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Convergence of Many Particle Approximations of Lighthill-Whitham-Richards Models with Total Variation Blow-Up

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Sammendrag

I denne prosjektrapporten studeres konvergens av Følg-Lederen (FL) modellen av første orden mot løsningen av Lighthill-Witham-Richards (LWR) modellen. Jeg studerer to generaliseringer av LWR modellen der løsningen har diskontinuerlige hopp i den totale variasjonen, og deres korresponderende FL modeller. I den første modellen har hastighetsfunksjonen en diskontinuerlig romlig avhengighet. I den andre er trafikkfluksen begrenset i et punkt. Det etableres et nytt variasjonsestimat på den numeriske fluksen i FL modellen, som brukes til å bevise at FL approksimasjonen er kompakt i L^p_{loc} for $1 \leq p < \infty$. Argumentet krever at fluksen er genuint ikke-lineær. Videre etableres det at grensene tilfredstiller entropiulikheter av Kružkov-typen. I den første modellen etableres unikhet av grensen når den romlig avhengigheten er en stegfunksjon. I den andre modellen etableres unikhet for stykkvis kontinuerlige fluksbegrensninger.

Abstract

The final report contains proofs which establish that the Follow-the-Leader (FtL) model converges towards the weak entropy solution the Lighthill-Witham-Richards (LWR) model. I am considering two generalisations of the LWR model where the total variation of the solution can blow-up immediately. The first model has a discontinuous space dependency in the velocity function. The second model has a unilateral local point constraint on the flux. A novel variation estimate on the numerical flux is established, which is used to prove compactness of the method in L^p_{loc} for $1 \leq p < \infty$. The argument requires a non-linearity condition on the flux. It is proven that the limits satisfy Kružkov-type entropy inequalities. For the first model, uniqueness of the limit is proven when the space dependency is piecewise constant. For the second model, uniqueness is proven for piecewise continuous flux constraints.

Preface

This report concludes my five year master's education at the Physics and Mathematics programme at NTNU. The work was carried out under the supervision of Professor Helge Holden, who also proposed the topic of the thesis. The work builds on previous work done in the spring of 2021, where I considered the convergence of the FtL model to the LWR model with smooth space dependency in the velocity function. I first set out to investigate convergence of the FtL method for the case with a discontinuous velocity function. It turned out that it was possible to establish that the FtL method converges to an LWR model with discontinuous flux. The most difficult part of the proof was to find the appropriate estimate to establish compactness. After finding it and completing the proof, I realised the same estimate could be used for other models which share the problem of total variation blow-up. Therefore, I decided to consider an LWR model with a point constraint on the flux. I was able to find the appropriate FtL model and prove convergence of the many particle limit, using a similar estimate. I would like to thank Helge, for suggesting an interesting topic for my masters project, for our fruitful discussions and for pushing me in the right direction. I would like to thank my parents, Anne-Berit and Tommy Charles, for their love and encouragement. I would also like to thank my fellow students at NTNU, my friends and my family.

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Chapter 1

The Many Particle Limit

The many particle approximation is based on the idea of approximating aggregated transport quantities using the dynamics of an increasing set of particles. In the field of traffic modelling, the topic has seen a growing interest in recent years. Many particle approximations, or micro-macro limits, have been used to bridge the gap between modelling paradigms. So-called microscopic models describe the traffic state as a collection of interacting particles, which are governed by a system of ODEs. For the purpose of this thesis, the vehicles are distributed on a single-laned road

$$x_{1/2}(0) < \ldots < x_{M-1/2}(0), \tag{1.1}$$

and do not overtake eachother. The prototypical example of a microscopic model is the first-order Follow-the-Leader model.

$$\begin{aligned}
\dot{x}_{i-1/2} &= v(\rho_i), \\
\dot{x}_{N-1/2} &= 1, \\
\rho_i &= \frac{l}{x_{i+1/2} - x_{i-1/2}}, \\
x_{i-1/2} &= x_{i-1/2}(0) \ \forall i \in \mathbb{C},
\end{aligned} \tag{1.2}$$

where $\mathbb{C}$ contains the set of cars and $l > 0$ is the length scale. The model is of first order, in that the velocity of a vehicle depends on the distance to the vehicle in front. The model offers a relatively high level of detail, in that it explicitly describes the trajectory of each vehicle. The quantities of interest in macroscopic models are statistical averages of the traffic density, velocity and flux. They offer a lower level of detail, compared to microscopic models. The simplest macroscopic model for dense traffic on a single lane road is the LWR model

$$\partial_t \rho + \partial_x (\rho v) = 0. \tag{1.3}$$

The model is a scalar conservation law, and the first of Euler's equations from gas dynamics [13, intro.]. The LWR model describes single laned traffic as a compressible fluid. An early application is [25], where kinematic wave theory is used to estimate how a region of high density propagates along a long crowded road. The energy and momentum equations of Euler hold no meaning in the traffic context. In the LWR model, the velocity is instead determined by an empirical relationship between the mean mass density and mean traffic speed. The prototypical model for the space mean speed is

$$v(\rho) = 1 - \rho, \tag{1.4}$$

which was proposed by Greenshield in the 30's. For an overview of the state of the art in traffic modelling, see [23]. The micro-macro limit is investigated by considering

$$\rho^l(x,t) = \sum_{i=1}^{M-1} \rho_i \mathbb{1}_{[x_{i-1/2}, x_{i+1/2})}, \tag{1.5}$$

as a numerical approximation for the solution of Eq. (1.3). The method contrasts popular shock capturing methods, such as the Godunov and Lax- Friedrich methods, in that it uses a dynamical grid. An early example is [15], in which is was proven that

$$\rho^l(x,t) \to \rho \text{ in } \mathscr{C}\left([0,T], L^1(\mathscr{R})\right) \text{ as } M \to \infty \text{ with } \left\|\rho^l\right\|_{L^1(\mathscr{R})} \text{ held constant} \tag{1.6}$$

assuming $v \in \mathscr{C}^1[0,1]$ and strictly decreasing. The limit ρ is the unique entropy solution of Kružkov. The mathematical interest in many particle limits stems from the fact that the FtL model is a discrete Lagrangian approximation. It approximates the continuum equation

$$x_t(M,t) = v(M,t), \tag{1.7}$$

using a finite set particles with positive mass l. In addition, the FtL model can be seen as the continuous time upwind scheme,

$$\frac{dy_i}{dt} = D_+(V_i), \text{ where } y_i = \frac{1}{\rho_i} \text{ and } V(y) = v\left(\frac{1}{y}\right), \tag{1.8}$$

for Eq. (1.3) in Lagrangian coordinates

$$y_t - (V(y))_x = 0. \tag{1.9}$$

The convergence of the FtL approximation to the solution of Eq. (1.3) can be proven by appealing directly to the theory of monotone, conservative and consistent methods in Lagrangian coordinates; see [20]. Several extensions and generalisations of FtL models have been considered in the research literature. Many particle approximations have been investigated for first order models with non-local velocity functions, in [19], for first order models with infinitely many vehicles ($\mathbb{C} = \mathscr{L}$), in [28], for second order models towards to ARZ models, in [10] and [18], to name a few. Recently, many particle approximations have been investigated outside the context of vehicular traffic, in [14] and [17]. Many articles concerning convergence of first order FtL models establish strong compactness of (1.5) with uniform total variation (T.V.) bounds in space.

$$\sum_{i=1}^{M-1} |\rho_{i+1} - \rho_i| \leq C < \infty, \text{ where } C > 0 \text{ is independent of } M. \tag{1.10}$$

Such bounds may not always exist. Consider the following generalisation of the space mean speed

$$v = v(x,\rho) = k(x)v(\rho), \tag{1.11}$$

which can be used to model changes in the condition of the road. This case is of interest in the study of discontinuous systems of conservation laws, see [31]. The difficulty in establishing strong compactness of the FtL model is determined largely by the assumptions on k. If k is sufficiently smooth, Ineq. (1.10) can be established. The case where $k \in W^{2,\infty}$ was proven in [27]. It was shown in [1, Thm. 2.9] that there exists conservation laws with discontinuous fluxes such that the T.V. of the exact solution is unbounded, for certain initial data. This can be understood as so-called nonlinear resonance, see [24]. If k is discontinuous, then a finite (or infinite) explosion of T.V. can occur. In fact, unbounded T.V. of numerical approximations may occur even when k is continuous, see for example [22, Ex. 8.13, p. 395]. In [4], the authors point out that the FtL approximation lacks the order preservation and finite acceleration properties, shared by the exact solution of Eq. (1.3) and well-known shock capturing schemes. This is even the case for $k = 1$. The lack of order-preservation can make strong compactness difficult to prove when uniform T.V. bounds cannot be established. The FtL method is a conservative method, and therefore not L^1-contractive. One cannot use the well-established convergence paradigm for L^1-contractive methods, which can be used to prove compactness of numerical approximations without T.V. bounds. See for example [35]. Steps towards establishing the many particle limit have been taken when k has a finite set of discontinuities. In [33], the existence of locally stable travelling wave solutions is proven. The authors propose developing a higher order method to deal with the oscillation near the discontinuities. [5] introduces a LWR model for pedestrian motion in a corridor with two exists. The model has a discontinuous coefficient

$$k(x,t) = \operatorname{sign}(x - \xi(t)), \tag{1.12}$$

where $\xi(t)$ is the position where the perceived cost to each corridor exit is equal, which depends non-locally on the solution itself. The authors prove convergence of the Follow-the-Leader scheme, using uniform B.V._{loc} estimates. In this thesis, two pairs of FtL/LWR models subject to blow-up of the total variation are considered. In Chapter 3, it is proven that the discontinuous velocity FtL model converges to

$$\rho_t + (k(x)\rho v(\rho))_x = 0, \tag{1.13}$$

when k is positive, has finitely many discontinuities and is smooth between discontinuities. In Chapter 4, it is shown that a flux constrained FtL model converges to the LWR model with a unilateral point constraint on the flux,

$$\begin{aligned} \rho_t + (\rho v(\rho))_x &= 0, \\ f(\rho(0^{\pm},t)) \leq q, \ q &\in (0, f_{\max}). \end{aligned} \tag{1.14}$$

In the context of vehicular traffic, point constraints can be used to model a tollgate along a highway, traffic lights or speed bumps. In pedestrian crowd dynamics, Prob. (4.1) can model doors, turnstiles, escalators and so on. The problem is closely related to conservation laws with discontinuous fluxes, see [11] and [12]. The compactness proofs in both chapters are inspired by the convergence analysis in [26], which uses the theory of compensated compactness.

Chapter 2

Background material and notation

The following theorem is taken from [32, 3.2.1, p. 82] and [36, Cor. 3.9, p. 79]

Theorem 2.1. *(existence and uniqueness of ODEs) Consider the Cauchy problem*

$$\frac{dx}{dt} = F(x), \quad x(0) \in \mathscr{R}^d \tag{2.1}$$

Let $U \subset \mathscr{R}^d$ be open and contain the initial data $x(0)$, and let $F : U \to \mathscr{R}^d$ be locally Lipschitz. Then there exists and interval $(-\eta, \eta)$ and a $\mathscr{C}^1$ function $x : (-\eta, \eta) \to U$ such that x solves (2.1). The solution x is unique. If $U = \mathscr{R}^d$ and F is bounded, one can set $\eta = \infty$.

The following definition and theorems are taken from [8].

Definition 2.2. (L^p-spaces) Let $(\Omega, \mathcal{M}, \mu)$ be a measure space.

$$L^p(\Omega, \mu) = \{f : \Omega \to \mathscr{R} | f \text{ is measurable and } \|f\|_{L^p} < \infty\}, \tag{2.2}$$

with norm

$$\|f\|_{L^p} = \begin{cases} \left(\int_X |f(x)|^p \, d\mu\right)^{\frac{1}{p}} & \text{for } \mathrm{p} \in [1, \infty) \quad (2.3) \\ \operatorname{essup}_{x \in X} |f(x)| & \text{if } \mathrm{p} = \infty \quad (2.4) \end{cases}$$

If the measure space is implied, we write L^p. The special case of $p = 1$ is called the space of *integrable functions*.

Theorem 2.3. *(Fischer-Riesz) L^p is a Banach space for any $p, 1 \le p \le \infty$.*

Theorem 2.4. *(Hölder's inequality) Assume that $f \in L^p$ and $g \in L^{p'}$ with $1 \le p \le \infty$, where p' is the Hölder conjugate of p. Then $fg \in L^1$ and*

$$\int |fg| \le \|f\|_p \, \|g\|_{p'} \,. \tag{2.5}$$

Theorem 2.5. *(dominated convergence theorem, Lebesgue) Let (f_n) be a sequence of functions in L^1 that satisfy*

- *$f_n(x) \to f(x)$ a.e. on Ω,*
- *there is a function $g \in L^1$ such that for all n, $|f_n(x)| \le g(x)$ a.e. on Ω.*

Then $f \in L^1$ and $\|f_n - f\|_1 \to 0$.

Theorem 2.6. *(Jensen's inequality) Assume $|\Omega| < \infty$ and $\phi : \mathscr{R} \to (-\infty, +\infty]$ be a convex l.s.c. function, $\phi \not\equiv +\infty$. Let $f \in L^1(\Omega)$ be such that $f(x) \in D(\phi)$ a.e. and $\phi(f) \in L^1(\Omega)$, then*

$$\phi\left(⨍_\Omega f\right) \leq ⨍_\Omega \phi(f). \tag{2.6}$$

The following lemma can be found in [22, p. 82].

Lemma 2.7. *(Crandall-Tartar) Let D be a subset of $L^1(\Omega)$, where Ω is some measure space. Assume that if ψ and ϕ are in D, then also $\psi \vee \phi = \max(\psi, \phi)$ is in D. Assume furthermore that there is a map $T : D \to L^1(\Omega)$ such that*

$$\int_\Omega T(\phi) = \int_\Omega \phi, \quad \phi \in D \tag{2.7}$$

Then the following statements, valid for all $\psi, \phi \in D$, are equivalent:

i) if $\phi \leq \psi$, then $T(\phi) \leq T(\psi)$.
ii) $\int_\Omega (T(\phi) - T(\psi))^+ \leq \int_\Omega (\phi - \psi)^+$, where $\phi^+ = \phi \vee 0$.
iii) $\int_\Omega |T(\phi) - T(\psi)| \leq \int_\Omega |\phi - \psi|$

The following compensated compactness lemma is taken from [26]. The assumptions are $k \in$B.V.($\mathscr{R}$), and $\alpha \leq k \leq 1$, $\alpha > 0$, a.e.,

$$\begin{cases} u \mapsto f(k,u) \in \mathscr{C}^2[a,b] \text{ for all } k \in [\alpha, 1], \\ k \mapsto f(k,u) \in \mathscr{C}^1[\alpha, 1] \text{ for all } u \in [0,1] \end{cases} \tag{2.8}$$

and

$$\partial^2_{uu} f(k(x), u) \neq 0 \text{ for a.e. } \quad u \in [0,1]. \tag{2.9}$$

Lemma 2.8. *Suppose $\{u^\epsilon\}_{\epsilon>0}$ is a sequence of measurable functions on $\mathscr{R} \times \mathscr{R}_+$ where,*

$$a \leq u^\epsilon(x,t) \leq b \text{ for all } (x,t) \in \mathscr{R} \times \mathscr{R}_+, \quad \epsilon > 0, \tag{2.10}$$

for $a, b \in \mathscr{R}$. If the two sequences

$$\begin{gathered} \{S_1(u^\epsilon)_t + Q_1(k(x), u^\epsilon)_x\}_{\epsilon>0}, \\ \{S_2(k(x), u^\epsilon)_t + Q_2(k(x), u^\epsilon)_x\}_{\epsilon>0}, \end{gathered} \tag{2.11}$$

belong to a compact subset of $W^{-1,2}_{loc}(\mathscr{R} \times \mathscr{R}_+)$, where

$$\begin{aligned} S_1(u) &= u - c, & Q_1(k,u) &= f(k,u) - f(k,c), \\ S_2(u) &= f(k,u) - f(k,c), & Q_2(k,u) &= \int_c^u (f_u(k,\xi))^2 \, d\xi, \end{aligned} \tag{2.12}$$

for any $c \in \mathscr{R}$. Then there exists a subsequence of $\{u^\epsilon\}_{\epsilon>0}$ that converges a.e. to a function $u \in L^\infty(\mathscr{R} \times \mathscr{R}_+)$.

2.1 Notation table

$\Delta_+(x_i)$	$\triangleq$	$x_{i+1} - x_i$ The forward difference operator.
$D_+(x_i)$	$\triangleq$	$\frac{x_{i+1}-x_i}{l}$ Lagrangian finite difference.
		Function spaces
Ω	$\triangleq$	An open subset of $\mathscr{R}^d, d \geq 1$.
$\mathscr{C}(\Omega)$	$\triangleq$	The continuous functions on Ω.
$\mathscr{C}^{0,\alpha}(\Omega)$	$\triangleq$	The α-Hölder continous functions on Ω. Lipschitz is $\alpha = 1$
$\mathscr{C}_c(\Omega)$	$\triangleq$	The continuous functions with compact support.
$\mathscr{C}^{(k)}(\Omega)$	$\triangleq$	The continuously differentiable functions on Ω.
$\mathscr{C}_c^{\infty}(\Omega)$	$\triangleq$	The infinitely differentiable functions with compact support.
$\mathscr{D}'(\Omega)$	$\triangleq$	The dual space of $\mathscr{C}_c^{\infty}(\Omega)$.
B.V.(Ω)	$\triangleq$	The functions with bounded total variation.
$L^p(\Omega)$	$\triangleq$	The functions in L^p.
		Miscellaneous
$\mathscr{R}, \mathscr{Q}, \mathscr{Z}, \mathscr{N}$	$\triangleq$	The real line, the rational numbers, the integers and the natural numbers
$\mathscr{R}^+/\mathscr{R}_0^+$	$\triangleq$	The intervals $(0, \infty), [0, \infty)$.
$⨍$	$\triangleq$	Integral average
a.e.	$\triangleq$	Almost everywhere.
Eq., Ineq., Exp., Prob.	$\triangleq$	Equation, Inequality, Expression, Problem

$\mathscr{C}^{(k)}\left(\overline{\Omega}\right)$ denotes the space where for each element $f \in \mathscr{C}^{(k)}\left(\overline{\Omega}\right)$, there exists $\tilde{\Omega}$ open, $\overline{\Omega} \subset \tilde{\Omega}$ such that $f \in \mathscr{C}^{(k)}\left(\tilde{\Omega}\right)$.

