Stian Nygård

Learning algebra with Minecraft

A comparative case study on the use of artifacts in pattern generalization activities

Masteroppgave i Matematikkdidaktikk
Veileder: Yael Fleischmann og Frode Rønning
Juni 2022

Masteroppgave

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Preface

First of all, I want to thank my wonderful supervisors, Yael Fleischmann and Frode Rønning, for the collaboration on my master thesis and for your rich knowledge and support throughout the work. The process would not have been the same without your professional tutoring through the vast plains of the didactical field. Thank you also, Heidi Strømskag, for many fruitful conversations and help finding relevant literature. I have had a great deal to learn from all three of you, and I appreciate all the enthusiasm you have shown during the last two years we have known each other.

Further, I want to thank my two amazing cats, Ellie and Gaea, for lying by my side during the most stressful of times. To my dear Julie: I am forever grateful for sailing the ocean with you! You are the wind in my sails, always pushing me forward. On stormy nights, you are the anchor keeping my ship at bay. To my brother Morten "Lektor" Nygård: We were born upon the tide, with the sea we did abide. With sword and pistol by our side, along the coach roads we did ride. Now we'll find a place to rest our spirit if we can, and perhaps we may become highwaymen again. Congratulations with the rank of Master!

Thank you to my fellow MasterChefs Emil and Bjarte, for prolonging life with a good laughter whenever needed. Thank you Sondre, for your everlasting love, support and dance shows. Also, a big thank you to Hans Zimmer for your inspiring music, giving an epic touch to any work process.

Lastly, the biggest thank you is owed to my parents Grete and Trond: For the greatest gift of all time, and for the endless support through my 25 years.

May the force be with you all! Always.

Stian Nygård
Trondheim
June 2022

Abstract

This study has aimed to extend knowledge on the use of artifacts in the work with algebraic patterns and generalization of patterns, with learning outcomes from lower-secondary school. The three artifacts Minecraft, centicubes and pen-and-paper were used in a mathematics class divided into three groups who participated in a work session of 45 minutes, each with one of the artifacts. The goal was to uncover how they engaged in the generalization activities, and how they used different semiotic representations and color coding to convey this generality. The framework of instrumental genesis has been used to identify how the pupils utilized the artifacts, and how the artifacts' constraints and potentialities influenced the pupils' knowledge and work methods.

What I have uncovered, is that Minecraft is a strong tool to utilize in this kind of mathematical activity, because of its ease of use, great visual representation, and because of its engaging nature. The centicubes were a bit harder for the pupils to control and master, but also offered good visual perspectives of the patterns. The pen-and-paper approach showed that it was hard for the pupils to draw three-dimensional structures, and that this constraint limited many of the pupils to engage in useful generalization of increasingly complex patterns.

Keywords: Algebraic pattern generalization, instrumental genesis, Minecraft, centicubes, drawing, game-based learning, semiotic representations.

Sammendrag

Målet med denne studien har vært å finne ut hvordan elever på ungdomsskolen arbeider med algebraisk mønster-generalisering ved hjelp av tre gitte artefakter, Minecraft, centikuber samt papir og blyant. Klassen ble delt i tre, der hver del ble gitt en av artefaktene som hjelpemiddel, og skulle arbeide med et gitt oppgavesett i 45 minutter. Målet var å finne ut hvordan disse elevene deltok i generaliseringsaktiviteter, samt hvordan de brukte ulike representasjoner og fargekoding for å formidle generaliteten i mønstrene. Rammeverket om instrumentell skapelse har blitt brukt for å bestemme ulike muligheter og begrensninger ved artefaktene, og hvordan disse egenskapene har formet elevenes kunnskap og arbeidsmetoder.

Det har blitt avdekket at Minecraft er et kraftig verktøy for å hjelpe elevene i oppgaver som handler om representasjon og generalisering av algebraiske mønstre. Spillet har vist seg å være enkelt å ta i bruk, gi gode visuelle representasjoner og være et engasjerende element i undervisningen. Centikubene var en del vanskeligere for elevene å bruke konstruktivt, men klossene ga også gode tredimensjonale representasjoner av mønstrene. Elevene som arbeidet med blyant og papir viste at det var svært vanskelig for mange å tegne tredimensjonale figurer, noe som igjen førte til at mange slet med å gjøre gyldige generelle antakelser.

Nøkkelord: Algebraisk generalisering av mønster, instrumentell skapelse, Minecraft, centikuber, tegning, spill-basert læring, semiotiske representasjoner.

Pass on what you have learned.

The Return of the Jedi (1983)

Contents

List of Figures

Chapter 1

Introduction

1.1 Background to the research

Ever since I was a little child, I have been obsessed with video games. I have spent countless hours in front of the computer, or in front of the TV with a PlayStation controller in my hand, playing all kinds of games. The reason why is very simple: The captivating nature of video games makes them fun, and very immersive. They can offer different perspectives of both real-world scenarios, like the endless trenches of World War I, or they could represent totally fictional worlds, like the magical realms in *World of Warcraft*. Games can also help simulate situations that otherwise would be dangerous to perform in real life, like explosive chemical reactions, or open-heart surgery. Common for all video games is that they want to give the player an experience of being part of something greater than themselves, and player enjoyment is the most important goal (Sweetser & Wyeth, 2005, p. 1).

In 2003, it was estimated that kids have played video games for 10.000 hours by the time they reach the age of 21 (Prensky, 2003, p. 2). This number is most likely a lot higher today, and the gaming often happens in sessions that last over several hours at a time. Many educators are in despair over pupils' short attention span, but their gaming habits prove that they can be focused if they find enjoyment in what they are doing. It is therefore claimed that "video games are not the enemy, but the best opportunity we have to engage our kids in real learning" (Prensky, 2003, p. 1). Kids are also likely to spend more time and effort on a task if they are intrinsically motivated for it (Malone, 1980). Although this is not a study of motivation, the aspects mentioned here are some of the reasons for wanting to experiment with Minecraft in the classroom – an idea that came to me while I was deeply immersed in the game myself and realized the huge potential it would have for building algebraic structures.

If included in a correct way into the teaching, video games can help engage and immerse the pupils in a way they are not used to. This research will examine how this can be achieved, by looking at *how pupils are using* the video game Minecraft in algebra learning. The game will be compared to centicubes – cubes measuring $1 \times 1 \times 1$ cm that connect to each other with pins and holes – and the traditional pen-and-paper approach. In addition to this, I will identify which strengths and limitations, later defined as potentialities and constraints, that are connected to the three approaches.

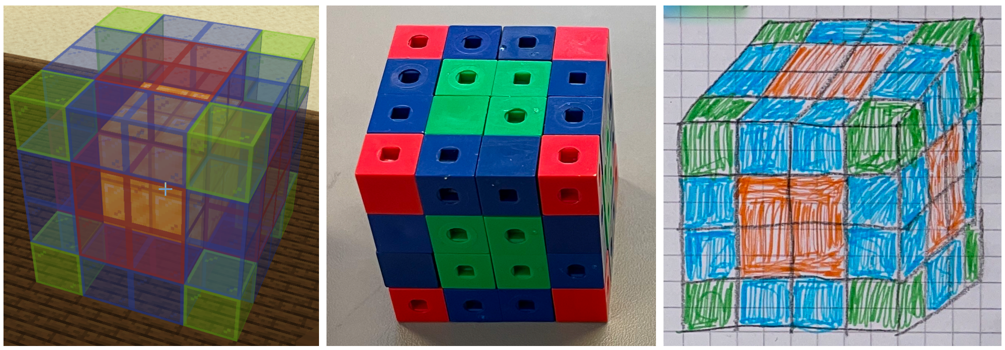


Figure 1.1: Examples of the tools utilized in the research. Minecraft (left), centicubes (middle) and drawing (right).

1.2 Research questions

According to Rivera (2010), a lot of previous educational research has contributed to understand how pupils gain knowledge in algebraic generalization, but less research has established how *teaching* can support the learning of pattern generalization. Further teaching experiments are required to generate more visual strategies for figural transformations (Rivera, 2010, p. 327). This master project aims to shed some light on that area, while also contributing with research in the intersection between mathematics and video games. This will be done by comparing three different visual strategies connected to the teaching and learning of algebra. The research questions that will be answered are the following:

> **RQ1:** How are pupils in lower secondary school using the three different artifacts Minecraft, centicubes and pen-and-paper for presenting and generalizing structures in cubic patterns, and what are the constraints and potentialities of the different artifacts?
>
> **RQ2:** To what extent do the different artifacts develop into instruments in the pupils' work with cubic patterns?

The first question addresses how the different artifacts are utilized in the three different situations. The term artifact is a part of the framework of *instrumental genesis*, which will be described in greater detail in section 3.2. In short, artifacts are material or abstract objects aiming to support pupils' activity in performing given tasks (Trouche, 2005, p. 144). The first research question seeks to describe how the pupils use these artifacts in generalization activities, by identifying the constraints and potentialities of the different approaches. I will give an account of how the pupils use semiotic representations and transformations of representations, and what this can tell about their understanding of algebraic pattern generalization.

Competence aims from the current curriculum, LK20, for lower secondary school have also been used in the development of the first research question. The competence aims that are relevant for 8^{th} and 9^{th} grade, are the following:

> The pupil is expected to be able to
>
> - describe and **generalize** [emphasis added] patterns in one's own words and algebraically, and
> - describe, explain and **present structures** [emphasis added] and developments in geometric and numerical patterns (Ministry of Education and Research, 2020).

The terms that are emphasized can be recognized in RQ1.

With the second research question, I want to establish to what extent the pupils can build learning instruments from the artifacts they are given, based on characteristics of the artifacts, and how they are utilized by the pupils. In the analysis and discussion of this question, I will do a comparison between the three ways of representing the mathematical object.

In other words, this study aims to compare three different approaches to visual representations in the learning of algebraic patterns. Since Minecraft is the novel element in the classroom, this approach will have a bit more focus than the other two in establishing a framework for the analysis. I want to see if Minecraft can offer *new* elements to the teaching process, that can be beneficial for the pupils.

All the three approaches involve interaction between the pupils and the representations of the elements that the patterns consist of. However, all these representations are quite different. Minecraft is a digital platform, where objects can be manipulated quite easily with a click of the mouse. The centicubes are physical objects that also can be manipulated by the pupils, in a similar but analog way. The pen-and-paper approach involves that the pupils need to create all the objects from scratch by drawing them.

To observe what the pupils did with the different tools, they were given a task I had made in advance. The first part consisted of five sub-tasks where the pupils were asked to generalize the pattern n^3 from the first two elements in the sequence. The second part, with three sub-tasks, dealt with the pattern $(n+2)^3$. Here, the pupils were supposed to recognize the connection between the terms from the expression, and the figures they had built/drawn with their respective artifacts. All the tasks are presented in Figure 4.1.

1.3 Thesis structure

The thesis is built up of ten chapters, with sections and subsections within each chapter. This first introduction chapter is meant to give an overview of what you can expect to find, by giving the motivation for the research and the overall structure of the report. Then a chapter will follow that introduces how video games can be a part of education. The term *serious games* will be defined and seen in connection with Minecraft. One section will also give an account of how the current curriculum supports the use of video games.

In chapter 3 and 5, I will give a detailed overview of the theoretical framework, and the methods used to carry out the research from beginning to end. Aspects like sociocultural learning, instrumental genesis, representations of algebraic objects and generalization will be discussed, before the focus is targeted at the research design and its methodology. In the method chapter, I will also discuss important aspects of research ethics and the validity of the qualitative approach. In chapter 4, the tasks given to the pupils will be presented, together with an epistemological analysis of cubes and the cubic numbers. In the following chapters, 6, 7 and 8, I will present the results from the conducted case study and an analysis of the collected data material. In chapter 9, I will present a discussion of the findings, compare the artifacts to each other and discuss my results in light of previous research in the field of algebraic generalization and video games in algebra teaching. The thesis will then be concluded at the end in chapter 10.

Before the introduction comes to an end, I want to give some attention to the nature of this research. The study is framed within the characteristics of a flexible design. It is planned, executed and interpreted by me, the researcher, with good guidance from my two supervisors. The results presented in this thesis are therefore based on subjective interpretations, assumptions and analyses of the collected data. I have however tried to minimize the subjectivity by seeking multiple sources for information on the case, using principles of data triangulation and theory triangulation. I will also give a detailed account of how and why everything has been done, to increase the credibility of the findings.

Chapter 2

Video games in education

In a recent article published in the Norwegian pedagogical trade magazine *Utdanning*, a group called “Didaktisk digitalt verksted”, affiliated with the teacher education department at the University of Stavanger, actualized the use of video games in school. They want to implement game pedagogy as a subject in all education of teachers. They predict that this will be a part of future teaching (Waksvik, 2022).

In this chapter, I want to bring attention to the nature of video games and how they can serve as purposeful tools in education. I will describe elements that make learning fun, define the terms *serious game* and *game-based learning*, justify how video games fit into the curriculum, and lastly explain what Minecraft is.

2.1 *What makes things fun to learn?*

Thomas W. Malone is an American professor of organization, information technology and management, at Massachusetts Institute of Technology (MIT). In 1980, he published a paper where he describes the characteristics of elements that make a video game fun. He also points out how these principles can be used in the designing of instructional games, that is, games designed for learning purposes. The paper has become widely acknowledged as a guide for good game design, with great emphasis on educational games. His research is based on experiments and existing theories on intrinsic motivation – what makes an activity fun and rewarding in itself (Malone, 1980).

In his report, Malone (1980) argues that motivation is an essential part of the learning process, and that when pupils are intrinsically motivated for a task, they are more likely to spend both time and effort on it. His results show that pupils can experience motivation through game elements such as rewards, feedback systems, sensory and cognitive curiosity, and challenges (Malone, 1980). With this outcome, it becomes interesting to test video games in education.

Regardless of the final goals, curiosity is the motivation to learn something. Video games can arouse curiosity by creating environments that trigger the player to explore the game world. Malone (1980) distinguishes between *sensory* and *cognitive* curiosity. Sensory elements can be everything from light, sound or other sensory stimuli of the environment. Cognitive elements on the other hand, trigger pupils' curiosity by giving them a sense that their existing knowledge in some ways seem incomplete or inconsistent. They therefore want to learn more, to get a better structure of their cognitive capabilities.

Computer games usually contain elements of *fantasy*, which make the games more interesting and fun to play and watch. Fantasy is the images that make up the visuals of the game, and it can vary from social or physical impossibilities to completely possible situations (Malone, 1980, p. 164). For example, Minecraft simulates many real-life situations, like fishing, shooting fireworks and animal husbandry, that follow the mechanics of the real world. To harvest wool from a sheep, you need to feed and grow the sheep, and make shears from iron bars, in order to cut the wool. On the other hand, it offers impossible actions like flying and traveling between dimensions.

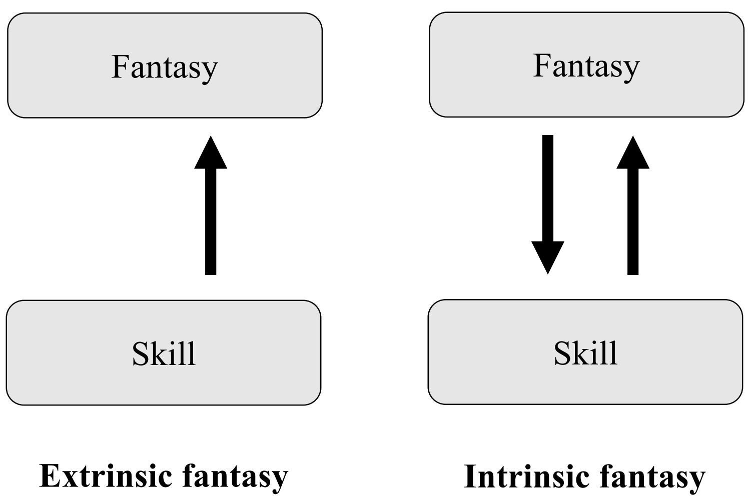


Figure 2.1: Extrinsic and intrinsic fantasy, as described by Malone (1980).

Fantasy is further divided into two different dimensions reflecting the interaction between the game's fantasy and the skill of the player. They are extrinsic and intrinsic fantasy – the difference can be seen in Figure 2.1. To explain *extrinsic fantasy*, Malone (1980) uses the word guessing game *Hangman*, where the players guess letter by letter to find a hidden word. The concept is the same as the traditional blackboard/pen-and-paper game. If the players guess correctly, the letter is added to the word. If their answer is wrong, a stickman is drawn closer to death, one step at a time. In this way, the skill of the player – good or bad – affects the fantasy, i.e., what happens on the screen, but the fantasy does not bring the players any more skill. The stickman does not give them any clue of what the next letter should be, so they will have to keep guessing the next letters, regardless of their previous answers and what fantasy that played out on the screen.

In the second dimension, *intrinsic fantasy*, the fantasy on the screen affects the skill of the player, and vice versa. Feedback from the fantasy will make the players adjust their strategy, to gain better results. Problems are often presented in terms of elements in the game world (Malone, 1980, p. 164), e.g., by blocks in Minecraft. When building a specific pattern in Minecraft, one will receive instant feedback from the visual fantasy if a block is placed correctly or not. Otherwise, if the players are supposed to build structures with a given number of blocks, they can easily count – or simply see – if they have used the correct number.

Figure 2.2: An example of how the fantasy can affect the skill of the player. The figure to the left has a misplaced block, which the player can notice immediately. The figure to the right has all blocks placed correctly.

A well-designed learning game utilizes most elements of intrinsic fantasy. If the player shall be able to receive constructive feedback on their work, they must receive something from the game telling them to which extent they have solved the task correctly. In the cases where extrinsic fantasies dominate, it often leads to methods where the player must guess how to solve tasks.

The main goal for Malone (1980) is to prove that computer technology can be used to make learning more efficient, interesting and enjoyable; in other words, learning can be *fun*. In general, games that are used for instructional purposes – such as Minecraft: Education Edition – are often called *serious games*. In the following two sections, I will define the broad spectrum of games that fall into this category, and how serious games have evolved into the field called game-based learning.

2.2 Serious games

A lot of definitions of the term *serious games* exist. Most of them have in common that they are games that offer other purposes than just entertainment. Based on different perspectives, Susi et al. (2007) give the definition: Serious games are "games that engage the user, and contribute to the achievement of a defined purpose other than pure entertainment (whether or not the user is consciously aware of it)" (p. 5). These types of games are widely used across industries like defense, education, scientific exploration, health care, emergency management, city planning, engineering, and politics ("Serious game," 2022). The purposes of the games can therefore vary greatly, but at the end of the day, they are all *games* that bring with them the joy of playing.

Susi et al. (2007) defines the term *game-based learning* as a "branch of serious games that deals with applications that have defined learning outcomes" (p. 2), a concept that will be elaborated in the following section.

2.3 Game-based learning

"A motivated learner can't be stopped" (Prensky, 2003, p. 1), but how can pupils be motivated into learning new things, especially when they find the content difficult to learn? As reported by several researchers – see for example Kiili (2005), Malone (1980), Prensky (2003) – video games can offer effective ways to keep pupils motivated while playing. The concept of game-based learning (GBL) takes advantage of the motivational factor, along with other elements of a game, discussed in section 2.2, to create a digital learning platform where the pupils can unfold. Prensky (2001) calls the new generation of pupils for *native speakers* in the language of digital media, so this is an arena where they can excel and feel safe. Children's brains are adapting easily to the technology, since they are surrounded by it from early age and spend so much time with it (Prensky, 2003, p. 2). This type of entertainment has therefore framed a lot of the pupils' preferences and abilities, and offers a great potential in their learning.

Some other key elements in video games that make them relevant in a classroom setting are engagement, immersiveness and repeatability. The latter makes it possible for pupils to try out different strategies, modify them and try again. So, what kind of learning can video games offer?

> On the surface, game players learn to do things – to fly airplanes, to drive fast cars, to be theme park operators, war fighters, civilization builders, and veterinarians. But on

> deeper levels they learn infinitely more: to *take in information* [emphasis added] from many sources and make decisions quickly; to deduce a game's rules from playing rather than by being told; to *create strategies* [emphasis added] for overcoming obstacles; to *understand complex systems through experimentation* [emphasis added]. (Prensky, 2003, p. 2)

When working with algebraic pattern generalization, the emphasized parts of the above quote are especially useful. The pupils need to take in the information they are given and make strategies that are useful for generalization. Patterns can often have very complex structures, that require some experimentation before one is able to make general assumptions about them.

Kiili (2005) speaks of *flow* as a mental state where a person, e.g., a video game player, is fully immersed and focused on a specific activity. He reasons for how the flow state can affect learning outcomes and compares it to Vygotsky's zone of proximal development (ZPD, see section 3.1). When playing a game, it is important that the skill of the player is matched with the challenge of the game. If the skill exceeds the challenge, it leads to boredom, and if the challenge is greater than the players' skill, it can lead to anxiety. If the skills and challenges match, the pupils can experience a good flow state where they are able to develop new skills, and thus expand their ZPD. In this state, the players also experience concentration, a sense of control and loss of self-consciousness (Kiili, 2005, p. 15).

2.4 Where do computer games fit into the curriculum?

The current Norwegian curriculum gives the teachers a lot of methodological freedom when teaching. This means that every teacher can choose for themselves how they want to convey the content of the curriculum to the pupils. One possible approach could therefore be to play games with them. In this section, I will highlight some aspects of the curriculum that support the use of video games.

One of the central values in the curriculum states that the pupils are supposed to develop abilities to work, both individually and together, through exploration and problem solving. "Giving the pupils the opportunity to solve problems and master challenges on their own contributes to developing perseverance and independence" (Ministry of Education and Research, 2020, p. 2). As a teacher you want to create an environment that fosters this opportunity to evolve skills in problem solving, and strategies for learning to learn. According to Malone (1980), overcoming such challenges is also a great motivation for learning.

Further, it is important for pupils to understand that mathematics exists everywhere around them, also outside of the classroom. Mathematics is the science of patterns (Strømskag, 2015, p. 474), and patterns can be found in nature, in your home, and even in video games. The work of a mathematician is to seek out these patterns in the different areas of mathematics. Examples can be numbers from arithmetic and number theory, or forms in geometry (Strømskag, 2015, p. 474). Mathematics is an essential subject for understanding that patterns and relationships like these exist within society and nature (Ministry of Education and Research, 2020, p. 2). Most kids today play video games in their spare time, so letting them explore with video games in school is a way to familiarize them with the fact that the surrounding world can be described with mathematics.

The competence aims concerning geometric and numerical patterns – from 8^{th} and 9^{th} grade mathematics – want the pupils to be able to describe, explain and generalize structures they are presented with. With the methodological freedom that teachers are offered, I think that utilizing a relevant three-dimensional computer software can be a very constructive way of teaching. In Minecraft, the pupils will have endless possibilities to arrange, rearrange, explore and experiment with different patterns built up of cubic blocks. One of the core elements in mathematics is *representation and communication*. A representation can be "concrete, contextual, visual, verbal and symbolic" (Ministry of Education and Research, 2020, p. 3). Minecraft mainly gives a visual representation of patterns, that is easily manipulated to the pupils' needs.

In the digital world we live in today, it is natural to include computer-based teaching into the pupils' lives. Computers are becoming very integrated in the classroom, where they can provide easy access to different forms of multimedia, presentations or computations. The learning offered in school is supposed to help evolve the pupils' *digital skills* – one out of five basic skills that are mentioned in LK20. Today's kids are mentioned by Prensky (2001) as *digital natives*, and therefore have the best prerequisites to adapt to digital environments. In mathematics, digital skills are about processing and presenting information using digital tools and also the skill of choosing relevant digital tools for exploration and problem solving (Ministry of Education and Research, 2020, p. 5). Working with computers is therefore essential in educational activity, and since games are generally viewed as fun, the combination – computer games – can be great for engaging.

Another core element in mathematics that is important in understanding why games can, and should be, used in education, is called *exploration and problem solving*. Exploration is the process where pupils use different open strategies to establish patterns and relationships (Ministry of Education and Research, 2020, p. 2). The strategies they choose become more important than the results, because learning to evolve and adapt strategies to new situations can be more useful than the answer itself. A well-constructed game environment opens for an immersive way of exploring strategies, because of its high replayability, that rarely leads to negative experiences. Minecraft is an example of a well-constructed learning game, and I will now explain why.

2.5 Minecraft: Education Edition

Minecraft is a Swedish-developed sandbox video game, that was released in November 2011. "A sandbox game is a video game with a gameplay element that provides the player a great degree of creativity to interact with, usually without any predetermined goal, or alternatively with a goal that the player sets for themselves" ("Sandbox game," 2022). The term *gameplay* is concerned with how the player is supposed to play the game, and how to interact with the elements within it. In other words, the gameplay is defined by the set of rules, challenges and plot of the game that meets the player. The concept of Minecraft revolves around three main actions, to *mine*, to *craft* and to *build*, with cubic blocks of different materials, each the same size. Mining refers to the removal of these blocks, while building refers to the action of placing the same blocks elsewhere. Crafting is the process where the players use blocks or materials to construct tools, items or other blocks from a defined recipe. Because it is a sandbox game, the players can freely choose which blocks they want to gather, and what they want to build from them. This freedom makes the possibilities for exploring and utilizing the game mechanics nearly endless, like a giant sandbox. The game even has a creative mode, where the players can build freely with all resources in the game, without having to gather them first. This is in most cases a more convenient mode to use in the classroom, because the building usually is the main activity.

Since its release, Minecraft has sold 238 million copies, and it is therefore the most sold video game ever, by far ("List of best-selling video games," 2022). There are 140 million monthly active users today, ranging from kids in kindergarten to grown-ups. The Education Edition was released in November 2016, after someone had seen the potential of using its features in the classroom. Since then, it has become widely popular to engage pupils by including this already beloved game in their school life. The teachers can create their own worlds with the elements they want to include, and the pupils can join this world to do whatever the teachers want them to, with predefined limitations. Minecraft: Education Edition is used across subjects, and across grades. In the following, I will explain some of the technical aspects of the game, letting the teachers create the learning

Figure 2.3: Screenshot from the Algebra desert.

environments they want.

As mentioned, there is almost nothing you cannot do in Minecraft[1]. When you start up the game you can choose to enter an existing world, made by someone else or by yourself, or create a new world. When a new world is created, the game generates an infinitely large world consisting of different types of biomes[2], monsters, villages, caves and other structures. Depending on what environment the teacher wants to create, there exists different modes that the game can be set to. The first one is *creative mode*, where the pupils have full access to all materials the game offers. The pupils can fly and the settings are peaceful, so that the monsters – that otherwise roam the world to make things more difficult – cannot harm them. The second is called *survival mode*, where they must mine all the resources themselves, and they cannot fly. In addition to these two, *adventure mode* and *hardcore mode* also exist. The world that I created for this project is called the *Algebra desert*, where the pupils were playing in survival mode. The reason for using this mode was for me to be able to control how many blocks of each type the pupils had available. How this was done, can be read about in subsection 5.3.1.

When setting up the world, the teacher has full control of what the pupils can and cannot do

[1]Throughout the thesis, I will refer to Minecraft: Education Edition as Minecraft.

[2]A biome is a distinct area with unique climate and environment characteristics, e.g., desert, forest or jungle.

when they are playing, with the help of three different types of special blocks that only the teacher can access in the “world builder mode”. These are called *deny*, *allow* and *border* blocks. The names speak for themselves, but in short: The deny blocks can be placed *underneath* anything that the teacher does not want the pupils to break. For example, in the Algebra desert, I put deny blocks underneath all the buildings that the pupils were not supposed to interact with or destroy. This included the giant gateway leading into the desert, the sandcastle where the pupils received their tasks, and the information blackboards placed around, to ensure that the pupils did not break vital items and information.

The allow blocks act as the opposite of deny blocks. These are only used if the teacher makes the world immutable, in which case the pupils are not allowed to build or break anything, anywhere. The allow blocks could then be placed in designated areas where it would be possible to build. I did not have to use allow blocks in the Algebra desert, since it was not immutable.

The third special block is the border block. When placed, these blocks create an invisible border which the pupils cannot cross. This keeps their space of movement restricted to a specified area. The borders are very useful to prevent the pupils from straying too far away from where they are supposed to work. I created a perimeter around the Algebra desert using these blocks, just to make sure that no one would escape and get lost in the otherwise infinitely large Minecraft world.

Chapter 3

Theoretical framework

The theoretical, or conceptual, framework of a research design is part of the main foundation for building a good study. According to Rienecker and Jørgensen (2013) it is *what* that you are *asking with*. In this chapter, I will give an overview of the theoretical concepts that will be used to analyze the data material. Aspects of sociocultural learning theory, instrumental genesis, semiotic representations and algebraic pattern generalization will be discussed. Later, in section 5.1, I will explain how this framework has been adopted in the planning and execution of the research.

3.1 Sociocultural learning theory and the mediation of tools

What is learning? A difficult question to answer, and yet a question with many different answers. From a sociocultural perspective, these answers must be seen in connection with the context that the learning takes place in. The social context *shapes* the learning, and learning is *created* in this context. In this section, I will look at how the sociocultural learning theory frames the context of this study.

The sociocultural learning theory mainly builds on the works of the Soviet psychologist Lev Vygotsky. The essence of his theory is that social interaction and speech play a fundamental role in the development of cognition and higher psychological functions in the individual (Vygotsky, 1978, p. 23). According to Wertsch (1985), these social processes must be seen in interrelation with the fact that the mental processes of pupils "can be understood only if we understand the tools and signs that mediate them" (pp. 14-15). Thus, Vygotsky's conclusion is that the pupils' use of speech is just as important as their use of hands and eyes, to solve a task: "Intellectual development [...] occurs when speech and practical activity [...] converge" (Vygotsky, 1978, p. 24). As children evolve, so does their function of speech. It gets a planning function, which means that their speech guides, determines and dominates their behavior (Vygotsky, 1978, p. 28). To solve tasks, the pupils use words to explain, either to themselves or out loud, *what* they are going to do, and *how* they are going to do it. This function is especially important while working in pairs, where the participants rely on explaining their mental processes to their partner.

According to Vygotsky (1978), learning happens in two different stages. First in interaction with other people, and then through creating internal mental structures. The first stage – which takes place in a social context – is called an interpsychological process, because learning is developed between the pupils while they interact with each other and with artifacts they are given (Vygotsky, 1978, p. 57). This could be pupils working in pairs or small groups, where they have the opportunity to read tasks out loud together and translate them into something they are able to understand. In discussion, they find answers that are later processed and developed internally, at the second stage of learning that Vygotsky (1978) calls intrapsychological.

"An underlying assumption of [sociocultural] research is that humans have access to the world only indirectly, or *mediately* [emphasis added], rather than directly, or immediately" (Wertsch et al., 1995, p. 21). Figure 3.1 shows an adaptation to this study on how Vygotsky (1978) pictured the relation between the learner and the learning outcome, mediated by an external artifact. The mediation is an active process, meaning that the artifacts that are involved do not *cause* action, but they play an important part in *shaping* the action (Wertsch et al., 1995, p. 22).

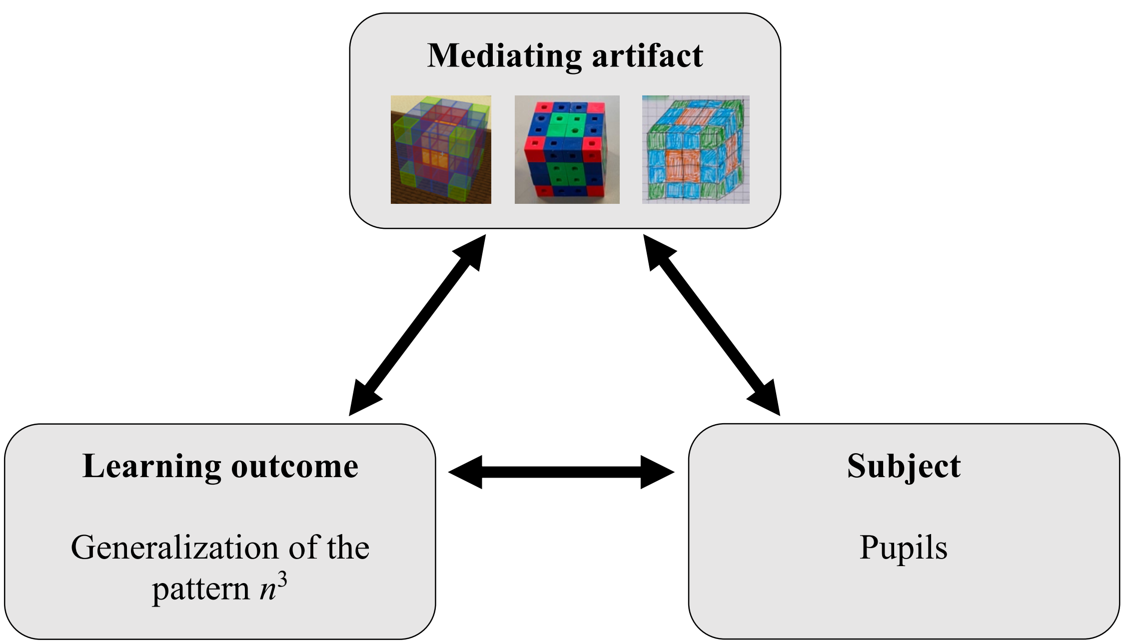


Figure 3.1: A model of the relationship between the pupils and the desired learning outcome, mediated through different artifacts. Based on Drijvers and Gravemeijer (2005, p. 166) and Vygotsky (1978, p. 54).

The use of artifacts seems to be playing a great part in many learning processes. According to Bruner (1996), Vygotsky embraced Francis Bacon's assertion that "[n]either hand nor intellect by themselves serve you much; tools and aids perfect (or complete) things" (p. 152). By this, it is meant that tools or artifacts that are given to a learner can help connect their mind with the physical world surrounding it, and that this interaction can create meaningful learning. Vygotsky (1978) further stresses this fact, that humans are active participants in their own existence and learning, and he therefore proclaims the importance of creating and using auxiliary, or artificial, stimuli. These could be tools presented to them, the language they are exposed to, or their own bodies. Common for all stimuli, is that they are actively adapted by the person who utilizes them. This means that it is a process that is individual for every person, based on their prior knowledge. This results in the definition of the *zone of proximal development*, which is the area that the pupils are cognitively prepared for, and that they need help through social interaction to fully develop (Vygotsky, 1978, p. 86).

From a sociocultural point of view, it becomes apparent that both speech and the mediation of tools are closely linked to any learning process. This mediation is part of the foundation that led Drijvers and Gravemeijer (2005) and Trouche (2005) to seek out a theoretical framework, called instrumental genesis – that will be accounted for in the following section – where the subjects of learning actively develop means to solve tasks using artifacts.

3.2 Instrumental genesis - the birth of an instrument through interaction between pupil and artifact

Because this research is aiming at introducing relatively new elements into an algebraic educational setting, it is necessary to identify how these elements can affect the learning processes of the pupils. Drijvers and Gravemeijer (2005) and Trouche (2005) explain the properties of a framework for understanding and describing the processes behind appropriating to new phenomena, called *instrumental genesis*. This structure distinguishes between two elements, called *artifacts* and *instruments*, and the definition of both is as follows:

- An *artifact* is a material or abstract object, aiming to sustain human activity in performing a type of task; it is *given* to a subject (Trouche, 2005, p. 144).

- An *instrument* is what the subject *builds* from the artifact (Trouche, 2005, p. 144). An artifact

> becomes an instrument when there is a meaningful relationship between the artifact and the user who is supposed to solve a particular task (Drijvers & Gravemeijer, 2005, p. 166).

An artifact can therefore be any tool or resource that is given to pupils in a particular learning situation. In my case, the artifacts are the three different tools, or environments, presented for the pupils: The game Minecraft, centicubes, and pen-and-paper. When the pupils are utilizing these artifacts, they are *building* the instruments. This building process – that Drijvers and Gravemeijer (2005) refer to as the birth of an instrument – is called *instrumental genesis* (Trouche, 2005, p. 144). In this section, I will present relevant terms from this framework that will later be used to identify how the pupils use the artifacts based on their characteristics, and to what extent the artifacts are developed into instruments.

Instrumental genesis is a very complex process, which takes the artifact's characteristics – its *potentialities* and *constraints* – into account. By using these two terms, Trouche (2005) defines an artifact's characteristics. The constraints are linked to the artifact's limitations in being used, while the potentialities are the different qualities, strengths or possibilities that the artifact offers in a given environment. The pupils' activities while working with the artifacts are therefore partly determined by these constraints and potentialities. They also play an important role in the pupils' process of adapting to the environments (Trouche, 2005, p. 139).

Trouche (2005) further suggests some categories related to the properties of artifacts in general, that could help identify their strengths and limitations. *Existence mode constraints/potentialities* are properties of the artifact as either a physical or electronic object, that the user cannot modify. This could for example be processor and memory capacity of a computer or calculator, or the structure of a screen composed of a finite number of pixels (Trouche, 2005, pp. 146-147). Other examples, related to this research, could be the size of the centicubes, or the color of a pen. These are fixed properties of the objects that can have impact on how they are used.

