf{D_N}$ is the demand realization over $L_R$ periods, $\mathbf{Q_N}$ is the order placed in response to this demand, and $n$ is the number of units among $\mathbf{Q_N}$ that require rework. This is because the i.i.d. demand implies that $Prob(\mathbf{D_A}) = Prob(\mathbf{D_B})$ for $\mathbf{D_A} = \mathbf{D_B}$ and the same is also true for the probability that $j = k$ of these units must be reworked and that $p_{0,t}(x) = p_{L_R,t}(x)$. The computational effort to find the expected cost per period using the exact equation in (4) can be substantial because all possible combinations of $D_0$ and $\mathbf{D_N}$ must be considered and an approximation is hence suggested and presented in Section A of the Supporting Information.

## 4.2 | Cost structure and optimal base-stock level under a perfect rework process

The optimal base-stock level $S^\star$ is, by definition, the base-stock level $S$ that minimizes the expected cost per period, $EC_P(S)$. Before discussing how to determine $S^\star$, we examine the behavior of $EC_P(S)$ by describing in detail the behavior of the cost for a good unit and then briefly discuss the behavior of the cost for a bad unit. The expected backorder cost for a good unit satisfying the $x:th$ demand occurring "after" period 0 is naturally nonincreasing toward zero in $x$. Typically, it is decreasing, but for low values of $x$, it is piecewise constant due to the periodic nature of the problem (see Figure 3). The length of the plateaus for the backorder cost when $x \leq 0$ (i.e., case 1 in Equation (1)) are equal to $D_0$, $D_{-1}$, and so forth, which is in line with the discussion about $C_G(x, D_0, \mathbf{D_B})$ in the description of Step 2 in the previous subsection.

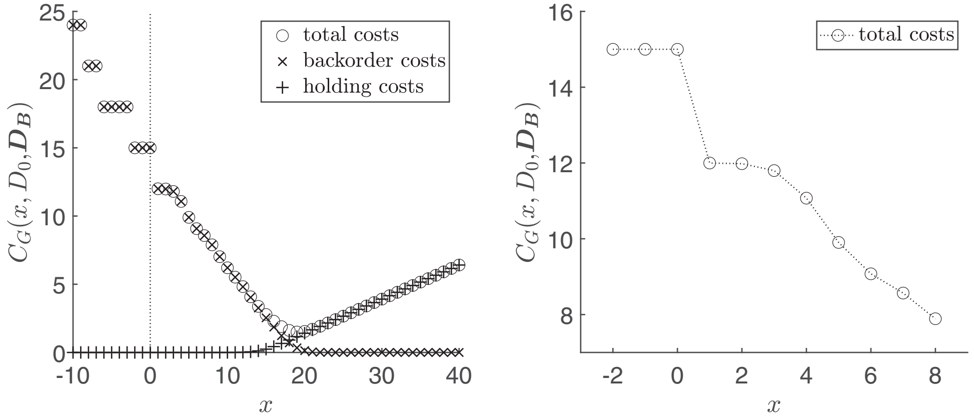


**FIGURE 3** Periodic review production–inventory system with random yield and nonperfect rework in period $t$

In contrast, the expected holding cost will be nondecreasing (typically increasing) in $x$ from a constant value of 0 for $x \leq 0$. The behavior of the expected holding-, backorder-, and total cost for a good unit $C_G(x, D_0, \mathbf{D_B})$ as a function of $x$ is illustrated in Figure 3. A figure for $C_{RW}(x, D_0, \mathbf{D_A})$ would look very similar, with the difference being that the plateaus would be of the length $D_{L_R}, D_{L_R-1}, \ldots$ .

From Figure 3 and the explanation above, it is apparent that the expected cost for a unit, either good or reworked, in Equation (4) is nonconvex in $x = IP - n$. This carries over to the expected cost per period as a function of the base-stock level, where the upper and lower limits in the summation of $C_G$ and $C_{RW}$ in Equation (4) are functions of $S$. This complicates the search for the optimal base-stock level but the decreasing and increasing characteristics of the backorder and holding cost, respectively, limit the required number of base-stock levels that need to be investigated in the search for $S^\star$.

## 4.3 | Approximation of the marginal cost under a perfect rework process

The unit-tracking/decomposition approach is traditionally based on the marginal cost of the last unit ordered rather than the expected cost per period. As previously mentioned, such an approach is problematic due to the possibility of units crossing each other in the pipeline. However, a good approximation of the marginal cost can be attained if one instead focuses on the last unit that becomes available at the end of period $L_P + L_R$ given that at least one unit becomes available in this period. The last unit that becomes available in this period will—in line with the discussion about $C_{RW}(x, D_0, \mathbf{D_A})$—be used to satisfy the $S - n$th demand occurring "after" period $L_R$. If the base-stock level is decreased by one unit, this demand will instead be satisfied by a unit that becomes available in a later period; it is not certain in which period due to the stochastic demand and yield. For a high-demand product, it is reasonable to assume that some units become available in each period and that the unit used to satisfy the $S - n$th demand thus will be available in the next period, that is, period $L_P + L_R + 1$. This would suggest the following approximate expression of the marginal cost of increasing the base-stock level by one unit from $S - 1$ to $S$:

$$\begin{aligned}\Delta EC_P(S) &= EC_P(S) - EC_P(S-1) \\ &\approx \sum_{n=0}^{L_R \cdot D_{\max}} Prob(n)\left(h \cdot \left(1 - P_{L_R, L_P+1}(S-n)\right)\right. \\ &\quad \left. - b \cdot P_{L_R, L_P+1}(S-n)\right), \end{aligned} \tag{5}$$

where $Prob(n)$ is the probability of $n$ bad units over $L_R$ periods and $P_{L_R, L_P+1}(S-n) = \sum_{t=1}^{L_P+1} p_{L_R,t}(S-n)$ is the probability that demand $x = S - n$ occurs in one of the $L_P + 1$ periods following period $L_R$. The optimal base-stock level, $\tilde{S}$, based on the approximation in (5) is the largest $S$ for which $\Delta EC_P(S)$ is negative:

$$\tilde{S} = \arg\max_{S}\{\Delta EC_P(S) | \Delta EC_P(S) < 0\}. \tag{6}$$

Note that if there are periods with no units becoming available then the delay might be more than one period, and (5) is thus an approximation of the marginal cost. Otherwise, $\tilde{S}$ equals the optimal base-stock level $S^\star$.

## 4.4 | Extension to a nonperfect rework process

The same general principle as in Sections 4.1 to 4.3 can be used to determine the expected cost per period and the optimal base-stock level even if the rework process is nonperfect and items might have to undergo rework several times. That is, we can still (i) match each unit ordered in period $t = 0$ with a demand for a given demand scenario, (ii) determine the expected cost for each such pair, and (iii) calculate the expected cost per period by summing up all these costs weighted with the probability of each scenario. To describe the matching process, we use the additional notation

**TABLE 2** Additional notation for nonperfect rework

| | |
|---|---|
| $Q_{R,t}$ | Total number of items entering the rework process in period $t$ |
| $\boldsymbol{Q_{R,0}}$ | Total number of units having entered the rework process in all periods before period 0 ($\boldsymbol{Q_{R,0}} = [Q_{R,-\infty}, \dots, Q_{R,-1}]$) |
| $i_m$ | Number of units ordered in period 0 that need to be reworked $m$ times |
| $I_m = I_{m-1} - i_m$ | Number of units ordered in period 0 that need to be reworked more than $m$ times |
| $J$ | Number of units ordered before period 0 that will arrive in or after period $L_P$ due to rework |
| $K_m$ | Number of units ordered before period 0 that will arrive in or after period $L_P + \sum_{k=1}^{m} L_R^k$ plus the number of units ordered in periods $(0, \sum_{k=1}^{m} L_R^k]$ that will arrive after period $L_P + \sum_{k=1}^{m} L_R^k$, all due to rework. |

introduced in Table 2. In line with Step 1 in Section 4.1, the $Q_0 - I_0$ good units ordered in period 0 will be used to satisfy "future" demand $S - Q_0 + 1 - J$ to $S - I_0 - J$ occurring "after" period $t = 0$. Furthermore, the $i_m$ units ordered in period 0 that need to be reworked $m$ times will be used to satisfy the $S - I_{m-1} + 1 - K_m$th to $S - I_m - K_m$th demand occurring "after" period $\sum_{k=1}^{m} L_R^k$.

