Simon Farstad Østraat

Gravitational Waves

An Introduction

Hovedoppgave i MSPHYS
Veileder: Jens Oluf Andersen
Januar 2022

Hovedoppgave

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Summary

In this thesis we will first the weak field limit of Einsteins field equations and the basic formalism for GWs in the linearised theory. Further we will take a step outside the linearised theory in order to able to find the stress-energy tensor of propagating waves. Next we consider non-relativistic binary systems to the lowest order moment, the quadrupole moment, and find the power and amplitude of the radiation which the system emits. Lastly we consider how experiments such as LIGO is able to detect the GW despite their extremely small amplitude.

Sammendrag

I denne oppgaven vil vi først se den svake feltgrensen til Einsteins feltligninger og den grunnleggende formalismen for gravitasjonelle bølger i den lineariserte teorien. Videre vil vi ta et skritt utenfor den lineariserte teorien for å kunne finne spennings-energitensoren til forplantende bølger. Deretter vurderer vi ikke-relativistiske binære systemer til momentet av laveste orden, kvadrupolmomentet, og finner kraften og amplituden til strålingen som systemet sender ut. Til slutt vurderer vi hvordan eksperimenter som LIGO er i stand til å oppdage gravitasjonelle bølger til tross for deres ekstremt små amplituder.

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Abbreviations

LFE	=	Linearised field equations
GW	=	Gravitational wave

Chapter 1

Introduction

Before Einstein would publish his theory of special relativity there was a contradiction between two seemingly fundamental physical theories, Newtonian mechanics and Maxwells equations. In Newtonian mechanics the addition of velocities in different inertial frames is additive, which poses a problem when one consider Maxwells equations which state that electro-magnetic waves travel at a constant speed regardless of inertial frame. Einstein solved this by combining the concepts of space and time and using Lorentz transformations to move from one reference frame to another. One shortcoming of this theory was the assumption of inertial frames of reference, it could not describe what happens in an accelerating frame. Thus the quest to expand the theory of special relativity into a more general theory began. For Einstein it started with him having the “happiest thought of his life”, he realised a person freely falling from a roof top will have no gravitational field in their vicinity, everything the person drops during the fall would stay next to them, the person can therefore interpret themselves to be at rest. This thought led to the equivalence principle, stating there is no difference between a gravitational and inertial mass, in other words being affected by gravity is the same as being accelerated. In 1915 Einstein published his theory of general relativity where he derived his field equations, which describes the effect of gravity as a curvature in space-time using Riemannian geometry.

Just one year later in 1916 Einstein solved the field equations approximately for a weak gravitational field, where he found the solution to be of the form of the wave equation, thus predicting the existence of gravitational waves [1][2]. The search for experimental proof of the existence of these waves begun the 1960’s with James Weber, where he hoped to detect them using the resonant frequency of a mass as the wave passed through, but it did not yield any conclusive results [3]. It was not until 2016, 100 years after Einstein predicted their existence that they would be proven to be real by the LIGO (Laser Interferomter Gravitational Observatory) [4]. A discovery that granted Rainer Weiss, Barry C. Barish, and Kip Thorne the Nobel prize in physics in 2017 “for their decisive contributions to the LIGO detector and the observation of gravitational waves”. The discovery and continued observation of gravitaional waves will allow astronomers to view our universe through a new lens, and further research some of the more extreme cases of gravity, like black holes and neutron stars. In this thesis we will elaborate on the properties of gravitational waves, especially how they propagate, how they are formed, and how they affect binary systems.

Chapter 2

The Einstein Field Equations

For a long time the laws that governed gravity were expressed in terms of forces as derived by Isaac Newton, it was sufficient for our lives on Earth but was incomplete. A more complete theory was derived by Albert Einstein, expanding on the theory of special relativity he formulated a geometric explanation of gravity, namely the general theory of relativity. The central result was the Einstein field equations, and from them arose the new concepts of black holes and gravitational waves. It is the latter we shall look at, which saw its light through the linear solutions of the field equations.

2.1 Derivation of The Field Equations

In order to find the linear solutions to Einsteins field equations we First need to derive them, which we will do using a variational approach. The derivation follow closely the work done in [5].

We begin by looking at the variational principle of classical mechanics, namely the principle of least action. It states that a systems trajectory between a configuration at an initial time and a final time is such that the action, S, of the system is stationary. This principle can be extended from one of discrete particles into one of continuous fields, as is a natural starting place for finding the equations for a gravitational field. For a set of fields, Φ^a, we have that the action, S, on a general space-time manifold is given by an integral over the Lagrangian density, $\mathcal{L}$, which depends on the fields and their derivatives. We thus have the following expression for the action over a region $\mathcal{R}$

$$S = \int_{\mathcal{R}} \mathcal{L}(\Phi^a, \partial_\mu \Phi^a, \partial_\mu \partial_\nu \Phi^a, \ldots) d^4x. \tag{2.1}$$

By demanding that the action is invariant under small variations of the fields, $\delta S = 0$, $\Phi^a \to \Phi'^a = \Phi^a + \delta\Phi^a$, and that the variation, $\delta\Phi^a$, vanishes on the boundary of the region $\mathcal{R}$, we are able to derive the Euler-Lagrange equations for a general field in space-time. By considering up to second order derivatives of $\mathcal{L}$ we get

$$\frac{\delta \mathcal{L}}{\delta \Phi^a} = \frac{\partial \mathcal{L}}{\partial \Phi^a} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \right] + \partial_\mu \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \Phi^a)} \right] = 0, \tag{2.2}$$

where $\delta\mathcal{L}/\delta\Phi^a$ is the variational derivative.

2.1.1 Field Equations in Vacuum

As alluded to above, in order to derive the field equations we need to find an action that will allow us to derive the equations of gravity in a vacuum. As physical theories should generally be covariant, the action, S, has to be a scalar under general coordinate transformations, which also means that the Lagrangian must be a scalar field at each point in the region $\mathcal{R}$. By using the invariant volume element $d^4V = \sqrt{-g}d^4x$, where g is the determinant of the metric tensor $g_{\mu\nu}$, we can express the Lagrangian density as $\mathcal{L} = L\sqrt{-g}$.
As $\sqrt{-g}d^4x$ is an invariant scalar field our task becomes to finding an L which is a scalar under coordinate transformations. In order to make sure this L describes a gravity we also want it to depend on the components of the metric tensor, $g_{\mu\nu}$. We thus end up with the Ricci scalar, R, which is the simplest non-trivial scalar that depends on the metric and its derivatives. It is also the only one derivable from the metric tensor which does not depend on derivatives higher than the second order. By putting this together we end up with the Einstein-Hilbert action, S_{EH}.

$$S_{\text{EH}} = \int\limits_{\mathcal{R}} R\sqrt{-g}d^4x. \tag{2.3}$$

So by defining the Lagrangian density as $\mathcal{L} = R\sqrt{-g}$, we see that (2.3) is of the same form as (2.1). This in turn means that we can write the variational derivative of the system, $\delta\mathcal{L}/\delta g_{\mu\nu}$, in the same form as (2.2). The resulting equation is a tedious procedure to calculate, so instead we look at the variation in the action the arises by varying the metric tensor:

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu},$$

where $\delta g_{\mu\nu}$ and its derivative vanish on the boundary, $\delta\mathcal{R}$, of the region $\mathcal{R}$.
By writing the Ricci scalar in terms of $g_{\mu\nu}$ we get the following expression for the variation in the Einstein-Hilbert action

$$\delta S_{\text{EH}} = \int\limits_{\mathcal{R}} \delta g_{\mu\nu} R_{\mu\nu}\sqrt{-g}d^4x + \int\limits_{\mathcal{R}} g_{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}d^4x + \int\limits_{\mathcal{R}} g_{\mu\nu}R_{\mu\nu}\delta(\sqrt{-g})d^4x. \tag{2.4}$$

This we can define as a sum of three variations, $\delta S_{EH} \equiv \delta S_1 + \delta S_2 + \delta S_3$. In order to make progress we would like to write all the terms as a product of $\delta g_{\mu\nu}$, as it is an arbitrary variation.
First we take a look at δS_2. We start by considering the variation in the Riemann curvature tensor $R^\sigma_{\mu\nu\rho}$, which we find in (A.3). As it is a sum of connections, we make an arbitrary variation in the connection coefficients,

$$\Gamma^\sigma{}_{\mu\nu} \to \Gamma^\sigma{}_{\mu\nu} + \delta\Gamma^\sigma{}_{\mu\nu}.$$

$\Gamma^{\sigma}_{\mu\nu}$ is the difference between two connections so it is a tensor. We therefore look at it in local geodesic coordinates at an arbitrary point P, where the neighbourhood of P is Euclidean, thus we have ${\Gamma^{\sigma}}_{\mu\nu}(P) = 0$. So at the point P we are only left with the first two terms of the Riemann curvature tensor, and thus we have the variation

$$\delta {R^{\sigma}}_{\mu\nu\rho} = \partial_\nu(\delta {\Gamma^{\sigma}}_{\mu\rho}) - \partial_\rho(\delta {\Gamma^{\sigma}}_{\mu\nu}).$$

At such a point P the partial derivatives and the covariate derivatives coincide so we can exchange the partial derivatives with their covariate counterparts, and thus we obtain the Palitini equation:

$$\delta {R^{\sigma}}_{\mu\nu\rho} = \nabla_\nu(\delta {\Gamma^{\sigma}}_{\mu\rho}) - \nabla_\rho(\delta {\Gamma^{\sigma}}_{\mu\nu}). \tag{2.5}$$

By contracting on σ and ρ we get the expression for the variation of the Ricci tensor

$$\delta R_{\mu\nu} = \nabla_\nu(\delta {\Gamma^{\sigma}}_{\mu\sigma}) - \nabla_\sigma(\delta {\Gamma^{\sigma}}_{\mu\nu}), \tag{2.6}$$

We can insert (2.6) back into the expression of δS_2, and by using that the covariant derivatives of the metric tensor, $g^{\mu\nu}$, vanish we get

$$\begin{aligned}\delta S_2 &= \int_{\mathcal{R}} g^{\mu\nu}[\nabla_\nu(\delta {\Gamma^{\sigma}}_{\mu\sigma}) - \nabla_\sigma(\delta {\Gamma^{\sigma}}_{\mu\nu})]\sqrt{-g}d^4x \\ &= \int_{\mathcal{R}} \nabla_\nu(g^{\mu\nu}\delta {\Gamma^{\sigma}}_{\mu\sigma} - g^{\mu\sigma}\delta {\Gamma^{\nu}}_{\mu\sigma})\sqrt{-g}d^4x\end{aligned} \tag{2.7}$$

This is of the form of the divergence theorem (A.10), which means we can rewrite it in terms of a surface integral over the boundary $\partial\mathcal{R}$. As with the variation in the metric, we assume the variation of the connection vanishes on the boundary. This gives us $\delta S_2 = 0$. Now it is time for δS_3, in this term we must express $\delta(\sqrt{-g})$ in terms of $\delta g^{\mu\nu}$. First we note that by using $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$ and that δ^μ_ν is invariant under variation, we get the following relation:

$$\delta g^{\mu\rho}g_{\rho\nu} + g^{\mu\rho}\delta g_{\rho\nu} = 0. \tag{2.8}$$

To be able to express δg we need to consider the derivation of g with respect to the metric tensor. As $g = det(g_{\mu\nu})$ its derivative is calculated using the Jacobi formula (A.11). By then using equation (2.8) we end up with

$$\begin{aligned}\delta g &= g g^{\mu\nu}\delta g_{\mu\nu} \\ &= -g g_{\mu\nu}\delta g^{\mu\nu}.\end{aligned} \tag{2.9}$$

Now we have what we need to express what we wanted. By using the chain rule and then inserting (2.9) we get

$$\begin{aligned}\delta(\sqrt{-g}) &= -\tfrac{1}{2}(-g)^{-1/2}\delta g \\ &= -\tfrac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\end{aligned} \tag{2.10}$$

Now we can finally combine these results and write the variation in the Einstein-Hilbert action as

$$\delta S_{\text{EH}} = \int_{\mathcal{R}} (R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu} \sqrt{-g} d^4 x \tag{2.11}$$

We now demand that the variation in the action, $\delta S_{EH} = 0$, as the action should be stationary, and that the variation in the metric tensor, $\delta g^{\mu\nu}$, is arbitrary. We then finally end up with Einstein's field equation in vacuum:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 0, \tag{2.12}$$

where we defined the tensor $G_{\mu}\nu$ which is known as the Einstein tensor.

2.1.2 Field Equations in the Presence of Matter

We have now derived the field equations in vacuum, so now we want to consider the field equations in the presence of other fields. All we have to do is add another term to the action that will account for additional fields.

$$S = \frac{1}{16\pi} S_{\text{EH}} + S_{\text{M}} = \int_{\mathcal{R}} \left(\frac{1}{2\kappa} \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{M}} \right) d^4 x \tag{2.13}$$

Here S_M is known as the "matter" action for any non-gravitational field present. In order to make the resulting equations be able to derive Newtons theory of gravity the factor $1/16\pi$ was chosen. We now want to vary the action with respect to the contravariant metric tensor. This gives us the following expression

$$\frac{1}{16\pi} \frac{\delta \mathcal{L}_{\text{EH}}}{\delta g^{\mu\nu}} + \frac{\delta \mathcal{L}_{\text{M}}}{\delta g^{\mu\nu}} = 0, \tag{2.14}$$

where by using (2.11) we can express the first term by the result we derived previously.

$$\frac{\delta \mathcal{L}_{\text{EH}}}{\delta g^{\mu\nu}} = \sqrt{-g} G_{\mu\nu}, \tag{2.15}$$

where $G_{\mu\nu}$ is as given in (2.12). So we see that the part not given by the field equations in vacuum is the definition of the energy-momentum tensor, $T_{\mu\nu}$. We thus arrive at the full Einstein equations:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.16}$$

where the energy-momentum tensor is given by:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{\text{M}}}{\delta g^{\mu\nu}}. \tag{2.17}$$

$$T^{(\phi)}_{\mu\nu} = (\nabla_\mu \phi)(\nabla_\nu \phi) - g_{\mu\nu} \left[\frac{1}{2} (\nabla_\sigma \phi)(\nabla^\sigma \phi) - V(\phi) \right] \tag{2.18}$$

2.2 Field Equations for Weak Gravitational Fields

The gravitational field equations are highly non-linear, this makes it almost impossible to find a general solution for them analytically. In order to find such a general analytical solution we apply the equations to a weak gravitational field, where the metric takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1. \tag{2.19}$$

This is known as the weak-field metric, where $\eta_{\mu\nu}$ is the Minkowski metric for flat space-time, and $h_{\mu\nu}$ is a perturbation that represents the curvature of space-time caused by a weak gravitational field. An example of such a gravitational field is the one in our solar system, in this case when considering a Newtonian gravitational field Φ we have the order of magnitude $|h_{\mu\nu}| \sim |\Phi| \lesssim M_\odot/R_\odot \sim 10^{-6}$ [6]. Because of this low order of magnitude a good approximation is to expand the field equations in powers of $h_{\mu\nu}$ and only keep the terms to the first order, thus linearising the field equations. The contravariant weak-field metric to the first order is obtained by defining that $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$, and we end up with equation (A.15)

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu},$$

where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$. Using this result we look at raising and lowering indices to the first order

$$g^{\mu\alpha}h_{\alpha\nu} = (\eta^{\mu\alpha} - h^{\mu\alpha})h_{\alpha\nu} = \eta^{\mu\alpha}h_{\alpha\nu},$$

this gives us that we can use $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ instead of $g^{\mu\nu}$ and $g_{\mu\nu}$ in order to raise and lower the indices of small quantities.

