Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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Representations and Character Varieties of Frieze Groups

Master's thesis in Mathematical Sciences
Supervisor: Markus Szymik
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Contents

1 Introduction

The goal of this master thesis is to understand the representation varieties, and the character varieties of the frieze groups. This means we want to understand how the symmetries of patterns on strips of infinite length (but not height) can be represented in the space of linear isomorphisms on a finite dimensional vector space. Up to isomorphism the frieze groups are:

$$\mathbb{Z}, \mathbb{Z}/(2) \times \mathbb{Z}, D_\infty, D_\infty \times \mathbb{Z}/(2).$$

The last of these is not discussed in this text. The representation variety of $\mathbb{Z}$ is just Gl_n itself which is well-studied. The representation variety of this is one of the classical Lie groups. We recap what the invariant polynomials of this group are under conjugation and that the character variety $X_n(Gl_n) \cong (Gl_1) \times \mathbb{C}^{n-1}$. We also explore $\mathbb{Z}/(2)$ and its representation varieties since it is crucial to understand this group before you can understand the latter three frieze groups.

In section 2 we briefly overview the some basic algebraic geometry. We touch on affine varieties, coordinate rings, linear algebraic groups and representation varieties.

In section 3 we discuss involutions in the general linear group of a vector space V over a field of characteristic not 2. We prove that they are in general exactly the linear operators A with $V = E_1^A \bigoplus E_{-1}^A$. We then prove that if V is a finite dimensional vector space over $\mathbb{C}$ then the space of involutions equals the disjoint union

$$\coprod_{i=0}^{n} Gl_n//(Gl_{n-i} \times Gl_i)$$

with the group action being left multiplication.

In section 4 we tackle $R_n(\mathbb{Z}/(2) \times \mathbb{Z})$. We define a collection of principal $Gl_{n-k} \times Gl_k$ bundles $IC_{n,k} \to Gl_n//(Gl_{n-k} \times Gl_k)$. We prove that they are n^2 dimensional complex smooth manifolds. We prove that $R_n(\mathbb{Z}/(2) \times \mathbb{Z})$ is a disjoint union of

$$\coprod_{i=0}^{n} IC_{n,i}.$$

In section 5 we give a brief introduction to Geometric Invariant Theory (GIT) and character varieties. We prove that $X_n(IC_{n,k}) \cong X_n(Gl_n) \cong (Gl_1) \times \mathbb{C}^{n-1}$.

In section 6 we give results that compute certain low dimensional components of $X_n(D_\infty)$. We use these to show that $X_2(D_\infty)$ and $X_3(D_\infty)$ are disjoint unions of some number of $\mathbb{C}$ and $\{*\}$.

In the last section we define group cohomology. We explain why it is useful for the computation of the tangent space at character varieties. We then give an equivalent condition for irreducible points of $X_n(D_\infty)$ and show these have a tangent space of dimension 1. We lastly find the invariant polynomials of Gl_k acting on $M_{l\times k} \times M_{k\times l}$ with $G \cdot (P, Q) = (PG^{-1}, GQ)$ when $k \geq l$.

We now discuss the what the frieze groups are, and why they are interesting in geometry.

1.1 Frieze Groups

In plane geometry we sometimes study the symmetries on a given 2-dimensional figure or pattern. A pattern that repeat infinitely in one direction is called a **frieze pattern**, the group of symmetries on a frieze pattern is called a Frieze group. Such patterns are common in both art and architecture. See section 3.4 of A Course in Modern Geometries by Judith M. Cederberg [2] for a more precise definition and breakdown of frieze groups. Our breakdown will certainly give an explanation as to why the groups stated are the appropriate symmetry groups, but we will largely rely on intuition in lieu of rigorous proofs.

We now give some examples of types of patterns that can arise, these examples are representative of all the frieze patterns.

Example 1.1 (Hop)**.** The first pattern type is the most restricted one. Hop refers to someone hopping forward on one leg.

. . . F F F F F F F F . . .

We can not rotate or reflect it vertically or horizontally. The only thing we can do is to shift the pattern forward or pull it back a finite number of times. This describes the infinite cyclic group $\mathbb{Z}$.

Example 1.2 (Step)**.** The name is step is from each piece of the pattern being opposite symmetric feet as if you are walking.

. . . L Γ L Γ L Γ L Γ . . .

The first symmetry here is the one where you slide each character forward two steps. We notice that we can not reflect it horizontally or rotate it. We can reflect it horizontally however if we then push it forward a character. Two of these motions will generate the first proposed symmetry. Thus, this group is also the infinite cyclic group $\mathbb{Z}$.

Example 1.3 (Slide)**.** This group is slightly more complicated.

$$\cdots \bigvee \bigvee \bigvee \bigvee \bigvee \bigvee \bigvee \bigvee \cdots$$

We cannot rotate or reflect horizontally, but we can reflect vertically either in middle of a pattern or between two of them, as well as pushing it horizontally. We call gliding the figure forward one step G, we fix on of the $\bigvee$ and call the reflection in the middle of it R_V. We notice that any vertical reflections in the middle of a $\bigvee$ can be expressed as $R_V G^n$ for reflecting between the $\bigvee$ $n-1$ and n steps to the right of the initial $\bigvee$. If the reflection is between the $\bigvee$ n and $n-1$ steps to the left you apply $G^n R_V$ If you want to reflect in the middle of a $\bigvee$ n steps to the right of $\bigvee$ you apply $G^n R_V G^{-n}$. It can be shown that $R_V G R_V = G^{-1}$. The group is freely generated by R, GR with relations that $R^2 = e, (GR)^2 = e$. Thus the symmetry group is $D_\infty := \langle a, b\rangle/(a^2, b^2)$.

Example 1.4 (Spinning Hop)**.** This group has a similar setup as the last one.

$$\ldots \text{S S S S S S S S} \ldots$$

Here it is allowed to glide it, but we cannot reflect in either direction. Rotations at or between the repeating patterns is fine though. In much the same approach as with Slide we can show that the symmetry group is $D_\infty = \langle a, b \rangle / (a^2, b^2)$.

Example 1.5 (Spinning Slide)**.** The final group with this symmetry group.

$$\cdots \vee \wedge \vee \wedge \vee \wedge \vee \wedge \cdots$$

This group is generated by glide-reflects and rotations between the patterns and vertical reflection in the middle of a pattern. The symmetry group is $D_\infty = \langle a, b \rangle / (a^2, b^2)$.

Example 1.6 (Jump)**.** This group is relatively simple compared to the last three.

$$\ldots \text{E E E E E E E E} \ldots$$

It is not permitted with rotations or vertical reflection, but the horizontal one is permitted. It is important to note that this reflection performed twice is the identity, and it also commutes with the glides. Thus the symmetry group is $\langle a, b \rangle / (a^2, ab = ba) = \mathbb{Z}/(2) \times \mathbb{Z}$.

Example 1.7 (Spinning Jump)**.** In the last one everything goes.

$$\ldots \text{H H H H H H H H} \ldots$$

We can rotate and vertically reflect between and in the middle of patterns. We can also reflect horizontally and glide. We note that rotations can be expressed by combining two different rotations at the same point, the ordering does not matter. The horizontal reflections commute with all the other symmetries. We can thus think of it as Slide, but with one extra generator that commutes with the others. The group of symmetries is thus $\langle a, b, c \rangle / (a^2, b^2, c^2, ac = ca, bc = cb) = D_\infty \times \mathbb{Z}/(2)$.

Figure 1: Nidarosdomen

Example 1.8. The church Nidarosdomen resides a few hundred meters from NTNU. There are multiple frieze patterns one could analyze here. On the roof between the two towers there is a railing, if it were to infinitely continue in both directions it would be a frieze pattern. This pattern allows both

Figure 2: Railing with a frieze pattern

horizontal and vertical reflections, it therefore also allows rotations. This is a pattern of type spinning jump, it has the frieze group $D_\infty \times \mathbb{Z}/(2)$. It is a good thing to know that I am not the only mathematician who has distracted themselves with math during a particularly boring sermon.

Example 1.9. Consider the following pattern which you can find at the glass doors of BUEN Kulturhus in Mandal. This pattern is selected since it

Figure 3: Frieze pattern on glass door

is printed on the door to the room this is being written. The pattern is the following stylized way of writing BUEN repeated:

$$\dots \supset \cup \subset \cap \supset \cup \subset \cap \supset \cup \subset \cap \dots$$

It is "$\supset \cup \subset \cap$" repeating over and over? Yes, but it is also the pattern "$\cup \subset \cap \supset$" repeated over and over. This reveals a subtlety in frieze patterns.

They are not just one pattern repeated over and over, you can look at it as an infinite set of patterns depending on where it is started. We want the midpoint of the pattern to be the midpoint of $\cup$ or $\cap$. We can vertically reflect in the middle and between those patterns. It is of type slide and has the symmetry group D_∞.

2 Representation varieties

In this section we are going to give a short introduction to some of the central concepts of this text.

2.1 Affine varieties

Roughly speaking, algebraic geometry is the study of geometric objects that can be expressed using polynomial equations. We take a collection of polynomial equations $S \subseteq \mathbb{K}[X_1, \dots X_n]$ and we compute the zeros of those equations. In the following definition we very quickly set up the basic framework for algebraic geometry this setup is due to Perrins book [8].

Definition 2.1. *Let $\mathbb{K}$ be any field. We define the following functions on the power sets*

$$V : \mathcal{P}(\mathbb{K}[X_1, \dots X_n]) \to \mathcal{P}(\mathbb{K}^n), V(S) = \{x \in \mathbb{K}^n | f(x) = 0, \forall f \in S\}$$
$$I : \mathcal{P}(\mathbb{K}^n) \to \mathcal{P}(\mathbb{K}[X_1, \dots X_n]), I(V) = \{f \in \mathbb{K}[X_1, \dots X_n] | f(x) = 0, \forall x \in V\}$$

A subset $V \subseteq \mathbb{K}^n$ is an ***Affine Algebraic Variety*** *if there exist some subset $S \subseteq \mathbb{K}[X_1, \dots X_n]$ such that $V = V(S)$. For the purposes of this text we will simply call these varieties.*

We also define the ***coordinate ring*** *of a variety:*

$$\mathcal{O}(V) := \mathbb{K}[X_1, \dots X_n]/(I(V))$$

The notation here might seem confusing since we are using the same letters for both functions and sets, but the idea is that you have an algebraic and geometric way of interpreting an object. If you have a subset of $\mathbb{K}[X_1, \dots X_n]$ you can turn it into a variety, and if you have a subset of $\mathbb{K}^n$ you can turn those into an ideal of $\mathbb{K}[X_1, \dots X_n]$, if you quotient out that ideal you get a $\mathbb{K}$ algebra called the coordinate ring. If I is the ideal generated by the set S then $V(I) = V(S)$. We can therefore think of the sets S as ideals, since they have the same variety as the ideal they generate. We will sometimes write $V(f)$ or $V(f = g)$ rather than the more formal $V(\{f\})$ or $V(\{f - g\})$.

What happens if you keep going? More precisely, what are the functions $V \circ I$ and $I \circ V$? If we have $x \in V$ then $\forall f \in I(V), f(x) = 0$, therefore $x \in V(I(V))$. If $f \in I$, then $\forall x \in V(I), f(x) = 0$ and therefore $f \in I(V(I))$. We can therefore conclude that we have inclusions $V \subseteq V(I(V)), I \subseteq I(V(I))$. The first of which is an equality if and only if V is a variety. Assuming that $\mathbb{K}$ is algebraically closed, the second is an equality if and only if I is radical as a consequence of Hilbert's Nullstellensatz, furthermore the induced coordinate ring is a reduced commutative $\mathbb{K}$ algebra. If we limit ourselves to radical ideals and varieties the functions V and I are inverses of each other.

When establishing a theory of mathematics we are generally interested in both objects as well as maps between those objects.

Definition 2.2. *Let $V \subseteq \mathbb{K}^n$ and $W \subseteq \mathbb{K}^m$ be varieties, a function $\phi : V \to W$ is called* ***regular*** *if there are polynomials $f_1, \dots f_m \in \mathbb{K}[X_1, \dots, X_n]$ such that $\forall x \in V$*

$$\phi(x) = (f_1(x), \dots, f_m(x))$$

Now we have established two categories. One is the category of reduced finitely generated $\mathbb{K}$ algebras. The other category is that of affine varieties, with morphisms being the regular maps. If $\mathbb{K}$ is algebraically closed, for every variety there is a reduced finitely generated reduced $\mathbb{K}$ algebra. The reason we are restricting ourselves to talking about regular maps is that they very neatly induce morphisms between algebras. Take a regular map $\phi : V \to W$. The induced map is $\phi^* : \mathcal{O}(W) \to \mathcal{O}(V), \phi^*(f) = f \circ \phi$. The construction is functorial.

2.2 The Zariski Topology

For both real and complex space we have a standard notion of topology on that space. This topology is often referred to as euclidean topology for real space, the notion can be expanded to complex space by noticing that $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. There is also a topology for affine sets.

Definition 2.3. *A subset $U \subseteq V$ of an affine variety is* ***Zariski closed*** *or just* ***closed*** *if it is itself an affine variety. We say that U is* ***Zariski open*** *or just* ***open*** *if $U \backslash V$ is Zariski closed.*

This defines a topology on V. The empty set is a variety since $\emptyset = V(S)$ for some set S that contains a non-zero constant polynomial, and V is a variety so $V = V(I(V)$. We can take arbitrary intersections of closed sets $\bigcap_{i \in \Lambda} V_i = V(\bigcup_{i \in \Lambda} I(V_i))$ and still get a variety. If you want to take the union of two closed sets you can multiply the polynomials in each set.