Action prestructuration constraints/potentialities are defined as the link between the artifact and the prestructuration of the user's actions. These are certain properties of the artifact that give the users a predefined area of application. Thus, the pupils are not completely "free" to use the artifacts as they want (Trouche, 2005, pp. 145, 148). For example, the centicubes used in the classroom experiment all have one pin, four holes and one plain face. Therefore, there are only four ways a centicube can be put together with another one, and the pin will always have to be used when connecting two cubes.

The constraints and potentialities of the artifacts play an important role in how they are utilized.

In the analysis of how the pupils are using Minecraft, centicubes and drawing, I will identify what physical and cognitive properties of the artifacts that affect the pupils' ability to build instruments from the tools and solve the given tasks.

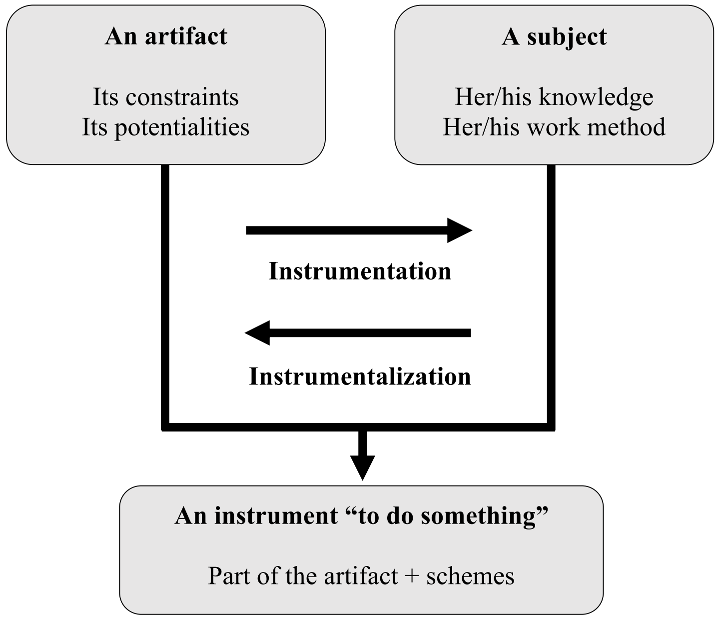


Figure 3.2: Instrumental genesis as described by Trouche (2005, p. 144).

In addition to being a process of building instruments, instrumental genesis is also a process with both *individual* and *social* aspects, that are dependent on material factors, availability of artifacts and the way the teacher includes them in the teaching (Trouche, 2005, p. 150). Material factors could be the related to the nature of computers. Working in pairs with one computer grants both pupils the ability to see the screen and discuss what they see, but only allows one of them to control the computer. In terms of controlling the environment, using a computer therefore favors individual work. Working with centicubes or drawing could to a greater extent favor the social aspects of working in pairs or small groups, because everyone would be able to participate. Factors regarding availability could for example be that centicubes are only available at school, while Minecraft and pencils could also be available outside of the school context.

Since the artifacts are mediators of human activity, the instrumental genesis is further associated with "the subject's activity, her/his knowledge and former work methods" (Trouche, 2005, p. 144). Figure 3.2 gives a model of how the interaction between the artifact and the subject is building the "instruments to do something" – to perform a type of task. This interaction consists of two opposite processes that Trouche (2005) calls *instrumentation* and *instrumentalization*. In the following, I will explain the difference between the two processes.

3.2.1 Instrumentation and instrumentalization

The two interconnected components of instrumental genesis, instrumentation and instrumentalization, is described as the "two-sided relationship between tool and learner as a process in which the tool [...] *shapes* the thinking of the learner [instrumentation], but also is *shaped by* his thinking [instrumentalization]" (Drijvers & Gravemeijer, 2005, p. 190).

Instrumentalization can be summarized as the expression of a subject's specific activity: "What a user thinks the tool was designed for and how it should be used" (Trouche, 2005, p. 148). In other words, this is a process defined by how the artifact is utilized and shaped by the subject. The subject's goal in this process is to master the artifact, modify it and make it useful.

The instrumentalization process contains three different stages. The first is a stage of *discovery*, where a selection of relevant attributes of the artifact is established. This could for example be a process where the pupils determine which commands or abilities in Minecraft that will be relevant for them to be able to solve the task. The second stage of instrumentalization is one of *personalization*, where the pupils "fit the tools to their hands" (Trouche, 2005, p. 148). This could be where they are getting comfortable with working with the artifact, by determining how they are best suited to do so. Lastly follows the stage of *transformation*, where features of the tool are modified to make it easier to use, or different processes connected to it are automated. This could for example include modification of the toolbar in Minecraft or adding new key binds to the software. These types of transformation were not intended or made easily available for the pupils in the experiment of this research, but the transformations would have been possible to carry out. Also, the centicubes do not offer many ways of being transformed in this sense. Sometimes the transformations take directions which were not planned by the designer of the artifact (Trouche, 2005, p. 148), but that is not necessarily a bad thing. Creative pupils can find ways to take advantage of features of the artifact that had not been thought about in the design.

Based on how the three stages explained above are executed by the pupils, the instrumentalization process can either lead to an *enrichment* or *impoverishment* of the artifact (Trouche, 2005). If the pupils can utilize most of the potentialities of the artifact, the position of the artifact in the classroom will be enhanced, and thereby can lead to its enrichment. However, if the constraints limit the pupils' action space so much that the pupils are unable to use the artifact to perform a task, it will lead to impoverishment of the artifact, weakening its position as a learning tool. In the analysis I will look at the pupils' instrumentalization process to identify *how* the different pupils are using the tools they are given, in order to discuss the strengths and limitations of the three artifacts and to what extent the pupils are able to build instruments from them, based on these properties.

Instrumentation is the opposite process of instrumentalization. It is directed towards the subject by the artifact's constraints and potentialities, and it shapes the subjects and their actions. The subjects experience possibilities and restrictions because of how the artifact is constructed. This process can be divided into two different stages. When pupils first meet an artifact, they can go through an *explosion phase*, where an explosion of various new strategies and techniques present themselves. In this phase, the pupils seem to be looking for a way to connect these new possibilities with their previous known techniques. Trouche (2005) describes different phenomena linked to the explosion phase, and one of them is *oscillation*, where pupils oscillate between different techniques. Another is called *zapping*, where they quickly change technique or representation without having time to analyze each approach. The last stage in instrumentation is identified as a *purification phase*, where the pupils reach a stabilization of their *instrumented techniques*. This is a stage where they often are fixated on a few strategies that they have learned how to use, and therefore have gotten rid of the techniques they failed with in the explosion phase.

3.2.2 Instrumented action schemes and instrumented techniques

To further understand the concept of instrumental genesis, Drijvers and Gravemeijer (2005) and Trouche (2005) elaborate on how *schemes* can be used to analyze the use of artifacts. A *scheme* is defined as a stable mental organization of activity (Drijvers & Gravemeijer, 2005, p. 167). The implicit knowledge within a scheme that enables a person to perform a task is called the *operational invariants*. *Utilization schemes* are connected to activities mediated by an artifact, and these schemes consider the technical and conceptual skills needed by the pupils to use the artifact for performing a particular task (Drijvers & Gravemeijer, 2005; Trouche, 2005). The authors distinguish between two types of utilization schemes, called usage schemes and instrumented action schemes.

A *usage scheme* is an elementary scheme directly related to the artifact and the specific activities the user engages in. An example of a usage scheme is the ability to turn on a computer, or to use the mouse and keyboard for specific operations with the computer. Since schemes are mental structures of individuals, they are invisible to an external observer. The operational invariants do however come through in the actions of the user. *Gestures* are the visible part of usage schemes.

Through the pupils' gestures one can get access to their usage schemes (Trouche, 2005, p. 151). It is not possible to see that a pupil is able to turn on a computer, before observing that he/she actually turns it on.

The usage schemes act as building blocks of the *instrumented action schemes*, which are related to the specific transformations of mathematical objects (Drijvers & Gravemeijer, 2005, p. 167). Both technical and mental skills connected to the artifact are required, in order to utilize the artifact to perform the tasks the pupils want to solve with it. The instrumented action schemes also rely mainly on invisible, mental structures, so they too can be hard to reach for external observers. However, parts of these schemes are accessible for observation, research and analysis. These parts are called *instrumented techniques* (Trouche, 2005, p. 151).

A *technique* is a set of gestures, closely connected to the usage schemes of performing a particular task. When artifacts are integrated into the technique, the technique is called instrumented. In other words, instrumented techniques are observable actions performed with the artifacts. While the instrumented action schemes stress mental and cognitive aspects, the instrumented techniques concern the external, visible and manifest parts of the schemes (Drijvers & Gravemeijer, 2005, p. 169). Since they are visible to an observer, they act as an entrance for analyzing instrumental genesis. Examples of such techniques could be the ability to draw cubes with pen and paper, or to use the flying ability in Minecraft to change perspective of the surroundings.

Figure 3.3 puts many of the concepts introduced in this section in relation to each other. The figure shows an example of what an instrumented action scheme *can* look like. It is observed that an instrumented action scheme is built from a collection of different usage schemes, and it consists of a visible part, and an invisible part. The visible part of an instrumented action scheme consists of the instrumented technique, which is a set of gestures – the visible parts of the involved usage schemes. The invisible part of the instrumented action scheme consists of the operational invariants, which are the invisible parts of the usage schemes. Further, it is noted that the operational invariants *guide* the gestures, and the gestures *institute* the operational invariants (Trouche, 2005, p. 154).

During the process of instrumental genesis, pupils develop the mental schemes discussed above that "organize both the problem solving strategy, the concepts and theories that form the basis of the strategy, and the technical means for using the tool" (Drijvers & Gravemeijer, 2005, p. 166). An instrument cannot exist without the mental schemes within the users of it. This is consistent with Figure 3.2, where an instrument is defined as the utilized parts of the artifact, together with the co-existing schemes that give these parts meaning. It is important to note that, even if instrumental

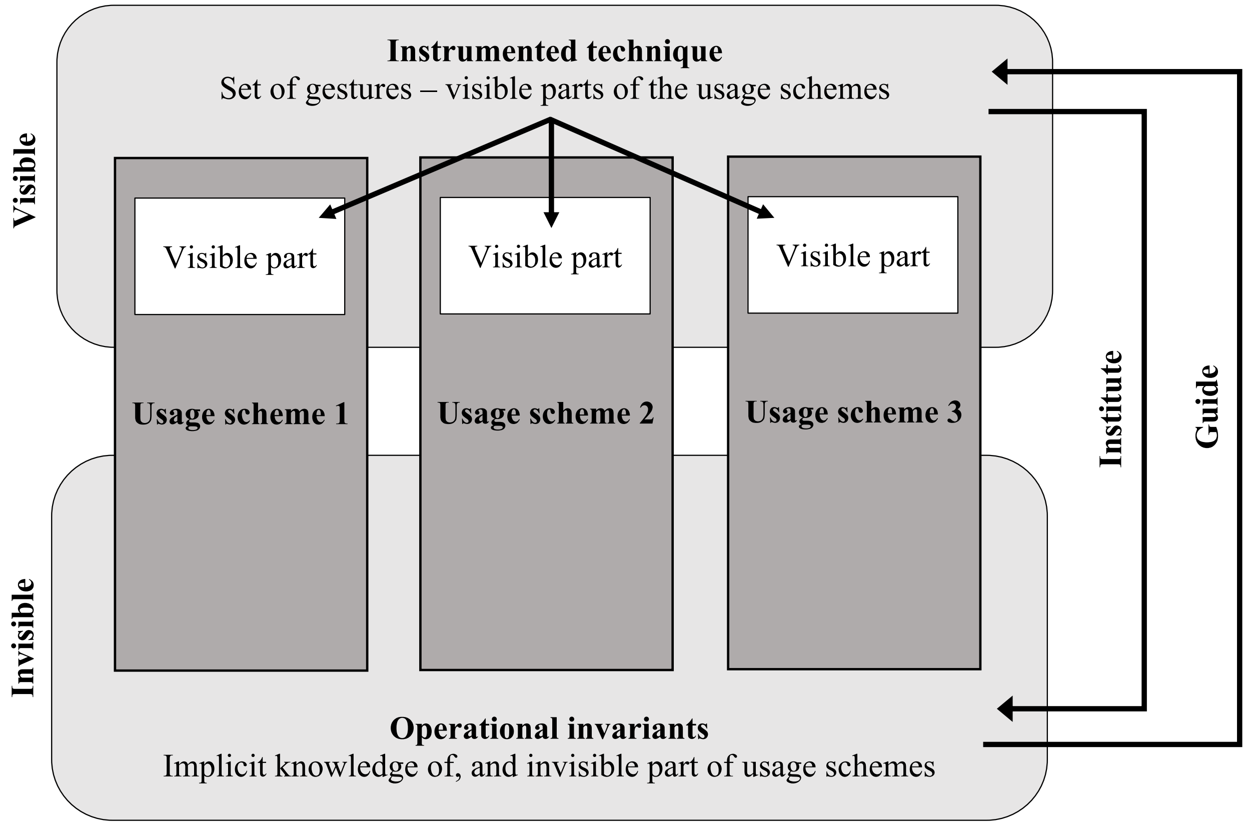


Figure 3.3: Example of an instrumented action scheme, showing the relationships between scheme and technique, gesture and operational invariants. The figure is based on Trouche (2005, p. 155).

genesis often is a social process, the development of utilization schemes is individual, so these schemes can differ from individual to individual (Drijvers & Gravemeijer, 2005, p. 168).

The instrumental genesis framework brings many useful concepts into the analysis of this case study, when it comes to describing and explaining how the artifacts were used, how their influencing properties can be characterized and how the utilization schemes of the pupils can be accessed. I will now move from the world of artifacts and into the world of mathematical objects, and how these can be represented in different ways.

3.3 Semiotic representations of mathematical objects

All mathematical objects are by nature abstract. It is the job of a mathematician to make meaning of them, and to represent them in ways that make them possible to perceive. This section will discuss how semiotic representations are used to obtain and develop knowledge about mathematical objects, and how transformations between representations are helpful – and even necessary – to do so. I will depend mostly on the work of Duval (2006) on this topic, but I will also add some perspectives from Trouche (2005).

When working with mathematical tasks, the representations used by the pupils can tell a lot about their understanding of the subject. The theory of semiotic representations will therefore be used to gain access to the pupils' cognitive apprehension of the objects they are working with. This theory makes it possible to characterize how the pupils present the mathematical objects, and how they translate between different registers of representations. These aspects will be important to answer the question (RQ1) of how the pupils use artifacts to present the algebraic patterns, and how they use these representations to engage in generalization activities, which will be further discussed in the next section.

The definition of a *representation* is something standing for something else. This can be "individuals' beliefs, conceptions or misconceptions to which one gets access through the individuals' verbal or schematic productions" (Duval, 2006, p. 104). In other words, representations are used to communicate or describe objects based on our knowledge and view of them. *Semiotic representations* are those representations that denote, communicate and work with abstract mathematical objects.

Duval (2006) recognizes four different registers of semiotic representations that are utilized in mathematical activity, presented in Table 3.1. A *register*, he defines as all semiotic systems "that permit a transformation of representations" (Duval, 2006, p. 111). The table is split into two different dimensions, one regarding the visual aspects of representations (columns), and the other regarding algorithmic aspects (rows). The non-visual representations are called discursive, and they consist of everything from verbal statements and arguments to algebraic symbols. The visual representations are called non-discursive, consisting of drawings, construction and coordinate systems.

Further, Duval (2006) distinguishes between multi- and monofunctional registers. The distinction lies in whether the representation can be converted into an algorithm or not. For example, a verbal statement cannot be made into an algorithm because of the nature of the representation. A set of equations in algebraic notation on the other hand could be algorithmized, i.e., made into a clear-cut computational process.

By combination of the discursive/non-discursive and multi-/monofunctional dimensions, Duval (2006) ends up with a total of four individual registers. These can be used to analyze the collected

	Discursive representations (Non-visual)	Non-discursive representations (Visual)
Multifunctional registers (Non-algorithmic)	Natural language: - verbal associations - reasoning - proving - describing situations	Shape configurations: - operative apprehension - construction with tools - drawings
Monofunctional registers (Algorithmic)	Numeral systems: - symbolic/algebraic notations - calculation	Cartesian graphs: - change of coordinate system - interpolation/extrapolation

Table 3.1: Examples from each of the four representational registers, based on Duval (2006, p. 110).

data material, by identifying how the pupils were representing the objects in the given tasks, and how they are transforming them. All the pupils worked with the same mathematical objects – the cubic numbers – but with three different tools to represent them. All three tools, or artifacts, were supposed to give visual representations of the mathematical objects, to guide the pupils into giving symbolic representations of the same.

According to the core elements in the curriculum for mathematics, the pupils must be able to switch between different kinds of representations, including the ability to utilize natural language in mathematical contexts (Ministry of Education and Research, 2020). For semiotic representations, Duval (2006) classifies two different types of *transformation* between the registers, namely *treatment* and *conversion*. Treatments – displayed as curved arrows in Figure 3.4 – takes place when a mathematical object is transformed within the same register. Treatments "depend mainly on the possibilities of semiotic transformation, which are specific to the register used" (Duval, 2006, p. 111). A conversion – displayed as straight arrows in Figure 3.4 – is present when the representational register is switched, without changing the mathematical object itself.

I will now look at two examples of treatment and conversion in an algebraic context, given the equation $2x + 4 = 8$. A treatment of this equation can involve the steps in solving it within

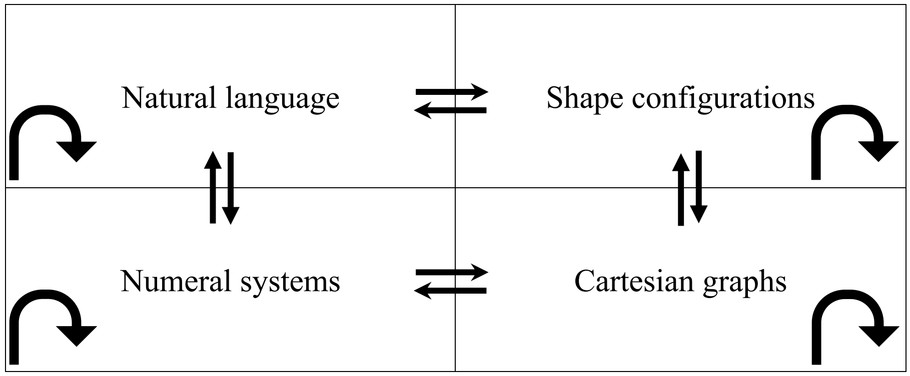


Figure 3.4: Duval's four registers, with curved arrows representing treatments within registers, and straight arrows representing conversions between registers.

algebraic symbol notation, using legal operations on each side:

$$2x + 4 = 8$$
$$2x = 4$$
$$x = 2$$

Treatments are identified between the three steps of the equation $2x+4 = 8$. In Figure 3.5 however, the representational register has been changed into graphs in a coordinate system. The two sides of the equation have been converted into two linear graphs, and the intersection between them represents the solution $x = 2$.

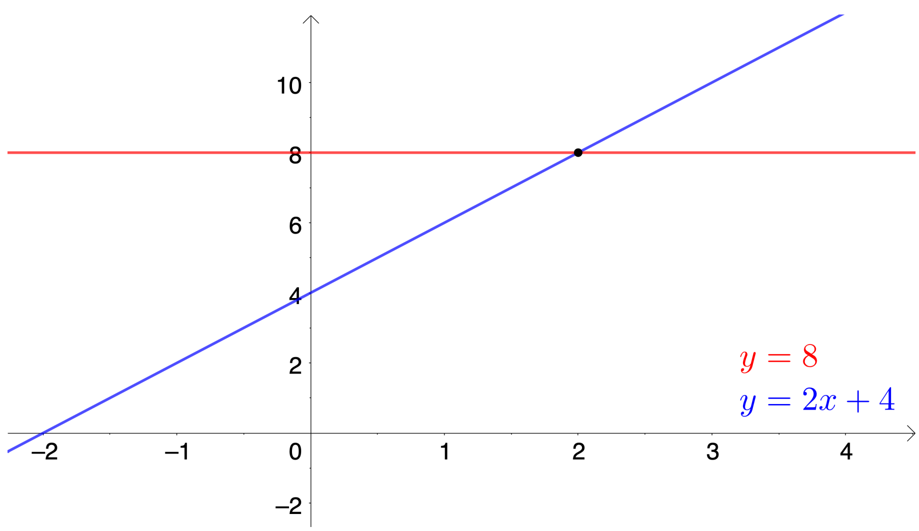


Figure 3.5: The equations $y = 2x + 4$ and $y = 8$ converted to a coordinate system.

Trouche (2005) addresses a didactic phenomenon regarding conversions of representations, that can appear as the pupils are adapting to a given environment, which he calls *localized determination*. This is connected to the difficulty of moving from one register to another. It happens when the pupils are unable to change their point of view. They keep repeating the same type of technique to receive an answer, while making slight adjustments between trials, even if the technique does not seem to be fruitful. Just like Duval (2006), Trouche (2005) also calls out the need to be able to represent a mathematical object in several registers, to form a complete notion of the object.

3.4 Algebraic pattern generalization

When analyzing how pupils use different tools to generalize patterns, it is important to understand how they think, and what processes they go through to justify their answers. They need to establish connections between the structure of a given pattern and how it can be described by a general algebraic formula (Rivera, 2010). To be able to characterize how the pupils engage in visual generalization activities, which was the main goal of the tasks they were given in this study, I will present and use theoretical concepts elaborated by Becker and Rivera (2006), Rivera (2010), Nilsson and Eckert (2019), and Mata-Pereira and da Ponte (2019). These concepts will help answering RQ1, about how the pupils are using artifacts to present and generalize algebraic structures. This previous research will also be considered in the discussion chapter, where I will compare my findings to the existing knowledge, in order to point out my own contribution to the field of algebraic generalization.

On the matter of pattern generalization, Becker and Rivera (2006) distinguish between two modes for communicating generality – figural and numerical. Pupils who are *predominantly numerical generalizers* construct formulas from available numerical elements but are unable to give valid reasons for them. They also seem to rely heavily on trial and error, and making their formulas fit to the information they have, even if they do not understand what the different components in the formula stand for. Because of this, they usually justify their formulas only by the little information they already have examined. Some pupils will be satisfied if they find a formula that fits the first two elements in a pattern.

Pupils who are *predominantly figural generalizers* have the ability to connect symbols and variables to the patterns that generate the figures they are working with. These pupils can detect invariant properties of the elements in the sequence, giving them a relational understanding of the patterns as fixed configurations. Therefore, figural generalizers have greater success than numerical generalizers in finding explicit formulas for arbitrary elements. This might have to do with the different ways they understand the meaning of variables. Figural generalizers view variables in the context of functional relationships, unlike numerical generalizers, who view them as placeholders with no meaning other than being a generator for certain number sequences (Becker & Rivera, 2006, pp. 465-466).

In their research, Becker and Rivera (2006), further elaborated by Rivera (2010), established some theoretical concepts connected to algebraic pattern generalization. The first one is *noticing invariant relationships figurally*. This is the pupils' ability to detect invariant properties of the figural sequence, and that the pattern evolves in a certain, fixed way. A visualization of the patterns often helps pupils notice these relationships (Nilsson & Eckert, 2019).

Secondly, Becker and Rivera (2006) point at *the significance of algebraically useful figural strategies.* The pupils need to choose a strategy that is useful for establishing general formulas, and that is convenient in both near (e.g., 5^{th} term) and far (e.g., 100^{th} term) generalization. The process behind the figural strategies Rivera (2010) calls *abductive-inductive action*. The abductive phase is where pupils form hypotheses from a known stage of a given pattern, to extend the pattern in near generalization tasks, e.g., find the next two elements in the pattern. When they establish a rule, they enter the inductive phase where they test their hypotheses. If they are successful, it can lead to a confirmation of the suggested rule, or else they will have to go through another abduction phase. When they finally confirm the general rule, they can tackle far generalization tasks, such as establishing the 100^{th} term. To achieve meaningful pattern generalization, the process of abductive-inductive action must be followed by a *symbolic action*, where the content is translated into algebraic expressions, preferably direct formulas (Rivera, 2010, pp. 300-301). Usually, the goal is to confirm a general assumption about an evolving sequence or pattern, and turn this into a symbolic representation, but in the final task of this experiment, it was opposite. There the pupils were presented with a general formula, $n^3 + 6n^2 + 12n + 8$, and were asked to decompose this into a visual representation using color codes for each term.

According to Mata-Pereira and da Ponte (2019), the way algebraic problems are presented to the pupils has something to say for the generalization activities they engage in. With a focus on supporting the pupils' abductive reasoning, the authors mainly report on the use of *teacher's actions* in classroom discussions, but the following elements could also apply in this setting, especially during the interviews: *Invitations* to engage in discussion or reasoning, *guiding* the pupils into continue their reasoning, *suggesting actions* by introducing them to information or arguments, and lastly *challenging* them into providing own arguments or solutions (Mata-Pereira & da Ponte, 2019, p. 4).

Further, it is noticed by Becker and Rivera (2006) that differences in the generalization process can be related to *additive vs. multiplicative relationships in pattern formation.* With an additive approach, an element in the sequence is seen as the result of *adding something* to the previous element. Often this strategy will lead pupils to the generalization of a recursive (implicit) formula, and often also leads them to difficulties in far generalization (Nilsson & Eckert, 2019). Take for example the evolving L shaped pattern in Figure 3.6. The first element consists of three black squares, the second has five, and the third has seven. An additive strategy will be to "add 2" to each element to get to the next. The recursive formula would be $a_n = a_{n-1} + 2$, where $a_1 = 3$. This is not a suitable strategy for far generalization. Multiplicative approaches on the other hand, often lead the pupils to establish direct (explicit) formulas. In the case of the L shapes, the direct formula would be $a_n = 2n + 1$. Becker and Rivera (2006) claim that the predominant figural pupils will see multiplicative structures easier (p. 470).

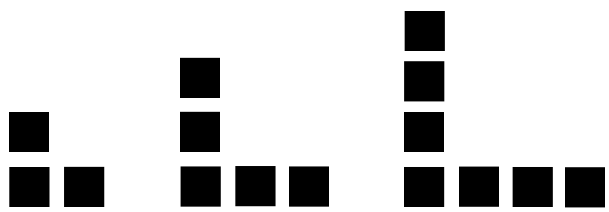


Figure 3.6: L shaped pattern where the n-th element is described by either the direct formula $a_n = 2n + 1$ or the recursive formula $a_n = a_{n-1} + 2$.

Becker and Rivera (2006) and Rivera (2010) distinguishes between two ways of developing general formulas called constructive and deconstructive generalization. *Constructive generalization* refers to when the pupils see the patterns as non-overlapping parts, that add together to a recognized shape. A distinction is also made between *constructive standard generalization* (CSG) and *constructive non-standard generalization* (CNG). The standard dimension denotes that the terms in the direct formula are written in a simplified form, whereas non-standard expressions contain terms that could be further simplified. *Deconstructive generalization* happens when pupils see the patterns as overlapping parts that can be decomposed (Rivera, 2010, p. 305). The distinction between constructive and deconstructive generalization can be made clear by looking at the pattern in Figure 3.6 again. A pupil with a constructive approach would most likely see one black square in the bottom left corner of each element, *and* two equally sized "arms" coming out of it, each with the same number of squares as the figure number. This would lead to a formula like $a_n = 1+n+n$ (CNG), $a_n = 1 + 2 \cdot n$ (CSG), or $a_n = 2n + 1$ (CSG). A deconstructive approach would be to see the figures as two equally sized "arms", that *overlap* in the corner. The corner square is therefore counted twice and must be accounted for by subtracting one of them. A formula could be $a_n = 2(n + 1) - 1$, or $a_n = n + 1 + n + 1 - 1$.

3.4.1 Visualization in the generalization of patterns

Strømskag Måsøval (2011) identified that one of the success factors in algebraic generalization of shape patterns was the pupils' ability to reason visually, i.e., identify, analyze and describe patterns (p. 97). This ability has to be fostered by the teacher through task design and teaching methods. A study performed by Nilsson and Eckert (2019) showed how color coding of elements could stimulate pupils' visualization of algebraic structures in tasks of generalizing patterns. They claimed that pupils have a hard time giving up their perception of a pattern once they have established a structure, and that visualization therefore plays an important part in supporting the symbolic representations. Like Becker and Rivera (2006), they also acknowledged that generalization tasks often are separated into near and far generalization, and that pupils often struggle with the latter. Nilsson and Eckert (2019) showed that the color-coded visual patterns enabled the pupils to make more sense of the variable parts of an expression. They were also able to communicate meaning making by "visually linking parts of the expression to the corresponding parts of the figure" (p. 7), which is exactly what the pupils were supposed to do in one of the tasks in this study.

Nilsson and Eckert (2019) pointed out the need for research to focus more on task and lesson design, and they encouraged a further exploration of how coloring could be used in algebra teaching. In their study, they concluded that to include color coding in the crucial first steps of pattern perception, could support their further generalization. They did not explore in depth which effect this would have in far generalization tasks, but they indicated a need to fill this gap. My study will shed some light on how color coding can support pupils' claims in the work of far generalization tasks, by making them connect colored blocks to the terms of a general cubic expression ($n^3 + 6n^2 + 12n + 8$). Strømskag (2015) refers to this diagrammatic isolation of parts of the pattern as *decomposition* and is valid because of the invariant properties of the pattern.

3.5 Algebra in video games

In 2012, a group of researchers affiliated with the University of Oslo (UiO) and the Norwegian University of Life Sciences (NMBU) set out to gain knowledge about the use of different types of educational resources in the subjects English, mathematics, natural science and social studies. Under the name *ARK&APP* – which translates to paper & application – they made their contribution to how paper-based and digital resources, hereunder video games, could serve in the organization of teaching. They did this through 12 different case studies across the subjects in 5^{th} grade, 8^{th} grade, and first year of upper secondary school (VG1) (Gilje et al., 2016). Through a teacher survey reporting on methods and resources used by the teachers, it emerged that digital resources were seldom used in mathematics, in favor of paper-based ones (Gilje et al., 2016, p. xxii). However, I want to highlight some of the findings from two of the case studies, conducted in a 5^{th} grade and an 8^{th} grade mathematics class where video games were used in the algebra teaching.

In the 5^{th} grade classroom, a variety of visual representations were used to lay the grounds for deeper conversations about the algebraic concepts the pupils were learning. Gilje et al. (2016) found that the use of multiple sources for stimulating algebraic activity would enhance the pupils' understanding and reasoning. The class was working with two different algebra games with very different solving strategies. One was called *Symbolenes verdi* and the other was called *Bike racing Math Algebra Game*. Without using a formal symbolic language, the former game gave the pupils the time and possibility to reflect upon algebraic concepts, structures and connections between them. The bike game on the other hand, was built up as a race between four bikes, where the speed of the player's bike would adjust according to the answers given to arithmetic problems. Wrong answers slowed down the bike, and correct answers sped it up. The bike would also slow down gradually if no answer was given. This added a strong element of competition to the game, which led the pupils to quickly start guessing for answers when they were stuck. Since the *gameplay* of these two games was so different, the way they were used differed accordingly. One of the games fostered reflection, while the other mainly fostered competition and engagement.

The pupils in the 8^{th} grade classroom worked in a similar way with two different digital representations of algebraic concepts. One of them was the game *DragonBox* and the other was the digital math problem solving program Kikora. The symbols used in *DragonBox* have little in common with the formal symbolic language used in mathematics, while Kikora used algebraic symbols. Gilje et al. (2016) observed in a similar way as with the 5^{th} grade, that the game *DragonBox* had the effect of engaging the pupils to a high degree, while Kikora enabled them to learn more of the mathematical content. The analysis did however uncover that the connection between the game worlds and the formal mathematical tasks in school were too underdeveloped (Gilje et al., 2016, p. 67).

Through all the case studies, Gilje et al. (2016) noticed increased engagement in all of the classes that spent a lot of time on video games in mathematics. The engagement was however prominent in the games that contained an element of competition. The competition element led the pupils to guessing for answers to a much higher degree than the games where they had time to reflect on their answers. The results of the case studies showed that the computer games created enthusiasm and curiosity, and the authors also found more interaction between pupils working in pairs with the games. The authors therefore problematize the connection between engagement and learning, because if one overshadows the other, the pupils may struggle to get anywhere. This is something I will examine further in the discussion of this research, where I will look at how Minecraft can connect these two aspects, and act as a more balanced learning game.

3.6 How will the theory be applied?

Before turning the attention towards the research methods, I want to summarize the content of the theoretical framework, to see the parts in connection with each other. How will these perspectives help answer the research questions?

The learning activities in this research take place in a social context, where the pupils are working with artificial stimuli, or artifacts. These adapt to each pupil individually, based on their prior knowledge of the artifact and the mathematical tasks. The sociocultural learning theory is framing the context of the learning situations, where intellectual development happens in the convergence between speech and practical activity.

The main part of the framework is the instrumental genesis. Both RQ1 and RQ2 are based on terms from this theory. It brings concepts like; artifacts and instruments to define what the learning tools are; potentialities and constraints to identify their strengths and limitations; instrumentation and instrumentalization to describe *how* pupils are using the artifacts and *how* the artifacts shape their actions; and instrumented action schemes of the pupils that are manifested through their instrumented techniques. RQ2 is concerned with identifying to what extent the pupils can build instruments from the artifacts in the work with cubic patterns. To do this, I will look at how the constraints and potentialities influences the instrumental genesis, and whether the instrumentalization processes lead to enrichment or impoverishment of the artifacts. This framework is therefore essential.

RQ1 is concerned with how pupils use the artifacts, but also with how they *present* and *generalize* algebraic patterns. While the instrumental genesis looks at how the artifacts are utilized by identifying their constraints and potentialities, the concepts of semiotic representations and pattern generalization will tell more about the cognitive access the pupils has towards the mathematical objects, through their representations and generalization processes. These two theories create a way to describe how fluent the pupils are to change semiotic registers – between mono- and multifunctional, discursive and non-discursive registers – and how this affects their generalizing of structures. Which figural strategies are the pupils using? Are they establishing additive or multiplicative relationships, and do they have a constructive or deconstructive approach? These are questions that can be answered with the theoretical framework of representation and generalization. I will also use the previous research in these fields to discuss my own contribution to the area of algebraic generalization.

Chapter 4

The tasks regarding cubic numbers

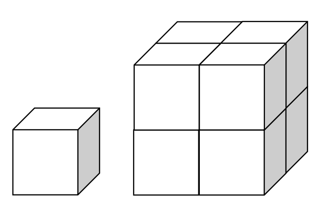
4.1 Worksheet containing the tasks

The pupils were given one task with eight sub-questions, connected to the cubic numbers. The tasks were divided into two parts, where Task a-e were concerned with the sequence defined in this thesis as $\{a_n\} = n^3$. The second part, Task f-h, regarded the sequence defined as $\{b_n\} = (n+2)^3$. In the task description all pairs were given a visual representation of the first two elements of the sequence $\{a_n\}$. The drawing and centicube pairs were given a picture of the first two elements (see worksheet in Figure 4.1), and the Minecraft pairs were given a designated work space where these elements had already been built for them.

The tasks given to the three different groups were the same. The only variations between them, were made so the wording would fit each of the artifacts. The tasks listed in Figure 4.1 are taken from the worksheet given to the drawing pairs. To keep the exact formulation given to the pupils, I have chosen to recite the tasks in Norwegian:

Figure 4.1: Worksheet with Task a to h, given to the pupils.

På bildet under ser dere begynnelsen på en følge med figurer som tenkes å fortsette i det uendelige. Bildet viser figurene med nummer $n = 1$ og $n = 2$.



a) Tegn hvordan figur 3 og 4 vil se ut.

b) Hvor mange kuber er det i hver av de fire figurene?

c) Hvor mange kuber tror dere det vil være i figur nummer 10 i følgen? Forklar hvorfor dere tror det.

d) Hvilken sammenheng kan dere se mellom nummeret n på figuren og antallet kuber i figuren?

e) Kan dere finne en formel for å regne ut hvor mange kuber det er i en vilkårlig figur nummer n?

Oppgave f) til h) er på neste side. Ikke begynn på disse før dere er ferdige med denne siden.

f) Dere skal nå utvide figur nummer 1 og 2. Tegn kuber med en annen farge (eventuelt skyggelegg) rundt begge figurene, slik at de passer inn i et mønster der figur nummer n har $(n+2)^3$ kuber (begge fargene til sammen).

g) Hvor mange kuber av hver type er det nå i figur 1 og 2? Fyll ut tabellen under:

	$n = 1$	$n = 2$
Antall kuber av den første fargen		
Antall kuber av den nye fargen		
Antall kuber til sammen		

h) Ved å gange sammen parentesuttrykk, kan vi se at uttrykket $(n+2)^3$ også kan skrives som summen av de fire leddene $n^3 + 6n^2 + 12n + 8$.
Klarer dere å gjenkjenne hva hvert av leddene i uttrykket tilsvarer i figur 1 og 2? Altså, hvor i figurene er n^3, hvor er $6n^2$, osv.? Vis dette ved å markere på tegningen hvilke kuber i det ytterste skallet som representerer hvert enkelt ledd.

4.2 Epistemological analysis of cubes and the cubic numbers

What is a cube and which terms are needed to describe it? What are cubic numbers, and what constitutes a figural pattern? Before the pupils' work with cubes and cubic numbers can be analyzed, the properties of these mathematical structures need to be established first.

