Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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Representation Schemes for Finitely Generated Groups

Master's thesis in Mathematical Sciences
Supervisor: Markus Szymik
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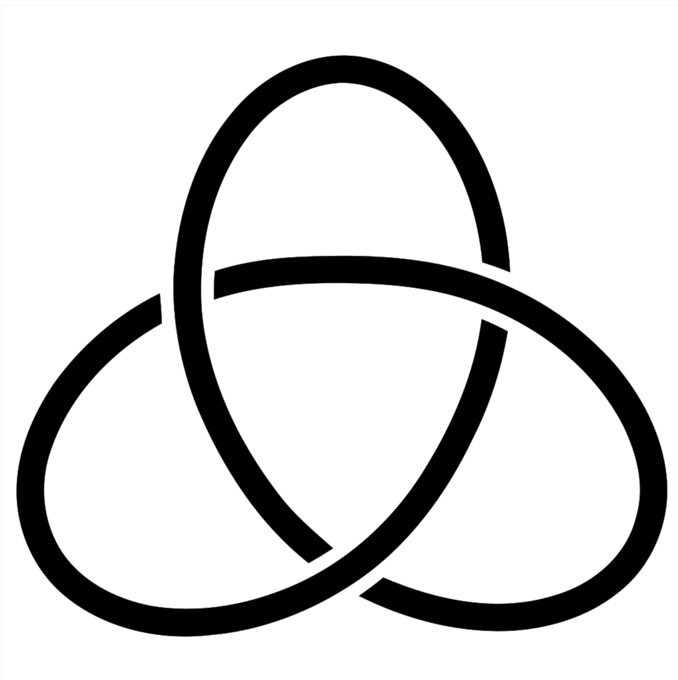

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Abstract

The representation space of a finitely generated group over a field k can be viewed as a point of an affine k-scheme. There is a categorical equivalence between affine k-schemes and commutative k-algebras, thus the coordinate k-algebra of a representation scheme characterises the k-scheme completely. The coordinate k-algebra of a scheme and the coordinate ring of the underlying variety are related via nilradical reduction. We define the moduli scheme through a geometric invariant theory quotient and show that the ring of invariants is finitely generated by traces and determinants. In addition, the moduli space parameterises equivalent semi-simple representations. The tangent space of an affine scheme can be realised as a 1-cocycle space and the representation space is completely reducible when the first cohomology vanishes everywhere.

Resume

Repræsentationsrummet for en endeligt genereret gruppe over et legeme k kan ses som et punkt på et affint k-skema. Der er en kategorisk ækvivalens mellem affine k-skemaer og kommutative k-algebraer, hvilket vil sige, at koordinat-k-algebraen for et repræsentationsskema karakteriserer k-skemaet fuldstændigt. Koordinat-k-algebraen for et skema og koordinatringen af den underliggende sort er relateret via nilradikal reduktion. Vi definerer modulskemaet gennem en geometrisk invariant teori kvotient og viser, at ringen af invarianter er endeligt genereret af spor og determinanter. Derudover parametriserer modulrummet ækvivalente semi-simple repræsentationer. Tangentrummet for et skema kan realiseres som et 1-cocyklusrum, og repræsentationsrummet kan reduceres fuldstændigt, hvis den første cohomologi forsvinder overalt.

Contents

Introduction

For a finitely generated group Γ, an n-dimensional **representation** of Γ is a group homomorphism $\rho : \Gamma \to \mathrm{GL}_n(k)$ where k will usually be take to be an algebraically closed field of characteristic zero. In this thesis we study the representation space $R(\Gamma, \mathrm{GL}_n(k))$, consisting of all n-dimensional representations of Γ. The aim is to provide an approachable introduction to representation schemes, with an emphasis on detailed proofs and worked examples. We assume a basic understanding of abstract algebra and some knowledge of elementary category theory is recommended, but not necessary. The primary source throughout is Lubotzky and Magid's book [LM85] and a more general overview of the subject can be found in Heusener's survey [Heu14].

For finite groups, representations are well understood. Maschke [Ste12, p. 23] proved that, for a finite group, any representation over a field k of characteristic not dividing the order of the group is a direct sum of simple representations. Thus, in the finite case, it is sufficient to study these simple representations since the representation space is completely reducible. However, in the infinite case, representations are not as well behaved. Take, for instance, the representation

$$\rho : \langle x, y \mid x^2 = y^3 \rangle \to \mathrm{GL}_2(\mathbb{C}) : x \mapsto \left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right) : y \mapsto \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$$

This representation is neither simple, nor is it a direct sum of linear representations, see example 1.1.10. In order to describe the structure of the representation space, we view the representation space as an affine variety $V \subset \mathbb{A}_m(k)$, constrained by relations on the finitely generated group Γ. More generally, we consider what happens to the representation space when we vary the base ring k. A functor $\mathfrak{X} : \mathbf{CRing} \to \mathbf{Set}$ is an **affine scheme** if, for some $A \in \mathbf{CRing}$, we have $\mathfrak{X}$ is naturally isomorphic to the functor

$$\mathrm{Spec}(A) : \mathbf{CRing} \to \mathbf{Set} : B \mapsto \mathrm{Hom}_{\mathbf{CRing}}(A, B)$$

In this case we call A the **coordinate ring** of the affine scheme $\mathfrak{X}$, denoted by $\mathcal{O}(\mathfrak{X})$. We want to ensure that each point of an affine scheme gives rise to an affine variety, so we work over a fixed base field k. Since we are only concerned with representation spaces defined over k-algebras, we obtain the notion of an **affine *k*-scheme** (2.1.3). These are affine schemes $\mathfrak{X}$ such that there exists a natural transformation from $\mathfrak{X}$ to $\mathrm{Hom}(k, \_)$.

Theorem A.
There exists an equivalence of categories

$$\mathbf{Comm}k\mathbf{Alg} \sim \mathbf{Aff}k\mathbf{S}^{op}$$

This is Theorem 2.1.6 in the main text and the proof is a direct consequence of Yoneda's Embedding [Lan71, p.61]. Note that affine schemes can be thought of as affine $\mathbb{Z}$-schemes since $\mathbb{Z}$ is terminal in $\mathbf{CRing}$. Thus, to prove statements about affine schemes, we can work on the level of coordinate k-algebras instead. If we consider the functor sending an arbitrary ring B to the representation space of Γ over that ring B we obtain:

Theorem B. *[LM85, p. 3]*
If Γ is a finitely generated group, and $n \in \mathbb{Z}_{>0}$, then

$$R(\Gamma, GL_n) : \mathbf{CRing} \to \mathbf{Set} : B \mapsto Hom(\Gamma, GL_n(B))$$

is an affine scheme.

This is proven in 2.2.2 and relies on finding a suitable candidate for the coordinate ring of $R(\Gamma, \mathrm{GL}_n)$ and using the equivalence in 2.1.2. On the level of affine varieties, we consider all regular maps from the variety $V \subset \mathbb{A}_m(k)$ to the base field k and call this the **coordinate ring** of the variety, denoted $\mathcal{O}(V)$. The coordinate k-algebra of a k-scheme and the coordinate ring of the underlying variety are related in the following sense:

Theorem C. *[LM85, p. 31]*
Let $\mathfrak{X} := R(\Gamma, GL_n)$, considered as a k-scheme. Then, on the level of points, we have that the coordinate ring of the variety $\mathfrak{X}(k)$ is given by

$$\mathcal{O}(\mathfrak{X}(k)) \cong \mathcal{O}(\mathfrak{X})^{red}$$

This is a consequence of Hilbert's Nullstellensatz [AM69, p. 85] and proven in 2.3.6. In particular, if the k-algebra representing an affine k-scheme is reduced, then the coordinate ring of the variety and the coordinate k-algebra of the scheme above coincide. For example, $\mathcal{O}(\mathrm{GL}_n)$ is reduced, see 2.3.7.

For group representations, we have the notion of equivalent representations, where two representations $\rho, \rho' : \Gamma \to \mathrm{GL}_n(k)$ are **equivalent** if they are equal up to conjugation by some element of $\mathrm{GL}_n(k)$. Similarly, we can talk about representations which are invariant under conjugation or, more generally, invariant under some group action. In order to extend this idea to affine schemes, we define affine group schemes, affine schemes which behave like groups, and affine group scheme actions induced by these. This then allows for a geometric invariant theory (GIT) quotient of our affine scheme $\mathfrak{X}$ by some group scheme $\mathfrak{G}$, where the quotient intuitively consists of elements of $\mathfrak{X}$ which are invariant under the action induced by $\mathfrak{G}$.

In the case of the representation k-scheme $\mathfrak{X} = R(\Gamma, \mathrm{GL}_n)$, we define the **moduli scheme** of $\mathfrak{X}$ as the GIT-quotient of $\mathfrak{X}$ by GL_n, denoted $M(\Gamma, \mathrm{GL}_n)$. The action of GL_n on $\mathfrak{X}$ descends to an action of $\mathrm{GL}_n(k)$ on the coordinate k-algebra $\mathcal{O}(\mathfrak{X})$.

Theorem D.
The projection

$$q_\Gamma : R(\Gamma, GL_n(k)) \to M(\Gamma, GL_n(k))$$

induces a bijection between points of $M(\Gamma, GL_n(k))$ and equivalence classes of semi-simple representations in $R(\Gamma, GL_n(k))$.

This is theorem 4.3.2 in the main text and proven using a technical result by [LM85, p. 25]. In this sense, the moduli space encapsulates the well behaved representations of the representation variety. Moreover, we can express the moduli space in terms of traces and determinants:

Theorem E. *[LM85, p. 27]*
Let k be our fixed ground ring and consider the affine k-scheme $\mathfrak{X} = R(\Gamma, GL_n)$. Then the coordinate k-algebra $\mathcal{O}(\mathfrak{X})^{GL_n(k)}$ of the moduli scheme $\mathfrak{X}//GL_n(k)$ is generated by trace and determinant elements. Moreover, the ring of invariants $\mathcal{O}(\mathfrak{X})^{GL_n(k)}$ is finitely generated.

The finite generation is proven in Theorem 3.3.5 and follows from GL_n being linearly reductive. The generation by traces and determinants is proven in Theorem 4.2.1 and relies on Procesi's result about invariants in matrix rings [Pro76]. Using this presentation of the moduli scheme, we show that the underlying moduli space of the free abelian group $\langle x, y \mid xy = yx\rangle$ into $\mathrm{SL}_2(\mathbb{C})$ is isomorphic to the surface (4.2.4):

$$M(\langle x, y \mid xy = yx\rangle, \mathrm{SL}_2(\mathbb{C})) \cong \{(a, b, c) \in \mathbb{C}^3 \mid a^2 + b^2 + c^2 - abc - 4 = 0\}$$

Returning to the representation scheme, we consider the tangent space of the scheme at a representation. The **tangent space** of $\mathfrak{X}$ at the point $\rho \in \mathfrak{X}(k)$ is defined as the fibre at ρ of the map $\eta^* : \mathfrak{X}(k[\epsilon]) \to \mathfrak{X}(k)$ induced by sending $\epsilon \mapsto 0$ (5.1.2). The tangent space can be expressed as a 1-cocycle space:

Theorem F. *[LM85, p.33]*
Let $\rho \in R(\Gamma, GL_n(k))$, then there exists a k-linear isomorphism

$$\begin{aligned} Z^1(\Gamma, \mathit{Conj} \circ \rho) &\xrightarrow{\ \cong\ } T_\rho(R(\Gamma, GL_n(\_))) \\ \tau &\longmapsto [\gamma \mapsto (I + \tau(\gamma)\epsilon)\rho(\gamma)] \end{aligned}$$

The proof can be found in Theorem 5.2.2. In the case of the trefoil knot group into $\mathrm{GL}_2(\mathbb{C})$, we calculate the tangent space and first cohomology groups of a family of semi-simple representations:

Example.
Let $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ and $\Gamma = \langle x, y \mid x^2 = y^3 \rangle$ and consider the family of representations $\rho : \Gamma \to GL_2(\mathbb{C})$ given by:

$$\rho : x \mapsto \left(\begin{smallmatrix} z_1^3 & 0 \\ 0 & x_2^3 \end{smallmatrix}\right) : y \mapsto \left(\begin{smallmatrix} z_1^2 & 0 \\ 0 & x_2^2 \end{smallmatrix}\right)$$

Then the tangent space of the representation variety $\mathfrak{X} = R(\Gamma, GL_n)$ at ρ is given by

$$\begin{aligned} T_\rho(\mathfrak{X}) \cong \mathbb{C}^4 \quad &\iff \quad z_1/z_2 \in \ \mathbb{C} \setminus \{0, e^{\pi i/3}, e^{5\pi i/3}\} \\ T_\rho(\mathfrak{X}) \cong \mathbb{C}^6 \quad &\iff \quad z_1/z_2 \in \ \{e^{\pi i/3}, e^{5\pi i/3}\} \end{aligned}$$

Moreover, the first cohomology is given by

$$\begin{aligned} H^1(\Gamma, \mathit{Conj} \circ \rho) \cong \mathbb{C}^2 \quad &\iff \quad z_1/z_2 \in \ \mathbb{C} \setminus \{0, 1, e^{\pi i/3}, e^{5\pi i/3}\} \\ H^1(\Gamma, \mathit{Conj} \circ \rho) \cong \mathbb{C}^4 \quad &\iff \quad z_1/z_2 \in \ \{1, e^{\pi i/3}, e^{5\pi i/3}\} \end{aligned}$$

The tangent space calculations are done in example 5.3.1 and the first cohomology groups are computed in 5.4.4. We obtain a sufficient condition for when the representation variety is completely reducible:

Theorem G. *[LM85, p.37]*
Let $\mathfrak{X} = R(\Gamma, GL_n)$, then:

$$H^1(\Gamma, \mathit{Conj} \circ \rho) = 0 \ \forall \ \rho \in \mathfrak{X}(k) \implies \rho \text{ semi-simple } \forall \ \rho \in \mathfrak{X}(k)$$

This is Theorem 5.4.6 in the main text and the proof consists of showing that the orbit of every representation is closed before invoking a technical result by Lubotzky and Magid [LM85, p. 25].

This thesis is divided into five chapters. The first and second chapter cover foundations and elementary results. The third and fourth chapter focuses primarily on moduli spaces, whereas the final chapter concerns itself with tangent spaces and cohomology.

Chapter 1 introduces some preliminary definitions of the representation space and affine varieties. We provide an introduction to category theory and define affine schemes in that setting.

Chapter 2 is used to prove the categorical equivalence between commutative k-algebras and affine k-schemes and show that the representation scheme is affine. We prove that the coordinate ring of a variety V is isomorphic to the coordinate k-algebra above V, modulo nilpotents.

Chapter 3 introduces the moduli space of representations, defined via the geometric invariant theory quotient. We show that the the coordinate k-algebra of the moduli space is finitely generated.

Chapter 4 is used to give a reformulation of the moduli space, proving that the coordinate k-algebra of the moduli space is generated by traces and determinants. We show that the moduli space parameterises equivalent semi-simple representations and that semi-simple representations are defined by their character, up to equivalence.

Chapter 5 provides a more geometric approach, showing that the tangent space of the representation scheme is given by 1-cocycles and describing the 1-coboundary space in terms of the orbit map. We calculate the tangent space and first cohomology of the trefoil knot group and give a sufficient condition for complete reducibility of the representation space.

Notation

Affine n-space	$\mathbb{A}_n$	Ex 1.3.10
Affine scheme	$\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$	Def 1.3.9
Affine scheme represented by A	Spec(A)	Def 1.3.9
Category	$\mathcal{A}, \mathcal{B}, \mathcal{C}$	Def 1.3.1
Coordinate ring	$\mathcal{O}(\mathcal{V})$	Def 1.2.3
Coordinate k-algebra	$\mathcal{O}(\mathfrak{X})$	Def 1.3.9
First cohomology group	$H^1(\Gamma, \text{Conj} \circ \rho)$	Def 5.4.3
Free group of rank n	$\mathbb{F}_n$	Def 1.1.2
General linear and Special linear group	$\text{GL}_n(k)$, $\text{SL}_n(k)$	Def 1.1.5
General linear and Special linear scheme	GL_n, SL_n	Prop 2.2.1
Geometric invariant theory quotient	$\mathfrak{X}//G$	Def 3.1.5
Matrix group scheme	M_n	Rem 4.1.3
Moduli scheme	$M(\Gamma, \text{GL}_n)$	Def 3.2.2
Moduli space	$M(\Gamma, \text{GL}_n(k))$	Def 3.2.2
Matrix ring	$\text{M}_n(k)$	Rem 4.1.3
Representation scheme	$R(\Gamma, \text{GL}_n)$	Th 2.2.2
Representation space	$R(\Gamma, \text{GL}_n(k))$	Def 1.1.11
Zariski tangent space	$T_\rho(\mathfrak{X})$	Def 5.1.2
1-coboundary space	$B^1(\Gamma, \text{Conj} \circ \rho)$	Th 5.4.2
1-cocycle space	$Z^1(\Gamma, \text{Conj} \circ \rho)$	Th 5.2.2

1 Representation Varieties

We begin with some elementary definitions, before introducing the space of representations $R(\Gamma, \mathrm{GL}_n(k))$ of a finitely generated group Γ into $\mathrm{GL}_n(k)$, which we view as an affine variety. We define the coordinate ring of an affine algebraic set $V \subset k^m$ as the set of all regular maps from V to the base ring k. In order to study the representation variety across different base rings, we define affine schemes; representable functors from the category of commutative rings $\mathbf{CRing}$ to the category of sets $\mathbf{Set}$. From this perspective, an affine variety over some base ring k is interpreted as an affine scheme at the point k. We will usually denote by R an arbitrary commutative ring and by k an algebraically closed field of characteristic zero. The general reference for representation theory and algebraic geometry is [Ste12] and [Har77], respectively.

1.1 The Space of Representations

Definition 1.1.1. A group $(\Gamma, \times)$ is said to be **finitely generated** if there exists some finite subset $S \subseteq \Gamma$ such that for all $g \in \Gamma$, there exist $s_i \in S,\ a_i \in \mathbb{Z}$ with $i \in \{1, ..., n\}$ satisfying $g = \bigtimes_{i=1}^{n} s_i^{a_i}$. Call the magnitude of the set S the **rank** of the group Γ. In this context, S is referred to as a **generating set** for Γ and each element in the generating set is a **generator** for Γ. A **presentation** of Γ is a set of generators S of Γ together with a set of relations R on those generators, denoted $\langle S \mid R \rangle$.

Definition 1.1.2. A **free group** $\mathbb{F}_n$ of rank n is a finitely generated group of rank n, with no relations imposed on the generators. Let $S = \{s_1, ..., s_n\}$ be a generating set for $\mathbb{F}_n$. An element of $\mathbb{F}_n$ is any word in the elements of S and two words are only equal if it follows from the group axioms. Note that, for Γ a finitely generated group of rank n, there exists a surjective homomorphism

$$\phi : \mathbb{F}_n \to \Gamma$$

sending generators to generators. In other words, every finitely generated group is a homomorphic image of a free group. Moreover, since we are free to choose where the generators of a free group are mapped to, we have the isomorphism

$$\psi : \mathrm{Hom}(\mathbb{F}_1, \Gamma) \cong \Gamma : (f : s_1 \mapsto \gamma) \mapsto \gamma$$

where s_1 is a generator for $\mathbb{F}_1$ and Γ is any finitely generated group. Usually, the general linear group is defined over a field but in our case it will be more convenient to consider commutative rings in general. To that end, we require the notion of a module, which can be thought of as a vector space where the scalar field is replaced by a ring.

Definition 1.1.3. Let R be a commutative ring, then a **R-module** M is an abelian group under addition with an operation $* : R \times M \to M$ such that

I) $r_1 * (m_1 + m_2) = r_1 * m_1 + k_1 * m_2$ for any $r_1 \in R$ and $m_1, m_2 \in M$

II) $(r_1 + r_2) * m = r_1 * m + r_2 * m$ for any $r_1, r_2 \in R$ and $m \in M$

III) $(r_1 r_2) * m = r_1 * (r_2 * m)$ for any $r_1, r_2 \in R$ and $m \in M$

IV) $1_R * m = m$ for any $m \in M$

Usually, a module is defined over a general ring with unity, where R acts from the left or the right, giving rise to the notion of a left and right R-module but since we always assume that our base rings are commutative rings, the two notions coincide. We call a module M **free** if it has a proper basis. That is, there exists some linearly independent generating set of M. If there exists a finite basis of M of magnitude n, then call M free of **rank** n.