Chapter 3

LWR with Discontinuous Flux

3.1 The Follow-the-Leader model with discontinuous velocity function

The LWR model with discontinuous flux is given as

$$\rho_t + (k(x)\rho v(\rho))_x = 0, \tag{3.1}$$

where k is assumed to be discontinuous. The velocity function in the Follow-the-Leader model is modified accordingly.

$$\dot{x}_{i-1/2} = k(x_{i-1/2})v(\rho_i) \text{ for } i \in \{1, ..., M-1\}, \tag{3.2}$$

with mass density

$$\rho_i := \frac{l}{x_{i+1/2} - x_{i-1/2}} \text{ for } i \in \{1, ..., M-1\}. \tag{3.3}$$

The leader is governed by the ODE

$$\dot{x}_{M-1/2} = k(x_{M-1/2}), \quad \rho_M := 0. \tag{3.4}$$

It is assumed that

$$0 \le \rho_0 \le 1, \quad \operatorname{supp} \rho_0 \text{ is compact.} \tag{3.5}$$

Let $x_{1/2}(0) = \overline{x}_{\min}$ and $x_{M-1/2}(0) = \overline{x}_{\max}$ be min and max of the support of ρ_0. For the remaining vehicles,

$$x_{i-1/2}(0) = \inf\left\{ x \in \mathscr{R} \,\middle|\, \int_{-\infty}^{x} \rho_0 dx = (i-1)\,l \right\}, \quad \text{for } i \in \{2, ..., M-1\} \tag{3.6}$$

where

$$l := \frac{\|\rho_0\|_{L^1(\mathscr{R})}}{M-1}. \tag{3.7}$$

This definition takes into account possible vacuum regions within the support of ρ_0. The free flow speed and jam density is set to one. In addition, it is assumed that

$$\begin{aligned} &v \in \mathscr{C}^{0,1}([0,1]), \text{ non-increasing, } v(0)=1, v(1)=0, \\ &v \geq 1-\rho^{\sigma-1} \text{ for some } \sigma > 1. \end{aligned} \tag{3.8}$$

The assumptions on k are

$$k \in \mathscr{C}^2\left(\mathscr{R}\backslash\{\xi_1,....,\xi_p\}\right), \ \|k'\|_\infty < \infty, k(x) \in [\alpha,1], \text{ for } 0 < \alpha \leq 1. \tag{3.9}$$

The coefficent k is allowed to have finitely many discontinuities, and is well-behaved between the discontinuities. The existence and uniqueness of Prob. (3.2)-(3.9) can be proven using standard ODE theory. Define

$$\Phi(z) := \int_0^z \frac{d\hat{z}}{k(\hat{z})}, \tag{3.10}$$

where we adopt the convention that $\int_y^x = -\int_x^y$ if $x < y$. Since k is bounded from above and away from zero, it follows that Φ is a bijection on $\mathscr{R}$. It is also Lipschitz continuous and has a Lipschitz continuous inverse, denoted by Ψ. The main utility of Φ is that it removes the multiplicative dependency of k in the Follow-the-Leader model. Assume that a Lipschitz continuous solution exists and let $\Phi_{i-1/2} = \Phi(x_{i-1/2})$. The solution satisfies the chain rule a.e.,

$$\dot{\Phi}_{i-1/2} = \frac{1}{k\left(x_{i-1/2}\right)}\dot{x}_{i-1/2} = v\left(\frac{l}{\Psi\left(\Phi_{i+1/2}\right)-\Psi\left(\Phi_{i-1/2}\right)}\right). \tag{3.11}$$

The coordinate map Φ gives an alternative formulation of the Follow-the-Leader model, by absorbing the space dependency into the argument of v. Since v, Ψ are both Lipschitz, the ODE system (3.11) has a Lipschitz continuous and bounded right-hand side, and therefore a uniquely and globally defined $\mathscr{C}^1$-solution, by Theorem 2.1. The solution of Prob. (3.2)-(3.9) is defined as

$$x_{i-1/2} = \Psi \circ \Phi_{i-1/2}, \tag{3.12}$$

which is Lipschitz continuous and satisfies Eq. (3.2) for a.e. $t \geq 0$. Uniqueness is inherited from the transformed system, since a solution to the original problem corresponds to a solution of the transformed system,

$$\Phi_{i-1/2} = \Phi \circ x_{i-1/2}, \tag{3.13}$$

which is unique. The Follow-the-Leader density is

$$\rho^l(x,t) := \sum_{i=1}^{M-1} \rho_i \mathbb{1}_{[x_{i-1/2}, x_{i+1/2})}. \tag{3.14}$$

Lemma 3.1. *(Distances between vehicles go to zero) Let $y_i = \frac{1}{\rho_i}$. Assume that (3.8) and (3.9) hold, and that $y_i(0) \geq 1$ for $i \in \{1,...,M-1\}$. Then,*

$$1 \leq y_i(t) \leq \left(\left(\frac{y_i(0)}{\alpha}\right)^\sigma + \frac{\sigma t}{\alpha^{\sigma-1} l}\right)^{\frac{1}{\sigma}}, \tag{3.15}$$

where σ is given in assumption (3.8). If $\kappa > \frac{1}{\sigma}$ and $y_i(0) \leq C$ for some positive constant C independent of $i \in \{1,...,M-1\}$ and l, then

$$\max_{t\in[0,T], i\in\{1,...,M-1\}} l^\kappa y_i(t) \to 0 \text{ as } l \to 0. \tag{3.16}$$

The lemma implies that vehicles do not collide and that distances between vehicles converge to zero, when the mass of each vehicle goes to zero. As $\sigma > 1$, set $\kappa = 1$ in Ineq. (4.20). Lemma 3.1 is a generalisation of [21, Lemma 2.1], which considers the case $\alpha = 1$.

Proof. A simple calculation shows that

$$\frac{dy_i}{dt} = k(z_{i+1/2})v\left(\frac{1}{y_{i+1}}\right) \geq 0, \tag{3.17}$$

for $y_i(t) \leq 1$. The lower bounds follows, as $y_i(0) \geq 1, \forall i$. To establish the upper bound, let

$$\hat{y}_i = \frac{\Phi_{i+1/2} - \Phi_{i-1/2}}{l}, \quad d(x,y) = \frac{\Psi(y) - \Psi(x)}{l}. \tag{3.18}$$

Then

$$\begin{aligned} \frac{d\hat{y}_i}{dt} &= \frac{v\left(\frac{1}{d(\Phi_{i+1/2},\Phi_{i+3/2})}\right) - v\left(\frac{1}{d(\Phi_{i-1/2},\Phi_{i+1/2}}\right))}{l} \\ &\leq \frac{1 - v\left(\frac{1}{\alpha\hat{y}_i}\right)}{l} \leq \frac{1}{l\,(\alpha\hat{y}_i)^{\sigma-1}} \end{aligned} \tag{3.19}$$

The first inequality follows from $v \leq 1$, v is decreasing and

$$\Psi(y) - \Psi(x) \geq \alpha(y - x) \quad \text{for } x < y. \tag{3.20}$$

The second inequality follows from the lower bound on v in (3.8). It has been shown that

$$\frac{d\left(\hat{y}_i^\sigma\right)}{dt} \leq \frac{\sigma}{\alpha^{\sigma-1}l}. \tag{3.21}$$

The bounds

$$\hat{y}_i^\sigma \leq \hat{y}_i(0)^\sigma + \frac{\sigma}{\alpha^{\sigma-1}l}t \text{ and } y_i \leq \hat{y}_i \leq \frac{1}{\alpha}y_i, \tag{3.22}$$

imply

$$y_i \leq \left(\left(\frac{y_i(0)}{\alpha}\right)^\sigma + \frac{\sigma t}{\alpha^{\sigma-1}l}\right)^{\frac{1}{\sigma}}. \tag{3.23}$$

□

In order to consider a larger class of ρ_0, the uniform upper bound $y_i(0) \leq C$, $\forall i \in \{1,\ldots,M-1\}$, $\forall l > 0$ of Lemma 3.1 can be relaxed. From Ineq. (3.15), it can be seen that

$$\max_{i\in\{1,\ldots,M-1\}} \left(x_{i+1/2}(0) - x_{i-1/2}(0)\right) \to 0 \text{ as } l \to 0, \tag{3.24}$$

implies

$$\max_{t\in[0,T], i\in\{1,\ldots,M-1\}} \left(x_{i+1/2}(t) - x_{i-1/2}(t)\right) \to 0 \text{ as } l \to 0. \tag{3.25}$$

Condition (3.24) is satisfied if $\operatorname{supp}\rho_0$ is an interval and for $K \subset\subset (\operatorname{supp}\rho_0)^\circ$ in subset topology, $\exists\, C_K > 0$ such that

$$\rho_0(x) \geq C_K > 0 \text{ for } x \in K, \tag{3.26}$$

where $\subset\subset$ denotes compact embedding. In addition, each boundary point of $\operatorname{supp}\rho_0$ is contained in an open neighbourhood of $\mathscr{R}$ where ρ_0 is monotone. The first and last distance, $x_{3/2}(0)-x_{1/2}(0)$ and $x_{M+1/2}(0)-x_{M-1/2}(0)$, converge slowest to zero out of all vehicles, and can be used as a uniform bound. Next, assume Ineq. (3.26) holds but $\operatorname{supp}\rho_0$ is the finite disjoint union of V closed intervals. For sufficiently small values of l, each interval has a rightmost vehicle $x_{i_j-1/2}$, for $j \in \{1,...,V\}$. The indices i_j themselves are allowed to depend on the constant l. For $j \in \{1,...,V-1\}$,

$$x_{i_j+1-2}(0)-x_{i_j+1-2}(0) \to d_j \text{ as } l \to 0, \tag{3.27}$$

where $d_j > 0$ is the length of the vacuum between interval j and $j+1$. In other words, $V-1$ distances fail to converge to zero. Assume that for each $x \in \partial(\operatorname{supp}\rho_0)$, there exists an open neighbourhood $O(x)$ of $\mathscr{R}$ such that

$$x \in O(x) \text{ and } \rho_0 \text{ is monotone on } O(x), \tag{3.28}$$

then the distances between vehicles converge uniformly to zero, possibly except the distances corresponding to vacuum regions.

$$\max_{i\in\{1,...,M-1\}\setminus\{i_1,...,i_{V-1}\}} \left(x_{i+1/2}(0)-x_{i-1/2}(0)\right) \to 0 \text{ as } l \to 0, \tag{3.29}$$

which implies

$$\max_{t\in[0,T],i\in\{1,...,M-1\}\setminus\{i_1,...,i_{V-1}\}} \left(x_{i+1/2}(t)-x_{i-1/2}(t)\right) \to 0 \text{ as } l \to 0. \tag{3.30}$$

Lemma 3.2. *Under assumption (3.5) and (3.6), the Follow-the-Leader density has the invariant region*

$$0 \le \rho^l \le 1 \ \ \forall (x,t) \in \mathscr{R}\times\mathscr{R}^+. \tag{3.31}$$

Furthermore,

$$\operatorname{supp}\rho^l(\cdot,t) \subset [\overline{x}_{\min}, \overline{x}_{\max}+T], \tag{3.32}$$

for any $t \in [0,T], l > 0$.

Proof. From Eq. (3.6), it follows that

$$\rho_i(0) = \frac{l}{x_{i+1/2}(0)-x_{i-1/2}(0)} = \fint_{x_{i-1/2}(0)}^{x_{i+1/2}(0)} \rho_0 dx \le 1, \tag{3.33}$$

from which Lemma (3.1) gives the invariant region. The left-limit of Ineq. (3.32) holds because the vehicles are moving rightwards. The leader vehicle has velocity is less than or equal to one, by assumptions (3.8) and (3.9), which gives the right-limit. □

3.2 Strong compactness of the Follow-the-Leader method

Recall the concept of the material derivative. Formally, it holds that

$$\frac{D\rho}{dt} = \rho_t + \rho_x \dot{x} = -\rho\left(kv(\rho)\right)_x, \tag{3.34}$$

when $\dot{x} = k(x)v(\rho(x,t))$ and ρ is solution of Eq. (3.1). The Follow-the-Leader model has a discrete equivalent,

$$\frac{d\rho_i}{dt} = -\rho_i^2 D_+ \left(k_{i-1/2} v_i\right). \tag{3.35}$$

Consider next the entropy flux

$$Q_\rho(k,\rho) = k\rho^2 v'(\rho). \tag{3.36}$$

Let $Q(k,0) = 0$, then

$$Q(k,\rho) = k\rho^2 v(\rho) - k\int_0^\rho 2\rho v(\rho) d\rho. \tag{3.37}$$

The entropy flux can be used to rewrite Eq. (3.35) as a marching formula with a source term.

$$\begin{aligned}\frac{d\rho_i}{dt} + D_+\left(Q\left(k_{i-1/2},\rho_i\right)\right) &= k_{i+1/2} v_{i+1} D_+\left(\rho_i^2\right) - D_+\left(2k_{i-1/2}\int_0^{\rho_i} \rho v(\rho) d\rho\right) \\ &= -\frac{2k_{i+1/2}}{l}\int_{\rho_i}^{\rho_{i+1}} \rho\left(v(\rho) - v_{i+1}\right) d\rho \\ &\quad - D_+\left(k_{i-1/2}\right)\int_0^{\rho_i} 2\rho v(\rho)\, d\rho.\end{aligned} \tag{3.38}$$

The first term on the right is of interest.

$$QV(x,y) := \int_x^y \rho\left(v(\rho) - v(y)\right) d\rho \tag{3.39}$$

Since the velocity is decreasing with respect to density, the integrand is an unsigned function for $x, y \geq 0$. In fact, for $x, y \geq 0$,

$$QV(x,y) \geq \frac{x^2\left(v(x) - v(y)\right)^2}{2\left(1 + L_v\right)}, \tag{3.40}$$

where L_v is the Lipschitz constant of v. If $y > x$, the integrand can at worst decrease with derivative $(1 + L_v)$ from the value at x,

$$x\left|v(x) - v(y)\right| \geq 0, \tag{3.41}$$

to zero, the value at y. The graph intercepts the first coordinate axis at

$$\frac{x\left|v(x) - v(y)\right|}{(1 + L_v)}, \tag{3.42}$$

to the right of x, and forms a right triangle. If $x > y$, the integrand can at worst remain zero until the argument is (3.42) to the left of x, from which it has to increase with $(1 + L_v)$ to attain (3.41). The area under these graphs is the right-hand side of Ineq. (3.40), in both cases. The lower bound can be realised if velocity suddenly becomes constant for small densities, such as the flattened Greenshield model

$$v(\rho) = \begin{cases} 1 - \dfrac{\rho}{\rho_f} & \text{if } \rho > \rho_f, \\ 1 & \text{if } \rho \leq \rho_f, \end{cases} \tag{3.43}$$

where $\rho_f \in (0,1)$ separates free and congested flow. Since $QV(x,y)$ is non-negative for non-negative arguments, it can be used to establish a variation estimate. By definition,

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\rho_i^2}{2} dx = \frac{l}{2}\rho_i, \tag{3.44}$$

which shows directly that

$$\frac{d}{dt}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\rho_i^2}{2} dx\right) = \frac{l}{2}\frac{d\rho_i}{dt}. \tag{3.45}$$

Let $QV_{i-1/2} = QV(\rho_{i-1},\rho_i)$ and insert Eq. (3.45) into Eq. (3.38). Take the sum over $i \in \{1,\dots,M-1\}$ and integrate in time.

$$\begin{aligned}\int_0^T \sum_{i=1}^{M-1} k_{i-1/2} QV_{i-1/2} dt &= -\int_0^T \sum_{i=1}^{M-1} \frac{d}{dt}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\rho_i^2}{2} dx\right) + \frac{1}{2}\Delta_+\left(Q(k_{i-1/2},\rho_i)\right) + \Delta_+\left(k_{i-1/2}\right)\int_0^{\rho_i} \rho v(\rho)\, d\rho \\ &\le -\frac{1}{2}\left\|\rho^l(T)\right\|^2_{L^2(\mathscr{R})} + \frac{1}{2}\left\|\rho_0\right\|^2_{L^2(\mathscr{R})} + \frac{1}{2}\int_0^T Q(k_{1/2},\rho_1) + \text{T.V.}(k)dt \\ &\le \frac{1}{2}\left\|\rho_0\right\|_{L^1(\mathscr{R})} + \text{T.V.}(k)\, T, \end{aligned} \tag{3.46}$$

The last inequality follows from $0 \le \rho^l \le 1$, by Lemma 3.1, and $Q(k,\rho) \le 0$ on $[\alpha,1]\times[0,1]$, by Eq. (3.36) and $Q(k,0) = 0$. Since $k(x) \ge \alpha > 0$ and Ineq. (3.40) holds, the following lemma has been proven.

Lemma 3.3. *(Variation estimates) For $0 \le T < \infty$,*

$$\begin{aligned}\int_0^T \sum_{i=1}^{M-1} QV_{i-1/2} dt &\le \frac{1}{2\alpha}\left\|\rho_0\right\|_{L^1(\mathscr{R})} + \frac{\text{T.V.}(k)}{\alpha} T < \infty, \\ \int_0^T \sum_{i=1}^{M-1} \rho_i^2 (v_{i+1} - v_i)^2 dt &\le \frac{1+L_v}{\alpha}\left(\left\|\rho_0\right\|_1 + 2\text{T.V.}(k)\, T\right) < \infty. \end{aligned} \tag{3.47}$$

The variation estimates are sufficient to prove the following compactness lemma.

Lemma 3.4. *($W^{-1,2}_{loc}$ compactness) For any function $S(k,\rho) \in \mathscr{C}^2([\alpha,1]\times[0,1])$, the sequence of distributions*

$$\left\{\partial_t S\left(k(x),\rho^\Delta\right) + \partial_x\left(Q\left(k(x),\rho^\Delta\right)\right)\right\}_{\Delta>0}, \tag{3.48}$$

lies in a compact subset of $W^{-1,2}_{loc}\left(\mathscr{R}\times\mathscr{R}_0^+\right)$, where $\partial_\rho Q(k,\rho) = \partial_\rho S(k,\rho)\,\partial_\rho(k\rho v(\rho))$.