A *cube* is a three-dimensional object or shape, with a *length*, *width* and *height* that are equally sized. It is shaped as a box with flat surfaces that are called *faces*. Each face is square, and the sides of these squares are called *edges*. Three faces meet each other at each vertex at $90°$ angles ("Cube," 2022). A cube consists of six faces, twelve edges and eight vertices.

Algebraic figural patterns consist of a sequence of stages following a certain configuration and development (Rivera, 2010, pp. 297-298). The constituents of such patterns are recognized by Strømskag Måsøval (2011) as elements and components. An *element* is the geometrical configuration formed at each stage of the sequence, and the *components* are the building blocks of the elements. The relationship between the terms is shown in Figure 4.2, which is based on the figural pattern from the tasks described in the previous section.

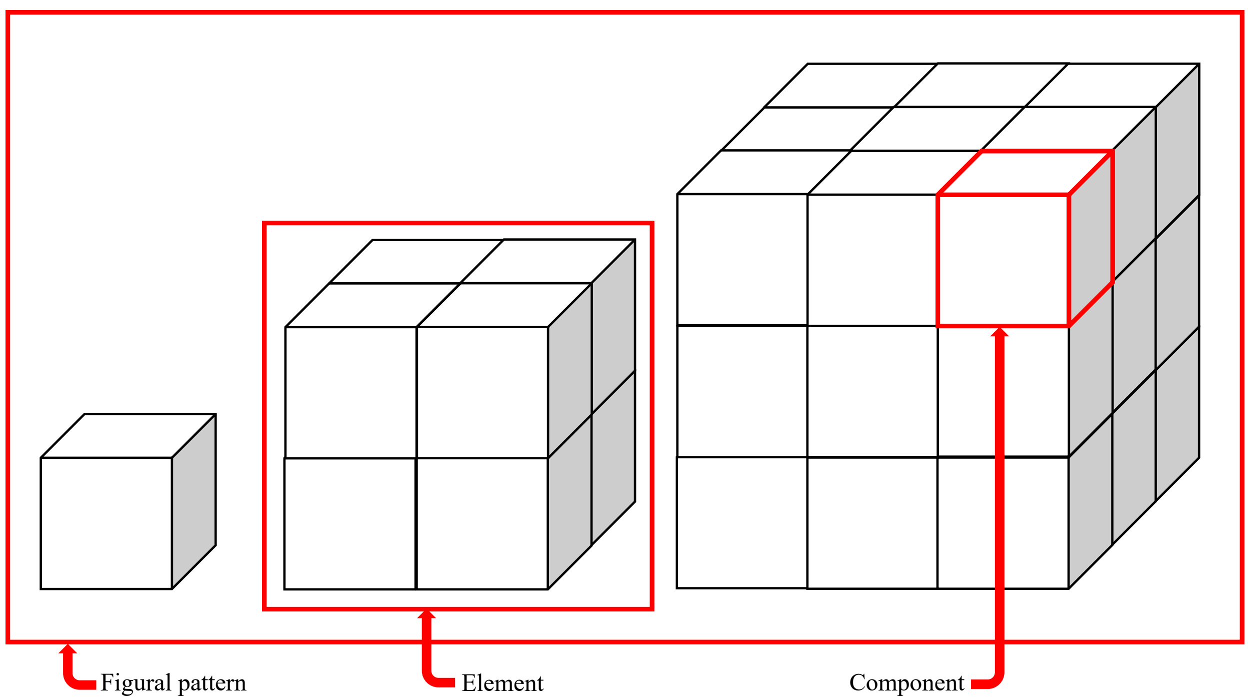


Figure 4.2: Constituents of the first figural pattern given in the worksheet. Terms are based on Strømskag Måsøval (2011, p. 140).

Each of the elements in a pattern sequence has a corresponding numerical value, based on the number of components within the element (Strømskag Måsøval, 2011, p. 140). Figural patterns are visual representations of *figurate numbers* (Kempen & Biehler, 2020). Figurate numbers are number sequences that can be illustrated by an arrangement of equidistant shapes, like the ones in Figure 4.2 which is a representation of the numbers

$$\{a_n\}_{n=1}^{\infty} = \{1, 8, 27, 64, ...\}$$

These mathematical objects are also known as cubic numbers, due to being formed as three-dimensional cubes in a visual representational register. The sequence above, given with a symbolic representation, can generally be written as $\{a_n\}_{n=1}^{\infty} = n^3$, where n is the ordered number of an arbitrary element in the sequence.

According to Rivera (2010), the learning goal of working with figurate numbers is to "engage in meaningful and mathematical viable *pattern generalization*", by constructing and justifying algebraic structures, for example as direct formulas (p. 298). Strømskag Måsøval (2011) distinguishes between two types of figural patterns. The first type (I) is *ordinary figural patterns*, where the goal is to determine an expression for an arbitrary element a_n in the number sequence mapped from the figural pattern, represented by either a direct or recursive formula, like the example given in Figure 3.6. The other type (II) she calls *equivalence patterns*, which aim to establish a mathematical theorem that represents the equivalence between two algebraic expressions of the n-th element in a figural pattern (Strømskag, 2017, pp. 73-74; Strømskag Måsøval, 2011, pp. 141-143). An example is the relationship between the n first odd numbers and the n-th square number, where the equivalence statement is written as $1+3+5+...+2n-1 = n^2$ or $\sum_{k=1}^{n}(2k-1) = n^2$ (Strømskag, 2017, p. 74).

The tasks in this research were of type I, where direct formulas were the optimal solutions. The first five sub-tasks in the worksheet (Figure 4.1) were concerned with near and far generalization of the sequence $\{a_n\} = n^3$. The tasks led up to finding a direct formula for the pattern. In the last three sub-tasks the situation was different. There, the pupils were asked to expand the first two elements from $\{a_n\}$ to fit the *given* expression of the sequence $\{b_n\} = (n+2)^3$. This led up to Task h, where the pupils were asked to decompose the expression $(n + 2)^3 = n^3 + 6n^2 + 12n + 8$ into general color coded parts of the elements. Like Nilsson and Eckert (2019) designed their tasks for near generalization, Task h required the pupils to engage in far generalization by decomposing and color coding each term of the expression on the cubes. This was not a typical design of a generalization task, because the pupils were supposed to make visual generalizations from symbolic expressions, and not the traditional way of going the other way.

The figurate numbers n^3 are an example of a pattern with *high Gestalt goodness*. This measure reflects “the orderly, balanced, and harmonious form of the pattern, which allows learners to specify an algebraically useful formula easily” (Rivera, 2010, p. 304). The pupils’ job is to establish this form, and to make the generalization. In the task they were given in this experiment, they were supposed to generalize from only the first two elements of the sequence. The possibility could therefore arise that some pupils would establish other patterns, that did not follow a cubic approach. Examples of this will be shown in the analysis. The task could have been made less ambiguous, by giving the pupils the first three elements. However, the reason for giving only the first two, was because it would be too demanding to make the pupils draw/build the fourth and fifth elements, containing respectively 64 and 125 cubes.

Chapter 5

Methods

In this chapter, I will describe and give reason for the choices of research methods that have been made in the execution of this case study. It will include a section discussing the research design, based on different perspectives, definitions and theory on case studies. The following section will address the design of the three different environments that the tasks took place in. After that, I will describe the methods that were used for collecting the data material, and how the data was analyzed. At the end of the chapter, two sections will follow that discusses the research ethics and the validity of the methods.

5.1 Research design of the case study

The design process of a study is concerned with turning good research questions into projects (Robson & McCartan, 2016, p. 71). One framework that Robson and McCartan (2016) discuss, takes five elements into account that are important for the planning and execution of the study. These are the purpose and conceptual framework of the research, the research question(s), and the methods and sampling strategies used to obtain the desired data material.

The *purpose* defines what the study is trying to achieve, and why. The *conceptual framework* is concerned with the existing theory of what is going on. This is used to analyze the case within the boundaries of relevant theoretical perspectives. The *research questions* stand in the middle and are meant to guide the whole process of generating and analyzing the data. The questions are supposed to provide answers that satisfy the purpose, while also limiting the research into what is achievable given the time and resources available. The *methods* are the specific techniques used to obtain and analyze the data. Lastly, the *sampling strategy* is the who, when and where of the project (Robson & McCartan, 2016, p. 72).

In Figure 5.1 this framework is adapted to my research. The figure suggests an order in which

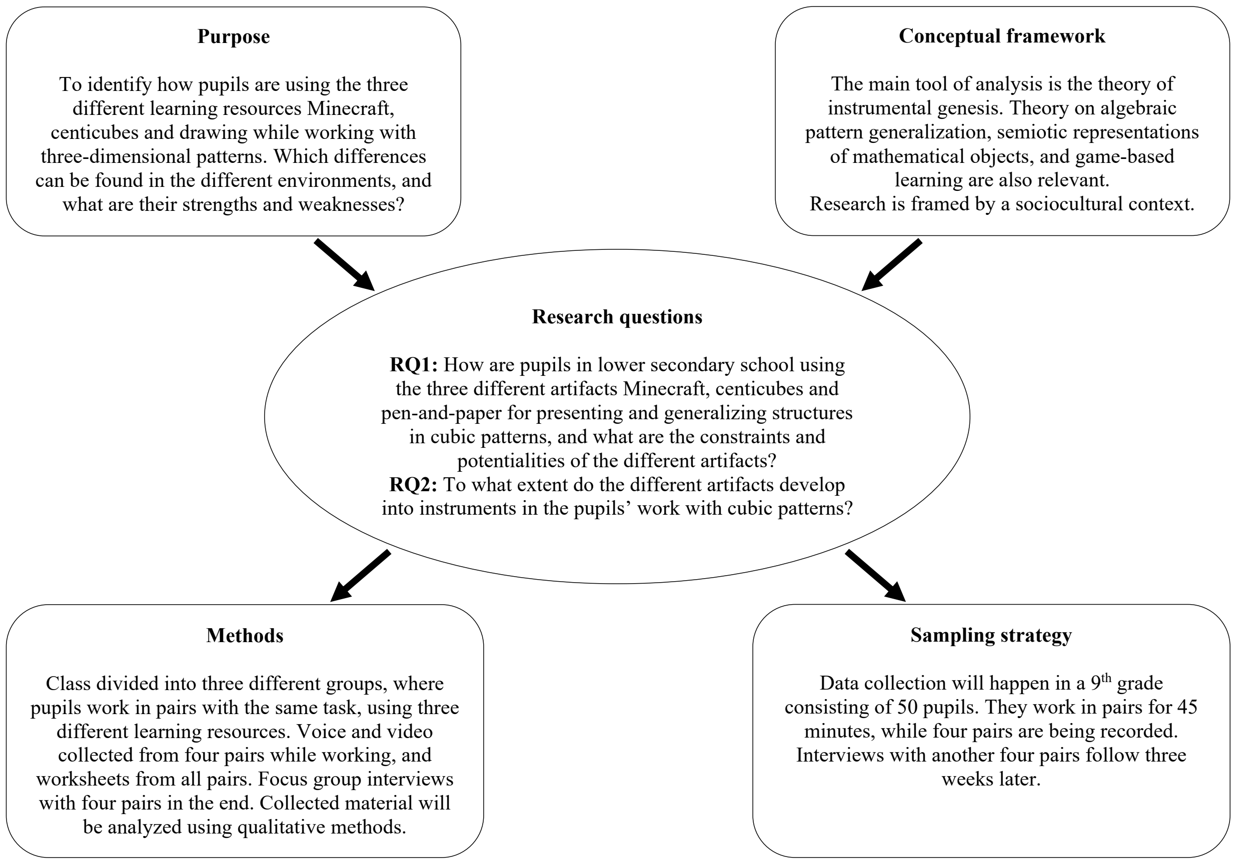


Figure 5.1: Research design framework, based on Robson and McCartan (2016, p. 73).

the study has been planned, starting by defining the purpose and the framework of the study. The process started with a desire to find out how the video game Minecraft could be used in mathematics education. Because the game is based on building with equally sized, cubic blocks, I wanted to make the research revolve around figurate numbers. After experimenting with different tasks to try out with the pupils, I decided to use only one task (eight sub-tasks), regarding three-dimensional patterns, namely the cubic numbers.

All this led to a formulation of the research question, that would define what I needed to find out to satisfy the purpose of the study. The initial question that I asked myself was: “How can Minecraft help pupils with visual representations of algebraic patterns?” After realizing that I would need to buy Minecraft Education licenses for every pupil that was going to enter the game, I had to go back and redefine the purpose to include some other elements that could be compared to Minecraft. At first, I settled for drawing, but after good suggestions from my supervisors, we decided to include centicubes as well. The purpose was then to compare these three tools for visual representations of cubic structures. As suggested by Robson and McCartan (2016), there should always be a repeated revisiting of the framework during the research process in a flexible design. Therefore, the research questions also had a lot of reworks, before landing on the formulation seen in Figure 5.1.

To put this research design into life, I decided to perform a case study approach – following a flexible design – of the pupils working with the different materials. In the following, I will define what a case study is, and describe how its characteristics apply to this project. I will also provide a brief explanation of why this is a flexible design.

5.1.1 What is a case study?

According to Simons (2009), a case study is a study of the unique, or the particular. A case study is rather an approach, than a method in itself. It is depending on a research intent and a methodological purpose, which again have an impact on the methods, like for example interviews and observation, that need to be chosen to gather data. MacDonald and Walker (1975) refer to the case as an *instance in action*. They deliberately use the word instance, to emphasize their belief that it is possible to draw general conclusions from the particular. "Case-study is the way of the artist, who achieves greatness when, through the portrayal of a single instance locked in time and circumstance, he communicates enduring truths about the human condition" (MacDonald & Walker, 1975, p. 2). What they conceive here is that an in-depth analysis of the particular case can gain knowledge of general importance. Nonetheless, it is important to state that whatever the instance might be – the goal, whether it is to generalize or not, is the same: "To present a rich portrayal of a single setting to inform practice, establish the value of the case and/or add to knowledge of a specific topic" (Simons, 2009, p. 24), and to "reveal properties of the class to which the instance belongs" (MacDonald & Walker, 1975, p. 2).

All the above lead towards a definition of what a case study can be. Before stating her own definition, Simons looks upon four different perspectives from other researchers. What they all have in common is their commitment to the study of a situation or phenomenon and its complexities, in a real-life context, independent of the methodology used. She then presents her definition as follows:

> *Case study* [emphasis added] is an in-depth exploration from multiple perspectives of the complexity and uniqueness of a *particular project* [emphasis added], policy, institution, programme or system in a "real life" context. It is research-based, inclusive of *different methods* [emphasis added] and is evidence-led. The primary purpose is to

> generate *in-depth understanding* [emphasis added] of a specific topic [...], or system to *generate knowledge* [emphasis added] and/or inform policy development, *professional practice* [emphasis added] and civil or community action. (Simons, 2009, p. 21)

Stake (1995) further suggests two different types of case study, based on the nature of the case under the scope. These are *intrinsic* and *instrumental* studies. The former is the study of a case for the intrinsic interest in the case itself. This is typically when the case is *given*, and one is interested in studying the particular attributes of that specific case. An instrumental approach occurs when a case is *chosen*, based on the need to explore and answer a research question determined on some other ground (Simons, 2009, p. 21; Stake, 1995, pp. 3-4). My research follows the path of an instrumental case study, which will be explained in the following section.

5.1.2 The case of: Minecraft vs. centicubes vs. drawing

The case in this thesis is a 9$^{\text{th}}$ grade mathematics class, consisting of 50 pupils, working with the three different learning instruments Minecraft, centicubes and drawing. This is an instrumental case study, because the class is an arbitrary class, that was chosen to perform tasks given by the research design. In other words, the case itself was *made* to gain insight into the research questions. The case study is used to "understand something else" (Stake, 1995, p. 3), and therefore it is no intrinsic interest in this class in particular. Instead, the research is based on a wish to explore how the instruments for learning are used by the chosen group. The class could have been any group of 9$^{\text{th}}$ graders in a Norwegian mathematics classroom that would match the design. For this project I chose a class that I knew well prior to the research. This made it easier for me to get in contact with the class, and to plan and carry out the research design.

Following up on the definition from Simons (2009) above, the *particular project* being researched, took place in the pupils' classroom during the school hours where they usually have mathematics. Because of this, all the sessions lasted for 45 minutes each. The *different methods* used to gather data about the case was a mixture of unstructured observation while they were working, collected worksheets from all pairs, video recordings of a few selected pairs, and video recorded interviews afterwards. These will all be explained in detail in the following section (5.2). The purpose of the methods was to gain *in-depth understanding* of how the pupils used the different artifacts to represent the mathematical structures in the specific task they were given, and to *generate knowledge* about the differences between the artifacts.

This research, like most case studies, is based on a flexible research design. This typically means that it involves a very flexible structure, in terms of methodology, research questions, theoretical framework and data gathering. While the main purpose should be clear from the beginning, the implementation of the research does not necessarily follow a linear path, as it is adapting and evolving along the way (Robson & McCartan, 2016, p. 146). A flexible design involves multiple techniques for qualitative data collection. It can also involve some collection of quantitative data, but that is not the case here. The social nature of the research makes it part of an *open system* (Robson & McCartan, 2016). It is a non-laboratory situation, where it is hard to predict what will be the outcome, because of external influences and the fact that "[p]eople, information and all other aspects of the situation are likely to change in ways that may or may not have anything to do with the focus of [the] investigation" (Robson & McCartan, 2016, p. 37). This means that the future cannot be predicted in an open system, but the past can be explained. The methods for doing so will be explained in the following.

5.2 Data collection methods

5.2.1 Sampling strategy

The data collection consisted of two different parts, taking place three weeks apart. The first part involved the implementation of the designed tasks in all three groups, and the second part involved interviews with four pairs. The whole class consisted of 50 pupils, and the distribution into the different groups[1] was as follows:

- **Drawing:** 22 pupils, 11 pairs
- **Centicubes:** 15 pupils, 6 pairs and 1 trio
- **Minecraft:** 6 pupils, 3 pairs

The varying sizes of the groups were mainly a consequence of the way the class is usually structured: The 50 pupils are separated into two parts, one consisting of 30 pupils, and one smaller, consisting of 20 pupils. When one of the parts has teaching in mathematics, the other one has some

[1]Throughout the thesis, I will refer to *groups* as all pupils participating within one approach, e.g., Minecraft group being the six pupils working with Minecraft, and *pairs* as the units within each approach consisting of two pupils that worked together.

other subject, and then they switch after 45, 60 or 90 minutes. The largest part of the class was chosen for drawing, because I thought that this would be the most difficult group to get suitable answers from. Accordingly, I wanted to increase the sample size of this group. The smaller part of the class was therefore chosen to work with the centicubes. Further, six pupils were chosen to work with Minecraft – four from the larger part of the class and two from the smaller part.

The reason for choosing only six pupils for Minecraft was purely practical, because I only had access to two computers and two licenses for Minecraft. I had bought these two licenses in advance, and the pupils were playing the game on two of my own computers, using these licenses. Because of the small sample size, the pupils in the Minecraft group were chosen based on fulfilling the criteria of having some experience with Minecraft, wanting to be recorded, and willingness to participate in interviews after. The six pupils were distributed across three pairs, each with one boy and one girl. The three boys all had a lot of experience from playing the game on PC, while the girls had played it on console, but not as much as the boys. In total, 43 out of 50 pupils participated in the project. Those who did not, did either not give their consent, or were not present during collection.

I wanted the pupils to work in pairs, in order to create some discussion between them while they were doing the tasks. In this way, I could make them interact with each other – as well as with the learning materials. When being forced to talk about the mathematics they were working with, many of the pupils' ideas and conceptions related to the mathematical objects became apparent. For me, this was an opportunity to gain more insight into what they were thinking when doing the tasks, and how they were using the different instruments. Because they were working in pairs, it was also easier to conduct group interviews with them in the second part of the data collection.

5.2.2 Sequence of events

For the first part of the data collection, I visited the class on a Monday and the following Tuesday. The drawing group was the first one to do the tasks. I rigged up the video camera and voice recorder in the back of the classroom and prepared for handing out the tasks and drawing sheets. I picked out a voluntary pair that wanted to be recorded, based on me knowing that they would take the tasks seriously. They were placed in the back of the classroom with the camera. After everyone had received the tasks, all the pupils were given 45 minutes to finish them, before I collected the worksheets. While collecting data from one group, the other two were either having another subject, or were taken out of class to do some other work.

After lunch the same day, the centicube group was ready for action. The implementation of this followed the same setup that was just described for the drawing group. The six pupils from the Minecraft group were doing the tasks the day after. When it was their turn, they were taken into a separate classroom, where I had set up two computers (see Figure 5.2), with an external mouse and keyboard. The screen-capture and voice recorder were ready to start recording when the pupils entered. They were placed inside the Minecraft world, and from there they had to find their way through themselves.

The three pairs who played Minecraft were divided into two sessions, so that the first two pairs played during the first session, and the third pair played in the last session. Two of the pairs that participated were screen- and voice-recorded – one from the first session, and one from the second session.

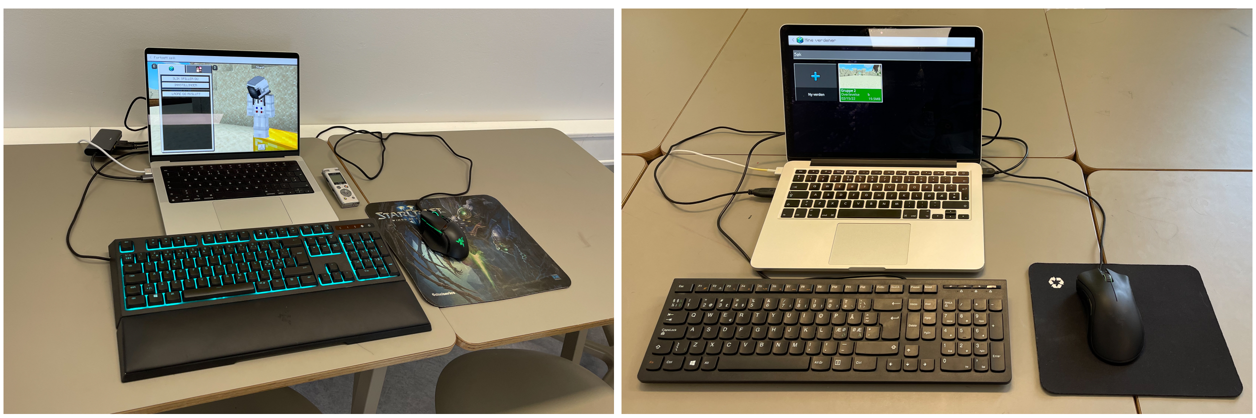


Figure 5.2: Setup of the two Macs used to play Minecraft.

As the last Minecraft pair had finished the tasks, the first part of the data collection was concluded. The interviews in the second part, were conducted three weeks later. In the meantime, I was doing a first round of analysis of the data, to get an idea of who I wanted to interview, and what I wanted to ask them, in order to gather viable evidence to answer the research questions. The interview process will be explained in greater detail soon, as the following subsections will describe how and why the different collection methods have been used.

5.2.3 Written worksheets

To be able to receive some response from all the pupils, even those who were not recorded, I wanted all of them to write down their answers, so that I could collect written material that would include additional perspectives to the research. The pupils working with centicubes wrote on regular, lined sheets, but those working with drawing got grid sheets, to make it easier to draw the cubes. The pupils working in Minecraft wrote their answers in the in-game book called *book and quill*, which is a book with a feather quill that the pupils can write with. In this book, they both receive the tasks, and answer them. In Figure 5.3, an example of a worksheet from each of the three groups is given.

5.2.4 Video, voice and screen recording

In all the sessions where I was collecting data, I used a video camera to capture one of the pairs, with the exception of Minecraft, where screen-recording was used instead. In addition to the camera, a voice recorder was used as a backup in case the video camera would malfunction, and to get better audio quality. The voice recorder was also used for the Minecraft pairs.

In all the situations, the video camera was placed in a stationary spot, capturing the pupils in a way that Bjørndal (2017) refers to as a *semi-total image*. This is a section that is focusing more on a main subject – in this case the objects on the pupils' desk – rather than the total image of the classroom. The camera was placed behind the pupils' backs, focusing on the worksheets and the learning resources they were using. From this perspective, I could get a full view of everything they did, while not focusing on their faces, or on the background noise from the rest of the class. By taking away the attention on the pupils, and targeting their solving strategies instead, they would maybe also feel a bit safer in front of the camera.

Since the Minecraft tasks were performed on a computer, I had the ability to capture a direct recording of the pupils' screens instead of using the video camera to record from an external point of view. I used the software Panopto to capture the screens. The computer camera and microphone were excluded from the recording for privacy reasons. Instead, the external voice recorder was used to capture their conversation. While the pupils were working in Minecraft, I did a screen recording of everything that happened in the game. I started the recording after the pupils had logged into the world.

The observational methods of voice, video and screen recording were very useful for finding out what was going on in the particular situations in this exploratory phase of the research. The recorded material acted as a way of testing the insights of how the learning resources were used, before these insights were explored further in the in-depth analysis of the interviews and the worksheets. By having everything on record, I would also eliminate a great deal of the observational biases that Robson and McCartan (2016) discusses, such as the researcher's selective attention and selective memory. The attention of the video is the same throughout each session, which grants the

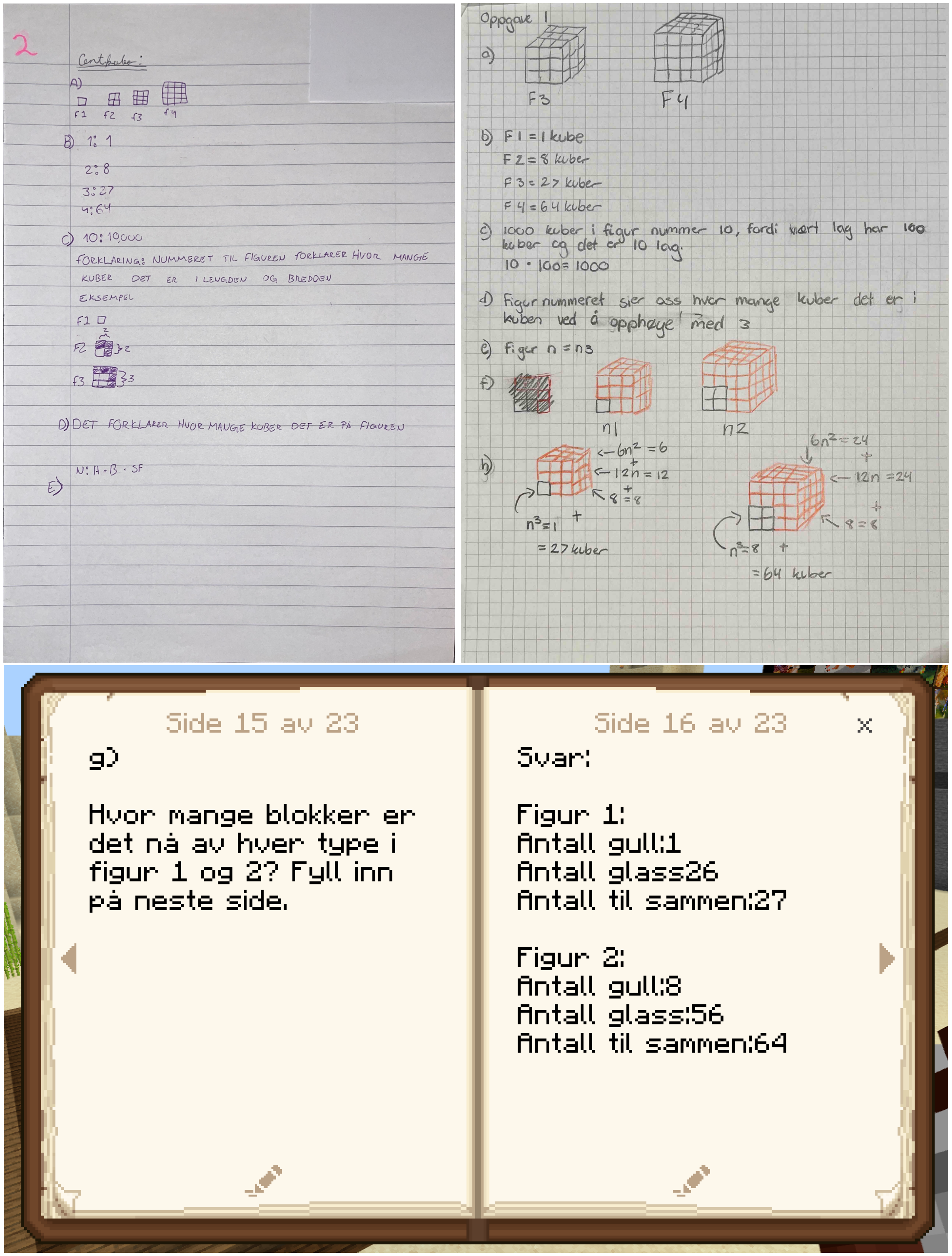


Figure 5.3: Example of worksheets collected from the different groups.

possibility to go over it again and again, in cases where the observed situation was overwhelmingly complex (Robson & McCartan, 2016, p. 331). Also, with regards to memory, the account of the situation was more complete with audio and video. If something turned out unclear, I could return to the video.

In the end, it was very important to transcribe all the videos, to get a clearer picture of certain aspects of the communication. The transcription provided a detailed account of what happened in the different situations. This overview was much easier to handle when coding. In addition to that, the process of writing everything down, really made me think about details that were not noticed before (Bjørndal, 2017, p. 101), details that could be crucial for the understanding of the researched topic.

5.2.5 Focus group interviews

To get a deeper understanding of what the pupils were thinking when performing the tasks, I wanted to interview some of them after collecting the written and digital data material. Since they were working in pairs, the interviews were also conducted with the same pairs, in *focus group interviews*. This is an open-ended group discussion with focus on *one specific topic* (Robson & McCartan, 2016, p. 300). The interviewees were chosen based on the answers they handed in during the first round of data collection. The interviews took place three weeks after.

Interviews are commonly used in qualitative studies, because it is a flexible and adaptable way of gathering information. Asking people directly about what is going on, is a great shortcut to answering the research question (Robson & McCartan, 2016, p. 286). In this case, I wanted to find answers to how the pupils were using their given resources to solve figural pattern tasks. When it came to establishing what they were doing and what they had done, the pupils themselves were in a "uniquely favourable position" to show and tell this (Robson & McCartan, 2016, p. 286).

I conducted a total of four proactive, semi-structured interviews. Proactive means that the interview was aiming at engaging the pupils into analyzing their own practice, regarding how they were working with the mathematical objects. My role as interviewer was to facilitate their reflection (Simons, 2009, p. 48). I therefore brought the pupils' written responses, as well as the artifacts they had used and the representations of the mathematical objects they had produced when doing the tasks, to provide a stimulus for them to recreate their story. The Minecraft pairs were placed back in the worlds they had played earlier, the centicube pair received cubes they could use, and the drawing pair received pens and paper sheets.

The interviews were semi-structured, which means that I had an interview guide (see Appendix B) with a set of questions and topics that were going to be discussed. The wording and order were by default the same for all the interviews, but they could be modified with follow-ups, rephrasing, and additional unplanned questions to stimulate the pupils to elaborate on their thoughts (Robson & McCartan, 2016, p. 285). I further used principles of Mata-Pereira and da Ponte (2019) to *guide* and *challenge* the pupils to contribute to their own reasoning. Guiding the pupils' actions means to, explicitly or implicitly, conduct them along the discussion to make more contributions. Challenging the pupils' actions means leading them to add information or arguments, or evaluate an argument or solution (Mata-Pereira & da Ponte, 2019).

Each focus group consisted of one pair that were also working together in the first part of the data collection. Two pairs from the Minecraft group were chosen, as well as one from the centicube group, and one from the drawing group. I wanted to conduct one interview with a pair from each group, but to get some additional perspectives on the Minecraft approach – being the novel element of the research – I chose an extra pair here. The first three pairs were chosen based on them giving similar answers to the tasks in the first part. Three of the pairs managed to do the first five tasks (regarding the formula $a_n = n^3$) correctly but struggled a bit with generalizing the pattern connected with the expression given as $(n+2)^3 = n^3 + 6n^2 + 12n + 8$. I therefore wanted to see if they could complete the generalization, with me guiding them in an interview setting. The additional Minecraft pair were chosen among those who managed to do all the tasks correctly, and I wanted them to elaborate on how they had approached the tasks. Everything that was done during the interviews, was video and voice recorded, to help in the analysis later.

5.3 Design of the three different environments

In this section, I will give a description of how the tasks about cubic numbers were turned into three completely different approaches, with distinct ways of representing the mathematical objects.

5.3.1 Creating the Algebra desert

For the Minecraft approach, I wanted the pupils to *feel* like they were playing a game. I therefore tried to utilize some different elements that make Minecraft a game, like rewards, sound and music, a clear goal, as well as interaction with the characters and environment. When the pupils first started their mission, they were granted with some good tools that would help them, as an initial

Figure 5.4: The gateway welcoming the pupils.

reward. When they were done with the tasks, they received fireworks in all shapes and colors above the patterns they had built. The sound image of the game was intended to create a nice atmosphere around the experience.

Everything that the players could see and interact with in the Algebra desert, I had made from scratch. I wanted it to have a grand entrance, to make the pupils feel welcome at once. I therefore created a huge gateway that they had to go through to enter the task. This can be seen in Figure 5.4. Once inside the gate, they were met with a blackboard that gave them their “mission”, that is, instructions of where to go. They were asked to travel to the nearby sandcastle, where the non-playable character (NPC) Snøfrid lived. She had a chest that gave the pupils some resources to start with, in addition to a *book and quill* with the tasks. They received a pickaxe, an axe, a shovel, 128 gold blocks, 128 glass blocks, and the book. In the book they could read the tasks and answer them (see Appendix D for content of the book).

I created the world using creative mode, but when it was finished, I switched over to survival mode. The pupils would then only have access to the items I had left for them in the chest at Snøfrid’s house. As described in section 2.5, one cannot usually fly in survival mode, but in Education Edition, the flying ability can be enabled anyway. This ability was given to the pupils, so that they could build the figures more easily, and be able to move around to spectate what they built.

Figure 5.5: Blackboard giving starting directions.

Figure 5.6: The NPC called Snøfrid, giving the pupils instructions.

After the pupils were finished with the tasks, they *signed* the book and quill to make it non editable, and *exported* it as a PDF file. Both these functions exist within the game and are very easy to use. This was a good way for me to be able to collect written material from the Minecraft pairs. An example of the result of this can be seen in Appendix D.

If the reader is interested in a more detailed and immersive experience of the Algebra desert, here is the URL to a YouTube video showcasing the world: https://youtu.be/r7VQh1IY_Ng

5.3.2 Centicubes and drawing

Since all the mathematical tasks had been made in advance, the centicubes and drawing sheets were relatively easy to prepare. For the drawing group, all I had to prepare was sheets that would be suitable for drawing. I printed out a bunch of grid sheets that I had made myself, so that the pupils would be able to draw cubes a bit easier. Therefore, this group only received the tasks – printed on one double-sided page – and enough drawing sheets to be able to solve all the tasks.

It was a bit more time consuming to prepare the tasks for the centicube group, because I wanted all the pairs to work under identical conditions. I therefore prepared one bag of 200 centicubes for each pair, which had to be sorted and counted, in addition to one page with the tasks and lined sheets for them to write on. Every bag consisted of cubes in four different colors. There were 100 building blocks of the first color, 40 of the second, 40 of the third, and 20 of the fourth. The reason for this was to make sure there were enough blocks of each color to first build the first four cubic numbers $\left(\sum_{n=1}^{4} n^3 = 100\right)$, in addition to the extension around the first two elements afterwards. Out of the first color, I had made the elements $\left\{a_n\right\}_{n=1}^{n=2} = n^3$ for them, and put them in the bag, so they only had to build elements a_3, a_4 themselves.

5.4 Analysis of the data material

After the collection of the data material from the three different groups, the qualitative analysis that followed was performed in steps that could be described as:

- Organizing the data
- Identifying a framework
- Sorting the data to the framework

- Using the framework for descriptive analysis

After each session of data collection, all the material was anonymized and sorted onto an external hard drive. The written worksheets were anonymized by removing all the names, and digitalized. All the video and audio recordings, which included the work sessions and interviews, were fully transcribed in the process of organizing. This was a good way for me to get acquainted with the data, while also making it easy to go through afterwards. Everything was organized in a way that was easy for me to navigate through, to find the desired responses.

For each of the three groups, I also produced a *document sheet* that clarified the context and significance of each piece of the collected data material, while at the same time summarizing the content (Robson & McCartan, 2016, p. 467). This process was important for the analysis because it heavily reduced the amount of data. As a part of this process, I also produced a spreadsheet where I made comments and color codes for each of the pupils' answers to every task. This spreadsheet can be seen in Appendix C.

The theoretical framework that was identified in chapter 3, defined the main tools for analyzing the data material. While the research questions and the collected data were used as a guide, I could identify all the theoretical concepts I would need to answer the questions. The data was then sorted to the framework, to have an idea of which parts of the questions that could be answered with each part of the framework. This sorting was used to justify each section in the theoretical framework.