Remark 2.4. If we are working over $\mathbb{C}$ then the Zariski topology is a coarser topology compared to the euclidian one, see chapter 1.10. of The Red Book [7]. This is a very useful piece of information that shall be used on several occasions.

2.3 Linear Algebraic Groups

An affine algebraic group is an object in the category of affine varieties that is also a group. Meaning that G needs to be both an affine variety as well as a group, the multiplication map $m : G \times G \to G$, and inversion map need to both be regular.

The most important example that we are going to be using is $Gl_n(\mathbb{K})$, for ease of notation we are going to be using Gl_n. Gl_n is the set of linear isomorphisms over $\mathbb{K}^n$, we typically represent this as the set of invertible $n \times n$ matrices over $\mathbb{K}$. From classic linear algebra we know that a matrix is invertible if and only if it has a nonzero determinant. We may be tempted to identfty Gl_n with some subset of $M_n := M_{n \times n}$, the set of all $n \times n$ matrices over $\mathbb{K}$. The problem is that $\det(M) \neq 0$ is not a polynomial equation. The solution to this problem is quite elegant, we instead think of it as the determinant being invertible. We can define $Gl_n \subseteq \mathbb{K}^{n^2+1}$ as $V(\det(M)t = 1)$. The t will be uniquely determined by the other entries $t = \det(M)^{-1}$. For this reason we will often not think that much of t when doing computations, what t is follows from the other $X_{i,j}$. Therefore Gl_n an n^2 dimensional space even if it is a subset of an n^2+1 dimensional space.

It may not be obvious that this group is an affine group, as it is not immediately clear that inversions are regular. By looking at how this works for Gl_2 we get a good idea at how the group works in general. Any $M \in Gl_2$ is of the form

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, ad - bc \neq 0$$

we can algebraically express the inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} td & -tb \\ -tc & ta \end{bmatrix} = \begin{bmatrix} t(ad-bc) & 0 \\ 0 & t(ad-bc) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

thus the inversion map is regular; $i(a,b,c,d,t) = (td, -tb, -tc, ta, ad - bc)$.

A linear algebraic group is a zariski closed subgroup of some Gl_n. For more details on linear algebraic groups and Gl_n, see Nolan Walach's book [14].

2.4 Representation varieties

Definition 2.5. *Let Γ be a group. A Gl_n* ***representation of*** *Γ is a group homomorphism $\rho : \Gamma \to Gl_n$.*

It is in general unclear that the set of group representations should be variety. We need to put some restrictions on Γ for this to be the case. One assumption could be that it should be a finite group. Much is to be said about the representations of finite groups, but we can actually widen our scope and talk about finitely generated groups. For the remainder of this thesis Γ will be used to refer to a finitely generated group. Let $a_1, \ldots, a_k$ be a set of generators for Γ. Then $Hom(\Gamma, Gl_n)$ can be described by using $(\rho(a_1), \ldots, \rho(a_k))$. The set of all these representations is a variety; we are asking for a collection of k matrices that satisfy a set of relations. Each relation simply amounts to n^2 equations. We shall denote the representation variety as $(\Gamma, Gl_n) := R_n(\Gamma)$.

3 Involutions in Gl_n

In this section we will classify the involutions on vector spaces. We pay special attention to the finite dimensional case and the matrices that are involutions. For reasons that shall be explained a little later we will assume that $\mathbb{K}$ is a field with characteristic different from 2. Towards the end of the chapter we will start assuming that $\mathbb{K} = \mathbb{C}$.

Definition 3.1. *An involution on a vector space V is a linear transformation $A : V \to V$ such that $A^2 = I$.*

This is equivalent to a linear transformation being its own inverse $A^{-1} = A$. This means that in particular involutions are invertible. The n-th representation variety of $\mathbb{Z}/(2)$ is the set of involutions in Gl_n, this description is of course inadequate for our purposes. We are interested in describing it from a geometric point of view. A good place to start would be to compute some of these varieties.

Example 3.2. $\hom(\mathbb{Z}/(2), Gl_1) = \{1, -1\}$.

Example 3.3. We can compute $Hom(\mathbb{Z}/(2), Gl_2)$ as the set of two by two matrices that satisfy:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By comparing the upper left and lower right entry we see that $a^2 = d^2$, either $a = d$ or $a = -d$. If we assume that they are equal and non-zero then we can look at the upper right and lower left entry to see that $2ac = 0$ and $2ab = 0$. We assumed that $a \neq 0$ so $b = c = 0$. We are left with $A = aI$, $I = A^2 = a^2 I$ so either $a = 1$ or $a = -1$. We thus obtain that either $A = I$ or $A = -I$.

If we then assume that instead $a = -d$ (this encapsulates $a = d = 0$), what then follows is that $ac + cd = ac - ac = 0 = ab - ab = ab + bd$. In this case the upper right and lower left are immediately zero and we only need $a^2 + bc = 1$. We can therefore express our variety as a coproduct $\hom(\mathbb{Z}/(2), Gl_2) = \{I\} \coprod \{a, b, c, d | a + d = 0, a^2 + bc = 1\} \coprod \{-I\}$. It seems as if the representation is two isolated points and a two dimensional conic.

If we want to learn more about A is always a good idea to find the eigenvalues of A. We know that all the eigenvalues are non-zero since A is invertible. An eigenvalue must satisfy

$$\begin{aligned} Ax &= \lambda x \\ x &= \lambda Ax \\ Ax &= \frac{1}{\lambda} x \\ \lambda x &= \frac{1}{\lambda} x \\ \lambda^2 &= 1 \end{aligned}$$

λ is either 1 or -1. What this means is that we have two eigenspaces E_1^A, E_{-1}^A. It would be most convenient for us if the eigenspaces spanned all of $\mathbb{K}^n$. This can be proved in general, not just of finite dimensional vector spaces. At this point we note that if $\mathbb{K}$ has char 2 then it only has one eigenvalue $\lambda = 1$.

Lemma 3.4. *If A is an involution on vector space V then $\frac{A+I}{2}$ is a projection onto E_1^A with kernel E_{-1}^A. Similarly $\frac{-A+I}{2}$ is a projection onto E_{-1}^A with kernel E_1^A.*

Proof. Let us verify that for all vectors $x \in V$ we have that $\frac{Ax+x}{2}$ and $\frac{-Ax+Ax}{2}$ are eigenvectors. By applying A we see that:

$$A\Big(\frac{Ax + x}{2}\Big) = \frac{Ax + x}{2}, A\Big(\frac{-Ax + x}{2}\Big) = \frac{Ax - x}{2}$$

they map onto the eigenspaces. They are also projections, let $x \in E_1^A, y \in E_{-1}^A$, then

$$\frac{A + I}{2}(x) = \frac{x}{2} + \frac{x}{2} = x, \frac{-A + I}{2}(y) = \frac{y}{2} + \frac{y}{2} = y$$

The final claim is that the kernels of each map is the other eigenspace. $\frac{A+I}{2}(x) = 0$ if and only if $Ax = -x$, similarly $\frac{-A+I}{2}(x) = 0$ if and only if $Ax = x$, which verifies the last claim. $\square$

We are going to denote $P_1^A = \frac{A+I}{2}, P_{-1}^A = \frac{-A+I}{2}$.

Theorem 3.5. *A linear transformation A is an involution on a vector space V if and only if $V = E_1^A \bigoplus E_{-1}^A$.*

Proof. We first assume that A is an involution. For any $x \in V$, we have that $x = I(x) = P_1^A(x) + P_{-1}^A(x)$. Combining this with $E_1^A \bigcap E_{-1}^A = 0$ we have $V = E_1^A \bigoplus E_{-1}^A$.

We then assume that $V = E_1^A \bigoplus E_{-1}^A$. For any $v \in V$ we can find unique $x \in E_1^A, y \in E_{-1}^A$ such that $v = x + y$. We get

$$A^2(v) = A(A(v)) = A(A(x+y)) = A(x-y) = x+y = v.$$

A is an involution. $\square$

This theorem says that involutions are linear maps that split a space into two subspaces where it eiter acts as the identity or the negative of the identity. The theorem also describes involutions in Gl_n.

Corollary 3.6. *A matrix $M \in Gl_n$ is an involution if and only if it is diagonalizable with entries 1 and -1.*

Proof. If $A = MDM^{-1}$ with D being a diagonal matrix with entries $1, -1$, then $A^2 = MDM^{-1}MDM^{-1} = MD^2M^{-1} = I$. If we assume that A is an involution then it is diagonalizable with entries $1, -1$ since the eigenspaces corresponding to those values span $\mathbb{K}^n$. $\square$

We are going to denote the $n \times n$ diagonal matrix with the last k entries being -1 and the rest 1 as J_k. For two dimensions it means that the involutions are the conjugacy classes of the following matrices

$$J_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, J_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, J_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since I and $-I$ commute with all matrices their conjugacy classes are only themselves. With both of these observations we can deduce the following

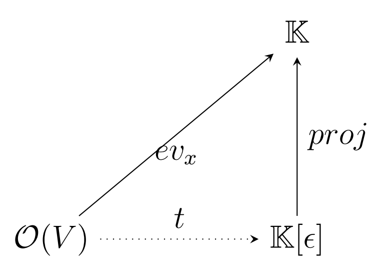
$$\{M \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} M^{-1}\} = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} | a + d = 0, a^2 + bc = 1\}$$

We are interested in finding out if we can describe the conjugacy classes of all involutions in such algebraic terms.

3.1 The tangent space of $R_n(\mathbb{Z}/(2))$

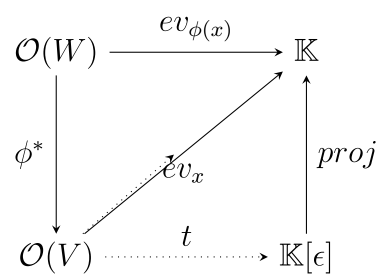
One of the tools of algebraic geometry is the correspondence between the geometric dimension of an object and the algebraic dimension of the corresponding algebraic object. The tangent space of a scheme at a point provides a bound for the dimension at that point, if we can also find a sequence of increasing prime ideals that has the same length as the tangent space at the point then we have found the dimension and proven that the point is smooth.

Definition 3.7. *For a field $\mathbb{K}$ the dual numbers $\mathbb{K}[\epsilon]$ are the polynomials over ϵ with the relation that $\epsilon^2 = 0$. Let V be an affine variety. A **tangent vector** at a point x is an algebra morphism $t : \mathbb{K}[X_i]/I(V) \to \mathbb{K}[\epsilon]$ making the following diagram commute:*



The morphism t is defined by where it sends the generators, $t(X_i) = X_i(x) + a\epsilon$. If u is another tangent tangent vector with $t(X_i) = X_i(x) + b\epsilon$, we can define addition by $(t + u)(X_i) = X_i(x) + (a + b)\epsilon$. We denote this space by T_xV. If $\phi : V \to W$ is a regular map of varieties, then we can define

$T\phi : T_{\phi(x)}W \to T_xV$ by sending a map $t : \mathbb{K}[X_i]/I(V) \to \mathbb{K}[\epsilon]$ to the map $t \circ \phi^*$.



The definition works since the top triangle commutes, and finding a t such that the bottom triangle commutes makes the entire square commute.

Example 3.8. Let us look at $V(a^2 + bc = 1)$ over $\mathbb{C}$, it has the coordinate ring $\mathbb{C}[a, b, c]/(a^2 + bc - 1)$. For simple curves and surfaces it is often wise to instead use a more classical approach. The kernel of the jacobian provides an equivalent formulation of the tangent space [8]. We compute the Jacobian

$$\begin{bmatrix} 2a & c & b \end{bmatrix}.$$

Since this is never a zero vector when $a^2 + bc = 1$, the kernel of this map and hence the dimension of the tangent space is 2. We also have this following sequence of prime ideals

$$(0) \subset (a, bc - 1)/(a^2 + bc - 1) \subset \mathbb{C}[a, b, c]/(a^2 + bc - 1)$$

which implies the dimension of the variety is at least 2, combining this with the tangent space dimension being 2 tells us that that it is a smooth variety of dimension 2.

Example 3.9. Another illustrative example is the tangent space of $Gl_n(\mathbb{C})$. This is a frieze group representation since $R_n(\mathbb{Z}) = Gl_n$. The tangent space can be computed directly from the definition. A tangent vector at a point M sends each generator $X_{i,j}$ to $M_{i,j} + N_{i,j}\epsilon$. Combining all $N_{i,j}$ we get a matrix such that $M + N\epsilon$ needs to be invertible inside of $M_n(\mathbb{C}[\epsilon])$. For any choice of N, this is invertible by

$$(M + N\epsilon)(M^{-1} - M^{-1}NM^{-1}\epsilon) = I + (NM^{-1} - NM^{-1})\epsilon = I.$$

Thus $T_M(Gl_n)$ is n^2 dimensional at all points.

