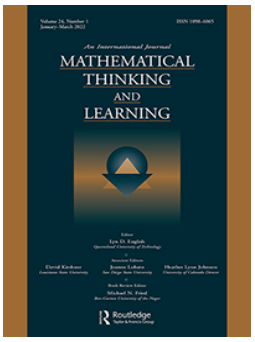

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The complexity of supporting reasoning in a mathematics classroom of shared authority

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ABSTRACT

The paper addresses the potential relationships between shared authority in mathematics classrooms and students' mathematical reasoning. Even though tensions and challenges related to shared authority are explicated in the literature, there are few examples of how these issues play out in mathematics teaching. We investigate the case of a mathematics teacher attempting to share authority as well as applying several moves recognized as supporting meaningful student learning. Data has been collected in a fourth-grade Norwegian classroom and is analyzed by means of open coding, inspired by literature. We identify the moves used by the teacher, and we rank these moves along two dimensions: (1) their potential to support mathematical reasoning and (2) their potential for sharing authority. From this, we uncover how a teacher's work of orchestrating mathematical discussions involves moves in all four quadrants, and we discuss how the interplay of moves affects the authority structures and the collaborative reasoning in the classroom.

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Mathematical reasoning; authority; teacher moves

Facilitating students' mathematical reasoning in whole-class discussions is considered a core practice in mathematics teaching (D.L. Ball & Forzani, 2009; Lampert et al., 2013). Accordingly, mathematics education research has, during the last decade, produced several studies that focus on the work of orchestrating whole-class discussions. These studies often make use of or develop frameworks for teachers' discursive moves (Conner et al., 2014; Drageset, 2014; Ellis et al., 2019; Hufferd-Ackles et al., 2004; Makar et al., 2015; Mata-Pereira & da Ponte, 2017; Mueller et al., 2014; Munson, 2019; Da Ponte & Quaresma, 2016). Moreover, they share an explicit or implicit aim of *shared authority* between students and the teacher, this being a desired feature of the mathematical discourse in the classroom. A potential definition of shared authority is provided by Otten et al. (2017): "Shared authority involves students' opportunities to be led and also to lead mathematical discourse, aligning with the goal of full participation in the discourse community" (p. 113). Shared authority has also been highlighted as a condition for successful mathematics teaching in general education research (see, e.g., Lampert, 1990; Schoenfeld, 2018; Stein et al., 2008). However, there is a lack of studies that delve deeply into the relationship between supporting mathematical reasoning and maintaining shared authority in the teaching of mathematics. Studies claiming to be guided by some conception of authority rarely explain or define the term (Wagner & Herbel-Eisenmann, 2014b), and supporting students' mathematical reasoning is rarely a focus in the research on authority. In this study, we focus on the intersection of shared authority and supporting mathematical reasoning in whole-class discussions, building on the existing literature on both topics. However, this intersection has certain innate intricacies and potential tensions.

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It is acknowledged that sharing authority while maintaining control of the mathematical content is a demanding task for the teacher. Ben-Zvi and Sfard (2007) identify a struggle in the mathematics teacher's role between practicing shared authority and acknowledging students' need to interact with a competent peer and follow in their footsteps. Stein et al. (2008) claim that there is an important balance to be maintained by the mathematics teacher between granting students authority over the lesson and ensuring that the lesson remains true to the canonical mathematical discourse. One example of struggling with this balance is shown in Otten et al.'s (2017) study of the collective authoring of geometry proofs in a tenth-grade classroom. Adopting an authority perspective on whole-class proving, the study shows how a teacher (Ms. Finely) offers structuring moves, clarifications, and requests for justifications, thus providing some guidance that supports the students' participation in a reasoning and proving discourse. However, Ms. Finley's leading of the discussion demonstrates that authority rests either with the textbook or with the teacher. Ng et al. (2020) provide an example of a discourse move initially described as facilitating collective thinking (i.e., requesting collective responses), thus reflecting shared authority. However, in Ms. L's classroom, this move serves "mainly as a collective way to answer trivial questions or validate responses which were put forward" (Ng et al., 2020, p. 22). These examples indicate that sharing authority in a whole-class mathematical discussion cannot be reduced to applying *talk moves* only. However, few studies have described teachers who succeed, to some extent, in sharing authority with their students, as well as the dilemmas they face in their quest to institute shared authority while also facilitating students' mathematical reasoning in whole-class discussions. We seek to extend the existing knowledge about how these dilemmas can play out in the classroom. To this end, we present a case study of a mathematics teacher in grade four who attempts to share authority, as well as applying several moves that can be recognized as supporting meaningful student learning. The study is guided by the following research question:

In a teacher's orchestration of a whole-class mathematics discussion, what relationships exist between managing mathematical reasoning and managing shared authority?

The study is framed by Sfard's commognitive theory (Sfard, 2008). To answer the research question, we begin by making it clear what we mean by mathematical reasoning and how mathematical reasoning can be facilitated in the classroom, and we outline the characteristics of authority structures used in analyzing whole-class mathematical discussions. In the Methods section, the case study is described, and we elaborate on the process of analysis: first, we identify the teacher's moves in the discussions. Then, we rank them according to their potential to support students' mathematical reasoning and the authority structure they reflect. The results of this double ranking are presented in the Findings section, together with chronological excerpts from the data. Finally, we discuss the potential relationships between managing mathematical reasoning and managing shared authority that surface from the case study.

Theoretical background and conceptual framework

In commognition, doing mathematics is defined as participation in mathematical discourse, and learning mathematics is defined as the process of individualizing mathematical discourse (Sfard, 2008). This theory enables the double lens needed to investigate our research question, in the sense that we must be explicit about mathematical reasoning and the interaction patterns that may communicate authority. Regarding mathematical reasoning, we use Jeannotte and Kieran's (2017) commognition-based conceptual model. Although no theories on authority are directly associated with commognition, we follow Otten et al. (2017), who claim that, in any discourse, there is a constant negotiation of authority. Thus, a discursive lens for studying the concept of authority appears. In this section, we define mathematical reasoning and elaborate on how teachers' actions can support students' mathematical reasoning. We describe how the teacher's actions can have a high or low potential to support students' mathematical reasoning. Moreover, we operationalize the concept of shared and teacher-led authority in the setting of teacher-orchestrated discussions involving mathematical reasoning based on descriptions and characteristics from the literature.

Facilitating students' mathematical reasoning in whole-class discussions

We follow Jeannotte and Kieran's (2017) conceptual model of mathematical reasoning, which is based on a synthesis of the current body of research on reasoning and proving in mathematics education, positioned within a commognitive frame (Sfard, 2008). Here, mathematical reasoning is conceptualized as two categories of meta-discursive processes: processes related to *the search for similarities and differences* and processes related to *validating mathematical statements*. Additionally, there is the process of *exemplifying*, which supports the two categories. Both categories of processes have sub-categories. For the processes related to the search for similarities and differences, the sub-categories are juxtaposed; for the validating processes, the sub-categories form a hierarchy, with each sub-category including the previous ones but being more rigorous. Table 1 provides an overview of the framework. All the categories are defined according to how the involved processes treat what, in commognitive terms, are called *narratives*. A narrative, in mathematical discourse, is a sequence of utterances that describes the properties of objects or the relationships between objects (Sfard, 2008). This includes but is not restricted to definitions, theorems, and the results of computations.

We provide some examples of what mathematical reasoning processes can look like. In the problem "odd one out," in which students are given a short list of numbers – here, 4, 20, 32, 16 – and the task is to determine which one does not fit, students can *compare* the numbers in the list, considering similarities and differences in the numbers' mathematical properties ("32 and 16 are both in the 8-times table," "32, 16, and 4 are powers of 2, but 20 is not"). When further asked to suggest another number that would fit in the list, the process of *classifying* can come into play when new numbers are determined to belong to the class "powers of 2." Pursuing the idea of powers of 2, if students are prompted to investigate more closely, they could *identify the pattern* that the numbers 2, 4, 8, 16, 32, 64, 128, 256 have a recurring 2, 4, 8, 6 as the last digits. This can be *generalized* to something that likely holds when we continue to compute powers of 2 and formalized as a *conjecture* that must be studied further. When the conjecture is made, it can be validated with a *justification* or a *proof*. Of course, not all tasks that are fruitful for fostering students' mathematical reasoning will include all the mathematical reasoning processes.

Table 1. Mathematical reasoning processes (Jeannotte & Kieran, 2017).