In Step 2, the expected cost for a good unit under nonperfect rework is much in line with the cost for such a unit under perfect rework, that is, Equation (1). The difference is that there is no upper bound to the delay for $x < 0$, which implies that one must track the demand in all past periods and not just the last $L_R^1$ periods. The expression for a unit that needs rework, $C_{RW_m}$, depends on the number of times, $m$, it needs to be reworked. The cost for a unit satisfying a "future" demand, that is, when $x > 0$, will be the same independent of the number of times it is being reworked as $p_{\sum_{k=1}^{m} L_R^k, t}(x) = p_{L_R^1, t}(x) = p_{0,t}(x)$. Again, the difference in cost occurs when the unit is used to satisfy a demand that has already occurred, that is, $x \le 0$, as the demand vector that dictates the delay is now $[D_{-\infty}, \dots, D_{\sum_{k=1}^{m} L_R^k}]$ rather than just $[D_0, \dots, D_{L_R^1}]$.

When computing the expected cost, $EC_{NP}(S)$, in Step 3, one must thus consider all possible outcomes of an infinitely long demand vector. This is computationally intractable and we will therefore use the following approximation:

$$
\begin{aligned}
AC_{NP}(S) = &\sum_{D_0=0}^{D_{\max}} Prob(D_0) \sum_{IP=S-Q_0+1}^{S} \left( \sum_{N=0}^{IP-1} Prob(N) \right. \\
&\cdot \left( b \sum_{t=1}^{L_P} (L_P + 1 - t) p_t(IP - N) + h \sum_{t=L_P+2}^{\infty} (t - (L_P + 1)) p_t(IP - N) \right) \\
&\left. + b \sum_{N=IP}^{\infty} Prob(N) \left( L_P + 1 + \frac{N - IP}{\mu_D} \right) \right) \\
&\approx EC_{NP}(S).
\end{aligned} \tag{7}
$$

To arrive at (7), it is first concluded that i.i.d. demand implies that the probability for $J = N$ is the same as the probability for $K_m = N$ for all $m$. The i.i.d. demand also implies that the probability of the $x : th$ demand after period $\tau$ occurring in period $\tau + t$ is independent of $\tau$ for all $m$ and we let $p_t(x)$ denote this probability. The sum $\sum_{N=0}^{IP-1} Prob(N) \dots$ in (7) thus provides the exact expression for all cases where $x = IP - N > 0$.

The probability that $x \le 0$, that is, the unit is used to satisfy a demand that has occurred before time $\tau$, is typically negligible due to positive lead-times and unit backorder cost that are significantly higher than unit holding cost. We approximate the expected cost for these scenarios using the sum $\sum_{N=IP}^{\infty} Prob(N) \dots$ in (7). The approximation is based on calculating the time before $\tau$ that the demand occurred by taking its ordinal number, $-x = N - IP$, and dividing it with the expected demand per period, $\mu_D$. A similar approximation was used for the case of a perfect rework process with no observable difference in true and approximately calculated cost (see Section A of the Supporting Information for the approximation under perfect rework).

Similarly as in the case of perfect rework process in Equation (5), we can estimate the marginal cost of increasing $S$ as

$$
\begin{aligned}
\Delta EC_{NP}(S) &= EC_{NP}(S) - EC_{NP}(S-1) \\
&\approx \sum_{N=0}^{L_R \cdot D_{max}} Prob(N) \Big( h \cdot \big(1 - P_{L_R, L_P+1}(S - N)\big) \\
&\quad - b \cdot P_{L_R, L_P+1}(S - N) \Big),
\end{aligned} \tag{8}
$$

and use this expression to find the optimal base-stock level.

## 5 | STATE-DEPENDENT POLICY

Remember that the base-stock policy discussed in the previous section is not the optimal policy under stochastic yield with perfect or nonperfect rework. Indeed, the optimal policy is unknown. Here, we present computationally efficient heuristics for state-dependent ordering policies that consider the current pipeline inventory instead of basing the decision on the long-run distribution of the same, as the base-stock policy does. Similar to Section 4, we will focus on a problem with a perfect rework process and then briefly explain how to extend the heuristic to a problem with a nonperfect rework process.

Inspired by the unit-tracking/decomposition approach, the state-dependent ordering policies are based on comparing the expected cost of unit $u$ if it is ordered in this period, period 0, or the next, period 1, assuming that one returns to the optimal base-stock policy with base-stock level $S = S^{\star}$ in period 1 after unit $u$ has been ordered. Under these heuristics, one should order unit $u$ now and raise the inventory position in period 0 to $S+u$ if the expected cost of doing this is less than the expected cost of ordering this unit in the next period. Note that $u$ might be either positive or negative because one might choose to order more or less than what is stipulated by the base-stock policy.

To reduce the computational complexity, it is assumed that if unit $u$ requires rework, it will do so independent of when it is ordered. This reduces the number of scenarios that need to be considered to two, namely, that $u$ is of good quality (Scenario 1) and that it needs rework (Scenario 2). In both of these scenarios, the decision whether or not to postpone the ordering of unit $u$ by one period will only affect the inventory level in two periods. In certain circumstances, it is plausible that unit $u$ will need rework if ordered in period 0 but not if it is ordered in period 1 or vice versa (Scenarios 3 and 4, respectively). The decision when to order the unit will then affect the inventory level in multiple periods ($L_R - 1$ periods in Scenario 3 and $L_R + 1$ periods in Scenario 4 to be precise), which substantially increases the computational complexity. The effect of incorporating these scenarios was investigated in a numerical study, but no significant difference in the cost performance was observed, wherefore excluding Scenarios 3 and 4 is reasonable. Using this assumption, the expected cost for unit $u$ if ordered in this period rather than the next can be expressed as

$$\Delta EC_{H,P}(S,u,\mathbf{Q'_B}) = g \cdot \Delta EC_{G,P}(S,u,\mathbf{Q'_B}) + (1-g)\cdot \Delta EC_{RW,P}(S,u). \quad (9)$$

In Equation (9), $\Delta EC_{G,P}(S,u,\mathbf{Q'_B})$ is the expected cost of ordering unit $u$ now rather than in period 1 if it is of good quality (i.e., Scenario 1), $\Delta EC_{RW,P}(S,u)$ is the corresponding expected cost if $u$ needs rework (i.e., Scenario 2) and $g$ is the probability of unit $u$ being of good quality. Two methods of estimating $g$ are presented later in this section, resulting in two different heuristics. Both heuristics entail ordering unit $u$ now if $\Delta EC_{H,P}(S,u,\mathbf{Q'_B})$ is negative, because this myopically constitutes an expected cost saving.