2.2.1 Linearised Gravitational Field Equations

In order to linearise the field equations we have to linearise its constituents, namely the Ricci scalar and Ricci tensor. They are written in terms of the connection coefficients (A.4), we thus insert the weak-field metric (2.19) and get

$$\Gamma^{\mu}_{\ \alpha\beta} = \frac{1}{2}(\eta^{\mu\nu} - h^{\mu\nu})[\partial_\alpha(\eta_{\nu\beta} + h_{\nu\beta}) + \partial_\beta(\eta_{\alpha\nu} + h_{\alpha\nu}) - \partial_\nu(\eta_{\alpha\beta} + h_{\alpha\beta})].$$

The derivatives of the Minkowski metric vanishes along with the second order terms of $h_{\mu\nu}$, so we are left with

$$\begin{aligned}\Gamma^{\mu}_{\ \alpha\beta} =& \frac{1}{2}\eta^{\mu\nu}[\partial_\alpha h_{\nu\beta} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}] \\ =& \frac{1}{2}[\partial_\alpha h^\mu_\beta + \partial_\beta h^\mu_\alpha - \partial^\mu h_{\alpha\beta}].\end{aligned} \tag{2.20}$$

We then substitute this expression in the definition of the Riemann curvature tensor (A.3), and since we only keep terms to the first order we get

$$\begin{aligned}R^{\alpha}_{\ \mu\nu\beta} =& \tfrac{1}{2}[\partial_\nu(\partial_\beta h^\alpha_\mu + \partial_\mu h^\alpha_\beta - \partial^\alpha h_{\mu\beta}) - \partial_\beta(\partial_\nu h^\alpha_\mu + \partial_\mu h^\alpha_\nu - \partial^\alpha h_{\mu\nu})] \\ =& \tfrac{1}{2}(\partial_\nu\partial_\mu h^\alpha_\beta + \partial_\beta\partial^\alpha h_{\mu\nu} - \partial_\nu\partial^\alpha h_{\mu\beta} - \partial_\beta\partial_\mu h^\alpha_\nu).\end{aligned} \tag{2.21}$$

By contracting over α and ν in the equation above we thus get the Ricci tensor

$$R_{\mu\nu} = \tfrac{1}{2}[\partial_\alpha\partial_\mu h^\alpha_\nu + \partial_\nu\partial_\alpha h^\alpha_\mu - \Box h_{\mu\nu} - \partial_\nu\partial_\mu h], \tag{2.22}$$

where we have used the d'Alembertian operator $\Box \equiv \partial_\alpha\partial^\alpha$ and have defined $h = h^\alpha_\alpha$. The Ricci scalar follows as it is just a contraction with the Ricci tensor (A.6), we thus get

$$\begin{aligned} R = \eta^{\mu\nu}R_{\mu\nu} &= \frac{1}{2}[\partial_\alpha\partial^\nu h^\alpha_\nu + \partial_\nu\partial^\alpha h^\nu_\alpha - \Box h - \partial_\nu\partial^\nu h] \\ &= \partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h. \end{aligned} \tag{2.23}$$

Now we just have to insert equation (2.22) and (2.23) into the field equations (2.16), so by only keeping terms of the first order we arrive at

$$\partial_\alpha\partial_\mu h^\alpha_\nu + \partial_\nu\partial^\alpha h_{\mu\alpha} - \Box h_{\mu\nu} - \partial_\nu\partial_\mu h - \eta_{\mu\nu}(\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h) = 16\pi T_{\mu\nu} \tag{2.24}$$

In order to simplify this rather long experssion we define the "trace reverse" of $h_{\mu\nu}$, which is defined as

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h,$$

where we also have that $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$, where we have defined $\bar{h} = \bar{h}^\alpha_\alpha$. Using these relations we can write the linearised field equations (LFE) as

$$\partial_\nu\partial_\alpha\bar{h}^\alpha_\mu + \partial_\mu\partial_\alpha\bar{h}^\alpha_\nu - \Box\bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta} = 16\pi T_{\mu\nu} \tag{2.25}$$

2.2.2 Linearised Gravity in the Lorenz Gauge

We start by introducing the gauge transformation for electromagnetic fields, which is when A_μ is a solution of the electromagnetic field equations then another solution that describes the same physical situation is given by $A^{(\text{new})}_\mu = A_\mu - \partial_\mu\xi$, where ξ is any scalar field. If we consider $h_{\mu\nu}$ as a tensor field defined on a background Minkowski space-time we can consider the infinitesimal general coordinate transformation (A.20) to be analogous to the gauge transformation in electromagnetism. So from equation (A.20) we see that if $h_{\mu\nu}$ is a solution to the LFE (2.25) then the same physical situation is also described by

$$h^{(\text{new})}_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu, \tag{2.26}$$

now it is considered a gauge transformation rather then a coordinate transformation. We now want to use this gauge transformation to simplify the LFE (2.25). We denote the gauge-transformed field as $h'_{\mu\nu}$, and perform a transformation on the trace-reverse form

$$\begin{aligned} \bar{h}'^{\mu\alpha} &= h'^{\mu\alpha} - \tfrac{1}{2}\eta^{\mu\alpha}h' \\ &= h^{\mu\alpha} - \partial^\mu\xi^\alpha - \partial^\alpha\xi^\mu - \tfrac{1}{2}\eta^{\mu\alpha}(h - 2\partial_\beta\xi^\beta) \\ &= \bar{h}^{\mu\alpha} - \partial^\mu\xi^\alpha - \partial^\alpha\xi^\mu + \eta^{\mu\alpha}\partial_\beta\xi^\beta. \end{aligned} \tag{2.27}$$

By taking the derivative with respect to x^α we find

$$\partial_\alpha\bar{h}'^{\mu\alpha} = \partial_\alpha\bar{h}^{\mu\alpha} - \Box\xi^\mu, \tag{2.28}$$

as the other terms cancel each other after some index tricks. As ξ^μ are arbitrary functions we are now free to choose $\Box\xi^\mu = \partial_\alpha \bar{h}^{\mu\alpha}$, such that we will have $\partial_\alpha \bar{h}'^{\mu\alpha} = 0$. By looking at the LFE (2.25) we see that in this new gauge all the terms but one vanish on the left hand side of the equation. This allows us to write the LFE much simpler, by dropping the primes we get

$$\Box\bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}. \tag{2.29}$$

This equation is valid as long as $\bar{h}^{\mu\nu}$ satisfies the gauge condition

$$\partial_\nu \bar{h}^{\mu\nu} = 0. \tag{2.30}$$

This gauge condition is valid for any further gauge transformations of the same form, given that

$$\Box\xi^\mu = 0, \tag{2.31}$$

is satisfied.

Chapter 3

Gravitational Waves

In the section above we were able to simplify the LFE by writing them in the Lorenz gauge, this led to an equation (2.29) that is written on the form of a wave equation. The equation and its gauge condition resemble what is found when applying the Lorenz gauge in electromagnetic theory, which led to the solution of electromagnetic waves. Following this analogy suggests the existence of gravitational wave solutions to the LFE.

3.1 Plane-Wave Solution

The simplest case is in the vacuum where the LFE in the Lorenz gauge reduce to

$$\Box \bar{h}^{\mu\nu} = 0, \tag{3.1}$$

with the gauge condition $\partial_\nu \bar{h}^{\mu\nu} = 0$. This is of the form of the wave equation, which means that we can find plane-wave solutions for $\bar{h}^{\mu\nu}$ of the form

$$\bar{h}^{\mu\nu} = A^{\mu\nu} e^{ik_\alpha x^\alpha}, \tag{3.2}$$

where $A^{\mu\nu}$ are constant complex components of the symmetric amplitude tensor and $k_\alpha = (\omega, \mathbf{k})$ are constant real components of a four-wavevector, where ω is the temporal angular frequency and $\mathbf{k}$ is the spatial wave vector. By substituting the plane-wave form (3.2) into the wave-equation (3.1) and using $\partial_\alpha \bar{h}^{\mu\nu} = k_\alpha \bar{h}^{\mu\nu}$ we get

$$\Box \bar{h}^{\mu\nu} = \eta^{\alpha\beta} \partial_\alpha \partial_\beta \bar{h}^{\mu\nu} = k^2 \bar{h}^{\mu\nu} = 0. \tag{3.3}$$

This means that in order for the wave equation to be satisfied k^2 must be a null vector. This means that the plane-wave solution describes a wave that propagates with a group and phase velocity at the speed of light, $k^0 = \omega = |\mathbf{k}|$. More generally for an observer moving at four-velocity U^μ, we have $\omega = -k_\mu U^\mu$ [7]. Finally we need to consider the gauge condition $\partial_\nu \bar{h}^{\mu\nu} = 0$, by inserting the plane-wave equation (3.2) into it we get

$$A^{\mu\nu} k_\nu = 0. \tag{3.4}$$

Thus for any k^μ that satisfies $k^2 = 0$ and equation (3.4) can be used to find a plane-wave solution (3.2) to the LFE in the Lorenz gauge in the vacuum. As these solutions are linear by design we can write any of them as a superposition of the plane-wave solutions of the form

$$\bar{h}^{\mu\nu}(x) = \int A^{\mu\nu}(\mathbf{k}) e^{ik_\alpha k^\alpha} d^3\mathbf{k}. \tag{3.5}$$

3.1.1 Transverse-traceless gauge (TT)

As mentioned above the amplitude tensor is symmetric, $A^{\mu\nu} = A^{\nu\mu}$, a property that reduces the number of independent components from 16 down to 10. From the gauge condition for plane-waves (3.4) and using that the wave vector must be null we see that a further four components are restricted and we are left with six independent components of the amplitude tensor, which is still four more than the two dynamical freedoms a gravitational field in general relativity has [6]. These four extra independent components were introduced with our gauge transformation, more specifically the arbitrary function ξ^μ. As the Lorentz gauge transformation is valid for any ξ^μ satisfying $\Box\xi = 0$ we can restrict this arbitrary function using its gauge condition. So similar to the plane-wave solution for $h_{\mu\nu}$, we choose the solution

$$\xi^\mu = B^\mu e^{ik_\alpha x^\alpha}, \tag{3.6}$$

where B^μ is a constant and k^α is the same 4-wavevector as earlier. By inserting this solution and the plane-wave solution of $h^{\mu\nu}$ (3.2) into the gauge transformation (2.27) we get

$$A'^{\mu\nu} e^{ik_\alpha x^\alpha} = A^{\mu\nu} e^{ik_\alpha x^\alpha} - \partial^\mu B^\nu e^{ik_\alpha x^\alpha} - \partial^\nu B^\mu e^{ik_\alpha x^\alpha} + \eta^{\mu\nu} \partial_\beta B^\beta e^{ik_\alpha x^\alpha}, \tag{3.7}$$

where prime denotes the gauge transformed field. The exponents cancel each other and we are left with an expression with only $A^{\mu\nu}$ and B^μ

$$A'^{\mu\nu} = A^{\mu\nu} - iB^\nu k^\mu - iB^\mu k^\nu + i\eta^{\mu\nu} B^\beta k_\beta. \tag{3.8}$$

By contracting we get

$$\begin{aligned} A'^{\mu}_{\ \mu} =& A^{\mu}_{\ \mu} - iB^\nu k_\nu - iB^\mu k_\mu + 4iB^\beta k_\beta \\ =& A^{\mu}_{\ \mu} + 2iB^\mu k_\mu, \end{aligned} \tag{3.9}$$

we now choose $B^\mu k_\mu = \frac{i}{2} A^{\mu}_{\ \mu}$ in order to get a new restriction for the gauge transformation,

$$A'^{\mu}_{\ \mu} = A' = 0, \tag{3.10}$$

which tells us that the trace of $\bar{h}'^{\mu\nu}$ must be zero and reduces the number of independent components down to three. We can impose a further restriction which acts as both a gauge condition and a choice of Lorentz frame [7]

$$A_{\mu\nu} U^\nu = 0, \tag{3.11}$$

where $U^\mu = (1,0,0,0)$ is a timelike four-velocity, which implies looking at a rest frame. For this equation to be zero the timelike components of the amplitude tensor, $A^{\mu\nu}$, must be zero so we get

$$A^{\mu 0} = 0. \tag{3.12}$$

This also means that $h'^{\mu 0} = 0$, and that we are only left with two independent components of the amplitude tensor, $A^{\mu\nu}$, just like we wanted. The restrictions we have introduced above are the gauge conditions for the transverse traceless gauge, we will denote the new gauge $\bar{h}'^{\mu\nu} = \bar{h}^{\mu\nu}_{\mathrm{TT}}$. Including the Lorentz gauge condition (2.30) the constraints in the transverse traceless gauge are thus

$$\bar{h}^{\mu 0}_{\mathrm{TT}} = 0, \tag{3.13a}$$

$$(\bar{h}_{TT})^i_i = 0, \tag{3.13b}$$

$$\partial_i \bar{h}^{ij}_{\mathrm{TT}} = 0. \tag{3.13c}$$

We thus have that in the transverse traceless gauge only the spatial components are non-zero and they are trace and divergence free. The equation (3.13b) also means that there is no difference between $\bar{h}^{\mu\nu}$ and $h^{\mu\nu}$ and they can be used interchangeably in this gauge.

3.1.2 Plane-wave in TT gauge

Let us now look at what happens to an arbitrary plane gravitational wave (3.2) in this gauge. By applying the gauge conditions (3.13) we get that the amplitude tensor have the following properties

$$A^{\mu 0}_{\mathrm{TT}} = 0, \qquad (A_{\mathrm{TT}})^i_i = 0, \qquad A^{ij}_{\mathrm{TT}} k_j = 0. \tag{3.14}$$

The last one gives us that the amplitude is perpendicular to the direction of wave propagation, which means that, like electromagnetic waves gravitational waves are also transverse waves. Given that we know the amplitude matrix $A^{\mu\nu}$ and the spatial wavevector $\mathbf{k}$ we want to be able to construct the tensor $A^{\mu\nu}_{\mathrm{TT}}$ that satisfies the above constraints. As mentioned, only the spatial components are non-zero so we only have to consider them. The spatial tensor we are left with thus have to be orthogonal to $\mathbf{k}$ and be traceless, we therefore introduce the spatial projection tensor

$$P_{ij} \equiv \delta_{ij} - n_i n_j, \tag{3.15}$$

which projects spatial tensor components onto the surface orthogonal to the unit spatial vector with components n^i, and it has the following properties $n_i P^i_j v^j = 0$ and $P^i_k P^k_j v^j = P^i_j v^j$, where v^j is an arbitrary vector component [5]. To find the components of the spatial amplitude matrix that are transverse to $\mathbf{k}$ we choose $n_i = \hat{k}^i$, and apply the projection tensor onto the spatial amplitude matrix

$$A^{ij}_{\mathrm{T}} = P^i_k P^j_l A^{kl}. \tag{3.16}$$

Now that we have found an expression for the transverse part we want to make it traceless, the trace of this tensor is given by $(A_T)^i_i = P_{kl} A^{kl}$ which we cannot say is traceless. By using $P^i_i = 3 - 1 = 2$ we can construct a tensor that fulfils the traceless constraint and is transverse to $\mathbf{k}$, such a tensor is [5]

$$A^{ij}_{\text{TT}} = (P^i_k P^j_l - \tfrac{1}{2} P^{ij} P_{kl}) A^{kl}. \tag{3.17}$$

This tensor is still transverse due to the definition of P_{ij} and the choice of unit vector n^i. For future reference we will define the terms in the parentheses as

$$\Lambda_{ij,kl}(\hat{\mathbf{n}}) = P_{ik} P_{jl} - \tfrac{1}{2} P_{ij} P_{kl}, \tag{3.18}$$

which we will call the Lambda tensor and using eq. (3.15) we can write it in terms of $\hat{\mathbf{n}}$

$$\begin{aligned} \Lambda_{ij,kl}(\hat{\mathbf{n}}) = \delta_{ik}\delta_{jl} - \tfrac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} \\ + \tfrac{1}{2} n_k n_l \delta_{ij} + \tfrac{1}{2} n_i n_j \delta_{kl} + \tfrac{1}{2} n_i n_j n_k n_l. \end{aligned} \tag{3.19}$$

From our definition of the Lambda tensor we see that it has the property of transforming a tensor to the TT-gauge,

$$A^{\text{TT}}_{ij} = \Lambda_{ij,kl} A_{kl}. \tag{3.20}$$

We can now verify that this transformation is in fact traceless by taking the trace of an arbitrary tensor in the TT-gauge,

$$(A_{\text{TT}})^i_i = (P^i_k P_{il} - \tfrac{1}{2} P^i_i P_{kl}) A^{kl} = (P_{kl} - \tfrac{2}{2} P_{kl}) A^{kl} = 0.$$

We thus have found a general formula for the amplitude tensor for a plane gravitational wave in the transverse traceless gauge. Let us now use equation (3.17) in order to find the amplitude of a plane wave travelling in the z-direction, we thus have the components of the wave vector $k^\mu = (\omega, 0, 0, \omega)$. From the gauge condition (3.4) we get that $A^{\mu 0} = A^{\mu 3}$, using this in order to construct a symmetric amplitude matrix we get

$$[A^{\mu\nu}] = \begin{pmatrix} A^{00} & A^{01} & A^{02} & A^{00} \\ A^{01} & A^{11} & A^{12} & A^{01} \\ A^{02} & A^{12} & A^{22} & A^{02} \\ A^{00} & A^{01} & A^{02} & A^{00} \end{pmatrix}. \tag{3.21}$$

We can now use equation (3.17) directly to compute the elements of the amplitude matrix in the transverse traceless gauge

$$[A^{\mu\nu}_{\text{TT}}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(A^{11} - A^{22}) & A^{12} & 0 \\ 0 & A^{12} & -\frac{1}{2}(A^{11} - A^{22}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.22}$$

which we see only depends on two independent components, which correlates with the two dynamical freedoms of a gravitational field in general relativity [6].

3.2 The Effect on Free Particles

Using the TT gauge we now want to consider what happens to a free particle in the presence of a gravitational wave, we choose a Lorentz frame such that the particle is initially at rest. We thus have four velocity $U^\alpha = (1, 0, 0, 0)$, which satisfies the gauge condition (3.13a). A free particle follows the geodesic equation (A.8), in this case we set the parameter λ as proper time τ and get

$$\frac{dU^\alpha}{d\tau} + \Gamma^\alpha_{\ \mu\nu} U^\mu U^\nu = 0. \tag{3.23}$$

As we look at a particle that is initially at rest, μ and ν must be zero, and using the linearised connection (2.20) we get

$$\frac{dU^\alpha}{d\tau} = -\Gamma^\alpha_{\ 00} = -\tfrac{1}{2}\eta^{\alpha\beta}(\partial_0 h_{\beta 0} + \partial_0 h_{\beta 0} - \partial_\beta h_{00}). \tag{3.24}$$

As we also know from the gauge condition $h_{\mu 0} = 0$ (3.13a), which gives us the acceleration

$$\frac{dU^\alpha}{d\tau} = 0.$$

This means that the particle will remain at rest forever, but at rest in this case means that the we have constant coordinates for the particle. Thus it is clear that the TT gauge defines a coordinate system that is attached to the particles. Thus the effect of a gravitational wave has no measurable consequence on a single particle.