Processes related to the search for similarities and differences	Processes related to validating mathematical statements
Generalizing: *Infers narratives about a set of mathematical objects or a relation between objects of the set from a subset of this set.* **Conjecturing**: *Infers a narrative about some regularity that has probable epistemic value and the potential for mathematical theorization.* **Identifying a pattern**: *Infers a narrative about a recursive relation between mathematical objects or relations.* **Comparing**: *Infers a narrative about mathematical objects or relations.* **Classifying**: *Infers a narrative about a class of objects based on mathematical properties and definitions.*	**Justifying**: *Aims to change the epistemic value (i.e., the likelihood or truth) of a mathematical narrative.* **Proving**: *Modifies the epistemic value of a narrative from* likely *to* true. *Constrained by the following*: • *the narratives that are accepted by the class community (the set of accepted narratives) that are true (from the viewpoint of the expert mathematician) and available without additional justification;* • *a final deductive restructuring;* • *the realizations (in the sense of* Sfard, 2008, p. 301) *that are appropriate and known, or accessible, to the class.* **Formal proving**: *Modifies the epistemic value of a narrative from* likely *to* true. *Constrained by the following*: • *the narratives that are accepted by the class community (the set of accepted narratives) that are true (from the viewpoint of the expert mathematician) and systematized in a mathematical theory;* • *a final deductive restructuring;* • *the realizations that are formalized and accepted by the class and mathematical communities.*
Exemplifying: *Supports other mathematical reasoning processes by inferring examples that aid in the search for similarities and differences and in the search for validation.*	

To support students' mathematical reasoning in a whole-class discussion, a teacher can use several moves, which the literature has attempted to describe. In-depth literature reviews regarding teacher move frameworks can be found in Drageset (2014) and Ellis et al. (2019). Here, we present one framework of particular interest to us, namely the *Teacher Moves for Supporting Student Reasoning* (TMSSR) framework (Ellis et al., 2019), which also makes use of Jeannotte and Kieran's (2017) definition of mathematical reasoning. The TMSSR framework lists teacher moves that can support students' mathematical reasoning. The moves are grouped into four categories based on their function: eliciting student reasoning, responding to student reasoning, facilitating student reasoning, and extending student reasoning. According to Ellis et al. (2019), there is a potential "ideal" relationship between the categories, which suggests that eliciting moves should be followed by responding to moves and/or facilitating moves and, finally, extending moves, which was often found to be "more effective in fostering the processes of searching for similarity and difference, validating, and exemplifying" (Ellis et al., 2019, p. 117). Moreover, a central structural feature of the TMSSR framework is its bipartite classification of moves: the teacher moves have a high or low potential to support students' mathematical reasoning. Often, the moves come in pairs, with a high-potential move corresponding to a low-potential move. Examples from the "elicit" category include the low-potential moves *eliciting answer* and *eliciting facts and procedures*, with the corresponding high-potential moves *eliciting ideas* and *eliciting understanding*. In the "responding" category, we find, e.g., *re-voicing* and *encouraging student re-voicing* (low potential) and *re-representing* (high potential). It is important to note that the idea of high and low potential to support students' mathematical reasoning does not mean that moves claimed to have low potential are unwanted or that using only high-potential moves will result in a more productive discussion. However, the high-potential moves have a greater "ability to emphasize a focus on the students' ideas, enabling teachers to provide students with a space to engage meaningfully in the processes of mathematical reasoning" (Ellis et al., 2019, p. 127). In our data analysis, we adopt the idea that teacher moves can have a high or low potential to support students' mathematical reasoning. However, as will be explained in the Methods section, we do not use the TMSSR framework itself as an analytical tool.

Shared authority and mathematical reasoning

Authority per se is a concept taken from the social sciences and defined by Weber (1947) as "the probability that a command with a given specific content will be obeyed by a given group of persons" (p. 139) – that is, a social mechanism that delegates to certain persons or institutions the power to command others, who are, in turn, willingly commanded (in a non-coercive way). In the setting of mathematics education, the teachers are granted at least some measure of authority over their students, being expert representatives of mathematical discourse (Amit & Fried, 2005; Otten et al., 2017). At the same time, shared authority is considered advantageous and even necessary for a successful reform-based teaching practice. However, it is not clear from the existing literature what is meant by *shared authority* or how shared authority manifests itself in mathematics classrooms. In the following, we conceptualize authority in the setting of whole-class discussions involving mathematical reasoning. We identify and compare the definitions and characteristics of authority structures in the existing body of literature. The resulting conceptual framework, which will inform the analysis of the case study, is presented toward the end of this section (see Table 2).

Some authors explain the concept of (shared) authority etymologically – that is, *authority* is derived from *authoring*: "The idea is that students create, or author, mathematical ideas and their justifications (thus becoming authorities)" (Schoenfeld, 2013, p. 617), or "[W]ho has the authority to author ideas and under what conditions" (Munson, 2019, p. 2). Based on this, we identify *authoring* as a dimension of authority. Other explanations focus, instead, on authority structures resulting from or connected to students' participation in a discourse: "Shared authority involves students' opportunities to be led and also to lead mathematical discourse, aligning with the goal of full participation in the discourse community" (Otten et al., 2017, p. 113), or "[S]tudents feel empowered to engage more fully in the

Table 2. Characteristics of authority structures in whole-class mathematics discussions on mathematical reasoning.

Teacher-led authority in mathematics	Shared authority in mathematics
Authoring • Questions related to mathematics are closed. There is an expected answer or an expected way to proceed. • The teacher evaluates students' mathematical suggestions. • The teacher demonstrates solutions or strategies, with minimal involvement on the part of students. • Things are "happening" without the need for discussion.	• *Authoring* Students are invited to choose and present strategies to solve mathematical problems. • Students are involved in evaluating mathematical ideas. Students are credited for their mathematical ideas. • Students are invited to evaluate others' suggestions. • There is a choice regarding what action to take; this choice can be discussed. *Participating* • Students' personal lives and experiences are integrated into mathematics teaching. • Student participation is invited, from an inclusive point of view. • The teacher explicitly discusses authority with students.
Participating • The teacher uses phrases that indicate that students should work to meet the teacher's demands ("I" and "you" together).	

learning" (Kinser-Traut & Turner, 2020, p. 9). From this, we identify *participating* as a dimension of authority. Others make both authoring and participating explicit: "[Learners] assume the role of *author* and thereby obtain a sense of *authority* through choosing to participate in the discourse with their own voice" (Ng et al., 2020, p. 4), or "Authority concerns the degree to which students are given opportunities to be involved in decision making about the interpretation of tasks, the reasonableness of solution methods, and the legitimacy of solutions. Authority is therefore about 'who's in charge' in terms of making mathematical contributions" (Cobb et al., 2009, p. 44). These nuances in the literature led us to distinguish between the *authoring* and *participating* characteristics of authority structures. We note that there is a subtle hierarchic distinction between authoring and participating: authoring is an aspect of participating in a discourse, but participating in a classroom discourse on mathematics involves more than authoring. Interpreting the concept of authority within a commognitive framework, we understand authoring as the aspects of authority that are related to the construction and validation of narratives in a discussion, while participating refers to the aspects of authority that are not explicitly concerned with narratives and are, instead, connected to other aspects of students' involvement in the discourse.

Having identified two aspects of authority in mathematical discourse, we turn to the various authority structures and how they can be recognized. Most studies focusing on facilitating students' mathematical reasoning in classroom discussions in which the concept of authority is discussed suggest an explicit or implicit dichotomy between an authority structure involving the teacher holding most of the authority and one involving the teacher and students sharing authority. Thus, we also make use of such a dichotomy. For the sake of simplicity, we will use the terms *teacher-led authority* and *shared authority*. Next, we review the characteristics of these two authority structures. In doing so, we will preserve the distinction between authoring and participating.

Shared authority structures are often separated from teacher-led authority structures by identifying who is allowed or invited to suggest, develop, and evaluate mathematics questions in the classroom. Mathematical questions that are closed (i.e., they have one correct response) are associated with teacher-led authority (Drageset, 2014; Hamm & Perry, 2002; Wagner & Herbel-Eisenmann, 2014a). The same is true of classrooms in which there is an expected way to solve problems (Hamm & Perry, 2002; Harel & Rabin, 2010; Kinser-Traut & Turner, 2020). Another characteristic of teacher-led authority is that the teacher evaluates students' mathematical suggestions (Hamm & Perry, 2002; Harel & Rabin, 2010; Otten et al., 2017) and demonstrates solutions or strategies with minimal student involvement (Drageset, 2014; Harel & Rabin, 2010). The characteristics of shared authority are typically opposite to the above-mentioned features: students are invited to choose and present strategies with which to solve mathematical problems and to take part in verifying mathematical ideas (Drageset, 2014; Hamm & Perry, 2002; Harel & Rabin, 2010; Kinser-Traut & Turner, 2020; Otten et al., 2017; Stein et al., 2008). Explicitly crediting students for their ideas (e.g., by naming the ideas after them) is another feature of shared authority (Engle & Conant, 2002; Hamm & Perry, 2002; Lampert, 1990), as is the teacher inviting students to evaluate other students' contributions (Drageset, 2014; Kinser-Traut & Turner, 2020). We claim that these features relate to authoring because they are strongly intertwined with the production and validation of narratives.

There are also characteristics of authority structures that seem to relate more to participating than to authoring. Kinser-Traut and Turner (2020) describe integrating students' personal lives and interests into mathematics teaching as a way of sharing authority, for example, by asking students how they use a particular mathematical skill at home or allowing them to define the parameters of a mathematical task based on their own experiences. Herbel-Eisenmann et al. (2013), discussing a framework for talk moves, suggest that "inviting student participation" is a social goal in its own right and is not only for eliciting mathematical contributions. Martin et al. (2005) describe explicitly delegating authority to students as a way of sharing authority, for example, by making it clear to the students that it is their responsibility to validate a statement (note that here we are referring to "meta-talk" about validation of statements, not to the actual validation of given narratives – issues related to the latter would belong to authoring aspects of authority). As an extension of this, Wagner and Herbel-Eisenmann show how a teacher discusses authority with his class, suggested that doing so "can help students come to terms with their mathematics" (Wagner & Herbel-Eisenmann, 2014a, p. 882). All these characteristics of authority are connected to participating rather than authoring because, while they relate to students' and teachers' participation in the discourse, they do not explicitly address mathematical narratives.