To determine $\Delta EC_{G,P}(S,u,\mathbf{Q'_B})$, we use the fact that the cost is independent of how one sequences the units that become available in a period. This allows us to sequence the units, so that unit $u$ will be used to satisfy the same demand irrespective of whether it is ordered in this or the next period. Thus, it is the last unit made available in period $L_P$ if it is ordered now and the first unit made available in period $L_P+1$ if ordered in the next period. If one orders unit $u$ in period 0 then, by definition, one will raise the inventory position to $S+u$ in that period. If unit $u$ is of good quality, it will be used to satisfy the $S+u-i'-j' : th$ demand occurring "after" period 0 with the suggested sequencing. This is because it will overtake the $i'$ units among the other $Q'_0$ units ordered in period 0 that need rework and the $j'$ units among $\mathbf{Q'_B} = (Q_{-L_R+1}, \ldots, Q_{-1})$ that need rework (see Figure 2). The expected cost of ordering unit $u$ now rather than in the next period can thus, in line with Equation (5), be calculated as

$$\Delta EC_{G,P}(S,u,\mathbf{Q'_B}) = \sum_{i'=0}^{Q'_0} Prob\left(i' \,\middle|\, Q'_0\right) \sum_{j'=0}^{\Sigma Q'_B} Prob\left(j' \,\middle|\, \mathbf{Q'_B}\right) \cdot \left(h\left(1 - P_{0,L_P+1}(S+u-i'-j')\right) - b \cdot P_{0,L_P+1}(S+u-i'-j')\right), \quad (10)$$

where $\Sigma Q'_B = Q_{-L_R+1} + \cdots + Q_{-1}$. It should be noted that the distribution of $j'$ is independent of $u$ and only depends on the pipeline inventory, $\mathbf{Q'_B}$, which is known.

To determine $\Delta EC_{RW,P}(S,u)$, we use a similar sequencing rule as above to ensure that unit $u$ will be used to satisfy the same demand irrespective of when it is ordered. If unit $u$ needs rework it will be passed by all the good units ordered in periods 1 to $L_R$ (see Figure 2). By assumption, a base-stock policy is applied in these periods, so the inventory position after the order has been placed in period $L_R$ will be $S + \max(0, u - \Sigma D_A)$, where $\Sigma D_A$ is the sum of the demand in the $L_R$ periods after period 0. With the chosen sequencing rule, unit $u$ will thus be used to satisfy the $S + \max(0, u - \Sigma D_A) - k' : th$ demand occurring "after" period $L_R$, where $k'$ is the number of units among $\mathbf{Q'_A} = (Q'_1, \ldots, Q'_{L_R})$ that need rework. The resulting expected cost of ordering unit $u$ in this period rather than the next, if it needs rework, is

$$\Delta EC_{RW,P}(S,u) = \sum_{\mathbf{D_A}\in \mathbf{D_{L_R}}} Prob(\mathbf{D_A}) \sum_{k'=0}^{\Sigma D_A - u} Prob\left(k' \,\middle|\, \mathbf{Q'_A}\right) \cdot \left(h\left(1 - P_{L_R,L_P+1}(S+\max(0,u-\Sigma D_A)-k')\right) - b \cdot P_{L_R,L_P+1}\left(S+\max(0,u-\Sigma D_A)-k'\right)\right). \quad (11)$$

In accordance with the assumption that a base-stock policy is used in period 1 and onwards, the order quantities will be set as $Q'_1 = \max(0, D_1 - u)$ in period 1 and as $Q'_t = \max(0, D_t - \max(0, u - \sum_{\tau=1}^{t} D_\tau))$ in period $2 \le t \le L_R$ .

If the probability of unit $u$ being of good quality is constant, then $g$ will be equal to the expected yield rate, that is, $g = \bar{G}$. However, for some yield processes this probability will depend on the batch size and the additional number of units in the batch that need rework. For these yield processes, we suggest the following two approximations:

1. $g \approx \bar{G}$,
2. $g \approx \frac{G(Q'_0, i') + \bar{G}}{2}$.

The second estimate is based on the average of the state-dependent probability that unit $u$ is of good quality if ordered in this period given $Q_0$ and $i'$, $G(Q'_0, i')$, or in the next, $\bar{G}$. Note that when the decision is made, one knows $Q'_0$ but not $Q'_1$;

thus, the probability of unit $u$ being of good quality after the production process if ordered in period 1 must be based on the average yield. The above approximations lead to two heuristics, $BS_1$ and $BS_2$, which are based on the cost of ordering unit $u$ now compared to the cost of ordering it in the next period.

Under a nonperfect rework process, the marginal cost of postponing the ordering of unit $u$ reads

$$\begin{aligned}\Delta EC_{H,NP}(S, u, \mathbf{Q'_B}, \mathbf{Q_{R,0}}) \\ = g \cdot \Delta EC_{G,NP}(S, u, \mathbf{Q'_B}, \mathbf{Q_{R,0}}) \\ + (1-g) \cdot \sum_{m=0}^{\infty} p_m \cdot \Delta EC_{RW_m,NP}(S, u, \mathbf{Q'_B}, \mathbf{Q_{R,0}}), \quad (12)\end{aligned}$$

where $\Delta EC_{G,NP}$ is the cost of delaying the order of unit $u$ one period if it passes the quality inspection directly after the production process. Furthermore, $p_m$ is the probability that unit $u$ needs to be re-reworked $m$ times and $\Delta EC_{RW_m,NP}$ is the corresponding marginal cost of having ordered unit $u$ one period later in this case.

# 6 | NUMERICAL STUDY

In this section, we first present in Section 6.1 the test series used for the numerical studies. Afterwards, we investigate in Section 6.2 the performance of the state-dependent ordering policies, $BS_1$ and $BS_2$, compared to the optimal base-stock policy and an existing approach from the literature. Finally, we discuss a counterintuitive finding, which we call the "Mean Yield Paradox" in Section 6.3.

## 6.1 | Instance characteristics

As a starting point for the numerical study, we use an extended version of the test series in Sonntag and Kiesmüller (2017), which assumes a perfect rework process. The production time $L_P$ equals either five or ten periods. The corresponding rework times can vary from one to $L_P$ and are therefore $L_R \in \{1,2,3,4,5\}$ for $L_P = 5$ and $L_R \in \{1,2,3,4,5,6,7,8,9,10\}$ for $L_P = 10$. The demand is assumed to follow a normal distribution with mean $\mu_D$ equal to 20 and three different coefficients of variation $\rho_D \in \{0.1, 0.2, 0.3\}$, which are defined as the ratio between the standard deviation $\sigma_D$ and the corresponding mean demand ($\rho_D = \sigma_D/\mu_D$). For the holding and backorder cost parameters $h$ and $b$, we set the ratio $b/(b+h) \in \{0.75, 0.85, 0.95\}$ with $h = 1$. To emphasize that our solution procedures for determining the optimal base-stock level and the state-dependent order quantities are not limited to a specific demand or yield distribution, we analyze the performance under the two most common yield models. Namely, the "most-widely studied" (Gupta & Cooper, 2005) stochastic proportional yield model as well as the binomial yield model. Under a stochastic proportional yield model, the output $Y_P(Q)$ of the production process equals a random fraction $Z$ of the input $Q$ with $Z$ being a random variable on the interval [0; 1]. The yield factor $Z$ of the production process follows a beta-distribution that is defined on the interval [0, 1] with a mean yield $\mu_{Z,P} \in \{0.5, 0.8, 0.9\}$, reflecting situations with low and high yield rates. The corresponding coefficient of variation $\rho_{Z,P}$ is set such that for $\mu_{Z,P} = 0.5$, $\rho_{Z,P} \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$, and for $\mu_{Z,P} \in \{0.8, 0.9\}$, $\rho_{Z,P} \in \{0.1, 0.2, 0.3\}$. Under a binomial yield model, each unit is of perfect quality with probability $g$ as defined in Section 5. For a given order quantity $Q$, the yield is binomially distributed with parameters $(Q, g)$ and probability mass function

$$P(Y(Q) = k) = g^k (1-g)^{Q-k} \binom{Q}{k}, \quad \forall k = 0, 1, \ldots, Q. \quad (13)$$

The success probability $g$ corresponds to the expected yield for which we have considered the same values as under stochastic proportional yield, that is, $g = \mu_{Z,P} \in \{0.5, 0.8, 0.9\}$. The standard deviation of the yield probability is a function of $g$ and can thus not be varied independently of $\mu_{Z,P}$. It should also be noted that the two heuristics will coincide under binomial yield as the probability of unit $u$ being of good quality is constant and equal to $g$, that is, $BS = BS_1 = BS_2$. We discretize the demand and yield distributions and refer for details to Section B of the Supporting Information. In total 1485 instances under stochastic proportional yield and 405 instances under binomial yield are included in this test series.