In advance of the analysis, I used a *deductive* approach to decide on a set of pre-existing categories from the theory to create a structure around it (Robson & McCartan, 2016, p. 19). The labels *potentialities*, *constraints*, *instrumentation* and *instrumentalization* were deduced from the theory of instrumental genesis and constituted the pre-existing categories in the analysis. Further, to create new ideas related to the research, I used an *inductive* approach to generate codes that emerged from the data material (Robson & McCartan, 2016, p. 20). These codes made up the induced categories of the analysis. In Figure 5.7, 5.8 and 5.9, organizational charts show the hierarchy of the different category levels. For example, a pre-existing category of the analysis could be *potentialities of Minecraft*, deduced from the theory of instrumental genesis. An induced sub-category of the potentialities could be that *objects are easy to manipulate*, emerged from findings in the data material. Robson and McCartan (2016) call this form of logic, where one cycles between a deductive and an inductive approach, for *abductive reasoning*. Instead of just moving from theory to observation, or from observation to theory, a mix of the approaches can help explain the nature of the open system in this social research.

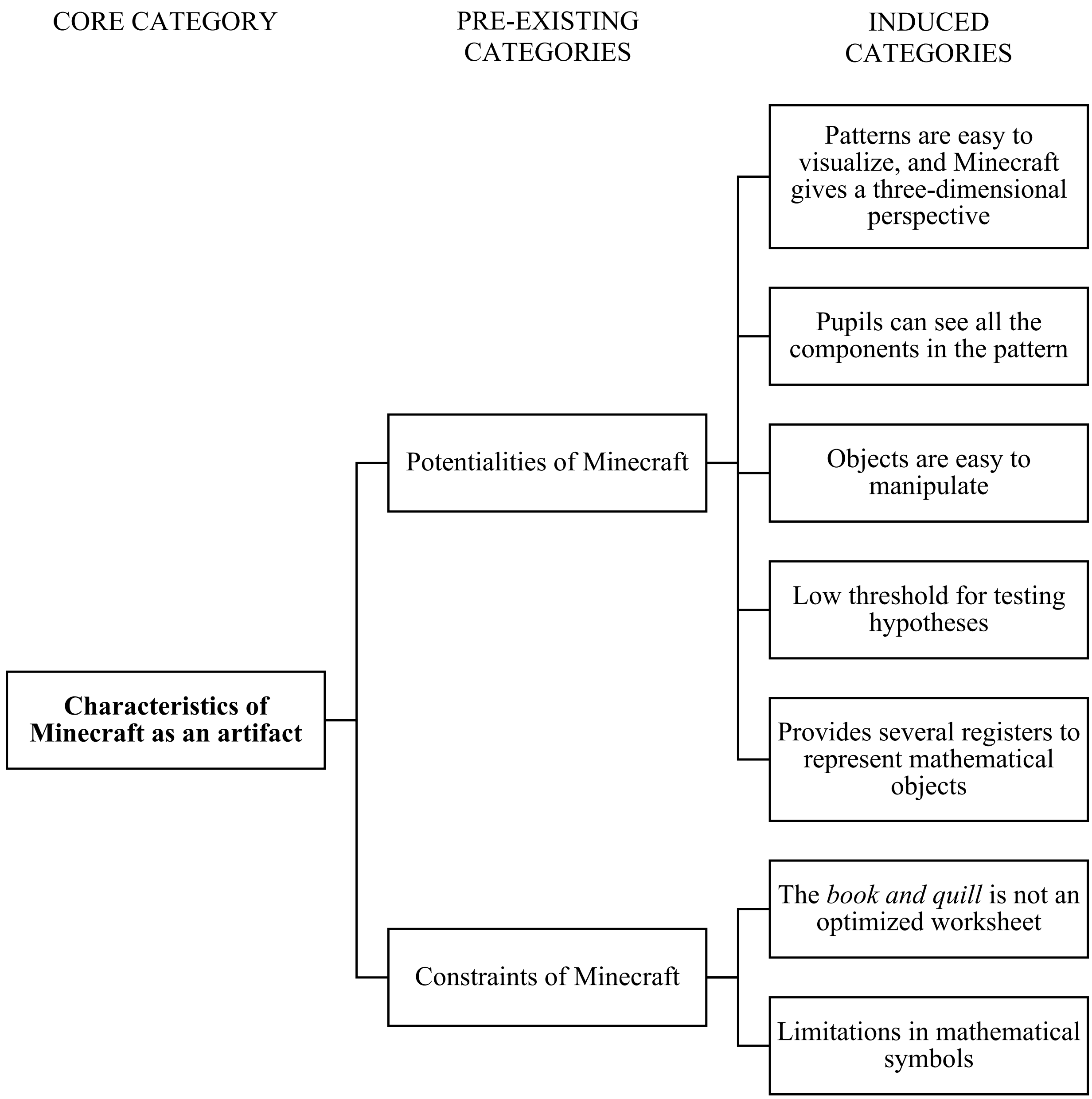


Figure 5.7: The core category *Characteristics of Minecraft as an artifact* with its pre-existing categories from instrumental genesis and induced categories from the data material, presented in chapter 6.

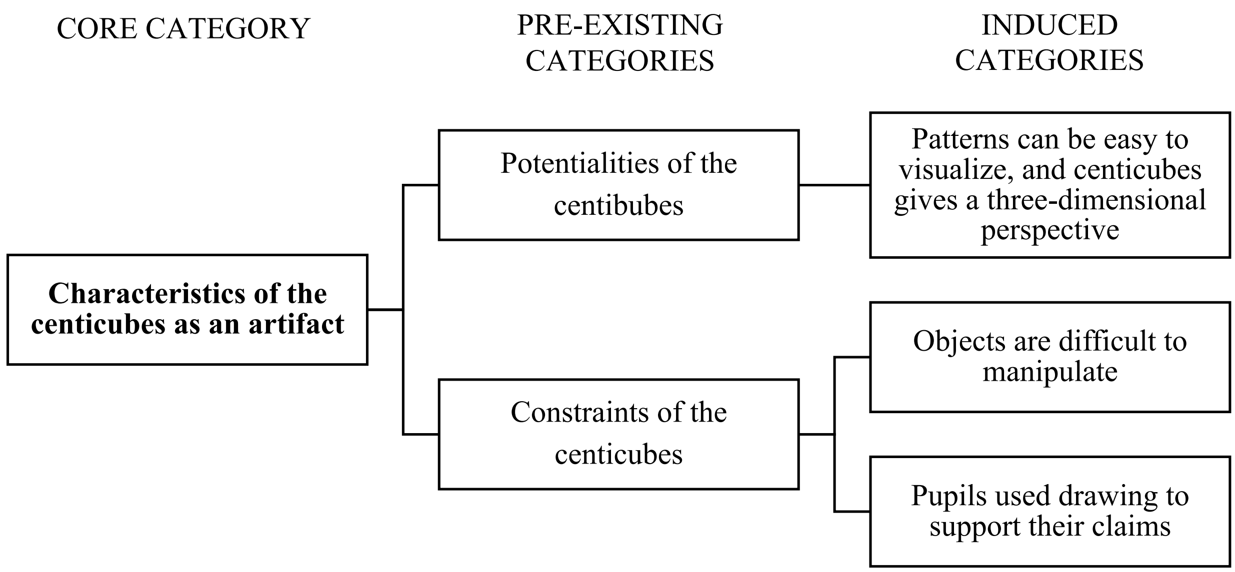


Figure 5.8: The core category *Characteristics of the centicubes as an artifact* with its pre-existing categories from instrumental genesis and induced categories from the data material, presented in chapter 7.

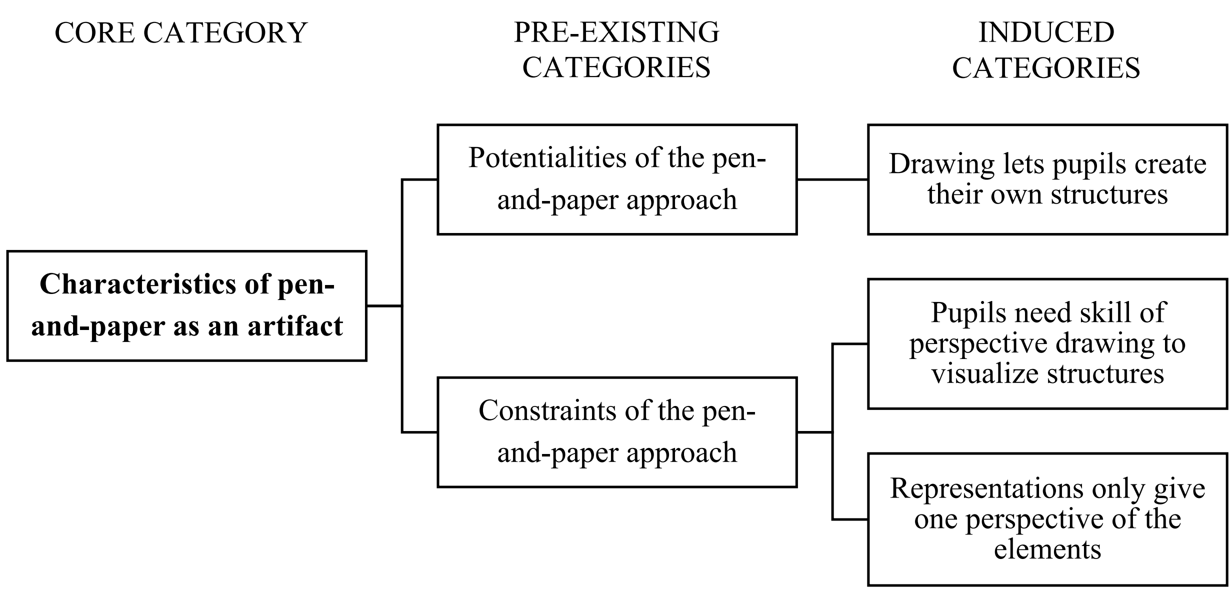


Figure 5.9: The core category *Characteristics of pen-and-paper as an artifact* with its pre-existing categories from instrumental genesis and induced categories from the data material, presented in chapter 8.

The descriptive analysis did not take place until after the theoretical framework was complete, because then I had a defined structure and the perspectives needed to answer the research questions. In the analysis I have looked at each of the three artifacts Minecraft, centicubes and pen-and-paper in its own chapter. For each approach I based the analysis upon the interesting findings that appeared in the data organization, transcription and creation of document sheets, including the color-coded result table (Appendix C). I have answered both RQ1 and RQ2 for each of the artifacts, by analyzing statements and written answers from the work process and interviews. Based on the artifacts' constraints and potentialities, I established how they were used, and to what extent they were developed into instruments in the hands of the pupils. All the results presented in the following chapters have been anonymized, and all of the names mentioned are completely fictional. To avoid too many names, the pairs that do not often recur in the analysis, will be referred to with the pair number they were assigned to when the data was anonymized. The pair numbers correspond with the numbers in Appendix C.

To analyze how Minecraft was used by the pupils to solve the tasks, I have looked at all three of the pairs that participated, since the sample size was so small for this approach. The first pair, consisting of David and Tess, were both screen and audio recorded while performing the tasks, and also participated in the interviews. The second pair, consisting of Jesse and Sarah, only participated in the work with the tasks, and their conversation was voice recorded during the process. Lastly, the third pair, consisting of Joel and Ellie, were neither screen nor voice recorded during the 45 minutes of task work, but they participated in the interviews afterwards. For all the three pairs – in addition to the recordings – I also had access to the exported *book and quill* and a copy of each of the Minecraft worlds that they worked in. Therefore, I had the opportunity to go back into the game and see what they had done, and to take screenshots of their results.

For both the centicube and pen-and-paper groups, I mainly focused on the results of the pairs that were recorded while working and being interviewed, but perspectives from other pairs were also drawn into the analysis for a broader understanding of how the artifacts were used, and to what extent they were developed into artifacts. The data material from the centicube group consisted of seven collected work sheets, one video recorded work session where the pupils Bill and Tommy were the main focus, and lastly a video recorded interview with the pupils Owen and Riley. In addition, some photographs were taken of the patterns that the pupils built during the work session. The data material from the pen-and-paper group consisted of eleven written/drawn worksheets, in addition to a video recorded work session and interview with the pupils Henry and Sam.

The dialogues that will be presented in the analysis chapter, are collected from the transcripts of the recorded data material. A quick guide to the reader on the use of symbols in these transcripts; (parentheses) are used to mark the researcher's own formulations of the quotation; ((double parentheses)) is marking comments from the researcher; [square brackets] are used to indicate an action performed by the speaker; ... dots are used to indicate pauses in the speech, one dot is lasting about two seconds; and finally “quotation marks” are used when the pupils speak in unison.

5.5 Research ethics

Throughout the study it has been very important for me to keep the pupils safe, by maintaining their autonomy and privacy privileges. In this section, I will explain the ethical considerations that have been made along the way, ranging from my appearance in the class, to the application and approval from NSD, in order to create these safe boundaries around the project.

Den nasjonale forskningsetiske komité for samfunnsvitenskap, humaniora, juss og teologi, also known as NESH, is actively working with revising the national guidelines for research ethics. They state that all researchers are subject to statutory duty of care, to ensure that the research follows the current norms of research ethics (NESH, 2021, p. 7). These norms tell the researcher to be honest and fair, and to represent the truth in their findings. Methodological norms demand the research to be verifiable and clear, in order to follow approved scientific methods. This method chapter should be comprehensive enough to be able to verify every step I have taken in the process.

As a researcher, one also has a responsibility towards all the people that are involved in the project. Basic principles such as human dignity, equality, freedom and autonomy, are very important to maintain during the process. Consent from the participants was therefore of the utmost importance. Children that are participating in research have particular requirements to be protected, and also have a very limited consent competence before they turn 15 years old. Because of this, I have used a lot of time to inform both the pupils and their legal parents about the purpose and nature of the research, in order to gain consent from them both. I prepared a combined information sheet and consent form (Appendix A), that the pupils brought home for their parents to read and sign. The pupils that were older than 15 years could sign for themselves but were advised to tell their parents about the project. The consent was voluntary, well informed, and unequivocally, as prescribed by NESH (2021).

To be allowed to do research that involves people, it is required to report the project, and gain permission from NSD – the Norwegian centre for research data. They need to know the details about the project, such as: Who is involved? How will data be collected and stored? When will data be deleted? Which rights do involved persons have? Will the participants be anonymized? All this information was provided to NSD through the same information sheet that was given to the pupils and their parents (Appendix A). I applied for this in the beginning of January and received their approval about three weeks later.

Tjora (2017) points to some general ethical aspects regarding the collection of qualitative data material. Trust, confidentiality, respect and reciprocity are four key elements that should be involved in all communication and interaction between the researcher and the participants (p. 46). These perspectives are also important in most social situations where people meet, so Tjora (2017) suggests that all social research should have higher standards for ethical approach, than in common interaction between people. For example, should the participants be able to withdraw from the research at any point in time, without that being a burden for them. They should also have the possibility to gain insight into what is being written about them, and any citations from them or their work (Tjora, 2017, pp. 47, 176–177). Even if this is rare for participants to do, the pupils' right to do so was clearly stated to them, both orally, and in the information sheet they received.

In Robson and McCartan (2016), it is also stated clearly that in any school-based research, there should be offered non-research alternatives to those who are not willing to participate in the research that happens in the classroom. This alternative should not be stigmatizing, and any credit or compensation offered for research participation should be comparable to that of non-research participation (pp. 226–227). Those who did not offer consent to be researched, were allowed to leave the classroom to work with homework activities. After finishing the data collection, I decided to bake a big cake to the whole class, to thank them for their time and effort in the project, regardless of who were being researched or not, or whether they were participating in videos or interviews.

A thumb rule of observation, and especially when audio and video is recorded, is that there should be minimal chances for the observed to be violated or harmed in any way. This was one of the reasons that I chose to make the video recordings from behind the pupils' backs, so that they would feel as safe as possible. At the same time, it is also important that the recordings are stored in a safe way, such that only those with consent to watch them has access (Bjørndal, 2017, p. 93). I have taken steps to ensure that the pupils' consent has not been abused. As mentioned earlier, all the written data were digitized after removing all their names. All the video and audio clips were transcribed and stored on NTNU devices that only I had access to. Every part of the data material that contains any personal information will be deleted when this project is finished, and this is something the pupils are aware of.

5.6 Validity of the methods

In this last section, I will discuss the validity of the methods chosen for this project. Is it possible to achieve reliable evidence in order to answer the research question? What are the strengths and what are the potential weaknesses of the collected data material? It is important to know the limitations of the research based on the choices made by the researcher, to determine if the presented results can be trusted. When doing a flexible case study, it important to acknowledge that it is the responsibility of the researcher to verify the accuracy of the data collection and analysis presented Robson and McCartan (2016). There are steps to take that assures that this reliability is kept.

First, it is necessary to give a full account of how the data has been collected, and to frame the study within the characteristics of a flexible design. These aspects include "an evolving design, the presentation of multiple realities, the researcher as an instrument of data collection, and a focus on participants' views" (Robson & McCartan, 2016, p. 147). Because of this, the research brings a lot of subjectivity into the picture, that the researcher needs to be aware of (Postholm, 2005, p. 86). He must therefore try to compensate with a rigorous approach, and a heads up to the reader that this is his own interpretation of the case.

Secondly, to make a more valid representation of the findings in the research, Robson and McCartan (2016) also suggest two strategies called *data triangulation* and *theory triangulation*. The former is recognized as the use of more than one method of data collection, in order to receive several perspectives on the same case (Robson & McCartan, 2016, p. 171). I have collected both written data, videos and interviews, to get an in-depth understanding and a more complete picture of the instance. Theory triangulation is the use of multiple theories or perspectives to enhance the rigor of the research (Robson & McCartan, 2016, p. 171). As described in chapter 3, the theoretical framework for this project is based upon both game-based learning principles, as well as theories about semiotic representations and pattern generalization in mathematics, all framed by the theory of instrumental genesis.

There are many strengths of conducting a case study as described in this chapter. According to Simons (2009), a case study can for example investigate and document several perspectives of the *instance in action* and explain how and why things happen. It is also a very flexible research design, in terms of time and methodology. Case studies are very responsive to change of focus area and methods (p. 23), so that in the case of unforeseen consequences of the instance, the framework can be shifted to suit the changes.

When it comes to interviews, it is suggested that “interview results can only be understood as products of the contingencies of the interview situation, and not, as is usually assumed, the unmediated expressions of respondents’ real opinions” (Robson & McCartan, 2016, p. 285). The pupils that participated in the interviews were chosen based on the interesting results they produced in the first part of the data collection. Since I knew these pupils from before, some biases would be expected in their answers, at least in those answers that did not directly concern the ways they had solved the mathematical tasks. If the pupils gave answers that they thought would be the “correct answer”, or an answer that would “satisfy” me as a researcher, this could raise concerns about reliability of the data (Robson & McCartan, 2016, p. 286). However, most of the interview was concerned with the pupils giving reason for how they had solved the mathematical tasks.

I want to conclude this chapter with a few words from Simons (2009), that sum up the issues concerning validity as discussed above:

> It is important to state that in many situations in which case study research is conducted, formal generalization for policy-making is not the aim. The aim is particularization – to present a rich portrayal of a single setting to inform practice, establish the value of the case and/or add to knowledge of a specific topic. (Simons, 2009, p. 24)

Chapter 6

Minecraft as a mediator of generalization

In this chapter, I will present an analysis of the most interesting observations made from the data collection of the three Minecraft pairs, made in light of the theoretical framework presented in chapter 3. The intention of the analysis in the three following chapters is to find answers to the two research questions:

> **RQ1:** How are pupils in lower secondary school using the three different artifacts Minecraft, centicubes and pen-and-paper for presenting and generalizing structures in cubic patterns, and what are the constraints and potentialities of the different artifacts?
>
> **RQ2:** To what extent do the different artifacts develop into instruments in the pupils' work with cubic patterns?

This chapter will be restricted to answer the questions regarding the Minecraft approach. It will be divided into three sections. The first and second section will present the identified potentialities and constraints of Minecraft, respectively. Each sub-section will be dedicated to specific codes related to these two categories, that have emerged from the data material. By identifying the specific potentialities and constraints of Minecraft, I will analyze how they have shaped the generalization activities connected to the cubic patterns. In other words, these two sections will be used to answer RQ1.

To answer RQ2, the third section will contain an analysis of the *instrumentation* and *instrumentalization* processes the pupils went through in the work with Minecraft. In this section, I will look at instances, or instrumented actions, found in the preceding sections, and discuss how they shaped the instrumental genesis.

To do this, I will rely on the work of all three pairs that participated in the Minecraft approach. Two of the pairs – consisting of David and Tess, and Jesse and Sarah – were able to solve all the tasks they were given. The last pair, Joel and Ellie, managed to do all the tasks except for the final one, but under *guidance* and *challenging* (Mata-Pereira & da Ponte, 2019) from me during the interviews, they were able to solve this as well.

In the analysis, I will use the term *element* to describe the geometrical configurations of the pattern sequence, and I will use the terms *component* and *block* interchangeably for the constituents that the elements are built from. The term block originates from Minecraft, where the building units are referred to as blocks.

6.1 Potentialities of Minecraft

In this section, I will present results from the analysis of the *potentialities* connected to algebraic activities in Minecraft. Among these are the abilities to easily manipulate objects in the game, visualize the patterns, having a three-dimensional perspective and a low threshold for testing hypotheses (sudden ideas that emerged in the pupils' work). In the analysis, I have also identified potentialities connected to Minecraft in general; that the game is generally perceived as fun and engaging. The potentialities that are not in direct connection with the algebraic context in which Minecraft was used, will later be used as arguments in the discussion chapter for how Minecraft can help engage pupils in learning.

6.1.1 Patterns are easy to visualize and Minecraft gives a three-dimensional perspective of the patterns

The first identified potentiality of Minecraft is connected to the visualization of patterns. Since everything the pupils are building exists within the game, on the screen in front of them, they can see and interact with it. The game world is three-dimensional, and the player can move around anywhere, so the pupils also gain a perspective on the mathematical objects they are building.

In the first five sub-tasks, the pupils were supposed to use gold blocks as components to build and generalize the sequence $\{a_n\} = n^3$. The sight that met the players in Task a, can be seen in Figure 6.1. When David and Tess came to solve Task a, they looked at the two elements and immediately noticed the invariant property of the sequence:

TESS: [Reading the task] Bruk gullblokkene og lag figur 3 og 4. Ehh, ja fordi at det her er liksom. [Looks around element number 2] Det her er to gange to gange to, [looks at element 1] og én gange én gange én.

Figure 6.1: Element number 1 and 2, that the pupils were met with in Task a.

DAVID: Og så tre gange tre gange tre!

TESS: Jah! [Tess starts building.]

DAVID: Husk på at du kan fly også nå!

TESS: Jah det var egentlig litt lurt å huske på. Nå holdte jeg på å glemme det.

It seemed that both David and Tess used the visuals of the game to quickly establish how the pattern would have to evolve. At the same time, David was guiding Tess through her discovery stage by reminding her of using the flying ability, as this would help her greatly in reaching different areas, and to get an overview of the elements in front of them. David already appeared to have established some usage schemes related to Minecraft, possibly based on his own experience from the game.

As they were about to start building element number 4, Tess was beginning to worry that they did not have enough gold blocks to finish it, because one of the stacks (1 stack = 64 blocks) in their inventory were almost empty, and she did not notice that they had another one:

TESS: Jeg føler vi har fått for lite gullblokker.

DAVID: Ehhh. Neida. [Thinking out loud] Fire gange fire gange fire, det vil jo være seksten gange fire...nei det vil være mer en nok (...)

TESS: Men jeg har bare fem igjen...

DAVID: Og så har du en hel stack med gullblokker til også.

Tess thought they would have to use glass blocks to finish building, but David made a quick calculation in his head that the 4^{th} element would require sixteen times four blocks, so they would have enough. Based on what he already had established about the evolution of the pattern, he could predict how many blocks they would need, and decided that they had enough.

Since David and Tess used the visuals to quickly establish the pattern, they opened the book and were able to calculate the number of blocks in both element number 3 $(3 \times 3 \times 3)$ and 4 $(4 \times 4 \times 4)$, without closing the book to look at the figures again. In this process they were independent of the visual representation of the elements and relied merely on *calculations* based on the invariant properties of the elements, rather than *counting* each of the blocks.

When Jesse and Sarah were supposed to build the 3^{rd} and 4^{th} element in the sequence, Jesse thought that he saw the invariant relationships in the pattern immediately and formed a hypothesis about how these next elements would look like. He went straight into an inductive phase, where he built the two elements to test this hypothesis, without any discussion with his partner. He first built a square with 3×3 blocks, and then extended this to include two extra layers on top, so that it contained $3 \times 3 \times 3$ blocks in total. He followed the same procedure with element 4 afterwards. Based on how the pattern sequence looked, both Jesse and Sarah agreed that they could see that the elements Jesse had built would fit to the evolution of the pattern. Figure 6.2 shows how these elements were built.

Figure 6.2: Elements 1 to 4, composed of gold blocks. All three pairs built identical structures.

JESSE: [Reading Task b] Hvor mange blokker er det i hver av de fire figurene? [Flies around to look at the figures, before opening the book and writing down the answers].

JESSE: [Mumbling] Figur tre, det er ni ganger...tjuesju.... Hva er seksten ganger fire?

To answer the Task b, Jesse first used the flying ability to get an overview of what all the four elements looked like. Then he *calculated* the number of blocks in the elements, using the property that they had an equal number of blocks in each direction, first $n \times n$, and then multiplied the answer with n. Sarah was then raising the question of how he did this:

SARAH: Men det er jo liksom blokker inni, hvordan fant du ut...?

JESSE: Ehh ja fordi da blir det å ta. Her så er det jo én i figur nummer én.

SARAH: Ja.

JESSE: I figur nummer to, så er det [counting] én, to i bredden. Og så tre og fire.

SARAH: Men hvordan, liksom hvor mange som er inni? På en måte. Det er jo blokker inni den.

From the building process, Sarah had clearly recognized that there were more blocks inside of the cubes than what was seen on the outside and was wondering how Jesse managed to calculate how many there were in total. Even though she had agreed that the structure looked correct, it did not seem like Sarah had noticed the invariant relationship yet and therefore wanted to *count* each of the blocks. Jesse tried to explain by showing on the figures they built how many blocks there were in each direction. He therefore stated that there was $3 \times 3 \times 3$ blocks in element number 3, and $4 \times 4 \times 4$ in element 4. Until now, she had adopted a more numerical strategy to the problem. However, as they were asked to find out how many components there would be in element number 10, Jesse used his reasoning from Task b to further convince Sarah of the pattern structure. With different registers – a combination of speech and pointing on the screen – he explained that there had to be an equal number of blocks in each direction of the elements, which seemed to make her see the pattern as well:

JESSE: Hvor mange vil det være i figur nummer ti?

SARAH: Er det ti gange ti gange ti da?

In their written explanation, they stated that since there was $2 \times 2 \times 2$ blocks in element number 2, there had to be $10 \times 10 \times 10$ in element number 10. This conclusion also led them to the following connection between the number of the element and the number of blocks in it (answer to Task d): "Vi kan se at nummeret n er det tallet man må gange med seg selv tre ganger for å finne figurtallet."

To the same task, David and Tess simply wrote down the answer "1000". Their discussion did however give insight into how they had established the multiplicative properties of the pattern:

DAVID: For at, det er jo høyde gange bredde, gange lengde, og da vil du jo, også at alle sidene er ti.

TESS: Ti, ti, ti. Hundre, hundre på hver. Og så ti sånne.

Their answer to the next task supports these statements, where they wrote that the connection between the number of the figure and the number of blocks in the element was: "Lengde bredde og høyde er det samme som figuren sitt nummer." In the interviews I asked them why they used these terms, where it came forward that they related the formula of an arbitrary element in the sequence to the volume of a cube:

TESS: Det var sånn, når man regner ut noe sånn volum, noe vi lærte på barneskolen, var at det da var lengde gange bredde gange høyde, og så tenkte jeg på det.

DAVID: Og at lengden, bredden og høyden [shows with the mouse on the element] er alle det samme som figurtallet.

I further asked them about the generality of these properties of the formula, when an interesting remark about the visual components emerged:

RESEARCHER: Fungerer denne formelen for absolutt alle figurene i denne følgen?

TESS: Ja! Ikke nullte, joo egentlig nullte og ja.

DAVID: Ja for at da vil det være null gange null gange null, og da vil det være ingen blokker.

TESS: Ja, det fungerer på alle.

Both David and Tess were determined that the formula would work for every element, and even for $n = 0$, which would exist, but consist of $a_0 = 0$ blocks. I also asked them if the formula would work for negative numbers, but then the pair concluded that it would not make physical sense to include negative element numbers.

6.1.2 Pupils can see all the components in the pattern

The glass blocks utilized by the pupils have the great feature of being transparent, which makes it easy to see every block in the structures that are built with them. In the following I will explain

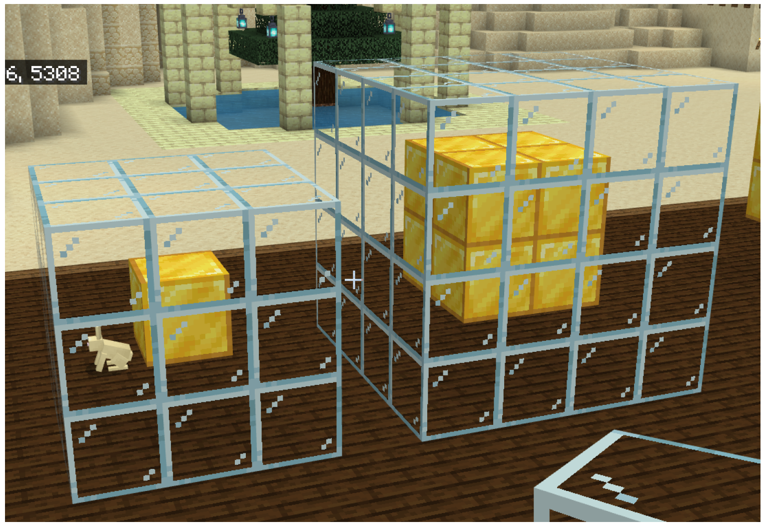


Figure 6.3: The first two elements of the sequence $\{b_n\} = (n+2)^3$ before color coding.

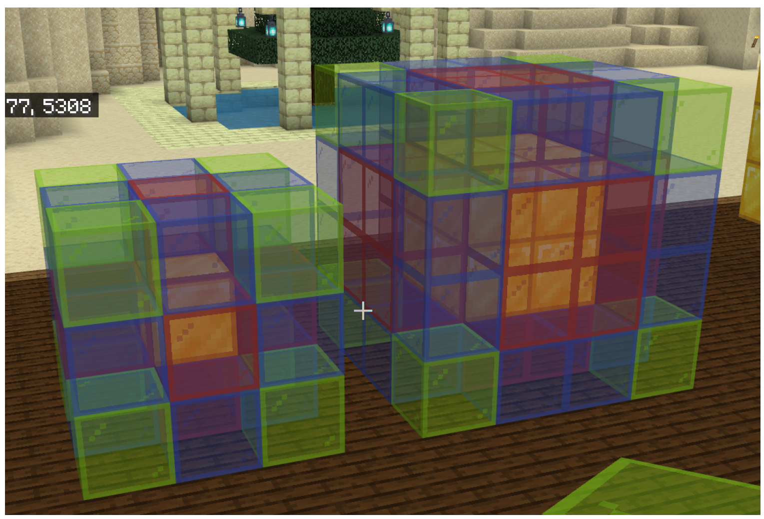


Figure 6.4: The first two elements of the sequence $\{b_n\} = (n+2)^3$ after color coding.

how the pupils found general structures of the patterns by utilizing the glass blocks to see all the components in the elements.

In Task g, the pupils were asked to fill a "table" with the numerical values of the glass components and the gold components in the structure $(n+2)^3$ that they had just built in Task f (Figure 6.3). All the pupils answered by first establishing that the number of gold blocks were the same as in Task a-e (n^3). They further knew that the total structure would contain $(n+2)^3$ number of blocks. To find out how many glass blocks there were in the structure, they therefore computed the difference between the total number of blocks and the gold blocks. Because they could see all the components at the same time, it was easy for the pupils to see that the cubes were composed of the sequence $\{a_n\} = n^3$ in the middle, surrounded by glass blocks that would then equal $(n+2)^3 - n^3$.

Figure 6.5: Above is the first element in the sequence $\{b_n\} = (n+2)^3$, consisting of 1 gold block surrounded by 26 glass blocks. Below is Task g, as solved by David and Tess.

In the interview with David and Tess, we talked about how they had utilized Minecraft in their work. During the first part, I did not uncover any aspects of the tasks that had not already been mentioned in the work session, but some interesting results regarding the generalization came forward at the end. I showed them a piece of paper where I had written the terms $n^3 + 6n^2 + 12n + 8$ with pen, so that I would be able to point at them when asking questions. First, I asked them if they could recognize the respective terms on the cubes they had built. Figure 6.6 shows the element they used as example when they explained their thoughts. They pointed at the element and explained where they had placed the different terms, and then I asked why it would have to be like that:

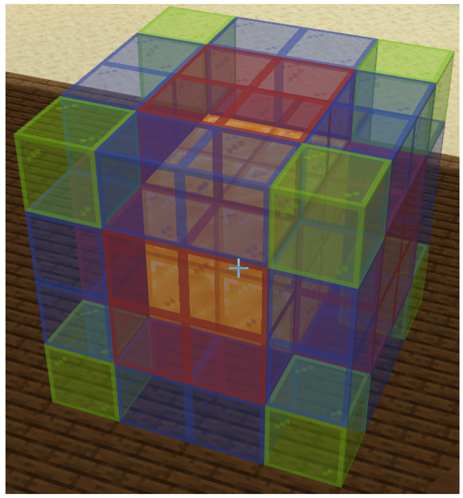


Figure 6.6: The color codes of David and Tess.

DAVID: Åtte det er jo kantene ((wrong term for vertices)), fordi at det vil alltid være åtte kanter i en firkant, nei en kube.

RESEARCHER: Når du sier kanter, mener du da..

TESS: "Hjørner!" For at jeg husker at man bruker å si kanter, men egentlig så er kanter sånne streker [points along the edges of the element].

RESEARCHER: Så, hvor mange kanter er det her da?

TESS: Eh, da har den, tolv.

DAVID: Seksten.

David pointed out that cubes always have eight vertices but used the wrong terminology. Tess stated that it is common to mix up edges and vertices, and this probably had to do with everyday concepts that the pupils have in their minds (Rønning & Strømskag, 2019). In this case, both David and Tess resolved the difference between the two concepts, and I asked them further to agree if it was twelve or sixteen edges in the element. They utilized the perspective that Minecraft gave them by flying around the element while counting. "Det ER tolv!" David exclaims. Afterwards, I wanted to engage them in an explanation of why their results would be valid as a general rule. I used principles uncovered by Mata-Pereira and da Ponte (2019), by *guiding* and *challenging* them into going deeper into their justifications:

RESEARCHER: Hvilken betydning har variablene og konstantene i leddene for figurene? ((Guiding them into a discussion of the distinction between variables and constants.))

DAVID: For eksempel [grabs the mouse and keyboard, and goes to look at element number 1], på her, når du har figurtall 1, [uses the example of the edges, and shows what he talks about on the figures] så vil du ha tolv blå, mens på figur 2 [looks at element 2] så vil jo de kantene bli dobbelt så stor. Og det endrer seg også med variabelen, at det er tolv gange to i stedet for tolv gange én. Og det vil jo bare fortsette å følge samme mønsteret. Her [looks at the $3 \times 3 \times 3$ gold element] så ville det jo ha vært tre blå (på hver kant), her [$4 \times 4 \times 4$ element] fire, [points ahead] fem, seks, sju.

David looked at the elements in the game to support his claims about how the cubes grow bigger with an increasing variable number. He noticed that the length of the edges doubled from element 1 to 2, and also made general assumptions about the nearby elements.

RESEARCHER: Hvordan tolker dere det at tallet åtte ikke har noe variabel tilknyttet seg?

TESS: Det er alltid åtte hjørner.

DAVID: I en kube.

RESEARCHER: Så det er alltid åtte uansett? ((Challenging them to add information about their argument.))

TESS: Ja, også, hjørnene blir ikke lengre. [Points to the vertices] Det er alltid bare sånn én på hjørnene.

They both agreed that it always would be eight vertices in a cube, and Tess also used the cubic element in front of them to conclude that the length of the vertices would always stay the same, and therefore this was a constant number.

RESEARCHER: Hvordan tolker dere leddet seks n i andre da?

DAVID: Det er de som rører...

TESS: Man kan se det som et slags kvadrat, da har du det i andre, fordi at kvadrat er kvadratcentimeter, ikke kubikk. Og da, vil du alltid ha seks sider i en terning. Og de gullblokkene kan man da se på som en terning som blir større og større [points at the elements], og den vil alltid ha lik form så du vil alltid ha seks sider... Da vil det alltid være seks sider i en terning, og så vil du ha det i andre, fordi det er et kvadrat på hver side [points at the face of element 2]. For du har liksom fire gullblokker på den ene siden [still talking about element number 2], da må det være fire glassblokker der.

RESEARCHER: Kan du gi samme resonnement for tolv n?