Thus far, we have considered papers that discuss or include whole-class discussions on mathematical reasoning. However, we note that the concept of authority itself is much wider than this. Authority is a central concept in sociology, anthropology, and politics, all of which are considered relevant in mathematics education (Fried, 2014), and many studies on authority in mathematics classrooms discuss sociocultural issues (Engle et al., 2014; Langer-Osuna, 2016, 2018; Tatsis et al., 2018; Vithal, 1999). In the literature concerning authority in mathematics classrooms more generally, however, authority is often conceptualized as much more complex than the dichotomy between teacher-led and shared authority. Amit and Fried (2005), drawing on Weber, as well as later ideas, describe authority as a "web of relations" that is "flexible and fluid" (Fried & Amit, 2008, p. 58). Perhaps the most extensive and applicable theory on the topic is that of (Herbel-Eisenmann & Wagner, 2010; Wagner & Herbel-Eisenmann, 2014a). Inspired by positioning theory and linguistic-discursive tools, Wagner and Herbel-Eisenmann connect subtle language patterns with authority structures (see also, Rowland, 1999), thus identifying four authority structures that may coexist in a mathematics classroom. We nevertheless choose the dichotomy as a basic construct in our study, as well as a means of fixing the limits of the study. However, we find it useful to add some of Wagner and Herbel-Eisenmann's characteristics of authority to our conceptual framework because they are compatible with and add information to our framework. In their framework, teacher-led authority[1] is characterized by phrasings indicating that things are happening simply because they have to; the actions seem predetermined or inevitable. Moreover, Wagner and Herbel-Eisenmann (2014a) claim that, whenever a teacher uses "I" and "you" in the same sentence when instructing students, they are seemingly asking the students to do something based on their own wish and that this is strongly associated with teacher-led authority. Shared authority structures, on the other hand, are recognized by the presence of choice and fact that decisions can be questioned, debated, and revised by every participant in the community. In our framework, we associate the characteristics related to choice with authoring, that is, being connected to the manipulation of narratives. As the characteristics related to teachers' use of "I" and "you" refer to the goal of student participation (i.e., to do as the teacher says), we associate them with participating.

While the authoring characteristics have a strong basis in the literature focusing on mathematical reasoning, the participating characteristics are less documented. The participating characteristics often arise from the literature connected to general research on authority structures or the sociocultural aspects of mathematics education (e.g., Herbel-Eisenmann et al., 2013; Kinser-Traut & Turner, 2020). Perhaps, this imbalance relates to Wagner's (2017) assertion that there is a gap "between scholars who attend to sociocultural issues and they who do not" (p. 296). Here, Wagner reflects on his personal experiences writing about mathematics education from the position of critical theory, which often results in the question "Where is the maths?" Presumably, this question is asked by researchers who do not "attend to sociocultural issues." According to our readings, there are few traces of sociocultural issues in most papers concerned with facilitating students' mathematical reasoning, although the term *authority* is commonly used. Within our conceptual framework, we acknowledge perspectives from both sides of the gap Wagner (2017) describes. In Table 2, we present a summary of the characteristics of teacher-led and shared authority structures in mathematics classrooms. This set of characteristics will be used as an analytical tool in the study.

Methods

To study the relationships between supporting students' mathematical reasoning and managing shared authority, we make use of a case study focusing on the fourth-grade teacher Tom (a pseudonym), who we followed for three consecutive mathematics lessons. While Tom demonstrates teaching moves that can be considered supportive of both sharing authority and promoting students' mathematical reasoning (e.g., his repeated attempts to encourage all the students to share their solutions, as well as his willingness to enable them to decide how to proceed in solving the task), he is, at the same time, facing what we assume to be common challenges when facilitating primary school students' work on a mathematical reasoning task. Thus, although the data in the study is limited, we consider it worthwhile to study Tom's practices, seeking to better understand the previously described intricacies and tensions regarding promoting shared authority during whole-class discussions while facilitating students' mathematical reasoning. Based on this, the study can be considered a *single instrumental case study* (Stake, 1995) because we deem Tom's mathematics classroom to be a "sharp focus of attention" (p. 5) but instrumental to the issue of shared authority in mathematical reasoning.

Participants and data collection

The teacher Tom had completed an undergraduate teacher education program for grades 1–10, with a specialization in mathematics. At the time of data collection, he had eight years of teaching experience in primary school. Data were collected about a month after Tom began teaching the fourth-grade class, but he had also taught the students when they were in first grade. With regard to his educational and professional background, we find Tom to be an ordinary mathematics teacher, unlike those who can be considered exemplary mathematics teachers, such as Deborah Ball (D. L. Ball & Bass, 2003; Stylianides, 2007, 2016), Magdalene Lampert (Lampert, 1990, 2001), and Vicky Zack (Reid, 2002; Zack, 1997). However, due to his seemingly strong effort to share authority in his mathematics classroom, Tom cannot be considered a typical Norwegian teacher. For example, Drageset's (2014) analysis of five Norwegian upper primary school teachers' orchestration of classroom discourse focused on mathematics teaching dominated by an Initiation-Response-Evaluation (IRE) discourse pattern (Cazden, 2001) and associated with teacher-led authority. Nevertheless, we acknowledge that the Norwegian context enables Tom's "style of teaching" because the Norwegian school system is described elsewhere as strongly egalitarian (Braathe & Ongstad, 2001).

Data were collected as part of a pilot study for a larger project on reasoning and proving in primary education (ProPrimEd[2]). The school was suggested as a partner in the project by the local government, which regulates university-school collaborations. Tom was encouraged by the school principal to participate, which he agreed to do. Two researchers made observations during one week of mathematics teaching (three lessons, each lasting 60–90 minutes). The aim of the pilot study was to gain insight into the mathematics teacher's current instructional practice and the students' learning environment. Hence, we did not interfere in either the planning or the implementation of the teaching and adopted the role of non-participant observers in the classroom. Observation was carried out using one fixed camera at the back of the classroom that was focused on the teacher, as well as an interactive whiteboard. One handheld camera was directed at the students' faces during whole-class discussions and at pairs of students during individual work. After the data collection, the videos were transcribed, with all participants being given pseudonyms (students we were unable to identify in the discussion were named "Student"). In addition, we sent a short questionnaire via e-mail to the teacher (see Appendix). In the questionnaire, Tom was asked to describe what characterizes a successful mathematics lesson with the class. His characterization of such a lesson was "that all students, regardless of the 'level' of strategies they choose, become engaged in the problem-solving process and feel that their thoughts are just as valuable as the others." Tom's answer confirms that he is conscious about aspects of both mathematical reasoning and shared authority in his teaching.

The context of Tom's classroom

The class worked on a proving task featuring ambiguous conditions (Stylianides, 2016), which demanded that the community of students and the teacher negotiated the task's conditions. Within a real-life context of a baker and his design for muffin boxes (Cameron & Fosnot, 2007), the students were invited to explore the arrays in shapes varying from 1 × 1 to 10 × 10 and argue regarding the total number of possible arrays. The task was given orally by Tom to the students twelve minutes into the first lesson: "You're going to make different types of boxes. It can't be larger than ten times ten. I can draw on the blackboard (. . .) [he draws a 10 × 10 rectangle] this is the largest box you can make. But, he [referring to the muffin baker] is wondering, what possibilities do I have? (. . .) And then there's another thing. And that is that when they're making muffin boxes they actually have to be quadrangular [Tom explains how a box folding machine works]. (. . .) So, what other sizes can you make? They don't have to be squares."

The number of muffin boxes can be found by systematically counting all arrays of type 1x1, 1x2, . . ., 1 × 10 and all arrays of type 2x1, 2x2, . . ., 2x10, up to 10x10, and then adjusting to account for the arrays that appear twice (e.g., 1 × 2 and 2 × 1). Thus, the class had to find a way to list all the arrays, as well as agree about the conditions for counting them, for example, whether equal arrays with different orientations were to be counted once or twice. The task can be considered to potentially promote several mathematical reasoning processes (see Table 1). In an exploration phase, students were given the chance to conjecture that the number of possible arrays was 10 + 10 + 10 + . . . + 10 = 100 (10 arrays of width 1, 10 arrays of width 2, and so on), or they could identify a pattern in that the number of arrays of width 1 is 10, the number of new arrays of width 2 is 10–1 = 9, the number of new arrays of width 3 is 9–1 = 8, and so on. In a validating phase, these conjectures or patterns could be established with an argument in the form of a *justification* or *proof* (the notion of *formal proof* is not relevant in a fourth-grade setting).

The students worked in groups to draw various arrays from 1 × 1 to 10 × 10. Because none of the groups developed a way to systematically draw all possible arrays, this work continued into the second lesson, followed by a whole-class discussion concerning how many arrays there could be. In the third lesson, the question concerning how to count equal arrays with different orientations arose. By the end of the third lesson, there was still no consensus in the class regarding the number of possible arrays. In the classroom, the activities shifted between students' work, individually or with a partner, and whole-class discussions. Our analysis focuses on the whole-class discussions, which were orchestrated by the teacher.

The process of analysis

We analyzed all whole-class discussions in the three mentioned lessons, focusing on the teacher's actions and utterances, in a two-step process. First, we identified the moves used by the teacher in the whole-class discussions. To maintain a double analytic focus – meaning that we aimed to uncover Tom's management of mathematical reasoning in relation to his management of shared authority – these moves were then classified according to the authority structure they reflected, as well as their potential to support mathematical reasoning.

The first analytic step was undertaken by means of open coding (Corbin & Strauss, 2008) in which the unit of analysis was single utterances by the teacher (or, sometimes, short series of subsequent single utterances when these seemed to serve the same goal). The coding was steered by the questions "What is the teacher doing when making such a statement/question/comment?" and "What might be the purpose of such a statement/question/comment?" The codes were developed and named through a process Corbin and Strauss (2008) describe as breaking apart data to search for the varied meanings of the phenomena being studied. We chose this inductive approach because it appeared to be advantageous as compared to using previously developed frameworks. Specifically, it enabled us to include, in the coding process, every utterance the teacher made. For example, a framework on supporting students' mathematical reasoning would "miss out" on many comments by the teacher that did not explicitly support or hinder mathematical reasoning, but these comments could still be important building blocks in a discussion that, as a whole, supports or hinders students' mathematical reasoning. Nevertheless, the analytic process was inspired by insights from the literature. Some of the codes were therefore given names inspired by previous literature, such as Sherin (2002), Drageset (2014), and Ellis et al. (2019). Note that, of these references, only the third provides a framework for moves to support students' mathematical reasoning.

