Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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Isomorphisms of L^p-operator algebras arising from projective group and groupoid representations

Master's thesis in MTFYMA
Supervisor: Eduardo Ortega
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Sammendrag

I denne avhandlingen ser vi hovedsakelig på isomorfiproblemet for den reduserte vridde gruppe- og gruppoide-L^p-operatoralgebraen. For en lokalkompakt gruppe G og en kontinuerlig 2-kosykel σ definerer vi den reduserte σ-vridde gruppe-L^p-operatoralgebraen $F^p_\lambda(G,\sigma)$. Vi viser at hvis $p \neq 2$, så er to slike algebraer isometriske isomorfe hvis og bare hvis gruppene er topologiske isomorfe og 2-kosykelene er kohomologe. For en vridning $\mathcal{E}$ over en étale gruppoide $\mathcal{G}$ definerer vi den reduserte vridde grupppoide-L^p-operatoralgebraen $F^p_\lambda(\mathcal{G};\mathcal{E})$. I avhandlingens hovedresultat viser vi at hvis gruppoidene er effektive og $p \neq 2$, så er to slike algebraer isometriske isomorfe hvis og bare hvis gruppoidene er isomorfe, og vridningene er ekte isomorfe.

Abstract

In this thesis we will mainly study the isomorphism problem for the reduced twisted group and groupoid L^p-operator algebra. For a locally compact group G and a continuous 2-cocycle σ we will define the reduced σ-twisted L^p-operator algebra $F^p_\lambda(G, \sigma)$. We will show that if $p \neq 2$, then two such algebras are isometrically isomorphic if and only if the groups are topologically isomorphic and the continuous 2-cocyles are cohomologous. For a twist $\mathcal{E}$ over an étale groupoid $\mathcal{G}$, we define the reduced twisted groupoid L^p-operator algebra $F^p_\lambda(\mathcal{G}; \mathcal{E})$. In the thesis main result, we show that if the groupoids are effective and $p \neq 2$, then two such algebras are isometrically isomorphic if and only if the groupoids are isomorphic and the twist are properly isomorphic

Preface

> Infinity! Törless knew the word from maths class. He had never imagined anything particular by it. It was forever returning; someone must have invented it once and since then it had been possible to calculate with it as surely as one did with anything solid... Now there was something terribly unsettling about the word. It struck him as a concept that had formerly been tamed, one with which he had performed his daily little tricks, and which had now suddenly been released. Something beyond understanding, something wild and destructive seemed to have been put to sleep by the work of a clever inventor, and had now been woken to life and grown terrible before him. There, in that sky, it now stood vividly above him and menaced and mocked
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> — Robert Musil, The Confusions of Young Törless

This thesis was written in spring of 2022, in my final year of the five year master program in applied physics and mathematics at NTNU, and builds upon a specialisation project the term prior with the same supervisor. The program was lenient enough to let me pursue pure mathematics in my final two years, where my fascination the infinities in the real analysis courses, combined with my long hate for differential equations after some tedious (shut up and) calculation based courses early in my studies, led me to do a master thesis within the field of operator algebra. I would like to thank my supervisor Eduardo Ortega, during the last year I never felt completely lost, and the path ahead was always clearer after our discussions.

Contents

1 Introduction

The study of operators on Hilbert space, in particular C^*-algebras, is one of the most active fields within functional analysis today and there is enormous literature on C^*-algebras. The term C^*-algebra was first coined in 1947 by Segal to describe norm closed self-adjoint subalgebras of the bounded operators on some Hilbert space. A natural generalisation of this is to replace the Hilbert space, which is an L^2-space, with an general L^p-spaces. Given $p \in [1, \infty]$, we define an L^p-operator algebra to be a Banach algebra that admits an isometric isomorphism to the space of bounded linear operators on some L^p-space. Hilbert spaces are very well behaved compared with general L^p spaces and the much more complex geometry of L^p-spaces due to the lack of inner product when $p \neq 2$, makes L^p-operator algebra usually much harder to study, even for L^p-operator algebras that "looks" like C^*-algebras. Many basic tools available in the C^*-algebra setting fail to hold for L^p-operator algebra. There exist for instance no general theory of L^p-operator algebras, and there is no abstract characterisation of when a Banach algebra is an L^p-operator algebra. For this reason the field has for the most part been studied through examples. However, more recent works have approached the field more systematically and abstractly, showing drips of a more general theory.