Under the assumption that $v \in \mathscr{C}^2([0,1])$, and $\rho v(\rho)$ is genuinely non-linear, Lemma 2.8 and Lemma 3.4 ensures ρ^l converges pointwise a.e. to a limit $\rho \in L^\infty\left(\mathscr{R}\times\mathscr{R}_0^+\right)$, possibly through a subsequence. By genuinely non-linear, it is meant that the second derivative of $\rho v(\rho)$ is non-zero almost everywhere,

$$2v'(\rho) + v''(\rho)\rho \ne 0 \quad \text{a.e. in } [0,1]. \tag{3.49}$$

Proof. (Lemma 3.4) Let Ω be an arbitrary open set of $\mathscr{R} \times \mathscr{R}_0^+$ of class $\mathscr{C}^1$, see [9, p. 298]. For any function which takes two-dimensional input, such as $S_{i,j-1/2} = S\left(k_{j-1/2}, \rho_i\right)$,

$$\begin{aligned} \Delta_+^1\left(S_{i,j-1/2}\right) &:= S_{i+1,j-1/2} - S_{i,j-1/2}, \\ \Delta_+^2\left(S_{i,j-1/2}\right) &:= S_{i,j+1/2} - S_{i,j-1/2} \\ \Delta_+\left(S_{i,j-1/2}\right) &:= S_{i+1,j+1/2} - S_{i,j-1/2}. \end{aligned} \tag{3.50}$$

The first operator increments with respect to ρ_i, the second with respect to $k_{i-1/2}$. A quantity depending on a single index should always be incremented, even if it appears in a product.

$$\Delta_+^1\left(k_{i-1/2}v_i\right) = \Delta_+^2\left(k_{i-1/2}v_i\right) = k_{i+1/2}v_{i+1} - k_{i-1/2}v_i. \tag{3.51}$$

Let $\tilde{S}_i = S(k(x), \rho_i)$ and $\tilde{Q}_i = Q(k(x), \rho_i)$. Let $\phi \in \mathscr{C}_c^\infty([-X, X] \times [0, T])$ for $X > 0, T > 0$. The distribution of interest is

$$\left\langle \mathcal{L}^\Delta, \phi \right\rangle := \int_{\mathscr{R}^+} \int_{\mathscr{R}} \left(S\left(k(x), \rho^l\right)\partial_t \phi + Q\left(k(x), \rho^l\right)\partial_x \phi\right) dx\, dt. \tag{3.52}$$

Consider the first term of the integrand

$$\int_0^T \sum_{i=0}^M \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{S}_i \phi_t \, dx\, dt. \tag{3.53}$$

Let $x_{-1/2} = -\infty$, $x_{M+1/2} = +\infty$ and $\rho_0 = \rho_M = 0$. Since the support of ϕ is compact,$\phi_{-1/2} = \phi_{M+1/2} = 0$. Leibniz rule for integration gives

$$\begin{aligned} \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{S}_i \phi_t \, dx &= \frac{d}{dt}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{S}_i \phi \, dx\right) - \Delta_+^2\left(S_{i,i-1/2}k_{i-1/2}v_i\phi_{i-1/2}\right) + \int_{x_{i-1/2}}^{x_{i+1/2}} \partial_\rho \tilde{S}_i \rho_i^2 D_+\left(k_{i-1/2}v_i\right)\phi(x)dx \\ &= \frac{d}{dt}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{S}_i \phi \, dx\right) - \Delta_+\left(S_{i,i-1/2}k_{i-1/2}v_i\phi_{i-1/2}\right) + \Delta_+^1\left(S_{i,i+1/2}\right)k_{i+1/2}v_{i+1}\phi_{i+1/2} \\ &+ \partial_\rho S_{i,i+1/2}\rho_i \Delta_+\left(k_{i-1/2}v_i\right)\phi_{i+1/2} + \partial_\rho S_{i,i+1/2}\rho_i \Delta_+\left(k_{i-1/2}v_i\right)\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\phi(x) - \phi_{i+1/2}\right)dx \\ &+ \int_{x_{i-1/2}}^{x_{i+1/2}}\left(\partial_\rho \tilde{S}_i - \partial_\rho S_{i,i+1/2}\right)\rho_i^2 D_+\left(k_{i-1/2}v_i\right)\phi(x)dx. \end{aligned} \tag{3.54}$$

The formula also holds for the edge cases $i \in \{0, M\}$.

$$\Delta_+^1\left(\partial_\rho S_{i,i+1/2}\rho_i\right) = \int_{\rho_i}^{\rho_i} \partial_{\rho\rho}^2 S\left(k_{i+1/2}, \rho\right)\rho\, d\rho + \Delta_+^1\left(S_{i,i+1/2}\right), \tag{3.55}$$

shows that right-hand side equals

$$\begin{aligned} &-\Delta_+\left(S_{i,i-1/2}k_{i-1/2}v_i\phi_{i-1/2}\right) + \partial_\rho S_{i,i+1/2}\rho_i\Delta_+\left(k_{i-1/2}v_i\right)\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\phi(x) - \phi_{i+1/2}\right)dx \\ &+ \int_{x_{i-1/2}}^{x_{i+1/2}}\left(\partial_\rho \tilde{S}_i - \partial_\rho S_{i,i+1/2}\right)\rho_i^2 D_+\left(k_{i-1/2}v_i\right)\phi(x)dx + \frac{d}{dt}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{S}_i \phi\, dx\right) \\ &+ \left(\Delta_+^1\left(\partial_\rho S_{i,i+1/2}\rho_i k_{i-1/2}v_i\right) - k_{i+1/2}v_{i+1}\int_{\rho_i}^{\rho_{i+1}} \partial_{\rho\rho}^2 S\left(k_{i+1/2}, \rho\right)\rho\, d\rho\right)\phi_{i+1/2} \end{aligned} \tag{3.56}$$

The sum over the first term is zero. Consider the second term,

$$\left\langle \mathcal{L}_1^l, \phi \right\rangle := \int_0^T \sum_{i=1}^{M-1} \partial_\rho S_{i,i+1/2}\rho_i \Delta_+ \left(k_{i-1/2}v_i\right) \fint_{x_{i-1/2}}^{x_{i+1/2}} \left(\phi(x) - \phi_{i+1/2}\right) dx dt, \tag{3.57}$$

An appliation of the Hölder inequality and using $S \in \mathscr{C}^1([0,1]\times[\alpha,1])$ gives

$$\begin{aligned}
\left|\left\langle \mathcal{L}_1^l, \phi \right\rangle\right| &\leq C \int_0^T \left\{ \sum_{i=1}^{M-1} \rho_i^2 \Delta_+ \left(k_{i-1/2}v_i\right)^2 \right\}^{\frac{1}{2}} \left\{ \sum_{i=1}^{M-1} \left(\fint_{x_{i-1/2}}^{x_{i+1/2}} \left(\phi(x) - \phi_{i+1/2}\right) dx \right)^2 \right\}^{\frac{1}{2}} dt \\
&\leq C_1 \int_0^T \left\{ \sum_{i=1}^{M-1} \rho_i^2 \Delta_+ \left(k_{i-1/2}v_i\right)^2 \right\}^{\frac{1}{2}} dt \max_{t\in[0,T], i\in\{1,\dots,M-1\}} (l y_i)^{\frac{2\alpha-1}{2}} \|\phi\|_{\mathscr{C}_0^{0,\alpha}(\Omega)} \\
&\leq C_1 T^{\frac{1}{2}} \left\{ \int_0^T \sum_{i=1}^{M-1} \rho_i^2 \Delta_+ \left(k_{i-1/2}v_i\right)^2 dt \right\}^{\frac{1}{2}} \max_{t\in[0,T], i\in\{1,\dots,M-1\}} (l y_i)^{\frac{2\alpha-1}{2}} \|\phi\|_{\mathscr{C}_0^{0,\alpha}(\Omega)} \\
&\leq C_3 \max_{t\in[0,T], i\in\{1,\dots,M-1\}} (l y_i)^{\frac{2\alpha-1}{2}} \|\phi\|_{\mathscr{C}_0^{0,\alpha}(\Omega)} \to 0 \text{ as } l \to 0, \ \forall \phi \in \mathscr{C}_0^{0,\alpha}(\Omega)
\end{aligned} \tag{3.58}$$

The constant $C_1 = C\left(\overline{x}_{\max} + T - \overline{x}_{\min}\right)^{\frac{1}{2}} > 0$, comes from application of Lemma 3.2. The third inequality is application of Jensen's inequality, for the function $x^{\frac{1}{2}}$ on $x \geq 0$. The final bound is established by invoking the second estimate of Lemma 3.3. A corollary of Morrey's theorem is that $W^{1,p}(\Omega) \subset C^{0,\alpha}(\Omega)$ is a continuous injection for $\alpha \in (0, 1-\frac{2}{p})$, when Ω is bounded and of class $\mathscr{C}^1$ [9, Cor. 9.14, p. 285]). Furthermore, [9, Thm. 9.17, p. 288] and Poincaré's inequality [9, Cor. 9.19, p. 290] imply that $W_0^{1,p}(\Omega) \subset C_0^{0,\alpha}(\Omega)$ is a continuous injection. This means that for $p > \frac{2}{1-\alpha}$, $\alpha \in \left(\frac{1}{2}, 1\right)$,

$$\left\{\mathcal{L}_1^l\right\}_{l>0} \text{ is compact in } W^{-1,q_1}(\Omega), \tag{3.59}$$

for $q_1 \in \left(1, \frac{2}{1+\alpha}\right), \alpha \in \left(\frac{1}{2}, 1\right)$. Consider next the third term of Exp. (3.56), which sum and integrate to

$$\left\langle \mathcal{L}_2^l, \phi \right\rangle := \int_0^T \sum_{i=1}^{M-1} \fint_{x_{i-1/2}}^{x_{i+1/2}} \left(\partial_\rho \tilde{S}_i - \partial_\rho S_{i,i+1/2}\right) \rho_i \Delta_+ \left(k_{i-1/2}v_i\right) \phi(x) dx dt. \tag{3.60}$$

Let $P = \{\xi_1, \dots, \xi_{|P|}\}$ be the set of discontinuities in k.

$$\fint_{x_{i-1/2}}^{x_{i+1/2}} \left|k(x) - k_{i+1/2}\right| dx \leq \frac{\left\|k'\right\|_\infty}{2} \left(x_{i+1/2} - x_{i-1/2}\right), \tag{3.61}$$

for $i \in \{1, \dots, M-1\} \backslash \{i_1, \dots, i_D\}$. The exceptions correspond to intervals which contain a least one discontinuity. The sum over Ineq. (3.61) is bounded uniformly, because of Ineq. (3.32). For the exceptional terms,

$$\fint_{x_{i-1/2}}^{x_{i+1/2}} \left|k(x) - k_{i+1/2}\right| dx \leq C_1, \tag{3.62}$$

since k is bounded. As $D \le |P|$, the sum over these terms is also uniformly bounded. Therefore,

$$\begin{aligned}\left|\left\langle \mathcal{L}_2^l, \phi\right\rangle\right| &\le \max_{[\alpha,1]\times[0,1]} \left|\partial^2_{\rho,k} S(k,\rho)\right| \left|\int_0^T \sum_{i=1}^{M-1} \fint_{x_{i-1/2}}^{x_{i+1/2}} \left|k(x) - k_{i+1/2}\right| |\phi(x)|\, dx dt \right.\\ &\le C_2\left(|P| + \left\|k'\right\|_\infty\right)\|\phi\|_\infty, \quad \forall \phi \in \mathscr{C}_0(\Omega).\end{aligned} \tag{3.63}$$

By summing over i and integrating in time, the fourth term is

$$\left\langle \mathcal{L}_3^l, \phi\right\rangle := -\int_{\mathscr{R}} S\left(k(x), \rho^l(0)\right)\phi(x,0)dx, \tag{3.64}$$

which satisfies

$$\left|\left\langle \mathcal{L}_3^l, \phi\right\rangle\right| \le C(X,T)\|\phi\|_\infty, \quad \forall \phi \in \mathscr{C}_0(\Omega). \tag{3.65}$$

Before dealing with the last term of Exp. (3.56), consider the second half of (3.52).

$$\begin{aligned}\int_{\mathscr{R}^+}\int_{\mathscr{R}} Q\left(k(x),\rho^l\right)\partial_x\phi\, dx dt &= \int_0^T \sum_{i=0}^{M} \int_{x_{i-1/2}}^{x_{i+1/2}} \left(Q(k,\rho_i) - Q\left(k_{i-1/2},\rho_i\right)\right)\phi_x dx dt\\ &+ \int_0^T -\sum_{i=0}^{M} \Delta_+\left(Q\left(k_{i-1/2},\rho_i\right)\right)\phi_{i+1/2} dx dt,\end{aligned} \tag{3.66}$$

where $k_{-1/2} = k_{1/2}$ and $k_{M+1/2} = k_{M-1/2}$ has been introduced. Consider the boundary terms of the first term on the right. Partial integration for piecewise $\mathscr{C}^1$ functions gives,

$$\begin{aligned}\int_0^T \left(\int_{-\infty}^{x_{1/2}} + \int_{x_{M-1/2}}^{\infty}\right) Q(k(x),0)\,\phi_x dx dt &= \int_0^T Q\left(k_{1/2},0\right)\phi_{1/2} - Q\left(k_{M-1/2},0\right)\phi_{M-1/2}\\ &- \left(\int_{-\infty}^{x_{1/2}} + \int_{x_{M-1/2}}^{\infty}\right)\partial_k Q(k(x),0)\,k'(x)\phi\, dx\\ &- \sum_{i=1}^{D_1}[Q](\xi_i)\phi(\xi_i,t),\end{aligned} \tag{3.67}$$

The first terms on the right cancel with corresponding terms in Eq. (3.66). The constants ξ_i are discontinuities of k in $\left(-\infty, x_{1/2}\right] \cup [x_{M-1/2}, \infty)$ and

$$[Q](\xi_i) = Q\left(k\left(\xi_i^+\right),0\right) - Q\left(k\left(\xi_i^-\right),0\right). \tag{3.68}$$

Let

$$\left\langle \mathcal{L}_4^l, \phi\right\rangle := -\int_0^T \left(\int_{-\infty}^{x_{1/2}} + \int_{x_{M-1/2}}^{\infty}\right)\partial_k Q(k(x),0)\,k'(x)\phi\, dx + \sum_{i=1}^{D_1}[Q](\xi_i)\phi(\xi_i,t)dt, \tag{3.69}$$

then

$$\left|\left\langle \mathcal{L}_4^l, \phi\right\rangle\right| \le C(X,T)\|\phi\|_\infty, \quad \forall \phi \in \mathscr{C}_0(\Omega). \tag{3.70}$$

Consider

$$\left\langle \mathcal{L}_5^l, \phi\right\rangle := \int_0^T \sum_{i=1}^{M-1} \int_{x_{i-1/2}}^{x_{i+1/2}} \left(Q(k,\rho_i) - Q\left(k_{i-1/2},\rho_i\right)\right)\phi_x dx dt \tag{3.71}$$

Since $Q \in \mathscr{C}^1$,

$$\left|\left\langle \mathcal{L}_5^l, \phi \right\rangle\right| \le C \int_0^T \sum_{i=1}^{M-1} \int_{x_{i-1/2}}^{x_{i+1/2}} \left|k(x) - k_{i-1/2}\right| |\phi_x| \, dx dt$$
$$\le C \left\|k^l - k\right\|_{L^p(\Omega)} \|\phi\|_{W_0^{1,q_2}(\Omega)}, \tag{3.72}$$

by the Hölder inequality. Again, let $|P|$ be the number of discontinuities in k. Then

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \left|k(x) - k_{i-1/2}\right|^p dx \le \frac{\left\|k'\right\|_\infty}{p+1} \left(x_{i+1/2} - x_{i-1/2}\right)^{p+1}, \tag{3.73}$$

for $i \in \{1, \ldots, M-1\} \backslash \{i_1, \ldots, i_D\}$, where the exceptions correspond to intervals which contain a least one discontinuity. For the exceptional cases,

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \left|k(x) - k_{i-1/2}\right|^p dx \le C \max_{t \in [0,T], i \in \{1,\ldots,M-1\}} \left(x_{i+1/2} - x_{i-1/2}\right), \tag{3.74}$$

Therefore,

$$\left|\left\langle \mathcal{L}_5^l, \phi \right\rangle\right| \le C \max_{t \in [0,T], i \in \{1,\ldots,M-1\}} \left((l y_i)^{\frac{1}{p}} + (l y_i)\right) \|\phi\|_{W_0^{1,q_1}(\Omega)} \to 0 \text{ as } l \to 0, \tag{3.75}$$

by Lemma 3.1 and Lemma 3.2. If $Q(k, 0) = 0 \ \forall k$, then

$$|Q(\rho, k_2) - Q(\rho, k_1)| \le C\rho \, |k_2 - k_1|, \tag{3.76}$$

which asymptotically improves Ineq. (3.75)

$$\left|\left\langle \mathcal{L}_5^l, \phi \right\rangle\right| \le \tilde{C} \left(l^{\frac{1}{p}} + l\right) \|\phi\|_{W_0^{1,q_1}(\Omega)} \to 0 \text{ as } l \to 0. \tag{3.77}$$