The open coding was undertaken by two researchers individually (the authors of this paper) before being compared and contrasted until agreement was reached. The set of codes was further applied to the transcripts from all three lessons. Again, this was undertaken individually by the two authors before the results were compared and contrasted. This stepwise process led to an adjustment of the list of codes and small changes to their wording. The complete list of codes (in alphabetical order), their characteristics and related examples from data are presented in Table 3. Because the codes assign words to the moves a teacher uses in his mathematics teaching, for simplicity, we refer to the codes as *teacher moves* throughout the remainder of the paper.

The second analytic step was undertaken by means of ranking Tom's moves in two ways. First, the moves were ranked according to their potential to support mathematical reasoning, inspired by Ellis et al.'s (2019) bipartite classification of moves in the TMSSR framework. For moves coinciding with or included in those described in the literature, we have agreed with their classification regarding a high/low potential for mathematical reasoning; for other moves, we mainly draw on the definition of mathematical reasoning provided by Jeannotte and Kieran (2017). Second, we applied the analysis of the characteristics of teacher-led authority versus shared authority (see Table 2) to this ranking. Thus, we determined whether the characteristics of each move along the mathematics reasoning continuum belonged to teacher-led authority or to shared authority. Simultaneously, we also noted whether each move related to the authoring or participating characteristics of authority. This second analytic step resulted in a four-cell table (see Table 4), with rows representing teacher-led and shared authority structures and columns representing low and high potential to support mathematical reasoning.

Discussing further the quality of the study, we draw on the criterion of *trustworthiness* (Lincoln & Guba, 1985), consisting of the four sub-criteria *credibility, transferability, dependability*, and *confirmability*. Credibility has to do with the extent to which claims and conclusions about Tom's mathematics teaching are believable to the reader. In both analytic steps, we had to interpret the teachers' intentions, e.g., when answering the question "What might be the purpose of such a statement/question/comment?" or determining whether a move was related to the authoring or participating characteristics of authority (was the teacher aiming to include students in the discussion or introduce

Table 3. Definition and examples of teacher moves in tom's mathematics classroom.

Teacher move	Definition	Examples from the data
Acknowledging contribution	The teacher acknowledges that he has heard a student's contribution, e.g., by repeating it or by giving a short "yes." Often, the teacher provides no evaluation.	Emil: *No, it can't. Then, we can take up to four times ten, then.* Tom: *Eight, nine, and, then, you said ten.*
Assessing student effort	Giving the students feedback on their effort on the task or on their involvement in the discussions, often followed by instructions or commands to improve effort.	Tom: *Now, I hear Pia a lot, and that's really good, and Pia is learning a lot from this, but the rest of you who are silent, do you learn anything? . . .Anna, what do you think?*
Closed progress detail	Inspired by Drageset (2014). "Questions typically request details needed to move the process forward. These details can be process answers (one step at a time) or details about how the process should go on to reach the answer" (p. 293).	Tom: *How many are there in the muffin box having four in width and eight in length, then?* Tom: *Håvar found out that if you make one [box] that is four wide and six tall, then how many muffins does it contain? How many muffins does it contain?*
Correcting question	Inspired by Drageset (2014): "The comments in this category typically include a question from the teacher to redirect the student towards another approach. The questions therefore act as corrections" (p. 290).	Tom: *Can both boxes be 32 when one of them is twice as big as the other?*
Demonstrating argument	Making explicit a connected sequence of assertions intended to verify or refute a mathematical claim. See also, Ellis et al. (2019) "providing conceptual explanation."	Tom: *If there are 10 boxes that are one in width, there are 10 boxes that are two in width, there are 10 boxes that are three in width, there are 10 boxes that are four, 10 boxes that are five, and so on up to 10, then that must mean there are 100 boxes.*
Demonstrating strategy	Providing and making explicit a strategy to systematize student responses or move the class forward in solving the task. See, also Ellis et al. (2019) "providing alternative solution strategies."	Tom: *We have 10 . . . I write ones, and 10 twos. Do you understand what I mean by ones and twos? Then, I'll draw. By ones, we mean the boxes that are of width one. . . And all the boxes, whether they are this one* (draws 1 × 1) *or that one* (draws 2 × 1) *or this one* (draws 8x1, without counting), *I call them ones. Yes?* (erases) *While the two-boxes are those that are two in width* (draws 2 × 1).
Devaluing own authority	The teacher expresses mathematical uncertainty or indicates/states that his opinions are not better than the students'.	Tom: *I can tell you what I think, but it's not as if what I think is any more correct than anyone else's ideas here.* Tom: *Yes, I don't know. I'm a bit unsure. That's what I find tricky.*
Eliciting answer	Inspired by Ellis et al. (2019). The teacher asks a student or a pair of students to share their answer on a task.	Tom: *Pia and Molly?*
Eliminating justification	Encouraging the students to make a guess about an answer so that they have something to share.	Tom: *You're allowed to make a guess.*
Filtering	Inspired by Sherin (2002). The teacher follows up on or focuses on one (or some) of the students' input or statements, thus directing the ensuing whole-class discussion. Often includes explicitly connecting students' names to the statement.	Tom: *The first thing I must grasp, which I thought was a little fun . . . Esten and Magnus? [Yes?] You had distributed the boxes between you? [Yes.]* Tom (to the whole class) *Do you think most boxes are squares or rectangles?*
Ignoring accusation of being an authority	Refusing to make decisions or give the answer to a mathematical question in order to make the students develop the decision/conclusion. Characterized by "what is not said."	Student: *Tell us the truth, tell us the truth.* Student: *Can you tell us the truth?* Student: *What's right?* Tom: *We must vote by the show of hands, then.*
Indicating relationship	Pointing toward or hinting about a mathematical relationship or a condition that is crucial to progressing on the task.	Tom: *Mari, would you say that. . . here, you have one box* (hands it to Mari), *and then I get it back* (rotates the box 90 degrees and gives it back), *and now you're having a completely different box!*

(*Continued*)

Table 3. (Continued).

Teacher move	Definition	Examples from the data
Making mathematical digression	The teacher halts the progress by making the students focus on a detail in their reasoning, typically involving terminology or a definition that is not crucial for bringing the mathematical work forward.	Tom: *What do you call those quadrilaterals that are just as long as they are tall?* Tom: *What is the name of this you are pointing at now?*
Making conclusion	Drawing a conclusion about a solution to a sub-task or about a proposed strategy in order to move the class forward in solving the main task.	Tom: *So, these are all the boxes we can make that are four wide. . . . We have found out that we can make 10 boxes that are four wide.*
Making democratic conclusion	Making a conclusion or decision about a mathematical question or an ambiguous condition of the task by using a poll.	Tom: *Of those who raised their hands, there were many more who thought that it was only worth ordering one type of box, and then, there were two who thought that you have to order two different types of boxes because they are different. Then, I think most of you think that these are the same box.*
Managing opinion poll	Asking for a show of hands to support the conclusion or make a decision in order to move forward in solving the task.	Tom: *Okay, then, I wonder how many people here think that there must be a hundred boxes? Raise your hands* (five or six students raise their hands). *Yes. And then, there are some here who think that there are not a hundred boxes. Raise your hands* (a few students raise their hands).
Mapping out student response	The teacher announces that he will ask all students to share their solutions to a task they have worked on.	Tom: *But first, I wonder. . . how many different boxes do you think it is possible to make? Hmm, I'm going to ask all of you, so you can take down your hands.*
Open progress initiatives	Inspired by Drageset (2014): "initiates progress but still leaves it at least partly open to the students to choose or suggest which path to follow. . . . The comments are also aimed at moving the process forward, but without pointing out the direction" (p. 294).	Tom: *So, how on Earth should we find out how many different boxes we can make?*
Providing opportunity to revise answer	When conflicting claims arise, the teacher addresses the class or selected peers to revise their answers so that the class can reach agreement.	Tom: *What do you say, should it be 24, as Karsten and Magnus are saying, or should it be 32, as Pia is saying? Hmm, Emil?* Emil: *I agree with 32.* Several students: *Me too.* Tom: *Karsten, what do you say?* Karsten: *I still say 24.*
Referring to the frames of the task	The teacher repeats information or rules stated in the task to approve or reject the students' input.	Tom: *But exactly this machine here cannot make, it cannot make hexagons. It only manages to make quadrilaterals.*
Requesting justification	Inspired by Ellis et al.'s (2019) "pressing for justification." Asking the students "to explain why something works or to justify a mathematical idea, strategy or solution" (p. 123).	Tom: *No, but you're sure you've drawn all [the different boxes]? How do you know you are drawing everyone, then?*
Requesting review of peers' narrative	Inspired by Drageset (2014). Asking students to review or consider peers' solutions or strategies.	Tom: *Mona and Siri, what do you think about Karsten's proposal?*
Suggesting de-sophistication of strategies	Leading the class back to initial strategies, instead of building on what has been discovered so far; for example, a longer discussion in the third lesson about how to count the various boxes devolves into chaos. Tom decides to return to the beginning and requires the students to draw all possible boxes.	Tom: *I wonder if you must draw all the boxes. . . . Now, you will get the following task, until the class is over. Can you draw all one-boxes, all two-boxes, three-boxes, four-boxes, five-boxes, sixes, sevens, eights, nines, and tens?*
Turn-and-talk	Asking the students to evaluate a given claim in pairs. Inspired by Kazemi and Hintz (2014).	Tom: *Talk together for one minute with the one you're seated with about whether, whether yes, whether you think there are a hundred boxes or not.*
Unraveling student input	Clarifying students' work or strategies. See, also Ellis et al.'s (2019) "figuring out student reasoning," described as "attempting to understand a student's solution, explanation, or reasoning." (p. 119)	Tom: *Square, yes. Who was going to draw the squares? You? . . . And then Esten was to draw the others? . . . So, one of you was going to make nine boxes and the other was going to make all the others?*

(*Continued*)

Table 3. (Continued).