Remark 3.10. Smoothness is a very desired property in algebraic geometry. A smooth point is one in which the dimension at a point is the same as the dimension of the tangent space. The definition of dimension at a point is something we do not deal with directly in this text, but can be found in any number of texts including Perrin [8]. If an irreducible variety is smooth and of the same dimension at every point, then it is a smooth real/complex variety depending on the field [10]. A common technique we shall use in this text is to show that a variety over $\mathbb{C}$ is smooth by showing that the tangent space at every point in the variety has the same dimension. In Harris' book chapter 14 the tools to show this are given. We combine two results, the dimension of the tangent space is an upper-semicontinuous function, and the set of smooth points is dense in the variety. The first tells us that the dimension will not locally "jump down", the second tells us that every neighbourhood contains a smooth point. If we assume that there is some point that is not smooth, then it must have dimension lower than the the tangent space dimension. It must be in the neighborhood of some smooth point with the same dimension as the tangent space, but that implies that the dimension has "jumped down" for the point that is not smooth.

Theorem 3.11. *Th tangent space of $R_n(\mathbb{Z}/(2))$ at an involution A which diagonalizes to J_k is a vector space of dimension $2k(n-k)$.*

Proof. We may think of a tangent vector to an involution A as another matrix B which satisfies the following computation in $\mathbb{K}[\epsilon]$.

$$I = (A + B\epsilon)^2 = A^2 + (AB + BA)\epsilon + B^2\epsilon^2 = I + (AB + BA)\epsilon \iff AB = -BA$$

We are going to once again exploit the diagonlization of A.

$$AB = -BA \iff MJ_kM^{-1}B = -BMJ_kM^{-1} \iff J_kM^{-1}BM = -M^{-1}BMJ_k$$

When you multiply with J_k from the left you preserve the top $n-k$ rows and invert the bottom k, multiplying form the right does the exact same thing but for columns. This means that $M^{-1}BM$ is a matrix of the form

$$F = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$$

with C being a $k \times (n-k)$ matrix and D a $(n-k) \times k$ matrix, we denote this space as $M_n/(M_{n-k} \times M_k)$. Thus the tangent matrices of A is the $2k(n-k)$ dimensional vector space $M(M_n/(M_{n-k} \times M_k))M^{-1}$. □

It would be nice if we could have all the points in our variety have the same dimension, this is obviously not the case. The question is if there is some way we could restrict $R_n(\mathbb{Z}/(2))$ to a specific conjugacy class of the variety. The answer is yes! We only need to recall that the trace of a matrix is equal to the sum of its eigenvalues. This means that if we want a smooth variety we need only consider $Inv_{n,k} := R_n(\mathbb{Z}/(2)) \cap V(\sum_{i=1}^{n} X_{i,i} = n - 2k)$. The individual $Inv_{n,k}$ are thus disjoint closed subsets of $R_n(\mathbb{Z}/(2))$, meaning that $R_n(\mathbb{Z}/(2))$ is in fact a disjoint union of all $Inv_{n,k}$. We also note that when $k < n/2$ the space $Inv_{n,n-k}$ contains exactly the negative of the matrices in $Inv_{n,k}$ and vice versa. Dividing the space like this tells us that the representation is a disjoint union of smooth spaces, hence it is itself smooth.

Corollary 3.12. $R_n(\mathbb{Z}/(2))$ *is a smooth variety.*

Proof. By theorem 3.11 the tangent space has the same dimension for every point in $Invn, k$, the variety must therefore be of dimension $2k(n-k)$. $R_n(\mathbb{Z}/(2)$ is smooth since it is a disjoint union of smooth spaces. □

3.2 The bundle structure on involutions

It seems natural that when we discuss the topology on $R_n(\mathbb{Z}/(2)$ we look at all the $n+1$ different conjugacy classes separately. Since we know that any involution is of the form MJ_kM^{-1} we could try to find a relation that determines when two such matrices are equal. When talking about the linear algebraic group $Gl_{n-k} \times Gl_k$ we are thinking about it as a subgroup of Gl_n in the following way $Gl_{n-k} \times Gl_k = Gl_n \bigcap_{i>k,j<n-k} V(X_{i,j} = 0) \bigcap_{i<n-k,j>k} V(X_{i,j} = 0)$. Note that this becomes a Zariski closed subset of Gl_n

Lemma 3.13. *$MJ_kM^{-1} = NJ_kN^{-1}$ if and only if there exists some $T \in Gl_{n-k} \times Gl_k \subseteq Gl_n$ such that $MT = N$*

Proof. We have that

$$MJ_kM^{-1} = NJ_kN^{-1} \iff N^{-1}MJ_k = J_kN^{-1}M$$

Let $T = N^{-1}M$, then

$$[TJ_k]_{i,j} = \begin{cases} [T]_{i,j} & \text{for} \quad j < k \\ -[T]_{i,j} & \text{for} \quad j \geq k \end{cases}$$

$$[J_kT]_{i,j} = \begin{cases} [T]_{i,j} & \text{for} \quad i < k \\ -[T]_{i,j} & \text{for} \quad i \geq k \end{cases}$$

This is the case if and only if T is of the form

$$\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$$

with $T_1 \in Gl_k$ and $T_2 \in Gl_{n-k}$. Thus we have $T \in Gl_{n-k} \times Gl_k$ such that $NT = M$. □

This theorem gives us a bijection from the quotient of Gl_n with the relation $g \cdot T \sim g, T \in Gl_{n-k} \times Gl_k$ to the conjugacy class of J_k. It should be noted that quotients of affine varieties are not varieties in general, not even when the relation is induced by a group action. It is however the case for this quotient. For the time being we will define $Gl_n//(Gl_{n-k} \times Gl_k)$ as the quotient of Gl_n using the relation defined above. The associated scheme $\mathcal{O}(Gl_n)^{Gl_{n-k} \times Gl_k} \subseteq \mathcal{O}(Gl_n)$ is defined as the polynomials left invariant by the multiplication action $(g \cdot f)(v) = f(g^{-1} \cdot v)$.

Gl_n is an irreducible variety and a smooth complex manifold. It being irreducible tells us that the coordinate ring is an integral domain. Thus the subalgebra $\mathcal{O}(Gl_n)^{Gl_{n-k} \times Gl_k} \subseteq \mathcal{O}(Gl_n)$ is also integral, hence $Gl_n//(Gl_{n-k} \times Gl_k)$ is also irreducible.

Let us define the following map $h_K : Gl_n//(Gl_{n-k} \times Gl_k) \to Inv_{n,k}, h_k(M) = MJ_kM^{-1}$. This map is a well defined regular bijection because of lemma 3.13. The map also respects the Gl_n action on both spaces:

$$h_k(g \cdot M) = h_k(gM) = gMJ_kM^{-1}g^{-1} = g \cdot MJ_kM^{-1} = g \cdot h_k(M).$$

This is surjection, thus $Inv_{n,k}$ is irreducible and a smooth manifold.

The fact that a bijection exists is not terribly interesting, but this one respects the following group actions. Let us consider the following Gl_n actions on $Gl_n//Gl_{n-k} \times Gl_k$ and $Inv_{n,k}$: let $g \in Gl_n$, $g \cdot M = gM$ and $g \cdot A = gAg^{-1}$. Recall that we defined the quotient og Gl_n using right multiplication, this means that multiplication on the left is a well defined action. The second group action could also be defined on the entire space $R_n(\mathbb{Z}/(2))$, but this action would not be transitive, which it is on $Inv_{n,k}$. We prove that it is an isomorphism.

Theorem 3.14. *Let $\mathbb{K} = \mathbb{C}$. The map $h_k : Gl_n//Gl_{n-k} \times Gl_k \to Inv_{n,k}, f(M) = MJ_kM^{-1}$ is an isomorphism of Gl_n spaces.*

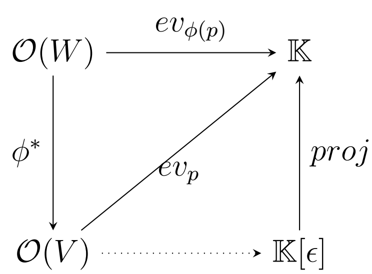
Proof. To prove this we are going to use the Inverse function theorem. We have already shown that h_k is a bijectiion that respects the group action. It is enough to verify that we have an isomorphism on the tangent space at a single element, since we have a transitive group action (see theorem 5.3.3. [13]).

$$T_I(Gl_n//Gl_{n-k} \times Gl_k) = T_I(Gl_n)/T_I(Gl_{n-k} \times Gl_k) = M_n/(M_{n-k} \times M_k)$$

Recall from theorem 3.11 that the tangent space at $f(I)$ is $M_n/(M_{n-k} \times M_k)$. If N is some tangent vector, then N is sent to $NJ_k - J_kN$. The tangent space linear map is a bijection. $\square$

Despite $Gl_{n-k} \times Gl_k$ not being compact it still acts nicely on Gl_n. By the fact that it is a stabilizer of each element is trivial it is a principal $Gl_{n-k} \times Gl_k$-bundle by [12] proposition 4.7.

Let us take some time to understand these constructions, as well as state some corollaries. The tangent space morphism is quite interesting in this context. If we have a regular map of varieties $\phi : V \to W$ we get an induced morphism of schemes $\phi^* : \mathcal{O}(W) \to \mathcal{O}(V)$ which is defined in the following way $\phi^*(g) = g \circ \phi$. This gives us a map of tangent spaces. Let the dashed arrow be a tangent at $p \in V$, the precomposing with ϕ^* is a tangent at $\phi(p)$



Example 3.15. Consider the algebras $\mathbb{C}[a, b, c]/(a^2+bc-1)$ and $\mathbb{C}[a, b, c, d, t]^{Gl_1 \times Gl_1}/((ad-bc)t - 1)$. The second is the collection of $Gl_1 \times Gl_1$ In this example we can explicitly compute the induced morphism. We just proved that the regular map

$$f\Big(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\Big) = t \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = t \begin{bmatrix} ad + bc & -2ab \\ 2cd & -(ad + bc) \end{bmatrix}$$

is an isomorphism of varieties. The induced map of algebras is thus also an isomorphism. This approach tells us that we can understand $\mathbb{C}[a, b, c, d, t]^{Gl_1 \times Gl_1}/((ad-$

$bc)t-1)$ by looking at the generators of $\mathbb{C}[a,b,c]/(a^2+bc-1)$.

$$h_1^*(a) = adt + bct$$
$$h_1^*(b) = -2abt$$
$$h_1^*(c) = 2cdt$$

These are generators. They are invariant under our group action. The group action is multiplying with a diagaonal matrix G, meaning we multiply each row with a scalar. The group action multiplies a and c with a with the first diagonal entry, b and d with the second diagonal entry, and t is multiplied with the inverse of both. Thus we get

$$G \cdot adt + bct = adt + bct$$
$$G \cdot abt = abt$$
$$G \cdot cdt = cdt.$$

Corollary 3.16. *The isomorphism from 3.14 tells us that the collection of* $h_k^*(X_{i,j})$ *is a set of generators of* $\mathcal{O}(Gl_n)^{Gl_{n-k}\times Gl_k}$.

Proof. This is a simple matter of using the functoriality property. It tells us that h_k^* is an isomorphism, thus it maps generators to generators. □

4 $\mathbb{Z}/(2) \times \mathbb{Z}$ representations

Now that we understand $R_n(\mathbb{Z}/2)$ we can apply this knowledge to compute the representation variety of the other frieze groups. $\mathbb{Z}/(2) \times \mathbb{Z}$ can be understood as a group with two generators and the relation that one of them is an involution and that the other must commute, meaning $\mathbb{Z}/(2) \times \mathbb{Z} \cong \langle a, b\rangle/(a^2, aba^{-1}b^{-1})$. We are going to denote

$$C_{Gl_n}(A) := \{B \in Gl_n | AB = BA\}.$$

This is a closed subset, in particular it is a linear algebraic group. The dimension in general depends on the jordan normal form of A (See Basic Abstract Algebra [1] for a deeper dive into jordan normal forms). The case for when A is an involution is simpler.

Proposition 4.1. *Let $A = MJ_kM^{-1} \in Inv_{n,k}$. Then $C_{Gl_n}(A) = M(Gl_k \times Gl_{n-k})M^{-1}$. Furthermore, it is a linear algebraic group of dimension $k^2 + (n-k)^2$.*

Proof. Let B be an invertible matrix that commutes with A, then

$$\begin{aligned} AB &= BA \\ MJ_kM^{-1}B &= BMJ_kM^{-1} \\ J_kM^{-1}BMJ_K &= M^{-1}BM, \end{aligned}$$

conjugating with J_k will change the sign on all elements not in the lower right $k\times k$ or upper left $(n-k)\times(n-k)$ box. Therefore all those entries must be zero in $M^{-1}BM$. Thus B commutes with if and only if $B \in M(Gl_k \times Gl_{n-k})M^{-1}$. This is a closed subspace in Gl_n of dimension $k^2+(n-k)^2$. It is an isomorphic conjugate to $Gl_{n-k} \times Gl_k$. □

The hypothesis is thus that for every point in $Inv_{n,k}$ we attach a fiber of dimension $k^2 + (n-k)^2$ to it. So the dimension at that point would be the sum of the dimension of the point in $Inv_{n,k}$ plus the dimension of the fiber: $k^2+(n-k)^2+2k(n-k) = (n-k+k)^2 = n^2$. The easiest way to demonstrate that the dimension is n^2 is through another tangent space computation.