The sequence $\left\{\mathcal{L}_5^l\right\}_{l>0}$ is compact in $W^{-1,q_2}(\Omega)$, for $q_2 \in (1,2]$ and $p = \frac{q_2}{q_2-1} \in [1, \infty)$. Consider sum of the last term of Exp. (3.56) and the second term on the right-hand side of Eq. (3.66). It can be seen from

$$\Delta_+\left(Q_{i,i-1/2}\right) = \Delta_+\left(Q(k_{i-1/2}, 0)\right) + \Delta_+\left(\partial_\rho S_{i,i-1/2} k_{i-1/2} \rho_i v_i\right)$$
$$- \Delta_+\left(k_{i-1/2} \int_0^{\rho_i} \partial_{\rho\rho}^2 S\left(k_{i-1/2}, \rho\right) \rho v(\rho) d\rho\right), \tag{3.78}$$

that the terms of their sum are

$$- \Delta_+^2\left(\partial_\rho S_{i,i-1/2}\right) k_{i-1/2} \rho_i v_i + \int_0^{\rho_i} \Delta_+\left(\partial_{\rho\rho}^2 S\left(k_{i-1/2}, \rho\right) k_{i-1/2}\right) \rho v(\rho) d\rho$$
$$- \Delta_+\left(Q(k_{i-1/2}, 0)\right) + k_{i+1/2} \int_{\rho_i}^{\rho_{i+1}} \partial_{\rho\rho}^2 S_{i,i+1}(\rho) \rho \left(v(\rho) - v_{i+1}\right) d\rho$$
$$= \partial_\rho S_{i,i+1/2} \Delta_+\left(k_{i-1/2}\right) \rho_i v_i - \int_0^{\rho_i} \Delta_+\left(\partial_\rho S\left(k_{i-1/2}, \rho\right) k_{i-1/2}\right) (\rho v(\rho))' \, d\rho$$
$$- \Delta_+\left(Q(k_{i-1/2}, 0)\right) + k_{i+1/2} \int_{\rho_i}^{\rho_{i+1}} \partial_{\rho\rho}^2 S_{i,i+1}(\rho) \rho \left(v(\rho) - v_{i+1}\right) d\rho, \tag{3.79}$$

with common factor $\phi_{i+1/2}$. Denote by $\left\langle \mathcal{L}_6^l, \phi \right\rangle$ the integral in time of the sum over Eq. (3.79). Since $S \in \mathscr{C}^2([0,1]\times[\alpha,1])$, the first three terms are $\mathcal{O}\left(\left|k_{i+1/2}-k_{i-1/2}\right|\phi_{i+1/2}\right)$, which sum up to $\mathcal{O}\left(\text{T.V.}(k)\left\|\phi\right\|_\infty\right)$. The integral over the sum of the last term is bounded by $C(T)\left\|\phi\right\|_\infty$, by Lemma 3.3.

$$\left|\left\langle \mathcal{L}_6^l, \phi \right\rangle\right| \leq \left(C(T) + C_2 \text{T.V.}(k)\right)\left\|\phi\right\|_\infty, \quad \forall \phi \in \mathscr{C}_0(\Omega). \tag{3.80}$$

For all the terms with $\left\|\phi\right\|_\infty$ bounds,

$$\left\{\left\|\mathcal{L}_i^l\right\|_{\mathcal{M}(\Omega)}\right\}_{i\in\{2,3,4,6\}} \leq C, \tag{3.81}$$

for some finite uniform constant $C > 0$. Let $\mathcal{M}(\Omega) = \left(\mathscr{C}_c(\Omega)\right)'$ be the space of signed radon measures of finite mass. From the embedding theorem [30, Lem. 2.55, p. 38], $\mathcal{M}(\Omega) \subset W^{-1,q_3}(\Omega)$ is a compact embedding for any $q_3 \in (1,2)$. Summing up all terms, the sequence $\left\{\mathcal{L}^l\right\}_{l>0}$ is compact in $W^{-1,q}(\Omega)$ for $1 < q := \min(q_1,q_2,q_3) < \frac{2}{1+\alpha} < 2$. As $0 \leq \rho^l \leq 1$, the sequence $\left\{\mathcal{L}^l\right\}_{l>0}$ is bounded in $W^{-1,r}(\Omega)$ for $r > 2$. By [26, Lem 3.3], $\left\{\mathcal{L}^l\right\}_{l>0}$ is compact in $W^{-1,2}(\Omega)$.

$$\left\{\mathcal{L}^l\right\}_{l>0} \text{ is compact in } W^{-1,2}_{\text{loc}}\left(\mathscr{R}\times\mathscr{R}_0^+\right), \tag{3.82}$$

as Ω is an arbitrary bounded open set in $\mathscr{R}\times\mathscr{R}_0^+$ of class $\mathscr{C}^1$. □

Remark 3.5. The proof can be extended to the case with vacuum regions without much difficulty. The assumption that distances between vehicles go to zero was used to show that $\mathcal{L}_1^l$ and $\mathcal{L}_5^l$ converges to zero in $W^{-1,q_1}(\Omega)$ and $W^{-1,q_2}(\Omega)$, respectively. In the case where $\operatorname{supp}\rho_0$ consists of V disjoint closed intervals, one can instead invoke Ineq. (3.30). First, remove the terms associated with each vacuum region from the sum in Def. (3.57), before using Hölder inequality. The sum over the extracted terms are compact in $W^{-1,q_1}(\Omega)$, as it can be bounded by $C(V-1)l\left\|\phi\right\|_{\mathscr{C}_0^{0,\alpha}(\Omega)}$, for $0 < C < \infty$. Similarly, extract the terms from Def. (3.71) and use the partial integration for piecewise smooth functions, as in Eq. (3.67). One obtains $V-1$ terms which can be bounded above by $\left\|\phi\right\|_\infty$, up to a uniform constant. The embedding theorem of [30, Lem. 2.55, p. 38] can be used to prove $W^{-1,q_3}(\Omega)$-compactness of these terms separately.

3.3 Convergence to weak solutions

Lemma 3.6. *(Convergence of initial value) For ρ_0 satisfying* (3.5) *and FtL initial value given in* (3.6)*, then*

$$\rho^l(x,0) \overset{\mathscr{D}'(\mathscr{R})}{\rightharpoonup} \rho_0. \tag{3.83}$$

Lemma 3.6 is implied by [5, Lem. 9].

Theorem 3.7. *(Convergence to a weak solution) Let $\rho^l = \rho(x,t)$ be the Follow-the-Leader scheme* (3.14)*. Assume v and k satisfy* (3.8) *and* (3.9)*. In addition, assume $v \in \mathscr{C}^2[0,1]$ and $\rho v(\rho)$ genuinely non-linear, that ρ_0 satisfies* (3.5) *and either Ineq.* (3.25) *or Ineq.* (3.30) *holds. There exists a subsequence such that, for any finite $T > 0$,*

$$\rho^l \to \rho \text{ in } L^p\left(\mathscr{R}\times[0,T]\right) \text{ as } l \to 0, \text{ for any } 1 \leq p < \infty, \tag{3.84}$$

and $\rho \in L^{\infty}\left(\mathscr{R}\times\mathscr{R}^{+}\right)$ is a weak solution ρ of Eq. (3.1), i.e. ρ is a bounded measurable function satisfying $\forall\phi \in C_c^{\infty}\left(\mathscr{R}\times\mathscr{R}^{+}\right)$,

$$\int_{\mathscr{R}^{+}}\int_{\mathscr{R}}\left(\rho\phi_t + k(x)\rho v(\rho)\phi_x\right)dxdt + \int_{\mathscr{R}}\rho_0(x)\phi(x,0)dx = 0. \tag{3.85}$$

Proof. The fact that ρ^l converges to ρ in $L^p_{\text{loc}}(\mathscr{R}\times[0,T])$ and a.e. follows from Lemma 2.8 and 3.4. By Lemma 3.31, convergence in L^p_{loc} and L^p is equivalent. If $S(k,\rho)=\rho, Q(k,\rho)=k\rho v(\rho)$ in Lemma 3.4 and $\phi\in\mathscr{C}_c^{\infty}\left(\mathscr{R}\times\mathscr{R}_0^{+}\right)$, If either Ineq. (3.25) or Ineq. (3.30) holds,

$$\left\langle \mathcal{L}_1^l,\phi\right\rangle \to 0 \text{ as } l\to 0. \tag{3.86}$$

In the latter case, one can extract the distances which do not converge to zero from the sum in Eq. (3.57), before invoking Hölder's inequality. The sum over the extracted terms can be bounded by $C(V-1)\|\phi\|_{\mathscr{C}_0^{0,\alpha}(\Omega)}l$, which converges to zero. Since $\partial^2_{\rho,k}S(k,\rho)=0$, Ineq. (3.63) gives

$$\left\langle \mathcal{L}_2^l,\phi\right\rangle = 0. \tag{3.87}$$

$$\left\langle \mathcal{L}_3^l,\phi\right\rangle \to -\int_{\mathscr{R}}\rho_0(x)\phi(x,0)dx, \tag{3.88}$$

from Lemma 3.6. In addition,

$$Q(k,0)=\partial_k Q(k,0)=0,\ \forall k\in[\alpha,1] \tag{3.89}$$

which implies that

$$\left\langle \mathcal{L}_4^l,\phi\right\rangle = 0. \tag{3.90}$$

Furthermore,

$$\left\langle \mathcal{L}_5^l,\phi\right\rangle \to 0 \text{ as } l\to 0, \tag{3.91}$$

with rate of convergence given in Ineq. (3.77). Eq. (3.89) and $\partial^2_{\rho\rho}S=0$, $\Delta^2_{+}\left(\partial_\rho S_{i,i-1/2}\right)=0$ together imply

$$\left\langle \mathcal{L}_6^l,\phi\right\rangle = 0. \tag{3.92}$$

As $\rho^l\to\rho$ in $L^1(\mathscr{R}\times[0,T])$,

$$\int_{\mathscr{R}\times\mathscr{R}^{+}}\rho\phi_t + k(x)\rho v(\rho)\phi_x dxdt + \int_{\mathscr{R}}\rho_0(x)\phi(x,0)dx = \lim_{l\to 0}\left\langle \mathcal{L}^l-\mathcal{L}_3^l,\phi\right\rangle = 0. \tag{3.93}$$

□

It is further assumed that

$$\left|\rho^l(x,0)-c\right| \overset{\mathscr{D}'(\mathscr{R})}{\rightharpoonup} |\rho_0-c| \text{ for any } c\in\mathscr{R}. \tag{3.94}$$

The limit also satisfies the following entropy inequality.

Theorem 3.8. *(A Kružkov type entropy inequality) For any* $0 \leq \phi \in \mathscr{C}_c^\infty\left(\mathscr{R}\times\mathscr{R}^+\right)$ *and* $c \in \mathscr{R}$, *the limit* ρ *of Theorem 3.7 satisfies*

$$\int_{\mathscr{R}\times\mathscr{R}^+} \left(|\rho-c|\,\phi_t + k(x)\operatorname{sign}(\rho-c)(\rho v(\rho)-cv(c))\,\phi_x\right)dx\,dt - \int_{\mathscr{R}\times\mathscr{R}^+} k' c v(c)\operatorname{sign}(\rho-c)\,\phi\,dx\,dt$$
$$+\int_{\mathscr{R}} |\rho_0-c|\,\phi(x,0)dx + \sum_{j=1}^{|P|}\left|[k]_j\right|\int_0^\infty |c|\,v(c)\phi(\xi_j,t)dt \geq 0. \tag{3.95}$$

Proof. Let

$$\mu(x) = |x-c|, \tag{3.96}$$

and consider first

$$\int_{\mathscr{R}\times\mathscr{R}^+} \mu^l\phi_t\,dx\,dt = \int_0^\infty \sum_{i=0}^{M}\int_{x_{i-1/2}}^{x_{i+1/2}} \mu_i\phi_t\,dx\,dt. \tag{3.97}$$

where $x_{-1/2} = -\infty$ and $x_{M+1/2} = +\infty$. As a result, $\phi_{-1/2} = \phi_{M+1/2} = 0$. Since ϕ is smooth and $\mu_i, x_{i-1/2}, x_{i+1/2}$ are Lipschitz, the following idenity holds a.e. [1]

$$\begin{aligned}
\int_{x_{i-1/2}}^{x_{i+1/2}} \mu_i\phi_t\,dx &= \frac{d}{dt}\left(\mu_i\int_{x_{i-1/2}}^{x_{i+1/2}}\phi_t\,dx\right) - \dot{\mu}_i\int_{x_{i-1/2}}^{x_{i+1/2}}\phi\,dx - \mu_i\Delta_+\left(\phi_{i-1/2}k_{i-1/2}v_i\right)\\
&= \frac{d}{dt}\left(\mu_i\int_{x_{i-1/2}}^{x_{i-1/2}}\phi_t\,dx\right) + \operatorname{sign}(\rho_i-c)\,\rho_i\Delta_+\left(k_{i-1/2}v_i\right)\phi_{i+1/2}\\
&+\operatorname{sign}(\rho_i-c)\,\rho_i\Delta_+\left(k_{i-1/2}v_i\right)\fint_{x_{i-1/2}}^{x_{i+1/2}}\left(\phi-\phi_{i+1/2}\right)dx\\
&-\Delta_+\left(\mu_i\phi_{i-1/2}k_{i-1/2}v_i\right) + \Delta_+(\mu_i)\,\phi_{i+1/2}k_{i+1/2}v_{i+1}
\end{aligned} \tag{3.98}$$

for $i \in \{0,\ldots,M\}$. The special cases $i \in \{0,M\}$ hold because $\rho_0 = \rho_M = 0$. Next, let

$$\begin{aligned}
F(x) :&= \operatorname{sign}(x-c)(xv(x)-cv(c))\\
&= \mu(x)v(x) - c\,|v(x)-v(c)|,
\end{aligned} \tag{3.99}$$

and consider

$$\int_{\mathscr{R}\times\mathscr{R}^+} k(x)F^l\phi_x - k' c v(c)\operatorname{sign}\left(\rho^l-c\right)\phi\,dx\,dt \tag{3.100}$$

Partial integration for piecewise $\mathscr{C}^1$ functions gives

$$\int_{\mathscr{R}} F^l k(x)\phi_x\,dx = \sum_{i=0}^{M} F_i\Delta_+\left(k_{i-1/2}\phi_{i-1/2}\right) - \sum_{j=1}^{|P|}[k]_j F^l\left(\xi_j\right)\phi(\xi_j,t) - \int_{\mathscr{R}} F^l k'(x)\phi\,dx, \tag{3.101}$$

[1] Let $\mu_i = \mu(\rho_i)$, then the a.e. derivative is given by

$$\dot{\mu}_i = -\operatorname{sign}(\rho_i-c)\,\rho_i^2 D_+\left(k_{i-1/2}v_i\right),$$

with the convention that sign(0) = 0. Since μ_i is lipschitz continuous, the fundamental theorem of calculus still holds [34, ex. 1.6.44, p. 169].

where $P = \{\xi_1, ..., \xi_{|P|}\}$ is the set of discontinuities in k, and

$$[k]_j = k\left(\xi_j^+\right) - k\left(\xi_j^-\right). \tag{3.102}$$

Therefore

$$\begin{aligned}\int_{\mathscr{R}\times\mathscr{R}^+} k(x)F^l\phi_x - k'cv(c)\,\mathrm{sign}\left(\rho^l - c\right)\phi\,dx\,dt = &\int_0^\infty \sum_{i=0}^{M} F_i\Delta_+\left(k_{i-1/2}\phi_{i-1/2}\right)dt \\ &- \int_{\mathscr{R}\times\mathscr{R}^+} \mathrm{sign}\left(\rho^l - c\right)\rho^l v\left(\rho^l\right)k'(x)\phi\,dx\,dt \\ &- \int_0^\infty \sum_{j=1}^{|P|}[k]_j F^l\left(\xi_j\right)\phi(\xi_j,t)dt.\end{aligned} \tag{3.103}$$

A straightforward calculation shows that

$$\Delta_+\left(F_i\right) \le \Delta_+\left(\mu_i\right)v_{i+1} + \Delta_+\left(v_i\right)\rho_i\,\mathrm{sign}\left(\rho_i - c\right). \tag{3.104}$$

Consider the first term on the right of Eq. (3.103).

$$\begin{aligned}\sum_{i=0}^{M} F_i\Delta_+\left(k_{i-1/2}\phi_{i-1/2}\right) &= \sum_{i=0}^{M-1} -\Delta_+\left(F_i\right)k_{i+1/2}\phi_{i+1/2} \\ &\ge -\sum_{i=0}^{M-1}\Delta_+\left(\mu_i\right)v_{i+1}k_{i+1/2}\phi_{i+1/2} \\ &\quad - \sum_{i=0}^{M-1}\Delta_+\left(v_i\right)\rho_i\,\mathrm{sign}\left(\rho_i - c\right)k_{i+1/2}\phi_{i+1/2}.\end{aligned} \tag{3.105}$$

It was used that $\phi_{M+1/2} = \phi_{-1/2} = 0$. Add Eq.(3.103) and the time integral of the sum over Eq. (3.98), and use Ineq. (3.105).

$$\begin{aligned}\int_{\mathscr{R}\times\mathscr{R}^+} &\mu^l\phi_t + k(x)F^l\phi_x - k'cv(c)\,\mathrm{sign}\left(\rho^l - c\right)\phi\,dx\,dt \\ &+ \int_{\mathscr{R}} \mu^l(x,0)\phi(x,0)dx \ge A_1 + A_2 + A_3,\end{aligned} \tag{3.106}$$

where we have defined

$$\begin{cases} A_1 := \int_0^\infty \sum_{i=1}^{M-1} \mathrm{sign}\left(\rho_i - c\right)\rho_i^2 v_i \int_{x_{i-1/2}}^{x_{i+1/2}} \phi\left(D_+^x\left(k_{i-1/2}\right) - k'\right)dx\,dt, \\ A_2 := \int_0^\infty \sum_{j=1}^{|P|} -[k]_j F^l\left(\xi_j\right)\phi(\xi_j,t)dt, \\ A_3 := \int_0^\infty \sum_{i=1}^{M-1}\left(\mathrm{sign}\left(\rho_i - c\right)\rho_i^2 k_{i+1/2} D_+\left(v_i\right)\right)\int_{x_{i-1/2}}^{x_{i+1/2}}\left(\phi - \phi_{i+1/2}\right)dx\,dt. \end{cases} \tag{3.107}$$

The function k has $T \in \{0, 1, ..., |P|\}$ discontinuities when restricted to $[x_{i-1/2}, x_{j+1/2}]$, possibly including the endpoints. Partition $[x_{i-1/2}, x_{j+1/2}]$, with respect to the discontinuities $\{\xi_1, ..., \xi_T\}$. Let $\xi_0 = x_{i-1/2}, \xi_{T+1} = x_{i+1/2}$,

$$\begin{aligned}&[k]_k = k\left(\xi_k^+\right) - k_-\left(\xi_k^-\right) \text{ for } i \in \{1, ..., T\}, \\ &[k]_0 = k\left(\xi_0^+\right) - k_{i-1/2} \text{ and } [k]_{T+1} = k_{i+1/2} - k\left(\xi_{T+1}^-\right).\end{aligned} \tag{3.108}$$