Teacher move	Definition	Examples from the data
Acknowledging contribution	The teacher acknowledges that he has heard a student's contribution, e.g., by repeating it or by giving a short "yes." Often, the teacher provides no evaluation.	Emil: *No, it can't. Then, we can take up to four times ten, then.* Tom: *Eight, nine, and, then, you said ten.*
Assessing student effort	Giving the students feedback on their effort on the task or on their involvement in the discussions, often followed by instructions or commands to improve effort.	Tom: *Now, I hear Pia a lot, and that's really good, and Pia is learning a lot from this, but the rest of you who are silent, do you learn anything? . . . Anna, what do you think?*
Closed progress detail	Inspired by Drageset (2014). "Questions typically request details needed to move the process forward. These details can be process answers (one step at a time) or details about how the process should go on to reach the answer" (p. 293).	Tom: *How many are there in the muffin box having four in width and eight in length, then?* Tom: *Håvar found out that if you make one [box] that is four wide and six tall, then how many muffins does it contain? How many muffins does it contain?*
Correcting question	Inspired by Drageset (2014): "The comments in this category typically include a question from the teacher to redirect the student towards another approach. The questions therefore act as corrections" (p. 290).	Tom: *Can both boxes be 32 when one of them is twice as big as the other?*
Demonstrating argument	Making explicit a connected sequence of assertions intended to verify or refute a mathematical claim. See, also Ellis et al. (2019) "providing conceptual explanation."	Tom: *If there are 10 boxes that are one in width, there are 10 boxes that are two in width, there are 10 boxes that are three in width, there are 10 boxes that are four, 10 boxes that are five, and so on up to 10, then that must mean there are 100 boxes.*
Demonstrating strategy	Providing and making explicit a strategy to systematize student responses or move the class forward in solving the task. See, also Ellis et al. (2019) "providing alternative solution strategies."	Tom: *We have 10 . . . I write ones, and 10 twos. Do you understand what I mean by ones and twos? Then, I'll draw. By ones, we mean the boxes that are of width one . . . And all the boxes, whether they are this one* (draws 1 × 1) *or that one* (draws 2 × 1) *or this one* (draws 8x1, without counting), *I call them ones. Yes?* (erases) *While the two-boxes are those that are two in width* (draws 2 × 1).
Devaluing own authority	The teacher expresses mathematical uncertainty or indicates/states that his opinions are not better than the students'.	Tom: *I can tell you what I think, but it's not as if what I think is any more correct than anyone else's ideas here.* Tom: *Yes, I don't know. I'm a bit unsure. That's what I find tricky.*
Eliciting answer	Inspired by Ellis et al. (2019). The teacher asks a student or a pair of students to share their answer on a task.	Tom: *Pia and Molly?*
Eliminating justification	Encouraging the students to make a guess about an answer so that they have something to share.	Tom: *You're allowed to make a guess.*
Filtering	Inspired by Sherin (2002). The teacher follows up on or focuses on one (or some) of the students' input or statements, thus directing the ensuing whole-class discussion. Often includes explicitly connecting students' names to the statement.	Tom: *The first thing I must grasp, which I thought was a little fun . . . Esten and Magnus? [Yes?] You had distributed the boxes between you? [Yes.]* Tom (to the whole class) *Do you think most boxes are squares or rectangles?*
Ignoring accusation of being an authority	Refusing to make decisions or give the answer to a mathematical question in order to make the students develop the decision/conclusion. Characterized by "what is not said."	Student: *Tell us the truth, tell us the truth.* Student: *Can you tell us the truth?* Student: *What's right?* Tom: *We must vote by the show of hands, then.*
Indicating relationship	Pointing toward or hinting about a mathematical relationship or a condition that is crucial to progressing on the task.	Tom: *Mari, would you say that . . . here, you have one box* (hands it to Mari), *and then I get it back* (rotates the box 90 degrees and gives it back), *and now you're having a completely different box!*

(*Continued*)

Table 3. (Continued).

Teacher move	Definition	Examples from the data
Making mathematical digression	The teacher halts the progress by making the students focus on a detail in their reasoning, typically involving terminology or a definition that is not crucial for bringing the mathematical work forward.	Tom: *What do you call those quadrilaterals that are just as long as they are tall?* Tom: *What is the name of this you are pointing at now?*
Making conclusion	Drawing a conclusion about a solution to a sub-task or about a proposed strategy in order to move the class forward in solving the main task.	Tom: *So, these are all the boxes we can make that are four wide. . . . We have found out that we can make 10 boxes that are four wide.*
Making democratic conclusion	Making a conclusion or decision about a mathematical question or an ambiguous condition of the task by using a poll.	Tom: *Of those who raised their hands, there were many more who thought that it was only worth ordering one type of box, and then, there were two who thought that you have to order two different types of boxes because they are different. Then, I think most of you think that these are the same box.*
Managing opinion poll	Asking for a show of hands to support the conclusion or make a decision in order to move forward in solving the task.	Tom: *Okay, then, I wonder how many people here think that there must be a hundred boxes? Raise your hands* (five or six students raise their hands). *Yes. And then, there are some here who think that there are not a hundred boxes. Raise your hands* (a few students raise their hands).
Mapping out student response	The teacher announces that he will ask all students to share their solutions to a task they have worked on.	Tom: *But first, I wonder. . . how many different boxes do you think it is possible to make? Hmm, I'm going to ask all of you, so you can take down your hands.*
Open progress initiatives	Inspired by Drageset (2014): "initiates progress but still leaves it at least partly open to the students to choose or suggest which path to follow. . . . The comments are also aimed at moving the process forward, but without pointing out the direction" (p. 294).	Tom: So, how on Earth should we find out how many different boxes we can make? Tom: What do you say, should it be 24, as Karsten and Magnus are saying, or should it be 32, as Pia is saying? Hmm, Emil? Emil: I agree with 32.
Providing opportunity to revise answer	When conflicting claims arise, the teacher addresses the class or selected peers to revise their answers so that the class can reach agreement.	Tom: Karsten, what do you say? Karsten: I still say 24.
Referring to the frames of the task	The teacher repeats information or rules stated in the task to approve or reject the students' input.	Tom: But exactly this machine here cannot make, it cannot make hexagons. It only manages to make quadrilaterals.
Requesting justification	Inspired by Ellis et al.'s (2019) "pressing for justification." Asking the students "to explain why something works or to justify a mathematical idea, strategy or solution" (p. 123).	Tom: No, but you're sure you've drawn all [the different boxes]? How do you know you are drawing everyone, then?
Requesting review of peers' narrative	Inspired by Drageset (2014). Asking students to review or consider peers' solutions or strategies.	Tom: Mona and Siri, what do you think about Karsten's proposal?
Suggesting de-sophistication of strategies	Leading the class back to initial strategies, instead of building on what has been discovered so far; for example, a longer discussion in the third lesson about how to count the various boxes devolves into chaos. Tom decides to return to the beginning and requires the students to draw all possible boxes.	Tom: I wonder if you must draw all the boxes. . . . Now, you will get the following task, until the class is over. Can you draw all one-boxes, all two-boxes, three-boxes, four-boxes, five-boxes, sixes, sevens, eights, nines, and tens?
Turn-and-talk	Asking the students to evaluate a given claim in pairs. Inspired by Kazemi and Hintz (2014).	Tom: Talk together for one minute with the one you're seated with about whether, whether yes, whether you think there are a hundred boxes or not.
Unraveling student input	Clarifying students' work or strategies. See also, Ellis et al.'s (2019) "figuring out student reasoning," described as "attempting to understand a student's solution, explanation, or reasoning." (p. 119)	Tom: Square, yes. Who was going to draw the squares? You? . . . And then Esten was to draw the others? . . . So, one of you was going to make nine boxes and the other was going to make all the others?

Table 4. Teacher moves organized according to their potential to support mathematical reasoning and their associated authority structure. The numbers in parentheses refer to the frequency of each move during all three lessons. Moves written in regular text are related to authoring, while moves written in *emphasized text* are related to participating.

	Low potential for supporting mathematical reasoning	High potential for supporting mathematical reasoning
Teacher-led authority structure	• Closed progress detail (20) • Correcting question (19) • Making mathematical digression (3) • *Assessing student effort (8)* (Total frequency: 57)	• Demonstrating argument (2) • Demonstrating strategy (3) • Indicating relationship (9) • Making conclusion (6) (Total frequency: 20)
Shared authority structure	• Eliciting answer (28) • Making democratic conclusion (1) • Managing opinion poll (6) • Providing opportunity to revise answer (7) • Unraveling student input (30) • *Acknowledging contribution (29)* • *Devaluing own authority (11)* • *Eliminating justification (2)* • *Ignoring accusation of being an authority (9)* • *Mapping out student response (3)* • *Suggesting de-sophistication of strategies (3)* (Total frequency: 129)	• Open progress initiative (8) • Requesting justification (10) • Requesting review of peers' narrative (11) • Turn-and-talk (3) • Filtering (31) (Total frequency: 63)

new narratives to push the reasoning further?). To this end, we considered the students' input and participation as a context, as well as the teacher's previous and further actions and answers on the questionnaire. Additionally, we used respondent validation by asking Tom to read a preliminary version of the paper. However, we acknowledge that the interpretations made are a possible threat to credibility and thus represent a limitation of the study.

Moreover, we emphasize that the aim of the analysis is to investigate *the case of Tom* in more detail, which affects the transferability of the study. Due to the limited data and timespan, we are unable to describe Tom's teaching as well as teacher moves and their potential to support mathematical reasoning and the associated authority structures more generally. Despite these limitations, we claim that our conceptually grounded case study analysis, in which we have accounted for connectedness to theory and previous research, illustrates or hint at dilemmas in teaching to support students' mathematical reasoning and shared authority. Thus, we assume that the context-dependent knowledge produced in this study enables "fuzzy generalizations" (Bassey, 1999) as "it reports that something has happened in one place and that it may also happen elsewhere" (p. 52).