TESS: Alltid tolv kanter i en kube, og så fordi at terningen inni, kuben inni blir større, så blir den hele også større i alle retninger.

RESEARCHER: Har figurnummeret noe å si for kantene da? ((Challenge.))

TESS: Det har noe å si for hvor lange kantene er.

It becomes clear that both David and Tess were able to connect the symbols and variables to the patterns they had visualized by Minecraft. They seemed to have a relational understanding of the invariant properties, which gave them the possibility to make general assumptions about the decomposition of the elements, consisting of edges, vertices, faces and a core. When giving their arguments, they relied on the ability to see all the components within the elements. For example, Tess could see the growing structure of the gold blocks within the glass blocks, when making the assumption of a growing die.

6.1.3 Objects are easy to manipulate

The next identified potentiality is connected to the manipulation of objects in the game. As mentioned, the two main activities in Minecraft are mining and building with blocks made of different materials. Utilizing the blocks is considered very easy, because it only requires one click of the mouse to either place or remove a block. The usage scheme connected to this action – being able to click a mouse – is therefore very simple, and something most pupils are familiar with. This gives the pupils a tool where they can express any structure they would like.

When David and Tess first entered the Algebra desert, Tess proceeded to the chest in Snøfrid's house to gather the diamond tools – a pickaxe, an axe, and a shovel – along with 128 gold blocks, 128 see-through glass blocks and the book and quill containing the tasks. As seen in Figure 6.7, it is very common to have the tools gathered at the left side of the toolbar, and the building blocks to the right, so that everything is quickly available. The toolbar has nine slots for tools, blocks and other items, which can be scrolled through with the mouse wheel. All the objects needed to complete the tasks were therefore possible to store simultaneously, and they were within immediate vicinity.

Figure 6.7: What the toolbar looked after Tess collected the items from Snøfrid.

In other words, the objects in the game require very little effort to place and replace. The pupils can carry everything with them at the same time and it is easy to switch between the items they want to use.

Pupils have a low threshold for testing hypotheses

JESSE: Jeg er helt lost. Jeg skal bare prøve noe.

Jesse's statement connects the fact that the objects are easily manipulated with one of the biggest potentialities with Minecraft in my eyes. Since it requires little effort to remove and replace blocks, it creates a low threshold for testing hypotheses and to enter the inductive stage in the abductive-inductive cycle, described by Rivera (2010). This is a crucial stage in the generalization process, that lets the pupils confirm or discard what they have abduced from their current knowledge.

This potentiality proved itself especially useful in Task f and h, where the pupils were asked to build the $(n+2)^3$ structure around the gold blocks, before replacing and color coding the blocks afterwards. In Task a-e, the Minecraft pairs did not have to rely so much on trial and error, because they all seemed to notice the n^3 pattern quickly. Task f, g and h were more demanding, so here it proved very useful to be able to seamlessly manipulate the blocks. When David and Tess were first trying to build the glass structure around the gold blocks, David did not quite get the point, but it seemed like Tess had an idea:

TESS: n pluss to vil da si at i (figur) 1 så blir det tre gange tre gange tre.
DAVID: Åååh, ja men da må det jo være sånn der [starts extending element 1].

David had not completely gotten it yet, so he built the structure seen in Figure 6.8. Then Tess took control of the game, walked over to look at the figure with $3 \times 3 \times 3$ gold blocks, and explained that she thought it would look like that one, only with glass blocks around it. David then proceeded to build a $3 \times 3 \times 3$ shape, but with the gold block in the corner of the element instead of in the center, as they were supposed to. Tess said that she thought that the glass blocks should surround the gold block, so they changed their approach, and proceeded to build both element number 1 and 2 correctly. Minecraft helped David build his initial thought of what the pattern would look like, while at the same time giving Tess an idea of his mental structure. She noticed that it would not be correct with the way she saw the pattern, so she took over the game to show David what it should look like. In a matter of seconds, they had exchanged thoughts through two different examples, and also established how to build the structure $(n+2)^3$.

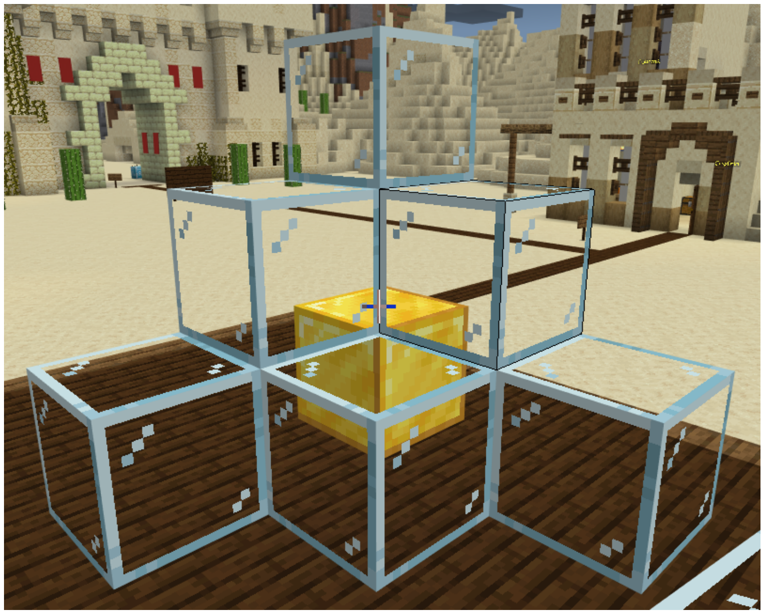


Figure 6.8: A step in the solution of Task f, by David and Tess.

As David and Tess reached Task h, they read it out loud, and discussed the connection between the expression $(n+2)^3$ and the four terms $n^3 + 6n^2 + 12n + 8$ for a minute. Then they went to look at element number 1 in $\{b_n\}$. Tess spotted, through the glass blocks, that the first term n^3 was the same as in the previous pattern sequence, i.e., the gold blocks $\{a_n\}$. While trying to explain this to David, she suddenly realized where $6n^2$ could fit as well:

TESS: [Looks around element 1] For at her har vi jo én, én to tre, fire fem seks sider, ikke sant? Og da ganges det med én, her to [flies around to look at element 2]. Det er [counting under her breath] seksten, det er seksten sider på den der, er det det?

TESS: Nei, det er én to tre fire, pluss én to tre fire, pluss én to tre fire, det er tjuefire sider ((with *sider* she refers to the blocks adjacent to the gold core blocks)). Men det gir også mening hvis vi [turns to the correct page in the book], hvis vi tar den [points to the $6n^2$ term], så blir det seks gange n gange n, og det blir tolv, tjuefire. Det blir de (blokkene) som er liksom ved siden av gullblokkene.

In these two statements, Tess went through an abductive stage connected to the term $6n^2$. They still wondered what the remaining two terms could be, so they decided to validate their current hypothesis by building what they thought they knew, in order to visualize it. Tess entered the inductive stage by replacing glass blocks with red glass blocks on element number 2, starting with the middle part of each face of the cube. She was still in the discovery stage of mining and building, because almost every time she was going to remove blocks, she placed out new ones instead. The player usually mines with the left mouse button, and places blocks with the right mouse button, so it can take some time to get used to doing it in the correct order.

Jesse and Sarah solved had a very similar approach to Task h, where they also tried to fit each new bit of information to the figures in front of them. Jesse acknowledged that he and Sarah struggled a bit in the beginning, but like David and Tess they managed to solve the task step by step. With some minor rephrasing of the task from me, they got along with trying out different strategies to make the expression fit to their cubes:

RESEARCHER: Hver enkelt del i dette uttrykket finnes en plass på begge figurene. Hvor kan dere finne dem?

JESSE: n i tredje, det må jo være gullblokkene.

SARAH: [Walks towards the elements they had built.]

RESEARCHER: Okei, du sier at n i tredje er inni figuren. Hvor er de andre tre leddene på figuren da?

JESSE: Seks n i andre...hver av sidene kanskje? [Opens the task in the book to see the terms again] (...)

JESSE: Hva det er tolv av? Hjørner! Én, to, tre, fire, fem, seks, sju, nei det er åtte hjørner!

After these statements, the pair proceeded to test some of the information they had come up with, to see if that would get them somewhere. In this process Jesse uttered the words that he felt lost, and just wanted to try something.

The processes that both Jesse and Sarah, and David and Tess went through by testing their hypotheses, showed that Minecraft has great potential as a visualization software. This becomes very clear in the following, because just as David and Tess finished representing the term $6n^2$ with red blocks, Tess got another revelation due to the visual cue. She looked at the figure, and started counting something:

TESS: Én to tre fire, fem seks sju åtte. Det! [She gets a sudden realization] Det her var jo enkelt. [She starts removing the blocks at the vertices in figure 2, as she counts] Én, to, tre, fire.

DAVID: [He understands now] Ååja, åtte kanter! ((He uses the wrong word *edge*, but I can see that they are speaking of the vertices))

TESS: Ja! [Starts filling in the vertices with green glass]

DAVID: Hva om du tar feil nå?

TESS: Da ødelegger vi bare, vi har sekstifire, så det går bra.

Tess was in a good flow state at this point and brightened up after each realization. She also addressed that it would be no problem if the assumptions were wrong either, because it would be easy to just remove and build new blocks. Since it is so easy to manipulate the blocks in Minecraft, it is a great artifact for visualizing sudden ideas, like the ones Tess was experiencing. As David and Tess were left with only one term in the expression, they were determined to show that the glass blocks that remained would fit to $12n$. Again, Tess stressed that it would be no problem to just test the hypothesis:

TESS: Da har vi igjen tolv x ((she uses the wrong variable name here)). Én, to, tre, fire... Nå bare prøver vi. [Starts breaking the blocks on the edges and replaces them with blue glass.] (...)

DAVID: [Counting the blue edges] Ni, ti, elleve, tolv.

TESS: [Proud] Jah!

DAVID: Men det sto jo tolv n da, gjorde det ikke?

TESS: Ja, tolv n, og det er det vi har gjort nå, fordi n er to.

Because they were able to fill in the available information piece by piece, both David and Tess, and Jesse and Sarah were able to solve the final task. They went through an abductive-inductive cycle for all the four terms and confirmed each of their hypotheses on the first try. As an example, the result of Jesse and Sarah can be seen in Figure 6.9.

6.1.4 Minecraft provides several registers to represent mathematical objects

All the pairs working with Minecraft had the ability to change registers to represent the pattern they were given in Task a-e. It seems like Minecraft enabled all of them to do the conversions without any trouble. When they wrote in the book, they used a combination of natural and symbolic language to express how they had solved the tasks. They were able to convert the visual cues into mathematical symbols.

In the game itself, I can recognize three different registers to represent the algebraic objects in. The book and quill offers – though without precise mathematical symbols – a way of representing the objects in both natural, written language (discursive, multifunctional register) and also

Figure 6.9: The result of Jesse and Sarah.

with symbolic representations (discursive, monofunctional register) – however not with formalized notation. Further, the game also provides a possibility for constructing visual representations of the objects with blocks (non-discursive, multifunctional register), as discussed extensively in the previous sections.

The discussion between the pairs in their natural language, mediated by Minecraft, seemed to help them understand each other. For example, when Sarah did not understand what was going on, Jesse tried to communicate his thoughts and arguments to her through speech, while also showing on the screen. His ability to use different representations did to some extent help Sarah identify the relationship between the pattern in front of them and the formula they had to establish. Together they seemed to be able to participate in both near and far generalization, and to present the structures in three different registers: Natural language, symbolic notation and construction with Minecraft blocks, see the conversion in Figure 6.10. What makes this unique to Minecraft, is that all three registers is reachable within the game, so the pupils do not have to rely on any external ways of representing the objects.

In their work, David and Tess also showed great mathematical competence when working with the tasks, because they were able to utilize all these three registers. In Task h they did a conversion between the symbolic notation provided in the task, a visual construction of the pattern, and also

Figure 6.10: A conversion between the visual Minecraft blocks, natural language and symbolic language, of the formula $a_n = n^3$.

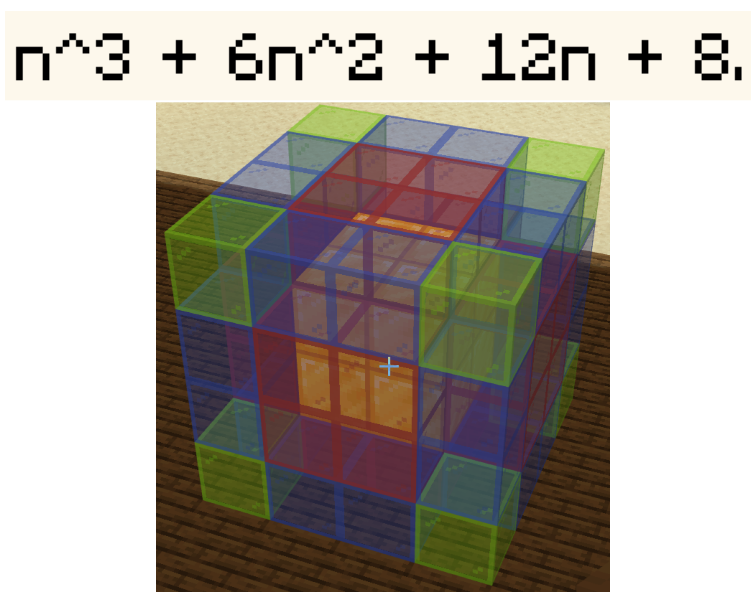


Figure 6.11: A conversion by David and Tess between two of the registers in Task h.

gave a verbal explanation of the structure. The arguments they provided did – to a greater extent than Jesse and Sarah – express the generality of the color coding. Figure 6.11 shows David and Tess' conversion between two of the registers.

6.2 Constraints of Minecraft

In this section, I will present results from the analysis of the *constraints* connected to algebraic activities in Minecraft. The analysis showed that there were few constraints directly concerned with the algebraic activity, but I will show some examples of shortcomings with the book and quill, and the representation of mathematical symbols in the game. Before I do this, I want to mention a few constraints of Minecraft in general that *can* make it difficult to use in some settings. I will not focus so much on them in the analysis of the data material, but they may be used as arguments during in the discussion chapter.

When using video games in school, which usually are used for entertainment purposes at home, it can sometimes be easy to get distracted by some of the game elements that are not related to the given tasks, e.g., algebraic generalization. This was for example seen when David and Tess entered the Algebra desert, where Tess became distracted by discovering that she was able to fly, so the pair missed the information blackboard at the beginning. Luckily the gameplay and the initiation of the tasks were intuitive enough to get them started anyways.

Joel and Ellie did experience other distractions that led them to some undesired utilization of the artifact, which often took them away from the tasks. Joel, being an experienced player with a *personalized* connection to the artifact, went through some *unplanned transformations* already in the beginning of the work process. There is an in-game item called a *lead* – a tool that can be used as a leash for animals. Joel had managed to apprehend one of these leads from a nearby non-playable character (NPC) whom he took it from. He then used the lead to capture a desert rabbit which he dragged along with him while solving the tasks. Every time his rabbit was killed, he went out to get a new one. Even if this was a harmless utilization of the world they were playing in, this transformation led them to use Minecraft more as a *game*, rather than a tool for algebra learning. It acted as a distraction from their work process, and they did never seem to get in the flow state that Kiili (2005) speaks of, where the players are fully immersed in the task and the challenge of the task matches the skill of the player. Joel and Ellie could not successfully solve all the tasks, and that led them to seek out other elements of the game.

6.2.1 The *book and quill* is not an optimized worksheet

There are some features of the *book and quill* that makes it suboptimal to use as a worksheet for reading and answering tasks. When the book is opened it fills the whole screen, so the players are unable to see the game environment and the book simultaneously. An example of this can be seen in Figure 6.12. When David and Tess were supposed to write down their answers to Task b, they encountered a problem with this as they lost the visual representation of the elements:

TESS: Det er åtte (blokker i figur 2), fordi at det er to gange to gange to.

DAVID: Ja.

TESS: Åh, nei, tulla. Det er "fire". Ja jeg glemte av at det var sånn kvadratblokker, da er det to gange to. Nei...

DAVID: Ehhh, ja men det er jo to gange to, gange to bortover der igjen.

TESS: Men så er det...

DAVID: Nei det er åtte, det er åtte!

With the book open, they did not see the elements in the sequence anymore, so they had to rely on their memory and imagination to picture them. What was interesting here, was that both David and Tess reconsidered how many blocks there were in the second element and said in unison that it had four blocks instead of eight. From her statement, it seemed like Tess only pictured the blocks in the square face of the element. David quickly noticed that something was not correct, because they had previously stated that there should be $2 \times 2 \times 2 = 8$ blocks, and he set them back on course without looking at the cubic structures.

Jesse also had some problems with the book taking up the whole screen. Especially in Task h, when he needed to check what the four terms looked like, he could not simultaneously look at the cubes. Jesse and Sarah said during the work session that they found the last task a bit confusing, and depended on some rephrasing of the question by me. They therefore used much time on this task, and had to rely on some trial and error to make the terms connect with the cubes. As I showed earlier, they were able to decompose the pattern into valid color codes, but it was extra time consuming for them to constantly open and close the book, and find the correct page for the task every time they needed to check their progress.

The problem with the book and quill filling the whole screen could have been countered by giving the pupils tasks on a separate, physical worksheet, where they could have read the tasks and written their answers without having to remove the view of the environment they were operating

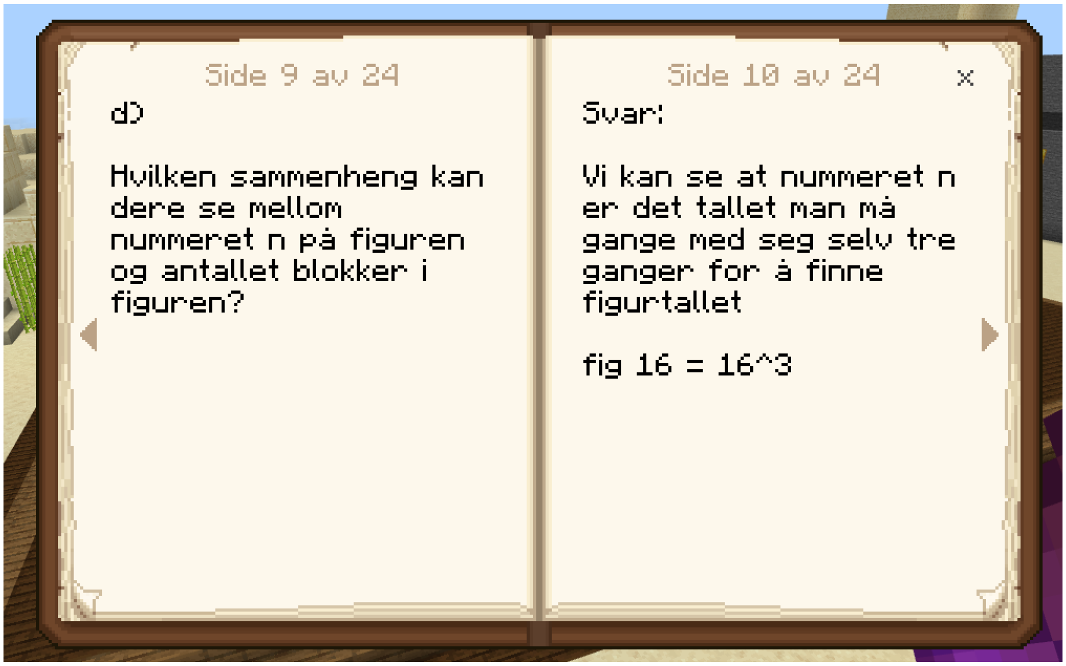


Figure 6.12: Example of how the screen looks like when the book and quill is open.

in. However, when the pupils were using Minecraft as a learning resource, I wanted them to be able to perform all the required tasks within the game in itself, without having to involve additional external resources – especially for the purpose of this research.

Lastly, I want to mention – even if this has nothing to do with algebra – that the book and quill can be a bit hard to navigate and edit, because every time it is opened, it opens at the first page instead of the page it was closed on. Also, if you want to edit some of the text on the middle of a page, you will have to scroll your way back in the text with the arrow keys, instead of just clicking where you would want to edit. This made the book unnecessarily time consuming to use. Nevertheless, the pupils adapted to this way of using it by the end of the session, as if it was a regular worksheet.

6.2.2 Limitations in mathematical symbols

The next constraint of Minecraft is that there is no dedicated tool for writing formulas with mathematical symbols in the book and quill, or anywhere else in the game. This would lead the pupils to represent some mathematical structures – for example fractions or powers – with regular symbols. When the pupils expressed powers in the book, they had to use a *caret* (^) to indicate exponentiation. They would have to write 10^3 instead of 10^3, and n^3 instead of n^3. Also, the pupils could not use italicized letters for variables such as n. See Figure 6.10 and 6.11 for examples of symbols utilized in the book and quill.

Furthermore, there is no calculator in the game to perform simple calculations, like multiplying $3 \times 3 \times 3$, because Minecraft is not a full-fledged mathematical software. Tess asked me at one point if they could use a calculator to do some of the calculations in the tasks, but David was ahead of her and had computed the answers in his head. In that case they managed without it, but it would be nice to see a calculator function in the game in the future. Without it, the pupils are led to compute such problems either with an external calculator or work it out in their heads. However, the calculations in the tasks discussed in this study should be manageable, and are good practice in mental arithmetic.

6.3 To what extent does Minecraft develop into an instrument?

The complex process of instrumental genesis – linking the potentialities and constraints of the artifact and the subject's activity (Trouche, 2005) – makes it difficult to determine exactly how far each of the pupils got in the building of instruments from the artifacts they were given. By looking at incidents of instrumented actions identified in the preceding sections, I will however try to establish how the pupils' instrumentation and instrumentalization processes affected the instrumental genesis of Minecraft.

6.3.1 Instrumentalization of Minecraft

Instrumentalization is the process where the pupils utilize their prior knowledge and work methods to manipulate an artifact (see Figure 3.2), here Minecraft. The first example of an *instrumented technique* is associated with the potentiality that patterns can be easy to visualize in Minecraft: When the learners managed to *control Minecraft*, they were able to *gain perspective* to perceive the elements in the pattern sequences from different angles. In other words, the pupils utilized the artifact to their benefit in order to notice the invariant pattern in the cubes. To be able to do it, there were certain steps the pupils had to master first. These were the usage schemes that together created the instrumented action scheme of the utilization.

From the scheme described in Figure 6.13, it is evident that the instrumented technique of gaining perspective of the pattern sequences only required basic usage schemes related to the use

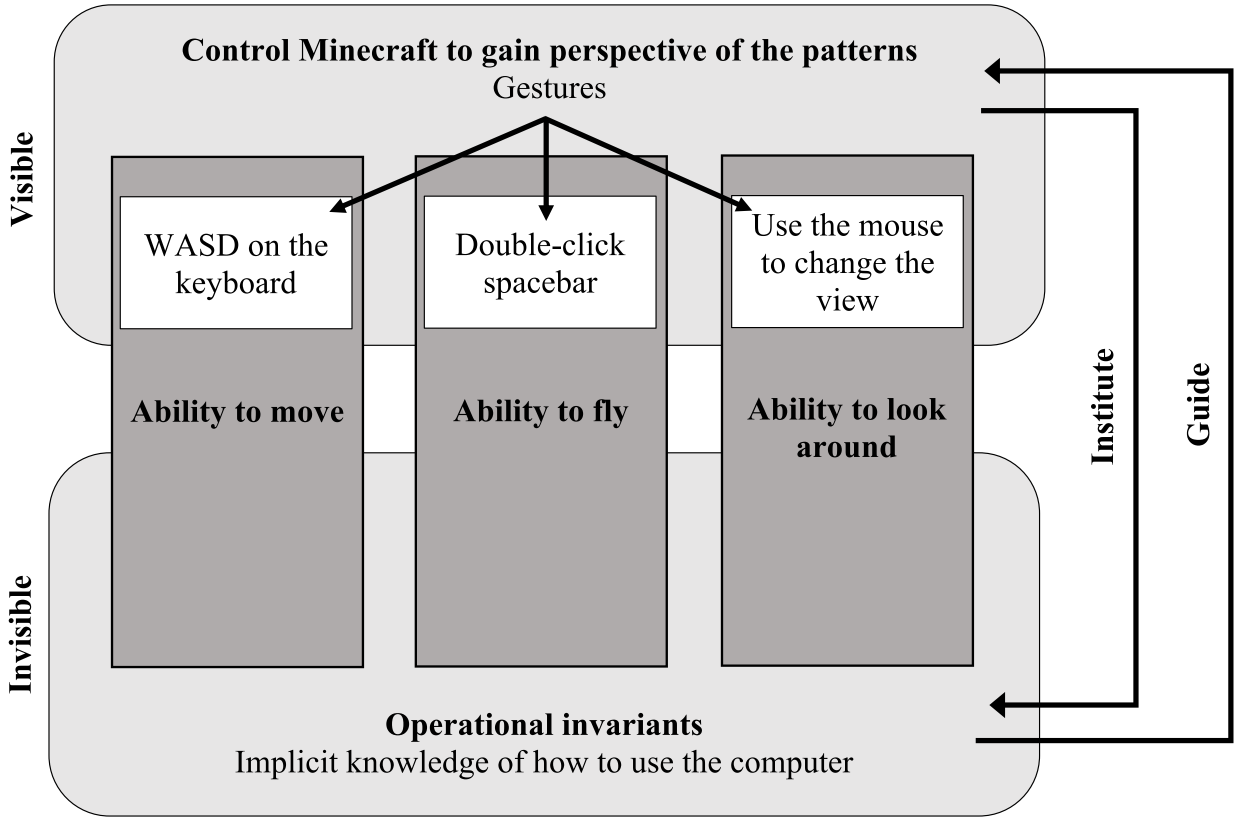


Figure 6.13: An instrumented action scheme connected to the visualization of objects with Minecraft. The instrumented technique is to control Minecraft to gain perspective of the patterns.

of a computer, such as using the mouse and WASD keys[1] on the keyboard. In the discovery phase of the instrumentalization, the pupils had to discover how to create these mental schemes, if they were not already known. Joel did for example not go through any discovery stage connected to this action scheme, because he was used to play Minecraft, and PC games in general. In the interviews, he stated that he had found the artifact very easy to use. When he was supposed to do something in the game, it was noticeable that he automatically found the position for his hands on the mouse and keyboard, before starting to play. To do the tasks, the pupils were not required to utilize any game elements that would not have been used to play regular Minecraft. The tasks only required basic usage schemes for mining, building, moving, flying, adding resources to the toolbar, and editing the book, which were all connected to the mouse and keyboard. Therefore, experienced Minecraft players like Joel did not have to go through an extensive discovery stage of the artifact, since the keyboard and mouse already were well personalized and *fitted to their hands*.

[1]The four keys W, A, S and D on a keyboard, usually used as directions to control movement in video games.

Sarah on the other hand, had previously played Minecraft on console, so she had to learn these schemes. She felt a bit insecure at the beginning, but Jesse guided her through the discovery stage, which went fast:

JESSE: Vet du hvordan man spiller Minecraft?

SARAH: Nei [laughs], jeg husker ikke. [Jesse proceeds to repeat the basics for her.]

JESSE: W for å gå frem. Der er det ei kiste. Og for å gå ut av fly, så tar du å dobbelttrykker på mellombar.

SARAH: Hvordan åpner jeg den? ((The chest opens)) Oi! Skal jeg ta alt?

JESSE: Ja! Du kan holde inne Shift når du trykker på dem.

In a minute, she learned how to walk, how to transfer items to the player's inventory and how to toggle flying. These were necessary schemes to establish for participating in any of the mathematical activity. Before she learned any of them, the only thing she would be able to do was to stand still and look around with the mouse.

When the pupils were given Minecraft to use, it came with a predefined set of key binds, such as the WASD keys being used to walk and Shift key being used to crouch, which are default for the game. Conflicts with existing schemes can arise when the key binds do not match the ones the pupils are used to. Jesse mentioned during the work session that he used different key binds when playing at home, related to running and crouching in the game. It did not make a big difference for him in this case, but it required him to establish new mental schemes about these gestures. He did however adapt to this quickly, since the usage schemes were very basic. Another option would be for him to transform the artifact, by changing the key binds in the game settings, but he chose to adapt to the environment instead.

The next instrumented technique is the *manipulation of objects*. The instrumented action scheme related to this technique also involved basic use of the keyboard and mouse but had more focus on the mouse actions – see Figure 6.14. The pupils needed to be able to move around using the WASD-keys on the keyboard, while at the same time clicking the correct mouse buttons – left click for mining, and right click for building.

The schemes that build up this technique could sometimes be confusing for the pupils, especially those that were in the discovery stage. Tess did for example struggle a bit with differentiating between the left and right click of the mouse, which often led her to place blocks when she was supposed to remove them. In the 45 minutes of the work session, she did not go beyond the dis-

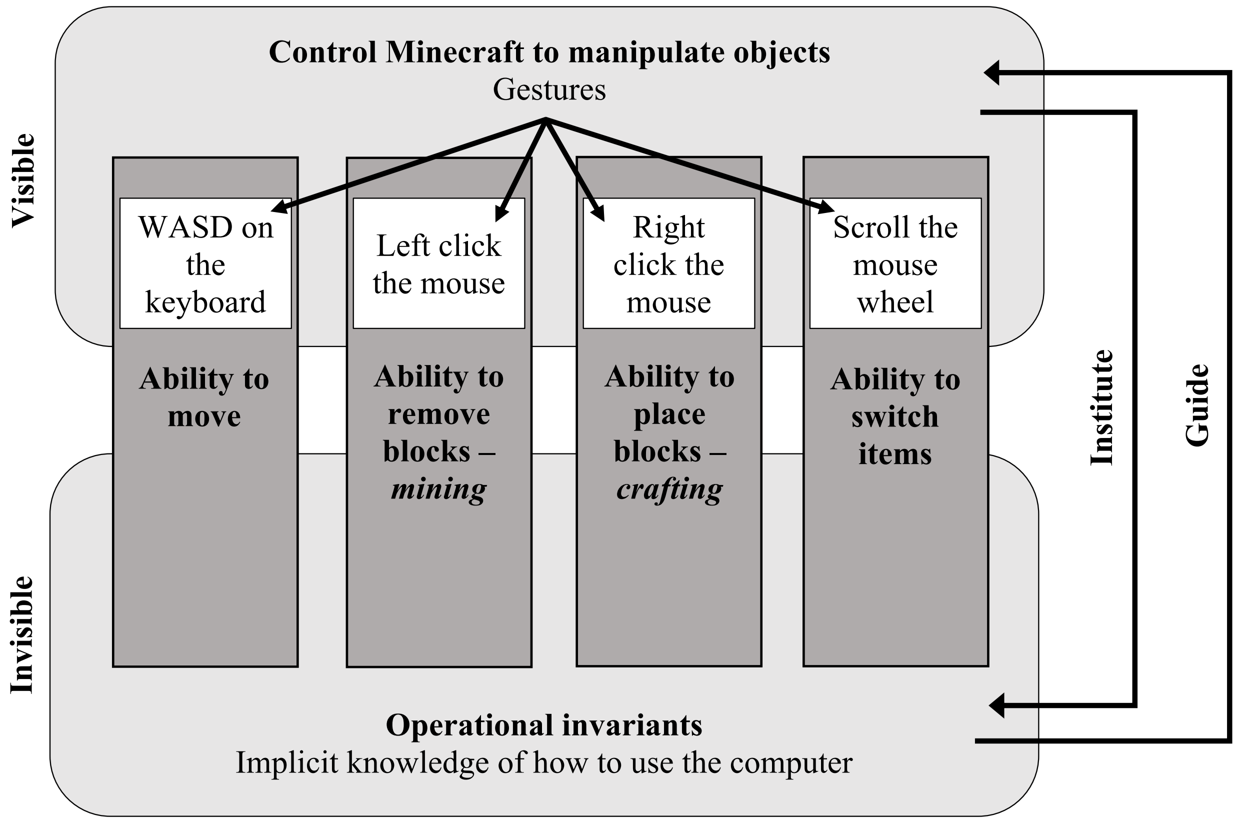


Figure 6.14: An instrumented action scheme related to the technique of controlling Minecraft for manipulating objects in the game.

covery stage of this technique, but that did not seem to impact her negatively in any way other than spending a few extra seconds when building.

In themselves, the usage schemes mentioned in this section do not offer any useful activity, but when the abilities were put together to create instrumented actions, they contributed to effective instrumentalization of Minecraft by the pupils in this algebra setting. Both the instrumented action schemes needed for manipulating objects in the game, and for gaining perspective on the objects, were directly connected to the utilization of the artifact, and not so much to the algebraic activity. Nevertheless, these were necessary to master, in order to use the full potential of the game to participate in the tasks.

6.3.2 Instrumentation of Minecraft

Instrumentation is the process directed towards the learners, who are shaped by the constraints and potentialities of Minecraft. In subsection 6.2.2, I identified the limitations in mathematical symbols as a constraint of the game as an artifact for algebraic teaching. For example, the pupils had to use the ^ symbol to express powers, instead of expressing them with exponents. In one way this can be good, because that is the way a computer expresses exponentiation, e.g., in *Computer Algebra Systems* or in some coding languages. On the other hand, it is not the formal mathematical symbolic language that the pupils are taught to use. However, I think the pupils were used to seeing this notation, because I never got any questions about it, and they all seemed to understand how to use the caret for both reading the tasks and answering them.

A great part of the instrumentation was influenced by the visuals of the game, the *intrinsic fantasy* described by Malone (1980). The pupils always received visual feedback on what they had built, so they would quickly be able to tell if the patterns they were building had the correct configuration or not. For example, when building the sequence $\{a_n\} = n^3$ with the gold blocks, all the pairs immediately confirmed that they had built the structure correctly, because they could see the pattern evolve with the correct invariant properties. The potentiality of visualizing the patterns, and also being able to see all the components in the pattern while building with glass blocks, were utilized by all the pupils to gain advantage in the generalization tasks.

As an example of how the pupils were influenced by the visuals; after David and Tess opened the book to answer Task b, they did not close it once until they reached Task f, where they were supposed to start building again. They answered Task b-e based only on how they remembered the pattern from Task a. In other words, the visual representation they were given, enabled them to early establish a mental structure of the algebraic objects, so they did not need to check what the elements looked like. Jesse and Sarah had a very different approach, where they always went back to get visual inputs from the cubes between the tasks. Also, when the pupils were testing hypotheses related to the sequence $\{b_n\} = (n+2)^3$, they built their knowledge step by step because the visuals enabled them to see new patterns, after they had switched out some of the blocks.

Chapter 7

Centicubes as mediators of generalization

This chapter will have a similar structure to the previous one with the Minecraft approach. I will first identify the potentialities and the constraints of the centicubes, before discussing the instrumental genesis of the artifact more deeply afterwards. To answer the research questions, I will analyze the work of the seven pairs participating with the centicubes. The pair that was video recorded in this process consisted of Bill and Tommy. Due to the classroom configuration, a second pair, Abby and Marlene, sometimes appeared in the video when they turned to Bill and Tommy to compare answers. Further, I will rely on the answers of Owen and Riley, a pair I had a conversation with after they had finished the tasks. Based on this conversation, I wanted them to elaborate further on their approach in the interviews. Aspects from the remaining four pairs will also be included for a broader perspective of the utilization of the centicubes.

As can be seen in Appendix C, only one of the pairs managed to do all the tasks correctly. Two of the pairs (Abby and Marlene, and Owen and Riley) managed to do the tasks with some misunderstanding or incomplete answers. The rest of the pupils struggled with many of the tasks, especially the last ones. This chapter will get deeper into these results. For the sake of simplicity, I will, similarly to the Minecraft section, refer to the centicubes as blocks or components when speaking of the building units, and the geometrical configuration in each pattern stage as elements.

7.1 Potentialities of the centicubes

7.1.1 Patterns can be easy to visualize and centicubes give a three-dimensional perspective of the patterns

The first thing that is noticed about the centicubes is that they can make it very easy to visualize patterns, if utilized correctly. When the pupils were given the bag with the centicubes, they also received the first two elements of the pattern sequence, which I had built for them in advance. The worksheet they were given also contained an illustration of the two elements (see Figure 4.1). They were asked to take these two elements out of the bag and based on the structure build the next two elements with the same color. An example is seen in Figure 7.1, where one of the pairs received two white elements.

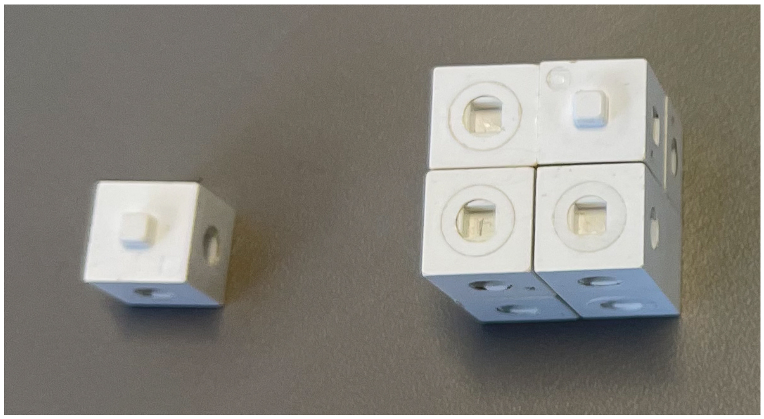


Figure 7.1: How the first two elements looked when taken out of the bag.