3.2.1 Response of two particles

We will now instead consider the relative motion of nearby particles in the presence of a gravitational wave, which is described by the equation of geodesic deviation (A.9). Let us now consider the two particles A and B, where we choose our coordinate system to be attached to the world line of A, which is known as the proper reference frame of A and is a local Lorentz frame [6]. We define both particles to be initially at rest, and as defined A is in the origin $x^i_a = (0, 0, 0)$ and particle B will be an arbitrary distance away from A, $x^i_B = (x_B, y_B, z_B)$, we let ξ^α be the components of the separation vector describing the distance from A to B. As both particles are initially stationary they therefore have the same four-velocity $U^\alpha = (1, 0, 0, 0)$, inserting this into the equation of geodesic deviation (A.9) simplifies the Riemann curvature tensor to $R^\alpha_{\ 00\beta}$, thus our equation reduces to

$$\frac{D^2\xi^\alpha}{D\tau^2} = R^\alpha_{\ 00\beta}\xi^\beta. \tag{3.25}$$

The left hand side includes the connection (A.4), which at particle A, where we do our calculations, will vanish due to the locally flat spacetime. So our left hand side ends up as just a derivative with respect to proper time,

$$\frac{d^2\xi^\alpha}{d\tau^2} = R^\alpha_{\ 00\beta}\xi^\beta. \tag{3.26}$$

As we saw in the section above in the TT-gauge there is a coordinate system that stays with the particle A and its reference frame, so it is a valid gauge for this scenario. By writing the Riemann curvature tensor in linear terms (2.21) and applying the gauge condition (3.13a) we are able to write equation (3.26) as

$$\frac{d^2\xi^i}{dt^2} = \frac{1}{2}\frac{\partial^2 h^{\mathrm{TT}i}{}_j}{\partial t^2}\xi^j, \tag{3.27}$$

where we used that to first order in h_{ij}^{TT} we have $\tau = t$ [6]. We thus have found that the acceleration relative between the two particles are proportional to the second derivative of $h_{\mu\nu}^{\mathrm{TT}}$ and the initial separation, so a larger initial separation will result in a bigger effect from the wave.

3.3 Polarisation of a Plane-Wave

To better understand the effects of equation (3.27) we consider the case of a gravitational plane wave (3.2) propagating in the z-direction, $k^\alpha = (\omega, 0, 0, k_z)$. We already calculated the amplitude matrix in the TT-gauge (3.22) earlier, and for future convenience we would like to rewrite it using

$$h_+ = \tfrac{1}{2}(A^{11} - A^{22}) \quad \text{and} \quad h_\times = A^{12}.$$

The amplitude matrix is now given by

$$[A_{\mathrm{TT}}^{\mu\nu}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.28}$$

which when applied to equation (3.27) makes it clear that only separation vector components orthogonal to the propagation will cause a disturbance, as is expected from a transverse wave. Our two amplitude components h_+ and $h_\times$ will have different effects on the separation vector, so we want to separate their contributions by first setting $h_\times = 0$. Then we end up with the two equations

$$\frac{\partial^2\xi^x}{\partial t^2} = \frac{1}{2}\xi^x\frac{\partial^2}{\partial t^2}(h_+ e^{ik_\alpha x^\alpha}), \tag{3.29a}$$

$$\frac{\partial^2\xi^y}{\partial t^2} = -\frac{1}{2}\xi^y\frac{\partial^2}{\partial t^2}(h_+ e^{ik_\alpha x^\alpha}). \tag{3.29b}$$

In the period before the wave passes the particles we have a wave-free region, $h_{ij}^{\mathrm{TT}} = 0$, in this period we define the static separation vector components as $\xi^i = \xi^i(0)$. Using these as our boundary conditions we are able to solve the two differential equations (3.29). Thus in the lowest order we have

$$\xi^x = (1 + \tfrac{1}{2}h_+ e^{ik_\alpha x^\alpha})\xi^x(0), \tag{3.30a}$$

$$\xi^y = (1 - \tfrac{1}{2}h_+ e^{ik_\alpha x^\alpha})\xi^y(0). \tag{3.30b}$$

From this we see that particles initially separated in the x-direction will oscillate in the x-direction and the same is the case for particles separated in the y-direction. A good way to visualise the effect a gravitational plane wave will have on a set particles is to organise them in a circle on the $x - y$ plane, the passing wave will then alternate stretching the circle in either the x- or y-direction, in a pattern +, making it into an ellipse. This makes the choice of introducing the subscript "+" more apparent.
We now return to the other case where $h_\times = 0$ and $h_+ \neq 0$, we then get the equations

$$\frac{\partial^2 \xi^x}{\partial t^2} = \frac{1}{2}\xi^y \frac{\partial^2}{\partial t^2}(h_\times e^{ik_\alpha x^\alpha}), \tag{3.31a}$$

$$\frac{\partial^2 \xi^y}{\partial t^2} = \frac{1}{2}\xi^x \frac{\partial^2}{\partial t^2}(h_\times e^{ik_\alpha x^\alpha}). \tag{3.31b}$$

By using the same conditions as for +, we get

$$\xi^x = \xi^x(0) + \tfrac{1}{2}\xi^y(0)h_\times e^{ik_\alpha x^\alpha}, \tag{3.32a}$$

$$\xi^y = \xi^y(0) + \tfrac{1}{2}\xi^x(0)h_\times e^{ik_\alpha x^\alpha}. \tag{3.32b}$$

These equations tell us that the particles will oscillate at a $45°$ angle to their respective x- and y-direction. A circle of particles would behave as the case above just at the aforementioned $45°$ angle and would move in the pattern of a $\times$.
We have now found the two independent modes of linear polarisation for a gravitational wave, given by h_+ and $h_\times$ [6]. We can however also consider circular polarisation of the wave, in this case we have a right handed, h_R, and a left handed, h_L, mode which is obtained by defining [7]

$$h_R = \frac{1}{\sqrt{2}}(h_+ + ih_\times), \tag{3.33a}$$

$$h_L = \frac{1}{\sqrt{2}}(h_+ - ih_\times). \tag{3.33b}$$

The effect of a circular polarised wave on the circle of particles we discussed earlier is the apparent rotation of the ellipse, where an h_R polarisation would rotate it in a right-handed sense.

3.4 The Energy Carried by a Gravitational Wave

The stretching of test particles by a passing gravitational wave do remind us of the effects of tidal forces which we know can generate a lot of heat in celestial bodies, one example in our own solar system is Jupiter's moon Europa which is able to sustain a liquid ocean beneath its icy surface due to the tidal heating caused by Jupiter's gravitational field[8]. So for a gravitational wave to carry energy should not be far fetched, even though that was not what Einstein and Rosen first argued from their cylindrical wave solutions they reached the conclusion that the waves are unstable and would collapse. They first tried to publish this result in Physical Review, but was rejected by a referee due to their solutions collapse existing due to a singularity of their coordinate system and not a physical one [9]. The idea of gravitational waves not being physical (having zero energy) was further pushed by Rosen in his attempt at calculate the stress-energy tensor for the waves which he found to be zero everywhere. At a conference in 1957 Feynman pointed out that a stick with two freely moving masses placed transversely to a GW would rub against the stick and create heat through the friction, the length of the stick would be unchanged due to atomic forces, thus GWs must carry energy[10]. This concept put forward by Feynman was later the same year elaborated on by Bondi, and Weber and Wheeler, proving that GWs carry energy [11] [12]. Two decades later an observation of the energy of GWs was found by monitoring the orbital period of a binary system, which they found to be decreasing at a rate similar to what one would expect from energy loss due to GWs [13].

If we want to derive an expression for the stress-energy tensor we have to mainly face two challenges, the first being the limitations of a linear theory. To see why we consider a simple aspect of the field equations (2.16), namely that energy is what causes the curvature of spacetime, thus it is natural to to consider how a GW will affect the curvature in order to find the stress-energy of the wave. In the linear case with the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ we are limited to only look at the perturbations on a flat background spacetime which in turn means the inability to consider the curvature and thus the energy of a wave. So we must step out of the comfort zone of the linear formulation of the field equations in order to find the stress-energy of a GW. The other challenge is the fact that it is impossible to find the energy of gravitational field in some local inertial frame, the gravitational fields are non-local, in other words there is no observable quantity of them in a some local region of spacetime. To see why this must be the case we can simply take Einsteins equivalence principle into consideration, as it gives us that we can always find in any given locality a frame of reference in which all local gravitational fields disappear (locally we have a flat spacetime) [6]. It tells us that will have a region with no gravitational fields and will thus neither have any local energy, which complicates the process of finding the energy of the energy that a GW will carry as we cannot determine the energy of single wave crest.

We have now gotten an overview of the two major obstacles in our path, in order to find an expression for the stress-energy of a wave we will follow the procedure as outlined by Maggiore and MTW [14] [6]. Let us start by stepping outside the linear theory by defining a new background, $\bar{g}_{\mu\nu}$, which is dynamical and can be curved, this gives us a new metric to work with

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1. \tag{3.34}$$

We now have a metric which allows for both the terms to be dynamical and have curvature, which raises the problem of how we can distinguish the two terms. We defined $\bar{g}_{\mu\nu}$ to represent the background curvature and $h_{\mu\nu}$ to represent the perturbations that arises from a GW, both of which we know the scale of. For a background curvature we consider the example of the scale of the Newtonian gravitational field in our solar system, $|\Phi| \lesssim M_{\odot}/R_{\odot} \sim 10^{-6}$, while the scale of GWs are of order $|h_{\mu\nu}| \sim 10^{-21}$, this difference in scale makes for a good way to distinguish the two terms in the metric. Let us thus introduce the variable L_B which represents the spatial variation in the background $\bar{g}_{\mu\nu}$ and for the perturbations we denote them with the wavelength λ such that

$$\lambda \ll L_B, \tag{3.35}$$

where $\lambda = \lambda/(2\pi)$ is the reduced wavelength which we get from an oscillating function $f(x) = e^{ikx}$, with $k = 2\pi/\lambda$. This is the spatial difference between the background and the perturbation, to consider the temporal difference we look at the equivalent statement but in frequency space

$$f \gg f_B, \tag{3.36}$$

where f is the peak of $h_{\mu\nu}$ and f_B is the peak of $\bar{g}_{\mu\nu}$. The method of separating a smooth background from more rapid fluctuations is known as the short-wave expansion. We can now move on to explain how this high-frequency (short wavelength) perturbation propagates in the background and how it affects the background itself.

3.4.1 GWs affect on the background

We want to understand how the perturbation $h_{\mu\nu}$ propagates through and affects the background $\bar{g}_{\mu\nu}$, to achieve that we will expand the field equations (2.16) around the background metric. In this expansion we will consider two small parameters, the amplitude $h \equiv \mathcal{O}(|h_{\mu\nu}|)$ and either λ/L_B or f_B/f. Either case can be treated in parallel given the appropriate notation, and both will be referred to as the short-wave expansion.

A convenient first step is to rewrite the field equations to the form

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T), \tag{3.37}$$

where T is the trace of the stress-energy tensor $T_{\mu\nu}$. Now we will expand the Ricci tensor to second order in $h_{\mu\nu}$

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + ..., \tag{3.38}$$

where $\bar{R}_{\mu\nu}$ is constructed only using $\bar{g}_{\mu\nu}$, $R^{(1)}_{\mu\nu}$ is linear in $h_{\mu\nu}$, and $R^{(2)}_{\mu\nu}$ is quadratic in $h_{\mu\nu}$. This gives us a relation between the terms and the two scales of frequency modes, $\bar{R}_{\mu\nu}$ can only contain low-frequency modes and $R^{(1)}_{\mu\nu}$ can only contain high-frequency modes. For $R^{(2)}_{\mu\nu}$ we can have both modes as a quadratic term $\sim h_{\mu\nu}h_{\alpha\beta}$ can consist as a combination of the two high-frequency wave-vectors k_1 and $k_2 \simeq -k_1$ which can combine into a low-frequency wave-vector. By expressing the field equations (3.37) in terms of the expanded Ricci tensor (3.38) we can can split the high- and low-frequency modes into two separate equations

$$\bar{R}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\text{Low}} + 8\pi(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T)^{\text{Low}}, \tag{3.39}$$

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\text{High}} + 8\pi(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T)^{\text{High}}, \tag{3.40}$$

here the superscripts would be switched had we instead looked at wave-lengths. The expression for the linear $R^{(1)}_{\mu\nu}$ has already been found earlier (2.22) when we derived linearised gravity, we do have to replace the derivatives with covariant derivatives (A.7) in order for it to apply to the curved background metric $\bar{g}_{\mu\nu}$, which we denote as $\bar{D}_\mu$. We thus get

$$R^{(1)}_{\mu\nu} = \tfrac{1}{2}[\bar{D}^\alpha\bar{D}_\mu h_{\nu\alpha} + \bar{D}^\alpha\bar{D}_\nu h_{\mu\alpha} - \bar{D}^\alpha\bar{D}_\alpha h_{\mu\nu} - \bar{D}_\nu\bar{D}_\mu h]. \tag{3.41}$$

The quadratic term of the Ricci tensor is found by the same procedure as for the LFE in section 2.2.1 but instead of using the weak-field metric (2.19) you use the one we have worked with in this section (3.34). It is a rather lengthy algebraic process to find, so we will just write down the end result [14]

$$\begin{aligned} R^{(2)}_{\mu\nu} = &\tfrac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}[\tfrac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}) \\ &+ h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma}) \\ &+ (\tfrac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\mu h_{\mu\nu})], \end{aligned} \tag{3.42}$$

where we note that $\bar{g}_{\mu\nu}$ is the metric that is used to raise and lower the indices, and will the be the case for other expressions as well. We will now go back and look at equation (3.39). For the case we have with the length scale of λ and L_B being vastly different there is a simple way to perform the projection on the long wavelength modes, we introduce a new scale $\bar{l}$ such that $\lambda \ll \bar{l} \ll L_B$ and average over a spatial volume with side $\bar{l}$. The wavelengths of the background L_B will be untouched during this procedure as they will be constant at the scale of $\bar{l}$. The wavelengths at the scale of λ will oscillate rapidly and average out to be zero. For the temporal approach we have used thus far we achieve the same by introducing a time scale $\bar{t}$, which is much larger than $1/f$ and much smaller than $1/f_B$, and average over $\bar{t}$. Equation (3.39) can thus be rewritten as

$$\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu}\rangle + 8\pi\langle T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T\rangle, \tag{3.43}$$

where $\langle ... \rangle$ denotes the averaging over either many wavelengths λ or several periods $1/f$. Through averaging we are also able to overcome our second challenge, namely the non-locality of gravitational energy. By looking at the effect of the perturbations on a macroscopic scale we are able to see the effects on the physical curvature in that macroscopic region, which enables us to talk about an effective "smeared-out" tensor for the stress-energy of the GWs. In other words, by doing the averaging the curvature in the region should give us a frame of reference such that will obtain a gauge invariant expression for the stress-energy. We denote this effective tensor by $\bar{T}^{\mu\nu}$ and define it from

$$\langle T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T\rangle = \bar{T}^{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\bar{T}, \tag{3.44}$$

where $\bar{T} = \bar{g}_{\mu\nu}\bar{T}^{\mu\nu}$ is the trace. We will also define another quantity $t_{\mu\nu}$ as

$$t_{\mu\nu} = -\frac{1}{8\pi}\langle R^{(2)}_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle, \tag{3.45}$$

where $R^{(2)} = \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}$, and its trace is defined as $t = \bar{g}^{\mu\nu}t_{\mu\nu} = \frac{1}{8\pi}\langle R^{(2)}\rangle$, here we used that $\bar{g}_{\mu\nu}$ is constant under the averaging. We are now able to write the field equations in the following way

$$\bar{R}_{\mu\nu} - \tfrac{1}{2}\bar{g}^{\mu\nu}\bar{R} = 8\pi(\bar{T}_{\mu\nu} + t_{\mu\nu}), \tag{3.46}$$

which gives us the dynamics of $\bar{g}^{\mu\nu}$, the long wavelength part of the metric and can be thought of as "macroscopic" field equations. The tensor $t_{\mu\nu}$ gives us the part of the stress-energy which is only dependent on the gravitational field itself and not any matter.

3.4.2 Wave propagation in a curved background

Before we consider expressing $t_{\mu\nu}$ in terms of $R^{(2)}_{\mu\nu}$ (3.42) we will first consider the propagation equation $R^{(1)}_{\mu\nu} = 0$, which is just the vacuum solution. As we did for the linearised case we define the new quantity

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2} h \bar{g}_{\mu\nu}. \tag{3.47}$$

Rewriting $R^{(1)}_{\mu\nu} = 0$ using $\bar{h}_{\mu\nu}$, equation (3.41) and the commutation relation (B.5), we get

$$\begin{aligned} \bar{D}_\alpha \bar{D}^\alpha \bar{h}_{\mu\nu} + \bar{D}_\beta \bar{D}_\alpha \bar{g}_{\mu\nu} \bar{h}^{\alpha\beta} - \bar{D}^\alpha \bar{D}_\nu \bar{h}_{\alpha\mu} \\ -\bar{D}^\alpha \bar{D}_\mu \bar{h}_{\alpha\nu} + 2\bar{R}_{\alpha\mu\beta\nu} \bar{h}^{\alpha\beta} - \bar{R}_{\alpha\mu} \bar{h}^{\alpha}_{\ \nu} - \bar{R}_{\alpha\nu} \bar{h}^{\alpha}_{\ \mu} = 0. \end{aligned} \tag{3.48}$$

We are able to simplify the above equation by applying a suitable gauge. Like for the case of the linearised theory we choose the infinitesimal coordinate transformation (A.20) as our gauge transformation. To apply it to our non-flat background we have to use covariant derivatives, thus we get

$$h'_{\mu\nu} = h_{\mu\nu} - D_\mu \xi_\nu - D_\nu \xi_\mu, \tag{3.49}$$

where prime denotes the system after the transformation. As we know, choosing the functions ξ^μ appropriately we are able to apply the Lorentz gauge condition $\bar{D}_\alpha \bar{h}^{\ \alpha}_{\mu} = 0$, thus reducing equation (3.48) to

$$\bar{D}_\alpha \bar{D}^\alpha \bar{h}_{\mu\nu} + 2\bar{R}_{\alpha\mu\beta\nu} \bar{h}^{\alpha\beta} - \bar{R}_{\alpha\mu} \bar{h}^{\alpha}_{\ \nu} - \bar{R}_{\alpha\nu} \bar{h}^{\alpha}_{\ \mu} = 0. \tag{3.50}$$

We note that normally the covariant derivative does not commute as of equation (B.5) and thus we should not be able to use the Lorentz gauge condition on some of the terms. However we see that the two terms involving the Riemann curvature tensor in equation (B.5) both are to third order in $h_{\mu\nu}$ so they vanish within our precision, and we can thus commute the covariant derivatives. The last two terms of equation (3.50) also disappear due to them being to third order in $h_{\mu\nu}$ as well, so we are left with

$$\bar{D}_\alpha \bar{D}^\alpha \bar{h}_{\mu\nu} + 2\bar{R}_{\alpha\mu\beta\nu} \bar{h}^{\alpha\beta} = 0, \tag{3.51}$$

which is the propagation equation for a GW on a curved background in the Lorentz gauge, which we see reduces to the form of the wave equation (3.1) in flat spacetime.