Example 1.1.4. Let R be a ring and consider the Cartesian product R^n for some $n \in \mathbb{Z}$. Define the scalar multiplication

$$* : R \times R^n \to R^n : (r, (r_1, ..., r_n)) \mapsto (rr_1, ..., rr_n)$$

Since R is a ring, the R-module structure of R^n follows from the distributivity and associativity of addition and multiplication in R. Moreover, R^n is a free R-module since

$$S := \{(1_R, ..., 0), (0, 1_R, ..., 0), ..., (0, ..., 1_R)\} \subset R^n$$

forms a basis for R^n as a module. Indeed, the set S is clearly linearly independent and, moreover, for any $(r_1, ..., r_n) \in R^n$, we have that

$$(r_1, ..., r_n) = r_1 * (1_R, ..., 0) + ... + r_n * (0, ..., 1_R)$$

as required. Note that, if R is a field, then we arrive at the usual definition of a vector space. That is, a R-module M is a vector space if and only if R is a field. Modules allow for a more general definition of the general linear group over a commutative ring:

Definition 1.1.5. Let R be a commutative ring, then the **general linear group** of degree n over R is the automorphism group of the free $R-$module R^n, denoted $\mathrm{GL}_n(R)$. That is, every element $g \in \mathrm{GL}_n(R)$ is a map $g : R^n \to R^n$ such that

I) $g(v + u) = g(v) + g(u)$ for any $v, u \in R^n$ *(additivity)*

II) $g(r_1 v) = r_1 g(v)$ for any $r_1 \in R$ and $v \in R^n$ *(homogeneity)*

III) There exists some $g^{-1} \in \mathrm{GL}_n(R)$ such that $g \circ g^{-1} = g^{-1} \circ g = \mathrm{id}_{R^n}$ *(bijection)*

Alternatively, $\mathrm{GL}_n(R)$ can be thought of as the group of invertable n by n matrices with coefficients in R. The **special linear group** is the subgroup $\mathrm{SL}_n(R) \leq \mathrm{GL}_n(R)$ of matrices with trivial determinant.

Definition 1.1.6. A **representation** of a group Γ into $\mathrm{GL}_n(R)$ is a map $\rho : \Gamma \to \mathrm{GL}_n(R)$ subject to:

$$\rho(ab) = \rho(a)\rho(b)\ \forall\, a, b \in \Gamma$$

That is, a representation is a group homomorphism. For two representations $\rho_1 : \Gamma \to \mathrm{GL}_a(R)$ and $\rho_2 : \Gamma \to \mathrm{GL}_b(R)$, we define the **direct sum** of representations [Heu14, p. 139] as a map:

$$(\rho_1 \oplus \rho_2) : \Gamma \to \mathrm{GL}_{a+b}(R) : \gamma \to \left(\begin{array}{c|c} \rho_1(\gamma) & 0 \\ \hline 0 & \rho_2(\gamma) \end{array}\right)$$

This is a representation since we have

$$\begin{aligned}
(\rho_1 \oplus \rho_2)(\gamma_1\gamma_2) &= \left(\begin{array}{c|c} \rho_1(\gamma_1\gamma_2) & 0 \\ \hline 0 & \rho_2(\gamma_1\gamma_2) \end{array}\right) \\
&= \left(\begin{array}{c|c} \rho_1(\gamma_1)\rho_1(\gamma_2) & 0 \\ \hline 0 & \rho_2(\gamma_1)\rho_2(\gamma_2) \end{array}\right) \\
&= \left(\begin{array}{c|c} \rho_1(\gamma_1) & 0 \\ \hline 0 & \rho_2(\gamma_1) \end{array}\right) \left(\begin{array}{c|c} \rho_1(\gamma_2) & 0 \\ \hline 0 & \rho_2(\gamma_2) \end{array}\right) \\
&= (\rho_1 \oplus \rho_2)(\gamma_1) \cdot (\rho_1 \oplus \rho_2)(\gamma_2)
\end{aligned}$$

If we consider a free sub-module $R^m \subseteq R^n$, we can define the **subrepresentation** ρ' of $\rho : \Gamma \to \mathrm{GL}_n(R)$ on R^m as a representation

$$\rho' : \Gamma \to \mathrm{GL}_m(R) : \gamma \mapsto \rho(\gamma)$$

Here, the automorphism given by $\rho(\gamma)$ on R^m is induced by the inclusion of $R^m \hookrightarrow R^n$. For this to be a valid representation, we require that the sub-module R^m is stable under ρ, such that the map ρ' is an automorphism of R^m. Denote such a subrepresentation of ρ by $\rho' \subset \rho$.

Definition 1.1.7. Two representations $\rho, \rho' : \Gamma \to \mathrm{GL}_n(R)$ are **equivalent** if and only if there exists some $g \in \mathrm{GL}_n(R)$ such that $\rho'(\gamma) = g\rho(\gamma)g^{-1}$ for every $\gamma \in \Gamma$. Denote this equivalence relation by $\rho \sim \rho'$.

In order to understand an arbitrary representation, it is often useful to study the building blocks of such a representation if it can be broken into 'smaller' parts. In the case of representation theory, we call such fundamental representations simple representations.

Definition 1.1.8. Let R be a commutative ring. A representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{GL}_n(R))$ is **simple** if the only sub-modules of R^n invariant under $\rho(\Gamma)$ are $\{0\}$ and R^n. If a representation is not simple, we say that it is **reducible**. ρ is said to be **semi-simple** if, for any $\rho' \subseteq \rho$, there exists $\rho^* \subseteq \rho$ such that $\rho = \rho' \oplus \rho^*$

Note that simple representations are precisely those representations with no non-trivial subrepresentations. If G is a finite group, there is a complete description of the representations from G to $\mathrm{GL}_n(k)$ when k is a field, in terms of simple representations.

Theorem 1.1.9 (Maschke)**.**
Let G be a finite group and k a field of characteristic m such that $m \not| \mathrm{Ord}(G)$, then every representation

$$\rho : G \to GL_n(k)$$

is semi-simple.

Proof: See Steinberg's book [Ste12, p. 23]

Unfortunately, this is not always the case for infinite groups. In particular, for Γ finitely generated, there exists some reducible representations which are not semi-simple.

Example 1.1.10. Consider the map $\rho : \langle x, y \mid x^2 = y^3 \rangle \to \mathrm{GL}_2(\mathbb{C})$ defined by

$$\rho : x \mapsto \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \qquad \rho : y \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

If we let $\rho(ab) = \rho(a)\rho(b)$ for any $a, b \in \langle x, y \mid x^2 = y^3 \rangle$, we have

$$\begin{aligned} \rho(x^2) &= \rho(x)\rho(x) \\ &= \left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 6 \\ 0 & 1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) \\ &= \rho(y)\rho(y)\rho(y) \\ &= \rho(y^3) \end{aligned}$$

That is, the map ρ respects the structure given by the relation in $\langle x, y \mid x^2 = y^2 \rangle$, so ρ is a representation. Moreover, note that

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}$$

Thus, we have

$$\rho(x)\rho(y^{-1}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

That is, for any $n \in \mathbb{Z} \setminus \{0\}$, we obtain

$$(\rho(x)\rho(y^{-1}))^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

It is clear then that the image of ρ is isomorphic to $\mathbb{Z}$:

$$\mathrm{Im}(\rho) = \{ \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C}) \mid n \in \mathbb{Z} \} \cong \mathbb{Z}$$

Consider the subset

$$U := \{(a, 0) \in \mathbb{C}^2 \mid a \in \mathbb{C}\} \subsetneq \mathbb{C}^2$$

For any $g \in \langle x, y \mid x^2 = y^3 \rangle$ we have $\rho(g) = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ for some $n \in \mathbb{Z}$. But then $\rho(g)u = u$ for any $u \in U$ so U is invariant under $\rho(\Gamma)$ and clearly $U \neq \{0\}$. Therefore, ρ is a reducible representation. Moreover, if we consider any non-trivial subrepresentation $\rho' \subset \rho$, then $\rho' : \Gamma \to \mathrm{GL}_1(\mathbb{C})$. But, if ρ is assumed to be semi-simple, then there must exist some other subrepresentation $\rho^* : \Gamma \to \mathrm{GL}_1(\mathbb{C})$ such that, for any $\gamma \in \Gamma$ and some $n \in \mathbb{Z}$, we have

$$\begin{aligned} \rho(\gamma) &= \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \\ &= (\rho' \oplus \rho^*)(\gamma) \\ &= \left(\begin{array}{c|c} \rho'(\gamma) & 0 \\ \hline 0 & \rho^*(\gamma) \end{array}\right) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This only holds for $n = 0$, a contradiction. Thus $\rho : \Gamma \to \mathrm{GL}_2(\mathbb{C})$ is a reducible representation which is not semi-simple. We show that, for finite groups, such a representation cannot exist. If G is a finite group, then the order of each element of G is finite. Let $g \in G$, then there exists some $m \in \mathbb{Z}_{>0}$ such that $g^m = 1_G$. Moreover, if $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ and $\rho(g) = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ for some $n \in \mathbb{Z} \setminus \{0\}$, then we have

$$\begin{aligned} I_2 &= \rho(1_G) \\ &= \rho(g^m) \\ &= \rho(g)^m \\ &= \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}^m \\ &= \begin{pmatrix} 1 & mn \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Since both $m, n \in \mathbb{Z} \setminus \{0\}$, we have $mn \neq 0$, a contradiction. Thus, such a representation ρ cannot exist for a finite group G. We note that the set of all representations from Γ to $\mathrm{GL}_n(k)$ is equipped with a natural structure endowed by the presentation of Γ.

Definition 1.1.11. Let $\Gamma = \langle x_1, ..., x_m | r_1, ..., r_s \rangle$ be finitely presented group with representation $\rho : \Gamma \to \mathrm{GL}_n(R)$ where R is a commutative ring. Now ρ is a group homomorphism, so ρ is completely determined by where it maps generators. Therefore, the tuple $(g_1, ..., g_m) := (\rho(x_1), ..., \rho(x_m))$, subject to relations $r_j(g_1, ..., g_m) = I_n$ for $1 \leq j \leq s$, characterises ρ entirely. The **space of representations** is defined as:

$$R(\Gamma, \mathrm{GL}_n(R)) := \{(g_1, ..., g_m) \in (\mathrm{GL}_n(R))^m | r_j(g_1, ..., g_m) = I_n, 1 \leq j \leq s\}$$

As we will see later, this presentation of the representation space defines an affine algebraic set when R is a field.

1.2 Affine Varieties

Definition 1.2.1. Let k be a field and $f_1, ..., f_m \in k[x_1, ..., x_n]$, then the **affine algebraic set** over the polynomials f_i is defined by

$$\mathcal{V}(f_1, ..., f_m) := \{(a_1, ..., a_n) \in k^n \mid f_i(a_1, ..., a_n) = 0 \; \forall \; i \in \mathbb{Z}_{1 \leq i \leq m}\}$$

That is, an affine algebraic set is the zero locus of a collection of polynomials. Moreover, if we define the ideal $I = (f_1, ..., f_m) \lhd k[x_1, ..., x_n]$, generated by f_i, then the affine algebraic set over I is given by

$$\mathcal{V}(I) := \mathcal{V}(f \in I) = \{(a_1, ..., a_n) \in k^n \mid f(a_1, ..., a_n) = 0 \; \forall \; f \in I\}$$

It is clear that $\mathcal{V}((f_1, ..., f_m)) = \mathcal{V}(f_1, ..., f_m)$ since each element f of the ideal generated by polynomials f_i necessarily contains a factor of some f_j, thus f vanishes when f_j vanishes. Moreover, for $I, J \lhd k[x_1, ..., x_n]$, we have:

$$\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(IJ) \quad \text{and} \quad \mathcal{V}(I) \cap \mathcal{V}(J) = \mathcal{V}(I + J)$$

In fact, affine algebraic sets are closed under arbitrary intersection and finite union. See [Har77, p. 2-3] for more details. Note that we will often refer to an affine algebraic set as an affine variety, without requiring that the set is irreducible. We define the **ideal of a variety** $\mathcal{I}(V)$ for some $V \subset k^n$ via

$$\mathcal{I}(V) := \{f \in k[x_1, ..., x_n] \mid f(v) = 0 \; \forall \; v \in V\}$$

That is, the ideal of a variety is the ideal generated by polynomials vanishing on the variety. For any ideal $I \lhd k[x_1, ..., x_n]$, we have $I \subseteq \mathcal{I}(\mathcal{V}(I))$.

Example 1.2.2. In general, we cannot expect $I \subseteq \mathcal{I}(\mathcal{V}(I))$ to be an equality. If we let $I = (x^2) \triangleleft \mathbb{C}[x]$, then

$$\begin{aligned}\mathcal{I}(\mathcal{V}(I)) &= \mathcal{I}(\mathcal{V}((x^2))) \\ &= \mathcal{I}(\{0\}) \\ &= (x)\end{aligned}$$

As a direct consequence of Hilbert's Nullstellensatz, we have that the states that the ideal of the variety of an ideal I is given by the radical of I, see [AM69, p. 85]. That is

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I} := \{f \in k[x_1, ..., x_n] \mid \exists\, m \in \mathbb{Z}_{>0},\ \text{st } f^m \in I\}$$

In the case of $I = (x^2)$, we have

$$\sqrt{I} = \{f \in \mathbb{C}[x] \mid \exists\, m \in \mathbb{Z}_{>0},\ \text{st } f^m \in (x^2)\}$$

Clearly $x \in \mathbb{C}[x]$ and $x^2 \in I$ so $(x) \subseteq \sqrt{(x^2)}$. Now, assume there exists some $f \in \mathbb{C}[x]$ such that $f^m \in (x^2)$. If f is constant then $f^m \in \mathbb{C}$ for any $m \in \mathbb{Z}_{>0}$ so $f \notin (x^2)$, a contradiction. Therefore f must be non-constant but then $f \in (x)$, so $\sqrt{(x^2)} \subseteq (x)$. We conclude that $\sqrt{(x^2)} = \mathcal{I}(\mathcal{V}((x^2)))$, as expected.

Definition 1.2.3. Let $V \subset k^n$ for some field k. We call $f : V \to k$ **regular** if there exists some $f' \in k[x_1, ..., x_n]$ such that $f(v) = f'(v)$ for any $v \in V$. The set of all regular functions on V is denoted by $\mathcal{O}(V)$ and referred to as the **coordinate ring** of the variety. Note that there is a natural surjection

$$\phi : k[x_1, ..., x_n] \to \mathcal{O}(V)$$

The kernel of ϕ is given by all polynomials in $k[x_1, ..., x_n]$ vanishing on V and thus

$$\mathcal{O}(V) \cong k[x_1, ..., x_n]/\mathcal{I}(V)$$

This shows that it is enough to study the ideal $\mathcal{I}(V)$, to describe $\mathcal{O}(V)$.

Example 1.2.4. Let $V = \{-1, 1\} \subseteq \mathbb{C}$. Then the coordinate ring of V is given by

$$\mathcal{O}(V) \cong \mathbb{C}[x]/\mathcal{I}(V) = \mathbb{C}[x]/\langle f \in \mathbb{C}[x] \mid f(-1) = 0 = f(1)\rangle$$

Choose arbitrary $f \in \mathcal{I}(V)$. Then $x-1, x+1 \mid f$ and since both $x-1$ and $x+1$ are linear polynomials, they are irreducible, thus $x^2 - 1 = (x-1)(x+1) \mid f$. But $x^2 - 1 \in \mathcal{I}(V)$ and $\mathcal{I}(V) \subset (x^2 - 1)$, so we conclude that

$$\begin{aligned}\mathcal{O}(V) &\cong \mathbb{C}[x]/(x^2-1) \\ &\cong \mathbb{C}[x]/(x-1) \oplus \mathbb{C}/(x+1) \\ &\cong \mathbb{C} \oplus \mathbb{C}\end{aligned}$$

1.3 Functorial Affine Schemes

We move to a more general approach, introducing affine schemes, which are representable functors from $\mathbf{CRing}$ to $\mathbf{Set}$. Note that there is also a topological approach to schemes, in which affine schemes are locally ringed spaces isomorphic to the prime spectrum of some ring [Har77, p. 74]. These two viewpoints are compatible in the sense that both the category of "topological" affine schemes and the category of "functorial" affine schemes are equivalent to the opposite category of commutative rings. We opt for the functorial approach since it is more convenient for the construction of the moduli space as a geometric invariant theory quotient and requires no background in topology. The general reference for category theory is [Lan71].

Definition 1.3.1. A **category** $\mathcal{C}$ consists of classes Obj$(\mathcal{C})$ and Hom$(\mathcal{C})$ consisting of **objects** and **morphisms**, respectively, together with class functions

$$\text{dom} : \text{Hom}(\mathcal{C}) \to \text{Obj}(\mathcal{C})$$

$$\text{cod} : \text{Hom}(\mathcal{C}) \to \text{Obj}(\mathcal{C})$$

and composition $\circ : \text{Hom}(B, C) \times \text{Hom}(A, B) \to \text{Hom}(A, C)$. Here $\text{Hom}(A, B)$ denotes the subclass of morphisms $f \in \text{Hom}(\mathcal{C})$ such that $\text{dom}(f) = A$ and $\text{cod}(f) = B$. We will write $f : A \to B$ to denote such an element $f \in \text{Hom}(A, B)$. These morphisms are subject to the following axioms:

I) $f : A \to B,\ g : B \to C$ and $h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$ **(associativity)**

II) For all $X \in \text{Obj}(\mathcal{C})$, there exists some $1_X \in \text{Hom}(X, X)$ such that for any $f : A \to X$ and any $g : X \to A$, we have that $1_X \circ f = f$ and $g \circ 1_X = g$ **(identity)**

In essence, a category consists of objects and maps between objects (morphisms) such that these maps behave in a natural way with composition and associativity. Note that we will often write $A \in \mathcal{C}$ for an object $A \in \text{Obj}(\mathcal{C})$ when the context is clear. In the case where we are working with several categories, we will include a subscript for the morphism classes $\text{Hom}_{\mathcal{C}}(A, B)$ to specify the category $\mathcal{C}$ to which the class belongs. A basic example of a category is sets together with total functions, denoted $\mathbf{Set}$.

Definition 1.3.2. Let $\mathcal{C}$ be a category. A morphism $f : A \to B \in \text{Hom}(\mathcal{C})$ is said to be a **monomorphism** if it satisfies the following: For any $C \in \text{Obj}(\mathcal{C})$ and any morphisms $g_1, g_2 : C \to A$, we have

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

Monomorphisms are categorical generalisations of injections. In particular, every injective morphism is a monomorphism. Similarly, f is an **epimorphism** if it satisfies: For any $C \in \text{Obj}(\mathcal{C})$ and any morphisms $g_1, g_2 : B \to C$, we have

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2$$

Epimorphisms are categorical generalisations of surjections, in the sense that every surjective morphism is an epimorphism. We say that a morphism $f : A \to B$ is an **isomorphism** if there exits a morphism $g : B \to A$ such that

$$f \circ g = 1_B \text{ and } g \circ f = 1_A$$

Definition 1.3.3. Let $\mathcal{C}$ be a category. If $\text{Hom}(A, B)$ is a set for any $A, B \in \text{Obj}(\mathcal{C})$, call $\mathcal{C}$ **locally small**. If both $\text{Obj}(\mathcal{C})$ and $\text{Hom}(\mathcal{C})$ are sets, we say that $\mathcal{C}$ is **small**. let $A \in \mathcal{C}$, then if we have

$$|\text{Hom}_{\mathcal{C}}(A, B)| = 1 \; \forall \; B \in \text{Obj}(\mathcal{C})$$

call A **initial** in $\mathcal{C}$. If instead

$$|\text{Hom}_{\mathcal{C}}(B, A)| = 1 \; \forall \; B \in \text{Obj}(\mathcal{C})$$

call A **terminal** in $\mathcal{C}$.

The empty set is initial in the category of sets since there is exactly one total function from the empty set, specifically the empty function. To move between categories we have functors, preserving identity morphisms and composition. We destinguish between two types of functors, covariant and contravariant functors. Covariant functors preserve the direction of morphisms whereas contravariant functors reverse the direction of morphisms.

Definition 1.3.4. Let $\mathcal{C},\ \mathcal{D}$ be categories, then a **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ consists of a pair of maps

$$F_{\text{Obj}} : \text{Obj}(\mathcal{C}) \to \text{Obj}(\mathcal{D})$$

$$F_{\text{Hom}} : \text{Hom}(\mathcal{C}) \to \text{Hom}(\mathcal{D})$$

satisfying the following axioms:

I) $F_{\text{Hom}}(f : A \to B) : F_{\text{Obj}}(A) \to F_{\text{Obj}}(B)$

II) $F_{\text{Hom}}(1_X) = 1_{F_{\text{Obj}}(X)}$

III) $F_{\text{Hom}}((g : B \to C) \circ (f : A \to B)) = F_{\text{Hom}}(g) \circ F_{\text{Hom}}(f)$

When the context is clear, we denote $F_{\text{Obj}} = F$ and also $F_{\text{Hom}} = F$. A **contravariant functor** is a functor such that $F(f : A \to B) : F(B) \to F(A)$ with appropriate composition. For a category $\mathcal{C}$, we can construct the **opposite category** $\mathcal{C}^{op}$ such that $\text{Obj}(\mathcal{C}) = \text{Obj}(\mathcal{C}^{op})$. Each morphism

$$(f : A \to B) \in \text{Hom}(\mathcal{C})$$

gives rise to a morphism

$$(f' : B \to A) \in \text{Hom}(\mathcal{C}^{op})$$

As such, we have that $\text{Hom}_{\mathcal{C}}(A, B) = \text{Hom}_{\mathcal{C}^{op}}(B, A)$ and it is clear that $(\mathcal{C}^{op})^{op} = \mathcal{C}$. We note that a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor from $\mathcal{C}^{op}$ to $\mathcal{D}$. Moreover, initial objects in $\mathcal{C}$ are terminal in $\mathcal{C}^{op}$. Similarly, monomorphisms in $\mathcal{C}$ are epimorphisms in $\mathcal{C}^{op}$ and vice versa. Whenever we write 'functor' we will assume that it is covariant since contravariant functors can be represented as covariant functors from the opposite category.