The field of L^p-operator algebras has seen a recent renewal in interest, spurred by new ideas and techniques taken from the field of operator algebra. There has been especially fruitful research on L^p-operator algebras that "looks" like C^*-algebras, sparked by Christoffer Phillips studies on the the L^p-analogue of the Cuntz algebra $\mathcal{O}_n$ and UHF-algebras. One has seen that the more complex geometry of the unit circle of L^p-spaces compared to L^2-spaces, lets one prove some interesting isomorphism results when $p \neq 2$. In this thesis we will follow in this line of research, following some of the recent work on L^p-analogue of the group and groupoid C^*-algebra. In particular we will follow and build upon some of the the recent work by Gardella et al.([5],[10],[10]).

For a locally compact group G, we denote the reduced group L^p-operator algebra by $F^p_\lambda(G)$, when $p = 2$ this is the the reduced $C^*_\lambda\lambda(G)$, we will therefore say $F^p_\lambda(G)$ is the L^p-analogue of the reduced C^*-algebra. This algebra was first introduced by Hertz in the 1970s, under the name p-pseudofunctions. In the last decade there has been a renewed interest in this algebra and other group algebras on L^p-spaces, in particular of our interest is the result by Gardella and Thiel in [10] which shows that, for $p \neq 2$, two such algebras are isometrically isomorphic if and only if the groups are toplogical isomorphic. Surprisingly this contrast with the reduced group C^*-algebras where two non-isomorphic groups can generate isomorphic algebras. For a continuous 2-cocycle σ on G, we will define the σ-twisted reduced L^p-operator algebra denoted $F^p_\lambda(G, \sigma)$, for $p = 2$ this is the σ twisted reduced group C^*-algebra, which is very well studied. In Theorem 3.7 we extend the isomorphism result of Gardella and Thiel to the reduced twisted group L^p-operator algebra, and show that two such algebras are isometrically isomorphic if and only if the groups are topological isomorphic and the 2-cocycles are cohomologous.

In [3] Choi, Gardella, and Thiel shows a similar L^p-rigidity result for the reduced groupoid L^p-operator algebra, dented $F^p_\lambda(\mathcal{G})$, for toplogical principal étale Hausdorff groupoids when $p \neq 2$. Following in a similar vein, we will extend this result to the twisted reduced groupoid L^p-operator algebra. A groupoid is an algebraic structure that behaves like groups but with multiplication only partially defined, and it is in this sense a generalisation of the notion of groups. The work on groupoid C^* algebras goes back to Renault in the 1980s [16], and has been studied intensively ever since. Similarly to the group algebra setting, given a normalised continuous cocycle σ, one can define the reduced σ-twisted groupoid L^p-operator algebra denoted $F^p_\lambda(\mathcal{G}, \sigma)$, but formally more general, one can define the notion of the twisted reduced groupoid L^p-operator algebra from a twist, based on Kumjian works on the groupoid C^*-algebra [12]. For a twist $\mathcal{E}$ over an effective étale groupoid $\mathcal{G}$ we define the twisted reduced groupoid L^p-operator algebra denoted $F^p_\lambda(\mathcal{G}; \mathcal{E})$. In [17], Renault shows that an effective groupoid and twist can be recovered its twisted reduced groupoid C^*-algebra together with the canonical abelian subalgebra $C_0(\mathcal{G}^{(0)})$. Following Renault´s and Choi, Gardella, and Thiele's paper, we will in our main result, Theorem 6.25, show that for a twist $\mathcal{E}$ over an effective étale groupoid $\mathcal{G}$, one can recover the twist and the groupoid generating the algebra when $p \neq 2$, and as a consequence of the more complex geometry of L^p spaces, this result means that if the groupoids are effective and $p \neq 2$, then two such algebras are isometrically isomorphic if and only if the groupoids are isomorphic and the twists are properly isomorphic.

2 Summary of the preliminary specialisation project

The master thesis build upon a specialisation project the term prior. In this section we will summarise some important results from the project. See [5] and [10] for more details, the papers on which the project was based on.