3.4.3 The stress-energy tensor of GWs

The expression for $t_{\mu\nu}$ (3.45) we defined in section 3.4.1 is difficult to interpret in terms of GWs at first glance, but through the use of $R^{(2)}_{\mu\nu}$ given by equation (3.42) we will be able to find an explicit expression that better suits our needs. In order to make this process simpler we introduce some properties of the averaging process denoted by the brackets $\langle ... \rangle$ [6]

1. Covariant derivatives commute; $\langle h \ D_\alpha D_\beta h_{\mu\nu} \rangle = \langle h \ D_\beta D_\alpha h_{\mu\nu} \rangle$. The fractional errors made by commuting are $\sim (\lambda / L_B)^2$, well below the accuracy of the averaging process itself.

2. Gradients average out to be zero; $\langle D_\beta (D_\alpha h h_{\mu\nu}) \rangle = 0$. The fractional errors are $\lesssim \lambda / L_B$.

3. One can freely integrate by parts, flipping derivatives from one h to the other; $\langle h \ D_\alpha D_\beta h_{\mu\nu} \rangle = \langle -D_\beta h \ D_\alpha h_{\mu\nu} \rangle$.

Using these properties we along with the expression for $R^{(2)}_{\mu\nu}$ (3.42), the definition of $\bar{h}_{\mu\nu}$ (3.47), and the propagation equation (3.51) on the expression of $t_{\mu\nu}$ (3.45) we get

$$t_{\mu\nu} = \frac{1}{32\pi}\langle \bar{D}_\mu \bar{h}_{\alpha\beta}\bar{D}_\nu \bar{h}^{\alpha\beta} - \tfrac{1}{2}\bar{D}_\mu \bar{h}\bar{D}_\nu \bar{h} - \bar{D}_\beta \bar{h}^{\alpha\beta}\bar{D}_\nu \bar{h}_{\alpha\mu} - \bar{D}_\beta \bar{h}^{\alpha\beta}\bar{D}_\mu \bar{h}_{\alpha\nu}\rangle, \quad (3.52)$$

the equation was also simplified by using $\langle R^{(2)}_{\mu\nu}\rangle = 0$. By choosing the Lorentz gauge, $\bar{D}_\alpha \bar{h}_\mu^{\ \alpha} = 0$, as well as a traceless gauge, $\bar{h} = 0$, we end up with a much simpler expression

$$t_{\mu\nu} = \frac{1}{32\pi}\langle \bar{D}_\mu \bar{h}_{\alpha\beta}\bar{D}_\nu \bar{h}^{\alpha\beta}\rangle. \quad (3.53)$$

For the case of a GW far away from a source we have flat spacetime and thus the covariant derivatives reduce to simple partial derivatives. Through a gauge transformation of the form of the infinitesimal coordinate transformation (3.49) we can verify that the stress-energy tensor (3.52) is gauge invariant and indeed a physical quantity as we set out to find. Through a more detailed study of the magnitude of the different terms of the expanded Ricci tensor (3.38) and Riemann tensor carried out by Isaacson one finds that the first order term is the dominant one for both tensors. By calculating the transformation of these tensors directly or by using the definition of the Lie derivative, one is able to verify that the expansion and thus the resulting stress-energy tensor are gauge invariant to a very good approximation. This is the case because of our short wave approximation where we assumed the waves to have a much higher frequency than the background [15].

One final thing we will note is the conservation of stress-energy, which arises from the Bianchi identity on the Einstein tensor (B.3) we have

$$\bar{D}^\mu(\bar{R}_{\mu\nu} - \tfrac{1}{2}\bar{g}^{\mu\nu}\bar{R}) = 0. \quad (3.54)$$

By taking the covariant derivative of eq. (3.46) we realise we have

$$\bar{D}^\mu(\bar{T}_{\mu\nu} + t_{\mu\nu}) = 0. \quad (3.55)$$

Thus it is the sum of the energy from the source and GWs that is covariantly conserved, rather than the individual terms. Reflecting the transfer of energy and momentum between the source and GWs.

3.4.4 The stress-energy tensor in the TT-gauge

We will now calculate the stress-energy in the TT-gauge of a single plane wave travelling in the z-direction in asymptotically flat space. The TT-gauge includes both the Lorentz gauge condition and the traceless condition already included in equation (3.52) so we only have to add the transverse gauge condition (3.13a) to it. so in the TT-gauge $t_{\mu\nu}$ takes the form

$$t_{\mu\nu} = \frac{1}{32\pi}\langle \partial_\mu h^{\mathrm{TT}}_{\alpha\beta}\partial_\nu h^{\alpha\beta}_{\mathrm{TT}}\rangle. \quad (3.56)$$

We take the real part of the plane wave (3.2) and set the phase such that we have a sine wave

$$h^{\mathrm{TT}}_{\alpha\beta} = A_{\mu\nu}\sin(k_\alpha x^\alpha). \quad (3.57)$$

Inserting this expression for $h_{\alpha\beta}^{\mathrm{TT}}$ into equation (3.56) we get

$$t_{\mu\nu} = \frac{1}{32\pi} k_\mu k_\nu A_{\alpha\beta} A^{\alpha\beta} \left\langle \cos^2(k_\lambda k^\lambda) \right\rangle. \tag{3.58}$$

The average of $\cos^2$ over several wavelengths is

$$\left\langle \cos^2(k_\lambda k^\lambda) \right\rangle = \frac{1}{2}. \tag{3.59}$$

To evaluate the other factors in equation (3.58) we use that the wave vector is given by

$$k_\mu = (-\omega, 0, 0, \omega), \tag{3.60}$$

and use equation (3.28) to evaluate the amplitude matrices

$$A_{\alpha\beta} A^{\alpha\beta} = 2(h_+^2 + h_\times^2). \tag{3.61}$$

Angular frequencies are not common to use when working with GWs so we instead use $f = \omega/2\pi$, inserting this back into equation (3.58) we get [7]

$$t_{\mu\nu} = \frac{\pi}{8} f^2 (h_+^2 + h_\times^2) \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \tag{3.62}$$

We will get a better understanding for the magnitude of the stress-energy tensor when we will direct our focus towards how GWs are generated and how we can detect them.

3.4.5 The energy flux

The energy flux of GW is the amount of energy carried by a GW through a unit surface far away from the source. In order to find the flux will first find the change of energy within a volume V with the shape of a spherical shell, which is placed far away from the source at which it is centred on. The energy within V is given by

$$E_V = \int_V d^3x \; t^{00}, \tag{3.63}$$

where t^{00} is the energy-density of the GWs in the volume. Differentiating with respect to time we get the change of energy

$$\frac{d}{dt} E_V = \int_V d^3x \; \partial_t t^{00}. \tag{3.64}$$

We can rewrite this using he conservation of energy derived from the Bianchi identity (3.55) which reduces to

$$\partial^\mu t_{\mu\nu} = 0, \tag{3.65}$$

when we are far away from the source in asymptotically flat space as both $T^{\mu\nu}$ and the Christoffel symbols vanish. Letting $\nu = 0$ we get

$$\partial_0 t^{00} + \partial_i t^{i0} = 0. \tag{3.66}$$

We substitute this into eq. (3.64) and get

$$\frac{d}{dt} E_V = -\int_V d^3\, x \partial_i t^{i0} = -\int_S dA\, n_i t^{0i}, \tag{3.67}$$

where in the last equality we used the divergence theorem (A.10) and n^i are the normal vector components to the outer surface S, and dA is a surface element. We denote the outer surface with a radius r which is still far away from the source, the surface element is thus $dA = r^2 d\Omega$, where $d\Omega$ is the solid angle element. Due to the spherical symmetry the normal vector becomes $\hat{n} = \hat{r}$, the unit vector in the radial direction, so we have

$$\frac{d}{dt} E_V = -\int_S dA\, t^{0r}. \tag{3.68}$$

The region we look at is in flat space so we can impose the TT-gauge, which has the following expression for the stress-energy tensor (3.56)

$$t^{0r} = \frac{1}{32\pi} \langle \partial^0 h_{ij}^{\mathrm{TT}} \partial_r h_{ij}^{\mathrm{TT}} \rangle. \tag{3.69}$$

So far we have not found an expression for h_{ij}^{TT}, which we shall look into in Section 4.1, where we will find that it can be expressed on the general form

$$h_{ij}^{\mathrm{TT}}(t, r) = \frac{1}{r} f_{ij}(t - r), \tag{3.70}$$

where f is a function of the retarded time. Therefore we have

$$\partial_r h_{ij}^{\mathrm{TT}}(t, r) = -\frac{1}{r^2} f_{ij}(t - r) + \frac{1}{r} \partial_r f_{ij}(t - r) \tag{3.71}$$

Functions on the form $f(t - r)$ have the property

$$\partial_r f_{ij}(t - r) = -\partial_t f_{ij}(t - r), \tag{3.72}$$

which gives us

$$\partial_r h_{ij}^{\mathrm{TT}}(t, r) = -\frac{1}{r^2} f_{ij}(t - r) - \partial_t f_{ij}(t - r). \tag{3.73}$$

Using that $\partial^r h_{ij}^{\mathrm{TT}}(t, r) = \frac{1}{r} \partial_t f_{ij}(t - r)$ we get

$$\partial_r h_{ij}^{\mathrm{TT}}(t, r) = \partial^0 h_{ij}^{\mathrm{TT}}(t, r) + \mathcal{O}(1/r^2), \tag{3.74}$$

where for large distances we can ignore the terms of order $\mathcal{O}(1/r^2)$, thus using eq. (3.69) we see that for large distances $t^{0r} = t^{00}$. Letting us write the total energy change inside the volume V as

$$\frac{d}{dt} E_V = -\int dA t^{00}, \tag{3.75}$$

as we can not have a negative energy density the energy travelling through the outer surface of V must be negative. This energy transport must be the energy carried away by the GW, whose energy per surface element dA must in turn be

$$\frac{d}{dtdA}E = t^{00} = \frac{1}{32\pi}\langle\partial^0 h_{ij}^{\mathrm{TT}}\partial^0 h_{ij}^{\mathrm{TT}}\rangle. \tag{3.76}$$

Thus for a surface element $dA = r^2 d\Omega$ we have the total energy change

$$\frac{dE}{dt} = \frac{r^2}{32\pi}\int d\Omega\langle\partial^0 h_{ij}^{\mathrm{TT}}\partial^0 h_{ij}^{\mathrm{TT}}\rangle. \tag{3.77}$$

However we still have to take care of the average $\langle\partial^0 h_{ij}^{\mathrm{TT}}\partial^0 h_{ij}^{\mathrm{TT}}\rangle$, which is over either the wavelength or the frequency of the wave. The most convenient choice for GW detectors is the frequency as it is done over time rather than space. For the case of an frequency average we can effectively remove the need for averaging by using eq. (3.76) and taking the integral from $t = -\infty \to t = \infty$, giving us the total energy flowing through an area dA,

$$\frac{d}{dA}E = \frac{1}{32\pi}\int_{-\infty}^{\infty} dt\langle\partial^0 h_{ij}^{\mathrm{TT}}\partial^0 h_{ij}^{\mathrm{TT}}\rangle. \tag{3.78}$$

After the integral has been performed we will be left with a time-independent average, which for the case of a purely temporal average will be over a constant. The averaging can thus be omitted and we get

$$\frac{d}{dA}E = \frac{1}{32\pi}\int_{-\infty}^{\infty} dt(\partial^0 h_{ij}^{\mathrm{TT}}\partial^0 h_{ij}^{\mathrm{TT}}), \tag{3.79}$$

which is the total energy carried by a GW through an area dA.

Chapter 4

Generation of Gravitational Waves

In section 3.4 we elaborated on and found an expression for the energy carried by a GW, but as we know energy can not be created from nothing so a natural next step is to look at the origin of GWs. To get an idea of where we are headed we can compare to the study of electromagnetic radiation, where one considered the contribution from electric monopoles, dipoles, and quadrupoles. Converting them to their gravitational analogues we have the mass monopole, which is the total mass-energy, which due to conservation of energy and mass do not radiate. For the dipole we have two equivalents, the electric and magnetic dipole moment, the radiation of which are given by their second time derivatives. The analogues for mass dipoles after differentiation is linear and angular momentum respectively, both of which are conserved quantities and will not result in radiation. This leaves us with the mass quadrupole moment which in its simplest form is generated by a binary system, which we would expect to give a non-zero result due to the observations of the Hulse-Taylor binary [13]. In this chapter we will follow the work of Maggiore [14].

4.1 Sources at Weak Fields

Our goal is to find an expression for the perturbation $h_{\mu\nu}$ in asymptotically flat space which would originate from a source. As we are far away from the source we return to the LFE in the Lorentz gauge (2.29)

$$\Box \bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}. \tag{4.1}$$

In this section we will follow the procedure of Maggiore [14]. As we know it is not very hard to solve it for a vacuum, but to find the effect from a source we naturally have to solve it while we keep the stress-energy tensor. We also note that $T_{\mu\nu}$ is conserved, so it obeys the flat space conservation law $\partial^\mu T_{\mu\nu} = 0$. Through the use of a Green's function $G(x^\mu - x'^\mu)$ we are able to solve equation (4.1), more specifically we want the Green's function for the d'Alembertian which gives us the solution of the wave equation in the presence of a delta source term [7]

$$\Box_x G(x^\mu - x^{\mu\prime}) = \delta^(4)(x^\mu - x'^\mu), \tag{4.2}$$

where $\Box_x$ is the d'Alembertian with respect to x^μ. Multiplying both sides of equation (4.2) with $T_{\mu\nu}(x^{\mu\prime})$ and integrating with respect to $x^{\mu\prime}$ we get

$$\int d^4x' \Box_x G(x^\mu - x'^\mu) T_{\mu\nu}(x'^\mu) = \int d^4x' \delta^(4)(x^\mu - x'^\mu) T_{\mu\nu}(x'^\mu). \tag{4.3}$$

The d'Alembertian is with respect to x^μ so we can take it outside the integral. It is worth mentioning the lack of the factor $\sqrt{-g}$ in the integral as we operate in flat spacetime. The right hand side is simply evaluated to be $T_{\mu\nu}(x^\mu)$, inserting into equation (4.1) we end up with

$$\bar{h}_{\mu\nu}(x^\mu) = -16\pi \int d^4x' G(x^\mu - x'^\mu) T_{\mu\nu}(x'^\mu). \tag{4.4}$$

So in order to find the perturbation produced by a wave $\bar{h}_{\mu\nu}(x^\mu)$ we have to find a function $G(x^\mu - x^{\mu\prime})$ which satisfies equation (4.2). This is a lengthy process so to not distract from the main goal of the section the calculations are shown in Appendix C.1. To summarise you use the Fourier transform of equation (4.2) and solve the resulting PDEs. From equation (C.20) we get

$$G(x^\mu - x'^\mu) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}\delta(x^0_\mathrm{r} - x'^0), \tag{4.5}$$

where $\mathbf{x}$ are spatial vectors and "r" denotes the retarded coordinates and we have defined the retarded time as

$$x^0_\mathrm{r} = t_\mathrm{r} = t - |\mathbf{x} - \mathbf{x}'|. \tag{4.6}$$

So the solution of equation (C.5) is

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4\pi \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{\mu\nu}\big(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}'\big), \tag{4.7}$$

where we integrated over x'^0. This general formula gives us that the perturbation is the sum of the stress-energy at the retarded time, which is the time at which the source started emitting radiation to an observer. Far away from the source we can transform this solution to the TT-gauge by using the Lambda tensor (3.18), where we use that it is traceless to get $h^{\mathrm{TT}}_{ij} = \Lambda_{ij,kl}\bar{h}_{kl}$. In the TT-gauge equation (4.7) becomes

$$h^{\mathrm{TT}}_{ij}(t, \mathbf{x}) = 4\Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{kl}\big(t - |\mathbf{x} - \mathbf{x}'|, x'\big), \tag{4.8}$$

where we let $\hat{\mathbf{x}} = \hat{\mathbf{n}}$ and will denote $|\mathbf{x}| = r$. The reason we are able to write the stress-energy tensor using only spatial components is because the spatial components and temporal ones are related through the conservation of stress-energy tensor. To see how h^{TT}_{ij} behaves at large distances from a source of radius d, where we have $r \gg d$, we expand

$$|\mathbf{x} - \mathbf{x}'| = r - \mathbf{x}' \cdot \hat{\mathbf{n}} + \mathcal{O}(d^2/r). \tag{4.9}$$

We now take the limit $r \to \infty$ at fixed time we are only left with r. So far away from the source we have

$$h_{ij}^{\mathrm{TT}}(t, \mathbf{x}) = 4\Lambda_{ij,kl}(\hat{\mathbf{n}})\frac{1}{r}\int d^3x'\; T_{kl}\big(t - r + \mathbf{x}' \cdot \hat{\mathbf{n}}, x'\big), \tag{4.10}$$

where we have neglected the terms of $\mathcal{O}(1/r^2)$. As with most things that oscillate we would like to to rewrite T_{kl} using the Fourier transform in order to have them in terms of the frequency

$$T_{kl}(t, \mathbf{x}) = \frac{1}{(2\pi)^4}\int d^4k\; \tilde{T}_{kl}(\omega, \mathbf{k})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}. \tag{4.11}$$