Example 1.3.5. Any locally small category $\mathcal{C}$ admits both a covariant and contravariant functor to $\mathbf{Set}$. Let $A, B \in \mathcal{C}$, then we have the two **Hom-functors**:

$$\text{Hom}(A, \_) : \mathcal{C} \to \mathbf{Set} : C \mapsto \text{Hom}_{\mathcal{C}}(A, C)$$

$$\text{Hom}(\_, B) : \mathcal{C} \to \mathbf{Set} : C \mapsto \text{Hom}_{\mathcal{C}}(C, B)$$

The notion of two categories being isomorphic is rarely practical since it is too restrictive. Instead, we define an equivalence of categories for locally small categories. Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exists some functor $F : \mathcal{C} \to \mathcal{D}$ between them which is bijective on hom-sets and essentially surjective on objects.

Definition 1.3.6. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ between locally small categories is an **equivalence of categories** if it satisfies:

I) $F : \text{Hom}_{\mathcal{C}}(A, B) \mapsto \text{Hom}_{\mathcal{D}}(F(A), F(B))$ is surjective **(full)**

II) $F : \text{Hom}_{\mathcal{C}}(A, B) \mapsto \text{Hom}_{\mathcal{D}}(F(A), F(B))$ is injective **(faithful)**

III) For any $D \in \text{Obj}(\mathcal{D})$ there exists $C \in \text{Obj}(\mathcal{C})$ such that $F(C) \cong D$ **(dense)**

Definition 1.3.7. Let F, G be functors from $\mathcal{C}$ to $\mathcal{D}$, then a **natural transformation** η from F to G is a family of morphisms satisfying:

I) For all $X \in \text{Obj}(\mathcal{C})$, we associate a morphism $\eta_X : F(X) \to G(X) \in \text{Hom}(\mathcal{D})$

II) For all $(f : X \to Y) \in \text{Hom}(\mathcal{C})$ we have $\eta_Y \circ F(f) = G(f) \circ \eta_X$

In the notation above, if η_X is an isomorphism in $\mathcal{D}$ for any $X \in \mathrm{Obj}(\mathcal{C})$, then call η a **natural isomorphism**.

Note that we could rewrite the definition of an equivalence of categories via natural transformations [Lan71, p. 91]. Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exists functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ such that we have natural isomorphisms:

$$F \circ G \cong 1_{\mathcal{C}} \text{ and } G \circ F \cong 1_{\mathcal{D}}$$

The advantage of this approach over 1.3.6 is that it is sometimes easier to verify that two functors do indeed give rise to an equivalence of categories whereas 1.3.6 is more useful in cases where there is not a natural candidate for an inverse.

Example 1.3.8. Commutative rings together with ring homomorphisms constitute a category, denoted $\mathbf{CRing}$. Indeed, every ring R has an automorphism given by

$$\mathrm{id}_R : R \to R : r \mapsto r$$

Moreover, if there exists ring homomorphisms

$$f : A \to B \qquad g : B \to C$$

then the composition $g \circ f : A \to C$ is also a ring homomorphism. Additionally, composition of ring homomorphisms is associative by construction.

Definition 1.3.9. A functor $\mathfrak{A} : \mathbf{CRing} \to \mathbf{Set}$ is a **(functorial) affine scheme** if it is naturally isomorphic to the functor

$$\mathrm{Spec}(A) : \mathbf{CRing} \to \mathbf{Set} : B \mapsto \mathrm{Hom}_{\mathbf{CRing}}(A, B)$$

for some commutative ring A. Explicitly, the functor $\mathfrak{A}$ is an affine scheme if, for any commutative rings R, S and any morphism $f : R \to S$, we have the isomorphisms

$$\eta_R : \mathfrak{A}(R) \to \mathrm{Hom}(A, R) \text{ and } \eta_S : \mathfrak{A}(S) \to \mathrm{Hom}(A, S)$$

such that the following diagram commutes:

$$\begin{tikzcd}
R \arrow[d, "f"] & \mathfrak{A}(R) \arrow[r, tail, two heads, "\eta_R"] \arrow[d, "\mathfrak{A}(f)"] & Hom(A, R) \arrow[d, "f\circ _"] \\
S & \mathfrak{A}(S) \arrow[r, tail, two heads, "\eta_S"] & Hom(A, S)
\end{tikzcd}$$

In other words, an affine scheme is a functor mapping commutative rings to sets which is essentially the same as a Hom-functor, for some fixed ring A. We say that A **represents** Spec(A) and that A is the **coordinate ring** of the scheme $\mathfrak{A}$, denoted $\mathcal{O}(\mathfrak{A})$. Note that, if we have a map of rings $g : A \to A'$, then this induces a map Spec$(A') \to$ Spec(A) via composition. For $(\beta : A' \to B) \in$ Hom(A', B) we have that $\beta \circ g : A \to A' \to B \in$ Hom(A, B). Thus

$$\_ \circ g : \text{Hom}(A', B) \to \text{Hom}(A, B)$$

Under these morphisms, affine schemes form a category, denoted **AffS**. We will exclusively concern ourselves with affine schemes throughout so the terms "affine scheme" and "scheme" will be used interchangeably.

Example 1.3.10. Consider the functor

$$\mathbb{A}^n : \mathbf{CRing} \to \mathbf{Set} : R \mapsto R^n$$

For any commutative ring R, we have $R \cong \text{Hom}_{\mathbf{CRing}}(\mathbb{Z}[x], R)$ given by $r \mapsto (x \mapsto r)$. Moreover, we have $R^n \cong \text{Hom}_{\mathbf{CRing}}(\mathbb{Z}[x_1, ..., x_n], R)$ given by $(r_1, ..., r_n) \mapsto (x_i \mapsto r_i)$. Thus, up to isomorphism of sets, we rewrite $\mathbb{A}^n$ as

$$\mathbb{A}^n : \mathbf{CRing} \to \mathbf{Set} : R \mapsto \text{Hom}_{\mathbf{CRing}}(\mathbb{Z}[x_1, ..., x_n], R)$$

That is, $\mathbb{A}^n$ is a scheme, represented by $\mathbb{Z}[x_1, ..., x_n]$.

2 Representation Schemes

We show that there are two categorical equivalences

$$\mathbf{CRing}^{\text{op}} \sim \mathbf{AffS} \qquad \mathbf{Comm}k\mathbf{Alg}^{\text{op}} \sim \mathbf{Aff}k\mathbf{S}$$

This allows us to parse to commutative rings when proving statements about affine schemes and vice versa. We show that the functor

$$R(\Gamma, \mathrm{GL}_n) : \mathbf{CRing} \to \mathbf{Set} : k \to R(\Gamma, \mathrm{GL}_n(k))$$

is an affine scheme. Additionally, we prove that there is a simple way of moving from the coordinate ring of the scheme $R(\Gamma, \mathrm{GL}_n)$ to the coordinate ring of $R(\Gamma, \mathrm{GL}_n(k))$ via nilradical reduction.

2.1 Categorical Equivalences

The Yoneda embedding provides a means to embed any locally small category $\mathcal{C}$ into the category of functors from $\mathcal{C}$ to $\mathbf{Set}$ such that the structure of the Hom-sets is preserved.

Theorem 2.1.1 (Yoneda Embedding)**.**
Let $\mathcal{C}$ be a locally small category, then the contravariant functor

$$F : \mathcal{C} \to \mathit{Fun}(\mathcal{C}, \mathbf{Set}) : B \mapsto \mathit{Hom}_{\mathcal{C}}(B, \_)$$

is full and faithful. In other words, it is bijective on Hom-sets.

Proof: See "Categories for the working mathematician" [Lan71, p.61]

Theorem 2.1.2.
There exists an equivalence of categories

$$\mathbf{AffS} \sim \mathbf{CRing}^{op}$$

Proof: It is sufficient to prove that there exists a full, faithful and dense contravariant functor from the category of commutative rings to the category of affine schemes. Consider the contravariant hom-functor:

$$G : \mathbf{CRing} \to \mathrm{Fun}\Big(\mathbf{CRing}, \mathbf{Set}\Big) : A \mapsto \mathrm{Hom}(A, \_)$$

$$G : \mathrm{Hom}(A_1, A_2) \to \mathrm{Hom}\Big(\mathrm{Hom}(A_2, \_), \mathrm{Hom}(A_1, \_)\Big) : f \mapsto \_ \circ f$$

Rings are sets with additional structure and the maps between two sets forms a set itself. Since ring morphisms are set maps with additional constraints, $\mathbf{CRing}$ is locally small. It follows immediately from the Yoneda Embedding, that G is both full and faithful. Moreover, the functors $H : \mathbf{CRing} \to \mathbf{Set}$ such that there exists $A \in \mathbf{CRing}$ with $G(A)$ naturally isomorphic to H are, by definition, affine schemes. Then G is dense onto the subcategory of affine schemes inside $\text{Fun}\Big(\mathbf{CRing}, \mathbf{Set}\Big)$, as required.

Q.E.D.

This justifies the emphasis on the ring A in the notation Spec(A) since the categories $\mathbf{CRing}^{op}$ and **AffS** are essentially the same. It is often useful to associate a base ring to our affine schemes and work over commutative k-algebras instead of all commutative rings. Unless otherwise specified, we assume that the commutative base ring we are working over is an algebraically closed field of characteristic zero. This is done in order to ensure that the points of our schemes are well-defined as affine varieties.

Definition 2.1.3. An affine scheme $\mathfrak{A} = \text{Spec}(A)$ is an **affine *k*-scheme**, for some commutative base ring *k*, if there exists a morphism of affine schemes $\text{Spec}(A) \to \text{Spec}(k)$. Denote the category of affine *k*-schemes by **Aff*k*S**.

Next, we note that, for any commutative ring A, there exits a unique ring morphism

$$\phi : \mathbb{Z} \to A : 1 \mapsto 1_A$$

That is, $\mathbb{Z}$ is initial in the category of commutative rings. Moreover, if we have a ring morphism $\phi : \mathbb{Z} \to A$, this induces an affine scheme morphism

$$\phi^* : \text{Spec}(A) \to \text{Spec}(\mathbb{Z})$$

In particular, Spec$(\mathbb{Z})$ is terminal in the category of affine schemes. Thus, every affine scheme is an affine $\mathbb{Z}$-scheme.

Definition 2.1.4. Let R and A rings, then A is an **R-algebra** if there exists a ring morphism $\alpha : R \to A$ such that

$$\text{Im}_\alpha(R) \subseteq \{a \in A \mid b \cdot a = a \cdot b \,\forall\, b \in A\} =: Z(A)$$

The set $Z(A)$ is called the **center** of ring A. α gives rise to a scalar multiplication:

$$\cdot : R \times A \to A : (r, a) \mapsto \alpha(r)a$$

Note that $Z(A) = A$ if and only if A is commutative. Thus, a commutative ring A is an R-algebra if and only if there exists a ring homomorphism $\alpha : R \to A$.

If commutative ring A is a R-algebra, for some commutative base ring R, we call A a **commutative *R*-algebra**. We denote the category of commutative k-algebras by $\mathbf{Comm}k\mathbf{Alg}$. Since $\mathbb{Z}$ is initial in $\mathbf{CRing}$, we have that any $A \in \mathbf{CRing}$ is a $\mathbb{Z}$-algebra. Since affine schemes are defined over commutative rings, we take 'k-algebra' to mean 'commutative k-algebra' throughout.

Example 2.1.5. Let $A = R[x_1, ..., x_n]$ for some commutative ring R, then A is trivially endowed with an R-algebra structure via the inclusion

$$\alpha : R \hookrightarrow A : r \mapsto rx_1^0$$

We define the scalar multiplication via

$$\cdot : R \times A \to A : (r, a) \mapsto \alpha(r)a = ra$$

The language of k-algebras and k-schemes allows us to reformulate 2.1.2 to be an equivalence between the category of affine $\mathbb{Z}$-schemes and the opposite category of commutative $\mathbb{Z}$-algebras. It is natural, then, to expect that the correspondence between affine schemes and commutative rings might extend to arbitrary base rings *k*. This is indeed the case:

Theorem 2.1.6.
There exists an equivalence of categories

$$\mathbf{Comm}k\mathbf{Alg} \sim \mathbf{Aff}k\mathbf{S}^{op}$$

Proof: Fix a base ring *k*, then consider the contravariant functor:

$$F : \mathbf{Comm}k\mathbf{Alg} \to \text{Fun}\Big(\mathbf{Comm}k\mathbf{Alg}, \mathbf{Set}\Big) : A \mapsto \text{Hom}_{\mathbf{Comm}k\mathbf{Alg}}(A, \_)$$

$$F : \text{Hom}(A_1, A_2) \to \text{Hom}\Big(\text{Hom}(A_2, \_), \text{Hom}(A_1, \_)\Big) : f \mapsto \_ \circ f$$

As in the proof of 2.1.2, Yoneda's embedding implies that F is both full and faithful. It remains to show that F is dense onto the subcategory of $\mathbf{Aff}k\mathbf{S}$. By the construction of F, each point $F(A)$ is a Hom-functor

$$\text{Hom}_{\mathbf{Comm}k\mathbf{Alg}}(A, \_) : \mathbf{Comm}k\mathbf{Alg} \to \text{Set}$$

Choose arbitrary $H \in \mathbf{Aff}k\mathbf{S}$, that is a functor $H : \mathbf{CRing} \to \mathbf{Set}$ such that there exists $B \in \mathbf{CRing}$ with $\text{Hom}_{\mathbf{CRing}}(B, \_) \sim^{nat} H$ and a morphism $H \to \text{Spec}(k)$. Since H is naturally isomorphic to $\text{Hom}_{\mathbf{CRing}}(B, \_)$, there must exist a scheme morphism

$$\phi^* : \text{Hom}_{\mathbf{CRing}}(B, \_) \to \text{Spec}(k)$$

By 2.1.2, we obtain a ring morphism

$$\phi : k \to B$$

This endows B with a k-algebra structure via the scalar multiplication

$$*_B : k \times B \to B : (k', b) \mapsto \phi(k') \cdot_B b$$

Thus, it is sufficient to find some $A \in \mathbf{Comm}k\mathbf{Alg}$ such that

$$F(A) := \mathrm{Hom}_{\mathbf{Comm}k\mathbf{Alg}}(A, \_) = \mathrm{Hom}_{\mathbf{CRing}}(B, \_) \sim^{nat} H$$

Let $A = B$ and choose arbitrary $C \in \mathbf{CRing}$. We claim that any ring morphism $\Phi : B \to C$ extends to a morphism of k-algebras. C is endowed with the structure of a k-algebra, via the ring morphism $\Phi \circ \phi$, given by

$$*_C : k \times C \to C : (k', c) \mapsto (\Phi \circ \phi(k')) \cdot_C c$$

Moreover, for arbitrary $k' \in k$ and $b \in B$, we obtain

$$\begin{aligned} \Phi(k' *_B b) &= \Phi(\phi(k') \cdot_B b) \\ &= \Phi \circ \phi(k') \cdot_C \Phi(b) \\ &= k' *_C \Phi(b) \end{aligned}$$

Since Φ is also a ring morphism by assumption, Φ is a morphism of k-algebras, as claimed. Every morphism of k-algebras is a morphism of rings so for any $B \in \mathbf{Comm}k\mathbf{Alg}$:

$$\mathrm{Hom}_{\mathbf{Comm}k\mathbf{Alg}}(B, \_) = \mathrm{Hom}_{\mathbf{CRing}}(B, \_)$$

Then, any arbitrary $H \in \mathbf{Aff}k\mathbf{S}$ is naturally isomorphic to $F(B)$ for some $B \in \mathbf{Comm}k\mathbf{Alg}$. Therefore, F is dense onto $\mathbf{Aff}k\mathbf{S}$, as required.

Q.E.D.

2.2 The Representation Scheme

From now on, we will assume that we are always working over some fixed base field k. From the proof of 2.1.6, we have shown that the coordinate ring of a k-scheme is a k-algebra which we will call the **coordinate k-algebra** of the scheme in order to avoid confusing the coordinate ring of varieties and schemes.

Proposition 2.2.1.
The functor GL_n given by

$$\mathrm{GL}_n : \mathbf{CRing} \to \mathbf{Set} : B \to \mathrm{GL}_n(B)$$

is an affine scheme.

Proof: It is sufficient to find a commutative ring representing GL_n. Let $B \in \mathbf{CRing}$. We note that, as a set, we have

$$\mathrm{GL}_n(B) = \{B_{i,j} \in B \mid \mathrm{Det}(B_{i,j}) \in B^*\}$$

where $i, j \in \{1, ..., n\}$. But then we have multiplicative inverse $y \in B$ such that

$$\mathrm{Det}(B_{i,j}) \cdot y = y \cdot \mathrm{Det}(B_{i,j}) = 1_B$$

Thus, since B is assumed to be commutative, we can characterise $\mathrm{GL}_n(B)$ via

$$\mathrm{GL}_n(B) = \{y, B_{i,j} \in B \mid \mathrm{Det}(B_{i,j}) \cdot y - 1 = 0\}$$

The existence of a multiplicative inverse to the determinant is precisely the relation imposed on $\mathrm{GL}_n(B)$. But this set is in 1 to 1 correspondence with the set

$$\mathrm{Hom}_{\mathbf{CRing}}(\mathbb{Z}[X_{i,j}, Y]/(\mathrm{Det}(X_{i,j}) \cdot Y - 1)\ ,\ B)$$

where coefficients are chosen from B. Therefore the ring

$$A := \mathbb{Z}[X_{i,j}, Y]/(\mathrm{Det}(X_{i,j}) \cdot Y - 1)$$

represents GL_n as a functor. We conclude that GL_n is indeed an affine scheme.

Q.E.D.

Note that, for any choice of commutative ring k, we obtain the morphism

$$\sigma : k \to A_k := k[X_{i,j}, Y]/(\mathrm{Det}(X_{i,j}) \cdot Y - 1)$$

given by the inclusion. Then, on schemes, this induces a morphism

$$\sigma^* : \mathrm{Spec}(A_k) \to \mathrm{Spec}(k)$$

In particular $\mathrm{GL}_n^k := \mathrm{Spec}(A_k)$ is an affine k-scheme given by

$$\mathrm{GL}_n^k : \mathbf{CRing} \to \mathbf{Set} : B \mapsto \mathrm{Hom}_{\mathbf{CRing}}(A_k, B)$$

Moreover, GL_n^k agrees with GL_n at all k-algebra points. This highlights the importance of tracking the base ring k, since 'the' coordinate ring of the scheme GL_n varies depending on if we are viewing it as an affine k-scheme or an affine $\mathbb{Z}$-scheme.

Theorem 2.2.2. *[LM85, p. 3]*
If Γ is a finitely generated group, then

$$R(\Gamma, GL_n) : \mathbf{CRing} \to \mathbf{Set} : B \mapsto R(\Gamma, GL_n(B))$$

is an affine scheme.

Proof: It is sufficient to find a ring A representing $R(\Gamma, \mathrm{GL}_n)$. Let $B \in \mathbf{Comm}k\mathbf{Alg}$. Consider the group $\Gamma = \langle x_1, ..., x_m | r_1, ..., r_t \rangle$ and $\rho(x_k) =: B^k \in \mathrm{GL}_n(B)$ for some representation $\rho \in R(\Gamma, \mathrm{GL}_n(B))$. Then we have the form for $R(\Gamma, \mathrm{GL}_n(B))$ we obtained in 1.1.11:

$$R(\Gamma, \mathrm{GL}_n(B)) = \{(B^1, ..., B^m) \in \mathrm{GL}_n(B)^m \mid r_s(B^1, ..., B^k) = \mathrm{id} \ , \ 1 \leq s \leq t\}$$

But each B^l for $l \in \{1, ..., m\}$ can be expressed as an $(n^2 + 1)$-tuple in B satisfying a relation:

$$B^l = \{B^l_{i,j}, y^l \in B \mid \mathrm{Det}(B^l_{i,j}) \cdot y^l = 1_B\}$$

Here $\mathrm{Det}(B^l_{i,j})$ denotes the formal determinant of the matrix $(B^l_{i,j})_{1 \leq i,j \leq n}$ for some fixed l. Since each relation r_s for $s \in \{1, ..., t\}$ is defined on the B^l matrices, we can define the same relations on $B^l_{i,j}$ and y^l instead, retaining the same information. That is, we define

$$r_s(B^l_{i,j}, y^l) := r_s(B^1, ..., B^k)$$

For $u, l \in \{1, ..., m\}$, we define:

$$r_{t+u}(B^l_{i,j}, y^l) := \mathrm{Det}(B^u_{i,j}) \cdot y^u$$

With the setup out of the way, we have:

$$\mathrm{Hom}(\Gamma, \mathrm{GL}_n(B)) = \{B^l_{i,j}, y^l \in B \mid r_s(B^l_{i,j}, y^l) = 1_B\}$$

Where $1 \leq i, j \leq n$, $1 \leq l \leq m$ and $1 \leq s \leq t + m$. Recall that n denotes the dimension of the matrices whereas m and t denote the amount of generators in Γ and relations in Γ, respectively. But this set is precisely:

$$\mathrm{Hom}(\Gamma, \mathrm{GL}_n(B)) = \mathrm{Hom}(k[X^l_{i,j}, Y^l]/(r_s(X^l_{i,j}, Y^l) - 1), B)$$

Given by sending $X^l_{i,j} \mapsto B^l_{i,j}$ and $Y^l \mapsto y^l$. Then $k[X^l_{i,j}, Y^l]/(r_s(X^l_{i,j}, Y^l) - 1)$ represents $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(\_))$, as required.