Theorem 4.2. *For any $\rho \in R_n(\mathbb{Z}/(2)\times\mathbb{Z})$ the dimension of $T_\rho R_n(\mathbb{Z}/(2)\times\mathbb{Z})$ is n^2.*

Proof. We do the standard tangent vectors to satisfy the relations on the generators of $\mathbb{Z}/(2) \times \mathbb{Z} = \langle a, b\rangle/(a^2, ab = ba)$. We see what this means for the tangent vectors:

$$\begin{aligned}
(A + C\epsilon)^2 &= I \\
AC &= -CA \\
(A + C\epsilon)(B + D\epsilon) &= (B + D\epsilon)(A + C\epsilon) \\
AB + (AD + CB)\epsilon &= BA + (DA + BC)\epsilon \\
AD - DA &= BC - CB.
\end{aligned}$$

From theorem 3.11 and proposition 4.1 we know that there are matrices $M \in Gl_n, F \in M_n/(M_k \times M_{n-k}), G \in Gl_k \times Gl_{n-k}$ such that $C = MFM^{-1}, A = MJ_kM^{-1}$ and $B = MGM^{-1}$. We expand on $AD - DA = BC - CB$

$$\begin{aligned}
&MJ_kM^{-1}D - DMJ_kM^{-1} = MGFM^{-1} - MFGM^{-1} \\
&J_kM^{-1}DM - M^{-1}DMJ_k = GF - FG.
\end{aligned}$$

Recall that multiplying with J_k from the left changes the sign on the bottom k rows, and multiplying from the right changes the sign on the k columns to the right. We denote the bottom left $k\times n-k$ box of $M^{-1}DM$ as $(M^{-1}DM)_{k\times n-k}$ and the top right $k \times n - k$ box of $M^{-1}DM$ as $(M^{-1}DM)_{n-k\times k}$. Thus

$$J_kM^{-1}DM - M^{-1}DMJ_k = \begin{bmatrix} 0 & 2(M^{-1}DM)_{n-k\times k} \\ -2(M^{-1}DM)_{k\times n-k} & 0 \end{bmatrix},$$

This tells us that the matrix $-M^{-1}DM$ is restricted in the lower left and upper right, but can be anything in the upper left and lower right. More formally, the map that conjugates a matrix with M^{-1} then flips the sign on the first $n - k$ columns is a linear isomorphism on M_n. The dimension of the space with the top left $(n - k) \times (n - k)$ and bottom right $k \times k$ box is

$k^2 + (n-k)^2$ dimensional. The space of permitted D is the preimage of this isomorphism. Adding the dimensions of permitted C and permitted D we get $k^2 + (n-k)^2 + 2k(n-k) = (n-k+k)^2 = n^2$. □

It is possible to think of the whole space as one n^2 dimensional variety (or a manifold that is not connected), but that would not quite capture the nature of what is going on here. As discussed earlier, each point is a combination of two matrices, one from an involution space, and one from the commutator space of that matrix. The higher the dimension of the involution space, the lower the dimension of the commutator space. We want to formalize this by showing that there is a map making

$$IC_{n,k} := R_n(\mathbb{Z}/(2) \times \mathbb{Z}) \bigcap Inv_{n,k} \times Gl_n \to Inv_{n,k}$$

is a principal $Gl_{n-k} \times Gl_k$-bundle.

Theorem 4.3. *The map $IC_{n,k} \to Inv_{n,k}$ that sends*

$$(A, B) \to A$$

is a principal $Gl_{n-k} \times Gl_k$-bundle.

Proof. We want to find a $Gl_{n-k} \times Gl_k$ action such that every point has trivial stabilizer, and such that

$$IC_{n,k}//Gl_{n-k} \times Gl_k \cong Inv_{n,k}.$$

We define the following $Gl_{n-k} \times Gl_k$ action. For $A = MJ_kM^{-1} \in Inv_{n,k}$ and $G, H \in Gl_{n-k} \times Gl_k$ with $B = MHM^{-1}$. Let

$$G \cdot (A, B) = (A, BMGM^{-1}) = (A, MHGM^{-1}).$$

The stabilizer at any point is trivial. The orbit of (A, B) is $A \times C_{Gl_n}(A)$. Thus

$$IC_{n,k}//Gl_{n-k} \times Gl_k \cong Inv_{n,k}.$$

We therefore have the desired result from proposition 4.7. in [12]. □

Corollary 4.4. *$IC_{n,k}$ is an n^2 smooth complex manifold.*

Proof. By theorem 4.2 we know that every point on $IC_{n,k}$ is smooth of dimension n^2. We only need for it to be irreducible. Since $Gl_n \times Gl_{n-k} \times Gl_k$ is irreducible we need only fin a surjective morphism from $Gl_n \times Gl_{n-k} \times Gl_k$ to $IC_{n,k}$. If $M \in Gl_n, G \in Gl_{n-k} \times Gl_k$ then the map that sends

$$(M, G) \to (MJ_kM^{-1}, MGM^{-1})$$

is surjective. □

5 GIT quotients and Character varieties

In mathematics we are often interested in quotients of mathematical objects. In particular we are interested in when the quotients are objects with a similar structure. From group theory we know that the quotient of a subgroup is a group if and only if the subgroup is normal. If we want to quotient out an arbitrary subgroup we would need to normalize it.

For the category of affine algebraic schemes (or varieties) we are also interested in taking quotients of relations. Here we will discuss what happens when that relations are orbits of a group action on our space. If we have the G space V, we want to define the quotient $V//G$ such that it is the scheme for $\mathcal{O}(V)^G$. The naive quotient could fail us, here is an example from Heusener[4]:

Example 5.1. Let Gl_1 act on $\mathbb{C}^2$ with multiplication: $g{\cdot}(a, b) = (ga, gb)$. The only polynomials in $\mathcal{O}(\mathbb{C}^2) = \mathbb{C}[X, Y]$ left invariant by the group action are the constant maps, so $\mathcal{O}(\mathbb{C}^2)^{\mathbb{C}^*} = \mathbb{C}$. We would therefore want $\mathbb{C}^2//\mathbb{C} \cong \{*\}$. We observe that the orbits are the lines passing through the origin, but excluding the origin. The closures of these points are the entire lines. All of these lines intersect.

We define the orbit of an element $v \in V$ as $O(v) := \{w \in V | \exists g \in G, w = g \cdot v\}$. We write $[v] := \{w | \overline{O(w)} \cap \overline{O(v)} \neq \emptyset\}$. We say that two elements are related under GIT if the closures of their orbit intersect to avoid confusion. $[v]$ are thus the points in the GIT-quotient It is trivial that everything in some orbit is related under GIT. In the above example everything is related under GIT.

For our purposes the naivé quotient has worked until now. Take $Gl_n//(Gl_{n-k} \times Gl_k)$ as an example. $Gl_{n-k} \times Gl_k$ is a closed subset, the orbit $O(M) = \{g \cdot M\}$ is thus also closed. It is however not the case for all G-spaces, including the representation varieties of frieze groups.

Example 5.2. Consider $M_2 \cong \mathbb{C}^4$ as a Gl_2 space by $g \cdot M = gMg^{-1}$. Every matrix is conjugate to its Jordan normal form [1]. The possible jordan normal forms are

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

The set of matrices with any given eigenvalues up to multiplicity is closed. In the two dimensional case the eigenvalues can be deduced by combining the trace and determinant. The matrices with eigenvalues λ_1, λ_2 are the same as $V(a+d = \lambda_1 + \lambda_2) \bigcap V(ad - bc = \lambda_1\lambda_2)$. The polynomials $a + d, ad - bc$ also generate the invariant algebra. We now want to show that any two matrices with the same eigenvalues are related under GIT.

$$\begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & \epsilon \\ 0 & \lambda \end{bmatrix}.$$

Thus the orbit of any matrix contains the diagonal matrix with corresponding eigenvalues in its closure, and the space of matrices of any given eigenvalues is closed. We use that fact that the standard closure is contained in the Zariski closure [7]. In summation, the class $[M]$ is entirely determined by its eigenvalues. $M_2//Gl_2 \cong \mathbb{C}^2$. This computation can be generalized to $M_n//Gl_n \cong \mathbb{C}^n$ (see example 1.2. [3]).

5.1 Character varieties

We are mainly interested in a speceific type of GIT-quotient called the character variety. If we have a representation variety $R_n(\Gamma)$, we can act on that variety via conjugation. If $\rho \in \hom(\Gamma, Gl_n), G \in Gl_n$ then we define $(G \cdot \rho)(x) = G\rho(x)g^{-1}$. Using this group action we define the character variety $X_n(\Gamma) := R_n(\Gamma)//Gl_n$. We have already described $R_n(\mathbb{Z}/(2))$ as the coproduct of different conjugacy classes, therefore $X_n(\mathbb{Z}/(2))$ is a set of $n+1$ elements. The goal for the remainder of this text is to compute the character varieties of frieze groups. The first one is well known, $R_n(\mathbb{Z})//Gl_n = Gl_n//Gl_n$.

Example 5.3. Let Gl_n act on itself by conjugation. We expand on what we did in example 5.2. The same idea applies, we want to find a closed set that captures the matrices with some given eigenvalues. For the two dimensional case we used the trace and determinant. This generalizes nicely to higher dimensions. If we look at the characteristic polynomial of a 2×2 matrix we get

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc).$$

This tells us that the trace and determinant are really just the coefficients of the characteristic polynomial. The characteristic polynomial, including all of its terms are left invariant by conjugation. They therefore give us the invariant polynomials, see example 1.2 in Lectures on invariant theory[3]. The coefficients also give us the polynomials that define the closed set of matrices with any given eigenvalues. We can conjugate on both sides to show that the orbit of any matrix contains a diagonal matrix in its closure. The polynomial coefficients can have any value other than the determinant. The determinant needs to be invertible, $Gl_n//Gl_n \cong Gl_1 \times \mathbb{C}^{n-1}$.

This example gives us the blueprint to compute the character variety of $\mathbb{Z}/(2) \times \mathbb{Z}$.

Theorem 5.4. *$X_n(\mathbb{Z}/(2)\times\mathbb{Z})$ is a disjoint union of $n+1$ copies of $Gl_1 \times \mathbb{C}^{n-1}$.*

Proof. We already know that we can rewrite the variety as

$$R_n(\mathbb{Z}/(2) \times \mathbb{Z}) = \coprod_{0 \le i \le n} IC_{n,i},$$

and also rewrite

$$X_n(\mathbb{Z}/(2) \times \mathbb{Z}) = \coprod_{0 \le i \le n} IC_{n,i}//Gl_n.$$

We shall compute $IC_{n,k}//Gl_n$ one at a time. Note that any element in $X_n(\mathbb{Z}/(2) \times \mathbb{Z})$ can by proposition 4.1 be written as $[J_k, G], G \in Gl_{n-k} \times Gl_k$.

Since the elements in $Gl_{n-k} \times Gl_k$ commute with J_k we can conjugate G such that it is arbitrarily close to a diagonal matrix like in example 5.2. The eigenvalues of G are preserved by the conjugation. The set of pairs $(A, B) \in R_n(\mathbb{Z} \times \mathbb{Z}/2)$ with B having a given set of n eigenvalues is closed, and any two pairs in this set are related under GIT.

Since this argument is independent of k we have that $X_n(\mathbb{Z}/(2) \times \mathbb{Z})$ is $n + 1$ copies of $Gl_1 \times \mathbb{C}^{n-1}$ $\square$

6 D_∞ Representations

In a previous section we discussed $R_n(\mathbb{Z}/(2))$, this is not a frieze group, however it is necessary to understand this if we want to understand $R_n(D_\infty)$. The reason for this is that D_∞ is the free product of two copies of $\mathbb{Z}/(2)$. What does that mean exactly? $\mathbb{Z}/(2)$ is defined as a group with one generator a where $a^2 = e$. Describing the free product can be a bit technical, the general idea is that you take the generators for each group and you add the relations that define each of them separately. Let us take two $\mathbb{Z}/(2)$ with generators a and b, then $D_\infty := (a, b)/(a^2, b^2)$. Something very interesting happens here. Despite it being a free product of two finite groups, the dihedral group is in fact infinite. Its elements are strings of a and b that alternate. $D_\infty = \{e, a, b, ab, ba, aba, bab, abab, baba, \ldots\}$. The strings that have odd length are involutions $(a(ba)^n)^2 = e = (b(ab)^n)^2$. The strings of even length greater than 0 are not involutions, their inverse is the other string of equal length, $(ab)^n(ba)^n = e$.

A Gl_n representation will send a to an involution A and b to an involution B. There are no additional requirements. Finding a representation of this group is simply a matter of finding two involutions. Thus

$$R_n(D_\infty) = R_n(\mathbb{Z}/(2)) \times R_n(\mathbb{Z}/(2)) = \coprod_{\substack{0 \le i \le n \\ 0 \le j \le n}} Inv_{n,i} \times Inv_{n,j}$$

We will write $\rho(a) = A, \rho(b) = B$. Every point is smooth and the dimension at (A, B) is $\dim_{(A,B)} R_n(D_\infty) = \dim_A R_n(\mathbb{Z}/(2)) + \dim_B R_n(\mathbb{Z}/(2))$.

6.1 The Character Variety of D_∞

On the surface this might not seem to be that much more difficult than to find the character variety of $\mathbb{Z}/(2)$, but that is not the case. We can describe $R_n(D_\infty)$ as an ordered pair of involutory matrices, (A, C) and (B, D) be two such pairs. We wish to discuss under what circumstances these pairs are congruent. Clearly A needs to be similar to B and C needs to be similar to

D, but that is not enough, there needs to be a $G \in Gl_n$ such that

$$G \cdot (A, C) = (GAG^{-1}, GBG^{-1}) = (B, D)$$

Let $A, B \in Inv_{n,k}$, we want to describe all $G \in Gl_n$ such that $g \cdot A = B$. We have an isomorphism described earlier $h_k : Inv_{n,k} \to Gl_n // Gl_{n-k} \times Gl_k$. We can therefore compute in a different space, $h(G \cdot A) = G \cdot h(A) = Gh_k(A) = h_k(B)$. G must be an element in $h_k(B)(Gl_{n-k} \times Gl_k)h_k(A)^{-1}$. Similarly if $C, D \in Inv_{n,l}$ then $G \cdot C = D$ if and only if $G \in h_l(D)(Gl_{n-l} \times Gl_l)h_l(C)^{-1}$.