We now want to evaluate the integrand of equation (4.10)

$$\begin{aligned}\int d^3x' T_{kl}\big(t - r + \mathbf{x}' \cdot \hat{\mathbf{n}}, x'\big) =& \frac{1}{(2\pi)^4}\int d^3x' \int d\omega d^3k \tilde{T}_{kl}(\omega, \mathbf{k})e^{-i\omega(t-r)}e^{i(\mathbf{k}-\omega\hat{\mathbf{n}})\cdot\mathbf{x}}\\ =& \frac{1}{(2\pi)^4}\int d\omega d^3k\; \tilde{T}_{kl}(\omega, \mathbf{k})e^{-i\omega(t-r)}(2\pi)^3\delta^3(\mathbf{k} - \omega\hat{\mathbf{n}})\\ =& \frac{1}{2\pi}\int d\omega\; \tilde{T}_{kl}(\omega, \omega\hat{\mathbf{n}})e^{-i\omega(t-r)},\end{aligned} \tag{4.12}$$

we thus get

$$h_{ij}^{\mathrm{TT}}(t, \mathbf{x}) = 4\Lambda_{ij,kl}(\hat{\mathbf{n}})\frac{1}{r}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\; \tilde{T}_{kl}(\omega, \omega\hat{\mathbf{n}})e^{-i\omega(t-r)}. \tag{4.13}$$

We have only made assumptions about the position relative to the source so we have found a general expression valid for both relativistic and non-relativistic speeds. By setting $dA = r^2 d\Omega$ in eq. (3.79) we get the total energy per unit solid angle

$$\frac{d}{d\Omega}E = \frac{r^2}{32\pi}\int_{-\infty}^{\infty} dt\; (\partial^0 h_{ij}^{\mathrm{TT}}\partial^0 h_{ij}^{\mathrm{TT}}). \tag{4.14}$$

By inserting eq. (4.13) we get an integral of the form $\int_{-\infty}^{\infty} dt\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega d\nu$, where ω and ν are frequencies. Through a plane wave expansion we are able to use the property $\tilde{T}_{kl}(-\omega, -\omega\hat{\mathbf{n}}) = \tilde{T}_{kl}^{*}(\omega, \omega\hat{\mathbf{n}})$ to write the integral over the frequencies the following way [14]

$$\begin{aligned}\int_{-\infty}^{\infty} d\omega\; \tilde{T}_{kl}(\omega, \omega\hat{\mathbf{n}})e^{-i\omega(t-r)} = \int_{0}^{\infty} d\omega\; \big(&\tilde{T}_{kl}(\omega, \omega\hat{\mathbf{n}})e^{-i\omega(t-r)}\\ &+ \tilde{T}_{kl}^{*}(\omega, \omega\hat{\mathbf{n}})e^{i\omega(t-r)}\big).\end{aligned} \tag{4.15}$$

Using this expression we differentiate with respect to time and multiply both frequency integrands together. We notice that only the exponential factors are time dependent and that they can be evaluated using the Fourier representation of the δ-function

$$\delta(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} d\xi e^{-i\xi x}, \tag{4.16}$$

as we have limited ourselves to positive frequencies the cross terms are the only ones that will have a non-zero δ-function, which is the case when $\omega = \nu$. The general equation for the energy per unit solid angle is thus

$$\frac{dE}{d\Omega} = \frac{\Lambda_{ij,kl}(\hat{\mathbf{n}})}{2\pi^2}\int\limits_{0}^{\infty} d\omega\, \omega^2 \tilde{T}_{ij}(\omega,\omega\hat{\mathbf{n}})\tilde{T}^*_{kl}(\omega,\omega\hat{\mathbf{n}}), \tag{4.17}$$

where we used that the lambda tensor is a projector, $\Lambda_{ij,ab}\Lambda_{ab,kl} = \Lambda_{ij,kl}$.

4.2 Low Velocity Expansion

One way of simplifying the equations for radiation is to assume that the velocities inside the source is much smaller than the speed of light. If we let the typical frequency inside the source be denoted by ω_s and the source have radius d we get the velocity $v \sim \omega_s d$ inside the source. Here the source is the region with a configuration of masses that will produce GWs. As the frequency of radiation ω will be dependent on the motion inside the source it is fair to assume it will be of order ω_s. Using that the wavelength of said radiation is $\lambda = 1/\omega$, it will be of order

$$\lambda \sim \frac{d}{v}. \tag{4.18}$$

So for velocities in the non-relativistic range, where $v \ll 1$, the reduced wavelength will be much larger than the radius of the source, $\lambda \gg d$. This means that the finer details of the motion inside the source do not contribute a lot to the overall radiation, so we care more about the courser features i.e. the lower order multipole moments. From our analysis in the introduction of this chapter we know that the lowest order multipole moment should be a quadrupole moment. We will start the multipole expansion by considering the expression for h^{TT}_{ij} in asymptotically flat space, which we recall is given by eq. (4.10)

$$h^{\mathrm{TT}}_{ij}(t,\mathbf{x}) = 4\Lambda_{ij,kl}(\hat{\mathbf{n}})\frac{1}{r}\int d^3x'\, T_{kl}\big(t - r + \mathbf{x}'\cdot\hat{\mathbf{n}}, x'\big), \tag{4.19}$$

where T_{kl} is given in terms of its Fourier transform

$$T_{kl}(t - r + \mathbf{x}'\cdot\hat{\mathbf{n}}, x') = \frac{1}{(2\pi)^4}\int d^4k\, \tilde{T}_{kl}(\omega,\mathbf{k})e^{-i\omega(t-r+\mathbf{x}'\cdot\hat{\mathbf{n}})+i\mathbf{k}\cdot\mathbf{x}'}. \tag{4.20}$$

The stress-energy tensor T_{ij} is only non-vanishing inside the source, so the integral in eq. (4.19) is restricted to the region $|\mathbf{x}'| \leq d$, where we remember that the primed coordinates denote a position of radiation within the source. If we let the peak of the non-relativistic source $\tilde{T}_{kl}(\omega, \mathbf{k})$ be around the frequency ω_s, the dominant contribution to h_{ij}^{TT} must be given by a frequency ω that satisfies

$$\omega \mathbf{x}' \cdot \hat{\mathbf{n}} \lesssim \omega_s d \ll 1. \tag{4.21}$$

Given this we can expand the exponential in (4.20) as follows

$$e^{-i\omega(t-r+\mathbf{x}'\cdot\hat{\mathbf{n}})+i\mathbf{k}\cdot\mathbf{x}'} = e^{-i\omega(t-r)}\Big[1 - i\omega x'^i n^i + \tfrac{1}{2}(-i\omega)^2 x'^i x'^j n^i n^j + ...\Big], \tag{4.22}$$

which we see is equivalent to expanding

$$T_{kl}(t - r + \mathbf{x}' \cdot \hat{\mathbf{n}}, x') \simeq T_{kl}(t - r, x') + x'^i n^i \partial_t T_{kl} + \tfrac{1}{2} x'^i x'^j n^i n^j \partial_t^2 T_{kl} + ..., \tag{4.23}$$

where the derivatives are evaluated at $(t - r, x')$. We insert this into eq. (4.19) and define the following tensors which are the momenta of the stress tensor T^{ij}

$$S^{ij}(t) = \int d^3x \; T^{ij}(t, \mathbf{x}), \tag{4.24a}$$

$$S^{ij,k}(t) = \int d^3x \; T^{ij}(t, \mathbf{x}) x^k, \tag{4.24b}$$

$$S^{ij,kl}(t) = \int d^3x \; T^{ij}(t, \mathbf{x}) x^k x^l. \tag{4.24c}$$

Using these momenta we can write h_{ij}^{TT} as

$$h_{ij}^{\mathrm{TT}}(t, \mathbf{x}) = 4\Lambda_{ij,kl}(\hat{\mathbf{n}}) \frac{1}{r} \Big[S^{kl} + n_m \dot{S}^{kl,m} + \tfrac{1}{2} n_m n_p \ddot{S}^{kl,mp}\Big]_{\mathrm{ret}}, \tag{4.25}$$

where "ret" means it is evaluated at retarded time. The tensors $S^{ij,\cdots}$ must be symmetric in the first two indices as they are defined in terms of the symmetric tensor T^{ij}. Each factor x^m is of order $\mathcal{O}(d)$ and each time derivative of S^{ij} gives a factor of order $\mathcal{O}(\omega_s)$, so $\dot{S}kl, m$ will give an additional factor of order $\mathcal{O}(\omega_s d = v)$, and thus $\ddot{S}^{kl,mp}$ a factor $\mathcal{O}(v^2)$. In order to get a clearer physical interpretation of the various terms in the above expansion we would like to replace the momenta of T^{ij} with the momenta of the energy density T^{00} and the momenta of the momentum density T^{0i}. The momenta of T^{00} we define as

$$M = \int d^3x \; T^{00}(t, \mathbf{x}), \tag{4.26}$$

$$M^i = \int d^3x \; T^{00}(t, \mathbf{x}) x^i, \tag{4.27}$$

$$M^{ij} = \int d^3x \; T^{00}(t, \mathbf{x}) x^i x^j, \tag{4.28}$$

$$M^{ijk} = \int d^3x \; T^{00}(t, \mathbf{x}) x^i x^j x^k. \tag{4.29}$$

The momenta of momentum density we define as

$$P^i = \int d^3x \; T^{0i}(t, \mathbf{x}), \tag{4.30}$$

$$P^{i,j} = \int d^3x \, T^{0i}(t, \mathbf{x}) x^j, \tag{4.31}$$

$$P^{i,jk} = \int d^3x \, T^{0i}(t, \mathbf{x}) x^j x^k. \tag{4.32}$$

We are working in the linearised theory and therefore we work with the assumption that $T^{\mu\nu}$ impacts the surrounding spacetime minimally such that we can assume flat-space and the relation

$$\partial_\mu T^{\mu\nu} = 0. \tag{4.33}$$

This means we are neglecting the effects the GWs have on the source. So for a volume V larger than the source with boundary ∂V we have

$$\begin{aligned} \dot{M} &= \int\limits_V d^3x \, \partial_0 T^{00} = - \int\limits_V d^3x \, \partial_i T^{0i} \\ &= - \int\limits_{\partial V} d\Sigma^i \, T^{0i} \\ &= 0, \end{aligned} \tag{4.34}$$

where Σ is the surface and used that $T^{\mu\nu}$ vanishes outside the source. So in the linearised theory system that radiates GWs will not lose mass, which shows some of the limitations with using a linear model, as a physical system that radiates GWs will lose mass [14]. For the other momenta of T^{00} we get

$$\begin{aligned} \dot{M}^i &= \int\limits_V d^3x \, x^i \partial_0 T^{00} = - \int\limits_V d^3x \, x^i \partial_j T^{0j} \\ &= \int\limits_V d^3x \, (\partial_i x^i) T^{0j} = \int\limits_V d^3x \, (\delta^i_j) T^{0j} \\ &= P^i, \end{aligned} \tag{4.35}$$

and by using a similar procedure we calculate the rest of the momenta

$$\dot{M} = 0, \tag{4.36a}$$

$$\dot{M}^i = P^i, \tag{4.36b}$$

$$\dot{M}^{ij} = P^{i,j} + P^{j,i}, \tag{4.36c}$$

$$\dot{M}^{ijk} = P^{i,jk} + P^{j,ki} + P^{k,ij}. \tag{4.36d}$$

For the momenta of T^{0i} we get

$$\dot{P}^i = 0, \tag{4.37a}$$

$$\dot{P}^{i,j} = S^{ij}, \tag{4.37b}$$

$$\dot{P}^{i,jk} = S^{ij,k} + S^{ik,j}. \tag{4.37c}$$

The equations $\dot{M} = 0$ and $\dot{P}^i = 0$ are the conservation of mass and momentum respectively, which is what we expected form our qualitative analysis earlier. Using the symmetry of S^{ij} we get from eq. (4.37b) the equation

$$\dot{P}^{i,j} - \dot{P}^{j,i} = S^{ij} - S^{ji} = 0, \tag{4.38}$$

which gives us conservation of the angular momentum of the source. We will now use these momenta to express the momenta of S^{ij} and $\dot{S}^{ij,k}$. Differentiating eq. (4.36c) and inserting eq. (4.37b) we get the identity

$$S^{ij} = \tfrac{1}{2}\ddot{M}^{ij}. \tag{4.39}$$

If we differentiate eq. (4.36d) twice and insert eq. (4.37c) we get

$$\dddot{M}^{ijk} = 2(\dot{S}^{ij,k} + \dot{S}^{ik,j} + \dot{S}^{jk,i}), \tag{4.40}$$

which we can insert the derivative of eq. (4.37c) into and get

$$\dot{S}^{ij,k} = \tfrac{1}{6}\dddot{M}^{ijk} + \tfrac{1}{3}(\ddot{P}^{i,jk} + \ddot{P}^{j,ik} - 2\ddot{P}^{k,ij}). \tag{4.41}$$

We have thus expressed the two lowest orders of momenta of the source in terms of its momentum and mass, which we will take advantage of in the next section where we will derive the radiation emitted from a quadrupole.

4.3 Mass Quadrupole Radiation

We will now consider the leading term of eq. (4.25) which we will rewrite using eq. (4.39)

$$h_{ij}^{\mathrm{TT}}(t, \mathbf{x}) = 2\Lambda_{ij,kl}(\hat{\mathbf{n}})\frac{1}{r}\ddot{M}^{kl}(t_{\mathrm{r}}). \tag{4.42}$$

The tensor M^{kl} can be decomposed into an irreducible representation

$$M^{kl} = (M^{kl} - \tfrac{1}{3}\delta^{kl}M_{ii}) + \tfrac{1}{3}\delta^{kl}M_{ii}, \tag{4.43}$$

here we effectively have created a traceless term, where M_{ii} is the trace of M_{kl}. The Lambda tensor when contracted will result in a traceless-transverse tensor, so when contracting with eq. (4.43) the last term will vanish. The traceless term is thus the only contributing term, which we recognise from electromagnetism as the quadrupole moment

$$Q^{ij} \equiv M^{ij} - \tfrac{1}{3}\delta^{ij}M_{kk} = \int d^3x\ \rho(t, \mathbf{x})(x^i x^j - \tfrac{1}{3}r^2\delta^{ij}), \tag{4.44}$$

where we use $\rho = T^{00}$ for the energy density of the system, a quantity which includes the contributions from the rest mass, and the kinetic and potential energy of the system. The lowest order contribution to GWs is thus the quadrupole moment, inserting it back into eq. (4.42) we get

$$[h_{ij}^{\mathrm{TT}}(t, \mathbf{x})]_{\mathrm{quad}} = 2\Lambda_{ij,kl}(\hat{\mathbf{n}})\frac{1}{r}\ddot{Q}^{kl}(t_{\mathrm{r}}). \tag{4.45}$$

For completeness sake we contract the Lambda tensor with the quadrupole moment and by design we get

$$[h_{ij}^{\mathrm{TT}}(t,\mathbf{x})]_{\text{quad}} \equiv 2\frac{1}{r}\ddot{Q}_{ij}^{\mathrm{TT}}(t_{\mathrm{r}}). \tag{4.46}$$

We have thus managed to show that, as we suspected, the lowest order momenta which contributes to the emittance of GWs is the quadrupole moment.

4.3.1 Quadrupole radiation in an arbitrary direction

To find the quadrupole radiation emitted in an arbitrary direction $\hat{\mathbf{n}}$, we will first find it for the z-direction in the TT-gauge. The first step is contracting the Lambda tensor with the quadrupole moment to get an explicit expression. From the definition (**??**) we see that only the term M^{ij} contribute as contracting with the Kroenecker δ-function results in it vanishing. It is thus equivalent to use $\ddot{M}^{ij}$ instead of $\ddot{Q}^{ij}$, for the sake of physical interpretation we will use the former. We have already calculated the effect of the Lambda tensor for the case of wave travelling in the z-direction in section 3.1.2, so using eq. (3.22) we get

$$\Lambda_{ij,kl}(\hat{\mathbf{n}})\ddot{M}_{kl} = \begin{pmatrix} \frac{1}{2}(\ddot{M}_{11}-\ddot{M}_{22}) & \ddot{M}_{12} & 0 \\ \ddot{M}_{12} & -\frac{1}{2}(\ddot{M}_{11}-\ddot{M}_{22}) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.47}$$

The two polarisation amplitudes follows naturally as

$$h_{+} = \frac{1}{r}[\ddot{M}^{11}(t_{\mathrm{r}}) - \ddot{M}^{22}(t_{\mathrm{r}})], \tag{4.48}$$

$$h_{\times} = \frac{2}{r}\ddot{M}^{12}(t_{\mathrm{r}}). \tag{4.49}$$

Now that we know the amplitudes for the z-direction we will would like to find the momenta for an arbitrary direction $\hat{\mathbf{n}}$ in a coordinate system (x, y, z). To accomplish this we will introduce the unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, which are orthogonal to $\hat{\mathbf{n}}$ and each other. Now we let $\hat{\mathbf{u}} \times \hat{\mathbf{v}} = \hat{\mathbf{n}}$ and define the vectors $(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{n}})$ to be the axes of a coordinate system (x', y', z'), thus our GW will propagate in the z'-direction. As the polarisation modes we are given in the plane transverse to the direction of propagation we can write them as

$$h_{+} = \frac{1}{r}[\ddot{M}'^{11}(t_{\mathrm{r}}) - \ddot{M}'^{22}(t_{\mathrm{r}})], \tag{4.50}$$

$$h_{\times} = \frac{2}{r}\ddot{M}'^{12}(t_{\mathrm{r}}). \tag{4.51}$$

The next step is to convert them to the unprimed frame, (x, y, z). A good place to start is how the vector $\hat{\mathbf{n}}$ is related in the two coordinate systems, in the primed frame it has the coordinates $n'_i = (0, 0, 1)$ and in the unprimed system it has the coordinates $n_i = (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$. The relation between them can thus be expressed through a rotation matrix $\mathcal{R}$ as $n_i = \mathcal{R}_{ij}n'_j$. A rotation matrix which takes the form

$$\mathcal{R} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}. \tag{4.52}$$

Our momenta in the unprimed frame will thus be

$$M_{ij} = \mathcal{R}_{ik}\mathcal{R}_{jl}M'_{kl}. \tag{4.53}$$

Using this relation we are able to write the polarisation modes in terms of polar coordinates which gives us the ability to calculate the angular distribution of the quadrupole moment.