Q.E.D.

Call the scheme $R(\Gamma, \mathrm{GL}_n)$ the **representation scheme** of Γ. The inclusion

$$k \hookrightarrow k[X^l_{i,j}, Y^l] \to \mathcal{O}(R(\Gamma, \mathrm{GL}_n))$$

shows that the representation scheme is a k-scheme.

Example 2.2.3. Let $\Gamma = \langle a, b \mid a^2 = b^3 \rangle$ and $\mathfrak{G} = \mathrm{SL}_2$. We present an explicit description of the coordinate $\mathbb{C}$-algebra representing $R(\Gamma, \mathfrak{G}(\mathbb{C}))$. The representation variety is given by:

$$R(\Gamma, \mathfrak{G}(\mathbb{C})) = \{(A, B) \in (\mathrm{SL}_2(\mathbb{C}))^2 \mid A^2 = B^3\}$$

The coordinate $\mathbb{C}$-algebra of the representation scheme is:

$$\mathcal{O}(R(\Gamma,\mathfrak{G})) = \mathbb{C}[A_{1,1},A_{1,2},A_{2,1},A_{2,2},B_{1,1},B_{1,2},B_{2,1},B_{2,2}]/R$$

$$\begin{aligned}
R &= (R_1 = R_2 = R_3 = R_4 = R_5 = R_6 = 1)\\
R_1 &= A_{1,1}\cdot A_{2,2} - A_{1,2}\cdot A_{2,1}\\
R_2 &= B_{1,1}\cdot B_{2,2} - B_{1,2}\cdot B_{2,1}\\
R_3 &= (A_{1,1}^2 + A_{1,2}A_{2,1}) - (B_{1,1}^3 + 2B_{1,1}B_{1,2}B_{2,1} + B_{1,2}B_{2,1}B_{2,2}) + 1\\
R_4 &= (A_{1,1}A_{1,2} + A_{1,2}A_{2,2}) - (B_{1,1}^2B_{1,2} + B_{1,1}B_{1,2}B_{2,2} + B_{1,2}^2B_{2,1} + B_{1,2}B_{2,2}^2) + 1\\
R_5 &= (A_{1,1}A_{2,1} + A_{2,1}A_{2,2}) - (B_{1,1}^2B_{2,1} + B_{1,2}B_{2,1}^2 + B_{1,1}B_{2,1}B_{2,2} + B_{2,1}B_{2,2}^2) + 1\\
R_6 &= (A_{1,2}A_{2,1} + A_{2,2}^2) - (B_{1,1}B_{1,2}B_{2,1} + 2B_{1,2}B_{2,1}B_{2,2} + B_{2,2}^3) + 1
\end{aligned}$$

As illustrated above, even for simple representation spaces, it is difficult to draw meaningful conclusions from the explicit description of the coordinate k-algebra. In chapter three, we will resolve this issue by introducing the moduli space of representations.

2.3 Reduction of the Coordinate *k*-algebra

Proposition 2.3.1. [LM85, p. 6]
Let $\mathfrak{X} = \text{Spec}(A)$ be an affine k-scheme over some fixed base ring k. Then we have the identification:

$$A \cong \text{Hom}_{\mathbf{Aff}k\mathbf{S}}(\mathfrak{X},\mathbb{A}^1)$$

Proof: Since a k-algebra morphism fixes the base ring k, we have the isomorphism

$$A \cong \text{Hom}_{\mathbf{Comm}k\mathbf{Alg}}(k[T],A)$$

induced by the map

$$\psi : A \to \text{Hom}_{\mathbf{Comm}k\mathbf{Alg}}(k[T],A) : a \mapsto \left(\sum_j b_jT^j \mapsto \sum_j b_ja^j\right)$$

But then, by the equivalence in 2.1.6, we obtain

$$A \cong \text{Hom}_{\mathbf{Comm}k\mathbf{Alg}}(k[T],A) \cong \text{Hom}_{\mathbf{Aff}k\mathbf{S}}(\text{Spec}(A),\text{Spec}(k[T]))$$

Since we are viewing $\text{Spec}(k[T])$ as a k-scheme, we have

$$\text{Spec}(k[T]) : \mathbf{Comm}k\mathbf{Alg} \to \mathbf{Set} : B \mapsto \text{Hom}_{\mathbf{Comm}k\mathbf{Alg}}(k[T],B) \cong B$$

Therefore, on k-algebras we have $\text{Spec}(k[T]) \cong \text{Spec}(\mathbb{Z}[T]) \cong \mathbb{A}^1$. We conclude that indeed

$$A \cong \text{Hom}_{\mathbf{Aff}k\mathbf{S}}(\mathfrak{X},\mathbb{A}^1)$$

Q.E.D.

Then each element a of the coordinate k-algebra $\mathcal{O}(\mathfrak{X})$ is in 1-1 correspondence with some k-scheme morphism $a^* : \mathfrak{X} \to \mathbb{A}^1$. On the level of points, this descends to a morphism $a^* : \mathfrak{X}(B) \to B$ for any $B \in \mathbf{Comm}k\mathbf{Alg}$. This is reminiscent of the coordinate ring of a variety and, as it turns out, these notions coincide whenever the coordinate k-algebra is a reduced ring.

Definition 2.3.2. Let $R \in \mathbf{CRing}$, then an element $r \in R$ is **nilpotent** if $r \neq 0$ and there exists some $n \in \mathbb{Z}_{>0}$ such that $r^n = 0$. A ring R is **reduced** if it contains no non-zero nilpotent elements. We define the **reduction** of B as

$$B^{\text{red}} = B \Big/ \bigcap_{\mathfrak{p} \lhd B,\ prime} \mathfrak{p}$$

Call the ideal $\bigcap_{\mathfrak{p} \lhd B,\ prime} \mathfrak{p}$ the **nilradical** of B.

We require a set-theoretical result in order to prove that the nilradical consists of all nilpotent elements of the ring.

Lemma 2.3.3 (Zorn)**.**
Suppose a partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element. That is, an element not less than any other element in P.

Proof: See ”Algebra” [Lan02, p. 884]

Proposition 2.3.4. [AM69, p.5]
Take $A \in \mathbf{CRing}$, then:

$$\bigcap_{\mathfrak{p} \lhd A,\ prime} \mathfrak{p} = \{a \in A \mid a^n = 0 \text{ for some } n \in \mathbb{Z}_{>0}\}$$

Proof: Assume $a \in A \setminus \{0\}$ nilpotent, that is $a^n = 0$ for some $n \in \mathbb{Z}_{>0}$. Since $0 \in \mathfrak{p}$ for any prime ideal $\mathfrak{p} \lhd A$, we have $aa^{n-1} = 0 \in \mathfrak{p}$ so $a \in A$ or $a^{n-1} \in \mathfrak{p}$. Continuing this process, we obtain $a \in \mathfrak{p}$ and since $\mathfrak{p}$ was an arbitrary prime ideal, a is in the intersection of all prime ideals of A. Thus

$$\{a \in A \mid a^n = 0 \text{ for some } n \in \mathbb{Z}_{>0}\} \subseteq \bigcap_{\mathfrak{p} \lhd A,\ prime} \mathfrak{p}$$

In order to prove the opposite direction, we use a counter-positive argument. Assume that non-zero $a \notin \{a \in A \mid a^n = 0 \text{ for some } n \in \mathbb{Z}_{>0}\}$. Then, we define the set

$$\Sigma := \{I \lhd A \mid a^m \notin I \; \forall \, m \in \mathbb{Z}_{>0}\}$$

The set Σ is partially ordered by $\subseteq$. The zero ideal is an element of Σ since a is chosen to be non-zero. Consider a chain of ideals $I_1 \subseteq I_2 \subseteq ...$ where $I_i \in \Sigma$ for all $i \in \mathbb{Z}_{>0}$. Then the union $I := \bigcup_i I_i$ is an ideal of A and an upper bound for the chain. We claim that $I \in \Sigma$. Assume, for a contradiction, that $a^m \in I$ for some $m \in \mathbb{Z}_{>0}$. Then $a^m \in I_j$ for some j, a contradiction. Thus $I \in \Sigma$. Since every chain in Σ has an upper bound in Σ, by Zorn's lemma, there exists some maximal element $\mathfrak{m} \in \Sigma$. We claim that $\mathfrak{m}$ is a prime ideal of A. Assume, for a contradiction, that there exists some $g, h \in A$ such that $gh \in \mathfrak{m}$ but $g, h \notin \mathfrak{m}$. Consider the ideals $\mathfrak{m} + (g)$ and $\mathfrak{m} + (h)$. Both are strict supersets of $\mathfrak{m}$, thus cannot be elements of Σ since $\mathfrak{m}$ is maximal. Then, there must exist some $r, s \in \mathbb{Z}_{>0}$ such that $a^r \in \mathfrak{m} + (g)$ and $a^s \in \mathfrak{m} + (h)$. Let $a^r = m_1 + g'$ and $a^s = m_2 + h'$ for some $m_1, m_2 \in \mathfrak{m}$ and $g' \in (g)$, $h' \in (h)$. Then

$$\begin{array}{rcl} a^{r+s} & = & a^r a^s \\ & = & (m_1 + g')(m_2 + h') \\ & = & m_1 m_2 + m_1 h' + g' m_2 + g' h' \\ & \in & \mathfrak{m} + (gh) \end{array}$$

But, this is a contradiction since $gh \in \mathfrak{m}$ meaning $\mathfrak{m} + (gh) = \mathfrak{m} \in \Sigma$. Thus $\mathfrak{m}$ is a prime ideal of A. We conclude that, for any choice of non-zero $a \notin \{a \in A \mid a^n = 0 \text{ for some } n \in \mathbb{Z}_{>0}\}$, there exists some prime ideal $\mathfrak{m}$ such that $a^m \notin \mathfrak{m}$ for any $m \in \mathbb{Z}_{>0}$ and, in particular, $a \notin \mathfrak{m}$. Thus, $a \notin \bigcap_{\mathfrak{p} \triangleleft A,\ prime} \mathfrak{p}$. Then

$$\{a \in A \mid a^n = 0 \text{ for some } n \in \mathbb{Z}_{>0}\} \supseteq \bigcap_{\mathfrak{p} \triangleleft A,\ prime} \mathfrak{p}$$

Since both containments hold, the sets are equal, as required.

Q.E.D.

Remark 2.3.5. For A noetherian, we do not require Zorn's lemma since every ascending chain of ideals in Σ stabilises. In the case where k is a field, thus noetherian, we have that $k[x_1, ..., x_m]$ is also noetherian by Hilbert's basis theorem [Hil90]. Moreover, if I is a two-sided ideal of $k[x_1, ..., x_m]$ then $k[x_1, ..., x_m]/I$ is noetherian. Thus, in the case of coordinate k-algebras over a field k, we do not require Zorn's lemma in order to prove the statement in 2.3.4.

The reduction A^{red} is the 'largest' quotient of A such that the quotient is reduced, where largest is in the sense of the argument above. This allows us to describe the exact relationship between the coordinate k-algebra of a scheme and the coordinate ring at a point of the scheme.

Theorem 2.3.6. *[LM85, p. 31]*
Let $\mathfrak{X} := R(\Gamma, GL_n)$ by a k-scheme. Then, on the level of points, we have that the coordinate ring of the variety $\mathfrak{X}(k)$ is given by

$$\mathcal{O}(\mathfrak{X}(k)) \cong \mathcal{O}(\mathfrak{X})^{red}$$

Proof: Recall that $\mathcal{O}(\mathfrak{X}) = k[X_{i,j}^l, Y^l]/(r_s(X_{i,j}^l, Y^l)-1)$. In order to simplify the notation, we set $k[x_1, ..., x_u] := k[X_{i,j}^l, Y^l]$ and $I = (r_s(X_{i,j}^l, Y^l) - 1)$. Using this notation, the coordinate k-algebra of $\mathfrak{X}$ becomes

$$\mathcal{O}(\mathfrak{X}) = k[x_1, ..., x_u]/I$$

Then, $\mathfrak{X}(k)$ is given by

$$\begin{aligned} \mathfrak{X}(k) &= \mathrm{Hom}_{\mathbf{Comm}k\mathbf{Alg}}(k[x_1, ..., x_u]/I, k) \\ &= \{(a_1, ..., a_u) \in k^u \mid f(a_1, ..., a_u) = 0 \; \forall \; f \in I\} \\ &= \mathcal{V}(I) \subseteq k^u \end{aligned}$$

Then, by Hilbert's Nullstellensatz, we have

$$\mathcal{I}(\mathfrak{X}(k)) = \mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$$

We obtain that $\mathcal{O}(\mathfrak{X}(k)) \cong \mathcal{O}(\mathfrak{X})/\sqrt{I}$ and it is sufficient to prove that $\mathcal{O}(\mathfrak{X})/\sqrt{I} = \mathcal{O}(\mathfrak{X})^{red}$. Note that

$$\bigcap_{\mathfrak{p} \triangleleft \mathcal{O}(\mathfrak{X}),\ \text{prime}} \mathfrak{p} = \{a \in \mathcal{O}(\mathfrak{X}) \mid a^n = 0 \text{ for some } n \in \mathbb{Z}_{>0}\}$$

But then, each element of the nilradical is an element of the radical of the zero ideal $\sqrt{(0)}$. For any element $b \in \sqrt{(0)}$, we have $b^n \in (0)$ for some $n \in \mathbb{Z}_{>0}$ and thus it is nilpotent. But the zero ideal of $\mathcal{O}(\mathfrak{X})$ is I, by construction, since $k[x_1, ..., x_u]$ is an integral domain. We conclude that

$$\mathcal{O}(\mathfrak{X}(k)) \cong \mathcal{O}(\mathfrak{X})/\sqrt{I} \cong \mathcal{O}(\mathfrak{X})^{\text{red}}$$

Q.E.D.

Example 2.3.7. Consider the affine k-scheme GL_n over a field k, represented by

$$\mathcal{O}(\mathrm{GL}_n) = k[X_{1,1}, X_{1,2}, ..., X_{n,n}, Y]/(\mathrm{Det}(X_{i,j}) \cdot Y - 1)$$

Then the coordinate ring of the variety $\mathrm{GL}_n(k)$ is given by

$$\mathcal{O}(\mathrm{GL}_n(k)) = k[X_{1,1}, X_{1,2}, ..., X_{n,n}, Y]/\sqrt{(\mathrm{Det}(X_{i,j}) \cdot Y - 1)}$$

But, from the definition of the determinant, the ideal $I := (\mathrm{Det}(X_{i,j})Y - 1)$ is given by

$$(\mathrm{Det}(X_{i,j})Y - 1) = (Y \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n^2} (X_{i,\sigma_i}) - 1)$$

Each summand in the determinant will be of the form

$$Y \cdot (X_{1,1})^{\alpha_{1,1}}(X_{1,2})^{\alpha_{1,2}} \cdot ... \cdot (X_{n,n})^{\alpha_{n,n}}$$

where $\alpha_{i,j} \in \{0,1\}$. Assume that there exists some polynomials $f, g \in k[X_{1,1}, X_{1,2}, ..., X_{n,n}]$ such that $f \cdot g = \text{Det}(X_{i,j})Y - 1$. If we write

$$f = \sum_{s=1}^{s'} k_s Y^{a_s} \prod_{i=1}^{n} \prod_{j=1}^{n} (X_{i,j})^{a_{i,j,s}} + k_{s'+1}$$

$$g = \sum_{t=1}^{t'} k'_t Y^{b_t} \prod_{i=1}^{n} \prod_{j=1}^{n} (X_{i,j})^{b_{i,j,t}} + k'_{t'+1}$$

for $k_s, k'_t \in k$ and $s', t', a_{i,j,s}, a_s, b_{i,j,s}, b_s \in \mathbb{Z}_{\geq 0}$ and assuming, without loss of generality, that not all $a_{i,j,s} = 0$ for any s and not all $b_{i,j,t} = 0$ for any t. These sums will be non-empty if f and g are both non-constant. Note also that, since k is integral, so is $k[X_{1,1}, X_{1,2}, ..., X_{n,n}, Y]$. Assume that $X_{1,1}$ is in a summand of f, then $X_{1,1}$ cannot be in a summand of g, otherwise we would have $(X_{1,1})^2$ as a factor of one of the summands in $\text{Det}(X_{i,j})$. Moreover, $X_{1,1}X_{1,j}$ is not a factor of any summand in $\text{Det}(X_{i,j})$ for any j so $X_{1,j}$ cannot be in a summand of g. Thus, every row of the matrix $(X_{i,j})$ is in either entirely in summands of f or g. I similar argument shows the same is true for the columns of $(X_{i,j})$. But then, if we assume the first row of $(X_{i,j})$ is entirely in f then there is one element of each column in f so all columns are in f and thus

$$g = k^* Y^b + k'_{t'+1}$$

for some $k^* \in k$ and $b \in \mathbb{Z}_{\geq 0}$. If $b > 1$ then a square factor of Y will appear in the product of f and g, a contradiction. Assume $b = 1$, then

$$\begin{aligned} fg &= (\sum_{s=1}^{s'} k_s Y^{a_s} \prod_{i=1}^{n} \prod_{j=1}^{n} (X_{i,j})^{a_{i,j,s}} + k_{s'+1})(k^*Y + k'_{t'+1}) \\ &= k^*Y \sum_{s=1}^{s'} k_s Y^{a_s} \prod_{i=1}^{n} \prod_{j=1}^{n} (X_{i,j})^{a_{i,j,s}} + k_{s'+1}k^*Y \\ &+ k'_{t'+1} \sum_{s=1}^{s'} k_s Y^{a_s} \prod_{i=1}^{n} \prod_{j=1}^{n} (X_{i,j})^{a_{i,j,s}} + k_{s'+1}k'_{t'+1} \end{aligned}$$

But then $k_{s'+1}k^*Y$ appears as a summand of $\text{Det}(X_{i,j})Y$ or $k_{s'+1}k^* = 0$. The first case is clearly a contradiction. Then $k_{s'+1}k^* = 0$ and, since k is assumed to be integral, $k_{s'+1} = 0$ or $k^* = 0$. If $k_{s'+1} = 0$ then $k_{s'+1}k'_{t'+1} = 0 \neq -1$, a contradiction. Thus, $k^* = 0$ and g is constant. We conclude that $\text{Det}(X_{i,j})Y - 1$ is an irreducible polynomial in $k[X_{1,1}, X_{1,2}, ..., X_{n,n}, Y]$. But the quotient of a ring by an irreducible element gives rise to an integral domain. Thus $\mathcal{O}(\text{GL}_n)$ is an integral domain and cannot contain any nilpotent elements. Therefore

$$\mathcal{O}(\text{GL}_n) = \mathcal{O}(\text{GL}_n)^{\text{red}} = \mathcal{O}(\text{GL}_n(k))$$

3 Moduli Schemes

We generalise the notion of a group and a group action to the setting of affine k-schemes. Since categorical objects are distinguished by the morphisms to and from the object, we define a group object in a category in terms of morphisms satisfying relations similar to the usual group axioms. This allows for the notion of an affine group scheme action in which a scheme acts on another scheme, analogous to a group acting on a set. From this perspective, we define the geometric invariant theory (GIT) quotient of a scheme $\mathfrak{X}$ under some action as the functor represented by the ring of invariants of $\mathcal{O}(\mathfrak{X})$ under the induced action on coordinate k-algebras. If we consider the conjugation action induced by GL_n on $\mathfrak{X} := R(\Gamma, \mathrm{GL}_n)$, we construct the moduli scheme $M(\Gamma, \mathrm{GL}_n)$ represented by

$$\mathcal{O}(\mathfrak{X})^{\mathrm{GL}_n(k)} = \{a \in \mathcal{O}(\mathfrak{X}) \mid a(A\rho A^{-1}) = a(\rho)\ \forall\, \rho \in \mathfrak{X},\ A \in \mathrm{GL}_n(k)\}$$

Here $a \in \mathcal{O}(\mathfrak{X})$ is thought of as a morphism $a : \mathfrak{X} \to \mathbb{A}_1$. Since the general linear group is linearly reductive, we conclude that $\mathcal{O}(\mathfrak{X})^{\mathrm{GL}_n(k)}$ is finitely generated. The general reference for invariant theory is [Muk03].