The only trio that participated in the research was also the only ones that managed to do all the tasks correctly in the centicube approach. Since I do not have any recording of them working, I am not completely sure of who contributed with what, so in the following I will refer to this trio as a unit, called the Fireflies. This trio was able to notice the invariant properties of the pattern and utilized the centicubes in their generalization activities. First, they built the four elements in $\{a_n\} = n^3$ with yellow centicubes, and were then able to expand the first two elements by blue centicubes, to match $\{b_n\} = (n+2)^3$. Figure 7.2 shows a step in this solution.

In Task h, the Fireflies were able to switch out the relevant blocks and connect different parts of the elements to the four different terms, each representing the edges, vertices, faces and the core. They also stated, with a combination of symbolic and natural language, how each of the terms fit to the cubes. In other words, they were able to convert between three different representational registers to convey their results (see Figure 7.3). The Fireflies did not express directly how this color coding would be valid for all elements, but based on their representations, they did seem to understand how all the terms were connected. The trio first identified the term n^3 as the core of the elements and $6n^2$ as the blocks in the middle of the outer layer. Then they saw the terms $12n$ and 8 in connection with each other. The pupils did not differentiate between edges and vertices, but collected them both under the term edges (“kanter” and “midten av kantene”), and denoted them as

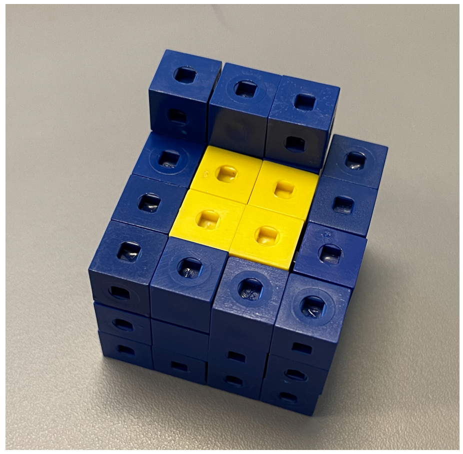


Figure 7.2: A step in the Fireflies' solution to Task f.

"opposite" of each other. When representing these two terms with the colored blocks, they mixed up the red and the blue color between the two elements. This could have been coincidental, but it could also have been a consequence of seeing the edges and vertices as one collective term, and therefore they ignored the difference between red and blue.

Even if the centicubes made it potentially easy to visualize the structures, I will now give an

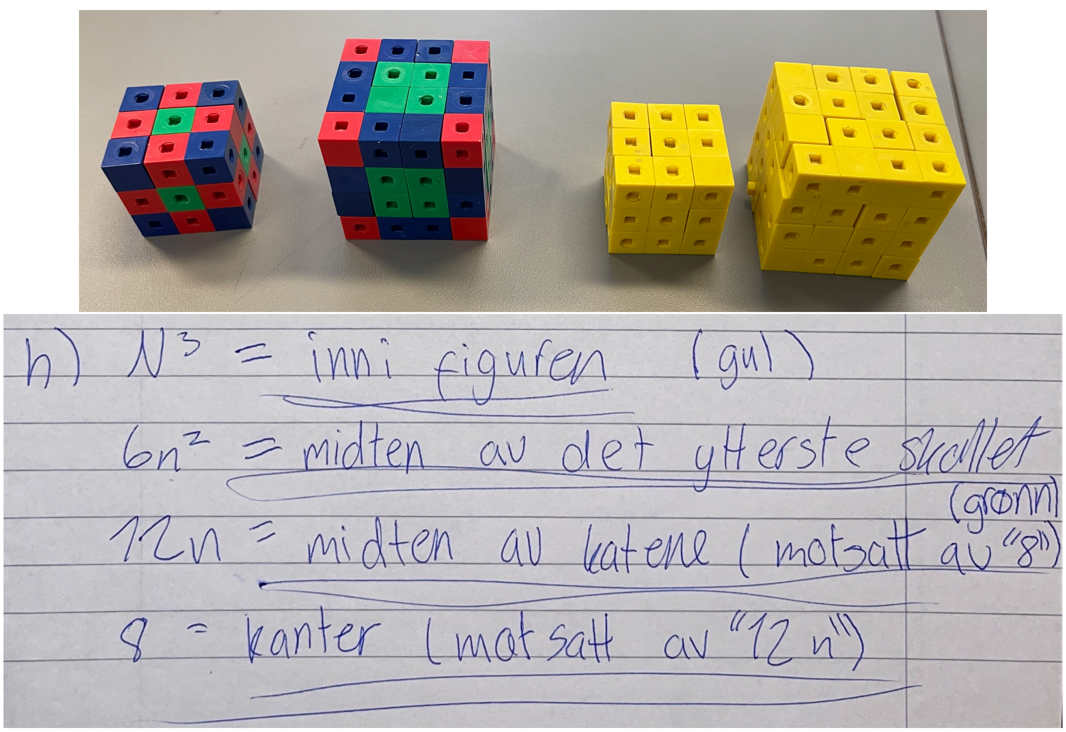


Figure 7.3: The Fireflies' answer to Task h. They made a conversion from the visual representation to natural language and symbols.

example of some pupils used them to notice different mathematical structures. When Bill and Tommy were given the first two elements in the sequence $\{a_n\}$, Bill did not notice the invariant properties right away and used the available numerical information to adopt an additive approach of "adding 7 to each new element". The additive relationship emerged right after Tommy had started building:

BILL: Jeg begynner å lage (figur) 4. Eller holder du på å lage 4 nå?

TOMMY: Jeg lager 3'ern.

BILL: 1'ern har én. Hvor mange har 2'ern? [Studies the element on the worksheet] 2'ern har åtte.

TOMMY: Ja.

BILL: Så det går opp med sju hver gang da.

TOMMY: 3'ern... Der er det jo to, to [points along the edges of element number 2], så blir det tre, tre [points along the edges of an imaginary element number 3], fire, fire, og fem, fem.

BILL: Ja det går opp med, det blir sju større hver gang.

TOMMY: Det blir jo ikke sju større hver gang?

Bill was convinced that the pattern was evolving as $+7$ between the stages and tried to persuade Tommy of his strategy. Tommy on the other hand, tried to convince Bill of *his* approach, which in this case was the correct one. Tommy had adopted a more figural strategy and used the first two elements to point and show Bill that the elements consisted of an equal number of components along each of the edges, but Bill was unyielding in his perception.

In the epistemological analysis of the tasks (section 4.2), I addressed a problem that could occur by designing the tasks ambiguously – that pupils could see other structures than those intended by the designer of the tasks. Bill saw the first two elements, identified that the numerical difference between the two was seven, adopted a numerical strategy and then concluded that it would have to be seven components between each of the elements in the sequence. He therefore stated that there should be fifteen blocks in the next element. Tommy had just finished building a $3 \times 3 \times 3$ cubic structure, and they started counting the number of blocks in it:

BILL: Det er ikke femten på den der [they both laugh].

TOMMY: Fjorten, femten, seksten, sytten [counts on the $3 \times 3 \times 3$ element]... Nei jeg teller jo samme kuben. [Starts over again] Én, to, tre...ni, ni [looks at the number of 3×3

layers in the element], tjuesju blir det faenmeg her. [Throws the cube in the air and drops it in the desk so that it falls apart. They both laugh.]

Bill did not think that the element Tommy had built was correct, because it did not contain 15 blocks, which he thought it should have. Right after, Abby and Marlene entered the conversation, who also had a disagreement on the invariant properties of the sequence. This gave rise to a discussion between the two pairs Abby and Marlene, and Bill and Tommy:

MARLENE: Er jeg den eneste som tenker at det bare er pluss sju?

BILL: Ja, det var det jeg også tenkte.

TOMMY: Nei jeg tror ikke det er pluss sju hver gang.

BILL: Det kan jo være det.

MARLENE: Det er akkurat det det kan være. Det er jo det letteste. Det må ikke være en kube. Det står figur, og da kan det være den figuren jeg lagde i sta (med femten kuber).

TOMMY: Men når det er én gange én gange én, to gange to gange to, tre gange tre gange tre...

MARLENE: Men det er flere mulige måter. Men vi kan ta n i tredje, det går fint.

BILL: Kanskje det gir mening. Vi har riktig, men deres gir mening. Neida, men da skriver vi det. n i tredje da. Eller x eller, ja. Sånn, det er formelen.

What Marlene indicated, was that there were several strategies that would fit to the pattern given by the first two elements, such as “add 7 to the previous element”, which could have been true if they would only look at the numerical values. If they had followed this strategy, it may have led them to the formula $a_n = 1 + 7(n-1)$, which would fit to the numerical values of the elements. It seemed like Marlene and Bill just ignored the visual cubic structure, because their approach would not have fitted to the visuals of the first two elements. It would not have been possible for the pupils to build cubic structures containing 15 components. However, Tommy was very eager to point and show on the cubic elements he had built ($3 \times 3 \times 3$ and $4 \times 4 \times 4$), to explain the pattern for Marlene and Bill. The visual and physical aspect of the centicubes helped Tommy explain the properties for the other two pupils, by showing and explaining with the elements in his hand. This shows a potential of the centicubes that was very useful in this situation. Both Marlene and Bill gave in to the arguments in the end, even though they were hard to convince, and together the two pairs proceeded with the expression n^3. The discussion between the pairs shows an example of pupils having a hard time giving up on their perception of a pattern once they have established a structure in their mind, as pointed out by Nilsson and Eckert (2019).

Another example of how the pupils used the centicubes to gain a visual advantage, emerged in the interview with Owen and Riley. In Task a, this pair had first built the four elements of the sequence $\{a_n\} = n^3$ with white blocks and seemed to quickly establish a multiplicative property of the pattern. They therefore *calculated* the number of blocks in them and wrote down their calculations ($1 \cdot 1 \cdot 1$ and $2 \cdot 2 \cdot 2$ etc.). When I asked the pair about the connection between the expressions n^3 and $(n+2)^3$, they used the physical elements in front of them (Figure 7.4) to lift and point while explaining how the first sequence evolved into the second one:

RESEARCHER: Hvordan tolket dere oppgave f?

OWEN: Vi må jo regne ut parentes først, så figur 1 [picks up the $1 \times 1 \times 1$ element] blir jo én pluss to som er tre [picks up the $3 \times 3 \times 3$ element], så den vil jo se ut som denne her. Og så figur nummer 2 [picks up $2 \times 2 \times 2$ element], blir jo to pluss to som er fire [picks up $4 \times 4 \times 4$ element] så vi bygget figur nummer 4.

By looking at the elements, Owen explained that element 1 would evolve to look like element 3, and 2 would evolve to 4, in the process of turning the expression n^3 into $(n+2)^3$. In the end of the interview, I asked the pair how they had utilized the centicubes in their work with the tasks. Owen answered that he had found it useful to be able to touch and visualize the mathematical structures with the centicubes.

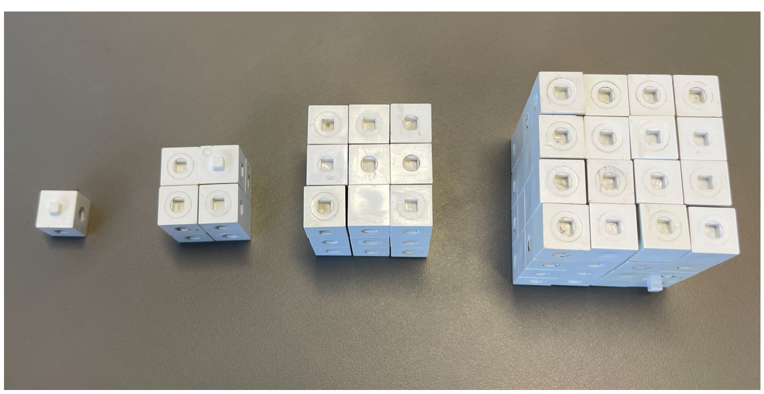


Figure 7.4: The four elements built by Owen and Riley in the first tasks.

The conclusion is that centicubes can make it easy for pupils to visualize patterns, but if the pupils rely on numerical rather than figural strategies, it can cloud their vision of the evolution of the pattern. The reason for saying that the patterns *can* be easy to visualize, is because that in order to do so, the pupils need the ability to actually build something useful with them. Later, I will show that this requires some usage schemes directed at the artifact that looks quite different from the Minecraft schemes, and therefore are a bit more complicated.

7.2 Constraints of the centicubes

7.2.1 Objects are difficult to manipulate

As mentioned, only one of the pairs (the Fireflies) were able to build the intended structures in Task f and h. A possible explanation for why the other pairs were unable to do them correctly, leads to the identification of a constraint of the centicubes; that they are difficult to manipulate effectively.

Bill and Tommy used the first 20 minutes to finish the building the elements $\{a_n\}_{n=1}^{n=4} = n^3$ in Task a. After that, they used another 10 minutes to build the elements $\{b_n\}_{n=1}^{n=2} = (n+2)^3$, but this part they did not do correctly. Instead of building a layer around the elements a_1 and a_2 with one color, they built new cubic structures from scratch, and used several colors with no system. In the first tasks, the elements took so much time to build that the pupils ended up solving the rest of the tasks before they finished building them. Throughout the work session, they uttered several times that the process was time consuming and boring, and that the centicubes were difficult to assemble and even more difficult to dismantle. The tasks, that were supposed to rely on teamwork, did not seem to be fairly distributed between Tommy and Bill:

TOMMY: Men kan jeg få lov til å skrive på neste oppgave? [Grabs the worksheet, but it is ripped back from him.]

BILL: Nei, nei nei nei. Du lager (figur) 4, jeg gidder ikke.

TOMMY: Men jeg orker ikke å lage alle figurene.

This resulted in a building process that took almost the entirety of the 45 minutes to finish. When they reached the last task, they said that they did not understand it, and therefore seemed to give up on trying to build the elements instead of testing different strategies in order to solve the task.

In Figure 7.5, I have presented an example of why the centicubes can be hard to manipulate. To remove the blocks marked in red, the users will have to remove every surrounding block, and most likely in the same order that they were built in. The blocks in the inner layers would be even harder to reach, because one would have to remove the outer layer first.

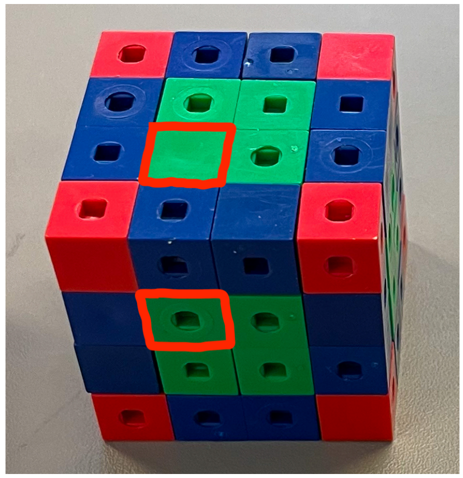


Figure 7.5: The centicubes marked in red are examples of blocks that can be hard to remove.

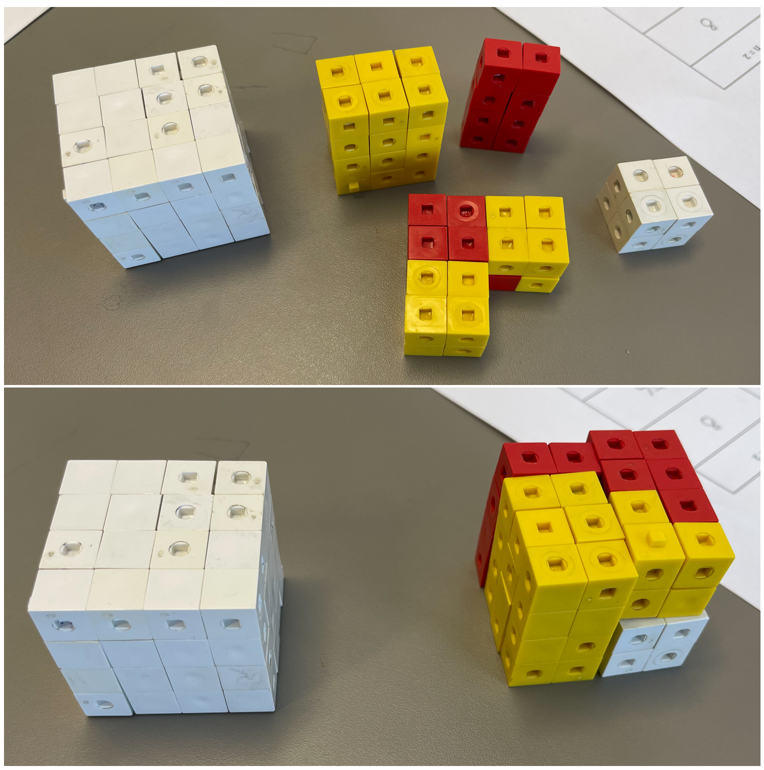


Figure 7.6: Owen and Riley almost solved all the tasks correctly, but struggled a bit with connecting the blocks.

As the 45-minute work session with the centicube group came to an end, I had a conversation with Owen and Riley about their approach of solving the last task. They wanted to get feedback from me on what they had done. Figure 7.6 shows their solution to the final task, where they were supposed to make a color-coded decomposition of the sequence $\{b_n\}$. Owen and Riley seemed to be very limited by the difficulty of manipulating the objects, but still managed to come up with reasonable and well-reflected solutions. When color-coding the terms in the expression $n^3 + 6n^2 + 12n + 8$, the pair were able to build four separate parts that each represented one of the terms in the expression, seen in the top picture of Figure 7.6, but as seen in the bottom picture, they were unable to connect the blocks in these parts together (they were just held together). This reflects a scheme that is necessary for participating in meaningful engagement with the centicubes, that they must be connected in a certain order to be assembled. It is therefore difficult to build four different structures first, and then try to make them fit together afterwards. It is easier to add the blocks one by one, to make sure all the pins and holes are utilized, but then again it is more difficult to test out different ideas that emerge, because the users will need to have a clear picture of what they are supposed to build before starting.

Owen and Riley used the specific example of $n = 2$ to solve the final task, but neither the color codes, nor the structure of the patterns they built could be generalized to match other elements than the specific example they presented. Even if they did not use the colors to differentiate their representation of each term, they argued in the interviews for what they had done:

RESEARCHER: Kan dere forklare hva dere har tenkt med disse fire figurene?

OWEN: Regnestykket, nei formelen, består av fire deler ikke sant. n i tredje, seks n i andre, tolv n og åtte, ikke sant? Så jeg bygget de fire ulike delene. Jeg brukte en sånn puslemåte, så hvis jeg setter dem sammen så [tries to connect the four structures]. Jeg mente å vise at hvis man setter de sammen så [struggles with finding the configuration that makes them form a cube]... Nå husker jeg ikke helt...

RILEY: Jo se her [takes the four structures and puts them together very quickly].

RESEARCHER: Okei, så hvilken figur er det vi har foran oss nå?

RILEY: Det er jo den her [takes out the white $4 \times 4 \times 4$ element and puts it beside the structure they just put together].

It was not intuitive for Owen how to put the four pieces together to form a cube, but Riley saw how they would fit. I tried to guide them into an explanation of how their puzzle approach would be a valid generalization:

RESEARCHER: Hvor finner dere da hvert av de fire leddene på figuren dere har laget?

OWEN: Hvis n er lik to, så to i tredje er lik åtte, så den må være den røde [picks up the red 2×4 structure ((Figure 7.6))]. Og så seks n i andre, så må du regne ut potens først, så

> det er seks gange fire så det er tjuefire, som er lik den gule figuren [picks up the yellow $3 \times 4 \times 2$ structure]. Tolv n er tolv gange to, som er tjuefire, som er den [picks up the L shaped structure with both yellow and red blocks]. Til slutt så er åtte lik denne [picks up the $2 \times 2 \times 2$ structure]. Og hvis du setter dem sammen så får du figur 2.

The strategy Owen described works for the case of $n = 2$, because the four terms have the correct numerical values, and when put together they generate a $4 \times 4 \times 4$ cube. I tried to challenge them to further explain if this would work for every element in the sequence, but they were not able to see how their solution could be generalized beyond element number 2.

Pupils used drawing to support their claims

This next identified constraint is not a constraint of the centicubes in themselves, but one I want to see in connection with the fact that the centicubes are hard to manipulate. It is difficult to say whether it was a reaction to the difficulty of manipulating the centicubes or not, but two of the pairs relied on drawing in addition to the use of centicubes when giving their answers on the worksheet. One could argue that when the pupils strayed away from the centicubes, that it led to a weakening of the the centicubes' position as a learning tool, because the pupils depended on different approaches than those offered by the artifact, to complete the tasks. At the same time, one could argue that the pupils used all available options to solve the task, and that it was good for the pupils to show that they were able to utilize several representational registers.

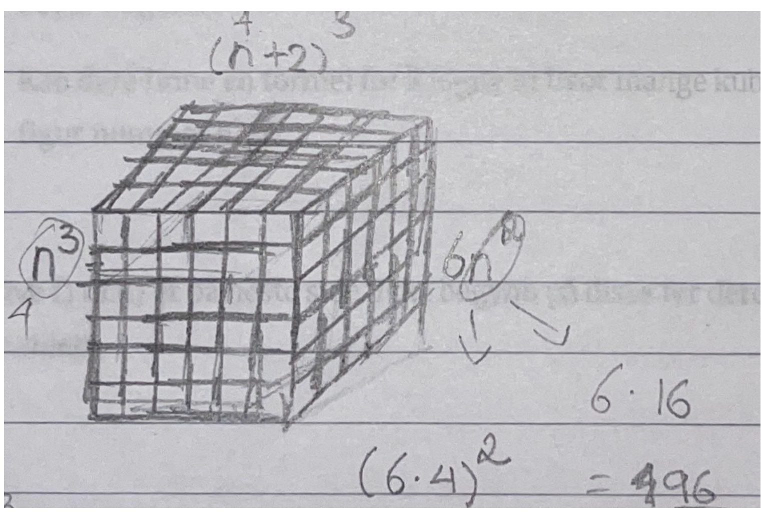


Figure 7.7: Owen and Riley's drawing, used to support their generalization of the expression $(n + 2)^3$, here with the example of $n = 4$.

In Figure 7.7, Owen and Riley's worksheet answer to Task h is seen. Here, they gave another example of an element in the sequence $\{b_n\} = (n+2)^3$, namely b_4. This is a demanding element to draw, because it contains so many components. Nevertheless, they drew it to further visualize the cubic structure. They also calculated the numerical values of the different terms in this specific example but did not say anything about how the terms could have been generalized.

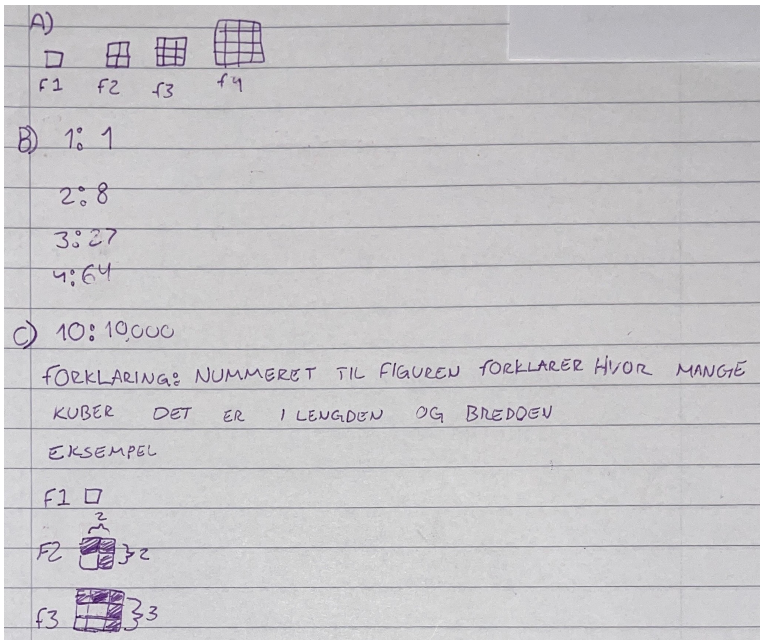


Figure 7.8: Pair 2 also used drawing to support their claims.

Figure 7.8 shows the example of Pair 2, that also used drawing in their work with the centicubes. They tried to draw the four elements in the first task, but with a simplified scheme limited to two dimensions, that led to an impoverishment of the centicubes' three-dimensional perspective. After adopting the two-dimensional structure, they also made some miscalculations in Task c, and saw the properties of the pattern as a square with growing length and width, and not a cube. Here, the two-dimensional character of the paper might have led them to a misunderstanding of the task. In Task e they did however go back to a three-dimensional world, because they stated the formula $N : H \cdot B \cdot SF$ for the number of blocks in a general element in the pattern sequence. Here, I do not understand what SF stands for, but the formula Pair 2 provided, showed that they still were able to notice the three-dimensional properties of the pattern.

7.3 To what extent do the centicubes develop into instruments?

7.3.1 Instrumentalization of the centicubes

In the previous sections, it was established that patterns *can* be easy to visualize with the centicubes, given the ability to participate in useful building strategies. To do so required some usage schemes (Figure 7.9) directed at the artifact that involved both some physical (grip) strength – because once the cubes had been connected, they were hard to remove from each other again – and a sense of order in which the building steps were executed in.

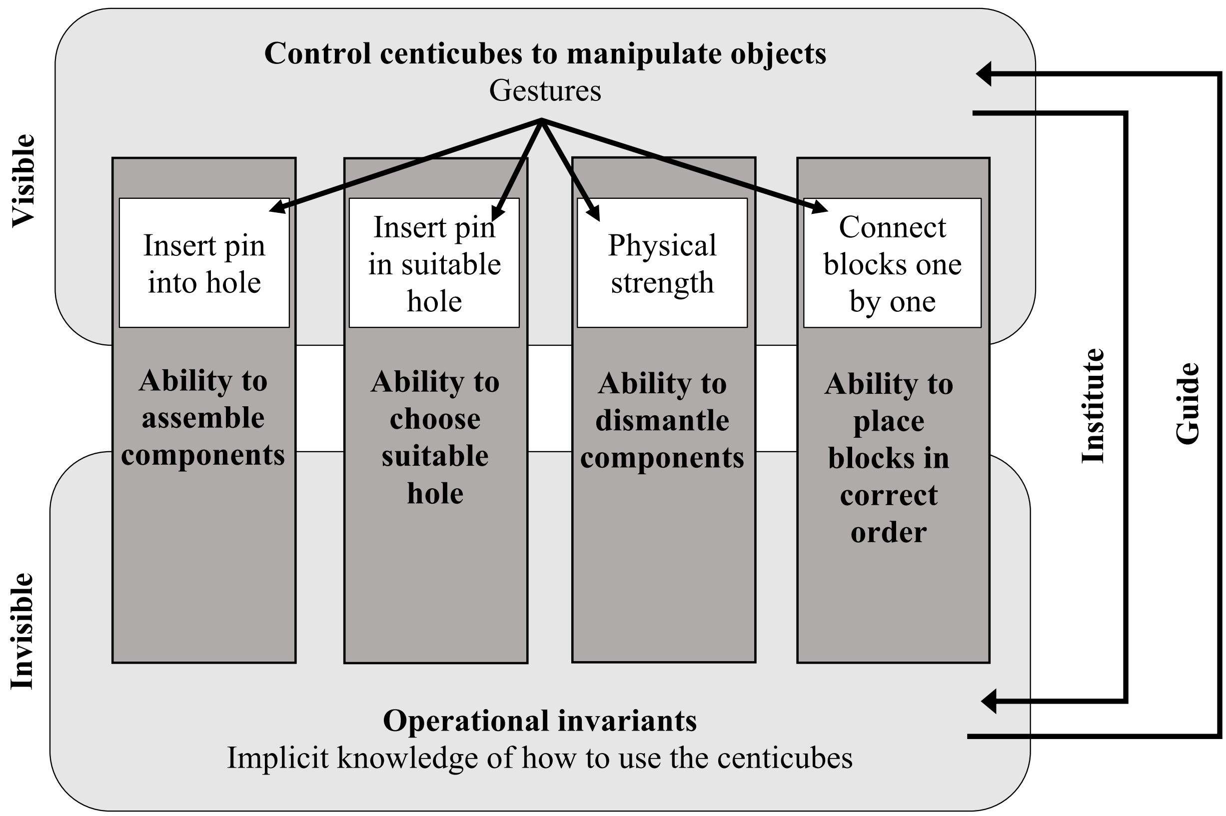


Figure 7.9: An instrumented action scheme for manipulating the centicubes. The instrumented technique is to control centicubes to manipulate objects.

Except for the Fireflies – the trio that managed to solve all the tasks – none of the pairs went far in the instrumentalization process. From the instrumented action scheme in Figure 7.9 described for the technique of manipulating objects, it can be quite demanding to use the centicubes for building cubic structures, especially when the tasks required the pupils to dismantle and assemble the components to build different structures. In order to connect two blocks, the pin on one block and the hole of the other block needed to be utilized. Because of this structure of the blocks, the connection had to be done in a way that left available holes for the next block to be placed. When the color coding was involved, the structures also required planning ahead of the building, so that they could be able to place the blocks in the correct order. Therefore, most of the pupils did not get reach the personalization stage of the instrumentalization, where they would have fit the centicubes to their hands, so that they could have been utilized in an effective way.

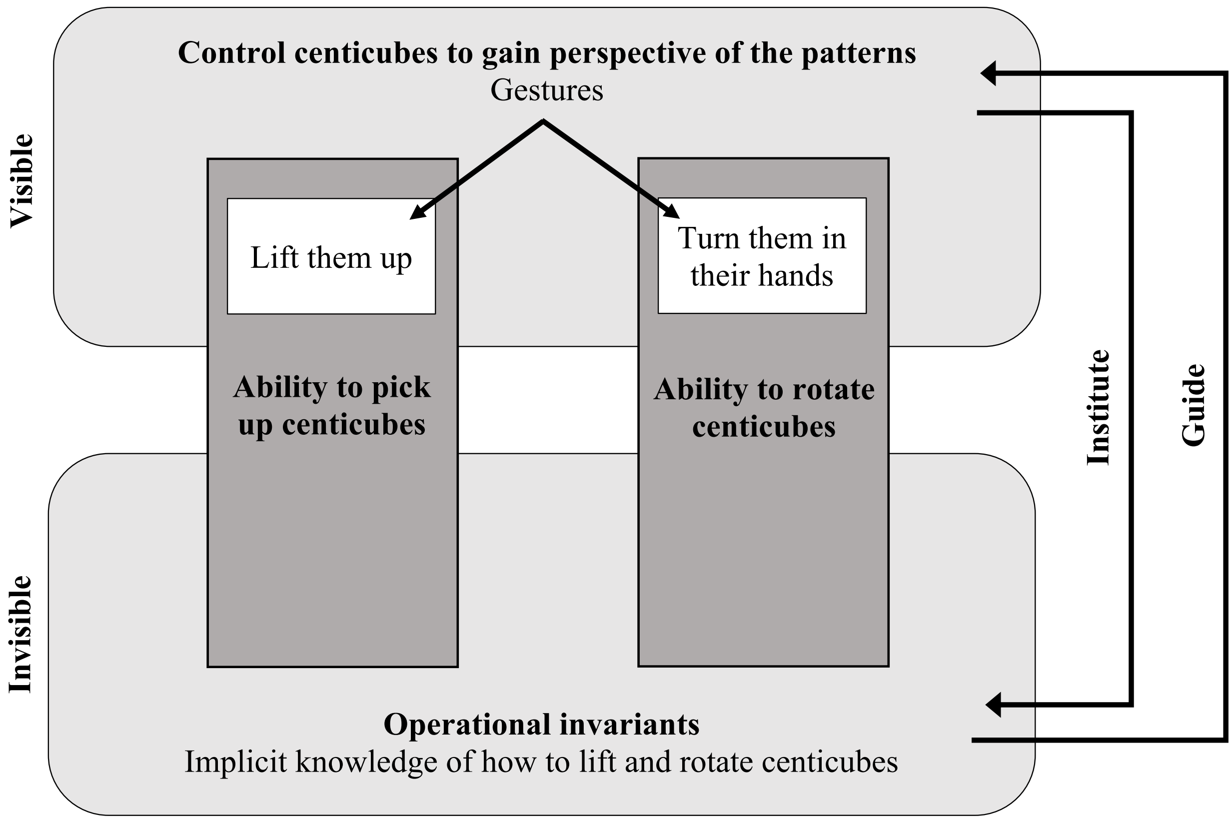


Figure 7.10: An instrumented action scheme for visualizing objects with centicubes. The instrumented technique is to control the centicubes to gain perspective of the patterns.

7.3.2 Instrumentation of the centicubes

The centicubes influenced the pupils' actions in many ways. What became most apparent in the classroom work session was that the centicubes were difficult to manipulate, so many of the pupils ended up doing things that were not intended. Some of the pupils became distracted by this and started throwing the cubes around and refused to participate in the building. During the recording, Tommy threw one of the elements he had built so high in the air that it fell apart on impact with the table. This was not related directly to the mathematical activities, but it did indirectly lead the pupils away from the tasks.

On the other hand, the centicubes were good to use for visualization because of their three-dimensional structure, and therefore gave perspective of every possible angle of the cubic patterns. When the pupils first had managed to build the patterns, the usage schemes, and thereby the technique, for gaining perspective were very simple (Figure 7.10). The pupils could just pick them up and turn them around. This instrumented technique was frequently used, as described in the previous sections, both in the work session and during the interviews. Owen did for example often pick up the cubes to *show* which elements he was talking about, compare elements with each other and to point to different parts of the elements. This helped him talk about the mathematics.

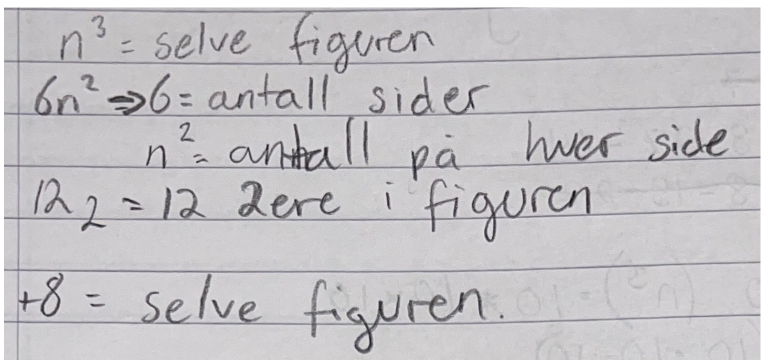


Figure 7.11: Abby and Marlene's attempt to decompose the structure of the expression $n^3 + 6n^2 + 12n + 8$.

Since most of the pupils were unable to build the patterns in useful ways that would have enabled them to see the invariant properties more clearly, the tasks became a struggle. The two pairs that chose to utilize drawing in their approach in addition to the centicubes, could have felt the need to also express themselves through another representation, because the centicubes limited them to do the same, due to the difficult manipulation of the blocks. Task f-h were especially difficult without the building of a useful pattern structure. Abby and Marlene built $3 \times 3 \times 3$ and $4 \times 4 \times 4$ structures in the last tasks, but they were not able to differentiate between any of the colors, so they did instead use symbols and words to express their thoughts, see Figure 7.11. They used very imprecise language but had some good points. First, they used the visualization of the cubes to establish that the cubes all had six faces, and that they in general had a square face containing n^2 number of blocks. Further, Abby and Marlene stated that there were "12 2ere" in the element. It looks like they chose to look at the specific element number 2, and they would then be referring to the blocks on the edges, but the pair was never able to connect $12n$ to the term edge.

Chapter 8

Drawing as a mediator of generalization

In this chapter, I will look at how the largest group of the class utilized a pen-and-paper approach to the generalization tasks of the cubic numbers. The drawing group consisted in total of 11 pairs. One of the pairs, Henry and Sam, were recorded during the work session, and they also participated in the interviews in the second part of the data collection. Their work will therefore be prominent in the analysis of this approach, but other pairs who provided interesting results will be considered as well.

Out of all the pairs in the drawing group, only one of them, Frank and James, managed to do all the tasks correctly. Most of the pairs were able to solve the first five sub-tasks, but struggled more with the last ones, and many left them completely blank. I will now analyze how the constraints and potentialities of this approach may have affected these results.

8.1 Potentialities of the pen-and-paper approach

8.1.1 Drawing lets pupils create their own structures

The ability to draw is very individual from person to person, something that potentially made the pen-and-paper approach very challenging for many pupils. To be able to represent the cubic patterns, some experience with drawing cubes or perspective drawing was required. The pen-and-paper in itself is perhaps the best-known artifact in a traditional classroom, so this is not a new approach for the pupils. They usually use pens or pencils to write down, solve, and present solutions to mathematical tasks they are given, and sometimes the solving strategies include drawing as well. In this section, I will look at some of the pupils that utilized the drawing to their advantage in algebraic generalization activities.

Both drawing and mathematics are concerned with turning invisible mental structures into something visual. A combination of the two disciplines can make pupils understand concepts better, but they need the ability to turn the abstract objects in their heads into something that can be interpreted on paper. The drawing approach therefore differed greatly from Minecraft and the centicubes, because the pupils had to create their own building blocks to express the patterns, instead of being given some predefined blocks either physically or digitally.