This does not like something one could easily compute by brute force, and we have yet to take into account that two elements are related not just if they are in the others orbit, but also if the closures of their orbits intersect.

Example 6.1. Let us attempt to compute $X_2(D_\infty)$. $R_2(D_\infty)$ is isomorphic to 4 copies of $Inv_{2,1}$, 4 copies of $\{*\}$ and one copy of $Inv_{2,1} \times Inv_{2,1}$. By taking the GIT quotient we get 8 copies of $\{*\}$ and one $Inv_{2,1} \times Inv_{2,1} // Gl_2$. The last space is the pairs

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, B = \begin{bmatrix} d & e \\ f & -d \end{bmatrix}, a^2 + bc - 1 = 0, d^2 + ef - 1 = 0$$

with two pairs related if the closures of their orbits intersect. We notice that the first element can be conjugated to J_1, (A, B) (J_1, C). If we want to conjugate further while preserving J_1 then we must conjugate with a diagonal matrix. We compute how this effects C

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & -a' \end{bmatrix} \begin{bmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{bmatrix} = \begin{bmatrix} a' & \lambda_1 \lambda_2^{-1} b' \\ \lambda_1^{-1} \lambda_2 c' & -a' \end{bmatrix},$$

the entries on the main diagonal are preserved. We want to find M such that $A = MJ_1M^{-1}$. This can be achieved by computing eigenvectors for the eigenvalues 1 and -1

$$\begin{bmatrix} a-1 & b \\ c & -a-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0, \begin{bmatrix} a+1 & b \\ c & -a+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0,$$

using this we can set

$$M = \begin{bmatrix} b & b \\ a-1 & a+1 \end{bmatrix}$$

if $b \neq 0$. We set $\rho' = M^{-1}\rho M$. $\rho'(a) = J_1$ It now remains to compute $\rho'(b)$

$$M^{-1}BM = \begin{bmatrix} \frac{2ad+bf+ce}{2} & g \\ h & -\frac{2ad+bf+ce}{2} \end{bmatrix}.$$

The g and h can be expressed, but they are not unique. What this means that if we have a pair (A, B) that is conjugate to (J_1, C) then the main diagonal on C is uniquely determined by the entries in (A, B).

$$\frac{2ad + bf + ce}{2} = a'.$$

If $b = 0$ then we know that

$$A = \begin{bmatrix} \pm 1 & 0 \\ c & \mp 1 \end{bmatrix}.$$

We can write it as one of

$$A = \begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ c/2 & 1 \end{bmatrix}^{-1}$$

$$A = \begin{bmatrix} -1 & 0 \\ c & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -c/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -c/2 \end{bmatrix}^{-1}.$$

In the first case we get

$$\begin{bmatrix} 1 & 0 \\ c/2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} d & e \\ f & g \end{bmatrix} \begin{bmatrix} 1 & 0 \\ c/2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2d+ce}{2} & h \\ g & -\frac{2d+ce}{2} \end{bmatrix}$$

In the second case we get

$$\begin{bmatrix} 0 & 1 \\ 1 & -c/2 \end{bmatrix}^{-1} \begin{bmatrix} d & e \\ f & g \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -c/2 \end{bmatrix} = \begin{bmatrix} \frac{-2d+ce}{2} & h \\ g & -\frac{-2d+ce}{2} \end{bmatrix}.$$

In either case we get

$$\frac{2ad+bf+ce}{2} = \frac{2d+ce}{2} = a'$$
$$\frac{2ad+bf+ce}{2} = \frac{-2d+ce}{2} = a'.$$

Which tells us the that the pairs (A, B) that conjugate to (J_1, C) with the upper left entry of C being a' is a Zariski closed set in $R_2(D_\infty)$.

If $b'c' \neq 0$ then it follows that (J_1, C) is conjugate to any other pair with the first matrix being J_1 and the second having the main diagonal be a' and $-a'$. Conversely $b'c' = 0$ is equivalent to $a' = \pm 1$. Then $b' = 0$ or $c' = 0$, they are related to

$$\begin{bmatrix} \sqrt{\frac{\epsilon}{b'}} & 0 \\ 0 & \sqrt{\frac{\epsilon}{b'}}^{-1} \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & -a' \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\epsilon}{b'}}^{-1} & 0 \\ 0 & \sqrt{\frac{\epsilon}{b'}} \end{bmatrix} = \begin{bmatrix} a' & \epsilon \\ 0 & -a' \end{bmatrix}$$
$$\begin{bmatrix} \sqrt{\frac{\epsilon}{b'}}^{-1} & 0 \\ 0 & \sqrt{\frac{\epsilon}{b'}} \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & -a' \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\epsilon}{b'}} & 0 \\ 0 & \sqrt{\frac{\epsilon}{b'}}^{-1} \end{bmatrix} = \begin{bmatrix} a' & 0 \\ \epsilon & -a' \end{bmatrix}$$

for arbitrary ϵ. Therefore $[(J_1, C)]$ contains either (J_1, J_1) or $(J_1, -J_1)$. In conclusion, the orbits of (J_1, C) where $a' \neq \pm 1$ are closed. When $a' = \pm 1$ then $(J_1, \pm J_1)$ is contained in the closure of that orbit. Therefore we can construct an isomorphism $\mathbb{C} \cong (Inv_{2,1} \times Inv_{2,1})//Gl_2$ that maps

$$a \rightarrow \left[\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right)\right]$$

Remark 6.2. The polynomial

$$\frac{2ad+bf+ce}{2}$$

which we used to find the closed sets that defined the equivalence classes under GIT is the invariant polyniomial, it does not change under conjugation.

Is it possible to compute $X_3(D_\infty)$ using similar techniques. The answer in part Yes! The thing to note is that the bottom left element is left invariant when we conjugate.

Theorem 6.3. *The space $Inv_{n,1} \times Inv_{n,1}//Gl_n \cong \mathbb{C}$ for all $n \geq 2$.*

Proof. We define

$$C = \begin{bmatrix} a & 1-a \\ 1+a & -a \end{bmatrix}.$$

The map $\phi : \mathbb{C} \to Inv_{n,1} \times Inv_{n,1}//Gl_n$ is defined as

$$a \to \left[\left(J_{n,1}, \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C \end{bmatrix} \right) \right]$$

Let $(A, B) \in Inv_{n,1} \times Inv_{n,1}//Gl_n$. By theorem 3.14 $Inv_{n,1} \times Inv_{n,1}//Gl_n \cong ((Gl_n//Gl_{n-1} \times Gl_1) \times (Gl_n//Gl_{n-1} \times Gl_1))//Gl_n$. We map (A, B) to (M, N) using this isomorphism.

$$\begin{aligned} (M, N) &\sim (I, M^{-1}N) \\ (A, B) &\sim (J_{n,1}, M^{-1}NJ_{n,1}N^{-1}M) = (J_{n,1}, A') \end{aligned}$$

Conjugating $(J_{n,1}, A')$ with $Gl_{n-1} \times Gl_1$ preserves the bottom right entry in A', therefore the map

$$[(A, B)] \to -a$$

where $-a$ is the entry on the bottom right is a regular map. Meaning it can be written as a polynomial expression using entries in (A, B). The set of pairs (A, B) that conjugate to some $(J_{n,1}, A')$ is therefore closed.

We will now assume that $A = J_{n,1}$ and that the bottom right entry of B is $-a$. We want to show that all such pairs are related under GIT, by showing that $(J_{n,1}, B) \in [\phi(a)]$. This will be proved using induction. Let $n > 2$, the statement is true for $n = 2$ from our computation of $X_2(D_\infty)$, we also assume the statement is true for $n-1$. We map this pair to (I, N) using the above isomorphism. We will not change the conjugacy class of this pair if we multiply either left or right using matrices in $Gl_{n-1} \times Gl_1$. In particular we can perform column operations such that the bottom right entry is 0. This is done by either performing an appropriate row operation if the entry to the right is non-zero, if it is zero we can instead swap the rows. There needs to be some entry on the first column that is non-zero. This is not the bottom entry, we swap a non-zero entry to the top of the column. We then multiply and row eliminate such that the first column is

$$e_1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}.$$

Now we map back to $(J_{n,1}, B') \in Inv_{n,1} \times Inv_{n,1}$. The first column of B' is e_1. We conjugate the pair by

$$\begin{bmatrix} \epsilon & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

The matrix B' but with the top row being e_1 is in the closure of the conjugacy class of $(J_{n,1}, B)$. The B' but where we remove the first row and column is in $Inv_{n-1,1}$. We have thus successfully reduced to a lower dimensional case, we can thus invoke the induction hypothesis and conclude that$(J_{n,1}, B) \in [\phi(a)]$.

In conclusion, we showed that the set of pairs (A, B) that are related to some $(J_{n,1}, B')$ with the bottom right entry of B' being $-a$ is a closed set and that any such pairs are related under GIT. We can thus conclude that ϕ is an isomorphism. □

This covers only some of the non 0-dimensional components of $X_3(D_\infty)$. We also need to think about $Inv_{3,1} \times Inv_{3,2}//Gl_3$. This can also be computed in general.

Theorem 6.4. *Let Gl_n act on $Inv_{n,1} \times Inv_{n,n-1}$ by conjugation, then the GIT-quotient $Inv_{n,1} \times Inv_{n,n-1}//Gl_n$ is isomorphic to $\mathbb{C}$.*

Proof. Similarly to 6.3 we want to conjugate (A, B) to the form $(J_1, MJ_{n-1}M^{-1})$, conjugating with $Gl_{n-1} \times Gl_1$ will preserve J_1 as well as the bottom right entry of the other matrix. We can multiply M with $Gl_{n-1} \times Gl_1$ from the left and $Gl_1 \times Gl_{n-1}$ from the right and still be in the same conjugacy class. This means that we can freely perform row operations on all but the bottom row, and perform column operations on all but the column furthest right. We want to reduce M to a matrix of the form

$$N = \begin{bmatrix} a & 0 & b \\ 0 & I & 0 \\ c & 0 & d \end{bmatrix}.$$

This is a matrix where the inner box is the identity and the all entries other then the ones in the corners are zero. The first step is to perform column operation such that the all but the first and last entry on the bottom row are 0. This is possible since if there is a non-zero entry among them this is possible, if all the entries are 0, we are already where we want to be. Afterwards we perform row operations such that the top left $(n-1) \times (n-1)$ box is diagonal. Now the matrix looks like this

$$\begin{bmatrix} a & 0 & b \\ 0 & D & \vdots \\ c & 0 & d \end{bmatrix}.$$

The entries in D have to be different from 0 for the matrix to be invertible. We invert the elements on the diagonal and perform column operations such that all the entries between b and d become 0. We conjugate J_{n-1} by N and get

$$NJ_{n-1}N^{-1} = \begin{bmatrix} -\frac{ad+bc}{ad-bc} & 0 & \frac{2ab}{ad-bc} \\ 0 & -I & 0 \\ -\frac{2cd}{ad-bc} & 0 & \frac{ad+bc}{ad-bc} \end{bmatrix}.$$

From here we take a similar approach as we did when computing $X_2(D_\infty)$ and show that $Inv_{n,1} \times Inv_{n,n-1}//Gl_n \cong \mathbb{C}$. $\square$

Corollary 6.5. *$X_3(D_\infty)$ is isomorphic to the disjoint union of 12 copies of $\{*\}$ and 4 copies of $\mathbb{C}$.*

Proof. We first write out the representation variety

$$R_3(D_\infty) = \coprod_{\substack{0\leq i\leq 3\\ 0\leq j\leq 3}} Inv_{3,i} \times Inv_{3,j}.$$

It contains 12 segments where either i or j equals 0 or 3. The GIT-quotients of these are $\{*\}$. Theorem 6.3 and 6.4 tells us that the remaining 4 segments are $\mathbb{C}$, □

We will not compute $X_n(D_\infty)$ for higher n, but we will give some results that could be generalized to compute these spaces.

7 Group Cohomology and $T_{[\rho]}X_n(D_\infty)$

A module can be described as an abelian group with a ring acting on it such that the action is distributive, associative and the identity acts trivially on the group. Similarly an abelian group with a group acting on it with the same axioms satisfied is called a group module. The notion of a group module is really just a specific case of the general idea of a module over a ring. For any G module M one can easily give M a $\mathbb{Z}[G]$ structure, where $\mathbb{Z}[G]$ is the group ring. The underlying abeilan group of the ring is the free abelian group on the group. Multiplication is defined in the following way $(\sum_{i=1}^{n} a_i g_i)(\sum_{i=1}^{m} b_i h_i) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j g_i h_j$. Any $\mathbb{Z}[G]$ module has an underlying G module structure. we can simply look at how g acts on our module. These constructions are equivalences of categories. Thus the category of G-modules is abelian.

Remark 7.1. For the star of this section we will use some notation that is very standard, but not used other places in this text. We use M for a module and G for a group, even if both these letters are used for different matrices other places in the text.

Example 7.2. Any abelian group M is a $\mathbb{Z}$ module. $n \cdot m = m + m + \ldots + m$, the addition is performed n times.