2.1 The Lamperti Theorem

The Lamperti theorem identifies the invertible isometries on an L^p-space when $p \neq 2$. In our setting we are working on locally compact groups, for which the left Haar measure is not necessarily σ-finite measure. We therefore need the Lamperti Theorem for the more general case of localizable measure spaces, which is given in [10]. The concept of a localizable measure is most natural defined in the setting of measure algebras. We will thus start, following Section 2 of [10] and [5] by defining the measure algebra and measureable function on an measure algebra. For a rigorous and detailed discussion on the topics see [19].

Definition 2.1. A *Boolean algebra* $\mathcal{A}$ is a set containing two distinct element 0 and 1, two commutative and associative operations $\vee$ and $\wedge$ and a notion of complement $a \mapsto a^c$ satisfying the following properties:

(1) $a \vee a = a \wedge a = a \quad \forall a \in \mathcal{A}$

(2) $a \vee (a \wedge b) = a \wedge (a \wedge b) = a \quad \forall a, b \in \mathcal{A}$

(3) $\forall a \in \mathcal{A}$, we have:

$$a \vee 0 = a = a \wedge 1, a \wedge 0 = 0 \text{ and } a \vee 1 = 1$$

(4) $a \vee a^c = 1$ and $a \wedge a^c = 0 \quad \forall a \in \mathcal{A}$

We write $a \leq b$ if a $a \vee b = a$. A *Boolean homomorphism* is a map between two Boolean algebras that preserves all the operation above and the distinct elements, 0 and 1.

A Boolean algebra $\mathcal{A}$ is *(σ-)complete* if every nonempty (countable)subset of $\mathcal{A}$ has a suprema and infima. A *measure*, μ, on $\mathcal{A}$ is map $\mu\colon \mathcal{A} \to [0, \infty]$ such that $\mu(0) = 0$, $\mu(a) > 0$ for all $a \neq 0$ and $\mu(\bigvee_{n\in\mathbb{N}} a_n) = \bigvee_{n\in\mathbb{N}} \mu(a_n)$ whenever a_n are pairwise orthogonal. We say that the measure μ is *semi-finite* if for every $a \in \mathcal{A}$, there exists $b \in \mathcal{A}$ with $b \leq a$ such that $0 < \mu(b) < \infty$. A *measure algebra* is pair $(\mathcal{A}, \mu)$ where $\mathcal{A}$ is a σ-complete Boolean algebra and μ is a measure on $\mathcal{A}$. If μ is semi-finite and $\mathcal{A}$ is complete we say that the measure algebra $(\mathcal{A}, \mu)$ is *localizable.*

Example 2.2. Let (X, Σ, μ) be a measure algebra, the null set $\mathcal{N} = \{E \in \Sigma\colon \mu(E) = 0\}$ is then a σ-ideal in Σ, and it follows that the quotient $\mathcal{A} = \Sigma/\mathcal{N}$ is a σ-complete Boolean algebra. Furthermore we can define a natural measure induced by this quotient $\tilde{\mu}\colon \mathcal{A} \to [0, \infty]$ given by $\tilde{\mu}(E+\mathcal{N}) = \mu(E)$ for all $E \in \Sigma$. The induced measure algebra $(\mathcal{A}, \tilde{\mu})$ is called the *measure algebra associated with* (X, Σ, μ)

Note that the measure space is localizable if and only if the associated measure algebra is localizable [19, 332B]. There is also a natural notion of measurable functions and integrals on a measure algebra, as we will define. Let $\mathcal{A}$ be a Boolean algebra, and let $\mathcal{B}$ be the Borel set of $\mathbb{R}$ (Which is also an Boolean algebra). A *measurable real function* on $\mathcal{A}$ is a Boolean homomorphism $f\colon \mathcal{B} \to \mathcal{A}$ which preserves sumprema of countable sets. We follow the notation of Fremlin [19] and write $\{f \in E\}$ for $f(E)$ and $\{f > t\}$ for $f(t, \infty)$. This is to put further emphasis on the connection to the usual measure spaces, as we will see in the following example.

Example 2.3. Let (X, Σ, μ) be a measure space, let $f\colon X \to \mathbb{R}$ be any measurable real function and let $(\mathcal{A}, \bar{\mu})$ be the measure algebra of (X, Σ, μ). We can then obtain a measurable real function on $\mathcal{A}$ from f, denoted $\tilde{f} : \mathcal{B} \to \mathcal{A}$ which is given by $\{\tilde{f} \in E\} = f^{-1}(E) + \mathcal{N}$ for every Borel subset $E \in \mathcal{B}$. This can also be written as $\{\tilde{f} \in E\} = \{x\colon f(x) \in E\} + \mathcal{N}$.