## 6.2 | Performance of the state-dependent policies

In this section, the performance of the state-dependent policies $BS_1$ and $BS_2$ derived in Section 5 are evaluated with respect to the relative cost savings compared to the base-stock policy with optimal base-stock level $S^{\star}$ derived in Section 4.1. Remember that the optimal policy is unknown, wherefore the base-stock policy is a reasonable benchmark for a performance analysis of the state-dependent policies. To highlight our methods contribution, we add a comparison to the "alternative base-stock policy," $SK$, proposed by Sonntag and Kiesmüller (2018), which is briefly described as follows.

***"Alternative base-stock policy" SK***
*In their paper, Sonntag and Kiesmüller (2018) propose an "alternative base-stock policy" based on an adjusted inventory position that is defined as the physical stock-on-hand minus backorders plus the expected number of units among the outstanding orders that will become available during the risk period* $[0, L_P]$*, rather than all outstanding orders. The reason for this is that during these* $L_P$ *periods, it is not possible to influence the amount of units available by placing new orders. Including only the expected outstanding units that become available during the risk period implies that the*

**TABLE 3** Relative cost savings $\Delta_i$ (in %) for the neglected instances in Table 4 with $i \in \{(5,1),(5,2),(10,1),(10,2)\}$ reflecting different combinations of $(L_P, L_R)$ and $\rho_D = 0.1$, $b/(b+h) = 0.95$, $\mu_{Z,P} = 0.9$, $\rho_{Z,P} = 0.3$

| $i$ (i.e., $(L_P, L_R)$) | $\Delta_i^{SK}$ | $\Delta_i^{BS_1}$ | $\Delta_i^{BS_2}$ |
|---|---|---|---|
| (5,1) | −4.96 | −54.17 | −7.96 |
| (5,2) | −5.50 | −39.72 | −3.49 |
| (10,1) | −1.92 | −39.79 | −3.95 |
| (10,2) | −2.27 | −14.43 | −2.20 |

*adjusted inventory position must be updated in each period based on the realized yield and, therefore, the exact numbers of units that enter the warehouse without requiring rework and those requiring rework.*

Note that the "alternative base-stock policy" $SK$ has been derived only for a stochastic proportional yield model and that an adjustment to a binomial yield model is not straightforward. Therefore, under binomial yield, we cannot compare the performance of the state-dependent policies with that of $SK$. To determine the relative cost savings of the state-dependent policies, $BS_1$ and $BS_2$, and the "alternative base-stock policy" $SK$ compared to the base-stock policy, the costs of the optimal base-stock policy $EC(S^\star)$ are calculated exactly using Equation (4). However, no expression exists for calculating the costs for the ordering policies $BS_1$, $BS_2$, and $SK$, wherefore these costs were determined via simulations. Specifically, a sequential sampling procedure was used, where each simulation run represented 5000 periods with a 1000-period warm-up phase. To guarantee a high validity, the simulation run was repeated until the half-width of the 95% confidence interval of the average cost per period was smaller than 0.1% of the corresponding sample average, with a minimum number of 10 runs.

The results are summarized in Table 4. Note that we omit four instances under stochastic proportional yield from the table because they lead to unusually high cost increases compared to the optimal base-stock policy and therefore create a false impression of the average performance of the heuristics. For completeness, the results for the four excluded instances are summarized in Table 3 and we refer back to these instances later in the section.

In Table 4, positive values indicate cost savings compared to the base-stock policy whereas negative values reflect cost increases. For each method and each parameter of our full factorial design, the average and the maximum relative cost savings, $\bar{\Delta}$ and $\Delta_{max}$, are displayed. To allow for an estimate of the absolute performance, the average expected cost $\overline{EC}(S^\star)$ of the base-stock policy with optimal base-stock level $S^\star$ is reported. The results show that the input parameters influence the potential to save costs in a structurally similar manner for all methods and both considered yield models. The relative savings compared to the base-stock policy are increasing with the demand uncertainty $\rho_D$ and the ratio of rework lead-time $L_R$ and production lead-time $L_P$ ($L_R/L_P$). For all other parameters, that is, the expected yield $\mu_{Z,P} = g$, the critical ratio ($b/(b+h)$), and in case of stochastic proportional yield the yield uncertainty $\rho_{Z,P}$, the relative savings are decreasing with an increase in these parameters. In the following, we focus on the most interesting findings and discuss them in more detail.

*The performance increases with the demand variability and the rework time*

The improvement in performance of the different methods is linked to the amount of information about the current pipeline inventory used when determining the order quantity. The base-stock policy uses the long-run distribution of the number of units ordered over $L_R$ periods to determine the order quantity based on the inventory position and thus no information about the current pipeline inventory. The method in $SK$ uses current information to correctly forecast the expected number of units ordered in periods $[-L_R, \dots, -1]$ that will need rework and adjust the ordering decision accordingly. The new heuristics $BS_1$ and $BS_2$ correctly forecast the full distribution function of the number of units needing rework based on the observed pipeline inventory. The accuracy of a forecast based on current information rather than long-run data naturally improves with the variability of the information available, that is, with the demand variability, $\rho_D$ and horizon considered, $L_R$. This improvement of the accuracy and detail of the forecast is the reason why $BS_1$ and $BS_2$ typically show a better performance than $SK$, and $SK$ a better performance than the base-stock policy and why the performance is improving in $\rho_D$ and $L_R$.

*The expected costs increase and the relative performance decreases with the yield uncertainty*

The increase in cost for the base-stock policy observed in Table 4 is expected as an increased uncertainty implies more safety stock and/or backorders. This also partly explains the observed decrease in the relative savings, as $\Delta_i$ is defined as the absolute savings divided with the expected cost for the base-stock policy. Another reason for the decreased performance with increasing yield is that the value of using more exact information about the pipeline inventory when forecasting the number of units that need rework is decreasing with the yield uncertainty. Comparing the test series for the binomial yield model with that of the stochastic proportional yield model, it can be concluded that the latter typically exhibits a significantly larger yield uncertainty. The differences between the results for the two yield models observed in Table 4 are, thus, in line with the conclusions above. The reason for lower yield uncertainty under binomial yield is that it is connected to the expected value through the variable $g$ and the yield uncertainty can, thus, not be set to the same high values used under stochastic proportional yield.

A closer look at the behavior of the heuristics under stochastic proportional yield and very high yield variability reveals that they become too near-sighted. This results in an overestimation of the cost impact of ordering unit $u$ now and downplaying the impact of decisions in subsequent periods.