Use the mean-value theorem

$$\begin{aligned}\Delta_+\left(k_{i-1/2}\right) &= \sum_{k=1}^{T+1}\left(k_-(\xi_k) - k_+(\xi_{k-1})\right) + \sum_{k=0}^{T+1}[k]_k \\ &= \sum_{k=1}^{T+1} k'(c_k)(\xi_k - \xi_{k-1}) + \sum_{k=0}^{T+1}[k]_k, \quad \text{for some } c_k \in (\xi_{k-1}, \xi_k). \end{aligned} \tag{3.109}$$

A_1 is decomposed into the following terms

$$\begin{aligned} A_1^1 &:= \int_0^\infty \sum_{i=1}^{M-1} \operatorname{sign}(\rho_i - c)\,\rho_i v_i \left(\sum_{k=1}^{T_i+1} \int_{\xi_{k-1}^i}^{\xi_k^i} \phi\left(k'(c_k^i) - k'(x)\right) dx \right) dt, \\ A_1^2 &:= \int_0^\infty \sum_{i=1}^{M-1} \sum_{j=0}^{T_i+1} [k]_j \operatorname{sign}(\rho_i - c)\,\rho_i v_i \fint_{x_{i-1/2}}^{x_{i-1/2}} \phi(x)\,dx\,dt. \end{aligned} \tag{3.110}$$

The first term goes to zero,

$$\begin{aligned} \left|A_1^1\right| &\le \int_0^T \sum_{i=1}^{M-1} \rho_i \left(\left\|k''\right\|_\infty \|\phi\|_\infty \sum_{k=1}^{T_i+1} \left(\xi_k^i - \xi_{k-1}^i\right)^2 \right) dt \\ &\le \int_0^T \sum_{i=1}^{M-1} \rho_i \left\|k''\right\|_\infty \|\phi\|_\infty \left(x_{i+1/2} - x_{i-1/2}\right)^2 dt \\ &\le C\,\|\phi\|_\infty\, l \to 0 \text{ as } l \to 0. \end{aligned} \tag{3.111}$$

The second inequality follows from $\sum_i x_i^2 \le \left(\sum_i |x_i|\right)^2$. The final constant is a consequence of Ineq. (3.32). Next, consider

$$\begin{aligned} A_1^2 + A_2 = &\int_0^\infty \sum_{j=1}^{|P|} [k]_j c v(c) \operatorname{sign}\left(\rho_{i_j} - c\right) \phi(\xi_j, t)\,dt \\ &+ \int_0^\infty \sum_{j=1}^{|P|} [k]_j \rho_{i_j} v_{i_j} \operatorname{sign}\left(\rho_{i_j} - c\right) \fint_{x_{i_j-1/2}}^{x_{i_j-1/2}} \phi(x) - \phi(\xi_j, t)\,dx\,dt. \end{aligned} \tag{3.112}$$

The density ρ_{i_j} corresponds to discontinuity ξ_j. The first term is bounded below by

$$-\sum_{j=1}^{|P|} \left|[k]_j\right| \int_0^\infty |c v(c)|\, \phi(\xi_j, t)\,dt. \tag{3.113}$$

The second term can be bounded in absolute value by

$$C\,\|\phi_x\|_\infty\, l \to 0 \text{ as } l \to 0. \tag{3.114}$$

Consider next

$$
\begin{aligned}
|A_3| &\leq \int_0^T \sum_{i=1}^{M-1} \rho_i \left|\Delta_+(v_i)\right| \left| \fint_{x_{i-1/2}}^{x_{i+1/2}} \left(\phi - \phi_{i+1/2}\right) dx \right| dt \\
&\leq \int_0^T \left\{ \sum_{i=1}^{M-1} \rho_i^2 \left(\Delta_+(v_i)\right)^2 \right\}^{\frac{1}{2}} \left\{ \sum_{i=1}^{M-1} \|\phi_x\|_\infty^2 \left(x_{i+1/2} - x_{i-1/2}\right)^2 \right\}^{\frac{1}{2}} dt \\
&\leq C_1 \int_0^T \left\{ \sum_{i=1}^{M-1} \rho_i^2 \left(\Delta_+(v_i)\right)^2 \right\}^{\frac{1}{2}} dt \max_{i\in\{1,\dots,M-1\}} (l y_i)^{\frac{1}{2}} \\
&\leq C \max_{i\in\{1,\dots,M-1\}} (l y_i)^{\frac{1}{2}} \to 0 \text{ as } l \to 0.
\end{aligned}
\tag{3.115}
$$

The second inequality is an application of Hölder's inequality. The constant

$$
C_1 = \|\phi_x\|_\infty \left(\overline{x}_{\max} + T - \overline{x}_{\min}\right)^{\frac{1}{2}} > 0,
\tag{3.116}
$$

comes from Lemma 3.2. The final estimate is an application of Jensen's inequality and Lemma 3.3. It has been shown that

$$
\begin{aligned}
&\liminf_{l\to 0} \int_{\mathscr{R}\times\mathscr{R}^+} \mu^l \phi_t + k(x) F^l \phi_x \, dx\, dt - \int_{\mathscr{R}\times\mathscr{R}^+} k' c v(c) \operatorname{sign}\left(\rho^l - c\right) \phi \, dx\, dt \\
&+ \int_{\mathscr{R}} \mu^l(x,0)\phi(x,0) dx + \sum_{j=1}^{|P|} \left|[k]_j\right| \int_0^\infty |c v(c)| \, \phi(\xi_j, t) dt \geq 0.
\end{aligned}
\tag{3.117}
$$

From here, one can use a technical lemma [22, Lem. 8.20, p. 410] and assumption (3.94) to show that Ineq. (3.95) holds. This was done in [22, p. 410-412], for the front-tracking algorithm. The arguments rely only on the L^1 and pointwise a.e. convergence of the method, and are therefore not repeated. □

The fact that the limit ρ satisfies the above entropy inequality is in-general not sufficient to establish uniqueness of the solution. In addition, existence of strong L^1-traces along the lines of discontinuities and $t = 0^+$ is needed. That is, measurable functions $\left(\gamma^\pm \rho\right)(t)$ such that

$$
\operatorname*{ess\,lim}_{\epsilon \downarrow 0} \int_0^T \left|\rho(\xi_i \pm \epsilon, t) - \left(\gamma^\pm \rho\right)(t)\right| dt = 0 \quad \text{for } i \in \{1, \dots, P\}, \text{ for any } T > 0.
\tag{3.118}
$$

In addition,

$$
\operatorname*{ess\,lim}_{\epsilon \downarrow 0} \int_{\mathscr{R}} |\rho(x, t+\epsilon) - \rho_0(x)| \, dt = 0.
\tag{3.119}
$$

If the flux is genuinely non-linear and k is piecewise constant, the following Lemma states that Ineq. (3.8) has a regularising effect near the lines of discontinuity of k.

Lemma 3.9. *A bounded measurable function ρ which satisfies the conditions of Theorem 3.7 and Theorem 3.8 admits strong (right and left) traces $\rho_m^\pm$ along each discontinuity $\xi_1, \dots, \xi_{|P|}$. Moreover, ρ admits a strong trace at $t = 0+$, so that the initial condition $\rho|_{t=0} = \rho_0$ is satisfied in the strong L^1-sense.*

A proof was given in [26, Lemma 6.1], under assumptions (2.8) and (2.9). The Lemma is adapted to the case with a space dependent discontinuous coefficient and initial data with compact support. The authors of [26] suggest that the result of Lemma 3.9 may hold for k which is smooth between the discontinuities, but this has to be assumed. Under the assumptions of Theorem 3.7 and assuming the the conclusion of Lemma 3.9 holds, Ineq. (3.8) implies uniqueness by the uniqueness theorem in [26, Theorem 6.1]. The uniqueness theorem requires a flux crossing condition. Said condition is satisfied, since the LWR flux is on multiplicative form

$$f(k,\rho) = k\rho v(\rho). \tag{3.120}$$

Since the limit is unique, the Follow-the-Leader scheme converges to the weak entropy solution.

3.4 Further research

It would be interesting to see whether assumptions on v and k can be relaxed, perhaps by using a different compactness technique. Common assumptions found in the literature are $v \in \mathscr{C}^{0,1}[0,1]$ and $k \in \text{B.V.}_{\text{loc}}(\mathscr{R})$. It seems reasonable that traces should exist when k is well-behaved between discontinuities, but the same cannot be said for more general k. The problem of trace existence has received considerable attention in the literature, as they have to be assumed when k has an infinite number of discontinuities. Many solution concepts for conservation laws with discontinuous fluxes have been developed in the literature, some of which do not assume traces to establish uniqueness; see [7], [29].

Chapter 4

LWR with a Unilateral Constraint on the Flux

4.1 The flux constrained Follow-the-Leader model

The Colombo-Goatin (CG) model is an LWR model with a unilateral local point constraint on the flux.

$$\begin{aligned} \rho_t + (\rho v(\rho))_x &= 0, \\ \rho(x,0) &= \rho_0, \\ f\left(\rho(0^{\pm},t)\right) &\leq q,\ q \in (0, f_{\max}), \end{aligned} \tag{4.1}$$

where $f_{\max}$ is the unique maximum of $\rho v(\rho)$. A Follow-the-Leader (FtL) model for approximating the solution of Prob. (4.1) is developed by controlling the velocities of the two vehicles which are closest to the interface $\{x = 0\}$. In the protypical Follow-the-Leader model, a calculation reveals that the approximate trace of the interface flux is

$$\left((1 - T(t))v_{k+1/2} + T(t)v_{k-1/2}\right)\rho_k, \tag{4.2}$$

where $v_{k-1/2}, v_{k+1/2}$ are the velocities of the next vehicle to pass (NVtP) and the nearest vehicle past (NVP) the interface, respectively. The function T takes values between zero and one, and meaures their relative closeness to the interface.

$$\rho_k = \frac{l}{x_{k+1/2} - x_{k-1/2}}, \tag{4.3}$$

is the mass density associated with NVtP. The flux at the interface is the product of a convex combination of the velocities of NVtP and NVP, and the mass density of NVtP. Therefore, the velocity function of the Follow-the-Leader model has to be on the form

$$v_{i-1/2} = v(\rho_{i-1}, \rho_i), \tag{4.4}$$

to constrain the flux directly. The proposed model is

$$v(\rho_{i-1}, \rho_i) = \begin{cases} \min(v(\rho_i), \hat{v}) & \text{if } x_{i-1/2} < 0, x_{i+1/2} \geq 0 \text{ and } \rho_i v(\rho_i) > q \\ \min(v(\rho_i), \hat{v}) & \text{if } x_{i-3/2} < 0, x_{i-1/2} \geq 0 \text{ and } \rho_{i-1} v(\rho_i) > q \\ v(\rho_i) & \text{otherwise,} \end{cases} \tag{4.5}$$

with

$$\hat{v}(t) = v(\hat{\rho}(t)), \tag{4.6}$$

where

$$\hat{\rho}(t) = \max\{\rho \in [0,1]; \rho v(\rho) = q(t)\}. \tag{4.7}$$

The first and second option of (4.5) represent the flux constraint for the NVtP and NVP, respectively. The right-most inequalities indicate whether the system is in the congested or free branch of the flow. If a vehicle falls into the congested branch, the velocity drops so that Exp. (4.2) is satisfied.

$$\begin{aligned} \rho_i \hat{v} \le \hat{\rho}\hat{v} = q(t) \text{ for } \rho_i \le \hat{\rho}, \\ \rho_i v_i \le q(t) \text{ for } \rho_i \ge \hat{\rho}. \end{aligned} \tag{4.8}$$

To ensure existence of a solution for the model, one may need to limit the number of times a vehicle can switch between congested flow and free flow. This can be done by introducing a hysteresis loop in the model, inspired by multi-regime models. For the NVtP and NVP, replace (4.5) by the multifunction

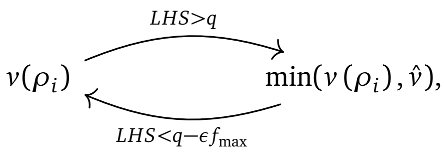

Figure 4.1: State transition diagram

The transition from the congested branch to free branch occurs for a stricter constraint level than the capacity drop. Their discrepancy is determined by $\epsilon > 0$. The idea is that a vehicle will need to traverse the hysteresis loop to transition back to the same state, which can be used to bound the number of state switches in finite time. Consider a measurable function ρ_0 such that

$$0 \le \rho_0 \le 1, \quad \operatorname{supp} \rho_0 \text{ is compact}, \tag{4.9}$$

The Flux-Constrained Follow-the-Leader (FC-FtL) model is defined as

$$\begin{aligned} \dot{x}_{i-1/2} &= v(x_{i-3/2}, x_{i-1/2}, x_{i+1/2}, \rho_{i-1}, \rho_i), && \text{for } i \in \{1, ..., M\}, \\ \rho_i &= \frac{l}{x_{i+1/2} - x_{i-1/2}}, \quad l := \frac{\|\rho_0\|_{L^1(\mathscr{R})}}{M-1} && \text{for } i \in \{1, ..., M-1\}, \\ \rho_0 &= \rho_M = 0. \end{aligned} \tag{4.10}$$

The right-hand side of the ODE is the multifunction given in (4.5) and Figure 4.1. Let $x_{1/2}(0) = \overline{x}_{\min}$ and $x_{M-1/2}(0) = \overline{x}_{\max}$ be min and max of the support of ρ_0. For the remaining vehicles,

$$x_{i-1/2}(0) = \inf\left\{ x \in \mathscr{R} \,\middle|\, \int_{-\infty}^{x} \rho_0 dx = (i-1)l \right\} \text{ for } i \in \{2, ..., M-1\}. \tag{4.11}$$

This definition takes into account possible vacuum regions within the support of ρ_0. For Eq. (4.5) to make sense for the edge-cases, $x_{-1/2} = -\infty, x_{M+1/2} = \infty$. In the LWR model, it is assumed that

$$\begin{aligned} &v \in \mathscr{C}^{0,1}([0,1]),\ \text{non-increasing},\ v(0) = 1, v(1) = 0, \\ &v \geq 1 - \rho^{\sigma-1} \text{ for some } \sigma > 1. \end{aligned} \tag{4.12}$$

For completeness,

$$v|_{(-\infty,0)} = 1\ v|_{(1,\infty)} = 0. \tag{4.13}$$

The control is piecewise continuous and has finitely many discontinuities

$$\begin{aligned} &q : [0,\infty) \to [0, f_{\max}) \\ &q \in \mathscr{C}(\mathscr{R} \backslash \{\zeta_1, ..., \zeta_D\}) \end{aligned} \tag{4.14}$$

The control is allowed to be zero on finitely many non-degenerate disjoint intervals, but is bounded below away from the discontinuities. There exist a $\tilde{\rho} \in (0,1)$ such that

$$q|_{[0,\infty)\backslash\{q^{-1}(\{0\})\}} \geq \tilde{\rho}\, v(\tilde{\rho}) > 0. \tag{4.15}$$

These assumptions covers for instance the application of traffic lights, which can be modeled as discontinuous jumps to and from zero. The Follow-the-Leader density is

$$\rho^l(x,t) := \sum_{i=1}^{M-1} \rho_i \mathbb{1}_{[x_{i-1/2}, x_{i+1/2})}. \tag{4.16}$$

Lemma 4.1. *(Vehicles do not cross for fixed states) Assume that (4.11) and (4.12) hold, and fix states in FC-FtL (4.10) model. I.e.fix which two vehicles can switch between states of Figure 4.1 and which regime either vehicle is in. Let* $y_i = \frac{x_{i+1/2} - x_{i-1/2}}{l}$ *for* $i \in \{1, ..., M-1\}$ *be computed for a fixed state, where* $y_i(0) \geq 1$. *Then*

$$y_i(t) \geq 1 \forall t \geq 0.\ \textit{for}\ i \in \{1, ..., M-1\} \tag{4.17}$$

Proof. A simple calculation shows that

$$\frac{dy_i}{dt} = \frac{v_{i+1/2} - v_{i-1/2}}{l} \geq \frac{v_{i+1/2}}{l} \geq 0 \tag{4.18}$$

for $y_i \leq 1$, since $v_{i-1/2} = v(\rho_i) = 0$. This implies Ineq. (4.17). □

Proposition 4.2. *(Existence) Prob. (4.10)-(4.15) has a globally defined forward solution, which is Lipschitz continuous.*

The solution is constructed by starting from the initial value and stitching together solutions of constant state. Continuity assumptions on q and v bound the number of state switches on compact intervals, for any $\epsilon > 0$. Details are given in the appendix.

Lemma 4.3. *(Distances between vehicles go to zero) Let* $y_i = \frac{1}{\rho_i}$ *where* ρ_i *is computed from FC-FtL with* $y_i(0) \geq 1$ *for* $i \in \{1, ..., M-1\}$. *Assume that (4.12) holds and* q *satisfies (4.14),(4.15). Then, there exists* $\epsilon_1 > 0$ *such that when the hysteresis parameter satisfies* $0 < \epsilon \leq \epsilon_1$

$$1 \leq y_i(t) \leq 2^{\frac{\sigma-1}{\sigma}} \left(\frac{\sigma t}{l} + (\hat{C} + y_i(0))^\sigma \right)^{\frac{1}{\sigma}}\ \textit{for}\ i \in \{1, ..., M-1\} \backslash \{i_1, ..., i_Q\}. \tag{4.19}$$

σ is given in assumption (4.12), Q is less than or equal to the number of intervals where q is zero and $\hat{C} > 0$ is a constant independent of l. If $\kappa > \frac{1}{\sigma}$ and $y_i(0) \le C$ for some positive constant C independent of $i \in \{1, ..., M-1\}$ and l, then

$$\max_{t\in[0,T], i\in\{1,...,M-1\}\setminus\{i_1,...,i_Q\}} l^\kappa y_i(t) \to 0 \text{ as } l \to 0. \tag{4.20}$$

The proof is left to the appendix.