4.3.2 Power radiated from a quadrupole

To find the energy contribution from the quadrupole moment gives to the radiated energy we just need to insert eq. (4.45) into the general eq. for the radiated energy (3.77)

$$P = \frac{r^2}{32\pi}\int d\Omega\ \Lambda_{ij,kl}\langle \dddot{Q}_{ij}\dddot{Q}_{kl}\rangle. \tag{4.54}$$

Only $\Lambda_{ij,kl}(\hat{\mathbf{n}})$ has angular dependence and is thus the only factor we have to evaluate in the integral, which we now from eq. (3.19) can be expressed as a product of the components of $\hat{\mathbf{n}}$. There are thus terms consisting of products of two and four vector components, which we can solve using the identities

$$\frac{1}{4\pi}\int d\Omega\ n_i n_j = \frac{1}{3}\delta_{ij}, \tag{4.55}$$

$$\frac{1}{4\pi}\int d\Omega\ n_i n_j n_k n_l = \frac{1}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{4.56}$$

Here we used that both tensors $n_i n_j$ and $n_i n_j n_k n_l$ are symmetric and therefore the integral can only depend on the symmetric product of Kronecker deltas [14]. Evaluating the angular integral of the Lambda tensor we get

$$\begin{aligned}\int d\Omega\ \Lambda_{ij,kl} = \int d\Omega\ \big(&\delta_{ik}\delta_{jl} - \tfrac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l\delta_{ik} - n_i n_k\delta_{jl} \\ &+ \tfrac{1}{2}n_k n_l\delta_{ij} + \tfrac{1}{2}n_i n_j\delta_{kl} + \tfrac{1}{2}n_i n_j n_k n_l\big),\end{aligned} \tag{4.57}$$

which after applying eq. (4.55) and (4.56) becomes

$$\int d\Omega\ \Lambda_{ij,kl} = \frac{2\pi}{15}(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}). \tag{4.58}$$

The radiated power from a quadrupole is thus

$$P_{\text{quad}} = \frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle. \tag{4.59}$$

This equation is known as the quadruopole formula, which was derived by Einstein. As we see it tells us that the power generated from a quadrupole is dependent on the jerk of the quadrupole averaged over time and evaluated at the retarded time $t - r$.

4.3.3 The reaction on non-relativistic sources

We would now like to direct our attention towards what happens to the source itself as it emits GWs. To do so we will evaluate what happens to a non-relativistic source that emits a GW at retarded time, $t - r$, which is observed at a large distance r from the source. For this we will stick with the linear theory of gravity, thus not taking into account the effects a curved space-time might have on the transfer of energy from the source to the distance r. In section 4.3.2 we found that the power observed at this large distance away from the source is given by eq. (4.59), as we are in flat spacetime the power lost from the source at time $t - radiationr$ must therefore be the negative of this

$$P_{\text{source}} = -\frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle. \tag{4.60}$$

It is worth noting that the averaging is evaluated at retarded time both for eq. (4.59) and eq. (4.60). As we have assumed a non-relativistic source we can safely use Newtonian mechanics to find the back-reaction on the source in terms of a force $\mathbf{F}$. From the definition of power we have

$$\frac{dE_{\text{source}}}{dt} = \langle F_i v_i \rangle, \tag{4.61}$$

which for the source we have written in terms of the averaged power at the retarded time, as the energy is only defined if we take the average over several periods. For an extended body we can write this as

$$\frac{dE_{\text{source}}}{dt} = \Big\langle \int d^3x \, \frac{dF_i}{dV} \dot{x}_i \Big\rangle, \tag{4.62}$$

dF_i/dV is the force per unit volume. To write this force in terms of the quadrupole moment we will rewrite eq. (4.60) using integration by parts such that we only have one factor be a first order derivative, using the properties laid out in Section 3.4.3 we get

$$\frac{dE_{\text{source}}}{dt} = -\frac{1}{5}\langle \frac{dQ_{ij}}{dt} \frac{d^5 Q_{ij}}{dt^5} \rangle. \tag{4.63}$$

The reason for this movement of derivatives will become apparent when we use the definition of the quadrupole moment (4.44) to find its time derivative

$$\frac{dQ^{ij}}{dt} = \int d^3x' \; \partial_t \rho(t, \mathbf{x}')(x'^i x'^j - \tfrac{1}{3} r^2 \delta^{ij}), \tag{4.64}$$

and observe that the term involving δ^{ij} will vanish when contracted with $d^5 Q_{ij}/dt^5$, as Q_{ij} is traceless. The conservation of the stress-energy tensor gives us the continuity equation

$$\partial_0 T^{00} + \partial_i T^{0i} = 0, \tag{4.65}$$

which for our Newtonian source is

$$\partial_t \rho + \partial_i(\rho v_i) = 0. \tag{4.66}$$

Using the continuity equation (4.66) we can substitute $\partial_t \rho$ in eq. (4.64) with $-\partial_i(\rho v_i)$, we can then integrate it by parts

$$\begin{aligned} \frac{dQ^{ij}}{dt} &= \int d^3x' \; \partial_t \rho(t, \mathbf{x}') x'^i x'^j = - \int d^3x' \; \partial_k [\rho(t, \mathbf{x}') \dot{x}_k] x'_i x'_j \\ &= - \, x'_i x'_j \int d^3x' \; \partial_k [\rho(t, \mathbf{x}') \dot{x}_k] + \int d^3x' \; \rho(t, \mathbf{x}') \dot{x}_k \partial_k (x'_i x'_j), \end{aligned} \tag{4.67}$$

the first term is a surface integral that vanishes as ρ vanishes at infinity, so we get

$$\frac{dQ^{ij}}{dt} = \int d^3x' \; \rho(t, \mathbf{x}') \dot{x}'_k (\delta_{ik} x'_j + \delta_{jk} x'_i). \tag{4.68}$$

Inserting eq. (4.68) into our expression for the power radiated from the source (4.63)

$$\begin{aligned} \frac{dE_{\text{source}}}{dt} &= - \, \frac{1}{5} \Big\langle \frac{d^5 Q_{ij}}{dt^5} \int d^3x' \; \rho(t, \mathbf{x}') \dot{x}'_k (\delta_{ik} x'_j + \delta_{jk} x'_i) \Big\rangle \\ &= - \, \frac{2}{5} \Big\langle \frac{d^5 Q_{ij}}{dt^5} \int d^3x' \; \rho(t, \mathbf{x}') \dot{x}'_k x'_j \Big\rangle. \end{aligned} \tag{4.69}$$

Comparing this equation to the one we found for the power given by the forces acting inside the source (4.62) and moving $d^5 Q_{ij}/dt^5$ inside the integral, we observe that we have found an expression of said forces

$$\frac{dF_i}{dV'} = -\frac{2}{5} \frac{d^5 Q_{ij}}{dt^5}(t) \rho(t, \mathbf{x}') x'_j. \tag{4.70}$$

Integrating the over the unit volume we get the total force

$$F_i = -\frac{2}{5} \frac{d^5 Q_{ij}}{dt^5}(t) \int d^3x' \; \rho(t, \mathbf{x}') x'_j, \tag{4.71}$$

which we can write in terms of the centre of mass coordinates

$$x_j(t) \equiv \frac{1}{m} \int d^3x' \; \rho(t, \mathbf{x}') x'_j, \tag{4.72}$$

thus we get

$$F_i = -\frac{2}{5} \frac{d^5 Q_{ij}}{dt^5}(t) m x_j(t) \tag{4.73}$$

We have thus found the self-force of the source as it radiates by using the energy balance of the emitted GW at infinity. The self-force is thus causing the system to inspiral at a rate proportional to the fifth time derivative of the quadrupole moment of the system, this inspiraling is what we expect as a gravitational system radiates. The motion of particles in a gravitational field is governed by the metric, so one should be able to find the self-force through the metric in the near-source region. To be able to achieve this one need to find the corrections to the geodesic motion of the source imposed by the self-force. As the metric in the near-source region must be taken as curved we must use the non-linear field equations for the calculations, which for products of point like source distributions does only have solutions under special circumstances [16]. There are ways around this which will need a deeper look into the non-linear field equations.

4.3.4 Radiation from a closed system of point masses

Let us consider a system of closed point masses in order to find what the radiation from such a system can look like. We start by considering a point-like particle moving on a trajectory $x_0(t)$ in flat spacetime, whose stress-energy tensor is given by [17]

$$T^{\mu\nu}(t,\mathbf{x}) = p^{\mu}(t)\frac{dx_0^{\nu}(t)}{dt}\delta^3(\mathbf{x}-\mathbf{x}_0(t)). \tag{4.74}$$

For a system of free point particles with the label *n*, which are moving on trajectories $x_n(t)$, we have the total stress-energy tensor

$$\begin{aligned} T^{\mu\nu}(t,\mathbf{x}) =& \sum_n p_n^{\mu}(t)\frac{dx_n^{\nu}(t)}{dt}\delta^3(\mathbf{x}-\mathbf{x}_n(t)) \\ =& \sum_n \gamma_n m_n \frac{dx_n^{\mu}(t)}{dt}\frac{dx_n^{\nu}(t)}{dt}\delta^3(\mathbf{x}-\mathbf{x}_n(t)), \end{aligned} \tag{4.75}$$

where we used that in flat space the momentum $p^{\mu} = mdx_0^{\mu}/d\tau)$, where τ is the proper time, becomes $p^{\mu} = \gamma m(dx_0^{\mu}/dt$, where $\gamma = (1-v^2)^{-1/2}$. The energy density and flux for this system is given as

$$T^{\mu 0}(t,\mathbf{x}) = \sum_n p_n^{\mu}(t)\delta^3(\mathbf{x}-\mathbf{x}_n(t)), \tag{4.76}$$

which we see reduces to an expression only dependent on the velocity, mass, and trajectory of the particle. Let us consider the conservation of energy using the Bianchi identity (B.3) in flat spacetime, $\partial_{\mu}T^{\mu\nu} = 0$, on eq. (4.75) where we will start with the case of one spatial component

$$\begin{aligned} \frac{\partial}{\partial x^i}T^{\mu i}(t,\mathbf{x}) =& \frac{\partial}{\partial x^i}\sum_n p_n^{\mu}(t)\frac{dx_n^{i}(t)}{dt}\delta^3(\mathbf{x}-\mathbf{x}_n(t)) \\ =& -\sum_n p_n^{\mu}(t)\frac{dx_n^{i}(t)}{dt}\frac{\partial}{\partial x_n^i}\delta^3(\mathbf{x}-\mathbf{x}_n(t)). \end{aligned} \tag{4.77}$$

which we can rewrite using the chain rule and then we compare the result to eq. (4.76)

$$\begin{aligned} \frac{\partial}{\partial x^i}T^{\mu i}(t,\mathbf{x}) =& -\sum_n p_n^{\mu}(t)\frac{\partial}{\partial t}\delta^3(\mathbf{x}-\mathbf{x}_n(t)) \\ =& -\frac{\partial}{\partial t}T^{\mu 0} + \sum_n \frac{\partial p_n^{\mu}(t)}{\partial t}\delta^3(\mathbf{x}-\mathbf{x}_n(t)) \end{aligned} \tag{4.78}$$

Moving the first term on the RHS we get the Bianchi identity we sought out to find

$$\begin{aligned} \partial_{\mu}T^{\mu\nu} =& \sum_n \frac{\partial p_n^{\mu}(t)}{\partial t}\delta^3(\mathbf{x}-\mathbf{x}_n(t)) \\ =& \sum_n \gamma_n m_n \frac{d^2 dx_n^{\mu}(t)}{dt^2}\delta^3(\mathbf{x}-\mathbf{x}_n(t)) = 0. \end{aligned} \tag{4.79}$$

The geodesic equation (A.8), in flat spacetime reduces to $d^2x^\alpha/d\tau^2 = d^2x^\alpha/dt^2 = 0$, we therefore see that every particle, n, must follow their geodesics in flat spacetime. Thus it does not make sense to use this stress-energy tensor (4.75) for interacting particles as the interactions would cause the particles to deflect from their geodesic. We need to look past the linear theory to be able calculate the corrections which are needed for interacting particles in a self-gravitating system, however we are still able to use the linear theory to calculate the leading terms of the quadrupole radiation (4.25) while ignoring the interaction terms. The reason is that in the non-linear theory one finds that the full stress-energy tensor of a self-gravitating system are of order $\mathcal{O}(v^2)$, so for a non-relativistic two particle system we have [14]

$$T^{\mu\nu}(t,\mathbf{x}) = \sum_{n=1}^{2} m_n \frac{dx_n^\mu(t)}{dt}\frac{dx_n^\nu(t)}{dt}\delta^3(\mathbf{x}-\mathbf{x}_n(t)) + \mathcal{O}(v^2). \tag{4.80}$$

This gives us $T^{00} = \mathcal{O}(v^0)$, $T^{0i} = \mathcal{O}(v)$, and $T^{ij} = \mathcal{O}(v^2)$, which is why we are able to calculate T^{00} and T^{0i} to the lowest order as they will not include the interaction terms. These lowest order terms

$$T^{00} = \sum_n m_n \delta^3(\mathbf{x}-\mathbf{x}_n(t)), \tag{4.81}$$

$$T^{0i} = \sum_n m_n \frac{dx_n^i(t)}{dt}\delta^3(\mathbf{x}-\mathbf{x}_n(t)), \tag{4.82}$$

do fulfil the conservation of energy, $\partial_0 T^{00} + \partial_i T^{0i} = 0$. Following the same calculations as for eq. (4.77) we get

$$\partial_0 T^{00} = -\sum_n m_n \frac{dx_n^i(t)}{dt}\frac{\partial}{\partial x_n^i}\delta^3(\mathbf{x}-\mathbf{x}_n(t)), \tag{4.83}$$

and

$$\partial_i T^{0i} = \sum_n m_n \frac{dx_n^i(t)}{dt}\frac{\partial}{\partial x_n^i}\delta^3(\mathbf{x}-\mathbf{x}_n(t)), \tag{4.84}$$

which we see satisfy $\partial_0 T^{00} + \partial_i T^{0i} = 0$ independently of the trajectory x_n^i. On the other hand the conservation of energy for $\partial_\mu \tau^{\mu j}$, where $\tau^{\mu\nu}$ is the stress-energy tensor with non-linear corrections, is dependent upon the gravitational potential of the system and therefore so must the trajectory of the particles [14]. We will not go further into $\tau^{\mu\nu}$ as it requires us to go past the linear theory, it is still worth mentioning as it tells us that due its dependence on the potential and by extension the interaction terms we can not calculate the leading terms of the quadrupole radiation (4.25), $S^{ij} + n_m \dot{S}^{ij,m}$, directly within our current framework. All is not lost as we can use energy-momentum conservation instead, as we did in section 4.3.3. Using eq. (4.39) and (4.41) we are able to replace $S^{ij} + n_m \dot{S}^{ij,m}$, with $\dddot{M}^{ijk}$ and $\ddot{P}^{i,jk}$, which are the momenta of energy and momentum density respectively. When we derived both of these quantities used the conservation of the stress-energy tensor. Thus by using them to calculate the GW amplitude for a quadrupole we are implicitly including the interaction terms of $T^{\mu\nu}$, while only needing to know that it is conserved and nothing else. A good tool to have which will allow us to calculate behaviour of GW in the non-relativistic limit.

4.4 The behaviour of a binary system

The framework for how a non-relativistic binary system should behave has been laid out in the previous sections, so it now time to put it together to see how a binary system of two compact stellar objects will give rise to GWs and the effect it has on the system. We already know from section 4.3.3 that they will spiral towards each other as they radiate, the details of which and the details GWs radiated will be worked out in this section. We are considering compact objects such as neutron stars or black holes so we can treat them as point like masses, with mass m_1 and m_2, and position $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. The non-relative nature of the chosen system allows us to use a Newtonian approximation in the center-of-mass frame, with coordinates

$$\mathbf{r}_{\text{CM}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}. \tag{4.85}$$

Using the reduced mass

$$\mu = \frac{m_1 m_2}{m}, \tag{4.86}$$

where $m = m_1 + m_2$, and also the relative coordinates $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. The second mass moment is thus

$$M^{ij} = m_1 r_1^i r_1^j + m_2 r_2^i r_2^j = m r_{\text{CM}}^i r_{\text{CM}}^j + \mu r^i r^j. \tag{4.87}$$

For an isolated the system the centre of mass will not be moving and we can thus choose a frame such that $r_{\text{CM}} = 0$, thus we have reduced the system to effectively only one particle with a mass μ and coordinate $\mathbf{r}$. So in the CM frame we have the mass density

$$\rho(t, \mathbf{x}) = \mu\delta^{(3)}(\mathbf{x} - \mathbf{r}), \tag{4.88}$$

and a second mass moment

$$M^{ij} = \mu r^i r^j. \tag{4.89}$$

From eq. (4.44) we also have the quadrupole moment of the system

$$Q^{ij}(t) = \mu(r^i(t) r^j(t) - \tfrac{1}{3} r^k r^k)\delta^{ij}. \tag{4.90}$$

This system has been reduced to a one-body problem whose equation of motion is

$$\ddot{\mathbf{r}} = -\frac{m}{r^3}\mathbf{r}, \tag{4.91}$$

where we recall that we have chosen units where $G = 1$. For a system with an orbital radius R the orbital frequency is [14]

$$\omega_s = \frac{m}{R^3}. \tag{4.92}$$

We will use this system in the next sections to find the amplitude of the GW produced by such a system and the power which is radiated as a consequence.