**TABLE 4** Relative performance of the myopic policies compared to the base-stock policy (in %)

| | | Stochastic proportional yield | | | | | | | Binomial yield | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | *SK* | | $BS_1$ | | $BS_2$ | | | $BS_1$ | |
| | | $\overline{EC}(S^*)$ | $\bar{\Delta}^{SK}$ | $\Delta^{SK}_{max}$ | $\bar{\Delta}^{BS_1}$ | $\Delta^{BS_1}_{max}$ | $\bar{\Delta}^{BS_2}$ | $\Delta^{BS_2}_{max}$ | $\overline{EC}(S^*)$ | $\bar{\Delta}^{BS_1}$ | $\Delta^{BS_1}_{max}$ |
| $(L_P, L_R)$ | (5,1) | 17.62 | −0.10 | 0.00 | 0.20 | 1.57 | 0.27 | 1.45 | 16.47 | 0.58 | 1.34 |
| | (5,2) | 18.92 | 0.26 | 0.88 | 0.58 | 2.68 | 0.65 | 2.70 | 16.92 | 1.04 | 2.46 |
| | (5,3) | 20.14 | 0.49 | 1.74 | 0.85 | 3.52 | 0.97 | 3.56 | 17.35 | 1.41 | 3.49 |
| | (5,4) | 21.20 | 0.68 | 2.49 | 1.12 | 4.54 | 1.25 | 4.56 | 17.76 | 1.74 | 4.45 |
| | (5,5) | 22.20 | 0.82 | 3.18 | 1.35 | 5.57 | 1.49 | 5.52 | 18.16 | 2.08 | 5.32 |
| | (10,1) | 22.91 | −0.03 | 0.00 | 0.19 | 0.83 | 0.15 | 0.83 | 22.01 | 0.32 | 0.80 |
| | (10,2) | 23.94 | 0.17 | 0.51 | 0.39 | 1.61 | 0.39 | 1.53 | 22.35 | 0.60 | 1.56 |
| | (10,3) | 24.92 | 0.32 | 1.00 | 0.55 | 2.10 | 0.58 | 2.07 | 22.68 | 0.86 | 2.16 |
| | (10,4) | 25.82 | 0.43 | 1.46 | 0.73 | 2.70 | 0.79 | 2.69 | 23.01 | 1.11 | 2.70 |
| | (10,5) | 26.66 | 0.52 | 1.88 | 0.88 | 3.34 | 0.96 | 3.29 | 23.32 | 1.30 | 3.23 |
| | (10,6) | 27.47 | 0.56 | 2.35 | 1.02 | 3.82 | 1.11 | 3.85 | 23.63 | 1.50 | 3.79 |
| | (10,7) | 28.24 | 0.59 | 2.72 | 1.16 | 4.51 | 1.27 | 4.45 | 23.93 | 1.70 | 4.32 |
| | (10,8) | 28.98 | 0.63 | 3.13 | 1.28 | 4.96 | 1.39 | 4.98 | 24.23 | 1.85 | 4.83 |
| | (10,9) | 29.70 | 0.63 | 3.54 | 1.40 | 5.47 | 1.51 | 5.55 | 24.52 | 2.03 | 5.32 |
| | (10,10) | 30.39 | 0.65 | 4.04 | 1.50 | 5.96 | 1.63 | 6.08 | 24.80 | 2.21 | 5.88 |
| $\rho_D$ | 0.1 | 16.13 | 0.22 | 3.15 | 0.69 | 5.06 | 0.68 | 5.16 | 12.00 | 1.09 | 3.40 |
| | 0.2 | 24.24 | 0.49 | 3.67 | 0.91 | 5.92 | 1.03 | 5.92 | 21.24 | 1.45 | 5.26 |
| | 0.3 | 33.40 | 0.62 | 4.04 | 1.04 | 5.96 | 1.17 | 6.08 | 30.98 | 1.52 | 5.88 |
| $b/(b+h)$ | 0.75 | 19.01 | 0.50 | 3.68 | 0.98 | 5.96 | 1.00 | 6.08 | 16.69 | 1.36 | 5.88 |
| | 0.85 | 23.41 | 0.51 | 3.83 | 0.93 | 5.92 | 0.97 | 5.85 | 20.42 | 1.36 | 5.65 |
| | 0.95 | 31.49 | 0.32 | 4.04 | 0.74 | 5.79 | 0.91 | 5.70 | 27.12 | 1.35 | 5.38 |
| $(\mu_{Z,P}, \rho_{Z,P})$ | (0.5,0.1) | 21.37 | 1.74 | 4.04 | 3.30 | 5.96 | 3.29 | 6.08 | | | |
| | (0.5,0.2) | 22.41 | 1.20 | 2.87 | 2.65 | 5.20 | 2.66 | 5.18 | | | |
| | (0.5,0.3) | 23.96 | 0.60 | 1.73 | 1.84 | 3.78 | 1.89 | 3.95 | 22.61 | 2.91 | 5.88 |
| | (0.5,0.4) | 25.88 | 0.06 | 0.88 | 0.99 | 2.44 | 1.13 | 2.68 | | | |
| | (0.5,0.5) | 28.09 | −0.50 | 0.43 | 0.08 | 0.91 | 0.40 | 1.49 | | | |
| | (0.8,0.1) | 20.98 | 0.64 | 1.62 | 0.86 | 1.67 | 0.86 | 1.71 | | | |
| | (0.8,0.2) | 23.73 | 0.57 | 1.58 | 0.50 | 1.15 | 0.51 | 1.18 | 21.12 | 0.90 | 1.80 |
| | (0.8,0.3) | 27.75 | 0.45 | 1.34 | −0.16 | 0.15 | −0.04 | 0.34 | | | |
| | (0.9,0.1) | 21.18 | 0.23 | 0.57 | 0.22 | 0.51 | 0.23 | 0.49 | | | |
| | (0.9,0.2) | 24.99 | 0.05 | 0.59 | 0.00 | 0.18 | −0.01 | 0.22 | 20.50 | 0.26 | 0.54 |
| | (0.9,0.3) | 30.61 | −0.17 | 0.39 | −0.64 | 0.10 | −0.38 | 0.00 | | | |
| | Total | 24.62 | 0.44 | 4.04 | 0.88 | 5.96 | 0.96 | 6.08 | 21.41 | 1.36 | 5.88 |

This leads to something resembling a bullwhip effect, where a small disturbance can lead to the ordering decisions starting to oscillate between high and low values with a large adverse effect on the expected costs per period. For example, for $BS_2$ with $(L_P, L_R) = (5, 1)$ in Table 3, the variance of the order quantity is six times higher than that of the demand, which explains the large cost increase compared to the base-stock policy, where the variance of the demand and order quantity are equal. *SK* shows a similar tendency for high variance in the order quantity, but to a smaller extent; under *SK*, the variance in the order quantity is up to four times higher than that of the demand.

*The best performance is observed for mean yields of* 50%
Because the base-stock policy is the optimal policy under perfect yield, the potential for cost improvements by myopically adjusting the base-stock policy is, as can be expected, decreasing with the expected production yield, $\mu_{Z,P}$. Therefore, the myopic approximations $BS_1$ and $BS_2$ lead to the highest improvements for the lowest mean yield. The

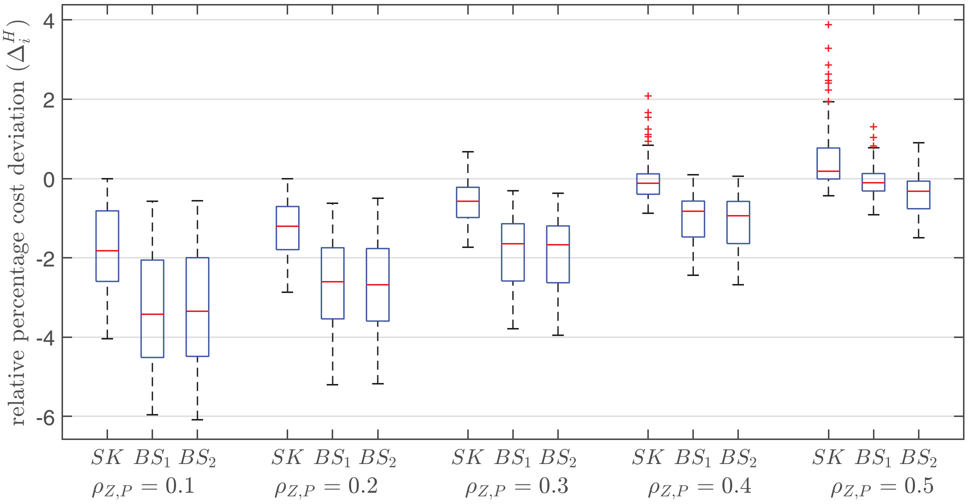


**FIGURE 4** Periodic review production–inventory system with random yield and nonperfect rework in period $t$ [Color figure can be viewed at wileyonlinelibrary.com]

boxplot in Figure 4 emphasizes the excellent performance of the myopic improvements under low mean yields and shows that $BS_2$, in particular, outperforms the approach $SK$ and the base-stock policy in almost all instances, particularly for low to medium yield variability.