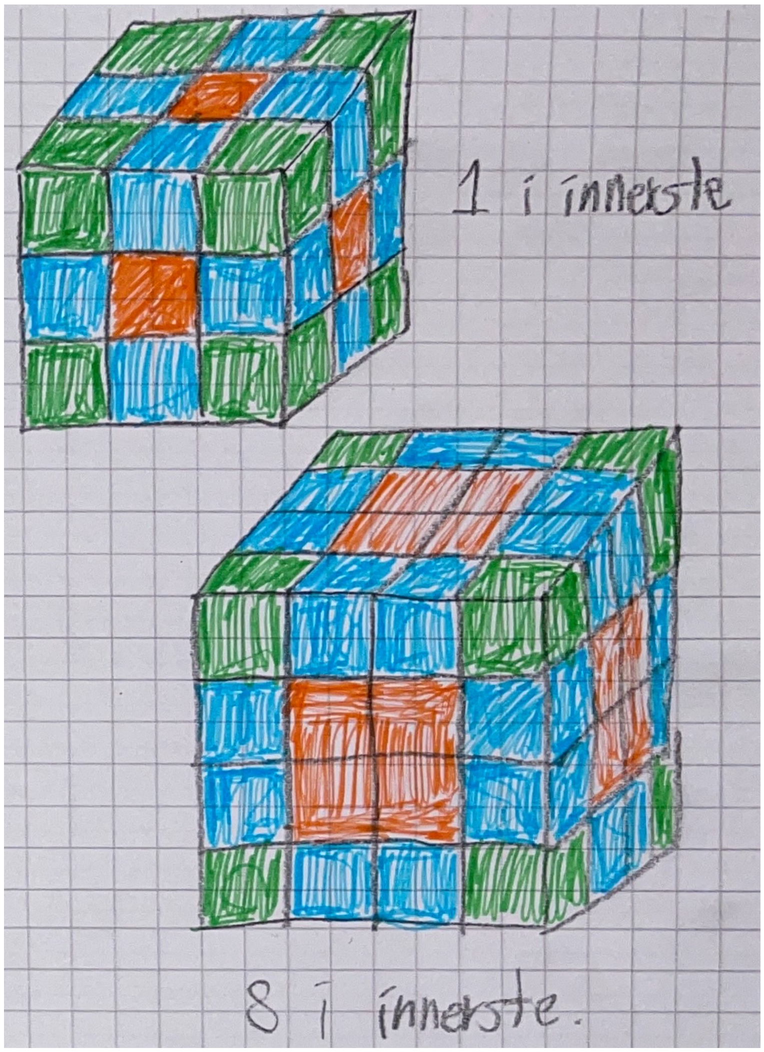


Figure 8.1: Frank and James' representations of the cubic pattern in the final task.

The first example is the work of Frank and James, who managed to see the invariant properties of the pattern, and also drew perfect cubes from the start. In Task a, they drew element number 3 and 4 correctly based on the initial structure they were given by the worksheet. They were able to calculate the number of blocks in each element in the sequence $\{a_n\} = n^3$ and stated that each element number were equal to the width, length and height of the respective elements, and that there were n blocks in each direction. It therefore seemed like the visual elements, even in a two-dimensional workspace, enabled them to notice the invariant properties, and adopt a multiplicative approach to the tasks. In Task h, Frank and James made a perfectly valid color coding of the first two elements of the sequence $\{b_n\} = (n + 2)^3$, seen in Figure 8.1.

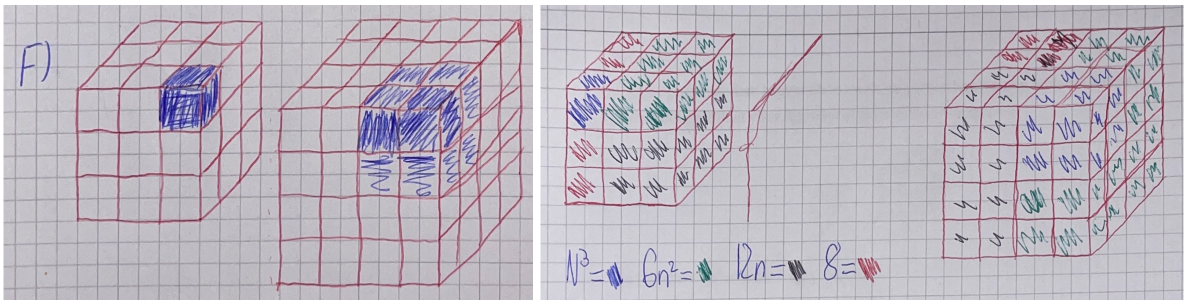


Figure 8.2: The representations of the cubic patterns in Task f and h, given by Pair 2.

Pair 2, seen in Figure 8.2, were also able to draw the elements but chose another approach than Frank and James when asked to draw the structure around the first sequence. By drawing the pattern from $\{a_n\} = n^3$ in the vertex of the sequence $\{b_n\}$, they were not able to see how the four terms $6n^2 + 12n + 8$ would fit to a general element. They were however able to color code the terms in the specific examples of $\{b_1, b_2\}$, by coloring the correct *number of blocks*, but without regard to the visual generality.

Pair 3 had the same strategy of drawing the term n^3 in the vertex, but they did not get so far as to color code any term. They simply wrote down the symbolic calculations of the terms and the numerical results, where they had inserted the values for $n = \{1, 2\}$. They then drew some arrows towards the elements, but they did not seem to point anywhere in particular.

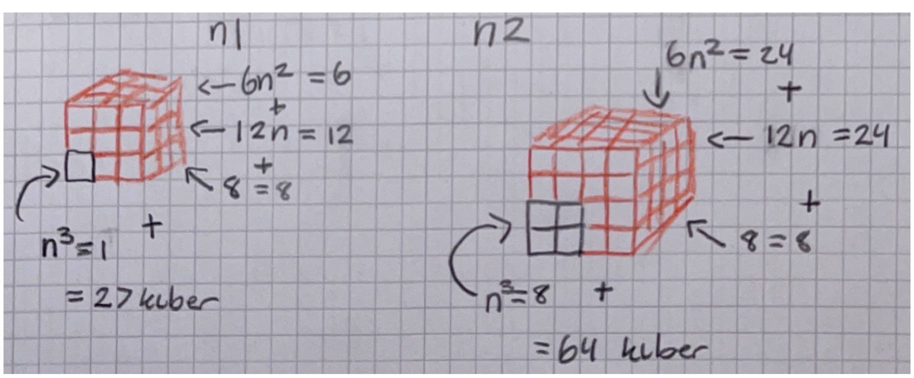


Figure 8.3: An attempt by Pair 3 to generalize the pattern in Task h.

Henry and Sam did also follow the same approach as the two pairs described above, but in the interviews, I challenged them to do the final task one more time, but this time with the n^3 component *inside* the rest of the blocks. The results can be seen in Figure 8.4.

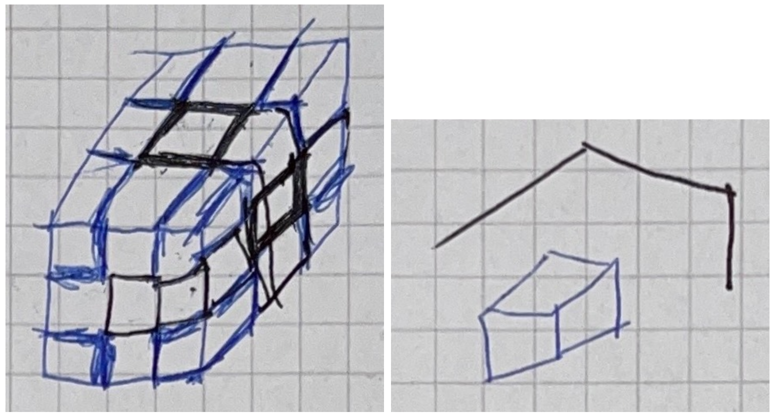


Figure 8.4: To the left is the element drawn by Henry with the peephole (middle block in front), and to the right is Sam's attempt to do the same before giving up.

RESEARCHER: Kan dere da prøve å tegne figurene på nytt, men da istedet for at vi får de gamle kubene i hjørnet, prøv å tegn de slik at de kommer inn i midten av kuben?

HENRY: Hvordan skal vi tegne det da? [Laughs.]

SAM: [Picks up a pen and a sheet of paper] Går det an å liksom, en kube...[draws a $1 \times 1 \times 1$ cube] sånn, en fin [ironical] kube, i midten, og så... [Picks up a pen with another color, and tries to draw around the first element, but realizes that he cannot do it. He therefore gives the pen to Henry.] Du kan ta å tegne, det var ikke en så bra kube.

Henry engaged in drawing a new element, while explaining his idea of creating a "peephole" in the cube, so that they were enabled to "see inside" the element:

HENRY: Okei så, siden den (1^3) er i midten, det er litt vanskelig å tegne den i midten, siden den er jo omringet av ting. Hvis vi har et kikkehull da! Hvis vi tenker at det her [draws a small square first] ser man inn i midten, og sååå...[draws a larger square around, which he expands into a $3 \times 3 \times 3$ cube around the "peephole"] tenker vi at det er egentlig et sånn, man bare ser inn til den som er under der nå [points to the middle of the element]... Og så er det egentlig en sånn her [draws a square underneath the element and points at it] som står foran (kikkehullet).

RESEARCHER: Ja, for det er ikke så enkelt å tegne den som er i midten eller?

HENRY: Nei!

Henry was able to draw the pattern with n^3 in the middle of the element, in a way that made him aware of what was inside of it, as well as outside. He also stated that the n^3 term had to be inside the peephole, so I then guided them into explaining where they could find the rest of the terms.

HENRY: Åtte, det kan vi kanskje legge i midten da [points around the top of the element] sånn i rundt den som er i midten.

RESEARCHER: På hvilken måte da?

HENRY: [Picks up the pen] Hvis vi ser bare på det midterste laget her, så tar vi det ut av kuben. Og den åtter'n, det er de åtte kubene som er i rundt sånn. [Marks the middle layer of 3×3 blocks with the pen.]

RESEARCHER: Vil dette fungere på figur nummer to?

HENRY: Ehhh... Nei siden det (8) er et fast tall, så da vil det ikke funke.

RESEARCHER: Hvor på kuben kan vi finne åtte da?

HENRY: [After 20 seconds of thinking] Kan det være hjørnene da? [Starts counting the vertices while pointing at the element] Det er åtte hjørner!

RESEARCHER: Og vil dere si at alle kuber har åtte hjørner?

SAM: Ja det blir en (blokk) på hvert hjørne.

RESEARCHER: [Guiding them] Og da har det ikke noe å si hvilken figur i følgen vi ser på eller?

HENRY: "Nei!"

When following this path, Henry and Sam were able to argue for each of the terms. There seems to be a connection between the ability to draw the cubic patterns, and the ability to generalize them. Henry was able to draw a structure that made both the pupils "see" the inner block of the element and decompose the different terms of the expression $n^3 + 6n^2 + 12n + 8$.

The pupils that were able to give viable pattern representations were able to notice structures easier than the pupils who did not, and therefore had an advantage in solving the tasks. In the following section, I will look at the constraints of the pen-and-paper, and look at examples of pupils who were limited by their drawing skills to participate in meaningful generalization of the patterns.

8.2 Constraints of the pen-and-paper approach

8.2.1 Pupils need skill of perspective drawing to visualize structures

Henry and Sam were often heard laughing during the recording, because they thought that their own drawing abilities were terrible. While trying to draw a $4 \times 4 \times 4$ structure, Henry noticed that he had missed one of the perspectives, so he had to erase parts of the pattern and draw again. After he finished, the pair concluded that it did not look pretty, but that it would be understandable. Figure 8.5 shows an example of Pair 8 and 9, who struggled a lot with drawing the patterns. They could not draw cubes that let them visualize the structures and they were unable to participate in useful mathematical activity. Pair 9 created the explicit formula $N = n \cdot n \cdot n$, but they were unable to give any written arguments for why it would be a valid generalization.

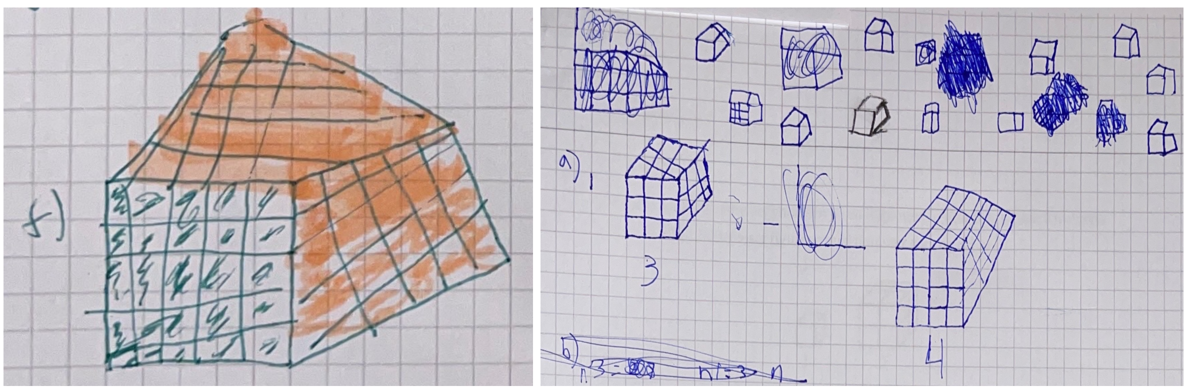


Figure 8.5: Example of Pair 8 (left) and 9 (right), that struggled very much with drawing the cubic patterns.

More interesting are the answers of Pair 8, who established the general formula $8 \cdot n = x$ for the arbitrary element they denoted x. Their answers leading up to this can be seen in Figure 8.6. In Task a, Pair 8 struggled with making the elements look like cubes. In Task b, when they were supposed to write down the number of components in the first four elements, they seemed to ignore the drawing they had made. They first wrote that element 1 had 1 component and element 2 had 8 components, which were correct answers. From there, the pupils appeared to utilize the numerical values of the elements instead of the figural elements they had drawn. The pair noticed that from element 1 to 2, the elements were "multiplied by 8", and therefore adopted a multiplicative relationship. They claimed that element number 3 had $3 \cdot 8 = 24$ components, and that element 4 had $4 \cdot 8 = 32$ components. The formula they were led to establish therefore became $8 \cdot n = x$, which did not fit the first two elements, but they took this into account when they wrote that "alt utenom i kube 1 er i 8 gangen, men formelen funker bare fra [figur] 3 og oppover".

The solution of Pair 8, is another example where the tasks' ambiguous design led the pupils towards an erroneous use of the numerical values in the pattern sequence. When the visual structure was ignored, the pupils were led to establish a formula based solely on numerical evolution. They were also unable to make their formula fit for all elements, where they had to exclude the first two.

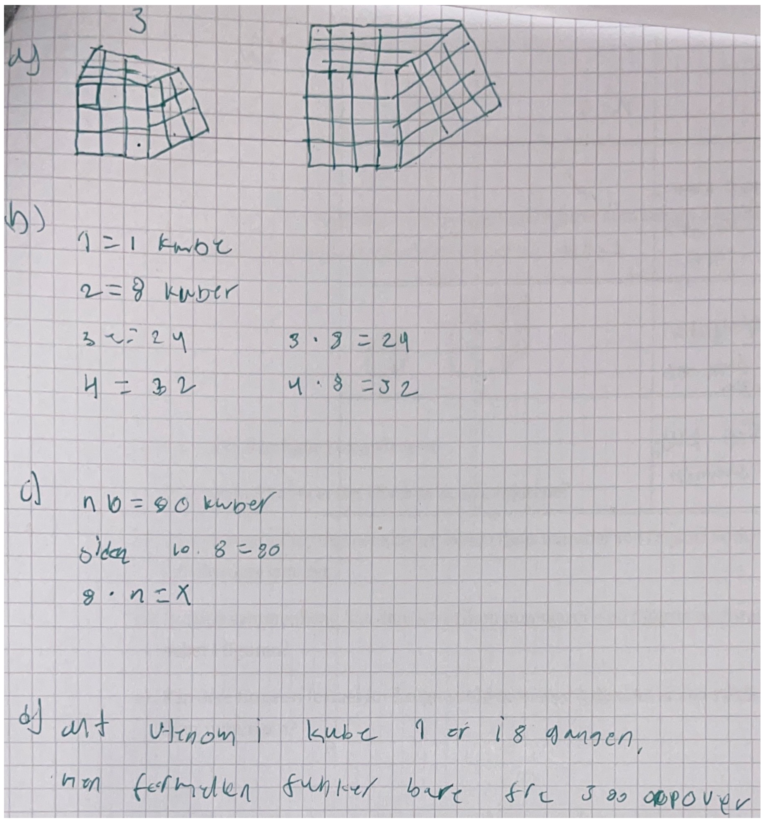


Figure 8.6: The solving strategy of Pair 8 in Task a to d.

Representations only give one perspective of the elements

As Henry uttered in the interviews, it is almost impossible to draw the cubic structures in a way that lets one see what is inside the outer layer. That was why he created the "peephole", through which he mentally visualized the inner components. Also, when they had drawn the elements, the pupils would be left with only one perspective of them. Regardless of how good the pupils were with drawing the structures, they could not just pick them up an get a three-dimensional view of each component.

Using drawings like the ones in Figure 8.7, the pupils tried to engage in meaningful generaliza-

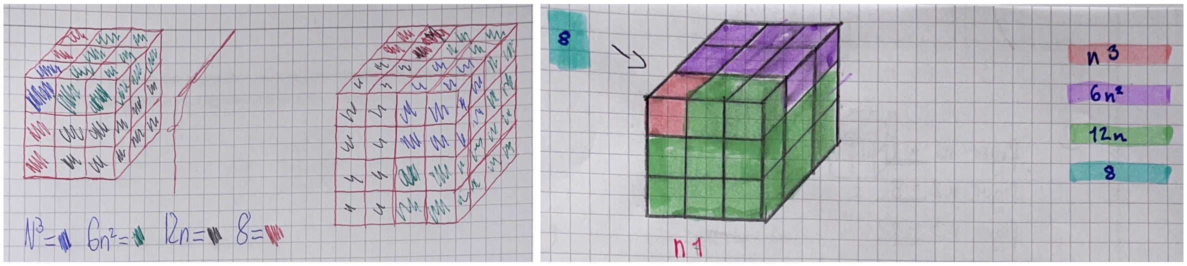


Figure 8.7: Pair 2 and Pair 10 presenting their color coded results. They had to represent the four terms without being able to change perspective.

tion, but they were unable to mark all the components because some of them were hiding behind the ones on the surface. Since many important points of view were missing, it is difficult to be sure of how the pupils had established the pattern behind the visible part, unlike the case of Frank and James (Figure 8.1), where it was very clear that they conveyed an unambiguous pattern, because all three visible faces consisted of the same pattern.

8.3 To what extent is the pen-and-paper developed into an instrument?

In this experiment, the pupils were building instruments for presenting and generalizing cubic patterns from the artifacts they were given. Even though the pen-and-paper approach was not new in the classroom, the pupils had not necessarily built this type of instrument from it before. They therefore went through processes of instrumentalization and instrumentation connected to the artifact, where they learned how to use it for the specific purpose in the tasks.

Since the pen-and-paper approach does not offer the same way of controlling the environment to gain perspective of the objects with a specific technique, as with Minecraft and the centicubes, I have not identified an instrumented action scheme for this aspect of the drawing approach.

8.3.1 Instrumentalization process of drawing

Most pupils already knew how to use a pen and paper, and many did probably not go through so much discovery connected to the artifact, but I have shown examples of pupils that were unable to draw cubic structures. These pupils were still trying to figure out how to use the pen as an artifact in this particular setting of drawing cubic patterns.

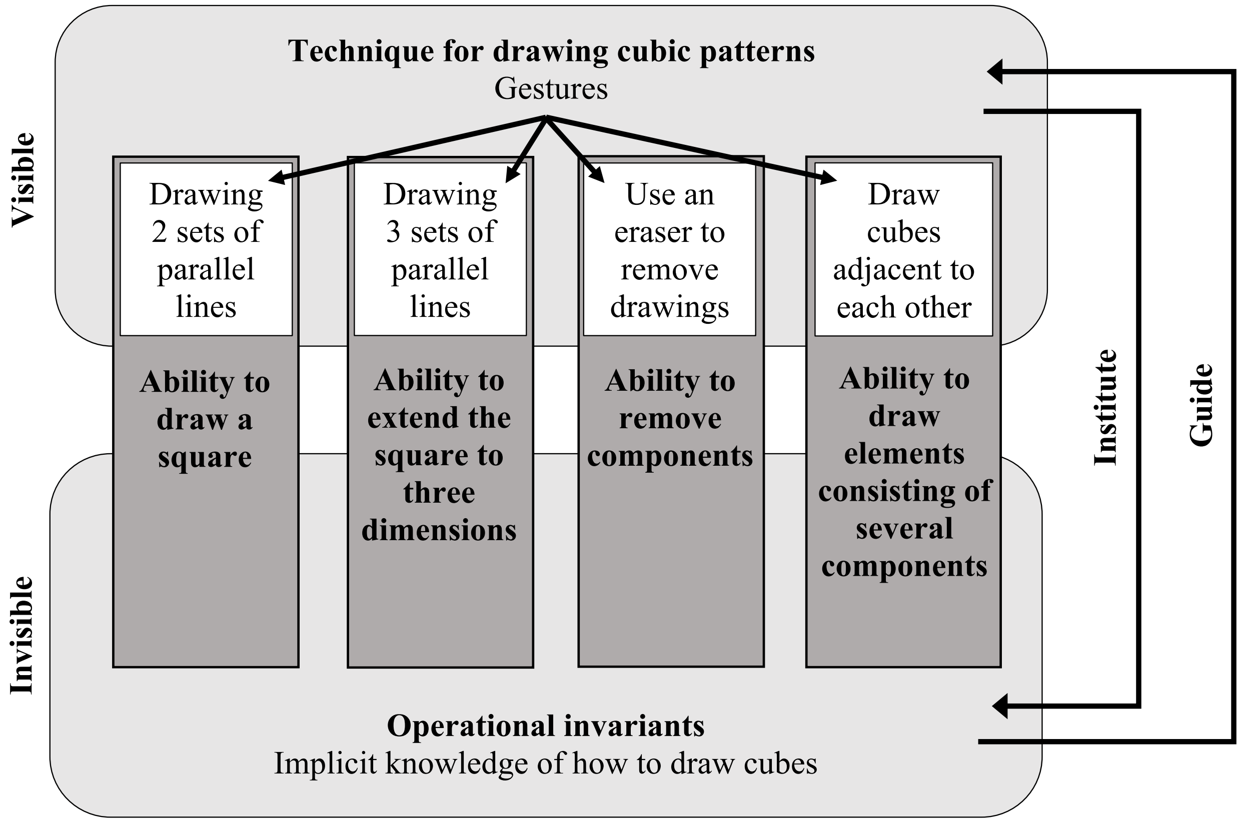


Figure 8.8: An instrumented action scheme for the technique of drawing cubic patterns.

The pupils used tools from their own pencil cases when solving the tasks. A variety of different tools were therefore mobilized. Some of them used ink pens or pencils, some used permanent markers, and one pair also used a ruler to draw straight lines. Those who chose to draw with pencils, in contrast to permanent markers and pens, had the advantage of accessing a usage scheme that the others did not, namely the ability to erase blocks they had drawn to remove or replace them. Henry did this when he noticed that his three-dimensional drawing was missing something. This made it easier for the pupils to manipulate the components and elements for visualization purposes.

The instrumented action scheme for the technique of drawing cubic patterns (Figure 8.8) states that it is required to draw a total of three sets of parallel lines[1] to constitute a single cube. To complement the scheme, I have identified these three sets in Figure 8.9. If the cubes are to be put together to form a larger structure of several cubes, it is additionally required to draw them next to each other, respecting the overlapping edges. When Henry was drawing cubic elements, he always

[1]I choose to ignore the principles of "vanishing point" perspective drawing, which would cause the lines not to be parallel.

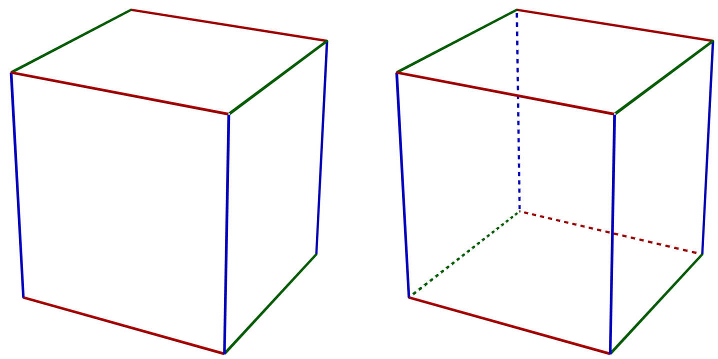


Figure 8.9: Color coding of the parallel edges in a cube. The twelve edges are parallel in three sets of four line segments.

started out with drawing a square, divided into a grid of $n \times n$ cells, corresponding to the first usage scheme in Figure 8.8. He then proceeded to extend the structures into the three-dimensional space by drawing parallel lines going out of the vertices of the square, which he then connected at the back. Henry's process of drawing the cubes therefore followed the instrumented technique described in Figure 8.8.

8.3.2 Instrumentation of the drawing

In the interview, when Sam was trying to draw similar cubes as Henry did, he would start by drawing a $1 \times 1 \times 1$ cube in the middle, and then try to draw a larger cubic structure around it (Figure 8.4). Because Sam only could utilize one of the identified usage schemes for drawing cubes, the instrumentation process led him to a simplified action scheme, together with an impoverishment of his operational invariants. The visual feedback he received from his own drawings, made Sam give up, and left Henry to contribute with the drawing in the pair. This is an example where the complexity of the artifact prevented the user from building a useful instrument from it. Henry was led to create a peephole where he could visualize the inside of the mathematical structure mentally. He used the artifact to receive a combination of mental and visual feedback, which enabled Henry and Sam to decompose the pattern.

As identified in section 8.2, it was not only Sam that were unable to draw cubes. Most of the pupils that struggled with drawing, were able to utilize the first usage scheme, leading them to draw the front face of the elements, but failed when trying to extend the two-dimensional squares into three dimensions by including the third set of parallel lines. In the cases where both pupils in the pair were unable to draw the cubes, I noticed that the lack of visualization stopped them from participating further with the tasks when they became increasingly difficult. Especially Pair 9 (Figure 8.5), that was identified as struggling a lot with the drawing, went through an explosion phase, trying to combine the first two usage schemes in Figure 8.8 in different ways. The pair was able to draw the front face of the cubes, but they were unable to reach a stable phase where they mastered the technique of drawing cubic structures by combining several usage schemes. Therefore, these two pupils never seemed to achieve any meaningful pattern construction with the pen-and-paper. Pair 8 seemed to abandon the visual representation they had made, because they could not establish the invariant properties of the elements they had drawn.

Both Pair 8 and Pair 9 struggled with *localized determination* (Trouche, 2005). Pair 8 were unable to convert their drawing into a symbolic representation that included the invariant properties of the pattern. Pair 9 had the opposite problem. They seemed to have an understanding of the invariant properties – because they were able to state the formula $N = n \cdot n \cdot n$ – but could not convert their mental structure into a visual one with the pen-and-paper, due to being unable to combine the usage schemes of drawing cubes.

I have also shown several examples of pupils who were able to utilize the usage schemes described in Figure 8.8. These pupils were able to draw good-looking cubic patterns, but most of them could not see the same general pattern that Frank and James did. It is difficult to say if the lack of perspective had something to do with it, but the drawing of the elements did not seem to shape the pupils into building useful instruments from the pen-and-paper.

Chapter 9

How do the artifacts compare to each other?

After analyzing each of the three approaches, some questions still remain unanswered. I have identified the constraints and potentialities of the artifacts and analyzed how these have affected the learners in situations related to the generalization of patterns. But how do these artifacts compare to each other? What are the similarities and differences between them, and what makes one approach favorable over another? Were any of the results surprising?

In this section I will zoom out and look at the three artifacts in relation to each other to answer these questions. First, I will compare the instrumented action schemes of each approach, to see how differences in the pupils' activities can be characterized, and what the strengths and weaknesses of the artifacts are. Then I will follow up with a discussion of how these artifacts could have been further developed to support the learning activities even better. After that, I will compare my results to some of the previous research that has been done in the field of algebraic generalization and on the position of video games in the mathematics classroom, mentioned in the theoretical framework in chapter 3.

9.1 Visualization and manipulation of mathematical objects – similarities and differences in instrumented action schemes

The two most important parts of participating in the mathematical activity highlighted by this study, have been the pupils' ability to *manipulate the objects* they were working with, and their ability to use the given artifact to *visualize the mathematical structures*. In this section, I want to compare the instrumented action schemes of the three artifacts, with respect to these two aspects. In Figure 9.1, the schemes of all three artifacts have been put together for comparison. The schemes to the left are regarding the manipulation, or creation, of objects, and the schemes to the right are concerned with gaining perspective. No instrumented action scheme has been identified for gaining perspective in the drawing approach, since this was not relevant for the artifact.

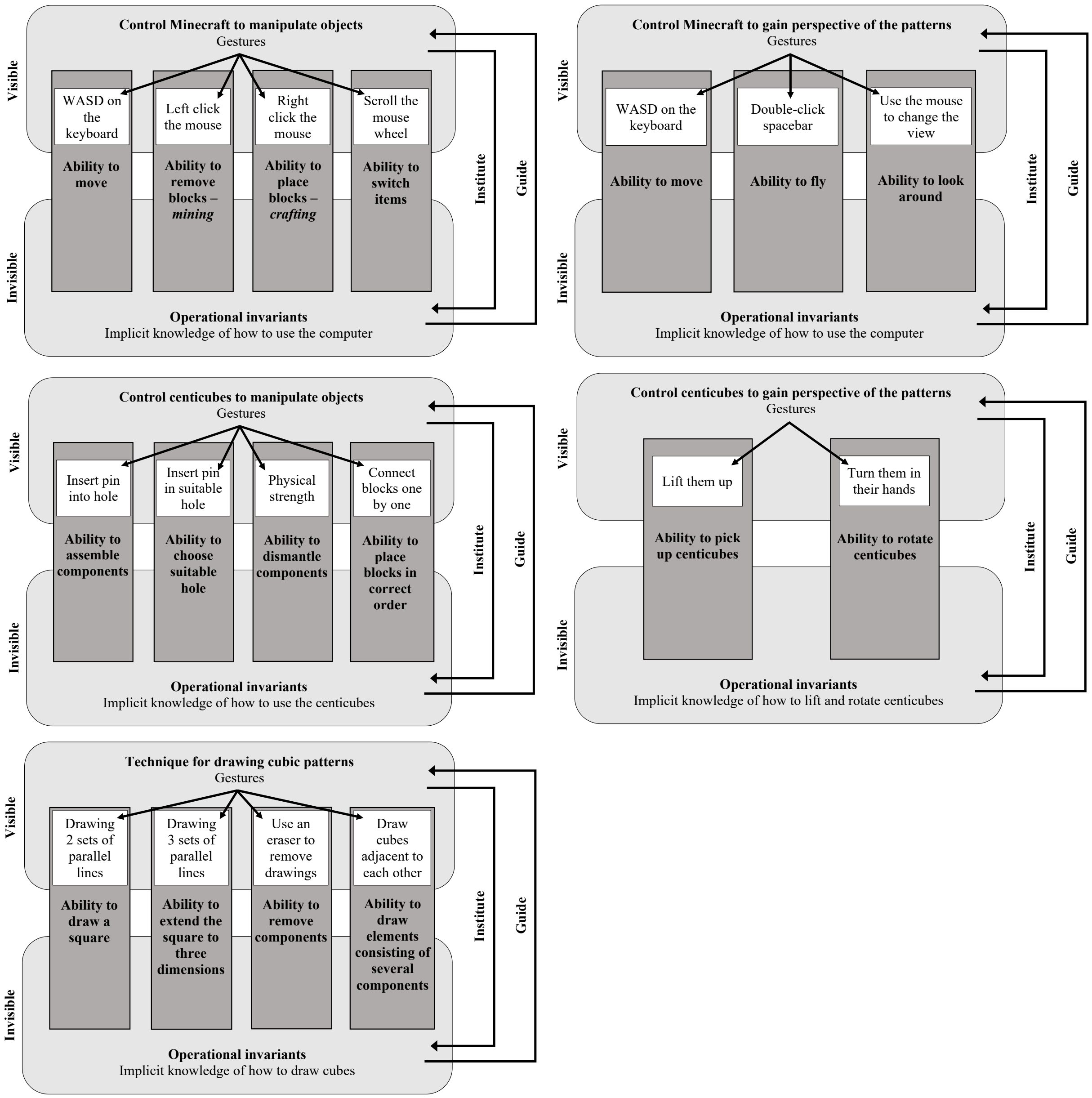


Figure 9.1: Instrumented action schemes for all three artifacts; Minecraft (top), centicubes (middle), and drawing (bottom).

As mentioned in subsection 8.1.1, mathematics is often concerned with turning invisible mental structures into something visible and accessible for external observation. I view the process of visualizing mathematical objects as a repeated interaction between the mental structures of a mathematician[1], and the representations used to express these. When pupils have mental ideas of a concept, it is important for them to be able to express the concept in some representation outside their heads, either through speech, writing, drawing or building. Then the learner can access these properties outside of his head, and the external representations can again lead to a further development of the mental structures of the concept. All the three different artifacts discussed in this thesis can help the pupils achieve active and purposeful use of the visualization process. With the artifacts, the pupils are able to represent their mental structures, each in their own unique way, guided by the operational invariants the pupils possess and evolve while using the artifacts. But which one makes it easiest for pupils to do so?

Concerning the instrumented action schemes for Minecraft, there is an apparent simplification, compared to the centicube and pen-and-paper environments, because of the simple manipulation of the artifact. In Minecraft, the control sequences for the different gestures have already been implemented in the software and are utilized by a click of the mouse or keyboard, whereas for the centicubes and the pen-and-paper, these have to be manually performed by the pupils. Engaging in building activities with the centicubes involves usage schemes directed at the artifact that require both physical strength and a sense of order in which the building steps are executed. Once the structures have been built, it would be easier to gain perspective of the patterns with the centicubes, because this technique only relies on the ability to pick them up and to rotate them in the hands. However, the visualization scheme for Minecraft is not much more complicated, and mainly relies on the ability to move around with the keyboard, and to change the view with the mouse. The pen-and-paper approach does not offer any kind of visual perspectives, so Minecraft seems to be the better choice for the combination of building *and* visualizing the structures.

The usage schemes of manipulating the centicubes hint towards a planning dimension of utilizing them for building three-dimensional structures. Because of the fact that the blocks would have to be placed and removed in a certain order to be utilized properly (see for example Figure 7.5), the artifact seemed to be utilized better by those pupils who already had noticed a structure they wanted to build and did not have to rely on removing blocks again. In the drawing approach, the usage schemes for creating cubes could be difficult to master if the pupils struggled with drawing skills, but the pupils were able to easily erase what they had drawn if they used a pencil. However, I could not find much evidence that the pupils actually used the ability to try out different strategies by manipulating the drawing.

[1] In this context, anyone who is concerned with mathematics, e.g., a pupil, is considered a mathematician.

Minecraft offered the easiest tool to handle, good for *testing* structures and solutions, because of the simple usage schemes related to the building activities in the game. All Minecraft pairs actively used the abilities to mine and build for easy visualization of the patterns, and they also used the simple manipulation schemes for going through series of abductive-inductive cycles, which seemed to help them in the repeated interaction between mental and visible structures mentioned earlier. The pupils that used Minecraft could therefore spend more time reflecting on the consequences of their manipulation of the algebraic structures. When the pupils were unsure of how to proceed, they could try out different assumptions instead of just giving up, unlike in the centicube and drawing approaches, where there seemed to be a higher threshold for this kind of testing, and a lower threshold for giving up. Also, the fact that Henry and Sam felt the need to make a "peephole" to *see* the inside of the structures they had made, points towards a clear advantage of Minecraft that provides clear glass blocks that the pupils are able to see through.

In the interviews, David and Tess stated that they had found it easy to *use* Minecraft, and that they found the visualization the game provided very useful. However, not all pupils have the same experience with Minecraft, and to make everyone in the classroom able to play the game would require some collective lessons. To include Minecraft in the teaching *could* therefore be a constraint in itself, if the time required to learn the game was more extensive than the time available. Nevertheless, Minecraft has a very basic gameplay, so that most *digital natives* would have an easy time to comprehend the usage schemes related to the game. Many of the teachers, the *digital immigrants*, would most likely struggle more with adapting to using Minecraft as a tool. Further, it seems after this project that the pupils also would have needed some basic instruction both on how to use the centicubes and how to draw cubes as well, because many of the pupils struggled with making useful interactions with these two artifacts. This would also have been time consuming, and the time aspect would have to be considered along the same lines as with Minecraft.