Lemma 4.4. *Under assumptions (4.9) and (4.11), the FtL density has the invariant region*

$$0 \le \rho^l \le 1 \ \forall (x,t) \in \mathscr{R} \times \mathscr{R}^+. \tag{4.21}$$

Furthermore,

$$\operatorname{supp} \rho^l(\cdot, t) \subset [\overline{x}_{\min}, \overline{x}_{\max} + T], \tag{4.22}$$

for any $t \in [0, T]$.

Proof. From (4.11), it follows that

$$\rho_i(0) = \frac{l}{x_{i+1/2}(0) - x_{i-1/2}(0)} = ⨍_{x_{i-1/2}(0)}^{x_{i+1/2}(0)} \rho_0 dx \le 1, \tag{4.23}$$

from which Lemma 4.3 gives the invariant region. The left-limit of (4.22) holds because the particles are moving rightwards. By (4.12), the velocity of the leader vehicle is less than or equal to one, which gives the right-limit. □

4.2 Strong compactness of the Follow-the-Leader method

Compactness of the FC-FtL model is with the same approach as in Chapter 3. Let

$$\begin{aligned} v_i &= v(\rho_i) && \text{the LWR velocity,} \\ v_{i-1/2} &= v(\rho_{i-1}, \rho_i) && \text{the FtL velocity.} \end{aligned} \tag{4.24}$$

Lemma 4.5. *(Variation estimates) For $0 \le T < \infty$,*

$$\begin{aligned} \int_0^T \sum_{i=1}^{M-1} \int_{\rho_i}^{\rho_{i+1}} \rho \left(v(\rho) - v_{i+1}\right) d\rho\, dt &\le \frac{1}{2} \|\rho_0\|_{L^1(\mathscr{R})} + \frac{1}{2} T, \\ \int_0^T \sum_{i=1}^{M-1} \rho_i^2 \left(v_{i+1/2} - v_{i-1/2}\right)^2 dt &\le (1 + L_v)\left(\|\rho_0\|_{L^1(\mathscr{R})} + 7T\right), \end{aligned} \tag{4.25}$$

where L_v is the Lipschitz constant of v.

The proof is similar to the one given for Lemma 3.3, and is left to the appendix. As the FC-FtL model reduces to the original Follow-the-Leader model away from the interface, the compactness proof is shorter than in Chapter 3.

Lemma 4.6. *($W^{-1,2}_{loc}$ Compactness) For any function $S(\rho) \in \mathscr{C}^2([0,1])$,*

$$\left\{\partial_t S\left(\rho^l\right) + \partial_x Q\left(\rho^l\right)\right\}_{l>0}, \tag{4.26}$$

lies in a compact subset of $W^{-1,2}_{loc}\left(\mathscr{R} \times \mathscr{R}_0^+\right)$, where $Q'(\rho) = S'(\rho)(\rho v(\rho))'$.

Proof. Let Ω be an arbitrary open set of $\mathscr{R}\times\mathscr{R}^+$ of class $\mathscr{C}^1$, see [9, p. 298]. Let $\phi \in \mathscr{C}_c^\infty([-X,X]\times[0,T])$, $x_{-1/2}=-\infty$, $x_{M+1/2}=\infty$, $\rho_0=\rho_M=0$ and $\phi_{M-1/2}=\phi_{-1/2}=0$. Consider first

$$\int_{\mathscr{R}\times\mathscr{R}^+} S\left(\rho^l\right)\partial_t\phi\, dx\, dt = \int_0^\infty \sum_{i=0}^{M}\int_{x_{i-1/2}}^{x_{i+1/2}} S_i\phi_t\, dx\, dt. \tag{4.27}$$

From the product rule and Leibniz rule for integration,

$$\begin{aligned}\int_{x_{i-1/2}}^{x_{i+1/2}} S_i\phi_t\, dx &= -S_i\Delta_+\left(v_{i-1/2}\phi_{i-1/2}\right) + \int_{x_{i-1/2}}^{x_{i+1/2}} S_i'\rho_i^2 D_+\left(v_{i-1/2}\right)\phi(x)dx + \frac{d}{dt}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} S_i\phi\, dx\right)\\ &= -\Delta_+\left(S_i v_{i-1/2}\phi_{i-1/2}\right) + S_i'\rho_i\Delta_+\left(v_{i-1/2}\right)\fint_{x_{i-1/2}}^{x_{i+1/2}}\left(\phi(x)-\phi_{i+1/2}\right)dx\\ &\quad+\left(S_i'\rho_i\Delta_+\left(v_{i-1/2}\right)+\Delta_+\left(S_i\right)v_{i+1/2}\right)\phi_{i+1/2} + \frac{d}{dt}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} S_i\phi\, dx\right). \end{aligned}\tag{4.28}$$

The formula also holds for the edge cases $i\in\{0,M\}$.

$$\Delta_+\left(S_i'\rho_i\right) = \Delta_+\left(S_i\right) + \int_{\rho_i}^{\rho_{i+1}} S''(\rho)\rho\, d\rho, \tag{4.29}$$

shows that the right-hand side equals

$$\begin{aligned}&-\Delta_+\left(S_i v_{i-1/2}\phi_{i-1/2}\right) + S_i'\rho_i\Delta_+\left(v_{i-1/2}\right)\fint_{x_{i-1/2}}^{x_{i+1/2}}\left(\phi(x)-\phi_{i+1/2}\right)dx\\ &+\left(\Delta_+\left(S_i'\rho_i v_{i-1/2}\right) - \int_{\rho_i}^{\rho_{i+1}} S''(\rho)\rho v_{i+1/2}\, d\rho\right)\phi_{i+1/2} + \frac{d}{dt}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} S_i\phi\, dx\right).\end{aligned}\tag{4.30}$$

The sum over the first term is zero. Let

$$\left\langle \mathcal{L}_1^l,\phi\right\rangle := \int_0^T \sum_{i=1}^{M-1} S_i'\rho_i\Delta_+\left(v_{i-1/2}\right)\fint_{x_{i-1/2}}^{x_{i+1/2}}\left(\phi(x)-\phi_{i+1/2}\right)dx\, dt. \tag{4.31}$$

An appliation of the Hölder inequality and using $S\in\mathscr{C}^1([0,1])$ gives

$$\begin{aligned}\left|\left\langle \mathcal{L}_1^l,\phi\right\rangle\right| &\le C\int_0^T\left\{\sum_{i=1}^{M-1}\rho_i^2\Delta_+\left(v_{i-1/2}\right)^2\right\}^{\frac12}\left\{\sum_{i=1}^{M-1}\left(\fint_{x_{i-1/2}}^{x_{i+1/2}}\left(\phi(x)-\phi_{i+1/2}\right)dx\right)^2\right\}^{\frac12} dt\\ &\le C_1\int_0^T\left\{\sum_{i=1}^{M-1}\rho_i^2\Delta_+\left(v_{i-1/2}\right)^2\right\}^{\frac12} dt \max_{t\in[0,T],i\in\{1,\dots,M-1\}}\left(l y_i\right)^{\frac{2\alpha-1}{2}}\|\phi\|_{\mathscr{C}_0^{0,\alpha}(\Omega)}\\ &\le C T^{\frac12}\left\{\int_0^T\sum_{i=1}^{M-1}\rho_i^2\Delta_+\left(v_{i-1/2}\right)^2 dt\right\}^{\frac12}\max_{t\in[0,T],i\in\{1,\dots,M-1\}}\left(l y_i\right)^{\frac{2\alpha-1}{2}}\|\phi\|_{\mathscr{C}_0^{0,\alpha}(\Omega)}\\ &\le C_3\max_{t\in[0,T],i\in\{1,\dots,M-1\}}\left(l y_i\right)^{\frac{2\alpha-1}{2}}\|\phi\|_{\mathscr{C}_0^{0,\alpha}(\Omega)} \to 0 \text{ as } l\to 0.\end{aligned}\tag{4.32}$$

The constant $C_1 = C\left(\overline{x}_{\max}+T-\overline{x}_{\min}\right)^{\frac12}>0$, comes from application of Lemma 4.4. The third inequality is application of Jensen's inequality, for the function $x^{\frac12}$ on $x\ge 0$. The final bound is established by invoking the second estimate of Lemma 4.5. A corollary of Morrey's theorem [9, Cor. 9.14, p. 285]) is that $W^{1,p}(\Omega) \subset C^{0,\alpha}(\Omega)$ is a continuous injection for $\alpha \in (0, 1-\frac{2}{p})$, when Ω is bounded and of class $\mathscr{C}^1$. Furthermore, [9, Thm. 9.17, p. 288] and Poincaré's inequality [9, Cor. 9.19, p. 290] imply that $W_0^{1,p}(\Omega) \subset C_0^{0,\alpha}(\Omega)$ is a continuous injection. Therefore, when $p > \frac{2}{1-\alpha}$, $\alpha \in \left(\frac{1}{2}, 1\right)$, $\left\{\mathcal{L}^l\right\}_{l>0}$ is compact in $W^{-1,q}(\Omega)$ for $q \in \left(1, \frac{2}{1+\alpha}\right), \alpha \in \left(\frac{1}{2}, 1\right)$. Since $0 \leq \rho^l \leq 1$, the sequence $\left\{\mathcal{L}^l\right\}_{l>0}$ is bounded in $W^{-1,r}(\Omega)$ for $r > 2$, and [26, Lemma 3.3] implies

$$\left\{\mathcal{L}_1^l\right\}_{l>0} \text{ is compact in } W^{-1,2}(\Omega). \tag{4.33}$$

Consider next

$$\begin{aligned} \int_0^T \sum_{i=0}^M \int_{x_{i-1/2}}^{x_{i+1/2}} Q_i \phi_x \, dx dt &= \int_0^T \sum_{i=0}^M Q_i \Delta_+ \left(\phi_{i-1/2}\right) dx dt \\ &= \int_0^T \sum_{i=0}^{M-1} -\Delta_+ (Q_i) \phi_{i+1/2} dx dt, \end{aligned} \tag{4.34}$$

where

$$\Delta_+ (Q_i) = \Delta_+ \left(S_i' \rho_i v_i\right) - \int_{\rho_i}^{\rho_{i+1}} S''(\rho) \rho v(\rho) \, d\rho. \tag{4.35}$$

Add the right-hand side to the sum over the third term of Exp. (4.30), and integrate in time. Let

$$\left\langle \mathcal{L}_2^l, \phi \right\rangle := \int_0^\infty \sum_{i=0}^{M-1} \left(\Delta_+ \left(S_i' \rho_i \left(v_{i-1/2} - v_i\right)\right) + \int_{\rho_i}^{\rho_{i+1}} S''(\rho) \rho \left(v(\rho) - v_{i+1/2}\right) d\rho \right) \phi_{i+1/2} dt. \tag{4.36}$$

The number of terms involving $v_{i-1/2}$ where

$$v_{i-1/2} \neq v_i, \tag{4.37}$$

can be bounded uniformly. Their sum is bounded by $C \|\phi\|_{L^\infty}$. For the remaining indices, the first term of the sum is zero. The sum over the second term is bounded by $C \|\phi\|_{L^\infty}$, by the first estimate of Lemma 4.5 and $S'' \in \mathscr{C}([0,1])$. Hence,

$$\left| \left\langle \mathcal{L}_2^l, \phi \right\rangle \right| \leq C \|\phi\|_{L^\infty(\Omega)} \ \forall \phi \in \mathscr{C}_0(\Omega). \tag{4.38}$$

The fourth term of Exp. (4.30) sum up to

$$\left\langle \mathcal{L}_3^l, \phi \right\rangle = -\int_{\mathscr{R}} S\left(\rho^l\right) \phi(x, 0) \, dx, \tag{4.39}$$

when integrated in time.

$$\left| \left\langle \mathcal{L}_3^l, \phi \right\rangle \right| \leq C(\Omega) \|\phi\|_{L^\infty(\Omega)} \ \forall \phi \in \mathscr{C}_0(X), \tag{4.40}$$

which shows that

$$\left\| \mathcal{L}_2^l \right\|_{\mathcal{M}(\Omega)}, \left\| \mathcal{L}_3^l \right\|_{\mathcal{M}(\Omega)} \leq C, \tag{4.41}$$

where $\mathcal{M}(\Omega) = (\mathscr{C}_c(\Omega))'$ is the space of signed radon measures of finite mass. Consider

$$\mathcal{L}^l = \mathcal{L}_1^l + \mathcal{L}_2^l + \mathcal{L}_3^l. \tag{4.42}$$

As $0 \leq \rho^l \leq 1$, the sequence $\{\mathcal{L}^l\}_{l>0}$ is bounded in $W^{-1,r}(\Omega)$ for $r > 2$. In addition, since Ineq. (4.33) and Ineq. (4.41) hold, $\mathcal{L}^l$ is a sum of a sequence of uniformly bounded measures and a precompact sequence in $W^{-1,2}(\Omega)$. An application of [16, Cor. 1, p. 8] shows

$$\{\mathcal{L}^l\}_{l>0} \text{ is compact in } W^{-1,2}(\Omega), \tag{4.43}$$

Since Ω was an arbitrary bounded open set of class $\mathscr{C}^1$, this proves the lemma. □

4.3 Convergence to weak solutions

From Lemma 3.6, it holds that

$$\rho^l(x,0) \overset{\mathscr{D}'(\mathscr{R})}{\rightharpoonup} \rho_0. \tag{4.44}$$

Up to a subsequence, the FC-FtL model converges to a weak solution of the unconstrained LWR model.

Theorem 4.7. *(Convergence to the weak solution) Let $\rho^l = \rho(x,t)$ be given in (4.49). Assume v satisfies (4.12). In addition, assume $v \in \mathscr{C}^2$, $\rho v(\rho)$ is genuinely non-linear and ρ_0 satisfies (4.9) and There exists a subsequence such that, for any finite $T > 0$,*

$$\rho^l \to \rho \text{ in } L^p(\mathscr{R} \times [0,T]) \text{ as } l \to 0, \text{ for any } 1 \leq p < \infty, \tag{4.45}$$

and $\rho \in L^\infty(\mathscr{R} \times \mathscr{R}^+)$ is a weak solution ρ of the Cauchy problem, i.e. ρ is a bounded measurable function satisfying $\forall \phi \in C_c^\infty(\mathscr{R} \times \mathscr{R}^+)$,

$$\int_{\mathscr{R}^+} \int_{\mathscr{R}} (\rho \phi_t + \rho v(\rho) \phi_x) \, dx dt + \int_{\mathscr{R}} \rho_0(x) \phi(x,0) dx = 0. \tag{4.46}$$

Proof. The fact that ρ^l converges to some $\rho \in L^\infty(\mathscr{R} \times [0,T])$ in $L^p_{\text{loc}}(\mathscr{R} \times [0,T])$ and pointwise a.e. follows from Lemma 2.8, by taking $k \equiv 1$. Ineq. (4.22) shows that convergence in L^p_{loc} implies convergence in L^p. A simple calculation reveals that

$$\begin{aligned} &\int_{x_{i-1/2}}^{x_{i+1/2}} \rho_i \phi_t + \rho_i v_{i-1/2} \phi_x dx \\ &\quad = \frac{d}{dt} \left(\int_{x_{i-1/2}}^{x_{i+1/2}} \rho_i \phi(x,t) dx \right) + \rho_i \Delta_+ \left(v_{i-1/2} \right) \fint_{x_{i-1/2}}^{x_{i+1/2}} \left(\phi(x) - \phi_{i+1/2} \right) dx, \end{aligned} \tag{4.47}$$

which shows that

$$\int_{\mathscr{R} \times \mathscr{R}^+} \rho^l \phi_t + \rho^l \tilde{v}^l \phi_x dx dt + \int_{\mathscr{R}} \rho^l(x,0) \phi(x,0) dx = \langle \mathcal{L}_1^l, \phi \rangle, \tag{4.48}$$

where

$$\tilde{v}^l := \sum_{i=1}^{M-1} v_{i-1/2} \mathbb{1}_{[x_{i-1/2}, x_{i+1/2})}. \tag{4.49}$$

The error associated with $v_{i-1/2} \neq v_i$ in $\tilde{v}^l$ vanishes in the limit. The right-hand side converges to zero, as was shown in Ineq. (4.32). As $\rho^l \to \rho$ in $L^1(\mathscr{R} \times [0,T])$ and (4.44) holds,

$$\int_{\mathscr{R}\times\mathscr{R}^+} \rho\phi_t + \rho v(\rho)\phi_x dxdt + \int_{\mathscr{R}} \rho_0(x)\phi(x,0)dx = 0. \tag{4.50}$$

□

It is assumed that

$$\left|\rho^l(x,0) - c\right| \overset{\mathscr{D}'(\mathscr{R})}{\rightharpoonup} |\rho_0 - c|, \text{ for any } c \in \mathscr{R}. \tag{4.51}$$

The limit is an entropy solution in the following sense

Theorem 4.8. *(The weak entropy solution) Assume (4.51) holds. For any $0 \le \phi \in \mathscr{C}_c^\infty(\mathscr{R} \times \mathscr{R}^+)$ and $c \in \mathscr{R}$, the limit ρ of Theorem 4.7 is bounded measurable function satisfying*

$$\begin{aligned} &\int_{\mathscr{R}\times\mathscr{R}^+} (|\rho - c|\phi_t - \operatorname{sign}(\rho - c)(\rho v(\rho) - cv(c))\phi_x)dxdt \\ &+ 2\int_0^\infty |c||v(c) - \hat{v}(t)|\phi(t,0)dt + \int_{\mathscr{R}} |\rho_0 - c|\phi(x,0)dx \ge 0. \end{aligned} \tag{4.52}$$

The unilateral constraint is satisfied in the sense of traces

$$f\left(\left(\gamma^-\rho\right)(0,t)\right) = f\left(\left(\gamma^+\rho\right)(0,t)\right) \le q, \ \textit{for a.e. } t \ge 0. \tag{4.53}$$

The proof is postponed to the end of the section. Condition (4.53) is well defined. For any test function ϕ such that $\phi(0,t) = 0$, Ineq. (4.52) shows that the limit is an entropy solution of

$$\rho_t + (\rho v(\rho))_x = 0 \text{ on } \Omega = (-\infty,0) \cup (0,\infty) \times (0,\infty), \tag{4.54}$$

in the sense of Kružkov. As the flux is $\mathscr{C}^2$ and genuinely non-linear, [3, Thm 2.2] ensures the existence of strong L^1-traces,

$$\lim_{h\downarrow 0} \frac{1}{h}\int_0^\infty \int_0^h \left|\rho(\pm x,t) - \left(\gamma^\pm\rho\right)(t)\right| \xi(t)dxdt = 0 \ \ \forall \xi \in \mathscr{C}_c^\infty([0,\infty)). \tag{4.55}$$

The definition of weak entropy solution given in Theorem 4.8 is very similar to the CG-entropy solution, which is obtained by replacing the second term of the left-hand side of Ineq. (4.52) with

$$2\int_0^\infty \left(1 - \frac{F(t)}{f_{\max}}\right) cv(c)\phi(t,0)dt. \tag{4.56}$$

The next result establishes that these notions are equivalent.