Example 7.3. The main example we are working with is $ad\rho$, where ρ is a Gl_n represantation of a group Γ. The abelian group is M_n and it has the Γ action $g \cdot M = \rho(g)M\rho(g)^{-1}$

We are going to be applying two different definitions. We us both the following and the one using derived functors, in particular it means that the construction below gives a chain complex. See [6] chapter 2 for a breakdown.

Definition 7.4. *Let M be a G module. We define $C^0(G, M) := M$ and for $n \geq 1$, we define the n-cochains $C^n(G, M)$ to be functions from G^n to M.*

They are abelian groups using addition. The coboundary maps are as follows:

$$d^1 m(g) = g \cdot m - m$$

and for $\phi \in C^n(G, M)$ we define

$$d^{n+1}\phi(g_1, \ldots, g_{n+1}) = g_1 \cdot \phi(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i \phi(g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1}\phi(g_1, \ldots, g_n)$$

The other cochains and coboundary maps are 0. *From here cohomology is defined in the usual way:*

$$H^n(G, M) := Ker(d^{n+1})/Im(d^n)$$

Let us unpack what this definition means for lower homology groups. $d^1 m = 0$ is equivalent to $g \cdot m = m$. The zeroth cohomology are the G-invariant elements of M. The 1-cocycles are maps $\phi : G \to M$ that satisfy $\phi(g_1 g_2) = g_1 \cdot \phi(g_2) + \phi(g_1)$. Therefore $\phi(e^2) = \phi(e) = e \cdot \phi(e) + \phi(e) = 2\phi(e)$, meaning that $\phi(e) = 0$. This identity also tells us that if we know what $\phi(g_1)$ and $\phi(g_2)$ are then we know what $\phi(g_1 g_2)$ is. More generally: we only need to understand what the generators of a group are mapped to know what the entire group is mapped to. We also kill maps of the form $\phi(g) = g \cdot m - m$.

It should be noted that the abelian group $H^n(G, M) \cong Ext^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$. This is useful for making computations, especially for higher dimensional cohomology groups. We give two examples to demonstrate the usefulness of this observation. The first example is also useful to demonstrate that certain cohomology groups are independent of the module M.

Lemma 7.5. *$H^n(\mathbb{Z}, M) = 0$ for all $n \geq 2$ regardless of how $\mathbb{Z}$ acts on M.*

Proof. Let x be a generator of $\mathbb{Z}$. Using this notation $\mathbb{Z}[\mathbb{Z}]$ looks like the ring of finite formal power series over $\mathbb{Z}$. We define $aug : \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}, aug(\sum_{i=m}^{n} a_i x^i) = \sum_{i=m}^{n} a_i$. The sum of the coefficients in a polynomial equals zero if and only if $x-1$ divides the polynomial. Thus $\ker(aug) = (x-1)\mathbb{Z}[\mathbb{Z}]$. We obtain the following projective resolution.

$$\ldots \longrightarrow 0 \longrightarrow (x-1)\mathbb{Z}[\mathbb{Z}] \xrightarrow{i} \mathbb{Z}[\mathbb{Z}] \xrightarrow{aug} \mathbb{Z} \longrightarrow 0$$

Therefore $Ext^n_{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z}, M) = 0$ for all $n \geq 2$. $\square$

Lemma 7.6. *If M is an abelian group where 2 is divisible. Then $H^n(\mathbb{Z}/(2), M) = 0$ for all $n \geq 1$, regardless of how $\mathbb{Z}/(2)$ acts on M.*

Proof. The 1-cocycles are the maps ϕ with $\phi(g_1 g_2) = g_1 \cdot \phi(g_2) + \phi(g_1)$. In particular the cochains need to send the identity to $0 \in M$ since $\phi(e) = e \cdot \phi(e) + \phi(e)$, e acts trivially on M, so $\phi(e) = 0$. If we kill the coboundaries we force all g to act trivially on M. $0 = \phi(e) = \phi(a^2) = \phi(a) + \phi(a) = 2\phi(a)$. Since 2 is invertible $\phi(a) = 0$.

For the higher cohomology groups we are going to compute $Ext^n_{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}, M)$. We can create a projective resolution by making the following observations. Firstly; $aug : \mathbb{Z}[\mathbb{Z}/(2)] \to \mathbb{Z}, aug(n_1 \cdot a + n_0) = n_1 + n_0$ is an epimorphism. The kernel of this map is $(a-1)\mathbb{Z}[\mathbb{Z}/(2)]$. We thus obtain an exact sequence

$$\mathbb{Z}[\mathbb{Z}/(2)] \xrightarrow{(a-1)\cdot} \mathbb{Z}[\mathbb{Z}/(2)] \xrightarrow{aug} \mathbb{Z} \longrightarrow 0$$

The kernel of $(a-1)\cdot$ is $(a+1)\mathbb{Z}[\mathbb{Z}/(2)]$, and the kernel of $(a+1)\cdot$ is $(a-1)\mathbb{Z}[\mathbb{Z}/(2)]$. Any ring viewed as a module over itself is projective. Now we are ready to make a projective resolution:

$$\ldots \xrightarrow{(a-1)\cdot} \mathbb{Z}[\mathbb{Z}/(2)] \xrightarrow{(a+1)\cdot} \mathbb{Z}[\mathbb{Z}/(2)] \xrightarrow{(a-1)\cdot} \mathbb{Z}[\mathbb{Z}/(2)] \xrightarrow{aug} \mathbb{Z} \longrightarrow 0$$

From this we conclude that the even degree homology for $n \geq 2$ are the same and that the odd degree homology are the same as $H^1(\mathbb{Z}/(2), M) = 0$. We apply $Hom(-, M)$ and compute homology at the even degree. $Ext^2_{\mathbb{Z}[\mathbb{Z}/(2)]}(\mathbb{Z}, M) = (a+1)M/(a-1)M = 0$ since 2 is divisible. $\square$

7.1 Cohomology and the Tangent space

Our interest in the cohomology of groups is that it is a great aid for computing the dimension of tangent spaces of representation and character varieties. In general there is an injective map from $T_\rho(R_n(\Gamma))$ to $Z^1(\Gamma, ad\rho)$, we will show what this map looks like later on. There is also an isomorphism $T_\rho(X_n(\Gamma)) \cong H^1(\Gamma, ad\rho)$ for irreducible representations.

Lemma 7.7. *There is an isomorphism between $T_\rho(R_n(D_\infty)$ and $Z^1(D_\infty, ad\rho)$, in particular they have the same dimension.*

Proof. The 1-cocycles are maps $\phi : \mathbb{Z}/(2) \to ad\rho$ that satisfy $\phi(gh) = g \cdot \phi(h) + \phi(g)$. This means that:

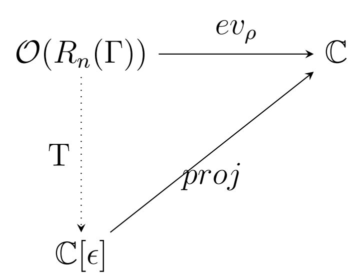
$$\begin{aligned} 0 = \phi(e) = \phi(a^2) &= a \cdot \phi(a) + \phi(a) \\ 0 = \phi(e) = \phi(b^2) &= b \cdot \phi(b) + \phi(b) \\ a \cdot \phi(a) &= -\phi(a) \\ a \cdot \phi(b) &= -\phi(b) \end{aligned}$$

This is in fact the exact same criteria placed on the term in front of ϵ from lemma 3.11. What this means is that all 1-cocycles are induced by tangent vectors, and that all tangent vectors can be induced by 1-cocycles. We construct a map that takes $A + B\epsilon$ and sends it to the map $\phi(a) = B$. Since this respects the linear structure of each space it is a linear isomorphism. $\square$

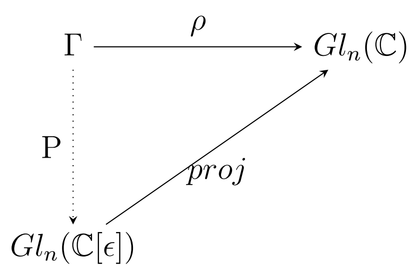
One might be tempted to try this exact same approach for all Γ. It does almost work, but it needs a slight modification.

Theorem 7.8. *There is an injective linear map $h : T_\rho(R_n(\Gamma)) \to Z^1(\Gamma, ad\rho)$*

Proof. Finding a tangent vector is the same as finding a dotted arrow making the diagram below commute.



And finding that arrow means we find an arrow making this diagram commute:



Thus $P(g) = \rho(g) + u(g)\epsilon$. Define $HT(g) := \phi(g) = u(g)\rho(g)^{-1}$. Then we compute:

$$\begin{aligned}
1 + \phi(g_1g_2)\epsilon &= P(g_1g_2)\rho(g_1g_2)^{-1} \\
&= P(g_1)P(g_2)\rho(g_2)^{-1}\rho(g_1)^{-1} \\
&= P(g_1)(1 + \phi(g_2)\epsilon)\rho(g_1)^{-1} \\
&= P(g_1)\rho(g_1)^{-1} + P(g_1)\phi(g_2)\rho(g_1)^{-1}\epsilon \\
&= 1 + \phi(g_1)\epsilon + (\rho(g_1) + \phi(g_1)\rho(g_1)\epsilon)\phi(g_2)\rho(g_1)^{-1}\epsilon \\
&= 1 + (\phi(g_1) + \rho(g_1)\phi(g_2)\rho(g_1)^{-1})\epsilon \\
&= 1 + (\phi(g_1) + g_1 \cdot \phi(g_2))\epsilon
\end{aligned}$$

which is equivalent to $\phi(g_1g_2) = g_1 \cdot \phi(g_2) + \phi(g_1)$. □

One may be tempted to invert the function by sending $\phi(g) \to \rho(g) + \phi(g)\rho(g)$. The problem is that this may not be well-defined. Example 2.18 in Heusner [4] is an instance when the inclusion is strict. It is generally the case that $\dim(Z^1(\Gamma, ad\rho))$ is greater than the or equal dimension at ρ We call the representations where $\dim_\rho R_n(\Gamma) = \dim(Z^1(\Gamma, ad\rho))$ **scheme smooth**, by lemma 7.7 all representations in $R_n(D_\infty)$ are scheme smooth.

What is clear is that the 1-cocycle group is closely related to the tangent space of a representation variety. What about the tangent space of character variety? There are strong results for certain types of representations. A Γ representation $\rho : \Gamma \to Gl_n$ gives a Γ action on $\mathbb{C}^n$ by $g \cdot v = \rho(g)v$. We say that a subspace $V \subseteq \mathbb{C}^n$ is Γ-**stable**, or **invariant** if $g \cdot v \in V, \forall v \in V$. We mainly use stable not to confuse it with the Γ-action on the coordinate ring of $R_n(\Gamma)$. If a representation has no non-trivial stable subspaces we say that it is **irreducible**. Heusener [4] gives the following result for irreducible representations.

$$T_{[\rho]}X_{Gl_n}(\Gamma) \cong H^1(\Gamma, ad\rho).$$

This result is a generalization of a stronger result about scheme smooth **semisimple** representations, which are representations that split $\mathbb{C}^n$ into irreducible representations. Heusner also states that ρ is semisimple if and only if $O(\rho)$ is closed, which is equivalent to $O(\rho) = [\rho]$.

Example 7.9. A representation that is not semisimmple is $\rho : \mathbb{Z} \to Gl_2$ that sends

$$a \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

It is not irreducible by $a \cdot (v, 0) = (v, 0)$, so it has a stable subspace, but it is also the only proper stable subspace. Therefore the space does not split. We can factor the matrix into involutions in the following way

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which tells us that the representation that sends the generators of D_∞ to those involutions is not semisimple. This corresponds to the fact that orbit of this representation is not closed as seen in the computation of $X_2(D_\infty)$.

We define the **centralizer** subgroup of Gl_n as $C_{Gl_n}(\rho(\Gamma)) = \{M \in Gl_n | M\rho(g) = \rho(g)M, \forall g \in \Gamma\}$. Adam Sikora [11] gives a proof that for scheme smooth semisimple Gl_n representations of finitely generated groups we have

$$T_{[\rho]}X_n(\Gamma) = T_0(H^1(\Gamma, ad\rho)//C_{Gl_n}(\rho(\Gamma))).$$

The group action on $H^1(\Gamma, ad\rho)$ by $C_{Gl_n}(\rho(\Gamma))$ is conjugation.

Remark 7.10. $C_{Gl_n}(\rho(\Gamma)) = H^0(\Gamma, ad\rho) \bigcap Gl_n$. Schur's lemma [9] tells us that that for irreducible representations $H^1(\Gamma, ad\rho)$ equals the diagonal matrices with all entries on the diagonal being the same, these will commute with any matrix. In particular $\dim H^1(\Gamma, ad\rho)$ will have its lowest possible value, namely 1. If this is the case then the centralizer will only include the matrices in Gl_n that commute with all matrices, meaning the group action is trivial and we get

$$T_{[\rho]}X_{Gl_n}(\Gamma) \cong T_0(H^1(\Gamma, ad\rho)).$$

Hopefully we have properly motivated why we are interested in group cohomology.