Let $L^0_{\mathbb{R}}(\mathcal{A})$ denote the set of all measurable real functions on $\mathcal{A}$. for $f, g \in L^0_{\mathbb{R}}(\mathcal{A})$ addition and multiplication is given by

$$\{f+g>t\} = \sup_{q\in\mathbb{Q}}\{f>q\}\wedge\{g>t+q\}$$

and

$$\{fg>t\} = \sup_{q\in\mathbb{Q}}\{f>q\}\wedge\{g>t/q\}$$

respectively, and absolute value is given by

$$\{|f|>t\} = \{f>t\}\vee\{f<t\}.$$

Example 2.4. Let $(\mathcal{A}, \bar{\mu})$ be the measure algebra of some measure space (X, Σ, μ), and let and let $f, g\colon X \to \mathbb{R}$ be any two measurable real functions. We then have that

$$(f+g)^{-1}(t,\infty) = \bigcup_{q\in Q}\left(f^{-1}(q,\infty)\cap g^{-1}(t-q,\infty)\right).$$

Note that that suprema in $(\mathcal{A}, \overline{\mu})$ is given by countable union. We therefore see that addition on $L^0_{\mathbb{R}}(\mathcal{A})$ is in line with what we would expect from Example 2.3.

Let $(\mathcal{A}, \mu)$ be a measure algebra, an let $L^0(\mathcal{A}) := \{f + ig\colon f, g \in L^0_{\mathbb{R}}(\mathcal{A})\}$ denote the vector space of measurable complex functions on $(\mathcal{A}, \mu)$. Using that $t \mapsto \mu(\{f > t\})$ is a decreasing function, and hence is Lebesgue measurable allows us to define the the following norm

$$\|f\|_1 = \int |f| d\mu = \int_t^\infty \mu(\{|f|>t\})dt$$

for all $f \in L^0(\mathcal{A})$. We can then define the space of integrable functions

$$L^1(\mathcal{A},\mu) := \{f\in L^0(\mathcal{A}) : \|f\|_1 < \infty\}$$

Let $f \in L^1_{\mathbb{R}}(\mathcal{A}, \mu)_+$ be a positive real functions, we then define the integral

$$\int f d\mu = \int_0^\infty \mu(\{f>t\})dt.$$

This extends linearly to $L^1(\mathcal{A}, \mu)$ in the usual manner. We then define the p-integrable functions for $p \in (1, \infty)$ as follows

$$L^p(\mathcal{A},\mu) := \{f\in L^0(\mathcal{A}) : |f|^p \in L^1(\mathcal{A},\mu)\}$$

For $p = \infty$ the norm is given by $\|f\|_\infty = \inf\{t \geq 0\colon \{|f| > t\} = 0\}$ and we define

$$L^\infty(\mathcal{A},\mu) := \{f\in L^0(\mathcal{A}) : \|f\|_\infty < \infty\}.$$

Let $(\mathcal{A}, \bar{\mu})$ be the measure algebra of some measure space (X, Σ, μ). From Example 2.3 recall that for every measurable function on (X, Σ, μ) we obtain a measurable function on $(\mathcal{A}, \bar{\mu})$. It turns out that the relation between (X, Σ, μ) and $(\mathcal{A}, \bar{\mu})$ is even stronger. We can actually identify $L^p(\mathcal{A}, \bar{\mu})$ with $L^p(X, \Sigma, \mu)$ for all $p \in [1, \infty]$ (Corollary 363I and Theorem 366B in [19]). This means that we can apply everything we know about $L^p(X, \Sigma, \mu)$ to $L^p(\mathcal{A}, \bar{\mu})$.

Localizable measure algebra is in fact the largest class for which Radon Nikodym Theorem is applicable, this allows one prove the Lamperti Theorem for localizable measure algebra,[10, Theorem 2.7.] which is usually only proved for σ-finite measure spaces.