4.4.1 GW amplitude from a binary system

To find the amplitude of the GWs produced by the system we will start with a simple case where we assume the motion of the relative coordinate, $\mathbf{r}$, is circular and we neglect the self-force from the gravitational radiation. The calculations will follow problem 3.2 and section 4.1 of Maggiore [14].

We choose a Cartesian coordinate system, $\mathbf{r} = (x, y, z)$, for the spatial coordinates such that the orbit lies in the (x, y) plane, we thus have

$$x_0(t) = R\cos\left(\omega_s t + \tfrac{\pi}{2}\right), \tag{4.93a}$$

$$y_0(t) = R\sin\left(\omega_s t + \tfrac{\pi}{2}\right), \tag{4.93b}$$

$$z_0(t) = 0, \tag{4.93c}$$

The second mass moment for the system can thus be found through eq. (4.89), of which the contributing components are

$$M_{11} = \mu R^2 \frac{1 - \cos\left(2\omega_s t\right)}{2}, \tag{4.94a}$$

$$M_{22} = \mu R^2 \frac{1 + \cos\left(2\omega_s t\right)}{2}, \tag{4.94b}$$

$$M_{12} = M_{21} = -\mu R^2 \frac{\sin\left(2\omega_s t\right)}{2}. \tag{4.94c}$$

In section 4.3.1 we found the amplitude of a propagating GW along the z-axis, eq. (4.50) and . Through a coordinate transformation of the moment (4.53)

$$M_{ij} = \mathcal{R}_{ik}\mathcal{R}_{jl}M'_{kl}, \tag{4.95}$$

the amplitude will be given for an arbitrary propagation direction in polar coordinates, where the primed coordinates are given along the z-axis and the rotation matrix $\mathcal{R}$ is given by eq. (4.52). So solving eq. (4.95) for M'_{ij} gives us

$$M'_{ij} = (\mathcal{R}^{\mathrm{T}} M \mathcal{R})_{ij}, \tag{4.96}$$

where $\mathcal{R}^{\mathrm{T}}$ is the transpose of $\mathcal{R}$. Taking the time derivative of the mass moments (4.94)

$$\ddot{M}_{11} = 2\mu R^2 \omega_s^2 \cos\left(2\omega_s t\right), \tag{4.97a}$$

$$\ddot{M}_{22} = -2\mu R^2 \omega_s^2 \cos\left(2\omega_s t\right), \tag{4.97b}$$

$$\ddot{M}_{12} = 2\mu R^2 \omega_s^2 \sin\left(2\omega_s t\right), \tag{4.97c}$$

we note that $\ddot{M}_{11} = -\ddot{M}_{22}$. Finally we can insert the above equations into eq. (4.50) and (4.4.1), and after some matrix multiplication we get the polarised amplitudes given by the non-vanishing momenta

$$
\begin{aligned}
h_{+} &= \frac{1}{r}[\ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2\theta \\
&\quad + \ddot{M}_{22}(\sin^2\phi - \cos^2\phi\cos^2\theta) \\
&\quad - \ddot{M}_{12}(\sin 2\phi(1+\cos^2\theta)] \\
&= \frac{1}{r}4\mu\omega_s^2 R^2 \frac{1+\cos^2\theta}{2}\cos(2\omega_s t_{\mathrm{r}} + 2\phi),
\end{aligned}
\tag{4.98}
$$

and

$$
\begin{aligned}
h_{\times} &= \frac{1}{r}[(\ddot{M}_{11} - \ddot{M}_{22})\sin 2\phi\cos\theta + 2\ddot{M}_{12}\cos 2\phi\cos\theta \\
&= \frac{1}{r}4\mu\omega_s^2 R^2\cos\theta\sin(2\omega_s t_{\mathrm{r}} + 2\phi).
\end{aligned}
\tag{4.99}
$$

Here r is the distance from the source and t_{r} is the retarded time. The quadrupole radiation thus has a frequency twice the frequency of the source, $\omega_{\mathrm{GW}} = 2\omega_s$. To more easily study these amplitudes we would like to use eq. (4.92) to replace the orbital distance R with $f_{\mathrm{GW}} = \omega_{\mathrm{GW}}/(2\pi) = \omega_s/\pi$, while also introducing the chirp mass

$$M_c = \mu^{3/5}m^{2/5}. \tag{4.100}$$

The amplitudes thus take the form

$$h_{+} = \frac{4}{r}M_c^{5/3}(\pi f_{\mathrm{GW}})^{2/3}\frac{1+\cos^2\theta}{2}\cos(2\pi f_{\mathrm{GW}}t_{\mathrm{r}} + 2\phi), \tag{4.101}$$

and

$$h_{\times} = frac4rM_c^{5/3}(\pi f_{\mathrm{GW}})^{2/3}\cos\theta\sin(2\pi f_{\mathrm{GW}}t_{\mathrm{r}} + 2\phi). \tag{4.102}$$

It is thus clear that to the lowest order the two factors that determine the amplitude are the masses of the two objects in the binary system along with their rotational frequency. We can further introduce two other quantities to help simplify the equations, the reduced wavelength $\lambda = 1/\omega_{\mathrm{GW}}$, and the Schwarzschild radius given the chirp mass

$$R_c \equiv 2M_c. \tag{4.103}$$

The Schwarzschild radius is a constant which arises from the static, spherically symmetric vacuum solution of Einstein's field equations, which for this case has been adopted for the chirp mass [7]. We combine λ, R_c, and the distance r into a new constant (r can be regarded as a constant as it is practically the distance from the Earth to the system)

$$\mathcal{A} = 2^{-1/3}\frac{R_c}{r}\left(\frac{R_c}{\lambda}\right)^{2/3} \tag{4.104}$$

The amplitudes thus take the form

$$h_{+} = \mathcal{A}\frac{1+\cos^2\theta}{2}\cos(2\pi f_{\mathrm{GW}}t_{\mathrm{r}} + 2\phi), \tag{4.105}$$

$$h_{\times} = \mathcal{A}\cos\theta\sin(2\pi f_{\mathrm{GW}}t_{\mathrm{r}} + 2\phi), \tag{4.106}$$

where we are left with only the angular dependence of the amplitude and the constant $\mathcal{A}$.

4.4.2 GW radiated power from a binary system

To calculate the power this system radiates we will consider the point of view of an observer trying to measure the GWs, which allows us to convert some variables into constants. The radiation that reaches the observer is whichever part of the system points toward it, therefore this direction makes a good choice for aligning the axis of our coordinate system. The angle θ thus becomes the angle between the orbit of the system and the normal to our line of sight, which will be a constant angle we name ι as long as the systems orbit stays fixed. As we have assumed a circular orbit any change in the angle ϕ can be written as $\Delta\phi = \omega_s \Delta t$, where Δt is the corresponding time it takes to rotate. Therefore one has to consider the combination $\omega_s t_{\mathrm{r}} + \phi$ to get the correct result, as both affect the angular position within the orbit. This allows us to shift the retarded time such that we have $\omega_s t_{\mathrm{r}} = \omega_s t + \alpha$, where $\alpha = \phi - \omega_s r$ is a constant. We can now choose our time origin such that $\cos(2\omega_s t + 2\alpha) \to \cos 2\omega_s t$. Implementing these relations we can write the amplitudes as

$$h_+ = \mathcal{A}\frac{1+\cos^2\iota}{2}\cos(2\omega_s t), \tag{4.107}$$

$$h_\times = \mathcal{A}\cos\iota\sin(2\omega_s t), \tag{4.108}$$

where we can note how the two different polarisations change in relative strengths as our angle of observation ι changes. This will result in different polarisation patterns such as circular polarisation when both polarisations has an equal contribution at $\iota = 0$.
We already have found the equation for power radiation earlier, it is given by eq. (4.59), which we will write on differential form

$$\frac{dP}{d\Omega} = \frac{r^2}{32\pi}\Lambda_{ij,kl}\langle \dddot{Q}_{ij}\dddot{Q}_{kl}\rangle. \tag{4.109}$$

Written in terms of the polarisation modes we get[14]

$$\frac{dP}{d\Omega} = \frac{r^2}{16\pi}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle, \tag{4.110}$$

we use that $\langle\sin^2 x\rangle = \langle\cos^2 x\rangle = 1/2$ and end up with

$$\frac{dP}{d\Omega} = \frac{2}{\pi}\Big(\frac{M_c\omega_{\mathrm{GW}}}{2}\Big)^{10/3} g(\theta). \tag{4.111}$$

The function $g(\theta)$ we have defined as

$$g(\theta) \equiv \Big(\frac{1+\cos^2\theta}{2}\Big)^2 + \cos^2\theta. \tag{4.112}$$

Finding the angular average of $g(\theta)$

$$\int d\Omega\,\frac{1}{4\pi}g(\theta) = \frac{4}{5}, \tag{4.113}$$

which leads us to finding the total power radiated

$$P = \frac{32}{5}\Big(\frac{M_c\omega_{\mathrm{GW}}}{2}\Big)^{10/3} \tag{4.114}$$

The radiated energy must have a source, which for a circular orbit is the sum of the kinetic and potential energy of the system

$$E_{\text{orbit}} = -\frac{m_1 m_2}{2R}. \tag{4.115}$$

The loss of energy must therefore result in a decreasing orbital radius, R, which given eq.(4.92) means that orbital frequency ω_s must increase. This increase in ω_s also means that total power radiated (4.114) will also increase as the system spirals inwards and will thus emit its largest burst of energy when the objects in the system collides.

Chapter 5

Detection of Gravitational Waves

The detection of gravitational waves is rather favourable here on Earth as our solar system has weak gravitational field, meaning that plane gravitational waves passing through the solar system stays more or less as plane waves [6]. So the effect they have on a collection of particles can be described by the equations (3.29) and (3.31), a +-polarised wave passing through some object will thus oscillate it as seen in Figure **??**. By observing the effect of these transverse oscillations and removing the possibility of other disturbances we are able to detect the effect of a passing wave. The first attempt at detecting a gravitational wave was done in the 1960's by Weber using a resonant mass detector [3]. But due to the generally small amplitude of gravitational waves, $h \sim 10^{-21}$, these kinds of detectors struggle reaching the required sensitivity [18].

Many years later in 2016 the first gravitational wave was confirmed detected at the Laser Interferometer Gravitational-Wave Observatory (LIGO) in USA, it confirmed yet another prediction made by the general theory of relativity[4]. It also marked the beginning for a new era of astronomy, one were we can add gravitational waves to our disposal for researching objects that are hard or impossible to detect with electromagnetic radiation, such as dark matter and black holes. A bit amusing is the fact that the experimental setup is rather similar to the very first interferometer, which was made by Albert Michelson of the famous Michelson-Morley experiment.

5.1 Laser Interferometers

Like the Michelson-Morley experiment, LIGO tries to detect the difference in time light takes takes to travel down two orthogonal arms, the light start out at as a single source which is split into each arm where it will be reflected back to the starting point by a mirror located at the end. When the beam is split the two resulting beams have correlated phases, if one arm is longer by half of the lights wavelength then the two beams will interfere destructively when they return, if there are no external disturbances. When a wave passes through the detector at a reasonable angle it will stretch one arm and shrink the other, thus

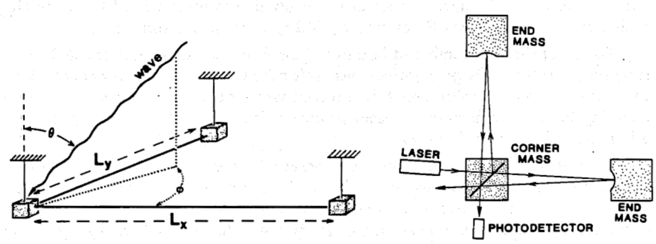

Figure 5.1: Schematic of the proposed design of LIGO, the angles mark the propagation direction for a gravitational wave passing through the detector. Taken from [19].

causing a relative phase shift in the beams and making them have constructive interference, giving us a measurable signal. The relative stretching of the two arms of length L is given by the linear combination of the two polarisation modes h_+ and $h_\times$ [19]

$$\frac{\delta L}{L} = [\tfrac{1}{2}(1+\cos^2\theta)\cos 2\phi]h_+ + [\cos\theta\sin 2\phi]h_\times \equiv h, \tag{5.1}$$

where the angles are as given in Figure 5.1, and $\delta L = L_x - L_y$. The quantity h is also known as the gravitational strain on a test mass, and is the combined amplitude of a passing gravitational wave. As mentioned, this amplitude is generally on the order of $h \sim 10^{-21}$, for a length L given in kilometers the change of distance δL will be of order

$$\delta L \sim 10^{-18}\text{m}. \tag{5.2}$$

When compared to the Bohr radius, $a_0 \sim 5 \times 10^{-11}m$, the challenge of detecting gravitational waves becomes clear. Nevertheless LIGO managed to create an interferometer that is sensitive enough to detect these minuscule changes, one of the techniques used was just making the arms long, as known from equation (3.27) the wave will cause a larger disturbance from a larger initial separation. This why LIGO is built with arms of length $L = 4$km, but that length still was not enough, to increase the effective length of the arms the laser beams are bounced back and forth many times, thus achieving a greater phase shift. If the beam is doing 100 round trips we get a phase shift of [7]

$$\delta\phi \sim 200\frac{2\pi}{\lambda}\delta L \sim 10^{-9}, \tag{5.3}$$

which is of an order that is easier to measure. The result of this feat of engineering bore fruits in 2016 when the signal in Figure 5.2 was detected, where we see the oscillations of a wave passing through the earth. The spike we see is a result of the increasing energy radiated as the two black holes collided, as we discussed at the end of section 4.4.2, at least for system

By having multiple of these types of interferometers across the globe we are able to cross check the results to verify that the signal was not just noise and also be able to more precisely find the origin of incoming waves. This is why LIGO consists of two locations across the USA, but LIGO is not alone some other are VIRGO in Italy [20], GEO600 in Germany [21] and KAGRA in Japan[22]. These projects leads the development of ground

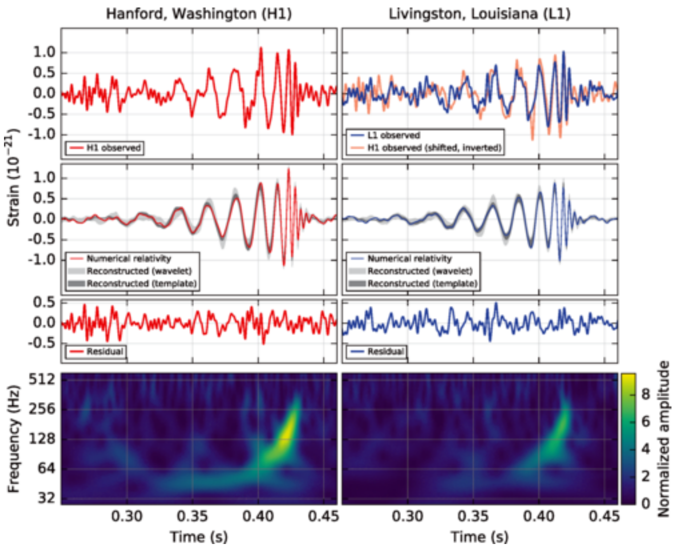

Figure 5.2: The first detection of gravitational wave by LIGO in 2016, showing the oscillations of a passing wave compared with the expected results calculated numerically. Courtesy of: Caltech/MIT/LIGO Lab [4].

based gravitational wave observatories. The next major step in gravitational wave astronomy will be to eliminate the disturbances and space restrictions we have here on Earth and launch an interferometer into space, this project is known as LISA (Laser Interferometer Space Antenna) and is carried out by ESA and NASA. The plan is to have an interferometer orbit the sun staying in the same orbit as the Earth, staying about 50-65 million kilometers behind, with three arms each being 2.5 kilometers long. LISA would be able to reach sensitivity in a band from below 10^{-4}Hz to above 10^{-1}Hz [23].

Chapter 6

Conclusion and outlook

In this thesis we have explored the framework of GWs from their mathematical origin as a perturbation upon a flat background spacetime to being able to predict the inspiraling of binary system at the lowest order and how that plays a role in our ability to detect GWs. We started by deriving the Eisntein field equations through the variational method, which we looked closer upon in the weak field limit and lead to us finding that the basic formalism for GWs in the linearised theory. A formalism we expanded upon in the TT-gauge and discovered the polarisation modes of GWs. Then we had to step outside the linearised theory in order to able to find the stress-energy tensor of propagating waves, a result which we used later to be able to find the power radiated in binary systems. The special case of binary systems we took a look at what was that of non-relativistic systems, where used the lowest order moment, the quadrupole moment, to find the power and amplitude of the radiation which the system emitted. Lastly we considered how experiments such as LIGO is able to detect the GW despite their extremely small amplitude.

Subjects that could be of further interest are a closer look into the non-linear aspects of GWs and consider their behaviours on a curved background, as well as a the angular momentum of GWs and the link to the graviton having to be a spin-2 particle.