With the centicubes, I experienced that a lot of pupils quickly became unfocused when the tasks became hard, and the centicubes also were hard to manipulate. Many of the pupils started playing with the cubes, and some were throwing them around the classroom instead. I think that the centicubes have more potential for building other structures than increasingly large cubes, because cubic structures are very "dense", i.e., they consist of several blocks in equally sized layers. This makes it hard to remove blocks in the middle of the cube – see for example Figure 7.5. Other structures like natural numbers, odd numbers, or maybe also square numbers are examples of algebraic patterns that would be much more easily built and manipulated with the centicubes, since they could have been represented as two-dimensional[2] patterns. The building blocks would be more accessible for reorganization if they were configured as a straight line or square of blocks, instead of as a dense cube. However, Minecraft could also have been used for a similar representation of two-dimensional figurate numbers, and the pen-and-paper approach would certainly be fit to the task as well.

The centicubes and Minecraft blocks are the most similar building blocks in this experiment, since they both are visual, prestructured[3] blocks that offer a full perspective when utilized. The main difference between them is that one exists physically in the real world, and the other one exists "physically" in the electronic game world. The centicubes are tiny, and it requires a lot of them to engage in comprehensive building. It is therefore possible to avoid a lot of mess in the classroom by choosing Minecraft, or drawing, as a work method. In Minecraft, the number of blocks are infinite, and I also believe that it is easier to *learn* how to play Minecraft, than it is to learn perspective drawing and how to use the centicubes purposefully, because the totality of the schemes connected to controlling Minecraft for algebraic purposes are much easier than those of the centicubes and drawing.

Potentially, the centicubes could have the advantage that both pupils could work on them at the same time, because they do not have to "share" the computer, like with Minecraft. This did not seem to be the case however, as the pupils rather tried to avoid to be responsible for the building, as seen in the dialogue of Tommy and Bill in subsection 7.2.1. The drawing approach also had the advantage of pupils being able to work simultaneously, on their own piece of paper. Many of the pupils in the drawing group did this, to compare their representations, which gave them additional perspectives of how their partner was thinking. The drawing therefore supported the mathematical conversation between the pupils to a larger degree. These conversations were also fostered by the Minecraft environment, where the pupils took turns of who controlled the game so that they could express their own mental structures for their partner.

[2]When building two-dimensional patterns with the centicubes, the measure of the pattern in the third dimension would always be equal to 1 block.

[3]The blocks do not have to be created by the pupils, as in the drawing approach.

9.2 How could the artifacts be further developed for use in the classroom?

Some of the constraints identified about Minecraft was connected to the use of the *book and quill* in the game. Since these constraints had nothing to do with algebra specifically, I did not create any instrumented action schemes connected to the book, but as I have hinted towards, it has some impractical usage schemes. The general constraints mentioned about the book and quill were related to reading tasks and writing answers in the book. There are three aspects I hope will become a part of the design of the book and quill in future updates of Minecraft: Education Edition. The first is a function that lets the player open the book where it was last closed. Secondly, it would have been preferable if the user could click anywhere on the page to place the edit marker, because as it is, one must use the arrow keys to navigate backwards or forwards in the written text. This resulted in a bit unnecessary use of time for the pupils, especially in Task b and g, where the pairs were supposed to fill values into the tables I had made in advance. Then they would have to navigate all the way back to the first line, write something there, and then navigate further with the arrow keys. The third function that would be good to see in the book and quill is a proper table function, which would have been great for the design of Task g. These are game elements that would be possible for the developers to fix in updates of the game. As Minecraft begins to have a greater place in the classroom, these are aspects that could make the game even more interesting to use. As of today, in a non-research setting, I would have preferred to use a separate worksheet to replace the book and quill, but for the purpose of the study, I wanted to explore the full potential of the game itself. Maybe this research can help fill some of the holes concerned with shortcomings like those identified here, and contribute to a positive development for the future.

Gilje et al. (2016) identified that the constraints with many learning games is that they lack a close connection between how mathematical structures are represented in the game, and the formal mathematical symbolization used to express concepts in textbooks and in teaching. In chapter 6, I identified some limitations concerning the use of mathematical symbols in Minecraft, where the pupils for example had to use a n^3 notation, instead of n^3. These limitations are fortunately not taking the pupils that far away from the formal notation, but it would still be good to have a way of expressing mathematical structures properly in the game, so that the pupils do not get confused by the lack of coherence between the representations.

The biggest constraint of the centicubes was that when the blocks first had been put together, they held on tight to each other. Since they were physically hard to remove, a suggestion that came to mind would be do produce the centicubes with magnets instead of pins and holes to stick them together. That would have made the centicubes more expensive to produce, but maybe a bit easier to reorganize and work with. It would still be a problem to get a grip on the blocks in the inner layers of the structures, but at least the outer layer would be easier to manipulate.

9.3 Color coding in far generalization tasks

One of the goals of this research has been to determine how pupils responded to three different learning environments for representing algebraic patterns visually. As stated by Strømskag Måsøval (2011), the ability to reason visually is one of the success factors for generalization of patterns. Further, Nilsson and Eckert (2019) and Rivera (2010) pointed out that pupils often struggle with far generalization tasks. Nilsson and Eckert (2019) suggested that color coding of components in the elements of a pattern sequence could support the pupils' claims in generalization tasks. In this study, I have designed a task (Task h, Figure 4.1) where the goal was to decompose the pattern of the sequence $\{b_n\} = (n+2)^3$ into four different color codes.

Nilsson and Eckert (2019) looked at how pupils responded to color codes mainly in near generalization tasks, and their study showed that this approach supported the flexibility of pupils' perception of the relationship between the algebraic expressions and the visual structures of the patterns. I developed a similar approach for this research, but with a goal of far generalization of the expression the pupils were given in Task h. I had the pupils generate their own color codes to give visual arguments that would be valid for every term in the general expression of the sequence.

The color-coding seemed to enable some of the pupils to change their interpretation of the variable n and turn it into visual expressions of the building blocks they were given. David and Tess – in the Minecraft approach – did for example see each of the terms isolated, and gave valid reasons for how each part of the visual cubic structure would evolve with an increased variable number, e.g., the blue edges would become longer, the core would become a bigger "die" and the vertices would stay the same because the term 8 had no variable connected to it.

During the analysis of the different approaches, it became clear that in order to visualize the patterns, the pupils would also have to be able to build them. I showed examples of pupils that managed to do so in all three environments, but I also showed that many of them did not. Minecraft seemed to provide the best solution to the building process, because the game made it very simple to place and replace blocks. The glass blocks in Minecraft also turned out to be even better tools for visualizing than I had pictured, because they were transparent and appeared in several colors. This enabled the pupils to see the core of the structures as well as the outside, something the pupils in the other two approaches were quite limited to do. Also, the lack of perspective from the drawn structures, limited the pupils to color only the components of maximum three faces of the cubes. However, when making a valid decomposition, the pupils would, strictly speaking, only have had to color one of the faces, because all six faces would have to look the same, as shown in the example of Frank and James (Figure 8.1).

I think that color coding in generalization tasks, like the ones discussed in this thesis, definitely can help pupils visualize algebraic structures. Decomposition of the patterns seems to be a useful strategy for noticing invariant properties of the structures and how different parts of the patterns evolve simultaneously. I believe that the content in Task h can be topic for many good classroom discussions of how to make general assumptions on the structure of cubes and how cubic pattern sequences evolve with an increasing variable number. This is a task I will take with me into my future life as a mathematics teacher.

9.4 Can engagement be a problem for learning?

It can be very hard to teach a classroom full of pupils if the pupils do not find the content interesting. It is the job of a teacher to introduce pupils for engaging learning environments and to make them active participants in their own learning. Engagement is important for including everyone in the teaching, but can the pupils ever get too engaged?

Gilje et al. (2016) indicate that the connection between engagement and actual learning in mathematics only makes sense when the pupils' experiences in a given environment can be converted to formal mathematical language. If the symbols or objects used in a game differ too greatly from the symbols the pupils are taught and tested in, the game quickly loses its purpose.

> Engasjement i aktiviteten bidrar gjerne til at elevene opplever innholdet og representasjonene i spillet som mer meningsfulle, og det bidrar igjen til økt motivasjon for å lære. Engasjement regnes derfor som en forutsetning for læring. Likevel er det viktig å spørre om et slikt engasjement i alle tilfeller fører til at elevene lærer mer og bedre. (Gilje et al., 2016, p. 66)

9.4.1 Engagement or reflection?

Gilje et al. (2016) showed in their case studies that digital visualization platforms could either result in deep reflection or wild guessing for pupils engaged with algebraic tasks, depending on how the gameplay was defined. Games that had a strong element of competition would more often result in pupils guessing for answers, because it was more important for them to win than to get correct answers, whereas a balanced gameplay fostered reflection rather than guessing.

When designing content for a teaching situation, it is important to think about which elements that need to go into it to make it engaging for pupils to participate. The Algebra desert was purposely created without any competition elements between the players or against the game world. Although it can be easy to add competition to a game environment, it does not necessarily have to be utilized all the time. The centicube and drawing approaches did not contain any elements of competition either, but an important part of the implementation of these approaches was that they gave the pupils new tools for representing different mathematical structures. These tools could help to broaden the pupils' horizons of choosing relevant registers to present and extend their knowledge with. In the process of gaining knowledge of abstract algebraic objects, it is imperative to have the means for representing the structures both visually and verbally (Duval, 2006), so giving the pupils different angles of approach could help them support their own reflection.

The pupils participating in the Minecraft group found the game approach very fun, and Sarah uttered that she would love to do homework in Minecraft rather than the traditional textbook approach. Joel said during the interviews that "Det er mye lettere å se hva du gjør [i Minecraft], enn når du sitter og skriver. Og så er det litt mer interessant enn å sitte å stirre i boka." This leads me to think that Minecraft can act as a low threshold gateway into engaging pupils that do not necessarily enjoy mathematics otherwise. I think that a balanced gameplay would be the key to success, where engaging elements play an important part, but the most important is that the pupils have time to contemplate on the mathematical content, which they do in Minecraft, at least in the Algebra desert. Therefore, it is also important that, in a non-research setting, the tasks that are presented and worked with in the game environment, are taken into a classroom discussion, where the pupils are able to reflect upon their experiences and discoveries. The games in themselves may not always be enough to bring meaningful learning, but when they give room for organized classroom discussions, directed by the teacher, the games can be catalysts for deep reflection.

As Gilje et al. (2016) pointed out, the element of competition will often lead pupils to great engagement in teaching situations, but often at the cost of the deep reflection that they need to gain new knowledge. Out of the two games that the authors used in the research of the 5^{th} grade classroom, one of them fostered engagement and competition, while the other fostered reflection, because of their different gameplay. The ideal learning environment, in my opinion, combines these two aspects and creates an environment where deep reflection is supported, while the pupils at the same time experience engagement and immersion in the tasks. It is therefore not a question of engagement *or* reflection, but rather a statement of engagement *and* reflection. This requires an environment designed to foster the interplay between both aspects.

I got a question once, regarding the design of the Algebra desert, the world that the pupils participated in. "Is not the word desert making the pupils associate algebra with something dry and boring?" I want to conclude this chapter with a short paragraph on this, because it is a very relevant question concerning task design, and something every teacher should bear in mind when creating learning environments for the pupils. In this case, I have the opposite opinion. Speaking of engagement, Minecraft offers many different *biomes* that the player can travel to, that all have their unique features. I agree that the word *desert* can create associations to something dry, and I could just as well have created an Algebra rainforest. However, the desert biome in Minecraft is a very relaxing and nice place, with a lot of oases, animals and fascinatingly viable plant species. I therefore created my world in the desert for reasons contrasting to boring.

Chapter 10

In retrospect

10.1 What have I learned?

This master project has aimed to gain more knowledge on how artifacts can be used in algebra education to mediate actions, leading the pupils towards knowledge of pattern generalization. The research questions I aimed to answer were the following:

> **RQ1:** How are pupils in lower secondary school using the three different artifacts Minecraft, centicubes and pen-and-paper for presenting and generalizing structures in cubic patterns, and what are the constraints and potentialities of the different artifacts?
>
> **RQ2:** To what extent do the different artifacts develop into instruments in the pupils' work with cubic patterns?

What I have uncovered is that Minecraft is a very good tool to utilize in the algebraic activity that this study has focused on, which has been generalization and decomposition of two different cubic patterns. Minecraft has turned out to be very easy to use, and the game offers great visual representation of the mathematical objects. Because of its engaging nature, the pupils participating also found this to be a pleasant alternative to traditional teaching. Seeing that engagement is a prerequisite for learning, Minecraft seemed to offer novel elements to the teaching process, that could be beneficial for the pupils. Most of the constraints identified with Minecraft were connected to the book and quill; more closely the editability of the book and the possible representations yielded. The book and quill will hopefully become a more comprehensive tool in the future of Minecraft: Education Edition.

The centicubes, on the other hand, were a bit harder for the pupils to control, but they also offered good visual perspectives of the patterns when they first were built. The pen-and-paper approach showed that it was hard for the pupils to draw three-dimensional structures, and that this constraint limited many of the pupils in their engagement with useful generalization of increasingly complex patterns.

In this project, all the pupils were given the same task, so there was no differentiation on the level of the task. That means that all the pupils, ranging from the least to the most skilled mathematicians, all participated in the same generalization activity. Regardless of the given artifact, it can be difficult for someone who struggles to understand the mathematical content to comprehend these kinds of tasks. What I have found in this study, however, has been an expression of engagement from the pupils concerning the algebraic tasks when presented with a game environment, where elements can be visualized and easily manipulated. Minecraft did not make the mathematics in itself any easier, but it definitely helped the pupils visualize the content. Even pupils who usually struggle a bit had a chance to perform in the generalization activities, by trial and error with the Minecraft blocks. If I were to perform these learning tasks as a full-scale classroom lesson – and the infrastructure around it had been in place – I would definitely have chosen Minecraft as the preferred way to teach.

If I was to conduct this study again, I would have applied some changes to the design of the tasks given to the pupils. After seeing how the pupils responded to the formulations, I would have rephrased two of the sub-tasks in the worksheet (Figure 4.1). Task f could have been formulated slightly different to make it more clear that the pupils were supposed to build the pattern sequence $\{b_n\} = (n+2)^3$ with the previous sequence $\{a_n\} = n^3$ in the core of the new structure. In this way, the new components would have surrounded the previous ones like a shell, and the components from n^3 would not have been placed in the vertex of $(n+2)^3$, like many pupils did. This clarification could possibly have enabled more of the pupils to see the point of Task h. However, Task h could also have been phrased a little more unambiguous to make more pupils understand that the expression $(n+2)^3$ was equal to the expression $n^3 + 6n^2 + 12n + 8$, and that the pupils were supposed to connect each of the terms with distinct parts of the pattern elements.

10.2 The road to come – future research and teaching

In this research, I have looked at the use of Minecraft in the very specific area of generalizing algebraic, cubic structures. In the study, six pupils with previous Minecraft experience were used to illuminate the area. However, it would also have been interesting to consider how pupils without any experience would work with the game in the same environment. Minecraft – being a giant sandbox – would also have been interesting to research in the context of other mathematical areas, like geometry or coding, because I believe that the potential is huge. For example, Minecraft: Education Edition contains a visual block-based coding program, that helps the pupils learn Python. Further, it would also have been interesting to see how teachers would work with and react to the use of Minecraft in the classroom, since it would be the their job to implement the game into the teaching.

Personally, I want to become a teacher who includes video games into the education of pupils. I think that even small elements of game-based learning will contribute positively to teaching situations, and the games are great ways to meet the pupils at their own playground. I have several good ideas of how to implement games across levels, subjects and topics, and working with this project has made me more confident to carry them out. It is something I think the pupils would appreciate from time to time.

In video games, the pupils have the possibility to express themselves in engaging and immersive worlds, and the pupils become active participants in their own learning. Robinson (2006) claims that today's schools kill the creativity of pupils. "We don't grow into creativity, we grow out of it. Or rather, we get educated out of it" (Robinson, 2006, 6:08). The world and its classrooms need to become areas where kids can unfold their creativity and explore different scenarios and approaches on their road to knowledge. Only then can they uncover new learning strategies, adapt to new environments and expand their horizons. Why not have fun at the same time? I really do believe Prensky (2003), when he says that video games are not the enemy, but the best opportunity we have to engage our kids in real learning, whether you want to teach them about mathematics, science, history or geography – or even *life itself*.

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Appendix A

Consent form

Appendix A shows the information sheet and consent form given to the pupils that participated in the research. The consent form on the last page, had to be signed by the pupils and/or their legal parents to confirm their consent. This is the same document that was provided to NSD for approval of the research.

Stian Nygård
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Trondheim, 3. februar 2022

Til elever og foresatte på 9. trinn ved xx ungdomsskole

Anmodning om tillatelse til innsamling av elevbesvarelser og lydopptak av intervju, samt videoopptak.

Mitt navn er Stian Nygård, og jeg er student på 5. året ved lektorprogrammet i realfag på NTNU. Nå i vår skal jeg skrive min masteroppgave i matematikkdidaktikk, og jeg ønsker derfor å besøke klassen deres for å samle inn forskningsdata til prosjektet. Prosjektet går ut på å finne ut hvordan man kan bruke det velkjente dataspillet Minecraft i algebraundervisning, og hvilke virkninger det kan ha for læringen i matematikk. Jeg kommer derfor til å gi ulike oppgaver til ulike grupper. En del av klassen skal bruke Minecraft, mens de andre får tilsvarende analoge oppgaver knyttet til temaet. Alle skal arbeide i par, i én skoletime. Jeg vil samle inn skriftlige besvarelser fra elevene, i tillegg til videoopptak av to (frivillige) par, for å analysere hvordan de har arbeidet med oppgavene. Selv om bare to av parene filmes, vil lyd fra andre grupper i klasserommet kunne bli fanget opp. Jeg vil i tillegg også ta opptak av PC-skjermene til to av parene i en spilløkt med Minecraft. Her brukes undertegnedes private PC og Minecraft-konto, så her kan ikke elevene identifiseres på noen som helst måte.

Jeg vil i etterkant ta kontakt med 3-4 av parene for et lite intervju på ca. 30 minutter, med lydopptak. Under samtalen kommer vi til å diskutere hva elevene har gjort, og hvordan de har tenkt for å komme frem til svarene sine. Samtalen kommer derfor kun til å komme inn på generelle spørsmål knyttet til elevenes besvarelser, og forståelse rundt oppgavene som har blitt besvart. Ingen personlige opplysninger vil komme til å bli nevnt under intervjuet. Det eneste jeg trenger av personlige opplysninger fra elevene, er navn på de skriftlige besvarelsene, slik at jeg kan finne dem igjen til et eventuelt intervju. Navnene vil selvfølgelig bli anonymisert når masteroppgaven min skal skrives. Alle andre personopplysninger vil også bli fullstendig anonymisert etter innsamlingen av videomaterialet.

Jeg trenger navn på elevene for å samtykke til innsamling av data, derunder et eventuelt intervju med lydopptak og/eller videoopptak av arbeidet. Jeg trenger også navn på besvarelsene for å finne igjen de elevene jeg ønsker å snakke med etter den skriftlige datainnsamlingen. Under intervjuet har jeg behov for lydopptak for å få så godt dokumenterte data som mulig. Med en mest mulig nøyaktig gjengivning av det som blir sagt, kan jeg igjen danne et bedre bilde av elevenes faglige utbytte. Siden jeg jobber alene, er det derfor mest hensiktsmessig å kunne høre besvarelsene i sin helhet i etterkant. Det samme gjelder også de to videoopptakene og skjermopptakene.

Jeg ber derfor om tillatelse til å samle inn skriftlig materiale produsert av elevene, samt til å kunne ta videoopptak/skjermopptak av arbeidet, og lydopptak av samtalene i etterkant. Det er da snakk om oppgavene som elevene har besvart først knyttet til matematikk, i tillegg til lydopptak av 3-4 intervjuer og to videoer av arbeid. Forutsetningen for tillatelsen er at alt innsamlet materiale blir behandlet med respekt og blir fullstendig anonymisert, og at prosjektet ellers følger gjeldende retningslinjer for etikk og personvern. **Det er selvfølgelig HELT frivillig å delta, og man kan til enhver tid trekke seg fra deltakelse UTEN å måtte oppgi noen grunn til det.** De som ikke ønsker å delta i prosjektet vil få et alternativt opplegg av læreren i den aktuelle skoletimen. Hvis du ikke vil delta, trenger du heller ikke å levere inn svarslippen på neste side.

Materialet som blir samlet inn vil kun bli sett og hørt av meg, og eventuelt av mine to masterveiledere ved NTNU. I det som presenteres fra prosjektet vil involverte personer bli anonymisert. Lyd- og videoopptakene vil bli tatt opp med opptakere fra NTNUs eiendom, og vil bli lagret på en ekstern harddisk. Skjermopptakene fra Minecraft vil bli lagret gjennom programvaren Panopto. Alle innsamlede data vil bli slettet etter at prosjektet er avsluttet, og senest 31. august 2022.

Så lenge du kan identifiseres i datamaterialet, har du rett til:

- innsyn i hvilke opplysninger vi behandler om deg, og å få utlevert en kopi av opplysningene
- å få rettet opplysninger om deg som er feil eller misvisende
- å få slettet personopplysninger om deg
- å sende klage til Datatilsynet om behandlingen av dine personopplysninger

Hvis du/dere vil vite mer om dette, eller hva det innsamlede materialet skal brukes til, så er det bare å ta kontakt med meg når som helst på telefon eller epost (se øverst i dokumentet for detaljer).

Faglig ansvarlig ved NTNU er [redacted]: tlf.: [redacted]; epost: [redacted]

NTNUs personvernombud er [redacted]: tlf. [redacted]; epost [redacted]

Hvis du/dere har spørsmål knyttet til NSD sin vurdering av prosjektet, ta kontakt med:

- NSD – Norsk senter for forskningsdata AS på epost (personverntjenester@nsd.no) eller på telefon: 55 58 21 17.

Jeg håper du synes forskningen min er av verdi, og at du er villig til å være med på den. Jeg ber om at svarslippen på neste side fylles hvis du gir tillatelse til deltakelse i prosjektet.

På forhånd takk!

Vennlig hilsen
Stian Nygård

Tillatelse

Som del av mitt masterprosjekt ved lektorutdanningen i realfag ved NTNU ber jeg om din tillatelse til innsamling av skriftlig besvarelse av matematikkoppgaver og videoopptak i klasserommet, samt en eventuell samtale med lydopptaker, og et eventuelt skjermopptak av PC. Jeg ber også om tillatelse til å kopiere/bruke besvarelsen(e) som du har produsert til å skrive en fagartikkel rundt elevers forståelse og representasjon av figurtall.

Forutsetningen for tillatelsen er at besvarelser og annet innsamlet materiale blir fullstendig **anonymisert** og behandlet med **respekt**, og at prosjektet følger gjeldende retningslinjer for **etikk** og **personvern**. Samtykke kan trekkes tilbake når som helst, og uten begrunnelse.

Sett kryss i den ruta som passer:

☐ Jeg (eleven) gir tillatelse (dersom eleven er over 15 år).

☐ Jeg/vi (foresatt(e)) gir tillatelse (dersom eleven er under 15 år). Jeg/vi har snakket med barnet vårt om dette, og hen har også gitt sitt samtykke.

Dato: ………………………

Elevens fornavn og etternavn: ………………….……………………….…………….............

Underskrift av eleven:

……………………………………………………………...

Underskrift av foresatt(e) (dersom eleven er under 15 år):

………………………………………………………………...

Vennligst returner svarslippen til læreren din så snart som mulig.

Appendix B

Interview guide

In Appendix B, I have provided the interview guide that was used by me during the interviews with the four pairs that participated. The guide shows the interview guide adapted to one of the Minecraft pairs, but the structure was similar for all other pairs.

Intervju, Minecraft:
Tusen takk for at dere er villige til å ta del i denne oppfølgingsdelen av prosjektet mitt. Jeg vil at dere skal være helt sikre på at alt som blir snakket om vil bli fullstendig anonymisert, og at opptaket vil bli slettet med en gang jeg er ferdig med analysen. Alt jeg ønsker å finne ut av er hvordan dere bruker disse hjelpemidlene for å løse matematiske oppgaver. Er det greit at jeg starter videokameraet da?

Del 1:
Husker dere hvordan oppgaven var og hvordan dere løste den?

Jeg tenkte at vi kan begynne med å friske opp hva dere har gjort i Minecraft. Jeg tenker at vi kan snakke oss gjennom oppgavene sammen.

1. I den første oppgaven skulle dere bygge de to neste figurene. Hva har dere gjort her?
    a. Har dere tenkt på…
    b. Hvorfor…
    c. Hvordan…
2. I den andre oppgaven skulle dere skrive ned hvor mange blokker det var i hver av figurene. Hvordan har dere gjort det?
    a. Har dere telt alle blokkene?
    b. Har dere regnet?
    c. Hvorfor…
    d. Er alle svarene deres rett?
3. I den tredje oppgaven blir dere bedt om å gjøre en antakelse om hvor mange blokker det vil være i en figur langt ute i følgen, nummer 10. Hva har dere gjort her?
    a. Forklar hvorfor…
    b. På hvilken måte bruker dere det dere allerede har bygget til å gjøre antakelsen?
4. I neste oppgave var spørsmålet om hvilken sammenheng dere ser mellom nummeret n på figuren og antallet blokker. Hva har dere svart?
    a. Kan dere vise på figurene?
    b. Hvorfor…
    c. La meg omformulere spørsmålet: Hva har nummeret på figuren å si for hvordan figuren ser ut?
5. I spørsmål e blir dere spurt om å finne en formel for antall blokker i figur nummer n. Hva har dere svart her?
    a. Fungerer den formelen for **alle** figurene i denne følgen?
6. Så får dere oppgaven å bygge rundt de to første kubene dere har foran dere, med glassblokker, slik at vi får en ny formel og forholde oss til. Hva har dere gjort her?
    a. Husker dere hvordan dere tolket denne oppgaven først?
        i. Gamle kuber i hjørnet, eller i midten?

        ii. Dere kan få bygge ut den andre figuren også nå hvis dere vil.

7. Så skal dere skrive ned hvor mange kuber det er av hver type i den nye figurfølgen. Hva har dere her?
    a. Ser dere noen sammenheng mellom antallet kuber i de to nye figurene, og de fire dere hadde før?
8. Så kommer den siste oppgaven, som kanskje var litt vanskelig formulert. Det som skjer er at vi ganger ut formelen dere har fått, slik (skriv ned regnestykket). Nå er det slik at hvert av disse fire leddene, kan knyttes til hver sin enkelte del av figuren. Hvordan kan vi gjøre det?
    a. Bruk spillet til å vis hva dere tenker.
    b. Start med det vi vet. Hva er det som er inni skallet vi har bygget rundt?
    c. Hva er det vi har 6 av på kuben?
    d. Hva er det vi har 12 av på kuben?
    e. Hva er det vi har 8 av på kuben?
    f. Hvilken betydning har konstantene i de ulike leddene?
        i. Tenk på ordet konstant.
        ii. Er det noe av det vi har snakket om, som ikke endrer seg fra figur til figur?
        iii. Uavhengig av figurnummeret. Noe som ikke endrer seg fra figur til figur
    g. Hvilken betydning har variablene i de ulike leddene?
        i. Hva betyr ordet variabel?
        ii. Er det noe av det vi har snakket om, som faktisk endrer seg fra figur til figur?
        iii. Avhengig av figurnummeret.
9. Vil dette være sant for alle kuber på formen (n+2)^2?
    a. Kan dere bruke spillet til å vise hva dere tenker?

Del 2:

1. Kan dere forklare **på hvilken måte** dere **brukte** Minecraft for å løse oppgaven?
2. Når dere arbeider med tredimensjonale mønster, slik som kubikktallene i oppgaven dere fikk, hvordan opplever dere det å ha muligheten til å bevege dere rundt i verdenen og se på mønsteret dere bygger?
3. I hvilken grad følte dere at Minecraft hjalp med å visualisere matematikken dere holdte på med?
    a. Hvordan da?
4. Når dere bare ble plassert rett inn i Minecraft-verdenen på den måten, syntes dere det var vanskelig å komme i gang med opplegget?
5. Var det vanskelig å bruke Minecraft?
    a. Hvor intuitivt/selvforklarende var bruken av Minecraft?
6. Igjen, tusen takk for tiden dere har ofret for denne forskningen! Før vi avslutter, er det noen aspekter rundt deres bruk av Minecraft dere føler at vi ikke har dekket, som dere har lyst til å si noe om?

Appendix C

Color coded result table

In the table presented in Appendix C, the color coded results from all pairs are shown. The pupils' pseudonyms are also given in the table. Green color denotes correct answers. Yellow denotes some kind of misunderstanding, inaccurate statement or consequential error. Red denotes wrong statements. The symbol Ø denotes a blank answer.

DRAWING

		Task a)	Task b)	Task c)	Task d)	Task e)	Task f)	Task g)	Task h)
Pair number		Color codes	Color codes	Color codes	Color codes	Color codes	Color codes	Color codes	Color codes
1	Henry					n^3			
	Sam								
2			n=4: 96			N*N*N		Cons. Error	
3						n^3			
4			Use n^2			(s*Le)*L	Ø	Ø	Ø
5						(n*n)*n			Ø
6			n=4: 77	250		n1*2*n3	Ø	Ø	Ø
7						N^3			
8						8*n=x		Ø	Ø
9			n=2: 4			n*n*n	Ø	Ø	Ø
10						n^3			
11						n^3			

CENTICUBES

		Task a)	Task b)	Task c)	Task d)	Task e)	Task f)	Task g)	Task h)
Pair number		Color codes	Color codes	Color codes	Color codes	Color codes	Color codes	Color codes	Color codes
1	Bill					n^3			
	Tommy								
2						h*b*sf			
3	Owen					n^3			
	Riley								
4						n^3			
5	Abby					n^3			
	Marlene								
6						h*b*l=n			
7						n^3			

MINECRAFT

		Task a)	Task b)	Task c)	Task d)	Task e)	Task f)	Task g)	Task h)
Pair number		Color codes	Color codes	Color codes	Color codes	Color codes	Color codes	Color codes	Color codes
1	Jesse					n^3			
	Sarah								
2	Joel		n=3: 18			n^3			
	Ellie								
3	David					n^3			
	Tess								

Appendix D

Book and quill example

After finishing the tasks, the pupils were asked to “sign” their book and quill (make it non-editable), and export it to a PDF file. Both the signing and exporting functions exists inside the game. The results from David and Tess can be seen in the example below.

Side 1 av 22

Denne boken inneholder alle delspørsmålene i oppgaven.

Ikke trykk på "Signer" før dere er ferdige med alle oppgavene!

Trykk "esc" for å komme ut av boka.

Side 2 av 22

...bla om

Side 3 av 22

Foran dere ser dere begynnelsen på en følge med figurer som tenkes å fortsette i det uendelige. Figurene dere ser har nummer n = 1 og n = 2.

Side 4 av 22

a)

Bruk gull-blokkene dere fikk fra Snøfrid, og lag figur nummer 3 og 4 ved siden av de to første.

Side 5 av 22

b)

Hvor mange blokker er det i hver av de fire figurene?

Side 6 av 22

Svar:

Figur1: 1
Figur 2: 8
Figur 3: 27
Figur 4: 64

Side 7 av 22

c)

Hvor mange blokker tror dere det vil være i figur nummer 10 i følgen? Forklar hvorfor dere tror det.

Side 8 av 22

Svar:
1000

Side 9 av 22

d)

Hvilken sammenheng kan dere se mellom nummeret n på figuren og antallet blokker i figuren?

Side 10 av 22

Svar:
Lengde bredde og høyde er det samme som figuren sitt nummer.

Side 11 av 22

e)

Kan dere finne en formel for å regne ut hvor mange blokker det er i en vilkårlig figur nummer n?

Side 12 av 22

Svar:
n^3

Side 13 av 22

f)

Dere skal nå utvide figur nummer 1 og 2. Legg glass-blokker rundt begge figurene, slik at de passer inn i et mønster der figur nummer n har (n+2)^3 blokker til sammen.

Side 14 av 22

Side 15 av 22

g)

Hvor mange blokker er det nå av hver type i figur 1 og 2? Fyll inn på neste side.

Side 16 av 22

Svar:

Figur 1:
Antall gull:1
Antall glass:26
Antall til sammen:27

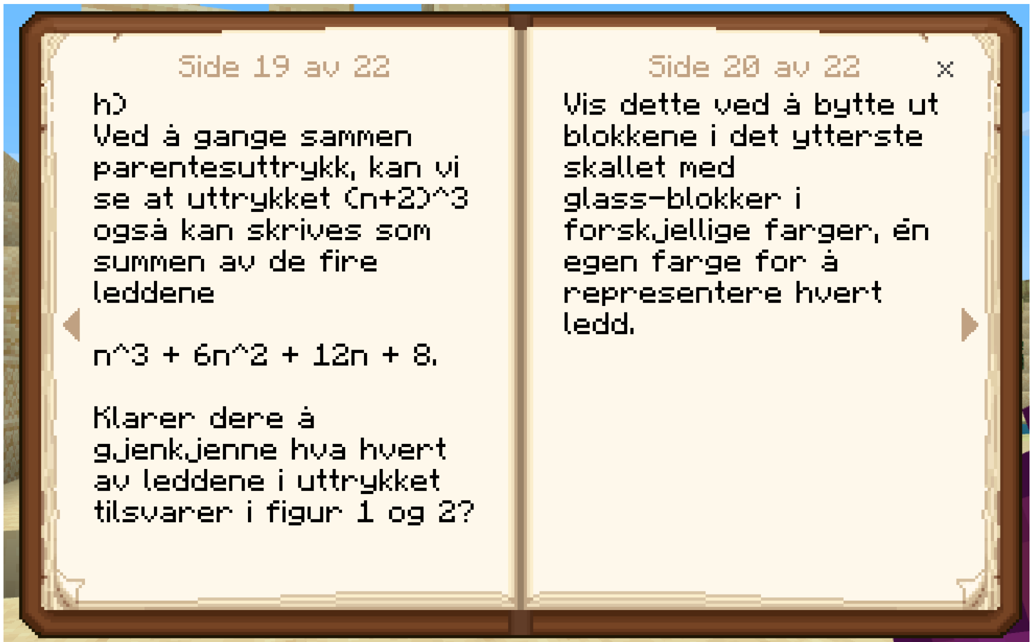
Figur 2:
Antall gull:8
Antall glass:56
Antall til sammen:64

Side 17 av 22

Før dere begynner på den siste oppgaven må dere gå bort til bassenget å hente det som ligger i kista der.

Side 18 av 22

Bla om når dere har hentet det.



Side 19 av 22

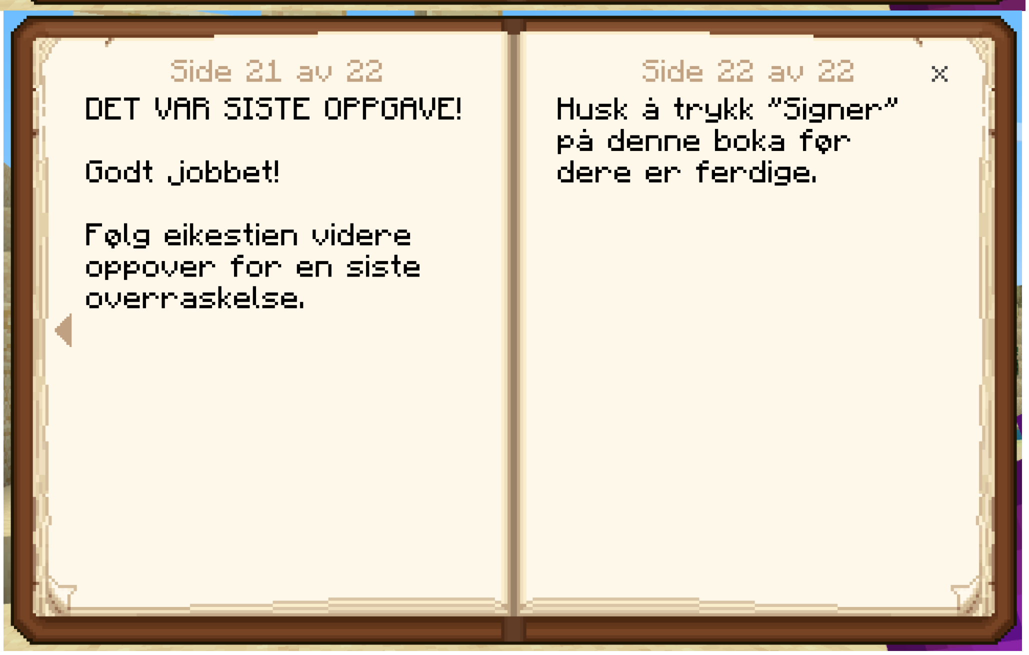
h)
Ved å gange sammen parentesuttrykk, kan vi se at uttrykket (n+2)^3 også kan skrives som summen av de fire leddene

n^3 + 6n^2 + 12n + 8.

Klarer dere å gjenkjenne hva hvert av leddene i uttrykket tilsvarer i figur 1 og 2?

Side 20 av 22

Vis dette ved å bytte ut blokkene i det ytterste skallet med glass-blokker i forskjellige farger, én egen farge for å representere hvert ledd.



Side 21 av 22

DET VAR SISTE OPPGAVE!

Godt jobbet!

Følg eikestien videre oppover for en siste overraskelse.

Side 22 av 22

Husk å trykk "Signer" på denne boka før dere er ferdige.

Stian Nygård

Learning algebra with Minecraft

NTNU
Kunnskap for en bedre verden