Theorem 2.5 (Radon-Nikodym)**.** *Let $\mathcal{A}$ be a complete Boolean algebra, and let μ and ν be two measures with $(\mathcal{A}, \mu)$ forms a localizable measure algebra. Then there exists a unique function $\frac{d\nu}{d\mu} \in L^0_{\mathbb{R}}(\mathcal{A})$, called the Radon-Nikodym derivative of ν with respect to μ, satisfying*

$$\int f d\nu = \int f\frac{d\nu}{d\mu}d\mu$$

for all $f \in L^1(\mathcal{A}, \mu)$.

Let $(\mathcal{A}, \mu)$ be a measure algebra, and let $p \in [1, \infty)$. We define the group of $\mathbb{T}$ valued functions on $\mathcal{A}$ as the following

$$\mathcal{U}(L^\infty(\mathcal{A}))\colon = \{f \in L^0(\mathcal{A})\colon \{|f| = 1\} = 1\}$$

This forms a group under pointwise multiplication. We denote the group of Boolean automorphisms on $\mathcal{A}$ by $\mathrm{Aut}(\mathcal{A})$. We denote the group of surjective isometries on $L^p(\mathcal{A})$ by $\mathrm{Isom}(L^p(\mathcal{A}, \mu))$. Given an function $f \in \mathcal{U}(L^\infty(\mathcal{A}))$, we define the map $m_f\colon L^p(\mathcal{A}, \mu) \to L^p(\mathcal{A}, \mu)$, given by $m_f(\xi) = f\xi$ for all $\xi \in L^p(\mathcal{A}, \mu)$. We need the following results from the specialisation project for reference. These results are taken from Section 3 in [10].

Lemma 2.6. *Let $(\mathcal{A}, \mu)$ be a measure algebra, and let $p \in [1, \infty)$. for every $f \in \mathcal{U}(L^\infty(\mathcal{A})$, the map $m_f\colon L^p(\mathcal{A}, \mu) \to L^p(\mathcal{A}, \mu)$, given by $m_f(\xi) = f\xi$ for all $\xi \in L^p(\mathcal{A}, \mu)$ is a surjective linear isometry on $L^p(\mathcal{A}, \mu)$. Moreover given $f, g \in L^p(\mathcal{A}, \mu)$, then $m_f m_g = m_{fg}$*

Lemma 2.7. *Let $(\mathcal{A}, \mu)$ be a localizable measure algebra, and let $p \in [1, \infty)$. Let $\varphi \in Aut(\mathcal{A})$. Then the map $u_\varphi\colon L^p(\mathcal{A}, \mu) \to L^p(\mathcal{A}, \mu)$, given by*

$$u_\varphi(\xi) = (\varphi \circ \xi) \left(\frac{d(\mu \circ \varphi^{-1})}{d\mu} \right)^{\frac{1}{p}}$$

is a surjective isometry on $L^p(\mathcal{A}, \mu)$. Moreover given another element $\phi \in Aut(\mathcal{A})$, we have that $u_\varphi \circ u_\phi = u_{\varphi \circ \phi}$

Lemma 2.8. *Let $(\mathcal{A}, \mu)$ be a localizable measure algebra. Let $\varphi \in Aut(\mathcal{A})$ and $f \in \mathcal{U}(L^p(\mathcal{A}, \mu)$. Then $u_\varphi m_f u_{\varphi^{-1}} = m_{\varphi \circ f}$*

And finally the Lamperti's theorem itself [10, Theorem 3.7].

Theorem 2.9. *Let $(\mathcal{A}, \mu)$ be a localizable measure algebra, let $p \in [1, \infty) \backslash \{2\}$ and let $T\colon L^p(\mathcal{A}, \mu) \to L^p(\mathcal{A}, \mu)$ be an surjective isometry, then there exists an unique $\varphi \in Aut(\mathcal{A})$ and $f \in \mathcal{U}(L^p(\mathcal{A}, \mu)$ such that $T = m_f u_\varphi$*

We also need Lemma 4.8 from [10], which lets us "lift" the automorphism of the measure algebra to the measure space. Let G be a locally compact, let Σ denote the Borel sets on G, and μ be the left Haar measure. Let $(\mathcal{A}, \overline{\mu})$ be the measure algebra of (G, Σ, μ). Let $s \in G$, the maps $M \mapsto sM$ and $M \mapsto Ms$ induces maps (under the canonical mapping $\Sigma \mapsto \mathcal{A}$) on $\mathcal{A}$ which we denote l_s and r_s respectively. Note that $r_s, l_s \in \mathrm{Aut}(\mathcal{A})$.