Finally, a trustworthy study also rests on dependable instruments and confirmability in the analysis of the data, which in this study is pursued by making the research process and related interpretations open to critical review. Accordingly, we have aimed for providing sufficient examples from data in both Table 3 (of teacher moves in Tom's mathematics teaching) and in the upcoming section on findings.

Findings

In this section, we first consider each of the four categories presented in Table 4, explaining why moves are placed in a given category. Then, we present chronological data excerpts from Tom's classroom to further illuminate how the different moves play out in the discussions. Moreover, the excerpts indicate some patterns that may be present in Tom's interactions with his students, which again gives insight into the relationships between supporting students' mathematical reasoning and shared authority. Before presenting and discussing the findings, we stress that a single move does not determine whether a longer sequence of a mathematical discussion generates a low or high potential to support students' mathematical reasoning. Importantly, we align with Ellis et al. (2019), who claim that different types of moves are critical in developing opportunities to extend student thinking and that high-potential moves do not always result in an improved mathematical discussion or deeper student thinking. Moreover, a move is assessed based on how it emerges in the data (as described in Table 3), not how it could play out when used differently (e.g., by a different teacher).

Overview of the four categories

In the two categories reflecting low mathematical reasoning potential, we note a common feature: although these moves might, in some situations, support or hinder mathematical reasoning, we consider them to be independent of the subject or activity being taught; they are likely to appear in whole-class discussions outside mathematics classrooms.

In the category of moves considered to have a low potential to support students' mathematical reasoning while reflecting a teacher-led authority structure, we identified five teacher moves. Four of them were related to the authoring characteristics of authority. With the moves *closed progress detail, correcting question*, and *making mathematical digression*, Tom took control of the process by steering the students in a certain direction, initiating a step-by-step process for solving the task, or evaluating the students' responses. Moreover, the questions Tom provided within each of these moves are closed (i.e., they have one correct response). By attending to details in the students' responses, these moves may reduce the problem's complexity, thus limiting their support for students' mathematical reasoning (see also, Drageset's (2014) commentary on his similar categories). The move of *referring to the frames of the task* was related to a mathematical task in this case; however, the instances of the move focus more on the story of Muffles and his box-making machine than on important mathematical relationships, so the move is classified as having a low potential to encourage mathematical reasoning. Because Tom chose the task and assigned it to the class – meaning that he authored the limitations presented for the students' investigation – the move reflects a teacher-led authority structure. The final move in this category was related to the participating characteristics of authority. Tom's move of *assessing student effort* involved instructions to do something so as to meet his own demands.

In the next category, we find moves considered to have a low potential to support students' mathematical reasoning that are related to a shared authority structure. Here, we identified eleven moves, making this the largest category in terms of number of moves, as well as the total frequency of observed moves. The moves related to authoring in this category are *eliciting answer, making democratic conclusion, managing opinion poll, providing opportunity to revise answer*, and *unraveling student input*. With these moves, the students are invited to choose and present their strategies, producing and evaluating narratives, and there is a choice of what actions to take next in solving the task. Thus, they reflect shared authority. However, we claim that none of them has a high potential to support mathematical reasoning. Regarding *making democratic conclusion* and *managing opinion poll*, we note that they are relatively subject-independent, as mentioned above. *Providing opportunity to revise answer* is not connected to justification and, thus, is not expected to bring forward any mathematical reasoning processes. The move *unraveling student input* seems to be used when Tom does not immediately understand a student's answer, and in contrast to *filtering*, the mathematics involved may be erroneous or unsuited for the class's collaborative reasoning. However, in other situations, we acknowledge that the moves *providing opportunity to revise answer* and *unraveling student input* could have a higher potential to support mathematical reasoning. The remaining six moves in the category are related to the participating characteristics of authority. *Acknowledging contribution* was present in Tom's consistent acknowledgment of students' contributions, often using their names. We interpret this move's main function to be to acknowledge students' participation in the discussion because it often happens that the students' contributions are not followed up on or subject to evaluation. Thus, the contributions' roles as narratives in the discourse seem minor compared to their roles as "contributions from someone." Moreover, the moves *devaluing own authority* and *ignoring accusations of being an authority* concern discussing (or obviously not discussing) authority with students – without using the word *authority* yet questioning who is in a position to make decisions – and we thus regard them as moves toward sharing authority. The moves *eliminating justification, mapping out student response* and *suggesting de-sophistication of strategies* are used to encourage students to participate in the discourse, with the mathematical content of the students' contributions seemingly being inferior to their participation.

In the two categories reflecting high mathematical reasoning potential, we note that all moves relate to the authoring characteristics of authority. Four of them, which rarely appear in Tom's management of the whole-class discussions, are moves with a high potential for mathematical reasoning. Here, we consider the moves *indicating relationship, demonstrating strategy, demonstrating argument*, and *making conclusion* to have a high potential to support mathematical reasoning because they offer hints, potential strategies, or justifications for the students to implement, preferably by focusing on why something might work (see also, Ellis et al., 2019). However, these moves reflect a teacher-led authority structure because the teacher is the main author of the strategies and justifications.

By contrast, we find that five of Tom's moves have a high potential to support mathematical reasoning and are associated with a shared authority structure: *filtering, open progress initiative, requesting justification, requesting review of peer's narrative*, and *turn-and-talk*. These are used more frequently than those in the previous category. The moves *open progress initiative* and *requesting review of peer's narrative* involve open, teacher-posed questions such as "How can we solve?" "How can we think or reason about?" and "How do we know?" (see also, Drageset, 2014), inviting the students to search for similarities and differences and validate mathematical statements (Jeannotte & Kieran, 2017). In addition, the students are invited to choose and present their strategies or evaluate others' suggestions, reflecting shared authority. The move *turn-and-talk* is only used in the data material whenever there is a given narrative the students should evaluate. Hence, we interpret its function to be to invite students to evaluate the narrative at hand, which points to an authoring characteristic of shared authority. Although the moves *filtering* and *requesting justification* may require a prominent teacher voice, they nevertheless refer to or invite the students to act as authors in the discussion; their contributions are valued and direct the whole-class discussion. Moreover, these moves focus the students' work on important ideas within the shared reasoning context and toward key mathematical reasoning processes (e.g., justification). Thus, we argue that they have a high potential to support mathematical reasoning.

Glimpses of Tom's classroom

Here, we choose to present excerpts from Lessons 2 and 3 because Lesson 1 mainly involved individual work or work in pairs and few whole-class discussions on the students' strategies or solutions. The excerpts illuminate characteristic features in Tom's use of moves. Lesson 2 began with a short recap of the task by Tom, followed by 30 minutes of students' continued work on the task. Then, a 30-minute whole-class discussion filled the remainder of the lesson. In Lesson 3, the class continued the discussion of the task for about 55 minutes. Then, they spent the last 15 minutes on work in pairs. That is, almost two-thirds of the two lessons were spent on whole-class discussions. In the following, we present data excerpts chronologically so that the reader can follow the progress of the class's work. The numbering in the transcripts continues throughout the lessons.

The first excerpt illustrates Tom's use of the move *mapping out student response* at the beginning of Lesson 2. This excerpt is from immediately after the students worked on the task for the first 30 minutes of the lesson. In the excerpt, we see consecutive uses of *eliciting answer* (Tom truly asks all of the students, as he states in line 136). This move is sometimes followed by the move *eliminating justification* (lines 156 and 168), in which students who do not have a ready answer are encouraged to make a guess about the number of arrays.

136	Tom:	I know that there are many of you who really, really want to share what you have found out, but first, I wonder . . . how many different boxes do you think it is possible to make? Hmm, I'm going to ask all of you, so you can take down your hands. Emil and Håvar?
137	Håvar:	Yes, we came up with two answers.
138	Tom:	Two answers. OK.
139	Håvar:	Since, if you count those having height, having similar, then . . .
140	Emil:	Can't we draw it? Can we draw it? Then, it's easier to explain.
141	Tom:	Yes, we can take [inaudible] for now, and then, we explain it afterward, since . . .
142	Emil:	OK, the numbers are 361 and . . .
143	Håvar:	Wait, I'm checking! I'm checking!
144	Emil:	I remember, Håvar
145	Håvar:	No, I'm checking. It's 361 and 181 (*Emil and Håvar in chorus*) [. . .]
156	Tom:	Mari, Ida, and Eline, what do you think? How many do you think there are? Boxes? . . . You can make a guess. . . . Should we move on?
157	Mari:	[inaudible]
158	Tom:	140? OK, Joakim and Maja, how many do you think there are?
159	Joakim:	Think, about what?
160	Tom:	How many boxes do you think it is possible to make, how many different boxes?
161	Joakim:	I don't know. I just made drawings [inaudible].
162	Tom:	Pia and Molly?
163	Pia:	129!
164	Tom:	Karsten and Morten?
165	Karsten:	Not sure.
166	Tom:	You are a little unsure? Mona and Siri?
167	Mona:	Hmm, we haven't thought about that.
168	Tom:	You haven't thought about that? You're allowed to make a guess.
169	Student:	Neither do we.
170	Tom:	Yes. . . . Esten and Magnus?
171	Esten:	181.

In this situation, many students take part in the discussion (probably more than if students had to volunteer their answers). However, it seems evident that the social goal of including all students is more important in this part of the discussion than providing explanations and justifications, as seen by Tom's unwillingness to let Emil and Håvar explain their numbers (line 141) and Tom's explicit promotion of guessing (lines 156 and 168). In the current episode, the discussion continues, with Tom addressing a student's strategy of distinguishing between square and rectangular boxes when drawing them (*filtering*). This leads to the move *managing opinion poll* (line 199) because he invites the students to vote on whether most of the boxes are squares or rectangles. The students disagree regarding this question, and many seem to think that the number of squares and the number of non-squares is the same. Tom provides a *correcting question* (line 201), but when the class does not respond to this, he hands the task back to the students (*open progress initiative*, line 206), without more help.