3.1 The Geometric Invariant Theory Quotient

Definition 3.1.1. A **Group object** in category $\mathcal{C}$ is an object $G \in \mathcal{C}$ such that the product $G \times G$ exists, with morphisms:

I) $m : G \times G \to G$ (group law)

II) $e : 1 \to G$ (identity)

III) $inv : G \to G$ (inverses)

Satisfying the following properties:

i) $m(m \times \mathrm{id}_G) = m(\mathrm{id}_G \times m)$

ii) For projections $p_1 : G \times 1 \to G$ and $p_2 : 1 \times G \to G$, we have $m(\mathrm{id}_G \times e) = p_1$ and $m(e \times \mathrm{id}_G) = p_2$

iii) For diagonal $d : G \to G \times G$ and $e_G : G \to G = e' \circ e$ where $e' : G \to 1$ then $m(\mathrm{id}_G \times inv)d = e_G$ and $m(inv \times \mathrm{id}_G)d = e_G$

When $G \in \mathbf{Grp}$ we have that these morphisms correspond to the ordinary definition of a group, in terms of elements.

Example 3.1.2. For a canonical example, consider the cyclic group with 5 elements, denoted $C_5 \in \mathbf{Grp}$, with generator x. Now the morphisms defining C_5 are given by

$$\begin{aligned}
&m : C_5 \times C_5 \to C_5 : x^n \times x^m \mapsto x^{m+n} \\
&e : 1 \to C_5 : 1 \mapsto e_{C_5} \\
&inv : C_5 \mapsto C_5 : x^n \mapsto x^{5-n}
\end{aligned}$$

Checking the properties, we have

$$\begin{array}{lrcl}
\text{i)} & m(m \times \mathrm{id}_{C_5})(x^a, x^b, x^c) & = & m(x^{a+b}, x^c) \\
& & = & x^{a+b+c} \\
& & = & m(x^a, x^{b+c}) \\
& & = & m(\mathrm{id}_{C_5} \times m)(x^a, x^b, x^c) \\
& & & \\
\text{ii)} & m(\mathrm{id}_{C_5} \times e)(x^a, x^b) & = & m(x^a, e_{C_5}) \\
& & = & x^{a+0} \\
& & = & p_1(x^a, x^b) \\
& & & \\
\text{iii)} & m(\mathrm{id}_{C_5} \times inv)d(x^a) & = & m(\mathrm{id}_{C_5} \times inv)(x^a, x^a) \\
& & = & m(x^a, x^{5-a}) \\
& & = & a^5 \\
& & = & e_{C_5}
\end{array}$$

The argument is identical for $m(e \times \mathrm{id}_G) = p_2$ and $m(inv \times \mathrm{id}_G)d = e_G$. Thus, $C_5 \in \mathbf{Grp}$ is a group object, as expected.

Definition 3.1.3. Let G be a group and X a set, then a **group action** of G on X is a map

$$\alpha : G \times X \to X$$

satisfying the following:

i) $\alpha(e_G, x) = x$ for all $x \in X$ **(identity)**

ii) $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for all $x \in X$ **(compatibility)**

A group acting on a set is, in essence, an endomorphism of the set in such a way that the group structure is compatible with the mapping. We can define a similar notion on schemes:

Definition 3.1.4. An object $\mathfrak{G}$ in the category of affine schemes is an **affine group scheme** if $\mathfrak{G}$ is a group object. A **left affine group scheme action** of $\mathfrak{G}$ on affine k-scheme $\mathfrak{X}$ is a morphism:

$$\sigma : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$$

satisfying the properties:

I) $\sigma \circ (1_{\mathfrak{G}} \times \sigma) = \sigma \circ (m \times 1_{\mathfrak{X}})$ with group law $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$

II) $\sigma \circ (e \times 1_{\mathfrak{X}}) = 1_{\mathfrak{X}}$ with identity $e : 1 \to \mathfrak{G}$

This action induces an action on the set $\mathfrak{X}(B)$ for any $B \in \mathbf{CRing}$ via:

$$\begin{tikzcd}
 & \mathfrak{G} \times \mathfrak{X} \arrow[r, "\sigma"] & \mathfrak{X} \\
 & B \arrow[d] \arrow[r, mapsto, "\sigma"] & B \arrow[d] \\
 & \mathrm{Hom}(\mathcal{O}(\mathfrak{G}), B) \times \mathrm{Hom}(\mathcal{O}(\mathfrak{X}), B) & \mathrm{Hom}(\mathcal{O}(\mathfrak{X}), B)
\end{tikzcd}$$

Moreover, if we take B to be a k-algebra, then there is a natural map $\phi : k \to B$. This map induces a map on the points of $\mathfrak{G}$ via:

$$\begin{tikzcd}
k \arrow[r, "\phi"] \arrow[d, mapsto, "\mathfrak{G}"] & B \arrow[d, mapsto, "\mathfrak{G}"] \\
\mathrm{Hom}(\mathcal{O}(\mathfrak{G}), k) \arrow[r, "\phi \circ _"] & \mathrm{Hom}(\mathcal{O}(\mathfrak{G}), B)
\end{tikzcd}$$

Then $\mathfrak{G}(k)$ acts on $\mathfrak{X}(B)$ for all $B \in \mathbf{Comm}k\mathbf{Alg}$, so $\mathfrak{G}(k)$ acts on the functor $\mathfrak{X}$. By the functoriality of the map $\mathcal{O} : \mathfrak{X} \to \mathcal{O}(\mathfrak{X})$, the action of $\mathfrak{G}(k)$ on $\mathfrak{X}$ induces an action of $\mathfrak{G}(k)$ on $\mathcal{O}(\mathfrak{X})$. A scheme $\mathfrak{G}$ equipped with the group scheme action of $\mathfrak{G}$ is called a **$\mathfrak{G}$-scheme**. This action will sometimes be abbreviated an affine action when the context is clear.

Definition 3.1.5. [Muk03, p. 164] Given affine k-scheme $\mathfrak{X} = \mathrm{Spec}A$ and affine group scheme action given by $\mathfrak{G}$, the **geometric invariant theory quotient** (**GIT quotient**) of $\mathfrak{X}$ by $\mathfrak{G}$ at point $B \in \mathbf{CRing}$, is defined as:

$$\mathfrak{X}(B)//\mathfrak{G} := \mathrm{Hom}(A^{\mathfrak{G}(k)}, B)$$

Here $A^{\mathfrak{G}(k)}$ denotes the fixed subring or the **ring of invariants** and is given by:

$$A^{\mathfrak{G}(k)} := \{a \in A \mid g \times a = a, \ \forall\, g \in \mathfrak{G}(k)\}$$

If A represents the k-scheme $\mathrm{Spec}(A) : \mathbf{CRing} \to \mathbf{Set}$, then the GIT quotient of $\mathrm{Spec}(A)$ by group scheme $\mathfrak{G}$ is the affine scheme $\mathrm{Spec}(A^{\mathfrak{G}(k)})$ represented by the k-algebra $A^{\mathfrak{G}(k)}$.

Example 3.1.6. [Heu16, p. 4] Consider the $\mathbb{C}$-scheme $\mathbb{A}^2 : A \mapsto \mathrm{Hom}(\mathbb{C}[x, y], A)$ and $G = \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Define the group scheme action of G on $\mathbb{A}^2$ at the point $\mathbb{A}^2(\mathbb{C})$ via:

$$\sigma : \mathbb{C}^* \times \mathbb{C}^2 \to \mathbb{C}^2 : (a, (z_1, z_2)) \mapsto (az_1, az_2)$$

The coordinate $\mathbb{C}$-algebra of the scheme $\mathbb{A}^2$ is given by

$$\mathcal{O}(\mathbb{A}^2) = \mathbb{C}[x, y]$$

But $\mathbb{C}$ is an integral domain and therefore $\mathbb{C}[x, y]$ is integral. Since $\mathcal{O}(\mathbb{A}^2)$ cannot contain any nilpotents, it is reduced. Thus the coordinate ring of the variety $\mathbb{A}^2(\mathbb{C}) \cong \mathbb{C}^2$ coincides with $\mathcal{O}(\mathbb{A}^2)$. We have

$$\mathcal{O}(\mathbb{C}^2) = \{f : \mathbb{C}^2 \to \mathbb{C} \mid \exists f' \in \mathbb{C}[x, y] \text{ st } f = f' \text{ on } \mathbb{C}^2\} \cong \mathbb{C}[x, y]$$

Now, if we choose arbitrary $(z_1, z_2) \in \mathbb{C}^2$, $g \in G$ and $f \in \mathcal{O}(\mathbb{C}^2)$, then:

$$(\sigma_g \times f)(z_1, z_2) = f \circ \sigma_g(z_1, z_2) = f(gz_1, gz_2)$$

The only polynomials f fixed by any choice of $g \in \mathbb{C}^*$, are the constant. Thus

$$\mathcal{O}(\mathbb{C}^2)^{\mathbb{C}^*} \cong \mathbb{C}$$

Then, we have that the GIT quotient of $\mathbb{A}^2$ by $\mathbb{C}^*$ is given by

$$\begin{aligned}
\mathbb{A}^2//G &\cong \mathrm{Hom}_{\mathbf{Comm}\mathbb{C}\mathbf{Alg}}(\mathcal{O}(\mathbb{A}^2)^{\mathbb{C}^*}, \mathbb{C}) \\
&\cong \mathrm{Hom}_{\mathbf{Comm}\mathbb{C}\mathbf{Alg}}(\mathbb{C}, \mathbb{C}) \\
&\cong \{f \in \mathrm{Hom}_{\mathbf{CRing}}(\mathbb{C}, \mathbb{C}) \mid f(bz) = bf(z) \; \forall \, b, z \in \mathbb{C}\} \\
&\cong \{f \in \mathrm{Hom}_{\mathbf{CRing}}(\mathbb{C}, \mathbb{C}) \mid f(b \cdot 1) = bf(1) = b \; \forall \, b \in \mathbb{C}\} \\
&\cong \{*\}
\end{aligned}$$

3.2 The Moduli Scheme of Representations

Returning to the specific case of $R(\Gamma, \mathrm{GL}_n(k))$, we can construct the moduli space of representations via the GIT-quotient.

Proposition 3.2.1. [LM85, p. 8]
Consider the map

$$\alpha : \mathrm{GL}_n \times R(\Gamma, \mathrm{GL}_n) \to R(\Gamma, \mathrm{GL}_n) : (A, \rho) \mapsto (A\rho A^{-1} : \gamma \mapsto A\rho(\gamma)A^{-1})$$

Then α is a morphism of schemes satisfying

i) $I_n \rho I_n^{-1} = \rho$ for any $\rho \in R(\Gamma, \mathrm{GL}_n)$

ii) $(A, (B, \rho)) = (AB, \rho)$ for any $A, B \in \mathrm{GL}_n$ and $\rho \in R(\Gamma, \mathrm{GL}_n)$

In other words, α is a group scheme action.

Definition 3.2.2. [MP16, p. 399] Let $\Gamma = \langle x_1, ..., x_m \mid r_1, ..., r_n \rangle$ be a finitely generated group and $\mathfrak{G} = \mathrm{GL}_r$. Then the **moduli scheme of representations** of Γ into $\mathfrak{G}$, denoted $M(\Gamma, \mathfrak{G})$, is defined as the GIT-quotient:

$$M(\Gamma, \mathfrak{G}) := R(\Gamma, \mathfrak{G}) // \mathfrak{G} = \mathrm{Hom}(\Gamma, \mathrm{GL}_r) // \mathrm{GL}_r$$

where $\mathrm{GL}_r(k)$ acts on $R(\Gamma, \mathfrak{G})$ via α in 3.2.1. On the level of coordinate k-algebras, we have that every $a \in \mathcal{O}(R(\Gamma, \mathrm{GL}_n))$ induces a map $a^* : R(\Gamma, \mathrm{GL}_n) \to \mathbb{A}_1$. Then a is stable under the action of GL_n by conjugation precisely when

$$\alpha(A, a^*)(\rho) := a^*(A\rho A^{-1}) = a^*(\rho)$$

That is, the coordinate k-algebra of the moduli scheme is given by

$$\mathcal{O}(R(\Gamma, \mathfrak{G}))^{\mathfrak{G}(k)} = \{a \in \mathcal{O}(R(\Gamma, \mathfrak{G})) \mid a^*(A\rho A^{-1}) = a^*(\rho) \text{ for all } \rho \in R(\Gamma, \mathfrak{G}),\ A \in \mathfrak{G}(k)\}$$

It is important to emphasise that we are viewing the representation scheme as a k-scheme, which is necessary for the conjugation by $\mathrm{GL}_n(k)$ to be well-defined. If we consider a point $M(\Gamma, \mathrm{GL}_n(k))$ of the moduli scheme, we call this the **moduli space of representations**.

Example 3.2.3. Let $\Gamma = \langle a, b \mid a^2 = b^3 \rangle$ and $\mathfrak{G} = \mathrm{SL}_2$. We have shown in 2.2.3 that the coordinate $\mathbb{C}$-algebra of the representation scheme is given by:

$$\mathcal{O}(R(\Gamma, \mathfrak{G})) = \mathbb{C}[A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2}, B_{1,1}, B_{1,2}, B_{2,1}, B_{2,2}]/R$$

Consider the element $b = A_{1,1} + A_{2,2} \in \mathcal{O}(R(\Gamma, \mathfrak{G}))$. Take some $\rho \in R(\Gamma, \mathfrak{G}(\mathbb{C}))$ and let $\rho(\gamma) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for some $\gamma \in \Gamma$. The induced map $b^* : R(\Gamma, \mathfrak{G}) \to \mathbb{A}_1$ is given, on the level of varieties, by

$$b^* : \rho(\gamma) \mapsto a + d$$

Now, choose arbitrary $B = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$, then

$$\begin{aligned}
b^*(B\rho(\gamma)B^{-1}) &= b^*\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} s & -q \\ -r & p \end{pmatrix} \right) \\
&= b^*\left(\begin{pmatrix} ap + cq & bp + dq \\ ar + cs & br + ds \end{pmatrix} \begin{pmatrix} s & -q \\ -r & p \end{pmatrix} \right) \\
&= b^*\left(\begin{pmatrix} aps + cqs - bpr - dqr & bp^2 + dpq - apq - cq^2 \\ ars + cs^2 - ar^2 - crs & bpr + dps - aqr - cqs \end{pmatrix} \right) \\
&= aps + cqs - bpr - dqr + bpr + dps - aqr - cqs \\
&= aps + dps - dqr - aqr \\
&= (a + d)(ps - qr) \\
&= a + d \\
&= b^*(\rho(\gamma))
\end{aligned}$$

Since ρ, B and γ were arbitrary, we conclude that $b \in \mathcal{O}(M(\Gamma, \mathfrak{G}))$. This is an example of a trace element of $\mathcal{O}(M(\Gamma, \mathfrak{G}))$ and, as we will prove later, every element of the ring of invariants is of this form.

3.3 Finite Generation of the Ring of Invariants

A priori, it is unclear as to if we should expect A^G to be finitely generated. In general, a subring of a finitely generated ring need not be finitely generated.

Example 3.3.1. Consider $\Gamma := \mathbb{C}[x, y]$, the polynomial ring in two variables over $\mathbb{C}$. This ring is finitely generated by the elements $x, y \in \Gamma$. We have the subring

$$\Gamma' := \mathbb{C}[x, xy, xy^2, ...] \subset \Gamma$$

By definition, we have that the set $G := \{x, xy, xy^2, ...\}$ generates Γ'. Assume, for a contradiction, that there exists some finite generating set S for Γ' given by

$$S := \{xy^{\alpha_1}, ..., xy^{\alpha_n}\} \subseteq G$$

with $\alpha_s \in \mathbb{Z}_{\geq 0}$. Choose an element $xy^m \in \Gamma' \setminus S$. Then, we must have

$$xy^m = c\prod_{s=1}^{n}(xy^{\alpha_s})^{\beta_s} = x^{\sum_{s=1}^{n}\beta_s}y^{\sum_{s=1}^{n}\alpha_s\beta_s}$$

For some $\beta_s \in \mathbb{Z}_{\geq 0}$ and some $c \in \mathbb{C}$. Comparing powers of x, we have

$$1 = \sum_{s=1}^{n}\beta_s$$

Since all β_s are non-negative, $\beta_t = 1$ for some $t \in \{1, ..., n\}$ and $\beta_s = 0$ for all $s \neq t$. Then $c = 1$ and

$$xy^m = cxy^{\alpha_t}$$

This is a contradiction since we assumed $xy^m \notin S$. Thus Γ' is a subring of a finitely generated ring Γ, by Γ' is not itself finitely generated.

Definition 3.3.2. A group object in the category of affine varieties is called an **algebraic group**. The group operation of such a group object is given by regular functions on the variety. An algebraic group G is **linearly reductive** if, for every epimorphism $\phi : V \to W$ of G-representations, the induced map on G-invariants $\phi^G : V^G \to W^G$ is surjective.

Proposition 3.3.3. The general and special linear group are both linearly reductive.

Proof: See ”An Introduction to Invariants and Moduli” [Muk03, p. 132-135]

Then, in the case of linearly reductive groups, the issue of finite generation is resolved by the following theorem:

Theorem 3.3.4 (Hilbert)**.**
Let G be an algebraic group acting on polynomial ring S. If G is linearly reductive, then the ring of invariants S^G is finitely generated.

Proof: See "Über die Theorie der algebraischen Formen" [Hil90] for the original proof, see also "An Introduction to Invariants and Moduli" [Muk03, p. 135].

Theorem 3.3.5. *The coordinate k-algebra of the moduli space*

$$\mathcal{O}(M(\Gamma, \boldsymbol{GL_r})) = \mathcal{O}(R(\Gamma, \boldsymbol{GL_r}))^{GL_r(k)}$$

is finitely generated.

Proof: This follows directly from the fact that $\mathrm{GL}_r(k)$ is linearly reductive by 3.3.3, Thus the ring of invariants is finitely generated by 3.3.4.

Q.E.D.

4 Character Varieties

We define trace and determinant maps from the general linear group to our base ring k. In the case of k-schemes, we define the analogous notion of trace and determinant elements in the coordinate k-algebra. For the representation scheme over a free group, we prove that

$$\mathcal{O}(R(\mathbb{F}_r, \mathrm{GL}_n)) \cong k[X_{i,j,1}, ..., X_{i,j,r}][\prod_{s=1}^{r} \det{}^{-1}(X_{i,j,s})] \cong \mathcal{O}(\mathrm{M}_n^r)[\prod_{s=1}^{r} \det{}^{-1}(X_{i,j,s})]$$

By applying Procesi's theorem [Pro76, p. 313], we show that the coordinate k-algebra of the moduli space $M(\mathbb{F}_r, \mathrm{GL}_n)$ is generated by trace and determinant elements. Moreover, this remains true for any representation scheme $R(\Gamma, \mathrm{GL}_n)$ over finitely generated Γ. Finally, we argue that the character variety parameterises equivalence classes of semi-simple representations and that two semi-simple representations are equivalent if and only if they have the same character.

4.1 Traces and Determinants

Definition 4.1.1. Recall that, for $R \in \mathbf{CRing}$, the **trace map** on $\mathrm{GL}_n(R)$ is a map

$$\mathrm{tr} : \mathrm{GL}_n(R) \to R : (x_{i,j}) \to \sum_{i=1}^{n} x_{i,i}$$

That is, the trace of a matrix is the sum of it's diagonal entries. The **determinant map** on $\mathrm{GL}_n(R)$ is a map

$$\det : \mathrm{GL}_n(R) \to R : (x_{i,j}) \to \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} (x_{i,\sigma_i})$$

Here S_n denotes the symmetric group of order n and sgn is the sign function of a permutation in S_n.

Consider the affine k-scheme $\mathrm{Spec}(A) := R(\Gamma, \mathrm{GL}_n)$ with the usual assumptions on Γ and a fixed base ring *k*. Then, for each element $a \in A$ there is a corresponding morphism $a^* : \mathrm{Spec}(A) \to \mathbb{A}^1$. We define the trace and determinant in the setting of schemes:

Definition 4.1.2. Let $\mathrm{Spec}(A) := R(\Gamma, \mathrm{GL}_n)$ and $\phi \in A$, then ϕ is a **trace element** if the corresponding morphism $\phi^* : \mathrm{Spec}(A) \to \mathbb{A}^1$ is the composition $\phi^* = \mathrm{tr} \circ \mathrm{ev}_\gamma$ where tr is the trace map and ev_γ is the evaluation at a point $\gamma \in \Gamma$ defined on the level of functors of points in the following way: Let $R \in \mathbf{Comm}k\mathbf{Alg}$ and $\gamma \in \Gamma$, then

$$\mathrm{ev}_\gamma : R(\Gamma, \mathrm{GL}_n(R)) \to \mathrm{GL}_n(R) : \rho \to \rho(\gamma)$$

That is, a scheme trace is an element $\phi \in A$ such that the diagram below commutes for some $\gamma \in \Gamma$.

$$\begin{tikzcd} R(\Gamma, \mathrm{GL}_n(R)) \arrow[r, "\phi^*"] \arrow[d, "\mathrm{ev}_\gamma"'] & R \\ \mathrm{GL}_n(R) \arrow[ur, "\mathrm{tr}"'] & \end{tikzcd}$$

Similarly, ϕ is a **determinant element** if $\phi^* : \mathrm{Spec}(A) \to \mathbb{A}^1$ is the composition $\phi^* = \det \circ \mathrm{ev}_\gamma$ where det is the determinant map.