Lemma 2.10. *Let $\pi \in Aut(\mathcal{A})$ such that $r_t \circ \pi = \pi \circ r_t$ for every $t \in G$. Then there exists a unique $s \in G$ such that $\pi = l_s$.*

2.2 The L^p-operator algebra

A *Banach algebra* is a Banach space that is also a associative complex algebra such that the norm is submultiplicative. We call the Banach algebra *unital* if it contains multiplicative identity with norm 1. A C^**-algebra* is a Banach algebra A, together with an involution such that $\|a^*a\| = \|a\|^2$ for all $a \in A$

Definition 2.11. Let A be a Banach algebra, we say that A is an L^p*-operator algebra* if there exists an L^p-space E and an isometric homomorphsim $A \to \mathcal{B}(E)$

Example 2.12. Let E be a Banach space, then $\mathcal{B}(E)$ is trivially an L^p-operator algebra. So if $E = l^p(1, .., n)$. Then $\mathcal{B}(E)$, which we can identify with M_n, is an L^p-operator algebra, we denote this algebra M_n^p

Note that the norm on an L^p-operator algebra is not unique in contrast with C^*-algebras. For instance let $s \in M_n^p$ be an invertable element, we can then define another L^p-operator norm on M_n given by $\|x\|_s = \left\|sxs^{-1}\right\|$.

Remark 2.13. By the Gelfand–Naimark theorem every C^*-algebra is an L^2-operator algebra. The converse is only true for self-adjoint L^2-operator algebras.

Definition 2.14. Let A be an L^p-operator algebra, and let E be some L^p-space. A *representation* of A on E is a contractive homomorphism $\varphi\colon A \to \mathcal{B}(E)$, we say it is *non-degenerate* if $\varphi(A)E := \operatorname{span}\{\varphi(a)\xi : a \in A, \xi \in E\}$ is dense in E.

L^p-operator algebras are generally much more complicated to study than C^*-algebras, due to the more complex geometry. There is still not any general theory for L^p-operator algebras, and many of the basic facts of C^*-algebras does not hold for L^p-operator algebras, even L^p analogous of well studied C^*-operator algebras. There is for instance no abstract characterisation of L^p-operator algebras or canonical way to obtain a representation on some L^p-space. We have also seen in example 2.12 that the norm is, in general not unique which means that injective homomorphisms are not necessarily isometric.

But the more complex geometry gives also rise to some surprising rigidity result for some L^p-analogous to well studied C^*-algebras which we do not have for the C^* algebras. For instance the L^p-analogous to the reduced C^*-algebra which is studied in [10], [3] and [5] among others. Interestingly, this is due to the lack of symmetry of the unit circle for L^p-spaces when $p \neq 2$. To illustrate this on can think of the unit circle of the p-norm on $\mathbb{R}^2$. When $p = 2$ this is a circle in the everyday meaning of the word and every rotation is therefor isometric, when $p \neq 2$ there is only four isometric rotation symmetries.

2.3 The twisted group L^p-operator algebra

Let G be as a locally compact group with the left Haar measure, it is well known that $L^1(G)$ forms a Banach space under convolution given by

$$(f * g)(s) = \int_G f(t)g(t^{-1}s)dt$$

We define the *left regular representation*, λ_p of $L^1(G)$ on $L^p(G)$ by convolution from the left

$$\lambda_p(f)(\xi(s)) = \int_G f(t)\xi(t^{-1}s)dt$$

We define *the reduced group L^p-operator algebra*, denote $F^p_\lambda(G)$ as the closure of $\lambda_p(L^1(G))$ in $\mathcal{B}(L^p(G))$ in the operator norm.

Definition 2.15. Let G be a locally compact group, and let $p \in [1, \infty)$. We define $F^p(G)$ the *full group L^p-operator algebra* as the completion of $L^1(G)$ in the norm

$$\|f\|_{F^p} = \sup\{\|\varphi(f)\| : \varphi\colon L^1(G) \to \mathcal{B}(E) \text{ contractive representation.}\}$$

for $f \in L^1(G)$,where E ranges over all L^p spaces.