Summarizing, the newly developed heuristics provide large potential for cost savings. Under stochastic proportional yield, the state-dependent policies $BS_1$ and $BS_2$ provide cost savings up to 6.08% compared to the base-stock policy and up to 4.5% compared to the $SK$ approach, while under binomial yield cost savings of up to 5.88% compared to the base-stock policy can be achieved. The highest savings are attained for low to medium production yields as emphasized in Figure 4. Such low yields occur, for example, in the LCD manufacturing or semiconductor industries (see Section 1). However, the savings are lower when production yields are high, and can be even negative under stochastic proportional yield. It is worth pointing out that the yield is almost a two-point distribution for parameter settings like $(\mu_{Z,P}, \rho_{Z,P}) = (0.9, 0.3)$, with either all units being good or all of them needing rework. In this case, a state-dependent ordering policy leads to an overreaction and, therefore, high variance in the order quantity and high costs. Therefore, under stochastic proportional yield the state-dependent ordering policies should not be used if a combination of the following is present:

- high yield uncertainty, that is, close to an all or nothing scenario (0% or 100% yield);
- high mean yield, that is, close to perfect yield (100%); and
- low demand uncertainty.

A policy using these rules along with the best heuristic $BS_2$ instead of purely using $BS_2$ increases the average cost saving relative to the base-stock policy from −0.08% to 1.00% (including all 1485 instances) and, more importantly, gives a maximum increase of just 0.23% compared to 7.96% displayed in Table 3. Correspondingly, for $BS_1$, the average cost savings increase from −0.25% to 0.95% and, more importantly, the maximum cost increase compared to the base-stock policy decreases from 54.17% as displayed in Table 3 to 1.13%. Thus, using the state-dependent ordering policies suggested with discretion (i.e., not when the mean yield or the yield uncertainty is very high) leads to large cost improvements especially in view of the high costs associated with, for example, the high-tech and automobile industry that have products incorporating substantial value.

## 6.3 | The mean yield paradox

Unexpectedly, the numerical study shows that the expected inventory-related costs are not always strictly decreasing in the expected yield, $\mu_{Z,P}$. For example, according to Table 4, the costs of the base-stock policy increase from 23.96 to 30.61 when $\mu_{Z,P}$ increases from 0.5 to 0.9 for $\rho_{Z,P} = 0.3$ under stochastic proportional yield—the most commonly used yield model—and the same effect can be observed for all solution methods. We refer to this counterintuitive effect as the "Mean Yield Paradox." Interestingly, no such effect occurred under binomial yield.

To analyze the "Mean Yield Paradox," we present the results from an extended test series for $L_P = 5$ (the results for $L_P = 10$ are similar and provide no additional insights). The test series is extended compared to the one described in Section 6.1 by using $\mu_{Z,P}$ equal to $\{0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9\}$. Instead of using $\rho_{Z,P}$ we have opted to use $\sigma_{Z,P}$ equal to $\{0.05, 0.1, 0.15, 0.2, 0.25\}$ to decouple the expected yield from the yield uncertainty. The resulting coefficients of variation are in line with those used in Section 6.2. The remaining parameters are kept as they were. While Section 6.2 focused on a perfect rework process to allow for comparisons with the approach $SK$ by Sonntag and Kiesmüller (2018), we now also analyze the system under nonperfect rework. For the test

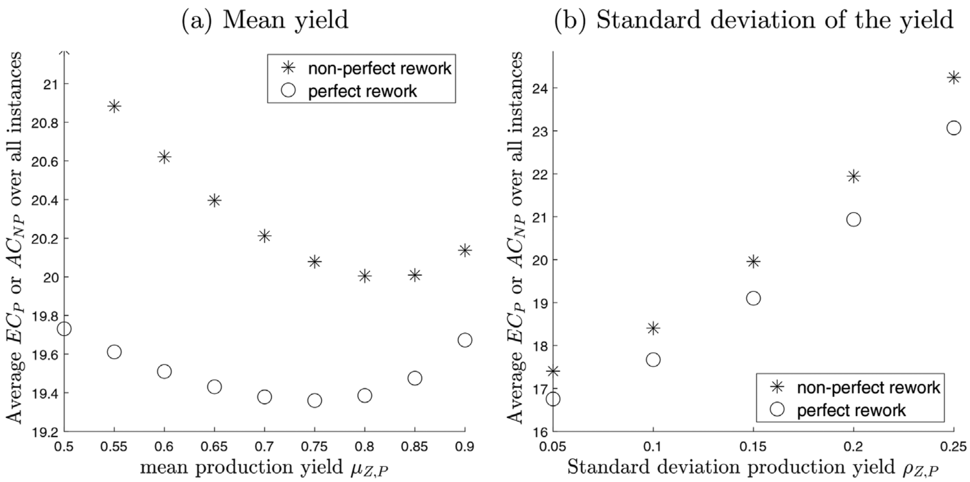


**FIGURE 5** Comparison of the cost performance of perfect and nonperfect rework systems depending on (a) the mean yield and (b) the standard deviation of the yield of the production process

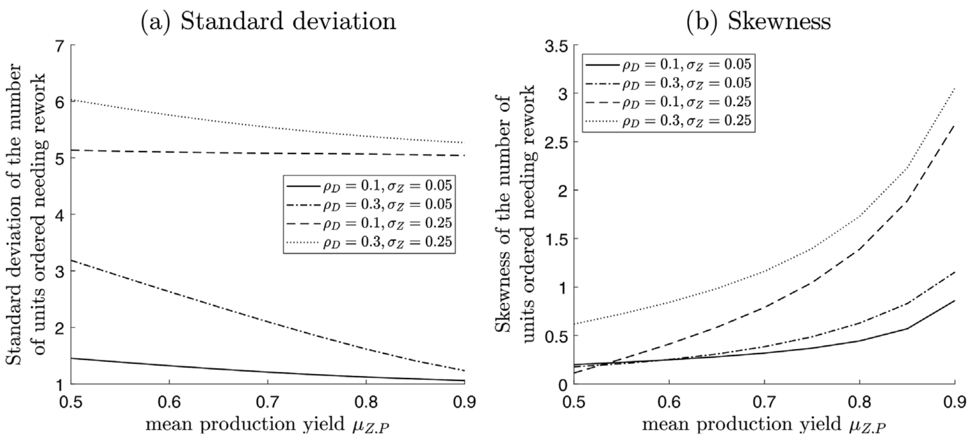


**FIGURE 6** Standard deviation (a) and skewness (b) of the number of units ordered over $L_R = 1$ periods needing rework under a perfect process

series with a nonperfect rework process, we define $\mu_{Z,R}$ and $\sigma_{Z,R}$ on the same range as $\mu_{Z,P}$ and $\sigma_{Z,P}$ and, for simplicity, assume the same distribution for each rework cycle. In total, we analyze 2025 instances under perfect rework and 91,125 instances under nonperfect rework.

The "Mean Yield Paradox" is illustrated by Figure 5a, which shows that the paradox exists under perfect and nonperfect rework. A similar paradox linked to the yield uncertainty cannot be observed as shown in Figure 5b. This figure verifies the conclusion from the previous section that the expected costs are increasing in the yield uncertainty. Figure 6a shows that the standard deviation of the number of units in a batch needing rework decreases with the expected yield, particularly when the demand uncertainty is high. This along with the decrease linked to lower yield uncertainties explains the initial decrease in expected cost observed in Figure 5a. However, a higher expected yield also implies that the yield distribution is—particularly under high yield variability—more anchored at the two extremes, that is, toward all units being of good quality or all units needing rework. This is illustrated by the skewness of the number of units in a batch that need rework in Figure 6b. The tendency of anchoring toward the extremes has a negative effect, which explains the increase of the expected cost for high values of $\mu_{Z,P}$ observed in Figure 5a. The positive effect of a decrease in variability of the number of units in a batch needing rework stemming from a higher expected yield increases in the rework lead time, whereas the negative effect of the tendency of all or nothing requiring rework decreases. This explains why the curve for the perfect rework process in Figure 5a is flatter than the one for nonperfect rework.