7.2 Cohomology of D_∞

Our goal now is to compute $H^m(D_\infty, ad\rho)$ for different ρ and m. The simplest of these to compute should be $H^0(D_\infty, ad\rho)$ as it is essentially the intersection of two centralizers, but even this one can be demanding as we increase the dimension. Computing the higher cohomology groups of D_∞ may be even more difficult to do directly, thankfully we can use the Lyndon-Hochschild-Serre spectral sequence (see Homology of Linear Groups Appendix 2 [5]) to compute the cohomology for $m \geq 2$. If you have a short exact sequence of groups with D_∞ as the middle term you can compute the homology of the middle term by using the homology of the adjacent groups in the sequence. What we mean by a short exact sequence is essentially the mapping of a normal subgroup into the middle term, then taking the quotient. The trick is to find a fitting short exact sequence.

We look at some of the subgroups that reside in D_∞. There are of course the cyclic subgroups generated by a and b, however these are not normal. The infinite cyclic subgroup generated by ab is a better candidate. This group can also be described as the elements which have an even amount of characters, conjugating would either add no characters, or add/subtract an even number of them, it is therefore a normal subgroup. The quotient is $\mathbb{Z}/(2)$. We obtain the following short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow D_\infty \longrightarrow \mathbb{Z}/(2) \longrightarrow 0$$

which is used to prove the following theorem:

Proposition 7.11. *$H^n(D_\infty, ad\rho) = 0$ for all $n \geq 2$.*

Proof. Using the above short exact sequence and the Lyndon-Hochschild-Serre spectral sequence we know there is a spectral sequence of cohomological type:

$$H^p(\mathbb{Z}/(2), H^q(\mathbb{Z}(ad\rho))) \Longrightarrow H^{p+q}(D_\infty, ad\rho)$$

From the lemmas 7.5 and 7.6 we know that $H^p(\mathbb{Z}/(2), H^q(\mathbb{Z}(ad\rho))) = 0$ for all $p \geq 1$ and $H^q(\mathbb{Z}(ad\rho)) = 0$ for all $q \geq 2$. The $E_2^{0,0}$ and $E_2^{0,1}$ are therefore the only two entries that do not have to be zero. $H^{p+q}(D_\infty, ad\rho) = 0$ for all $p + q \geq 2$. □

Remark 7.12. While this result is not used directly further on in this text it is still a result that could be useful further on if we want to prove further results about $X_n(D_\infty)$.

Now we want to introduce some additional notation. If it is clear what group representation we are discussing we may use the following:

$$\begin{aligned}
c^m &:= \dim C^m(\Gamma, ad\rho)) \\
b^m &:= \dim B^m(\Gamma, ad\rho)) \\
z^m &:= \dim Z^m(\Gamma, ad\rho)) \\
h^m &:= \dim H^m(\Gamma, ad\rho)).
\end{aligned}$$

We have computed $h^m = 0, m \geq 2$, now we want to demonstrate the a relationship that exists between h^0 and h^1.

Theorem 7.13. *Let $\rho : D_\infty \to Gl_n$ be a homomorphism such that $\rho(a) = A \in Inv_{n,k}, \rho(b) = B \in Inv_{n,l}$, then $h^0 - h^1 = n^2 - 2k(n-k) - 2l(n-l)$*

Proof. Recall that $h^0 = z^0$, $h^1 = z^1 - b^1$ by the definition of group cohomology and the fact that $z^0 + b^1 = c^0 = n^2$, which follows from them being the kernel and the image of the same linear map. From lemma 7.7 we also know that $z^1 = \dim_\rho(R_n(D_\infty)) = 2k(n-k) + 2l(n-l)$.

$$h^0 - h^1 = z^0 - z^1 + b^1 = z^0 - z^1 + n^2 - z^0 = n^2 - 2k(n-k) - 2l(n-l)$$

□

What this tells us is that h^1 can be computed by computing h^0. This can be a much simpler task. In the instance of ρ being irreducible we will demonstrate that $\dim T_{[\rho]} X_n(D_\infty) = 1$.

7.3 Irreducible D_∞ Representations

The definition of an irreducible representation is one which the Γ-stable subspaces under $g \cdot v = \rho(g)v$ are zero and the whole space. Schur's lemma furthermore says that $h^0 = 1$ for these, since specifically

$$H^0(\Gamma, ad\rho) = \{M \in ad\rho | \rho(g)M\rho(g)^{-1} = v, \forall g \in \Gamma\} = \{v \in ad\rho | \rho(g)M = M\rho(g), \forall g \in \Gamma\},$$

and Schur's lemma says that for irreducible representations $\lambda \cdot Id$ are the only linear maps that commute with the Γ action. This is why we want to find irreducible representations. The induced D_∞ action from ρ is essentially multiplication with two involutions $\rho(a) = A, \rho(b) = B$. We begin by finding an equivalent restating of a space being stable under one involution.

Remark 7.14. These next few theorems about stable subspaces use no assumptions about $\mathbb{C}$ other than it having characteristic different from 2. The assumption about $\mathbb{K} = \mathbb{C}$ is necessary if we want to use Schur's lemma (and most other results).

Lemma 7.15. *Let A be an involution on a finite-dimensional vector space V. The subspace U is A-stable if and only if there is a basis of eigenvectors spanning U.*

Proof. If such a basis exists: $\{x_1, \dots, x_m\}$ then for any vector in U:

$$A\Big(\sum_{i=1}^{m} a_i x_i\Big) = \sum_{i=1}^{m} \pm a_i x_i \in U$$

Let us now assume that U is A-stable. Let $\{x_1, \dots, x_m\}$ be a basis for U. The set:

$$S = \{Ax_1 + x_1, Ax_1 - x_1, \dots, Ax_m + x_m, Ax_m - x_m\}$$

is a set of eigenvectors that spans U by virtue of $\frac{Ax_i + x_i - (Ax_i - x_i)}{2} = x_i$ and $A + I, A - I$ mapping onto their corresponding eigenspaces by lemma 3.4. Also the set is contained in U since it is A-stable. We can therefore find a subset pf S which is a basis of U. □

Proposition 7.16. *Let A and B be linear transformations on a vector space V with $\dim V \geq 2$. If V only has trivial subspaces that are both A-, and B-stable, then A and B have no common eigenvectors. If A and B also are involutions over an n dimensional space then n is an even number and all the eigenspaces of A and B have dimension $n/2$.*

Proof. If A and B have some common non-trivial eigenvector x, then x spans a stable subspace. Therefore A and B can not have common non-zero eigenvectors.

Assuming that A and B are involutions over a vector space of dimension n. From theorem 3.5 we know that

$$E_1^A \bigoplus E_{-1}^A = V = E_1^B \bigoplus E_{-1}^B,$$
$$\dim E_1^A + \dim E_{-1}^A = n = \dim E_1^B + \dim E_{-1}^B.$$

This in combination with A and B having no non-zero common eigenvectors tells us that the intersection of any combination of these four spaces must be 0.

We want to use this fact to tell us that all of E_1^A, E_{-1}^A and E_1^B, E_{-1}^B must have dimension equal to $n/2$. If $\dim(E_1^A) > n/2$ then at least one of

$$\dim(E_1^A \bigcap E_1^B) > n/2 + \dim E_1^B - n$$
$$\dim(E_1^A \bigcap E_{-1}^B) > n/2 + \dim E_{-1}^B - n$$

is greater than greater than 0, a contradiction. If $\dim(E_1^A) < n/2$ then $\dim(E_{-1}^A) > n/2$ and the exact same argument would apply. We would also make an identical argument if $dim(E_1^B) > n/2$ or $\dim(E_{-1}^B) > n/2$. Thus $\dim(E_1^A) = \dim(E_{-1}^A) = \dim(E_1^B) = \dim(E_{-1}^B) = n/2$ and n is even. □

This gives us the tools to classify the irreducible representations.

Corollary 7.17. *The representation $\rho : D_\infty \to Gl_n, \rho(a) = A, \rho(b) = B$ is irreducible if and only if $n = 1$ or n is even and there is no subspaces that can both be generated from eigenvectors of A and eigenvectors of B.*

Proof. 1 dimensional representations are trivially irreducible. If ρ is irreducible then there are no non trivial stable subspaces by the definition. Lemma 7.15 tells us that no subspace can both be generated from eigenvectors of A and eigenvectors of B. Proposition 7.16 tells us that n is even if it is not 1.

Lemma 7.15 tells us that if no subspace can both be generated from eigenvectors of A and eigenvectors of B then there are no non trivial stable subspaces, thus ρ is irreducible. □

Remark 7.18. One may be tempted to use the condition that the four eigenspaces $E_1^A, E_{-1}^A, E_1^B, E_{-1}^B$ having no intersection is a strong enough criteria, this is not the case. Consider

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}^{-1}.$$

There is no intersection between the eigenspaces, but the subspace $(v_1, 0, v_3, 0)$ is stable.

Corollary 7.19. *If $\rho : D_\infty \to Gl_n$ is an irreducible representation then $\dim T_{[\rho]} X_n(D_\infty) = 1$.*

Proof. From theorem 7.13 and proposition 7.16 we know that

$$h^0 - h^1 = 1 - h^1 = n^2 - 2\frac{n}{2}(n - \frac{n}{2}) - 2\frac{n}{2}(n - \frac{n}{2}) = n^2 - \frac{n^2}{2} - \frac{n^2}{2} = 0.$$

Sikora [11] tells us that $\dim T_{[\rho]} = h^1 = 1$. □

For $n = 1$ every representation is trivially irreducible. What about for $n = 2$.

Example 7.20. We look at

$$\rho \to \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right)$$

The subspaces that are J_1-stable are the ones generated by e_1 or e_2. In order for the other matrix to have one of these as stable either b or c needs to be 0. As discussed in example 7.9, if only one is 0 then we have a reducible representation that is not semisimple. If both are 0 then the representation is reducible and semisimple. Lastly, if neither are 0 then the repåppresentation is irreducible (and semisimple).

7.4 The tangent space at $[(J_k, J_l)]$

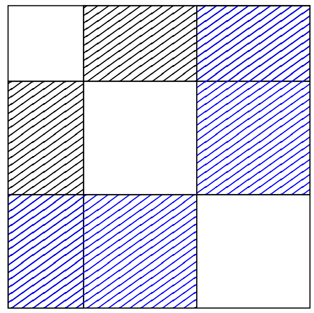
The goal of this subsection is to prove that the tangent space at the point $[(J_k, J_l)]$ is of dimension $\min\{n-k, k, n-l, l\}$. We do this by using the construction from Sikora [11]. We first want to understand the space $H^1(D_\infty, ad\rho)$ by using the definition.

Lemma 7.21. *Let $\rho : D_\infty \to Gl_n, \rho(a) = J_k, \rho(b) = J_l$. Then the space $H^1(D_\infty, ad\rho)$ is identified with the space of $n \times n$ matrices with all entries except the ones in the bottom left $\min\{k,l\} \times \min\{n-k, n-l\}$ box and the upper right $\min\{n-k, n-l\} \times \min\{k,l\}$ box (see figure in the proof). Furthermore, the space $H^1(D_\infty, ad\rho)//C_{Gl_n}(\rho(D_\infty))$ can be identified with the space*

$$M_{\min\{k,l\}\times\min\{n-k,n-l\}} \times M_{\min\{n-k,n-l\}\times\min\{k,l\}}//Gl_{\min\{n-k,n-l\}} \times Gl_{\min\{k,l\}},$$

the group action being $(G, H) \cdot (M, N) = (HMG^{-1}, GNH^{-1})$.

Proof. The first step is in finding $Z^1(D_\infty, ad\rho)$. These are functions that map a, b respectively to tangent vectors of J_k, J_l, we call these respectively K, L. We then notice that we can the 1-coboundaries are functions that map a, b respectively to $J_kMJ_k - M, J_lMJ_l - M$ for some matrix $M \in M_n$. We choose $\psi(g) = g \cdot M - M$ such that the bottom left $(n-k) \times k$ box and the top right $k \times (n-k)$ box of M has the exact same entries as those in K. We then do the same for the remaining entries that overlap with the non-zero entries in L. This function is identified with zero since ψ is a co-boundary, so it can be added to both K and L. $K - \psi(a) = 0$, and $L - \psi(b)$ is zero but for the bottom left $\min\{k,l\} \times \min\{n-k, n-l\}$ box and the upper right $\min\{n-k, n-l\} \times \min\{k,l\}$ box. In the figure below the black box is one of K, L and the blue box is the other one, the overlapping area is where $(\phi - \psi)(b)$ is not 0.



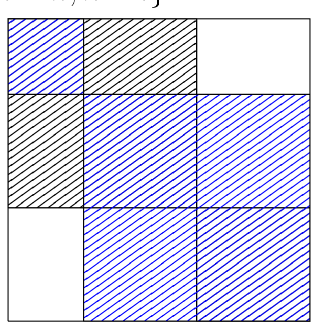
To compute the GIT-quotient we need to compute the centralizer,

$$C_{Gl_n}(\rho(D_\infty)) = C_{Gl_n}(\rho(J_k)) \bigcap C_{Gl_n}(J_l) = (Gl_{n-k} \times Gl_k) \bigcap (Gl_{n-l} \times Gl_l).$$

If an entry is 0 in either $(Gl_{n-k} \times Gl_k)$ or $(Gl_{n-l} \times Gl_l)$ then it must be zero in the intersection. The entry in both the bottom left $k \times (n-k)$ and $l \times (n-l)$ boxes as well as the top $(n-k) \times k$ and $(n-l) \times l$ boxes need to be zero. Thus

$$C_{Gl_n}(\rho(D_\infty)) = Gl_{\min\{n-k,n-l\}} \times Gl_h \times Gl_{\min\{k,l\}}$$

where h is the integer such that $n = \min\{n-k, n-l\} + h + \min\{k, l\}$. The figure below illustrates this, the areas that are not 0 are the ones shaded twice.