Remark 4.1.3. We note that

$$R(\Gamma, \mathrm{GL}_n) := \mathrm{Hom}(\mathbb{F}_r, \mathrm{GL}_n) \cong \mathrm{GL}_n^r$$

On the level of points, this follows from freely choosing where each generator of $\mathbb{F}_r$ is mapped. Let M_n denote the affine k-scheme given by:

$$M_n : \mathbf{CRing} \to \mathbf{Set} : R \to \mathrm{M}_n(R)$$

Here, $\mathrm{M}_n(R)$ denotes the ring of n by n matrices over R, referred to as the **matrix ring** of rank n over R. GL_n is a sub-scheme of M_n and we have diagram:

$$\begin{tikzcd} \mathrm{Hom}(\mathbb{F}_r, \mathrm{GL}_n) \arrow[d, "\wr"] & & \\ \mathrm{GL}_n^r \arrow[r, hook] \arrow[d] & \mathrm{M}_n^r \arrow[r, "\sim"] \arrow[d] & \mathbb{A}^{n^2 r} \\ \mathrm{GL}_n^r//\mathrm{GL}_n \arrow[r, hook] & \mathrm{M}_n^r//\mathrm{GL}_n & \end{tikzcd}$$

On the level of coordinate k-algebras, we have [LM85, p. 27]

$$\begin{aligned} \mathcal{O}(\mathrm{GL}_n) &= k[X_{i,j}, Y]/(\det(X_{i,j}) \cdot Y - 1) \\ &= k[X_{i,j}][\det^{-1}(X_{i,j})] \\ &= \mathcal{O}(\mathrm{M}_n)[\det^{-1}(X_{i,j})] \end{aligned}$$

If we instead consider GL_n^r, we obtain

$$\begin{aligned} \mathcal{O}(\mathrm{GL}_n^r) &= k[X_{i,j,1}, ..., X_{i,j,r}][\det^{-1}(X_{i,j,1}), ..., \det^{-1}(X_{i,j,r})] \\ &= k[X_{i,j,1}, ..., X_{i,j,r}][\textstyle\prod_{s=1}^r \det^{-1}(X_{i,j,s})] \\ &= \mathcal{O}(\mathrm{M}_n^r)[\textstyle\prod_{s=1}^r \det^{-1}(X_{i,j,s})] \end{aligned}$$

Theorem 4.1.4. *[Pro76, p. 313]*
Let $A_s = (a_{i,j,s})_{1 \le i,j \le n}$ where $s \in \{1, ..., r\}$ and $a_{i,j,s} \in k$ for some base ring k. Consider a polynomial $P \in k[a_{i,j,s}]$, invariant under the action of group G on the matricies A_s. Then $P \in k[tr(A_{\alpha_1} \cdot ... \cdot A_{\alpha_m})]$ ranging over all monomials $A_{\alpha_1} \cdot ... \cdot A_{\alpha_m}$ with $\alpha_t \in \{1, ..., r\}$.

In essence, the theorem states that an invariant in the entries of a set of matrices, is generated by the traces of monomials in these matrices. The immediate consequence of this is that the coordinate k-algebra of quotient $\mathrm{M}_n^r//\mathrm{GL}_n$ is generated by traces since the coordinate k-algebra consists of precisely the polynomials which are invariant under the action of $\mathrm{GL}_n(k)$. That is, we obtain a surjection:

$$\mathcal{O}(\mathrm{M}_n^r) \cong k[a_{i,j,s}] \twoheadrightarrow k[\mathrm{tr}(A_{\alpha_1} \cdot ... \cdot A_{\alpha_m})] \twoheadrightarrow \mathcal{O}(\mathrm{M}_n^r)^{\mathrm{GL}_n(k)}$$

Moreover, since the determinant of a matrix is invariant under conjugation, we have that:

$$\mathcal{O}(\mathrm{GL}_n^r)^{\mathrm{GL}_n(k)} = \left(\mathcal{O}(\mathrm{M}_n^r)[\prod_{s=1}^{r} \det{}^{-1}(X_{i,j,s})]\right)^{\mathrm{GL}_n(k)} = \mathcal{O}(\mathrm{M}_n^r)^{\mathrm{GL}_n(k)}[\prod_{s=1}^{r} \det{}^{-1}(X_{i,j,s})]$$

This observation, combined with the surjection above, gives rise to the surjection:

$$k[a_{i,j,s}, \prod_{s=1}^{r} \det{}^{-1}(a_{i,j,s})] \twoheadrightarrow k[\mathrm{tr}(A_{\alpha_1} \cdot ... \cdot A_{\alpha_m}), \det{}^{-1}(A_1), ..., \det{}^{-1}(A_r)] \twoheadrightarrow \mathcal{O}(\mathrm{GL}_n^r)^{\mathrm{GL}_n(k)}$$

Recall that $\mathrm{Hom}(\mathbb{F}_r, \mathrm{GL}_n) \cong \mathrm{GL}_n$ so, in fact, we have a surjection

$$k[\mathrm{tr}(A_{\alpha_1} \cdot ... \cdot A_{\alpha_m}), \det{}^{-1}(A_1), ..., \det{}^{-1}(A_r)] \twoheadrightarrow \mathcal{O}(M(\mathbb{F}_r, \mathrm{GL}_n))$$

4.2 Reformulating the Ring of Invariants

Theorem 4.2.1. *[LM85, p. 27]*
Let k be our ground ring and consider the affine k-scheme $\mathfrak{X} = Hom(\Gamma, GL_n) = Spec(A)$. Then the coordinate k-algebra $A^{GL_n(k)} = \mathcal{O}(X)^{GL_n(k)}$ of the moduli scheme $X//GL_n(k)$ is generated by trace and determinant elements. That is, there exists a surjection

$$k[tr(A_{\alpha_1} \cdot ... \cdot A_{\alpha_m}), det^{-1}(A_1), ..., det^{-1}(A_r)] \twoheadrightarrow \mathcal{O}(M(\Gamma, GL_n))$$

Proof: Assume that Γ is generated by r generators. Consider $\mathrm{M}_n^r(k)$, generated by the matrices $A_1, ..., A_r \in \mathrm{M}_n(k)$. We have already shown that there exists a surjection

$$k[\mathrm{tr}(A_{\alpha_1} \cdot ... \cdot A_{\alpha_m})] \twoheadrightarrow \mathcal{O}(\mathrm{M}_n^r)^{\mathrm{GL}_n(k)}$$

That is, the ring $\mathcal{O}(\mathrm{M}_n^r)^{\mathrm{GL}_n(k)}$ is generated by traces. Moreover, we have a surjection:

$$k[\mathrm{tr}(A_{\alpha_1} \cdot ... \cdot A_{\alpha_m}), \det{}^{-1}(A_1), ..., \det{}^{-1}(A_r)] \twoheadrightarrow \mathcal{O}(M(\mathbb{F}_r, \mathrm{GL}_n))$$

Thus, it is sufficient to prove that there exists a surjection

$$\mathcal{O}(M(\mathbb{F}_r, \mathrm{GL}_n)) \twoheadrightarrow \mathcal{O}(M(\Gamma, \mathrm{GL}_n))$$

Since Γ is generated by r generators, Γ is the image of $\mathbb{F}_r$ under some group morphism

$$\phi : \mathbb{F}_r \twoheadrightarrow \Gamma$$

This is a surjection by construction. Moreover, every surjection is an epimorphism. By the contravariance of the functor Hom(_, C), this induces a monomorphism

$$\Phi : \mathrm{Hom}(\Gamma, \mathrm{GL}_n) \hookrightarrow \mathrm{Hom}(\mathbb{F}_r, \mathrm{GL}_n) : \rho \mapsto \rho \circ \phi$$

Then, by the equivalence of categories between **AffS** and **CRing**^{op}, we obtain an epimorphism

$$\Phi^* : \mathcal{O}(\mathrm{Hom}(\mathbb{F}_r, \mathrm{GL}_n)) \twoheadrightarrow \mathcal{O}(\mathrm{Hom}(\Gamma, \mathrm{GL}_n)) = A$$

Recall that $\mathrm{GL}_n(k)$ is linearly reductive. Thus, by definition of linear reductiveness, we obtain

$$\Phi^* : \mathcal{O}(\mathrm{GL}_n^r) \twoheadrightarrow A, \text{ epimorphism } \implies \bar{\Phi}^* : \mathcal{O}(\mathrm{GL}_n^r)^{\mathrm{GL}_n(k)} \twoheadrightarrow A^{\mathrm{GL}_n(k)}, \text{ surjective}$$

Here $\bar{\Phi}^*$ denotes the induced map on invariants of Φ^*. Since the composition of surjective ring morphisms is surjective, we conclude that

$$k[\mathrm{tr}(A_{\alpha_1} \cdot \ldots \cdot A_{\alpha_m}), \det{}^{-1}(A_1), ..., \det{}^{-1}(A_r)] \twoheadrightarrow \mathcal{O}(M(\mathbb{F}_r, \mathrm{GL}_n)) \twoheadrightarrow \mathcal{O}(M(\Gamma, \mathrm{GL}_n))$$

Q.E.D.

Definition 4.2.2. Let $\rho \in R(\Gamma, \mathrm{GL}_n(k))$, then the **character** of ρ is the map

$$\chi_\rho : \Gamma \to k : \gamma \mapsto \mathrm{tr}(\rho(\gamma))$$

Theorem 4.2.3.
If we define the function

$$\Psi : R(\Gamma, GL_n(k)) \to k^p : \rho \mapsto (\chi_\rho(\gamma_1), ..., \chi_\rho(\gamma_m), det(\rho(\gamma_{m+1}))^{-1}, ..., det(\rho(\gamma_p))^{-1})$$

then we obtain the 1-1 correspondence:

$$\Psi(R(\Gamma, GL_n(k))) \cong M(\Gamma, GL_n(k))$$

Proof: See "Varieties of Representations of finitely generated groups" [LM85, p. 28]

Call the image of Ψ the **character variety** of Γ into $\mathrm{GL}_n(k)$. Since $\mathrm{SL}_n(k)$ is also linearly reductive we can do a similar computation for the special linear group [MP16, p. 400].

Example 4.2.4. Consider the finitely generated free abelian group $\Gamma = \langle x, y \mid xy = yx \rangle$ and the representation variety $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$. Then there exists some $\gamma_1, ..., \gamma_p \in \Gamma$ such that, for the function

$$\Psi : R(\Gamma, \mathrm{SL}_2(\mathbb{C})) \to \mathbb{C}^p : \rho \mapsto (\chi_\rho(\gamma_1), ..., \chi_\rho(\gamma_m), \det(\rho(\gamma_{m+1}))^{-1}, ..., \det(\rho(\gamma_p))^{-1})$$

we have

$$\Psi(R(\Gamma, \mathrm{SL}_n(\mathbb{C}))) \cong M(\Gamma, \mathrm{SL}_n(\mathbb{C}))$$

but the determinant of a matrix in $\mathrm{SL}_2(\mathbb{C})$ has trivial determinant by definition. Thus the character variety is given by the image of

$$\Psi^* : R(\Gamma, \mathrm{SL}_2(\mathbb{C})) \to \mathbb{C}^m : \rho \mapsto (\chi_\rho(\gamma_1), ..., \chi_\rho(\gamma_m))$$

Let $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right), B = \left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$, then

$$\begin{aligned}
\mathrm{tr}(A^{-1}) &= \mathrm{tr}(\left(\begin{smallmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{smallmatrix}\right)) \\
&= a_4 + a_1 \\
&= \mathrm{tr}(A)
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}(AB) &= \mathrm{tr}(\left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)\left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right)) \\
&= \mathrm{tr}(\left(\begin{smallmatrix} a_1b_1+a_2b_3 & a_1b_2+a_2b_4 \\ a_1b_2+a_2b_4 & a_3b_2+a_4b_4 \end{smallmatrix}\right)) \\
&= a_1b_1 + a_2b_3 + a_3b_2 + a_4b_4 \\
&= \mathrm{tr}(\left(\begin{smallmatrix} b_1a_1+b_2a_3 & b_3a_1+b_4a_3 \\ b_1a_2+b_2a_4 & b_3a_2+b_4a_4 \end{smallmatrix}\right)) \\
&= \mathrm{tr}(\left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)) \\
&= \mathrm{tr}(BA)
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) &= (a_1b_1 + a_2b_3 + a_3b_2 + a_4b_4) + (a_1b_4 - a_2b_3 - a_3b_2 + a_4b_1) \\
&= a_1b_1 + a_4b_4 + a_1b_4 + a_4b_1 \\
&= a_1(b_1 + b_4) + a_4(b_4 + b_1) \\
&= (a_1 + a_4)(b_1 + b_4) \\
&= \mathrm{tr}(A)\mathrm{tr}(B)
\end{aligned}$$

It can be shown that the character variety is given by the image of

$$\Psi^* : \rho \mapsto (\chi_\rho(x), \chi_\rho(y), \chi_\rho(xy))$$

See [MO09, p. 2378]. In particular, if we set $\rho(x) = X$, $\rho(y) = Y$ for some $\rho \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, then, via the identities above, we have

$$\begin{aligned}
4 + \mathrm{tr}(X)\mathrm{tr}(Y)\mathrm{tr}(XY) &= 4 + \mathrm{tr}(X)(\mathrm{tr}(YXY) + \mathrm{tr}(Y(XY)^{-1})) \\
&= 4 + \mathrm{tr}(X)(\mathrm{tr}(XY^2) + \mathrm{tr}(X)) \\
&= 4 + \mathrm{tr}(X)^2 + \mathrm{tr}(X)\mathrm{tr}(XY^2) \\
&= 4 + \mathrm{tr}(X)^2 + \mathrm{tr}(X^2Y^2) + \mathrm{tr}(Y^2) \\
&= 4 + \mathrm{tr}(X)^2 + \mathrm{tr}(XY)^2 - \mathrm{tr}(I_2) + \mathrm{tr}(Y)^2 - \mathrm{tr}(I_2) \\
&= \mathrm{tr}(X)^2 + \mathrm{tr}(Y)^2 + \mathrm{tr}(XY)^2
\end{aligned}$$

Then, the moduli space is given by

$$M(\Gamma, \mathrm{SL}_2(\mathbb{C})) \cong \{(a, b, c) \in \mathbb{C}^3 \mid a^2 + b^2 + c^2 - abc - 4 = 0\}$$

4.3 Parameterising the Moduli Space

Proposition 4.3.1. Consider the canonical projection:

$$q_\Gamma : R(\Gamma, \mathrm{GL}_n(k)) \to M(\Gamma, \mathrm{GL}_n(k))$$

Then each fibre $p^{-1}(x)$ contains a semi-simple representation and if $\rho, \rho' \in R(\Gamma, \mathrm{GL}_n(k))$ semi-simple with $q_\Gamma(\rho) = q_\Gamma(\rho')$, then $\rho \sim \rho'$.

Proof: See "Varieties of Representations of finitely generated groups" [LM85, p. 25]

Then, each fibre $p_\Gamma^{-1}(x)$ of representations in $R(\Gamma, \mathrm{GL}_n(k))$ can be identified with some equivalence class of semi-simple representations in that class:

Theorem 4.3.2.
the projection

$$q_\Gamma : R(\Gamma, GL_n(k)) \to M(\Gamma, GL_n(k))$$

induces a bijection between points of $M(\Gamma, GL_n(k))$ and equivalence classes of semi-simple representations in $R(\Gamma, GL_n(k))$.

Proof: The restriction of q_Γ to equivalence classes of semi-simple representations is well-defined since the points of $M(\Gamma, \mathrm{GL}_n(k))$ are $\mathrm{GL}_n(k)$-stable by definition. q_Γ is surjective by construction and since every fibre of q_Γ contains a semi-simple representation, the restriction of q_Γ to semi-simple representations is also surjective. Let $\rho \in R(\Gamma, \mathrm{GL}_n(k))$, semi-simple, and set $x = q_\Gamma(\rho)$. Assume there exists some other $\rho' \in R(\Gamma, \mathrm{GL}_n(k))$ with $\rho' \in q_\Gamma(x)$. Then, by 4.3.1, $\rho' \sim \rho$ as required.

Q.E.D.

Proposition 4.3.3. Assume that $\rho, \rho' \in R(\Gamma, \mathrm{GL}_n(k))$ are both semi-simple representations, then we have sufficient and necessary conditions:

$$\rho \sim \rho' \iff \chi_\rho = \chi'_\rho$$

Proof: The trace map is invariant under conjugation. Therefore we have

$$\rho \sim \rho' \implies \chi_\rho = \chi'_\rho$$

Moreover, in the case when the representations ρ and ρ' are semi-simple, by 4.2.1, we have

$$\chi_\rho = \chi'_\rho \implies \mathrm{tr}(\rho(\gamma)) = \mathrm{tr}(\rho'(\gamma))\ \forall\, \gamma \in \Gamma \implies q_\Gamma(\rho) = q_\Gamma(\rho')$$

By 4.3.2, we obtain $q_\Gamma(\rho) = q_\Gamma(\rho') \implies \rho \sim \rho'$, completing the argument.

Q.E.D.

See [CS83, p. 120] for an elementary proof in the special case of $\rho, \rho' \in R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ and [LM85, p. 28] for the generalised result. Since every equivalence class of semi-simple representations agrees on characters, we can reformulate 4.3.2 as follows:

Proposition 4.3.4. There exists a bijection

$$\begin{array}{ccc}
\{\chi_\rho : \Gamma \to k \mid \rho \in R(\Gamma, \mathrm{GL}_n(k)),\ \text{semi-simple}\} & \rightleftharpoons & M(\Gamma, \mathrm{GL}_n(k)) \\
\chi_\rho & \longmapsto & q_\Gamma(\rho) \\
\chi_{\rho'} \ \text{ for any } \rho' \in q_\Gamma^{-1}(x),\ \text{semi-simple} & \longleftarrow\!\shortmid & x
\end{array}$$

5 Tangent Spaces

We define the Zariski tangent space $T_r(\mathfrak{X})$ of $\mathfrak{X}$ at a point $r \in \mathfrak{X}(k)$ as the fibre at r of the morphism

$$\eta^* : \mathfrak{X}(k[\epsilon]) \to \mathfrak{X}(k)$$

induced by sending $\epsilon \mapsto 0$. Intuitively, the points of the tangent space will be points of the augmentation $\mathfrak{X}(k[\epsilon])$ which tend linearly to r when ϵ vanishes. We prove that there is a k-linear isomorphism

$$Z^1(\Gamma, \text{Conj} \circ \rho) \cong T_\rho(R(\Gamma, \text{GL}_n))$$

By considering the vanishing of the first cohomology group, we obtain a sufficient condition for a representation space to consist entirely of semi-simple representations.