It follows from Proposition 4.6. in [5] that this is in fact an L^p-operator algebra.

Gardella and Thiel showed the following rigidity result in [10], which was one of our main interests in the specialisation project, and is a consequence of the Lampertis Theorem.

Theorem 2.16. *Let G and H be two locally compact groups, and let $p \in [1, \infty) \setminus \{2\}$. Then there is an isometric isomorphism $F^p_\lambda(G) \cong F^p_\lambda(H)$ if and only if there is an isomorphism of toplogical groups $G \cong H$*

Definition 2.17. Let G be a locally compact group, a *continuous 2-cocycle* is a map $\sigma\colon G \times G \to \mathbb{T}$ satisfying the following:

(1) $\sigma(x_1, x_2)\sigma(x_1x_2, x_3) = \sigma(x_1, x_2x_3)\sigma(x_2, x_3)$
(2) $\sigma(x, e) = \sigma(e, x) = 1$

for all $x, x_1x_2, x_3 \in G$ and where e i the unit of G.

For the rest of the section G will denote a locally compact group, and the measure will be the left Haar measure. Using the cocycle we can define something called the *twisted convolution product* on $L^1(G)$. Let σ be a continuous 2-cocycle on G, the σ-twisted convolution product is defined as the following

$$(f *_\sigma g)(x) = \int_G f(y)g(y^{-1}x)\sigma(y, y^{-1}x)dy.$$

for $f, g \in L^1(G)$, we call it the σ-twisted to emphasis the dependence of the choice of 2-cocycle. Note that $(L^1(G), *_\sigma)$ forms a Banach algebra, and we will denote this algebra by $L^1(G, \sigma)$. $L^1(G, \sigma)$ has approximate identity and is unital if and only if G is compact.

We denote *σ-twisted left regular representation* $\lambda_p^\sigma \colon G \to (\mathcal{B}(L^p(G))$ given by the following

$$\lambda_p^\sigma(y)(f)(x) = \sigma(y, y^{-1}x)f(y^{-1}x), \quad y \in G, f \in L^p(G).$$

Similarly we have the σ-twisted right regular representation $\rho_p^\sigma \colon G \to \mathcal{B}(L^p(G))$ which is given as the following

$$\rho_p^\sigma(y)(f)(x) = \sigma(x, y)f(xy), \quad y \in G, f \in L^p(G).$$

The integrated form of the σ-twisted left regular representation $\lambda_p^\sigma \colon L^1(G) \to \mathcal{B}(L^p(G))$ is the action given by left convolution

$$\lambda_p^\sigma(f)(\xi)(x) = \int_G f(y)\xi(y^{-1}x)\sigma(y, y^{-1}x)dy.$$

Proposition 2.18. *Let $p \in [1, \infty)$, let G be a locally compact group and let σ be a continuous 2-cocycle on G. We then have that*

$$\lambda_p^\sigma(z)\rho_p^{\overline{\sigma}}(y) = \rho_p^{\overline{\sigma}}(y)\lambda_p^\sigma(z)$$

for all $y, z \in G$.

Proof. Fix $y, z \in G$, and let $f \in L^p(G)$, writing out the expressions we have

$$(\lambda_p^\sigma(z)\rho_p^{\overline{\sigma}}(y))(f)(x) = \sigma(z, z^{-1}x)\overline{\sigma(z^{-1}x, y)}f(z^{-1}xy)$$

and

$$(\rho_p^{\overline{\sigma}}(y)\lambda_p^\sigma(z))(f)(x) = \overline{\sigma(x, y)}\sigma(z, z^{-1}xy)f(z^{-1}xy)$$

Setting $x_1 = z$, $x_2 = z^{-1}x$ and $x_3 = y$ and using (2) in Defintion 2.17, we have that

$$\sigma(z, z^{-1}x)\sigma(x, y) = \sigma(z^{-1}x, y)\sigma(z, z^{-1}xy)$$

which can be rewritten to

$$\sigma(z, z^{-1}x)\overline{\sigma(z^{-1}x, y)} = \overline{\sigma(x, y)}\sigma(z, z^{-1}xy)$$

and the assertion follows. ☐

Definition 2.19. Let G be a loaccly compact group and let σ be a continuous 2-cocycle on G. Let $p \in [1, \infty)$, we define the *reduced σ-twisted group L^p-operator algebra* as the following

$$F_\lambda^p(G, \sigma) = \overline{\lambda_p^\sigma(L^1(G, \sigma))}^{\|\cdot\|} \subseteq \mathcal{B}(L^p(G)).$$