199	Tom:	Do you think most boxes are squares or rectangles? Raise your hand, those of you who think most boxes are squares. Hmm, raise your hand, those of you who think most boxes are rectangles. 50/50?
200	Student:	It's just as much.
201	Tom:	Are there as many [squares as rectangles]?
202	Student:	No!
203	Student:	Well, in just this one.
204	Tom:	Hmm, let's see.
205	Student:	There are just as many [squares as rectangles]!
206	Tom:	So, how on Earth should we find out how many different boxes we can make? Is there anyone having an idea how?

We claim that the move *managing opinion poll* does not facilitate mathematical reasoning processes ("guessing" is not a mathematical reasoning process, and in this case, many students are likely to guess because this question is not something they have worked on previously). In this situation, the opinion poll yields no conclusion and is followed by the move *open progress initiative*, concerning the counting of all possible boxes (line 206). This move was identified above as a move with a high potential to support mathematical reasoning while reflecting a shared authority structure. However, one could wonder whether the open progress initiative, in this example, is perhaps a bit too open to effectively foster students' reasoning. The lesson ends with a systematic counting task on the interactive whiteboard involving all arrays that are four units wide, and the student Pia suggests that there will be 100 unique arrays.

The work of finding a strategy to systematically count the arrays continues into Lesson 3. The question of how to count equal arrays with different orientations, such as 4×3 and 3x4, arises, and the class spends most of the lesson discussing the issue and considering what consequences the potential conclusions will have regarding the total number of arrays. However, despite making several attempts to involve his students in negotiating the conditions of the task, the class made limited progress and began demanding the teacher's decision regarding how to proceed, as shown in this excerpt from the third lesson:

496	Tom:	What Emil and Håvar are asking is whether these are different boxes, or are they the same box?
497	Håvar:	It is the same, but then, we wondered, should we count as if they were different, in a way? Then, we didn't get any answer, so that's why we had to find two answers.
498	Tom:	Yes, so that is the big question, is this one, or are they two different kinds of boxes? (*Tom observes several hands in the air*) Talk to the person next to you, and if you think this is two, tell why, and if you think this is one, tell why.

Here, Tom starts by *filtering* the question raised by Emil and Håvar, focusing the class's attention on this issue. Håvar comments that they "didn't get any answer" (from the teacher, when they worked in a previous lesson), so they "had to find two answers" (line 497). Tom does not address this at all in his next turn, thus *ignoring accusation of being an authority*. Instead, he asks the students in the class to determine the answer, using *turn-and-talk* (line 498) and also *requesting justification* (line 498). After the students have worked on this question for some minutes, Tom uses the move *mapping out student response* and, again, *eliciting answer* from all pairs of students. Again, this does not yield a consensus in the class, and finally, Tom asks Håvar and his partner in this lesson, Magnus, a question.

549	Tom:	Magnus and Håvar, what do you think?
550	Student:	We don't really think anything. Then, you can just find both answers, and then, you can just go and ask your teacher, and then, the teacher is supposed to answer . . .
551	Tom:	Since everything the teacher says is true?
552	Magnus:	[holds up a pair of erasers] It's the same! True, true, true, done! [turns to the teacher]
553	Tom:	You mean it's the same? [Magnus holds the erasers up to the teacher's face] If the teacher doesn't say something else, because, then, it's the teacher who tells the truth, and then, that's fine.
554	Student:	Tell us the truth, Tom! I've been asking you for days!
555	Student:	Tell us the truth! Tell us the truth! [. . .]
558	Student:	Yes, what's correct? Now that we've been working so hard, you must tell us!

In this part of the discussion, many students are shouting at the same time, so we were not able to accurately transcribe the students' names. Tom does not "tell them the truth." Instead, he is repeatedly *devaluing own authority* (lines 551 and 553). He seems to be very determined to let this decision rest with the students because, next, he uses the move *managing opinion poll* (beginning of line 586), this time leading to *making democratic conclusion* (toward the end of line 586). Because no mathematical explanation or justification is requested in making this conclusion, we see that this move reflects a low potential to support mathematical reasoning.

582	Tom:	Does Muffles need to order 50 boxes that are one wide and two long and 50 boxes that are two wide and one long?
583	Several students:	No.
584	Tom:	Yes, do we all agree on that?
585	Several students:	Yes. (*a few students disagreeing*)
586	Tom:	How many of you think that he needs to order different boxes (*waiting while some students are raising their hands*)? How many of you think that he can order just one box and, then, he can turn around those that he wants to turn around (*several students raising their hands*)? . . . Of those who raised their hands, at least, there were many more who thought that it was only worth ordering one type of box, and then, there were two who thought that you have to order two different types of boxes because they are different. Then, I think most of you think that these are the same box in that you think it's okay to order just one. Is there any difference for those who get the muffins, then?
587	Several students:	(*meekly*) No . . .
588	Tom:	No-o.

It seems to be Tom's intention to establish that differently oriented boxes should be counted as the same box (mathematically, this makes sense because the shapes are congruent, but within the setting of the task one could also argue – practically – for the other option). However, instead of telling the students from the beginning or the first time he was asked, Tom chooses to spend a great deal of time on this discussion. We interpret this as genuine effort to establish shared authority in the classroom using moves such as *mapping out student response, eliciting answer, managing opinion poll, turn-and-talk*, and *making democratic conclusion*. In the next excerpt, we provide examples of moves reflecting teacher-led authority (Tom states, "I think we need to, to get a little ahead" in line 679, thus indicating that he will take more authority). After agreeing that orientation does not matter, the class returns to the original task, now with the conditions set. The next question that arises is whether squared arrays should be added to the 100 arrays conjectured by Pia in Lesson 2. In Tom's contributions to this discussion, we find examples of *demonstrating strategy* (lines 679 and 681) and *demonstrating argument* (line 691). Despite Tom's attempt to demonstrate an argument showing that the squared arrays are included in the 100 conjectured arrays, some of the students insist that the squared arrays are not included (although they are now correcting 110 arrays to 109). Thus, the students seem to hold to their own (incorrect) ideas rather than adopting Tom's argument.

678	Student:	Now, I don't know whether there are 109 or 110 [boxes].
679	Tom:	We must find that out (*moving toward the whiteboard*). However, I think we need to, to get a little ahead. We have ten . . . I write ones, and ten twos (*writing "10 ones" and "10 twos" on separate lines on the whiteboard*). Do you understand what I mean by ones and twos (*students mumbling*)? Then, I'll draw . . . By ones, we mean the boxes that are of width one (*Tom drawing a 4 × 1 box*). For example, this box is one here and four here. And all the boxes, whether they are this one (*drawing 1 × 1*) or that one (*drawing 2 × 1*) or this one (*drawing 8x1, without counting*), I call them ones. Yes (*erasing the drawings*)? While the two-boxes are those that are two in width, if it looks like this (*drawing 2 × 1*).
680	Student:	But this we have made!
681	Tom:	Or this long (*drawing 4 × 2*) or this long (*drawing 9 × 2*). Then, they are still twos, since they are two [units] wide (*erasing the drawings*). Because it's something about . . . let me see. Now, I just write . . . one, two, three, four, five, six, seven, eight, nine, ten . . . twos, threes, fours, fives, sixes, sevens, eights, nines, tens. And then Pia says that this is not 100 [boxes] because they have already made one. You drew all the squares too. So, the squares come in addition to these?
682	Pia:	I think there are 109 [boxes].
		[. . .]
685	Tom:	Emma, what do you think?
686	Emma:	(*not responding*)
687	Tom:	About the boxes, we know that we have 10 different boxes for each width, but what do we do now?
688	Emma:	Hmm . . . I don't know.
689	Tom:	Don't know? Because, if that's what we have, then we will have 100 boxes [in total]. And then Pia says that she has already drawn this one (*drawing 1 × 1*) when she drew all the square boxes. Should it be . . . how many square boxes do we have, boxes that are as long as they are wide (*drawing 4 × 4*)?
690	Student:	We have 10!
691	Tom:	We have 10? Yes. It must be, since it is one and two and three and four, five, and six and so on. Do the others come in addition?
692	Student:	Yes.
693	Student:	So, then, we have 109 [boxes].

Toward the end of the lesson, Tom attempts to steer the students toward realizing that the squared arrays are among the 100 conjectured arrays, and he uses the move *providing opportunity to revise answer* (line 736). However, because this does not work (Pia stays with her conclusion), he decides to return to the initial drawing of the various arrays. This is coded as *suggesting de-sophistication of strategies* because the arguments and conclusions made at this stage are put aside while the class, in a way, returns to the beginning of the lessons.

731	Tom:	I wonder if we must draw every [box] on the board in the end.
732	Student:	Yes, do it!
733	Student:	Do you know the answer yourself?
734	Tom:	You will never know.
735	Student:	I have seen the answer.
736	Tom:	But Pia, do you still think that the 10 square boxes are 10 boxes other than those already standing over there (*pointing toward the list of boxes at the whiteboard*)?
737	Pia:	Yes (*shrugging her shoulders*).
738	Tom:	Yes, even if the one that is four times four is among those fours. Hmm, I don't think we get much more from you today. I wonder if I should let you draw all the boxes . . .
739	Student:	Can't you just say the answer?
740	Tom:	. . . in your book. No, you learn nothing from that, I think.
741	Student:	But we have learned so much already.
		[. . .]
743	Tom:	Now, there's 15 minutes left of the lesson – exactly 15 minutes. Now, you will get the following task, until the class is over. Can you draw all the one-boxes, all the two-boxes, the three-boxes, the four-boxes, the five-boxes, sixes, sevens, eights, nines, and tens?