The reason that the "Mean Yield Paradox" is observed under stochastic proportional yield but not under binomial yield lies in the yield models themselves. While, the stochastic proportional yield model is a batch-based yield model, the binomial yield model is an item-based yield model. That means, that under stochastic proportional yield, the probability of an item being of good quality depends on the batch size and the number of other units in the batch that pass the quality inspection. Under binomial yield, the probability of an item being of good quality is always equal to $g$ and the tendency to be anchored at the two extremes—all units being good or needing rework—does not exist.

## 7 | DISCUSSION OF THE ASSUMPTIONS

In this section, we discuss the practical implication of the assumptions made in Section 3.

### 7.1 | $L_P$ and $L_{R,m}$ are constant

The processing time $L_P$ and the rework time $L_{R,m}$ per rework cycle are both assumed to be constant and independent of the batch size, which is a common assumption under periodic planning, for example in material requirements planning (MRP) systems, to enable coordinated decisions even when some variability exists. From an industry perspective, processing times may be constant, for example, in the chemical industry the duration of chemical reactions is independent of the amount being produced (see, e.g., Blomer & Gunther, 2000; Dessouky & Kijowski, 1997; Grunow et al., 2002).

### 7.2 | No quality differentiation

Note that the quality of a reworked item passing the quality inspection is the same as that of an item produced correctly without rework, which is commonly the case in, for example, the automobile or pharmaceutical industries. Therefore, all items entering the warehouse can be sold to customers without quality differentiation. However, depending on the industry, quality differentiation can be reasonable and is very common. One common example is the semiconductor industry and the microchips production, where the speed of the chips can vary and based on their quality be used to satisfy different customer segments (see, e.g., Bitran & Gilbert, 1994; Gallego et al., 2006; Hsu & Bassok, 1999; Nahmias & Moinzadeh, 1997).

### 7.3 | Discrete instead of continuous time

The proposed model in this paper considers a periodic review policy which implies a discrete instead of a continuous time model. Based on our experience with industry, this assumption is reasonable as periodic production planning is used. Note that the length of a period can be chosen arbitrarily small in the model and one can, hence, asymptotically approach a continuous time model.

### 7.4 | Items are never disposed of

We assume that items can always be reworked even though it might take several rework cycles. The reason is that the unit-tracking approach is based on matching each ordered unit with a demand and vice versa, which is not possible if some units stochastically leave the system. Extending the solution process so that it can be applied to these scenarios is an interesting venue for future research as mentioned in Section 8.

### 7.5 | Defective items are reworked immediately

Reworking defective items immediately can be reasonable for several reasons: First, reworking costs are arguably lower than production costs as the rework process is carried out on a product that already has been processed. This makes it economically preferable to rework defective units rather than initiating production of a new unit. Second, the value of keeping the units as finished products is higher than keeping them as defective products because finished products can be used to serve customer demand. Reworking defective units immediately has the additional benefit of reducing the uncertainty about the warehouse's future inventory level. Third, rework is usually required at the end of the production process, which means that the holdings costs for work in process inventory and finished goods inventory do not differ too much. However, there are situations where an immediate rework is not optimal, for example, if one has plenty of finished goods in stock because of high yield and/or low demand over a prolonged period. Of course, in such a situation one typically does not have many units to rework. Such a setting is particularly true for high demand products with relatively stable demand and demand in each period as considered in this paper. However, for products with low and erratic demand, this assumption might be an issue.

### 7.6 | Focus on inventory-related costs

In this paper, we focus on the inventory-related cost of different ordering policies because these are the costs that can be reduced by altering the ordering policy. The expected production, rework and holding cost for work in process are constant if all defective units are reworked as the average batch size is the same as the average demand per period. However, to be able to determine if defective items should be reworked or scraped, as analyzed in Sonntag and Kiesmüller (2018), or to determine the value of improving the yield, one must consider additional costs such as production and rework costs.

## 8 | CONCLUSIONS AND FUTURE RESEARCH

This paper discussed how to determine order quantities in a periodic make-to-stock production–inventory system with random yield and rework by deriving an exact expression for the expected cost per period under a base-stock policy and then myopically improve upon the same.

Given that the optimal ordering policy is unknown, the numerical study revealed that it is reasonable to rely on a base-stock policy if the mean yield and coefficients of variation of the yield are high. In such a case, the system is close to a perfect inventory system without random yield, for which the base-stock policy is the optimal policy. A state-dependent ordering policy adds little value under such parameter settings. However, under low mean yields of 50%, which are commonly observed in the high-tech industry, the myopic improvements of the base-stock policy outperform not only the base-stock policy, by up to 6% in terms of costs, but also an existing approach by up to 4.5% in terms of costs. Such cost reductions may lead to sizable savings, especially when considering that products may incorporate substantial value, and that holding and backorder costs are relatively high in the high-tech industry.

In contrast to earlier presented research, the approaches presented in this paper have the advantage that they are applicable independent of the input parameters and can even handle various other yield models and other demand distributions, such as Poisson distributions, which emphasizes the contribution of this paper. Our paper does not only contribute to random yield problems under rework but is also highly relevant to other inventory systems where order-crossing occurs. As explained in Section 2.2, order-crossing is always prohibited in existing papers using the unit-tracking approach. Thus, the insights generated in this paper can be used, for example, to determine order quantities in an inventory system with continuous review and stochastic lead times where orders can cross each other. A highly interesting topic for future research is the consideration of nonstationary demand and yield distributions, because demand and yield distributions usually change over the life cycle of a product. Another worthwhile extension to the considered problem is to allow for products to be scrapped after a number of rework cycles if one has not managed to reach the required quality by then. A combination of the current heuristic and an inflation policy could be an interesting alternative to investigate in such a setting.

## ACKNOWLEDGMENTS

The authors thank the editor, senior editor, and anonymous reviewers for their detailed feedback and constructive comments. They also thank Victor Martínz de Albéniz and Moritz Fleischmann for their valuable feedback on earlier versions of the paper.

## REFERENCES

Axsäter, S. (1990). Simple solution procedures for a class of two-echelon inventory problems. *Operations Research*, *38*(1), 64–69.

Axsäter, S. (1993a). Exact and approximate evaluation of batch-ordering policies for two-level inventory systems. *Operations Research*, *41*(4), 777–785.

Axsäter, S. (1993b). Optimization of order-up-to-s policies in two-echelon inventory systems with periodic review. *Naval Research Logistics* (*NRL*), *40*(2), 245–253.

Berling, P., & Martínez-de-Albéniz, V. (2011). Optimal inventory policies when purchase price and demand are stochastic. *Operations Research*, *59*(1), 109–124.

Berling, P., & Martínez-de-Albéniz, V. (2016a). A characterization of optimal base-stock levels for a multistage serial supply chain. *Naval Research Logistics* (*NRL*), *63*(1), 32–46.

Berling, P., & Martínez-de-Albéniz, V. (2016b). Dynamic speed optimization in supply chains with stochastic demand. *Transportation Science*, *50*(3), 1114–1127.

Bitran, G. R., & Gilbert, S. M. (1994). Co-production processes with random yields in the semiconductor industry. *Operations Research*, *42*(3), 476–491.

Blomer, F., & Gunther, H.-O. (2000). LP-based heuristics for scheduling chemical batch processes. *International Journal of Production Research*, *38*(5), 1029–1051.

Bollapragada, S., & Morton, T. E. (1999). Myopic heuristics for the random yield problem. *Operations Research*, *47*(5), 713–722.