5.1 Zariski Tangent Spaces

Definition 5.1.1. Let $k \in \mathbf{CRing}$ and consider the ring

$$k[\epsilon] = \{a + b \cdot \epsilon \mid a, b \in k \ , \ \epsilon^2 = 0\}$$

Then we define the **augmentation map**

$$\eta : k[\epsilon] \to k : \epsilon \mapsto 0$$

Definition 5.1.2. Let $\mathfrak{X} = \text{Spec}(A)$ be an affine k-scheme, and $r \in \mathfrak{X}(k)$. Define the induced k-scheme morphism

$$\eta^* : \mathfrak{X}(k[\epsilon]) \to \mathfrak{X}(k) : \rho \mapsto \eta \circ \rho$$

The **Zariski tangent space** to $\mathfrak{X}$ at r is the fibre

$$T_r(\mathfrak{X}) := (\eta^*)^{-1}|_r$$

Let $r : A \to k$ and $\tau \in T_r(\mathfrak{X})$. Then, by assumption $\eta^*(\tau) = r$. Moreover, $\tau : A \to k[\epsilon]$ and we write $\tau(a) = \tau_1(a) + \tau_2(a)\epsilon$ for each $a \in A$. Thus

$$\tau \in T_r(\mathfrak{X}) \iff \tau(a) = r(a) + \tau_2(a)\epsilon \ \ k\text{-algebra morphism}$$

Therefore, it is equivalent to define $T_r(\mathfrak{X})$ via [LM85, p. 31]:

$$T_r(\mathfrak{X}) \cong \{\tau \in \text{Hom}(A, k) \mid a \mapsto r(a) + \tau(a)\epsilon \in \text{Hom}(A, k[\epsilon]), \ k\text{-algebra morphism}\}$$

Moreover, if we consider a morphism of k-schemes

$$F : \mathfrak{X} \to \mathfrak{Y}$$

Then there is an induced morphism of k-algebras, given by

$$F^* : \mathcal{O}(\mathfrak{Y}) \to \mathcal{O}(\mathfrak{X})$$

Then, for $(x : \mathcal{O}(\mathfrak{X}) \to k) \in \mathfrak{X}(k)$, we have $(x \circ F^* : \mathcal{O}(\mathfrak{Y}) \to k) \in \mathfrak{Y}(k)$. On $k[\epsilon]$-points, choose $t \in T_x(\mathfrak{X}) \subset \mathfrak{X}(k[\epsilon])$, then $t \circ F^* \in \mathfrak{Y}(k[\epsilon])$. In particular, since $\eta \circ t = x$ we have $(\eta \circ t) \circ F^* = x \circ F^* = \eta \circ (t \circ F^*)$. That is, a morphism F of k-schemes, induces a map

$$(DF)_x : T_x(\mathfrak{X}) \to T_{x \circ F^*}(\mathfrak{Y}) : t \mapsto t \circ F^*$$

The tangent space of the functor of points $\mathfrak{X}(k)$ at $r \in \mathfrak{X}(k)$ is given by

$$T_r(\mathfrak{X}(k)) := T_r(\mathrm{Spec}(\mathcal{O}(\mathfrak{X})^{\mathrm{red}}))$$

5.2 Tangent Spaces as 1-cocycles

Proposition 5.2.1. [LM85, p. 32]
Let $\mathrm{GL}_n(k)$ act on $\mathrm{M}_n(k)$ by conjugation, that is

$$\mathrm{Conj} : \mathrm{M}_n(k) \times \mathrm{GL}_n(k) \to \mathrm{M}_n(k) : (b, a) \mapsto aba^{-1}$$

Consider the associated semi-direct product $\mathrm{M}_n(k) \rtimes \mathrm{GL}_n(k)$. Explicitly this is the group $(\mathrm{M}_n(k) \times \mathrm{GL}_n(k), \cdot)$ where the group operation is given by:

$$(b_1, a_1) \cdot (b_2, a_2) := (b_1 + \mathrm{Conj}(a_1, b_2), a_1 a_2)$$

Then there exists an isomorphism

$$\psi : \mathrm{GL}_n(k[\epsilon]) \cong \mathrm{M}_n(k) \rtimes \mathrm{GL}_n(k)$$

such that, for the induced map η^*, the projection onto the second component

$$p_2 : \mathrm{M}_n(k) \rtimes \mathrm{GL}_n(k) \to \mathrm{GL}_n(k) : (b, a) \to a$$

yields $\eta^* = p_2 \circ \psi$.

Proof: Any matrix $a \in \mathrm{GL}_n(k[\epsilon])$ can be written in the form $a = p + q\epsilon$ for $p, q \in \mathrm{M}_n(k)$ and evaluating $\epsilon = 0$ shows that $p \in \mathrm{GL}_n(k)$. Then we obtain an isomorphism

$$\psi : \mathrm{GL}_n(k[\epsilon]) \cong \mathrm{M}_n(k) \rtimes \mathrm{GL}_n(k) : p + q\epsilon \mapsto (qp^{-1}, p)$$

This is clear since p is invertible and for any desired (b, a), we can find a $c = ba \in \mathrm{M}_n(k)$ such that

$$\begin{aligned}\psi(a + c\epsilon) &= (ca^{-1}, a)\\ &= (baa^{-1}, a)\\ &= (b, a)\end{aligned}$$

Moreover, ψ is a homomorphism via

$$\begin{aligned}\psi(p + q\epsilon)\psi(r + s\epsilon) &= (qp^{-1}, p)(sr^{-1}, r)\\ &= (qp^{-1} + psr^{-1}p^{-1}, pr)\\ &= ((qr + ps)(pr)^{-1}, pr)\\ &= \psi(pr + (qr + ps)\epsilon)\\ &= \psi((p + q\epsilon)(r + s\epsilon))\end{aligned}$$

Finally, we require that $\ker(\psi) = I_n$. Assume that there exists some $a + b\epsilon \in \mathrm{GL}_n(k[\epsilon])$ such that $\psi(a+b\epsilon) = (0, I_n)$. Then $(ba^{-1}, a) = (0_n, I_n)$ and $a+b\epsilon = I_n+0_n\epsilon$ as required.

Q.E.D.

Returning to representation spaces, we set $\mathfrak{X} = R(\Gamma, \mathrm{GL}_n(\_))$. Then we obtain

$$\mathfrak{X}(k[\epsilon]) = R(\Gamma, \mathrm{GL}_n(k[\epsilon])) = \mathrm{Hom}(\Gamma, \mathrm{GL}_n(k[\epsilon])) = \mathrm{Hom}(\Gamma, \mathrm{M}_n(k) \rtimes \mathrm{GL}_n(k))$$

For some choice of $\rho \in \mathfrak{X}(k)$, computing the tangent space $T_\rho(\mathfrak{X})$ involves finding the fibre of

$$\eta^* : \mathfrak{X}(k[\epsilon]) \to \mathfrak{X}(k) : \rho \mapsto \eta \circ \rho$$

To illustrate this, we have the diagram

$$\begin{tikzcd} \mathrm{Hom}(\Gamma, \mathrm{GL}_n(k[\epsilon])) \arrow[r, "\psi\circ\_"] \arrow[dr, "\eta^*\circ\_"'] & \mathrm{Hom}(\Gamma, \mathrm{M}_n(k) \rtimes \mathrm{GL}_n(k)) \arrow[d, "p_2\circ\_"] \\ & \mathrm{Hom}(\Gamma, \mathrm{GL}_n(k)) \end{tikzcd}$$

By 5.2.1, we have that there exists some isomorphism ψ such that the fibre $(\eta^*)^{-1}(\rho)$ is given by the set of homomorphisms $f \in \mathrm{Hom}(\Gamma, \mathrm{M}_n(k) \rtimes \mathrm{GL}_n(k))$ where $p_2 \circ f = \rho$. Similar to the argument in 5.1.2, assume

$$\tau : \Gamma \to \mathrm{M}_n(k) \rtimes \mathrm{GL}_n(k) \text{ with } p_2 \circ \tau = \rho$$

We write $\tau(\gamma) = (\tau_1(\gamma), \tau_2(\gamma)) \in \mathrm{M}_n(k) \rtimes \mathrm{GL}_n(k)$ for some $\gamma \in \Gamma$. Thus we can characterise the tangent space $T_\rho(\mathfrak{X})$ by functions $\tau_1 : \Gamma \to \mathrm{M}_n(k)$ such that:

$$\tau : \Gamma \to \mathrm{M}_n(k) \rtimes \mathrm{GL}_n(k) : \gamma \to (\tau_1(\gamma), \rho(\gamma)), \text{ a homomorphism}$$

See [LM85, p.33]. Moreover note that

$$\begin{aligned}\tau(\gamma_1)\tau(\gamma_2) &= (\tau_1(\gamma_1), \rho(\gamma_1))(\tau_1(\gamma_2), \rho(\gamma_2)) \\ &= (\tau_1(\gamma_1) + \rho(\gamma_1)\tau_1(\gamma_2)\rho(\gamma_1)^{-1}, \rho(\gamma_1)\rho(\gamma_2))\end{aligned}$$

Then the map τ is a homomorphism if and only if

$$\tau(\gamma_1)\tau(\gamma_2) = \tau(\gamma_1\gamma_2)$$

$$\iff (\tau_1(\gamma_1) + \rho(\gamma_1)\tau_1(\gamma_2)\rho(\gamma_1)^{-1}, \rho(\gamma_1)\rho(\gamma_2)) = (\tau_1(\gamma_1\gamma_2), \rho(\gamma_1\gamma_2))$$

But $\rho : \Gamma \to \mathrm{GL}_n(k)$ is a homomorphism by assumption, so we conclude that

$$T_\rho(\mathfrak{X}) \cong \{\tau : \Gamma \to \mathrm{M}_n(k) \mid \tau(\gamma_1\gamma_2) = \tau(\gamma_1) + \rho(\gamma_1)\tau(\gamma_2)\rho(\gamma_1)^{-1}\}$$

on the level of sets. The set $\{\tau : \Gamma \to \mathrm{M}_n(k) \mid \tau(\gamma_1\gamma_2) = \tau(\gamma_1) + \rho(\gamma_1)\tau(\gamma_2)\rho(\gamma_1)^{-1}\}$ is denoted $Z^1(\Gamma, \mathrm{Conj} \circ \rho)$ and referred to as the **1-cocycle space** of Γ with coefficients in $\mathrm{Conj} \circ \rho$.

Theorem 5.2.2. *[LM85, p.33]*
let $\rho \in R(\Gamma, GL_n(k))$, then there exists a k-linear isomorphism

$$Z^1(\Gamma, \mathit{Conj} \circ \rho) \xrightarrow{\quad\cong\quad} T_\rho(R(\Gamma, \mathit{GL}_n))$$

$$\tau \longmapsto [\gamma \mapsto (I_n + \tau(\gamma)\epsilon)\rho(\gamma)]$$

Proof: The argument above shows that we already have an isomorphism of sets. It remains to be shown that the specific map given above is a bijection. We define the function

$$P_\tau : \Gamma \to \mathrm{GL}_n(k[\epsilon]) : \gamma \mapsto (I_n + \tau(\gamma)\epsilon)\rho(\gamma)$$

and denote the map above by

$$\Theta : Z^1(\Gamma, \mathrm{Conj} \circ \rho) \to T_\rho(R(\Gamma, \mathrm{GL}_n)) : \tau \to P_\tau$$

Since $\rho(\gamma)$ is invertible for any choice of γ, we have that $\tau(\gamma)\rho(\gamma) = 0$ if and only if τ is the zero map, thus Θ is injective. Moreover, if we choose some arbitrary

$$(P^* : \gamma \mapsto \rho(\gamma) + \tau^*(\gamma)\epsilon) \in T_\rho(R(\Gamma, \mathrm{GL}_n))$$

Then, Θ is surjective if there exists some $\tau' \in Z^1(\Gamma, \mathrm{Conj} \circ \rho)$ with $\Theta(\tau') = P^*$. By the invertability of $\rho(\gamma)$, we can define $\tau(\gamma) := \tau^*(\gamma)\rho(\gamma)^{-1}$ and it is sufficient to check that τ is indeed a 1-cocycle. Let $a, b \in \Gamma$, then:

$$P^*(ab) = \rho(ab) + \tau(ab)\rho(ab)\epsilon \iff I_n + \tau(ab)\epsilon = P^*(ab)\rho(ab)^{-1}$$

By assumption, P^* is a representation. Therefore $P^*(ab) = P^*(a)P^*(b)$, and we obtain

$$\begin{aligned}
I_n + \tau(ab)\epsilon &= P^*(ab)\rho(ab)^{-1} \\
&= P^*(a)P^*(b)\rho(b)^{-1}\rho(a)^{-1} \\
&= P^*(a)(I_n + \tau(b)\epsilon)\rho(b)\rho(b)^{-1}\rho(a)^{-1} \\
&= P^*(a)\rho(a)^{-1} + P^*(a)\tau(b)\epsilon\rho(a)^{-1} \\
&= (I_n + \tau(a)\epsilon)\rho(a)\rho(a)^{-1} + (I_n + \tau(a)\epsilon)\rho(a)\tau(b)\epsilon\rho(a)^{-1} \\
&= I_n + \tau(a)\epsilon + \rho(a)\tau(b)\rho(a)^{-1}\epsilon + \tau(a)\rho(a)\tau b\rho(a)^{-1}\epsilon^2 \\
&= I_n + (\tau(a) + \rho(a)\tau(b)\rho(a)^{-1})\epsilon
\end{aligned}$$

Equating coefficients, we obtain

$$\tau(ab) = \tau(a) + \rho(a)\tau(b)\rho(a)^{-1}$$

This shows that τ is a 1-cocycle and thus $P_\tau = P^*$, as required.

Q.E.D.

5.3 Trefoil Knot Tangent Spaces

Example 5.3.1. Consider the trefoil knot group $\Gamma = \langle x, y \mid x^2 = y^3 \rangle$ into $\mathrm{GL}_2(\mathbb{C})$ and the representation given by

$$\rho : \langle x, y \mid x^2 = y^3 \rangle \to \mathrm{GL}_2(\mathbb{C}) : a \mapsto \begin{pmatrix} z_1^{\bar{a}} & 0 \\ 0 & z_2^{\bar{a}} \end{pmatrix}$$

where $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ and $\bar{a} \in \Gamma^{\mathrm{ab}}$ is the image of $a \in \Gamma$ under the abelianisation map

$$\mathrm{ab} : \langle x, y \mid x^2 = y^3 \rangle \to \langle x, y \mid x^2 = y^3 \rangle^{\mathrm{ab}}$$

We calculate the tangent space of Γ at ρ. Note that $\langle x, y \mid x^2 = y^3 \rangle^{\mathrm{ab}} \cong \langle x, y \mid x^2 = y^3, xy = yx \rangle$ since we require that all elements commute. Moreover, in the abelianisation, we have

$$\begin{aligned}
(xy^{-1})^3 &= xy^{-1}xy^{-1}xy^{-1} \\
&= x^3y^{-3} \\
&= xy^3y^{-3} \\
&= x
\end{aligned}
\qquad
\begin{aligned}
(xy^{-1})^2 &= xy^{-1}xy^{-1} \\
&= x^2y^{-2} \\
&= y^3y^{-2} \\
&= y
\end{aligned}$$

Thus, the element xy^{-1} in the abelianisation generates Γ^{ab}. That is

$$\phi : \langle x, y \mid x^2 = y^3 \rangle^{\mathrm{ab}} \to \mathbb{Z} : xy^{-1} \mapsto 1$$

is an isomorphism. Under ϕ we have $\phi(x) = 3$ and $\phi(y) = 2$. That is

$$\rho(x) = \begin{pmatrix} z_1^3 & 0 \\ 0 & z_2^3 \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} z_1^2 & 0 \\ 0 & z_2^2 \end{pmatrix}$$

It is clear that $\rho(x^2) = \rho(y^3)$, so ρ respects the structure imposed by Γ and ρ is therefore a representation. Then, let $\mathfrak{X} = R(\Gamma, \mathrm{GL}_2)$ and assume $\tau \in T_\rho(\mathfrak{X})$. Set

$$\tau(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \tau(y) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

Then, calculating $\tau(x^2)$, we have

$$\begin{aligned}
\tau(x^2) &= \tau(x) + \rho(x)\tau(x)\rho(x)^{-1} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + z_1^{-3}z_2^{-3}\begin{pmatrix} z_1^3 & 0 \\ 0 & z_2^3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_2^3 & 0 \\ 0 & z_1^3 \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + z_1^{-3}z_2^{-3}\begin{pmatrix} z_1^3 a & z_1^3 b \\ z_2^3 c & z_2^3 d \end{pmatrix}\begin{pmatrix} z_2^3 & 0 \\ 0 & z_1^3 \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + z_1^{-3}z_2^{-3}\begin{pmatrix} z_1^3 z_2^3 a & z_1^6 b \\ z_2^6 c & z_1^3 z_2^3 d \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & (\frac{z_1}{z_2})^3 b \\ (\frac{z_2}{z_1})^3 c & d \end{pmatrix} \\
&= \begin{pmatrix} 2a & (1+(\frac{z_1}{z_2})^3)b \\ (1+(\frac{z_2}{z_1})^3)c & 2d \end{pmatrix}
\end{aligned}$$

Similarly, for $\tau(y^3)$, we have

$$\begin{aligned}
\tau(y^3) &= \tau(y) + \rho(y)\tau(y^2)\rho(y)^{-1} \\
&= \tau(y) + \rho(y)(\tau(y) + \rho(y)\tau(y)\rho(y)^{-1})\rho(y)^{-1} \\
&= \tau(y) + \rho(y)\tau(y)\rho(y)^{-1} + \rho(y)^2\tau(y)\rho(y)^{-2} \\
&= \begin{pmatrix} p & q \\ r & s \end{pmatrix} + z_1^{-2}z_2^{-2}\begin{pmatrix} z_1^2 & 0 \\ 0 & z_2^2 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} z_2^2 & 0 \\ 0 & z_1^2 \end{pmatrix} \\
&\quad + z_1^{-4}z_2^{-4}\begin{pmatrix} z_1^4 & 0 \\ 0 & z_2^4 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} z_2^4 & 0 \\ 0 & z_1^4 \end{pmatrix} \\
&= \begin{pmatrix} p & q \\ r & s \end{pmatrix} + z_1^{-2}z_2^{-2}\begin{pmatrix} z_1^2 p & z_1^2 q \\ z_2^2 r & z_2^2 s \end{pmatrix}\begin{pmatrix} z_2^2 & 0 \\ 0 & z_1^2 \end{pmatrix} \\
&\quad + z_1^{-4}z_2^{-4}\begin{pmatrix} z_1^4 p & z_1^4 q \\ z_2^4 r & z_2^4 s \end{pmatrix}\begin{pmatrix} z_2^4 & 0 \\ 0 & z_1^4 \end{pmatrix} \\
&= \begin{pmatrix} p & q \\ r & s \end{pmatrix} + z_1^{-2}z_2^{-2}\begin{pmatrix} z_1^2 z_2^2 p & z_1^4 q \\ z_2^4 r & z_1^2 z_2^2 s \end{pmatrix} \\
&\quad + z_1^{-4}z_2^{-4}\begin{pmatrix} z_1^4 z_2^4 p & z_1^8 q \\ z_2^8 r & z_1^4 z_2^4 s \end{pmatrix} \\
&= \begin{pmatrix} p & q \\ r & s \end{pmatrix} + \begin{pmatrix} p & (\frac{z_1}{z_2})^2 q \\ (\frac{z_2}{z_1})^2 r & s \end{pmatrix} + \begin{pmatrix} p & (\frac{z_1}{z_2})^4 q \\ (\frac{z_2}{z_1})^4 r & s \end{pmatrix} \\
&= \begin{pmatrix} 3p & (1+(\frac{z_1}{z_2})^2+(\frac{z_1}{z_2})^4)q \\ (1+(\frac{z_2}{z_1})^2+(\frac{z_2}{z_1})^4)r & 3s \end{pmatrix}
\end{aligned}$$

Thus, since $x^2 = y^3$, we require $\tau(x^2) = \tau(y^3)$, implying

$$\begin{pmatrix} 2a & (1+(\frac{z_1}{z_2})^3)b \\ (1+(\frac{z_2}{z_1})^3)c & 2d \end{pmatrix} = \begin{pmatrix} 3p & (1+(\frac{z_1}{z_2})^2+(\frac{z_1}{z_2})^4)q \\ (1+(\frac{z_2}{z_1})^2+(\frac{z_2}{z_1})^4)r & 3s \end{pmatrix}$$

Setting $f := \frac{z_1}{z_2} \in \mathbb{C} \setminus \{0\}$ and $g := f^{-1} \in \mathbb{C} \setminus \{0\}$, we obtain the four relations

$$\begin{aligned} 2a &= 3p \\ (1+f^3)b &= (1+f^2+f^4)q \\ (1+g^3)c &= (1+g^2+g^4)r \\ 2d &= 3s \end{aligned}$$

Assume $z \in \mathbb{C}$ such that $z^3 + 1 = 0$, then $z = -e^{2\pi ik/3}$ for some $k \in \mathbb{Z}$. Then

$$\begin{aligned} z^4 + z^2 + 1 &= -e^{2\pi ik/3}z^3 + e^{4\pi ik/3} + 1 \\ &= e^{2\pi ik/3} + e^{4\pi ik/3} + 1 \\ &= \cos(2\pi k/3) + i\sin(2\pi k/3) + \cos(4\pi k/3) + i\sin(4\pi k/3) + 1 \\ &= 1 + \cos(2\pi k/3) + \cos(4\pi k/3) + i(\sin(2\pi k/3) + \sin(4\pi k/3)) \end{aligned}$$

For $k \equiv 0 \pmod 3$ we have

$$\begin{aligned} z^4 + z^2 + 1 &= 1 + \cos(0) + \cos(0) + i(\sin(0) + \sin(0)) \\ &= 1 + 1 + 1 + i(0 + 0) \\ &= 3 \end{aligned}$$

For $k \equiv 1 \pmod 3$ we have

$$\begin{aligned} z^4 + z^2 + 1 &= 1 + \cos(2\pi/3) + \cos(4\pi/3) + i(\sin(2\pi/3) + \sin(4\pi/3)) \\ &= 1 - \tfrac{1}{2} - \tfrac{1}{2} + i\left(\tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{2}\right) \\ &= 0 \end{aligned}$$

For $k \equiv 2 \pmod 3$ we have

$$\begin{aligned} z^4 + z^2 + 1 &= 1 + \cos(4\pi/3) + \cos(8\pi/3) + i(\sin(4\pi/3) + \sin(8\pi/3)) \\ &= 1 + \cos(4\pi/3) + \cos(2\pi/3) + i(\sin(4\pi/3) + \sin(2\pi/3)) \\ &= 0 \end{aligned}$$

Note that $-e^{2\pi i/3} = e^{5\pi i/3}$ and $-e^{4\pi i/3} = e^{\pi i/3}$. The calculation above shows that

$$z \in \{e^{\pi i/3}, e^{5\pi i/3}\} \iff z^3 + 1 = 0 \text{ and } z^4 + z^2 + 1 = 0$$