Theorem 4.9. *The weak entropy solution defined in (4.8) is the unique CG-entropy solution of (4.1).*

Since CG-entropy solutions are unique, the entire Follow-the-Leader sequence converges strongly to the limit. Theorem 4.9 can be proven by showing that the entropy solution is a $\mathcal{G}$-entropy solution [3, Def. 2.8, Prop. 2.6 A]. In $L^\infty(\mathscr{R} \times [0,T])$, the $\mathcal{G}$-entropy solution of Prob. (4.1) is the unique CG-entropy solution [3, Thm 2.9, 2.11].

Proposition 4.10. *($\mathcal{G}$-entropy solution) The entropy solution ρ of Theorem 4.8 is a Kružkov entropy solution for $x<0$ and $x>0$, i.e. for all non-negative test functions $\phi \in \mathscr{C}_c^{\infty}(\mathscr{R}\times[0,T]\backslash\{x=0\})$ and all $c\in\mathscr{R}$,*

$$\int_{\mathscr{R}\times\mathscr{R}^+} (|\rho-c|\,\phi_t - \operatorname{sign}(\rho-c)(\rho v(\rho)-cv(c))\,\phi_x)\,dxdt + \int_{\mathscr{R}} |\rho_0-c|\,\phi(x,0)dx \geq 0. \tag{4.57}$$

In addition, for a.e. $t>0$,

$$\big((\gamma^-\rho)(t),(\gamma^+\rho)(t)\big)\in\mathcal{G}(q(t)), \tag{4.58}$$

where $\mathcal{G}(q)=\mathcal{G}_1(q)\cup\mathcal{G}_2(q)\cup\mathcal{G}_3(q)\subset[0,1]^2$ is the admissibility germ of Prob. (4.1). For $q\in[0,f_{\max}]$,

$$\begin{aligned}
&\bullet\,\mathcal{G}_1(q)=\{(c_l,c_r)\in[0,1]^2; c_l>c_r, f(c_l)=f(c_r)=q\},\\
&\bullet\,\mathcal{G}_2(q)=\{(c,c)\in[0,1]^2; f(c_r)\leq q\},\\
&\bullet\,\mathcal{G}_3(q)=\{(c_l,c_r)\in[0,1]^2; c_l<c_r, f(c_l)=f(c_r)\leq q\}.
\end{aligned} \tag{4.59}$$

The singelton $\mathcal{G}_1(q)$ is the concrete extension of the classical LWR model, which corresponds to an admissable non-classical shock at the interface.

Proof. (Theorem 4.9/ Proposition 4.10) The proof is an adaptation of the proof of [3, Prop 2.5]. The Ineq. (4.57) follows from Ineq. (4.52). Condition (4.53) implies Rankine-Hugoniot across the interface $\{x=0\}$. If

$$\big((\gamma^-\rho)(t),(\gamma^+\rho)(t)\big)\notin\mathcal{G}(q(t)), \tag{4.60}$$

the only possibility is that

$$f\big((\gamma^-\rho)(t)\big)=f\big((\gamma^+\rho)(t)\big)<q \;\text{ and }\; (\gamma^-\rho)(t)>(\gamma^+\rho)(t). \tag{4.61}$$

Consider the cut function

$$\psi_\epsilon(x)=\begin{cases}1 & \text{if } |x|<\epsilon\\ 2-\dfrac{|x|}{\epsilon} & \text{if } \epsilon\leq|x|<2\epsilon\\ 0 & \text{if } |x|\geq 2\epsilon,\end{cases} \tag{4.62}$$

which is Lipschitz. Using a standard mollifier, it follows by approximation that Ineq. (4.52) is satisfied for $\phi=\psi_\epsilon(x)\xi(t)$, where $\xi\in\mathscr{C}_c^{\infty}((0,\infty))$.

$$\begin{aligned}
&I(\epsilon)+J(\epsilon)\geq 0,\\
&I(\epsilon)=\int_0^\infty\int_{\mathscr{R}}(|\rho-c|)\,\xi'\psi_\epsilon\,dxdt\\
&J(\epsilon)=\int_0^\infty\int_{\mathscr{R}}F(\rho)\xi\psi'_\epsilon\,dxdt+2\int_0^\infty|c|\,|v(c)-\hat{v}(t)|\,\phi(t,0)dt,
\end{aligned} \tag{4.63}$$

where

$$\begin{aligned}F(x):&=\operatorname{sign}(x-c)(xv(x)-cv(c))\\ &=|x-c|\,v(x)-c\,|v(x)-v(c)|,\end{aligned} \tag{4.64}$$

$I(\epsilon)\to 0$ as $\epsilon\to 0$. Furthermore[1],

$$\lim_{\epsilon\to 0} J(\epsilon)=\int_0^\infty \left(F\left(\left(\gamma^-\rho\right)(t),c\right)-F\left(\left(\gamma^+\rho\right)(t),c\right)+|c|\,|v(c)-\hat v(t)|\right)\xi(t)dt, \tag{4.65}$$

which implies

$$F\left(\left(\gamma^-\rho\right)(t),c\right)-F\left(\left(\gamma^+\rho\right)(t),c\right)+|c|\,|v(c)-\hat v(t)|\geq 0, \tag{4.66}$$

for a.e. $t\geq 0, \forall c\in\mathscr{R}$. Eq. (4.65) implies Ineq. (4.66) for a.e. $t\geq 0$, for a given $c\in\mathscr{R}$. Therefore it holds a.e. $\forall c\in\mathscr{Q}$. The left-hand side is continuous with respect to c, which implies a.e. $\forall c\in\mathscr{R}$. Let

$$\hat\rho(t)=\max\{\rho\in[0,1];\rho v(\rho)=q(t)\}, \qquad \hat v(t)=v(\hat\rho(t)). \tag{4.67}$$

Let $K\subset[0,\infty)$ be the set where (4.61) holds. On K, $\left(\gamma^+\rho\right)(t)\leq\hat\rho(t)\leq\left(\gamma^-\rho\right)(t)$ and

$$\begin{aligned}0 &\leq F\left(\left(\gamma^-,\rho\right)(t),\hat\rho(t)\right)-F\left(\left(\gamma^+\rho\right)(t),\hat\rho(t)\right.\\ &=f\left(\left(\gamma^-\rho\right)(t)\right)+f\left(\left(\gamma^+\rho\right)(t)\right)-2\hat\rho v(\hat\rho)\\ &=2\left(f\left(\left(\gamma^-\rho\right)(t)\right)-q(t)\right),\end{aligned} \tag{4.68}$$

which is a contradiction. K must be a null set, which establishes

$$\left(\left(\gamma^-\rho\right)(t),\left(\gamma^r\rho\right)(t)\right)\in\mathcal{G}(q(t)) \text{ for a.e. } t\geq 0. \tag{4.69}$$

□

The section is concluded with a proof of Theorem 4.8.

Proof. (Theorem 4.8) Consider first

$$\int_{\mathscr{R}\times\mathscr{R}^+}\mu^l\phi_t dxdt=\int_0^\infty\sum_{i=0}^M\int_{x_{i-1/2}}^{x_{i+1/2}}\mu_i\phi_t dxdt. \tag{4.70}$$

where $x_{-1/2}=-\infty$, $x_{M+1/2}=+\infty$ and

$$\mu(x)=|x-c|. \tag{4.71}$$

Since ϕ is smooth and $\mu_i, x_{i-1/2}, x_{i+1/2}$ are Lipschitz, the following idenity holds a.e. [2]

$$\int_{x_{i-1/2}}^{x_{i+1/2}}\mu_i\phi_t dx=\frac{d}{dt}\left(\mu_i\int_{x_{i-1/2}}^{x_{i-1/2}}\phi_t dx\right)-\dot\mu_i\int_{x_{i-1/2}}^{x_{i+1/2}}\phi dx-\mu_i\Delta_+\left(\phi_{i-1/2}v_{i-1/2}\right), \tag{4.73}$$

[1] For a continuous function θ on $[0,1]$, the trace of the composition is the composition of traces.

$$\lim_{h\downarrow 0}\frac{1}{h}\int_0^\infty\int_0^h\left|\theta\circ\rho(\pm x,t)-\theta\circ\left(\gamma^\pm\rho\right)(t)\right|\xi(t)dxdt=0\ \forall\xi\in\mathscr{C}_c^\infty([0,\infty)).$$

A proof is given under [3, Eq. 9].

[2] Let $\mu_i=\mu(\rho_i)$, then the a.e. derivative is given by

$$\dot\mu_i=-\operatorname{sign}(\rho_i-c)\rho_i^2 D_+\left(v_{i,i-1/2}\right), \tag{4.72}$$

with the convention that $\operatorname{sign}(0)=0$. Since μ_i is lipschitz continuous, the fundamental theorem of calculus holds [34, ex. 1.6.44, p. 169].

for $i \in \{0, \ldots, M\}$. The formula holds for the end-cases, since $\phi_{-1/2} = \phi_{M+1/2} = 0$ and $\rho_0 = \rho_M = 0$. As a convention, let $v_{M+1/2} = v_{-1/2} = 0$.

$$\begin{aligned}
\int_0^\infty \sum_{i=0}^{M} \int_{x_{i-1/2}}^{x_{i+1/2}} \mu_i \phi_t dx dt &= -\int_{\mathscr{R}} \mu^l(x,0)\phi(x,0)dx - \sum_{i=0}^{M} \mu_i \Delta_+ \left(\phi_{i-1/2} v_{i-1/2}\right) \\
&+ \int_0^\infty \sum_{i=0}^{M} \operatorname{sign}(\rho_i - c)\, \rho_i^2 D_+ \left(v_{i-1/2}\right) \int_{x_{i-1/2}}^{x_{i+1/2}} \phi dx dt \\
&= -\int_{\mathscr{R}} \mu^l(x,0)\phi(x,0)dx + \int_0^\infty \sum_{i=0}^{M-1} \Delta_+ (\mu_i)\, \phi_{i+1/2} v_{i+1/2} dt \\
&+ \int_0^\infty \sum_{i=0}^{M-1} \operatorname{sign}(\rho_i - c)\, \rho_i \Delta_+ \left(v_{i-1/2}\right) \fint_{x_{i-1/2}}^{x_{i+1/2}} \left(\phi - \phi_{i+1/2}\right) dx dt \\
&+ \int_0^\infty \sum_{i=0}^{M-1} \operatorname{sign}(\rho_i - c)\, \rho_i \Delta_+ \left(v_{i-1/2}\right) \phi_{i+1/2} dt. \qquad (4.74)
\end{aligned}$$

Combine the second and fourth term

$$\begin{aligned}
&\sum_{i=0}^{M-1} \left(\operatorname{sign}(\rho_i - c)\, \rho_i \Delta_+ \left(v_{i-1/2}\right) + v_{i+1/2} \Delta_+ (\mu_i)\right) \phi_{i+1/2} \\
&= \sum_{i=0}^{M-1} \left(\operatorname{sign}(\rho_i - c)\, c \Delta_+ \left(v_{i-1/2}\right) + \Delta_+ \left(\mu_i v_{i-1/2}\right)\right) \phi_{i+1/2} \\
&= \sum_{i=0}^{M-1} \left(c \Delta_+ \left(\operatorname{sign}(\rho_i - c) \left(v_{i-1/2} - v(c)\right)\right) + \Delta_+ \left(\mu_i v_{i-1/2}\right)\right) \phi_{i+1/2} \\
&+ \sum_{i=0}^{M-1} -c \Delta_+ (\operatorname{sign}(\rho_i - c)) \left(v_{i+1/2} - v(c)\right) \phi_{i+1/2} \\
&= \sum_{i=0}^{M-1} \Delta_+ \left(F_{i-1/2}\right) \phi_{i+1/2} + \sum_{i=0}^{M-1} -c \Delta_+ (\operatorname{sign}(\rho_i - c)) \left(v_{i+1/2} - v(c)\right) \phi_{i+1/2}. \qquad (4.75)
\end{aligned}$$

Consider $i \in \{1, \ldots, M-1\} \setminus \{k-1, k\}$, where $x_{k-1/2}$ is NVtP. Then

$$\begin{aligned}
&-c \Delta_+ (\operatorname{sign}(\rho_i - c)) \left(v_{i+1/2} - v(c)\right) \\
&= c \left|v_{i+1} - v(c)\right| \left(1 - \operatorname{sign}(\rho_{i+1} - c) \operatorname{sign}(\rho_i - c)\right) \geq 0. \qquad (4.76)
\end{aligned}$$

The case $c \leq 0$ is implied by the fact that $\rho_i \geq 0\ \forall i \in \{1, \ldots, M-1\}$. For $i \in \{k-1, k\}$, the terms can be bounded as in (4.76), if $v_{i+1/2} = v(\rho_{i+1})$. If not, then either (or both) of the terms equal

$$\begin{aligned}
&-c \Delta_+ (\operatorname{sign}(\rho_k - c)) (\hat{v} - v(c)) \text{ for } i = k, \\
&-c \Delta_+ (\operatorname{sign}(\rho_{k+1} - c)) (\hat{v} - v(c)) \text{ for } i = k+1, \qquad (4.77)
\end{aligned}$$

respectively. In any case, the second sum on the right-hand side of (4.75) can be bounded below by

$$-2\,|c|\,|\hat{v} - v(c)|\,\phi(0,t) - C\left|\phi(0,t) - \phi_{k+1/2}\right| - C\left|\phi(0,t) - \phi_{k-1/2}\right|, \qquad (4.78)$$

for some uniform finite positive $C>0$. Next, let $F^l = F\left(\rho^l\right)$ and consider

$$\int_{\mathscr{R}\times\mathscr{R}^+} F^l \phi_x dx dt = \int_0^\infty \sum_{i=0}^{M} F_i \Delta\left(\phi_{i-1/2}\right)$$
$$= \int_0^\infty \sum_{i=0}^{M-1} -\Delta_+\left(F_i\right)\phi_{i+1/2}. \tag{4.79}$$

Add (4.74) and (4.79), and use Eq. (4.75) and Ineq. (4.77).

$$\int_{\mathscr{R}\times\mathscr{R}^+} \mu^l \phi_t + F^l \phi_x dx dt + \int_{\mathscr{R}} \mu^l(x,0)\phi(x,0)dx + 2\int_0^\infty |c|\,|\hat{v}-v(c)|\,\phi(0,t)dt \geq A. \tag{4.80}$$

The right-hand side can be split

$$A = A_1 + A_2, \tag{4.81}$$

with

$$\begin{cases} A_1 & := \int_0^\infty \sum_{i=1}^{M-1} \operatorname{sign}\left(\rho_i - c\right)\rho_i \Delta_+\left(v_{i-1/2}\right) \fint_{x_{i-1/2}}^{x_{i+1/2}} \left(\phi - \phi_{i+1/2}\right) dx dt, \\ A_2 & := \int_0^\infty \sum_{i=0}^{M-1} \left(\Delta_+\left(F_{i,i-1/2} - F_i\right)\phi_{i+1/2}\right) - C\left|\phi(0,t) - \phi_{k+1/2}\right| - C\left|\phi(0,t) - \phi_{k-1/2}\right| dt. \end{cases} \tag{4.82}$$

First, consider the case where q is bounded away from zero.

$$\begin{aligned} |A_1| &\leq \int_0^T \sum_{i=1}^{M-1} \rho_i \left|\Delta_+\left(v_{i-1/2}\right)\right| \left|\fint_{x_{i-1/2}}^{x_{i+1/2}} \left(\phi - \phi_{i+1/2}\right) dx\right| dt \\ &\leq \int_0^T \left\{\sum_{i=1}^{M-1} \rho_i^2 \left(\Delta_+\left(v_{i-1/2}\right)\right)^2\right\}^{\frac{1}{2}} \left\{\sum_{i=1}^{M-1} \|\phi_x\|_\infty^2 \left(x_{i+1/2} - x_{i-1/2}\right)^2\right\}^{\frac{1}{2}} dt \\ &\leq C_1 \int_0^T \left\{\sum_{i=1}^{M-1} \rho_i^2 \left(\Delta_+\left(v_{i-1/2}\right)\right)^2\right\}^{\frac{1}{2}} dt \max_{i\in\{1,\ldots,M-1\}} \left(l y_i\right)^{\frac{1}{2}} \\ &\leq C \max_{i\in\{1,\ldots,M-1\}} \left(l y_i\right)^{\frac{1}{2}} \to 0 \text{ as } l \to 0. \end{aligned} \tag{4.83}$$

The second inequality is an application of Hölder's inequality. The constant

$$C_1 = \|\phi_x\|_\infty \left(\overline{x}_{\max} + T - \overline{x}_{\min}\right)^{\frac{1}{2}} > 0, \tag{4.84}$$

comes from application of Lemma 4.4. The final estimate is an application of Jensen's inequality and Lemma 4.5. Let q be zero on Q non-degenerate intervals, where Q is a finite number. For each $l>0$, there may exists a finite number of vacuum regions $i \in \{i_1, \ldots, i_Q\}$. Any term in the sum of A_1 goes to zero, as can be seen by

$$\left|\rho_i \Delta_+\left(v_{i-1/2}\right) \fint_{x_{i-1/2}}^{x_{i+1/2}} \left(\phi - \phi_{i+1/2}\right) dx\right| \leq \|\phi_x\|_{L^\infty}\, l \to 0. \tag{4.85}$$