We conjugate $H^1(D_\infty, ad\rho)$ with an element in $Gl_{\min\{n-k,n-l\}} \times Gl_h \times Gl_{\min\{k,l\}}$,

$$\begin{bmatrix} G & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & H \end{bmatrix} \begin{bmatrix} 0 & 0 & N \\ 0 & 0 & 0 \\ M & 0 & 0 \end{bmatrix} \begin{bmatrix} G^{-1} & 0 & 0 \\ 0 & F^{-1} & 0 \\ 0 & 0 & H^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & GNH^{-1} \\ 0 & 0 & 0 \\ HMG^{-1} & 0 & 0 \end{bmatrix}.$$

We notice that $F \in Gl_h \times Gl_h$ has no impact on $H^1(D_\infty, ad\rho)$, therefore $H^1(D_\infty, ad\rho)//C_{Gl_n}(\rho(D_\infty))$ can be identified with the space stated in the theorem. □

Before we compute this quotient in general we shall look at it in a 2-dimensional case.

Example 7.22. If ρ maps both generators to J_1 we are able to compute the tangent space at those points by using the construction from lemma 7.21. The 1-cocycles are the pairs

$$\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}, \begin{bmatrix} 0 & a \\ d & 0 \end{bmatrix}.$$

the 1-coboundaries are the pairs

$$\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}, \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}.$$

Thus the first cohomology group is identified with the two dimensional space of matrices

$$\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$$

being acted on by $C_{Gl_n}(\rho(D_\infty)) = Gl_1 \times Gl_1$. We already worked with this setup when we computed $X_2(D_\infty)$, but now we are going to look at it from a different angle, that being the algebraic point of view as opposed to the geometric point of view. Recall that $H^1(D_\infty, ad\rho)//C_{Gl_2}(\rho(D_\infty))$ is the scheme with the coordinate ring $\mathbb{C}[b,c]^{Gl_1\times Gl_1}$, we want to understand this algebra. Conjugating the cohomolgy group with $Gl_1 \times Gl_1$, as this is the centralizer of J_1.

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \begin{bmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{bmatrix} = \begin{bmatrix} 0 & \lambda_1\lambda_2^{-1}a \\ (\lambda_1\lambda_2^{-1})^{-1}b & 0 \end{bmatrix},$$

We see that this reduces to Gl_1 acting on $\mathbb{C}^2$ with $\lambda\cdot(b,c) = (\lambda b, \lambda^{-1}c)$. What we are interested in are polynomials f generated by b and c that are left invariant by the group action. Since $f(v,0) = f(\lambda v, 0)$ and $f(0,v) = f(0,\lambda^{-1}v)$ it is necessary for there to be no terms with only b's or c's. Also, $f(1,\lambda) = f(\lambda,1)$ for all λ. This means that the coefficient in front of b^ic^j is the same as the one in front of b^jc^i, we call this coefficient $\gamma_{i,j}$. Let f be a $Gl_1 \times Gl_1$ invariant polynomial with $\gamma_{i,i} = 0$,

$$f(\alpha,\alpha) = \sum_{i,j=0}^{N} \gamma_{i,j}\alpha^{i+j} = 2\sum_{i,j=0,i<j}^{N} \gamma_{i,j}\alpha^{i+j}$$
$$f(\alpha\lambda, \alpha\lambda^{-1}) = \sum_{i,j=0,i<j}^{N} \gamma_{i,j}\alpha^{i+j}(\lambda^{j-i} - \lambda^{i-j}),$$

we then define the polynomial

$$g(\alpha,\lambda) := \lambda^N f(\alpha\lambda,\alpha\lambda^{-1}) - \lambda^N f(\alpha,\alpha) = 0.$$

All the coefficients of this polynomial must be 0, in particular the coefficient in front of $\alpha^{i+j}\lambda^{N+j-i}$ must be zero. This coefficient is $\gamma_{i,j}$. This tells us that the polynomials

$$\sum_{i=0}^{N} \gamma_i (bc)^i$$

are the ones left invariant by $Gl_1 \times Gl_1$. It is a subalgebra generated by bc. This certainly looks similar to one of the invariant polynomials of M_2 when acted on Gl_2 by conjugation. The invariant polynomials are the coefficients of the characteristic polynomial

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - bc.$$

The invariant polynomials are $a+d, ad-bc$. If we reduce these by sending a and d to zero we get the same polynomial we computed that was invariant under $Gl_1 \times Gl_1$, which was bc. By computing

$$\begin{vmatrix} -\lambda & b \\ c & -\lambda \end{vmatrix} = \lambda^2 - bc.$$

This idea can be generalized. We look at the characteristic polynomial of matrices with only zero entries in the upper left and lower right. Here, non-trivial means the coefficients that are not $0, 1, -1$. The goal is to prove that if $k \geq l$, then the $Gl_k \times Gl_l$ action on $M_{l\times k} \times M_{k\times l}$ by $(g,h)\cdot(P,Q) = (hPg^{-1}, gQh^{-1})$ then the GIT quotient has dimension l. The way we do this is that we first compute the invariant polynomials of a convenient subgroup.

Proposition 7.23. *Set $k \geq l$. Let Gl_k act on $V = M_{l\times k} \times M_{k\times l}$ by $G \cdot (P,Q) = (PG, G^{-1}Q), G \in Gl_k$. The invariant algebra $\mathcal{O}(V)^{Gl_k}$ is freely generated by the entries in PQ.*

Proof. The general idea is to find a subspace of $S \subseteq V$ such that the $Gl_k \cdot S$ is dense in V in the Zariski topology. This is the the space is dense in V in a euclidean sense. Recall that the euclidean closure of a set is contained in the Zariski closure, if the euclidean closure is the whole space then the Zariski closure must contain the whole space, meaning it is the whole space. It is the case that any invariant polynomial on V is uniquely determined by how it acts on a dense set. This dense set is furthermore determined by how Gl_k acts on S. After proving this we then find the invariant polynomials on S and verify that those are induced by the entries in PQ.

We set

$$S = (P_l, \lambda I_l),$$

P_l has all entries not in the first l columns zero. Let $(P, Q) \in V$. It is clear that if $l = k$ this is true by simply choosing some G close to Q, we then see that $G^1 \cdot (PG, I) = (P, G)$.

We now assume $k > l$. There exists some G such that the first l columns (which we denote G_l) are linearly independent, in addition none are in the orthogonal complement of P, and lastly that $\|G_l - Q\| < \epsilon$. This is possible since that both the sets of matrices with linearly independent vectors and the set of vectors not in the orthogonal complement have non-trivial are dense in the euclidean sense, hence their intersection is also dense, but the matrices which have a distance greater than ϵ from Q is not dense. Here we need the assumption that $k > l$ for the orthogonal to have dimension at least $k - l$. The remaining columns of G are chosen so that they are linearly independent and in the orthogonal complement to P. Thus PG only has non-zero entries on the first l columns and $G^{-1} \cdot (PG, I) = (P, G)$ which is close to (P, Q).

Now we think of (P, Q) as a matrix of polynomial variables. The entries are invariant polynomials since $PQ = PGG^{-1}Q$. We evaluate the elements of S by the entries in PQ. They correspond to the polynomials generated by $P_{i,j}\lambda$.

The invariant polynomials of the space S are the ones where the combined degree of $P_{i,j}$ and λ are the same, hence the subalgebra generated by $P_{i,j}\lambda$. By simply multiplying G with a scalar you can vary the value of any polynomial that has entries where this condition does not hold true, see example 7.22 for how this can be done. Clearly there is an isomorphism

$$\mathbb{C}[X_{i,j}] \cong \mathbb{C}[X_{i,j}\lambda] \cong \mathcal{O}(V)^{Gl_k},$$

thus the entries in PQ freely generate the invariant algebra $\mathcal{O}(V)^{Gl_k}$. □

Theorem 7.24. *Let $V = M_n/(M_{n-k} \times M_k)$ be the space of $n \times n$ matrices with the upper left $(n-k) \times (n-k)$ and lower right $k \times k$ box only having zero entries. The characteristic polynomial has at most $Min\{k, n-k\}$ non-trivial coefficients. Furthermore, if we let $G = Gl_{n-k} \times Gl_k$ act on the space by conjugation, then $\mathcal{O}(V)^G$ is generated by the* $\min\{k, n-k\}$ *polynomial coefficients in* $\det(M - \lambda I)$*.*

Proof. Let $M \in M_n/(M_{n-k} \times M_k)$ Since the upper left and lower right box are empty we can write the characteristic polynomial as

$$det(M - \lambda I) = \begin{vmatrix} -\lambda I & Q \\ P & -\lambda I \end{vmatrix} = \sum_{\sigma \in S_n} (sgn(\sigma)\Pi_{i=1}^{n}(M - \lambda I)_{i,\sigma(i)}).$$

S_n is the group of permutations over n characters. Each term of the determinant is either plus or minus the product of an entry from every row and every column in the matrix. For every column, you either take the product of an entry from M or λ. If a term contains $M_{i,j}$ then this term can not include the λ from either row i or row j. The diagonal in M is 0 so $i \neq j$.

Each term also has an equal number of λ and entries from M. The highest number of entries from M that are possible is $2\min\{k, n-k\}$. This is due to the fact that you are limited from selecting entries from P and Q a number of times equal to the number of rows or columns in each, depending on which is lower. The lowest of these is $\min\{k, n-k\}$. The non-trivial terms are in front of

$$\lambda^{n-2}, \lambda^{n-4}, \ldots, \lambda^{n-2Min\{k,n-k\}}.$$

There are $\min\{k, n-k\}$ of these.

If we now think of the entries in M as the generators of $\mathcal{O}(V)$, we want to demonstrate that the polynomials in front of λ are G-invariant, and that they freely generate $\mathcal{O}(V)^G$. The Gl_n invariant polynomials over M_n are the coefficients in front the λ^i in the characteristic polynomial, see example 1.2. in Lectures on Invariant Theory[3]. Since the entries outside of P and Q are zero before and after G acts on any matrix in V, it is the case that these polynomial coefficients in front of λ^i are left invariant by G.

Secondly, we want to show that that the polynomials are algebraically independent if we say that the entries in M are polynomial variables. We note that $\lambda I - M$ is equivalent to both

$$\begin{bmatrix} \lambda I_{n-k} & 0 \\ 0 & \lambda I_k - (-P)(\lambda I_{n-k})^{-1}(-Q) \end{bmatrix} = \begin{bmatrix} \lambda I_{n-k} & 0 \\ 0 & \lambda I_k - (\lambda I_{n-k})^{-1} PQ \end{bmatrix}$$

$$\begin{bmatrix} \lambda I_{n-k} - (-Q)(\lambda I_k)^{-1}(-P) & 0 \\ 0 & \lambda_k \end{bmatrix} = \begin{bmatrix} \lambda I_{n-k} - (\lambda I_k)^{-1} QP & 0 \\ 0 & \lambda I_k \end{bmatrix}$$

We may assume that $k \leq n-k$ since if $k > n-k$ we may work with the second one and instead use QP, all the same arguments would hold. By computing the determinant of the first we obtain

$$det(\lambda I_n - M) = det(\lambda I_k - PQ)$$

which is the characteristic polynomial of PQ. From proposition 7.23 we know that the entries in PQ are algebraically independent, the theorem also tells us that any invariant polynomial in $\mathcal{O}(V)^G$ can be expressed using entries in PQ since they are the invariant polynomials of the group action by the subgroup $Gl_{n-k} \subseteq Gl_{n-k} \times Gl_k$. We also know that the characteristic polynomials over the entries in PQ are algebraically independent from the fact that the invariant polynomials of PQ over conjugation are algebraically independent, see $M_n//Gl_n$ [3]. It is therefore the case that the characteristic polynomials in are algebraically independent over the entries in P and Q since the entries in PQ are algebraically independent over the entries in P and Q.

There are no additional invariant polynomials since that would imply the existence of other invariant polynomials over the entries in PQ, but the characteristic polynomial coefficients are all the invariant polynomials over conjugation. Thus we have that the proposed $\min\{k, n-k\}$ polynomials freely generate the invariant algebra. □

Corollary 7.25. *The tangent space* $T_{[(J_k,J_l)]}X_n(D_\infty)$ *has dimension* $\min\{k, l, n-l, n-k\}$.

Proof. Lemma 7.21 and theorem 7.24 tells us that that for $\rho(a) = J_k, \rho(b) = J_l$ we have

$$\dim(T_0 H^1(D_\infty, ad\rho)//C_{Gl_n}(\rho(D_\infty)) = \min\{k, l, n-l, n-k\}).$$

We only need to demonstrate that ρ is semisimple, then we can invoke the article by Sikora which states that the tangent space dimension is $\min\{k, l, n-l, n-k\}$. We observe that for the standard unit vectors e_i that $g \cdot e_i = \pm e_i$. We therefore can spit $\mathbb{C}^n$ into n 1-dimensional spaces, which completes the proof. $\square$

If we combine the findings of this section we know that the tangent space dimension of irreducible points is 1, we know that the tangent space dimension at $[(J_{n/2}, J_{n/2})]$ is $n/2$. This tells us that at most two out of the three following statements can be true for even $n \geq 4$.

- $Inv_{n,n/2} \times Inv_{n,n/2}$ is a smooth variety.
- The dimension of every point is the same (more particularly $Inv_{n,n/2} \times Inv_{n,n/2} \cong \mathbb{C}^N$).
- There exists irreducible representations.

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