Discussion

In the previous section of this paper, we presented findings regarding the following research question: In a teacher's orchestration of a whole-class mathematics discussion, what relationships exist between managing mathematical reasoning and managing shared authority?

To answer the research question, we have made use of a case study that we have determined to be novel and significantly distinct from comparable earlier studies. Like Hamm and Perry (2002), Otten et al. (2017), and Ng et al. (2020), we identify how authority structures are at work in a mathematics teacher's classroom. Hamm and Perry's (2002) study is a multiple case study of six first-grade teachers who all except one positioned themselves as the sole mathematical authority in their classrooms. The authors claim that their findings support the assumption that children learn very early in their educations and that mathematics is a discipline to which students have little to contribute. Similarly, Otten et al.'s (2017) case study, which focuses on the ways that authority can manifest during whole-class proving, reveals similar and somewhat expected findings: authority is concentrated in the teacher or the textbook. Moreover, Ng et al. (2020) describe a teacher who was previously very authoritarian but later employed well-known teacher moves in her teaching. They observed a shift toward a shared authority structure, yet not all the teacher moves worked toward that aim. By contrast, we present, in our study, a teacher who almost never assumes authority. Indeed, Tom explicitly attempts to avoid doing so on some occasions. Also, he employs several moves that are well described in the literature on mathematical reasoning, and he purposely attempts to work toward a shared authority structure. Thus, we claim the case of Tom to be a contribution to the research literature, illustrating the complexities of teaching mathematical reasoning from a new perspective. Consequently, the case also provides insights into the relationships we seek to study.

In the three lessons, Tom and his group of fourth graders struggle to reach a conclusion to the task at hand, a lack of mathematical progress that can be explained based on the relationships queried in the research question. We find evidence that Tom recognizes student contributions with a potential for moving the mathematical work forward, but when the students themselves are unable to recognize and take advantage of their contributions, the class's collaborative reasoning stalls. In light of our double analytic approach, we suggest three reasons for this stagnation of progress: the sparse use of high-potential mathematical reasoning moves, not enough moves that reflect teacher-led authority, and too much use of moves that relate to the participating characteristics of authority. We discuss these potential causes in detail before summarizing the study.

From Table 4, we see that 83 of the moves employed by Tom reflect a high potential to support mathematical reasoning, as compared to 186 moves reflecting a low potential to support mathematical reasoning. Although we did not analyze the data with respect to patterns of eliciting, responding to, facilitating, and extending moves (Ellis et al., 2019), we assume that there will be cycles of "ideal" patterns in the material. Tom usually begins by eliciting many answers and then filters these so as to move toward ideas that will push the reasoning forward. Then, he attempts to facilitate reasoning and mathematical agreement. However, he often hands the decision-making over to the students, employing moves that reflect low support for mathematical reasoning (e.g., *making democratic conclusion* or *suggesting de-sophistication of strategies*). Only occasionally does he interfere as a pronounced mathematical participant in the discussion by *demonstrating argument, demonstrating strategy*, or even *making conclusion*. One could imagine that the extended use of these moves (or others serving the same goal) could help to move the discussion forward. These moves reflect a teacher-led authority structure, so using them could "fix" both the problem of too few moves with a high potential for mathematical reasoning and the problem of too little teacher-led authority, allowing the teacher to be what Ben-Zvi and Sfard (2007) describe as a more competent doer of mathematics. Thus, the few instances of these moves may imply an imbalance between encouraging students' authoring and ensuring their progress in mathematical work. On the other hand, an approach to teaching using more moves related to teacher-led authority could also result in undermining the shared authority structure in the classroom. In any case, the relationship between supporting students' mathematical reasoning and shared authority is more complex than a zero-sum-game, meaning that pursuing mathematical reasoning will not weaken the shared authority structure accordingly, and *vice versa*. This is made apparent when we turn to the third potential reason for Tom's struggles: the use of moves that relate to the participating characteristics of authority too frequently.

Sixty-five of the moves Tom employs are related to the participating characteristics of authority. This is approximately one-quarter of the total moves. In particular, the moves *suggesting de-sophistication of strategies* and *eliminating justification* seem to have the potential to *hinder* reasoning, although they are used only five times in total. Moreover, it seems somewhat natural that the authoring moves are, in general, more productive in terms of fostering mathematical reasoning because they relate to the construction and substantiation of narratives – what mathematical reasoning is all about (Jeannotte & Kieran, 2017). According to Sfard's commognitive theory and other discourse-centered theories of learning, participating in mathematical discourse is how one learns mathematics, eventually individualizing the discourse and becoming "able to employ the discourse agentively" (Sfard, 2019, p. 90). One well-known aspect of commognition is the idea that, along the way to this goal, learners must imitate more competent participants' actions and, when doing so, the goal of the learner is usually to be accepted in the community, rather than working with narratives (Sfard, 2008). It is possible to view Tom's use of moves that relate to the participating characteristics of authority as something that can support students in such imitation by pushing them to participate in the discussion. This means that when teaching novices in terms of mathematical reasoning, we must accept that "pretending" to do mathematical reasoning is a step toward learning it and, indeed, that ensuring participation is a prerequisite for ensuring mathematical reasoning. At the same time, in the case of Tom's classroom, one could object that Tom's sparse use of his own authority makes it more difficult for the students to know what to imitate. Of course, one could argue that Tom is stressing the participation issues too rigidly: when orchestrating mathematical discussions, there is a distinction between providing students with a chance to be heard and demanding that they be heard. Nevertheless, we claim that moves that reflect the participating characteristics of authority are important in the process of learning mathematical reasoning, rather than only being important as social goals.

Looking beyond what is evident in the data, we find a fourth potential reason for Tom's struggles, namely that he is working to establish norms as a new teacher. We have previously commented on an episode in Tom's teaching (see lines 549–558 in the previous section) in which the students demand a final settlement from him after they have "been working so hard" (line 558). On several occasions, Tom is *ignoring accusation of being an authority* or *devaluing own authority*, as seen in this short episode. Tom has recently begun teaching this group of students. Like the teacher Mark in the case study by Wagner and Herbel-Eisenmann (2014a), he may be particularly aware of issues of authority when establishing norms and expectations in this situation. From our limited data, we cannot tell how this will ultimately play out, but while Tom can work toward modifying his role as the expert authority in the classroom (e.g., by refusing to judge the validity of his students' mathematical claims), he is unlikely to avoid being perceived by his students as an authority figure. As Amit and Fried (2005) point out, authority is a complex web, which also includes the Weberian component of *charismatic authority*, typically held by teachers. This is distinct from the more obvious teacher characteristics of expert and traditional authority, and it is not obviously connected to subject matter. Thus, we assume that these kinds of conflicts may continue to arise in Tom's classroom, although their frequency and content may change over time.

We summarize that several relationships exist between managing mathematical reasoning and managing shared authority. The authority aspects that relate to authoring have obvious connections to mathematical reasoning, but the participating characteristic of authority may also support mathematical reasoning. A lack of moves associated with a teacher-led authority structure may slow the discussion and give the students less chance to interact with a competent doer; at the same time, such a situation can support shared authority and help to ensure that all students, at some level of imitation, take part in the discussion. Mathematical reasoning, like all mathematics, is not something that is learned here and now but something that takes time, and our limited study provides just a glimpse into these complex mechanisms.

Conclusion

In this study, we have extended the previous literature on supporting students' mathematical reasoning by co-focusing on the topic of authority. We claim that this has resulted in new perspectives on social aspects of teaching mathematical reasoning as compared to usually present in the literature. It has provided new insights into the complex relationships between managing mathematical reasoning and shared authority. We suggest that the distinction between the authoring and participating characteristics of authority is useful to describe and discuss these relationships. At the same time, we admit that our approach to authority is simplified because we do not include models of authority that extend the teacher-student dichotomy. Moreover, we have chosen to focus on the teacher and what he does to manage the mathematical reasoning and shared authority, but naturally, most participants in the discourse are not the teacher. They are students, all with their own voices in the classroom. Our study, with its relatively small amount of data, is too limited to capture these aspects. Nor does the study provide enough information about how to support students' mathematical reasoning while establishing and maintaining a shared authority structure. Thus, more studies are needed to expand our findings into a more robust framework of teacher moves that can support both of these ends. Nevertheless, we believe the case of Tom to be a valuable contribution to the research field because the double lens of mathematical reasoning/authority is novel and exhibits tensions that were already indicated but not well-documented in the literature. Because the consensus goal among mathematics education researchers – and thus, among mathematics teacher educators – is to promote student-centered teaching (Franke et al., 2007; Hufferd-Ackles et al., 2004), Tom's struggles are expected to manifest themselves in several mathematics classrooms.

Notes

1. Wagner and Herbel-Eisenmann do not describe teacher-led authority. Their category *personal authority* fits nicely, however, into our construct of teacher-led authority. Moreover, we view their categories *discursive inevitability* and *discourse as authority*, when employed by the teacher, as teacher-led authority structures. We associate their category *personal latitude* with shared authority.
2. ProPrimEd – Reasoning and Proving in Primary Education. The project is a collaboration between the Norwegian University of Science and Technology (NTNU) and Trondheim municipality and is partly funded by the Norwegian Research Council. For more information about the research project: https://www.ntnu.edu/ilu/proprimed.

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Appendix

Questions sent to the teacher

- Can you make a short summary of your education and teaching experience in mathematics?
- For how long have you been the mathematics teacher for the class we visited?
- How would you describe your mathematics teaching in fourth grade (before the COVID-19 lockdown)?
- How did you plan your mathematics teaching (before the COVID-19 lockdown)?
- How do you think your students would describe you as a mathematics teacher?
- Think of a successful or good mathematics lesson with the class. What characterizes such a lesson?