The sum over $\{i_1, \ldots, i_Q\}$ therefore converges to zero, and can be removed before using Hölder's inequality in (4.83),

$$|A_1| \leq \left(C \max_{i\in\{1,\ldots,M-1\}\setminus\{i_1,\ldots,i_Q\}} \left(l y_i\right)^{\frac{1}{2}} + Q l\right) \|\phi_x\|_{L^\infty} \to 0 \text{ as } l \to 0. \tag{4.86}$$

Next, consider

$$|A_2| \le \int_0^\infty \sum_{i=k}^{k+1} \left|\Delta_+\left(\phi_{i-1/2}\right)\left(F_{i,i-1/2} - F_i\right)\right| dt + C\left|\phi(0,t) - \phi_{i-1/2}\right| dt, \tag{4.87}$$

where $k, k+1$ correspond to NVtP and NVP at time $t \ge 0$, respectively. First,

$$\left|\Delta_+\left(\phi_{i-1/2}\right)\left(F_{i,i-1/2} - F_i\right)\right| = \left|\rho_i\left(v_{i-1/2} - v_i\right)\Delta_+\left(\phi_{i-1/2}\right)\right| \le \|\phi_x\|_\infty l. \tag{4.88}$$

If q is zero, both $v_{k-1/2} = v_{k+1/2} = 0$. A vacuum region is always created between NVP and the next vehicle. Therefore, the estimate in Lemma 4.3 holds for all distances between vehicles to the left of NVP, for any $t \ge 0$.

$$|A_2| \le 2\|\phi_x\|_\infty\left(l + C \max_{t\in[0,T], i\in\{1,\dots,k(t)\}}(l y_i)\right) \to 0 \text{ as } l \to 0. \tag{4.89}$$

It has been shown that

$$\begin{aligned}\liminf_{l\to 0}\int_{\mathscr{R}\times\mathscr{R}^+} \mu^l\phi_t + F^l\phi_x dx dt + 2\int_0^\infty |c|\,|\hat{v} - v(c)|\,\phi(0,t)dt \\ + \int_{\mathscr{R}} \mu^l(x,0)\phi(x,0)dx \ge 0.\end{aligned} \tag{4.90}$$

Let ρ be the weak solution of Lemma 4.7. As $\rho^l \to \rho$ in $L^1(\mathscr{R}\times[0,T])$, $\mu^l \to \mu(\rho)$ and $F^l \to F(\rho)$ in $L^1(\mathscr{R}\times[0,T])$, by continuity of μ and F. Furthermore, if (4.51) holds, then Ineq. (4.52) is proven. To prove (4.53), a weak characterisation of the flux trace at $\{x=0\}$ is used.[3] It is to be shown that

$$\mp\int_0^T\int_{\mathscr{R}^\pm} \rho\partial_t(\psi\xi) + f(\rho)\partial_x(\psi\xi)\,dx dt \le \int_0^T q\xi(t)dt, \tag{4.91}$$

where $\xi \in \mathscr{C}_c^\infty(\mathscr{R}^+)$, $\xi(t)\ge 0$ and $\psi \in \mathscr{C}_c^\infty(\mathscr{R}), \psi(x) \ge 0$, $\psi(0) = 1$. For $i \in \{0,\dots,M\}$,

$$\begin{aligned}&\int_{x_{i-1/2}}^{x_{i+1/2}} \rho_i\phi_t + \rho_i v_{i-1/2}\phi_x dx \\ &\quad = \frac{d}{dt}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} \rho_i\phi(x,t)dx\right) + \rho_i\Delta_+\left(v_{i-1/2}\right)\fint_{x_{i-1/2}}^{x_{i+1/2}}\left(\phi(x) - \phi_{i+1/2}\right)dx,\end{aligned} \tag{4.92}$$

for a general test function $\phi \in \mathscr{C}_c^\infty\left(\mathscr{R}\times\mathscr{R}_0^+\right)$. Let $x_{k+1/2}$ be NVP for some $t \ge 0$. After some work, it can be seen that

$$\begin{aligned}-\int_0^{x_{k+1/2}} &\rho_k\phi_t + \rho_k v_{k-1/2}\phi_x dx \\ &= -\frac{d}{dt}\left(\int_0^{x_{k+1/2}} \rho_k\phi(x,t)dx\right) - \rho_k v_{k-1/2}\left(\phi_{k+1/2} - \phi(0)\right) \\ &+ \left((1 - T_k(t))v_{k+1/2} + T_k(t)v_{k-1/2}\right)\rho_k\phi_{k+1/2} \\ &- \rho_k\Delta_+\left(v_{k-1/2}\right)\frac{x_{k+1/2}}{x_{k+1/2} - x_{k-1/2}}\fint_0^{x_{i+1/2}}\left(\phi(x) - \phi_{i+1/2}\right)dx,\end{aligned} \tag{4.93}$$

[3] See [3, Rmk. 2].

where the relative closeness function T_k has been introduced

$$T_k(t) = \frac{x_{k+1/2}(t)}{x_{k+1/2}(t) - x_{k-1/2}(t)} \in [0,1]. \tag{4.94}$$

The sum over Eq. (4.92) for $i = \{k+1, ..., M\}$ and Eq. (4.93) gives

$$\begin{aligned} &-\int_{\mathscr{R}^+} \rho^l \phi_t + f(\rho^l)\phi_x dx \\ &\qquad = \frac{d}{dt}\left(-\int_{\mathscr{R}^+} \rho^l \phi dx\right) + \big((1 - T_k(t))v_{k+1/2} + T_k(t)v_{k-1/2}\big)\rho_k \phi(0,t) + r(l), \end{aligned} \tag{4.95}$$

where $r(l)$ goes to zero as $l \to 0$, by Ineq. (4.86) and Ineq. (4.89). Let $\phi(x,t) = \xi(t)\psi(x)$. Since $\xi(0) = 0$, the integral over the first term is zero. A similar calculation shows that the integral over $\mathscr{R}^-$ gives the same second term as in Eq. (4.95), with a remainder term which converges to zero. This proves that the Rankine-Hugoniot condition holds across the interface. The velocity function (4.5) was chosen such that second term of Eq. (4.95) is bounded above by $q(t)\xi(t)$.

$$-\int_{\mathscr{R}^+ \times \mathscr{R}^+} \rho^l (\xi\psi)_t + f(\rho^l)(\xi\psi)_x \, dx dt \le \int_0^\infty q\xi dt + r(l). \tag{4.96}$$

Since the convergence to the limit is strong, Ineq. (4.96) implies Ineq. (4.53). □

4.4 Further research

Several recent papers have investigated phenomenon related to capacity drop at exists, self-organisation and other related concepts, by considering non-local point constraints on the flux. See for example [2] and [6]. Two possible extensions of the results of this thesis is to consider a constraint which depends non-locally on the solution, and a constraint which is not bounded away from zero. The FC-FtL model would be improved if existence of the solution fo Prob. (4.10)-(4.15) was established without explicitly bounding the number of state switches. The current FC-FtL model also lacks a uniqueness result.

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Appendix A

Proofs from Chapter 4

Proof of Prop. 4.2

Proof. Two vehicles are adjacent to the bottleneck at $t = 0$, t, due to (4.11).The initial state is defined as uncongested. By fixing this configuration, the FC-FtL reduces to a system of ODEs with a Lipschitz continuous and bounded right-hand side, which has a unique globally defined solution. By Lemma 4.1, one of three things may happen when t increases. One or both vehicles can switch state, either immediately or for $t > 0$, the vehicle $x_{k-1/2}$ may cross $\{x = 0\}$, or neither happens on $(0, \infty)$. In the former case, let

$$\tilde{t} := \inf\{\, t \geq 0 \,|\, \text{NVP or NVtP switches state}\}, \tag{A.1}$$

switch the appropriate state(s) at $\tilde{t} \geq 0$ and take the solution with switched state for $t \geq \tilde{t}$, with the solution at the previous state(s) as initial value. If q is continuous until the next stitching time, then the constraints associated with state switches are uniformly continuous on compact intervals. A cycle in transition diagram corresponds to a difference in the constraint by at least $\epsilon f_{\max}$, and change in time by at least some $\delta > 0$, where δ is independent of time. Hence, the number of state switches is bounded on compact intervals. Since the number of discontinuous jumps in q is bounded, the number of extra switches can be bounded as well. This process can be continued until NVtP crosses the bottleneck or $t \to \infty$. In the former case, k can decremented by one and the process can be repeated, possibly until all vehicles have crosses the bottleneck. In the latter case, the solution can be defined on all of $\mathscr{R}$ by stitching together solutions of constant state. If all vehicles cross the bottleneck, the solution can be globally defined the end-state that all vehicles are uncongested. Since $v(\rho_{i-1}, \rho_i)$ is bounded by one, the vehicle paths are globally Lipschitz continuous. □

Proof of Lemma 4.3

Proof. The lower bound was proven in Lemma 4.1. Each time q jumps to zero, the NVP is stopped and the next vehicle has an unconstrained velocity. The space density between the NVP and the next vehicle cannot bounded above, a priori. The number of such occurrences are bounded by Q, and are not considered further. Let $x_{k-1/2}$ be the NVtP at $t = 0$. For $i \in \{1, ..., k\}$, define the crossing time $t_{i-1/2} \geq 0$ for $x_{i-1/2}$ at the interface. For $0 \leq t < t_{i+1/2}$,

$$\frac{dy_i}{dt} = \frac{v(\rho_i, \rho_{i+1}) - v(\rho_i)}{l} \leq \frac{1}{l y^{\sigma-1}}, \tag{A.2}$$

by assumption (4.12). Hence,

$$\frac{d\,(y_i)^\sigma}{dt} \le \frac{\sigma}{l}. \tag{A.3}$$

Consider $M_i = (t_{i+1/2}, t_{i-1/2})$ and let $\tilde{y}_\epsilon$ correspond to the largest value of y which would trigger a right-to-left transition in Figure 4.1, for constraint level $q = \tilde{\rho} v(\tilde{\rho})$ given in (4.15). For a fixed $\tilde{\rho}$, such a $\tilde{y}_\epsilon$ exists for sufficiently small $\epsilon > 0$. Hence, if $y_i(t) \ge \tilde{y}_\epsilon$ on M_i, then the vehicle is in the low-density regime and $v(\rho_{i-1}, \rho_i) = v(\rho_i)$. Assume $y_i(\hat{t}_1) > \max\left(\tilde{y}_\epsilon, y_i(t_{i+1/2})\right)$ for some $\hat{t}_1 \in M_i$. Then, by the intermediate value theorem, there exists some interval $I = (t_1, t_1 + \Delta t_1) \subset M_i$, $\Delta t_1 \ge 0$ such that $y_i(t_1) = \max(\tilde{y}_\epsilon, y_i(t_{i+1/2}))$, $y_i(t_1 + \Delta t_1) = y_i(\hat{t}_1)$ and $y_i(t) \ge \max(\tilde{y}_\epsilon, y_i(t_{i+1/2}))$ on I. Therefore, $v(\rho_{i-1}, \rho_i) = v(\rho_i)$, which implies

$$\frac{d\,(y_i)^\sigma}{dt} \le \frac{\sigma}{l} \text{ on } I. \tag{A.4}$$

Hence

$$(y_i)^\sigma(\hat{t}_1) = (y_i)^\sigma(t_1 + \Delta t_1) \le \frac{\sigma \Delta t_1}{l} + \max(\tilde{y}_\epsilon, y_i(t_{i+1/2}))^\sigma, \tag{A.5}$$

which together with Ineq.(A.3) gives

$$(y_i)^\sigma(t_{i-1/2}) \le \frac{\sigma t_{i-1/2}}{l} + \max(\tilde{y}_\epsilon, y_i(0))^\sigma. \tag{A.6}$$

Assume that $x_{i-1/2}$ in the left state of Figure 4.1 $\forall t \in M_{i-1}$, then

$$\frac{d\,(y_i)^\sigma}{dt} \le \frac{\sigma}{l} \text{ on } M_{i-1}. \tag{A.7}$$

and Ineq. (A.6) holds for $t_{i-1/2} \mapsto t_{i-3/2}$. If not, then by assumption (4.15), there exist $\tilde{t} \in M_{i-1}$ such that,

$$\rho_{i-1} v(\rho_i) \ge q(t) - \epsilon f_{\max} \ge \alpha > 0, \tag{A.8}$$

for $\epsilon, \alpha > 0$ sufficiently small. This can be used to bound the distance between NVtP and the interface.

$$0 - x_{i-3/2}(\tilde{t}) \le \frac{l}{\alpha}. \tag{A.9}$$

If there exists $I = (t_2, t_2 + \Delta t_2) \subset M_{i-1} \cap [\tilde{t}, \infty)$ satisfying $y_i(t_2) = \max(y(\tilde{t})), \tilde{y}_\epsilon)$ and $y_i(t) \ge y_i(t_2)$ for $t \in I$, then there exists $\beta > 0$ sufficiently small such that if $1 - \beta \le \rho_{i-1} \le 1$, then

$$\frac{d\rho_{i-1}}{dt} \le -\rho_{i-1}^2 \frac{\hat{v} - v(\rho_{i-1})}{l} \le -(1-\beta)^2 \frac{\hat{v} - v(1-\beta)}{l} < 0. \tag{A.10}$$

By the choice of $\tilde{y}_\epsilon$, the corresponding velocity satisfies $\tilde{v}_\epsilon \ge \hat{v}$. Due to (4.15), $\beta > 0$ can be chosen sufficiently small such that $v(\rho_{i-2}, \rho_{i-1}) = v(\rho_{i-1})$ for $\rho_{i-1} \ge 1 - \beta$. In addition, $\hat{v} - v(1-\beta) \ge \tilde{\alpha} > 0$ for sufficiently small β. This shows that $\rho_{i-1} \le 1 - \beta$ and $\dot{x}_{i-1/2}(t) \ge v(1-\beta)$, after $t_2 + \Delta t_3$ where

$$\Delta t_3 \le \frac{l\beta}{(\hat{v} - v(1-\beta))(1-\beta)^2} \le C_1(\beta) l, \tag{A.11}$$

for some uniform $C_1(\beta) > 0$. Therefore,

$$\Delta t_2 \leq \frac{l}{\alpha v(1-\beta)} + \Delta t_3 \leq Cl, \tag{A.12}$$

otherwise the NVtP will have crossed the bottleneck and $I \not\subset M_{i-1}$.

$$\frac{dy_i}{dt} \leq \frac{1}{l}, \tag{A.13}$$

which implies that

$$y_i(t) \leq C + \max(y(\tilde{t}), \tilde{y}_\epsilon) \text{ for } t \in I \tag{A.14}$$

Since $\sigma > 1$, the general inequality $(x+y)^\sigma \leq 2^{\sigma-1}(x^\sigma + y^\sigma)$ for $x, y \geq 0$ gives

$$\begin{aligned} y_i^\sigma(t_{i-3/2}) &\leq 2^{\sigma-1}(C^\sigma + \max(y_i(\tilde{t}), \tilde{y}_\epsilon)^\sigma) \\ &\leq 2^{\sigma-1}\left(\frac{\sigma t_{i-3/2}}{l} + (C + y_i(0) + \tilde{y}_\epsilon)^\sigma\right). \end{aligned} \tag{A.15}$$

The estimate also holds when Ineq. (A.7) holds. For $t \geq t_{i-3/2}$, Ineq. (A.3) holds, which gives the global bound

$$y_i^\sigma(t) \leq 2^{\sigma-1}\left(\frac{\sigma t}{l} + (C + y_i(0) + \tilde{y}_\epsilon)^\sigma\right). \tag{A.16}$$

The upper bound of Ineq. (4.19) is proven. □

Proof of Lemma 4.5

Proof. The proof is essentially the same the proof of Lemma 3.3.

$$\begin{aligned} \frac{d\rho_i}{dt} + D_+\left(Q_{i-1/2}\right) &= v_{i+1/2}D_+\left(\rho_i^2\right) - D_+\left(2\int_0^{\rho_i} \rho v(\rho)d\rho\right) \\ &= \frac{-2}{l}\int_{\rho_i}^{\rho_{i+1}} \rho\left(v(\rho) - v_{i+1}\right)d\rho \\ &\quad + \left(v_{i+1/2} - v_{i+1}\right)D_+\left(\rho_i^2\right), \end{aligned} \tag{A.17}$$

where

$$Q_{i-1/2} = \rho_i^2 v_{i-1/2} - \int_0^{\rho_i} 2\rho v(\rho)d\rho \leq Q(\rho_i) \leq 0 \tag{A.18}$$

For $x, y \geq 0$,

$$QV(x,y) = \int_x^y \rho\left(v(\rho) - v(y)\right)d\rho \geq \frac{x^2(v(x) - v(y))^2}{2(1+L_V)} \geq 0, \tag{A.19}$$

Insert

$$\frac{d}{dt}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\rho_i^2}{2}dx\right) = \frac{l}{2}\frac{d}{dt}\rho_i, \tag{A.20}$$

in (A.17), sum over $i \in \{1, ..., M-1\}$ and integrate in time.

$$\int_0^T \sum_{i=1}^{M-1} QV_{i-1/2} dt = -\int_0^T \sum_{i=1}^{M-1} \frac{d}{dt} \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\rho_i^2}{2} dx + \frac{1}{2}\Delta_+ \left(Q_{i-1/2}\right) + \frac{\left(v_{i+1/2} - v_{i+1}\right)}{2} \Delta_+ \left(\rho_i^2\right)$$

$$\le -\frac{1}{2} \left\|\rho^l(T)\right\|_{L^2(\mathscr{R})}^2 + \frac{1}{2} \|\rho_0\|_{L^2(\mathscr{R})}^2 + \frac{1}{2} \int_0^T Q_{1/2} - Q_{M-1/2} dt + \frac{1}{2} T$$

$$\le \frac{1}{2} \|\rho_0\|_{L^1(\mathscr{R})} + \frac{1}{2} T. \tag{A.21}$$

The last inequality follows from $0 \le \rho^l \le 1$, by Lemma 4.4 and $Q_{1/2} \le 0, Q_{M-1/2} = 0$. To show the second estimate, use (A.19) and replace v_i by $v_{i-1/2}$ for the two vehicles which are adjacent the bottleneck. For the remaining vehicles, $v_i = v_{i-1/2}$. The potential error terms can be bounded by $(1 + L_v) 6T$. □