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Appendix **A**

Appendix A

A.1 Notation and conventions

Units

Throughout this thesis we use geometrised units, where the speed of light is set to unity, $c = 1$, and the gravitational constant is also set to unity, $G = 1$. The metric signature we use is of the form (-,+,+,+). The Einstein summation convention is employed, where summation over repeating indices is assumed. Indices that are written using the Greek alphabet signify time and space components, while indices using the Latin alphabet signify only spatial components. On a pseudo-Riemannian manifold we use the interval:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{A.1}$$

where $g_{\mu\nu}$ is the covariant metric tensor and dx^μ are the components of an infinitesimal displacement of a four-vector.
The new metric for a coordinate transformation $x^\mu \to x'^\mu$ is given by

$$g'_{\mu\nu} = g_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu}. \tag{A.2}$$

Tensors and relativity

The Riemann curvature tensor is written as

$$R^\rho{}_{\mu\nu\sigma} = \partial_\nu \Gamma^\rho{}_{\mu\sigma} - \partial_\sigma \Gamma^\rho{}_{\mu\nu} + \Gamma^\rho{}_{\lambda\nu} \Gamma^\lambda{}_{\mu\sigma} - \Gamma^\rho{}_{\lambda\sigma} \Gamma^\lambda{}_{\mu\nu}, \tag{A.3}$$

where Γ are the Christoffel symbols. The Christoffel symbols written as derivatives of the metric tensor is also known as the metric connection, it is written as follows

$$\Gamma^a_{\ bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}). \tag{A.4}$$

By using the symmetry properties of the Riemann curvature tensor, and contracting the first and last indices we end up with a new tensor, the Ricci tensor, which is defined as:

$$R_{\mu\nu} \equiv R^\alpha_{\ \mu\alpha\nu} = \partial_\alpha \Gamma^\alpha_{\ \mu\nu} - \partial_\nu \Gamma^\alpha_{\ \mu\alpha} + \Gamma^\alpha_{\ \beta\alpha}\Gamma^\beta_{\ \mu\nu} - \Gamma^\alpha_{\ \beta\nu}\Gamma^\beta_{\ \mu\alpha} \tag{A.5}$$

The Ricci scalar is given by the contracting the Ricci tensor, which means we get

$$R \equiv g^{\mu\nu} R_{\mu\nu} = R^\mu_{\ \mu}. \tag{A.6}$$

The derivative of the components of a vector, v^α, along a curve parametrised by λ is given by

$$\frac{Dv^\alpha}{D\lambda} \equiv \frac{dv^\alpha}{d\lambda} + \Gamma^\alpha_{\ \beta\gamma} v^\beta \frac{dx^\gamma}{d\lambda} \tag{A.7}$$

To find the geodesic $x^\alpha(\lambda)$, the parametrised path of freely falling particles, we use the equation [6]

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\ \gamma\beta} \frac{dx^\gamma}{d\lambda}\frac{dx^\beta}{d\lambda} = 0, \tag{A.8}$$

which is known as the geodesic equation, where λ is a scalar parameter of motion. The distance between to points on two different geodesics is given by the separation vector $\xi^\alpha(\lambda)$, the equation that describes how this distance evolve is given by the equation of geodesic deviation [7]

$$\frac{D^2 \xi^\alpha}{D\lambda^2} = R^\alpha_{\ \mu\nu\beta} \dot{x}^\mu \dot{x}^\nu \xi^\beta. \tag{A.9}$$

Divergence theorem

For an arbitrary vector field, V^μ, we get the divergence theorem:

$$\int_{\mathcal{R}} (\nabla_\mu V^\mu)\sqrt{|g|} d^4x = \int_{\partial\mathcal{R}} n_\mu V^\mu \sqrt{|\gamma|} d^3y \tag{A.10}$$

Here γ is the determinant of the induced metric on the boundary, and n_μ is the unit normal on the boundary.
The differential of the determinant of a matrix A, is given by Jacobi's formula:

$$\mathrm{d}\det(A) = \mathrm{Tr}(\mathrm{adj}(A)\mathrm{d}A)) \tag{A.11}$$

A.2 Calculations weak-field metric

Contravariant weak-field metric

An alternative way of writing the contravariant weak-field metric is $g^{\mu\nu} = g'^{\mu\nu} + \delta g^{\mu\nu}$. We use the definition $g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}_{\nu}$, in order to find an expression for the variation of the contravariant metric tensor.

$$\delta(g^{\mu\nu} g_{\mu\nu}) = g^{\mu\nu}\delta(g_{\mu\nu}) + \delta(g^{\mu\nu})g_{\mu\nu} = \delta(\delta^{\mu}_{\nu}) = 0, \tag{A.12}$$

by contracting with $g^{\mu\nu}$ and rearranging we get

$$\delta(g^{\mu\nu}) = -g^{\mu\alpha}\delta(g_{\alpha\beta})g^{\beta\nu}. \tag{A.13}$$

By choosing $g'^{\mu\nu}$ to be the Minkowski metric we get $g'^{\mu\nu} = \eta^{\mu\nu}$, we also choose the variance in the metric to be $\delta(g_{\mu\nu}) = h_{\mu\nu}$. We thus get to the first order of $\delta(g^{\mu\nu})$

$$\begin{aligned} \delta(g^{\mu\nu}) &= -(\eta^{\mu\alpha} + \delta g^{\mu\alpha})h_{\alpha\beta}(\eta^{\beta\nu} + \delta g^{\beta\nu}) \\ \delta(g^{\mu\nu}) &= -\eta^{\mu\alpha} h_{\alpha\beta}\eta^{\beta\nu} = -h^{\mu\nu}. \end{aligned} \tag{A.14}$$

We thus end up with the following expression for the contravariant field metric

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{A.15}$$

A.3 Gauge and coordinate transformations in linearised theory

The basic equations for a linearised theory of gravity for a coordinate system that is nearly globally Lorentz are the equation for the weak-field metric (2.19) and the linear field equations (2.25). The coordinate transformations that connect nearly globally Lorentz systems to each other are the global Lorentz transformation and the infinitesimal coordinate transformation [6], both of which will be introduced here.

A.3.1 Global Lorentz transformations

Global Lorentz transforms are of the form [5]

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}, \qquad \text{where} \quad \eta_{\mu\nu} = \Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}\eta_{\alpha\beta},$$

the coefficients $\Lambda^{\mu}_{\ \nu}$ are constant everywhere. By using the Jacobian we see that the metric tensor transforms as follows

$$g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} = \Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}(\eta_{\alpha\beta} + h_{\alpha\beta}) = \eta_{\mu\nu} + \Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu} h_{\alpha\beta}, \tag{A.16}$$

we see that the transformed metric tensor is still of the same form as the weak-field metric (2.19). The difference being that we have

$$h'_{\mu\nu} = \Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu} h_{\alpha\beta}. \tag{A.17}$$

Thus we see that the change that occurs under a transformation of the metric in the weak-field case is only given by the change in the perturbation term $h_{\mu\nu}$. Instead of a considering a slightly curved space-time representing the general relativistic weak field, we can view it in means of $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ as symmetric rank-2 tensors defined on a flat Minkowski space-time [5]. This gives us the property that we only need to consider the change in $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ as opposed to the whole metric, thus this is a special relativistic field akin to how the 4-potential, A_μ, describes electromagnetic fields on a flat Minkowski space-time.

A.3.2 Infinitesimal general coordinate transformations

Infinitesimal general coordinate transformation takes the form [5]

$$x'^\mu = x^\mu + \xi^\nu(x), \tag{A.18}$$

where $\xi^\nu(x)$ are four arbitrary functions with the same dimensions as $h_{\mu\nu}$. This type of transformation makes tiny changes in the forms of scalar, vector and tensor fields, but can be ignored in all of them except for the metric tensor as the tiny deviations from $\eta_{\mu\nu}$ contains all the information about gravity [5]. We want to express the metric tensor using this transformation, so we need to find the expression for the Jacobian. So from equation (A.18) we get

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu \xi^\mu,$$

where we have the inverse form

$$\frac{\partial x^\mu}{\partial x'^\nu} = \delta^\mu_\nu - \partial_\nu \xi^\mu.$$

Both ξ^μ and $h_{\mu\nu}$ are small quantities, so by working to the first order of these quantities we end up with the metric transformation

$$\begin{aligned} g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} &= (\delta^\alpha_\mu - \partial_\mu \xi^\alpha)(\delta^\beta_\nu - \partial_\nu \xi^\beta)(\eta_{\alpha\beta} + h_{\alpha\beta}) \\ &= \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \end{aligned} \tag{A.19}$$

where we have defined $\xi_\mu = \eta_{\mu\nu}\xi^\nu$. We also see here that the transformation of the metric keeps the same form as the weak-field metric (2.19). So the transformation of the perturbation $h_{\mu\nu}$ is given by

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu. \tag{A.20}$$

Appendix **B**

Riemann Geometry

B.1 The Bianchi Identity

By considering the permutations of the covariant derivative of the Riemann tensor along with its anti-symmetry $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$ one arrive at what is known as the Bianchi identity [7]

$$\nabla_{[\lambda} R_{\alpha\beta]\mu\nu} = 0, \tag{B.1}$$

where the square brackets denotes the permutation of the components within. By contracting twice on the Bianchi identity (B.1) one arrives at the relation

$$\nabla^{\mu} R_{\alpha\mu} = \tfrac{1}{2}\nabla_{\alpha} R, \tag{B.2}$$

which in turn gives us the covariant derivative of the Einstein tensor (2.12)

$$\nabla^{\mu} G_{\mu\nu} = 0. \tag{B.3}$$

B.2 Non-commutation of covariant derivatives

For a vector field with components B^{μ} we have the following commutation property for the covariant derivatives [6]

$$D_{\alpha} D_{\beta} B^{\mu} = D_{\beta} D_{\alpha} B^{\mu} + R^{\mu}{}_{\nu\beta\alpha} B^{\nu}. \tag{B.4}$$

For a second-rank tensor field with components $S^{\mu\nu}$ we have the following commutation property for the covariant derivatives [6]

$$D_{\alpha} D_{\beta} S^{\mu} = D_{\beta} D_{\alpha} S^{\mu} + R^{\mu}{}_{\rho\beta\alpha} S^{\rho\nu} + R^{\nu}{}_{\rho\beta\alpha} S^{\mu\rho} \tag{B.5}$$

Appendix C

Green's Function

C.1 Green's Function for the Wave Equation

One way to solve the inhomogeneous wave equation of the form

$$\Box\Psi(\mathbf{x},t) = f(\mathbf{x},t), \tag{C.1}$$

is through the use of a Green's function G, here Ψ is an arbitrary wave function and f is an arbitrary source term. Here $\Box$ is the d'Alembertian defined as

$$\Box = -\partial_t^2 + \nabla^2, \tag{C.2}$$

where ∇^2 is the spatial Laplacian. We start by considering a Green's function as a solution to the wave equation for a δ-function source term

$$\Box_x G(\mathbf{x},t;\mathbf{x}',t') = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta(t-t'), \tag{C.3}$$

where the primed coordinates are the location of the source. By multiplying both sides of equation (C.3) with $f(\mathbf{x}',t)$ and integrating with respect to $x^{\mu\prime}$ we get

$$\int d^4x' \Box_x G(\mathbf{x},t;\mathbf{x}',t')f(\mathbf{x}',t) = \int d^4x' \delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta(t-t')f(\mathbf{x}',t). \tag{C.4}$$

Evaluating the RHS to simply be $f(\mathbf{x},t)$, which we can insert back into eq. (C.1) and after integration we end up with the general solution for Ψ

$$\Psi(\mathbf{x},t) = \int d^4x' G(\mathbf{x},t;\mathbf{x}',t')f(\mathbf{x}',t). \tag{C.5}$$

Finding the solution to the Green's function G will in turn give us the solution to Ψ. A process we start by taking the Fourier transform with respect to the time coordinate of equation (C.3) we convert our time-dependent equation to the time-independent Helmholtz equation

$$(\nabla^2 + k^2)G(\mathbf{x},\mathbf{x}',\omega) = \delta^{(3)}(\mathbf{x}-\mathbf{x}')e^{i\omega t'}, \tag{C.6}$$

where $k = \omega$, which for the limit $k \to 0$ reduces to the Poisson equation. We will consider this system with the spatial boundary condition $G(\mathbf{x}, \mathbf{x}', \omega) \to 0$ as $|\mathbf{x} - \mathbf{x}'| \to \infty$. As we are considering a point source the resulting waves will be spherical spatially and the system will thus have spherical symmetry. To make it easier for us we will first evaluate eq. (C.13) for $\mathbf{x} \neq \mathbf{x}'$, which results in the homogeneous Helmholtz equation

$$(\nabla^2 + k^2)G(\mathbf{x}, \mathbf{x}', \omega) = 0. \tag{C.7}$$

As this system is spherically symmetric, there are no angular dependence in G, we thus introduce $r = |\mathbf{x}|$ which leaves us with

$$\frac{1}{r^2}\partial_r(r^2\partial_r G) + k^2 G = 0, \tag{C.8}$$

simplifying we get

$$\partial_r^2(rG) + k^2(rG) = 0. \tag{C.9}$$

This is just an ODE with the solutions

$$G = \frac{1}{r}Ae^{ikr} + \frac{1}{r}Be^{-ikr}. \tag{C.10}$$

The more interesting result we gain by ignoring the constants A and B, instead we put $\frac{1}{r}e^{\pm ikr}$ into the homogeneous Helmholtz equation (C.7)

$$\begin{aligned}(\nabla^2 + k^2)\tfrac{1}{r}e^{\pm ikr} &= \nabla \cdot \Big(\pm\, ik\tfrac{1}{r}e^{\pm ikr}\hat{\mathbf{r}}\Big) + \nabla e^{\pm ikr} \cdot \nabla \tfrac{1}{r} \\ &\quad + e^{\pm ikr}\nabla^2 \tfrac{1}{r} + \tfrac{1}{r}k^2 e^{\pm ikr} \\ &= e^{\pm ikr}\nabla^2 \tfrac{1}{r} \\ &= -\, 4\pi e^{\pm ikr}\delta^3(\mathbf{x}).\end{aligned} \tag{C.11}$$

We note that the $e^{\pm ikr}\delta^3(\mathbf{x})$ is the same as $\delta^3(\mathbf{x})$ due to the δ-function forcing $\mathbf{x} = 0$, so we get

$$(\nabla^2 + k^2)\tfrac{1}{r}e^{\pm ikr} = -4\pi\delta^3(\mathbf{x}) \tag{C.12}$$

With this result in the back of our minds we will direct our attention back towards the general Green's function for the wave equation (C.3). For the following calculations we will introduce the variables $R = |\mathbf{x} - \mathbf{x}'|$ and $\tau = (t - t')$. Through the Fourier transform we get the following relations; the Green's function $G(R, \tau)$ as the inverse Fourier transform with respect to τ

$$G(R, \tau) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} d\omega G(R, \omega)e^{-i\omega\tau}, \tag{C.13}$$

and the Fourier representation of the δ-function

$$\delta(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} d\omega e^{-i\omega t}. \tag{C.14}$$

By substituting eq. (4.2) and (C.14) into equation (C.3), we get

$$
\begin{aligned}
\Box_x\left(\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} d\omega G(R,\omega)e^{-i\omega\tau}\right) =&\delta^3(\mathbf{x}-\mathbf{x}')\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} d\omega e^{-i\omega t} \\
\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} d\omega(\nabla^2+k^2)G(R,\omega)e^{-i\omega\tau} =&\delta^3(\mathbf{x}-\mathbf{x}')\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} d\omega e^{-i\omega t}.
\end{aligned}
\tag{C.15}
$$

Moving them to the same side of the equation we get

$$
\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} d\omega\Big[(\nabla^2+k^2)G(R,\omega)-\frac{1}{\sqrt{2\pi}}\delta^3(\mathbf{x}-\mathbf{x}')\Big]e^{-i\omega\tau}=0. \tag{C.16}
$$

We see that the integrand must be zero, which is the case when

$$
(\nabla^2+k^2)G(R,\omega)=\frac{1}{2\pi}\delta^3(\mathbf{x}-\mathbf{x}') \tag{C.17}
$$

From the homogeneous solution (C.12) we know an expression for $G(R,\omega)$ that will satisfy the above equation, we thus have the two solutions

$$
G_\pm(R,\omega)=-\frac{1}{4\pi\sqrt{2\pi}}\frac{1}{R}e^{\pm ikR}. \tag{C.18}
$$

Now we just have to take the inverse Fourier transform

$$
\begin{aligned}
G_\pm(R,\tau)=&-\frac{1}{4\pi\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} d\omega\frac{1}{R}e^{\pm ikR}e^{-i\omega\tau} \\
=&-\frac{1}{8\pi^2}\frac{1}{R}\int\limits_{-\infty}^{\infty} d\omega e^{-i\omega(\tau\mp R)} \\
=&-\frac{1}{4\pi}\frac{1}{R}\delta(\tau\mp R),
\end{aligned}
\tag{C.19}
$$

where we used $k=\omega$. We now insert back the original expressions for R and τ and end up with the retarded and advanced Green's functions respectively

$$
G_\pm(\mathbf{x},t;\mathbf{x}',t')=-\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\delta\big[(t-t')\mp|\mathbf{x}-\mathbf{x}'|\big]. \tag{C.20}
$$

The difference between the two solutions have an important physical distinction as the retarded solution is non-vanishing for events after the time of the source, t', while the advanced solution is non-vanishing for events before t'. The retarded solution is thus the more interesting solution for most physical systems. A more rigorous approach involves applying the Sommerfeld radiation condition

$$
\lim_{R\to\infty} R\big(\partial_R-ik\big)G(R,\omega)=0, \tag{C.21}
$$

which ensures we have a sink and not a source at infinity. Inserting the solutions for the Green's function in the frequency domain (C.18) we get for the retarded Green's function $G_+(R,\omega)$

$$\begin{aligned}\lim_{R\to\infty} R\big(\partial_R - ik\big)G_+(R,\omega) &= \lim_{R\to\infty} R\Big(\frac{ie^{ikR}(kR+i)}{R^2}\Big) - ik\frac{e^{ikR}}{R} \\ &= -\lim_{R\to\infty}\frac{e^{ikR}}{R} = 0.\end{aligned} \tag{C.22}$$

As we expected from our physical interpretation the retarded Green's function do fulfil the Sommerfeld radiation condition. On the other hand, when we do the same for the advanced Green's function $G_-(R,\omega)$ we get

$$\begin{aligned}\lim_{R\to\infty} R\big(\partial_R - ik\big)G_-(R,\omega) &= \lim_{R\to\infty} R\Big(\frac{ie^{-ikR}(-1-ikR)}{R^2}\Big) - ik\frac{e^{-ikR}}{R} \\ &= -\lim_{R\to\infty}\frac{e^{-ikR}}{R} + ie^{-ikR}(k+1) \neq 0,\end{aligned} \tag{C.23}$$

which as expected does not satisfy the radiation condition. The advanced solution could thus be interpreted as energy coming from infinity and into a sink, which is not what we are after when considering the source of a wave equation.

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