Buscher, U., & Lindner, G. (2007). Optimizing a production system with rework and equal sized batch shipments. *Computers & Operations Research*, *34*(2), 515–535.

Chen, K., & Yang, L. (2014). Random yield and coordination mechanisms of a supply chain with emergency backup sourcing. *International Journal of Production Research*, *52*(16), 4747–4767.

Chiu, S. W., Ting, C.-K., & Chiu, Y.-S. P. (2007). Optimal production lot sizing with rework, scrap rate, and service level constraint. *Mathematical and Computer Modelling*, *46*(3-4), 535–549.

Choi, H.-c. P., Blocher, J. D., & Gavirneni, S. (2008). Value of sharing production yield information in a serial supply chain. *Production and Operations Management*, *17*(6), 614–625.

Dessouky, M. M., & Kijowski, B. A. (1997). Production scheduling of single-stage multi-product batch chemical processes with fixed batch sizes. *IIE Transactions*, *29*(5), 399–408.

Dettenbach, M., & Thonemann, U. W. (2015). The value of real time yield information in multi-stage inventory systems—Exact and heuristic approaches. *European Journal of Operational Research*, *240*(1), 72–83.

Ehrhardt, R., & Taube, L. (1987). An inventory model with random replenishment quantities. *International Journal of Production Research*, *25*(12), 1795–1803.

Erdem, A. S., & Özekici, S. (2002). Inventory models with random yield in a random environment. *International Journal of Production Economics*, *78*(3), 239–253.

Federgruen, A., & Zipkin, P. (1986). An inventory model with limited production capacity and uncertain demands I. The average-cost criterion. *Mathematics of Operations Research*, *11*(2), 193–207.

Gallego, G., Katircioglu, K., & Ramachandran, B. (2006). Semiconductor inventory management with multiple grade parts and downgrading. *Production Planning & Control*, *17*(7), 689–700.

Gates, D. (2021). *Boeing returns to red ink, projects $1 billion lost from* 787 *quality issues*. https://www.seattletimes.com/business/boeing-aerospace/boeing-projects-1-billion-lost-from-787-quality-issues/?amp=1

Gavirneni, S. (2004). Supply chain management at a chip tester manufacturer. In Terry P. Harrison, Hau L. Lee, John J. Neale. *The practice of supply chain management: Where theory and application converge* (pp. 277–291). Springer.

Gotzel, C., & Inderfurth, K. (2005). Approximate policies for hybrid production and rework systems with stochastic demand and yield. In Hein Fleuren, Dick den Hertog, Peter Kort. *Operations Research Proceedings* 2004 (pp. 41–49). Springer.

Grunow, M., Günther, H.-O., & Lehmann, M. (2002). Campaign planning for multi-stage batch processes in the chemical industry. *OR Spectrum*, *24*(3), 281–314.

Gupta, D., & Cooper, W. L. (2005). Stochastic comparisons in production yield management. *Operations Research*, *53*(2), 377–384.

Gurnani, H., Akella, R., & Lehoczky, J. (2000). Supply management in assembly systems with random yield and random demand. *IIE Transactions*, *32*(8), 701–714.

Henig, M., & Gerchak, Y. (1990). The structure of periodic review policies in the presence of random yield. *Operations Research*, *38*(4), 634–643.

Hsu, A., & Bassok, Y. (1999). Random yield and random demand in a production system with downward substitution. *Operations Research*, *47*(2), 277–290.

Huh, W. T., & Nagarajan, M. (2010). Linear inflation rules for the random yield problem: Analysis and computations. *Operations Research*, *58*(1), 244–251.

Inderfurth, K., & Kiesmüller, G. P. (2015). Exact and heuristic linear-inflation policies for an inventory model with random yield and arbitrary lead times. *European Journal of Operational Research*, *245*(1), 109–120.

Inderfurth, K., & Vogelgesang, S. (2013). Concepts for safety stock determination under stochastic demand and different types of random production yield. *European Journal of Operational Research*, *224*(2), 293–301.

Janakiraman, G., & Muckstadt, J. A. (2009). A decomposition approach for a class of capacitated serial systems. *Operations Research*, *57*(6), 1384–1393.

Kazaz, B. (2004). Production planning under yield and demand uncertainty with yield-dependent cost and price. *Manufacturing & Service Operations Management*, *6*(3), 209–224.

Kiesmüller, G. P., & Inderfurth, K. (2018). Approaches for periodic inventory control under random production yield and fixed setup cost. *OR Spectrum*, *40*(2), 449–477.

Kutzner, S. C., & Kiesmüller, G. P. (2013). Optimal control of an inventory-production system with state-dependent random yield. *European Journal of Operational Research*, *227*(3), 444–452.

Leachman, R. C., & Hodges, D. A. (1996). Benchmarking semiconductor manufacturing. *IEEE Transactions on Semiconductor Manufacturing*, *9*(2), 158–169.

Lopez, L. (2018). *Internal documents reveal the grueling way tesla hit its* 5,000 *model* 3 *target*. https://www.businessinsider.com/tesla-hit-model-3-target-by-reworking-thousands-of-cars-2018-8?r=DE&IR=T

Moussawi-Haidar, L., Salameh, M., & Nasr, W. (2016). Production lot sizing with quality screening and rework. *Applied Mathematical Modelling*, *40*(4), 3242–3256.

Muharremoglu, A., & Tsitsiklis, J. N. (2008). A single-unit decomposition approach to multiechelon inventory systems. *Operations Research*, *56*(5), 1089–1103.

Nahmias, S., & Moinzadeh, K. (1997). Lot sizing with randomly graded yields. *Operations Research*, *45*(6), 974–989.

Öztürk, H. (2017). A note on "production lot sizing with quality screening and rework." *Applied Mathematical Modelling*, *43*, 659–669.

Sarker, B. R., Jamal, A., & Mondal, S. (2008). Optimal batch sizing in a multi-stage production system with rework consideration. *European Journal of Operational Research*, *184*(3), 915–929.

Sonntag, D., & Kiesmüller, G. P. (2016). The shape of the yield and its impact on inventory decisions. 4*OR —A Quarterly Journal of Operations Research*, *14*(4), 405–415.

Sonntag, D., & Kiesmüller, G. P. (2017). The influence of quality inspections on the optimal safety stock level. *Production and Operations Management*, *26*(7), 1284–1298.

Sonntag, D., & Kiesmüller, G. P. (2018). Disposal versus rework–inventory control in a production system with random yield. *European Journal of Operational Research*, *267*(1), 138–149.

Voelkel, M. A., Sachs, A.-L., & Thonemann, U. W. (2020). An aggregation-based approximate dynamic programming approach for the periodic review model with random yield. *European Journal of Operational Research*, *281*(2), 286–298.

Wang, Y., & Gerchak, Y. (1996). Periodic review production models with variable capacity, random yield, and uncertain demand. *Management Science*, *42*(1), 130–137.

White III, C. C., & Cheong, T. (2012). In-transit perishable product inspection. *Transportation Research Part E: Logistics and Transportation Review*, *48*(1), 310–330.

Widyadana, G. A., & Wee, H. M. (2012). An economic production quantity model for deteriorating items with multiple production setups and rework. *International Journal of Production Economics*, *138*(1), 62–67.

Yano, C. A., & Lee, H. L. (1995). Lot sizing with random yields: A review. *Operations Research*, *43*(2), 311–334.

Yu, Y. & Benjaafar, S. (2008). A customer-item decomposition approach to stochastic inventory systems with correlation. In IIE Annual Conference. Proceedings, (p. 1964). Institute of Industrial and Systems Engineers (IISE).

## SUPPORTING INFORMATION

Additional supporting information may be found in the online version of the article at the publisher's website.

**How to cite this article:** Berling, P., & Sonntag, D. R. (2022). Inventory control in production–inventory systems with random yield and rework: The unit-tracking approach. *Production and Operations Management*, 1–18. https://doi.org/10.1111/poms.13706