If instead we consider $z \in \mathbb{C}$ such that $z^4 + z^2 + 1 = 0$ and set $z = e^{\theta i}$, then

$$\begin{aligned} 0 &= z^4 + z^2 + 1 \\ &= e^{4\theta i} + e^{2\theta i} + 1 \\ &= \cos(4\theta) + i\sin(4\theta) + \cos(2\theta) + i\sin(2\theta) + 1 \\ &= 1 + \cos(4\theta) + \cos(2\theta) + i(\sin(4\theta) + \sin(2\theta)) \end{aligned}$$

Let $\Theta = 2\theta$. By equating imaginary parts, we obtain

$$\begin{aligned}
0 &= \sin(2\Theta) + \sin(\Theta) \\
&= 2\sin(\Theta)\cos(\Theta) + \sin(\Theta) \\
&= \sin(\Theta)(2\cos(\Theta) + 1) \\
\implies & \Theta \in \{0, \pi, 2\pi/3, 4\pi/3\}
\end{aligned}$$

Next, we check which values for Θ satisfy the equation given by equating real parts:

$$\begin{array}{lclclcl}
\Theta = 0 & \implies & 1 + \cos(2(0)) + \cos((0)) & = & 1+1+1 & = & 3 \\
\Theta = \pi & \implies & 1 + \cos(2(\pi)) + \cos(\pi) & = & 1+1-1 & = & 1 \\
\Theta = 2\pi/3 & \implies & 1 + \cos(2(2\pi/3)) + \cos(2\pi/3) & = & 1 - \frac{1}{2} - \frac{1}{2} & = & 0 \\
\Theta = 4\pi/3 & \implies & 1 + \cos(2(4\pi/3)) + \cos(4\pi/3) & = & 1 - \frac{1}{2} - \frac{1}{2} & = & 0
\end{array}$$

We conclude that $\Theta \in \{2\pi/3, 4\pi/3\}$ if any only if $z^4 + z^2 + 1 = 0$. Then, for θ we have

$$\theta \in \{\pi/3, 4\pi/3\} \iff \Theta = 2\theta \equiv 2\pi/3 \ (\mathrm{mod}\ 2\pi)$$

$$\theta \in \{2\pi/3, 5\pi/3\} \iff \Theta = 2\theta \equiv 4\pi/3 \ (\mathrm{mod}\ 2\pi)$$

Thus, the zeroes of the equation are as follows:

$$z \in \{e^{\pi i/3}, e^{2\pi i/3}, e^{4\pi i/3}, e^{5\pi i/3}\} \quad \iff \quad z^4 + z^2 + 1 = 0$$

For ease of notation, we define the complex subsets

$$\begin{aligned}
A_1 &:= \{e^{3\pi i/3}\} \\
A_2 &:= \{e^{2\pi i/3}, e^{4\pi i/3}\} \\
A_3 &:= \{e^{\pi i/3}, e^{5\pi i/3}\} \\
A_4 &:= \mathbb{C} \setminus \{e^{\pi i/3}, e^{2\pi i/3}, e^{3\pi i/3}, e^{4\pi i/3}, e^{5\pi i/3}\}
\end{aligned}$$

With the notation above, we have four cases which can arise

$$\begin{array}{llclc}
\textbf{I)} & z \in & A_1 & \iff & z^3 + 1 = 0 \text{ and } z^4 + z^2 + 1 \neq 0 \\
\textbf{II)} & z \in & A_2 & \iff & z^3 + 1 \neq 0 \text{ and } z^4 + z^2 + 1 = 0 \\
\textbf{III)} & z \in & A_3 & \iff & z^3 + 1 = 0 \text{ and } z^4 + z^2 + 1 = 0 \\
\textbf{IV)} & z \in & A_4 & \iff & z^3 + 1 \neq 0 \text{ and } z^4 + z^2 + 1 \neq 0
\end{array}$$

An important point is that each such set A_i is closed under multiplicative inverses:

$$e^{3\pi i/3}e^{3\pi i/3} = 1 \qquad e^{\pi i/3}e^{5\pi i/3} = 1 \qquad e^{2\pi i/3}e^{4\pi i/3} = 1$$

Thus, $f \in A_i$ if any only if $g \in A_i$ for all $i \in \{1, 2, 3, 4\}$. Returning to our original four relations given by $\tau(x^2) = \tau(y^3)$, we have

$$\begin{aligned}
2a &= 3p \\
(1 + f^3)b &= (1 + f^2 + f^4)q \\
(1 + g^3)c &= (1 + g^2 + g^4)r \\
2d &= 3s
\end{aligned}$$

If $f, g \in A_4$ then none of the polynomials in f and g vanish so each arbitrary tuple $(a, b, c, d) \in \mathbb{C}^4$ determines the tuple $(p, q, r, s) \in \mathbb{C}^4$ and vice versa. If instead $f, g \in A_1$, then

$$\begin{aligned} 2a &= 3p \\ 0b &= (1 + f^2 + f^4)q \\ 0c &= (1 + g^2 + g^4)r \\ 2d &= 3s \end{aligned}$$

Thus both $q = 0$ and $r = 0$. Since b and c can be chosen freely, we have that τ is defined by an arbitrary tuple $(a, b, c, d) \in \mathbb{C}^4$. A similar argument shows that, in the case where $f, g \in A_2$, then τ is defined by an arbitrary tuple $(p, q, r, s) \in \mathbb{C}^4$. Finally, in the case where $f, g \in A_3$, we have

$$\begin{aligned} 2a &= 3p \\ 0b &= 0q \\ 0c &= 0r \\ 2d &= 3s \end{aligned}$$

Thus, there are no relations imposed on b, c, q and r. Then τ is defined by an arbitrary tuple $(a, b, c, d, q, r) \in \mathbb{C}^6$. We conclude that, for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, and the representation

$$\rho : \langle x, y \mid x^2 = y^3 \rangle \to \mathrm{GL}_2(\mathbb{C})$$

$$x \mapsto \begin{pmatrix} z_1^3 & 0 \\ 0 & z_2^3 \end{pmatrix} \quad \text{and} \quad y \mapsto \begin{pmatrix} z_1^2 & 0 \\ 0 & z_2^2 \end{pmatrix}$$

the tangent space at ρ is determined up to isomorphism by the ratio z_1/z_2 and in particular

$$\begin{aligned} z_1/z_2 \in \ & \{e^{\pi i/3}, e^{5\pi i/3}\} & \iff \ & T_\rho(\mathfrak{X}) \cong \mathbb{C}^6 \\ z_1/z_2 \in \ & \mathbb{C} \setminus \{0, e^{\pi i/3}, e^{5\pi i/3}\} & \iff \ & T_\rho(\mathfrak{X}) \cong \mathbb{C}^4 \end{aligned}$$

For a specific example, consider the trivial representation

$$\rho_I : \langle x, y \mid x^2 = y^3 \rangle \to \mathrm{GL}_2(\mathbb{C}) : a \mapsto I_2$$

We calculate the tangent space of $\mathfrak{X} = R(\Gamma, \mathrm{GL}_2)$ at the point ρ_I:

$$\begin{aligned} T_{\rho_I}(\mathfrak{X}) &\cong \{\tau : \Gamma \to \mathrm{M}_2(\mathbb{C}) \mid \tau(ab) = \tau(a) + \rho_I(a)\tau(b)\rho_I(a)^{-1}\} \\ &\cong \{\tau : \langle x, y \mid x^2 = y^3 \rangle \to \mathrm{M}_2(\mathbb{C}) \mid \tau(ab) = \tau(a) + \tau(b)\} \end{aligned}$$

Setting $x' := \tau(x) \in \mathrm{M}_2(\mathbb{C})$, we have

$$\begin{aligned} \tau(x^2) &= \tau(x) + \tau(x) \\ &= 2x' \\ &= \tau(y^3) \\ &= \tau(y) + \tau(y) + \tau(y) \\ &= 3\tau(y) \end{aligned}$$

Then $\tau(y) = 2/3x'$. In particular, each element of the tangent space corresponds to a choice of an element $x' \in \mathrm{M}_n(\mathbb{C})$, that is

$$T_{\rho_I}(R(\langle x, y \mid x^2 = y^3 \rangle, \mathrm{GL}_2)) \cong \mathrm{M}_2(\mathbb{C}) \cong \mathbb{C}^4$$

5.4 The First Cohomology

Definition 5.4.1. Let $\mathfrak{X} := R(\Gamma, \mathrm{GL}_n)$ and choose $\rho \in \mathfrak{X}(k)$. Then the **orbit map** $\Psi_\rho : \mathrm{GL}_n \to \mathfrak{X}$ is defined, on the level of points, by

$$\Psi_\rho^k : \mathrm{GL}_n(k) \to \mathfrak{X}(k) : g \mapsto (\Psi_\rho^k(g) : \gamma \mapsto g\rho(\gamma)g^{-1})$$

The **orbit** of ρ over k is the image $O(\rho) := \mathrm{Im}(\Psi_\rho^k)$. The **1-coboundary space** of Γ, with coefficients in Conj $\circ\, \rho$, is defined as

$$B^1(\Gamma, \mathrm{Conj} \circ \rho) := \{\tau : \Gamma \to \mathrm{M}_n(k) \mid \exists\, a \in \mathrm{M}_n(k) \text{ st } \tau(\gamma) = \rho(\gamma)a\rho(\gamma)^{-1} - a \;\forall\, \gamma \in \Gamma\}$$

Theorem 5.4.2. *[LM85, p.34]*
Let $\rho \in R(\Gamma, GL_n(k))$. Consider the map

$$(D\Psi_\rho)_{I_n} : T_{I_n}(GL_n) \to T_\rho(R(\Gamma, GL_n))$$

Then, under the identification given by the isomorphism in 5.2.2, we have

$$Im((D\Psi_\rho)_{I_n}) \cong B^1(\Gamma, Conj \circ \rho)$$

Proof: We begin by proving, for consistency, that

$$\tau \in B^1(\Gamma, \mathrm{Conj} \circ \rho) \implies \tau \in Z^1(\Gamma, \mathrm{Conj} \circ \rho)$$

Assume $\tau \in B^1(\Gamma, \mathrm{Conj} \circ \rho)$. Then there exists some $a \in \mathrm{M}_n(k)$ such that, for any $\gamma \in \Gamma$, we have $\tau(\gamma) = \rho(\gamma)a\rho(\gamma)^{-1} - a$. Let $\gamma_1, \gamma_2 \in \Gamma$, then

$$\begin{aligned} \tau(\gamma_1) + \rho(\gamma_1)\tau(\gamma_2)\rho(\gamma_1)^{-1} &= (\rho(\gamma_1)a\rho(\gamma_1)^{-1} - a) + \rho(\gamma_1)(\rho(\gamma_2)a\rho(\gamma_2)^{-1} - a)\rho(\gamma_1)^{-1} \\ &= \rho(\gamma_1)\rho(\gamma_2)a\rho(\gamma_2)^{-1}\rho(\gamma_1)^{-1} - a \\ &= \rho(\gamma_1\gamma_2)a\rho(\gamma_1\gamma_2)^{-1} - a \\ &= \tau(\gamma_1\gamma_2) \end{aligned}$$

Thus, every 1-coboundary is a 1-cocycle. Consider $T_{I_n}(\mathrm{GL}_n)$, that is the fibre of $\eta^* : \mathrm{GL}_n(k[\epsilon]) \to \mathrm{GL}_n(k)$ over the identity matrix $I_n \in \mathrm{GL}_n(k)$. Let $m \in T_{I_n}(\mathrm{GL}_n)$, then

$m = I_n + m'\epsilon$ for some $m' \in \mathrm{M}_n(k)$. We have

$$\begin{aligned}
\Psi_\rho^k(m)(\gamma) &= (I_n + m'\epsilon)\rho(\gamma)(I_n + m'\epsilon)^{-1} \\
&= (I_n + m'\epsilon)\rho(\gamma)(I_n - m'\epsilon) \\
&= \rho(\gamma) - \rho(\gamma)m'\epsilon + m'\epsilon\rho(\gamma) - m'\epsilon\rho(\gamma)m'\epsilon \\
&= \rho(\gamma) + (m'\rho(\gamma) - \rho(\gamma)m')\epsilon \\
&= (\rho(\gamma)\rho(\gamma)^{-1} + m'\rho(\gamma)\epsilon\rho(\gamma)^{-1} - \rho(\gamma)m'\epsilon\rho(\gamma)^{-1})\rho(\gamma) \\
&= (I_n + (m' - \rho(\gamma)m'\rho(\gamma)^{-1})\epsilon)\rho(\gamma)
\end{aligned}$$

Then, via the the identification in 5.2.2, for the 1-coboundary τ, defined by

$$\tau : \Gamma \to \mathrm{M}_n(k) : \gamma \mapsto m' - \rho(\gamma)m'\rho(\gamma)^{-1}$$

we obtain $\Theta(\tau) = \Psi_\rho^k(m)$. Thus, $\Theta^{-1}(\mathrm{Im}(\Psi_\rho^k)) \subset B^1(\Gamma, \mathrm{Conj} \circ \rho)$. It remains to prove that, for an arbitrary 1-coboundary $\tau \in B^1(\Gamma, \mathrm{Conj} \circ \rho)$, there exists some $m = I_n + m'\epsilon \in T_{I_n}(\mathrm{GL}_n)$ such that $\Psi_\rho^k(m) = \Theta(\tau)$. If $\tau \in B^1(\Gamma, \mathrm{Conj} \circ \rho)$, then $\tau : \Gamma \to \mathrm{M}_n(k) : \gamma \mapsto \rho(\gamma)a\rho(\gamma)^{-1} - a$ for some fixed $a \in \mathrm{M}_n(k)$. Thus, we conclude that

$$\begin{aligned}
\Psi_\rho^k(I_n - a\epsilon) &= (I_n + (-a + \rho a \rho^{-1})\epsilon)\rho \\
&= (I_n + \tau\epsilon)\rho \\
&= \Theta(\tau)
\end{aligned}$$

as required.

Q.E.D.

Definition 5.4.3. The **first cohomology group** of Γ with coefficients in $\mathrm{Conj} \circ \rho$ for some $\rho \in R(\Gamma, \mathrm{GL}_n(k))$ is defined as

$$H^1(\Gamma, \mathrm{Conj} \circ \rho) := Z^1(\Gamma, \mathrm{Conj} \circ \rho)/B^1(\Gamma, \mathrm{Conj} \circ \rho)$$

That is, the 1-cocycles modulo the 1-coboundaries of Γ. Combining 5.2.2 and 5.4.2, we obtain the identification

$$H^1(\Gamma, \mathrm{Conj} \circ \rho) \cong T_\rho(R(\Gamma, \mathrm{GL}_n))/\mathrm{Im}((D\Psi_\rho)_{I_n})$$

Example 5.4.4. Consider the trefoil knot group $\Gamma = \langle x, y \mid x^2 = y^3 \rangle$ into $\mathrm{GL}_2(\mathbb{C})$ and the representation given by

$$\rho : \langle x, y \mid x^2 = y^3 \rangle \to \mathrm{GL}_2(\mathbb{C})$$

$$x \mapsto \begin{pmatrix} z_1^3 & 0 \\ 0 & z_2^3 \end{pmatrix} \quad \text{and} \quad y \mapsto \begin{pmatrix} z_1^2 & 0 \\ 0 & z_2^2 \end{pmatrix}$$

for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ and define $f = \frac{z_1}{z_2}$ and $g = f^{-1}$ as in 5.3.1. We compute the first cohomology group $H^1(\Gamma, \text{Conj} \circ \rho)$. Choose arbitrary $\tau \in B^1(\Gamma, \text{Conj} \circ \rho)$. Then, there exists some $A = \left(\begin{smallmatrix} t & v \\ w & z \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{C})$ such that

$$\begin{aligned} \tau(x) &= \rho(x)A\rho(x)^{-1} - A \\ &= z_1^{-3}z_2^{-3}\begin{pmatrix} z_1^3 & 0 \\ 0 & z_2^3 \end{pmatrix}\begin{pmatrix} t & v \\ w & z \end{pmatrix}\begin{pmatrix} z_2^3 & 0 \\ 0 & z_1^3 \end{pmatrix} - \begin{pmatrix} t & v \\ w & z \end{pmatrix} \\ &= \begin{pmatrix} 0 & (f^3-1)v \\ (g^3-1)w & 0 \end{pmatrix} \end{aligned}$$

In the case of $\tau(y)$, a similar calculation yields:

$$\begin{aligned} \tau(y) &= \rho(y)A\rho(y)^{-1} - A \\ &= z_1^{-2}z_2^{-2}\begin{pmatrix} z_1^2 & 0 \\ 0 & z_2^2 \end{pmatrix}\begin{pmatrix} t & v \\ w & z \end{pmatrix}\begin{pmatrix} z_2^2 & 0 \\ 0 & z_1^2 \end{pmatrix} - \begin{pmatrix} t & v \\ w & z \end{pmatrix} \\ &= \begin{pmatrix} 0 & (f^2-1)v \\ (g^2-1)w & 0 \end{pmatrix} \end{aligned}$$

We compute the zeroes of $z^3 - 1$ and $z^2 - 1$. For ease of notation, define the sets

$$\begin{aligned} B_1 &:= \{1\} \\ B_2 &:= \{e^{2\pi i/3}, e^{4\pi i/3}\} \\ B_3 &:= \{-1\} \\ B_4 &:= \mathbb{C} \setminus \{1, -1, e^{2\pi i/3}, e^{4\pi i/3}\} \end{aligned}$$

With the notation above, we have four cases which can arise

$$\begin{array}{lrlcl} \textbf{I)} & z \in & B_1 & \iff & z^3 - 1 = 0 \text{ and } z^2 - 1 = 0 \\ \textbf{II)} & z \in & B_2 & \iff & z^3 - 1 = 0 \text{ and } z^2 - 1 \neq 0 \\ \textbf{III)} & z \in & B_3 & \iff & z^3 - 1 \neq 0 \text{ and } z^2 - 1 = 0 \\ \textbf{IV)} & z \in & B_4 & \iff & z^3 - 1 \neq 0 \text{ and } z^2 - 1 \neq 0 \end{array}$$

Moreover, B_i are all closed under multiplicative inverses. In case $z \in \mathbb{C} \setminus B_1$, then the 1-coboundary space is two-dimensional since either $\tau(x)$ or $\tau(y)$ is parameterised by $(v, w) \in \mathbb{C}^2$. If instead $z = 1$, then the coboundary space collapses. From 5.3.1, the tangent space is four-dimensional when $z = 1$. Thus, the 1-cohomology group of Γ with coefficients in $\text{Conj} \circ \rho$ is given by

$$\begin{array}{rlcl} z_1/z_2 \in & \{1, e^{\pi i/3}, e^{5\pi i/3}\} & \iff & H^1(\Gamma, \text{Conj} \circ \rho) \cong \mathbb{C}^4 \\ z_1/z_2 \in & \mathbb{C} \setminus \{1, e^{\pi i/3}, e^{5\pi i/3}\} & \iff & H^1(\Gamma, \text{Conj} \circ \rho) \cong \mathbb{C}^2 \end{array}$$

Definition 5.4.5. A representation $\rho \in R(\Gamma, \mathrm{GL}_n(k))$ is called **scheme rigid** if

$$H^1(\Gamma, \mathrm{Conj} \circ \rho) = 0$$

A finitely generated group Γ is **scheme n-rigid** if every representation in $R(\Gamma, \mathrm{GL}_n(k))$ is scheme rigid. Γ is **n-reductive** if every representation $\rho \in R(\Gamma, \mathrm{GL}_n(k))$ is semi-simple.

Theorem 5.4.6. *[LM85, p.37]*

$$\Gamma \textit{ scheme n-rigid} \implies \Gamma \textit{ n-reductive}$$

Sketch of proof

Let $\mathfrak{X} = R(\Gamma, \mathrm{GL}_n)$ and suppose that Γ is scheme n-rigid. Then, any $\rho \in \mathfrak{X}(k)$ is scheme rigid and we have

$$H^1(\Gamma, \mathrm{Conj} \circ \rho) = 0$$

We define the Zariski topology on the representation variety via the subspace topology

$$R(\Gamma, \mathrm{GL}_n(k)) \subset k^m$$

Then it can be shown that, for any $\rho' \in R(\Gamma, \mathrm{GL}_n(k))$ such that

$$H^1(\Gamma, \mathrm{Conj} \circ \rho') = 0$$

we have $O(\rho')$ is open with respect to the topology. This was originally proven in [Wei64], see also [LM85, p. 36]. Moreover, $R(\Gamma, \mathrm{GL}_n(k))$ is quasi-compact. In particular, every open cover of the space admits a finite subcover and the open orbits form a cover for the representation variety. Thus $R(\Gamma, \mathrm{GL}_n(k))$ consists of a finite disjoint union of these open orbits and since they are disjoint, the orbits are also closed with respect to the topology. Finally, a representation ρ is semi-simple if and only if its orbit $O(\rho)$ is closed [LM85, p. 25]. Since every representation $\rho \in \mathfrak{X}(k)$ is in some closed orbit, the orbit $O(\rho)$ is also closed. Thus, every representation is semi-simple and Γ is indeed n-reductive, as required.

Q.E.D.

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