We define σ-*twisted* p-*convolvers*, $CV_p(G,\sigma) = \lambda_p^\sigma(G)''$. This is the L^p-equivalent of the σ-twisted Von Neumean algebra, $W^*(G,\sigma)$ generated by $\lambda_2^\sigma(G)$. Note that by Proposition 2.18, the elements in $CV_p(G,\sigma)$ commutes with $\rho_p^{\overline{\sigma}}(x)$ for $x \in G$.

For a locally compact group G and cocycle on σ on G, we construct the *Mackey group* associated to G and σ, denoted G_σ, which as a topological space is the usual product space $\mathbb{T} \times G$, but with product given by

$$(\gamma_1, x_1)(\gamma_2, x_2) = (\gamma_1\gamma_2\sigma(x_1, x_2), x_1x_2)$$

and the inverse given by

$$(\gamma, x)^{-1} = \left(\overline{\gamma\sigma(x^{-1}, x)}, x^{-1}\right).$$

Definition 2.20. Let G be a locally compact group, and let σ, ρ be be two continuous 2-cocycles on G, we say that σ is *cohomologous* to ρ, denoted $\sigma \sim \rho$ if there exists a function $\gamma\colon G \to \mathbb{T}$ such that $\sigma(x_1, x_2)\overline{\rho(x_1, x_2)} = \gamma(x_1)\gamma(x_2)\overline{\gamma(x_1x_2)}$ for all $x_1, x_2 \in G$.

The main result of the specialisation project was showing the following.

Theorem 2.21. *Let G and H be two discrete groups, let $p \in [1,\infty) \setminus \{2\}$, let σ be a continuous 2-cocycle for G and ρ be a continuous 2-cocycle for H. Then there is an isometric isomorphism $F_\lambda^p(G,\sigma) \cong F_\Lambda^p(H,\rho)$ if and only there is an isomorphism of topological groups $G \cong H$ and $\sigma \sim \rho$.*

In the next section we will generalise this result to locally compact groups in general.

3 L^p-rigidity for the reduced σ-twisted group L^p-operator algebra

In this section we want to extend the results from the specialisation project given, in the last chapter, from the discrete groups, to all locally compact groups. First we need to define the twisted algebra of measures.

Let G be a locally compact group, and let σ be a continuous 2-cocycle on G. The Banach space of complex-valued Radon measures on G with bounded variation, denoted $M(G)$, becomes a unital Banach algebra under twisted convolution, where the twisted convolution product, for two measures $\nu_1, \nu_2 \in M(G)$, is given by

$$\int_G f(z)d(\nu_1 *_\sigma \nu_2)(z) = \int_G \int_G f(xy)\sigma(x,y)d\nu_1(x)d\nu_2(y)$$

for every $f \in C_c(G)$. We will denote the algebra by $M(G,\sigma)$ and call it the *σ-twisted algebra of measures*. Note that for $\nu \in M(G,\sigma)$ the norm can be given on the form $\|\nu\| = \sup_{|f|\leq 1} \left|\int_G f d\nu\right|$. Let $f \in C_c(G)$ be any function with $|f| \leq 1$, we then have that

$$\begin{aligned}
\left|\int_G f d(\nu_1 *_\sigma \nu_2)\right| &= \left|\int_G \int_G f(xy)\sigma(x,y)d\nu_1(x)d\nu_2(y)\right| \\
&\leq \int_G \left|\int_G f(xy)\sigma(x,y)d\nu_1(x)\right| d|\nu_2|(y) \\
&\leq \|\nu_1\| \, \|\nu_2\| \, .
\end{aligned}$$

It follows that $\|\nu_1 *_\sigma \nu_2\| \leq \|\nu_1\| \, \|\nu_2\|$. Let $\nu_1, \nu_2, \nu_3 \in M(G,\sigma)$, we then have that the following:

$$\begin{aligned}
\int_G f(w)d(\nu_1 *_\sigma (\nu_2 *_\sigma \nu_3))(w) &= \int_G \int_G f(xy)\sigma(x,y)d\nu_1(x)(d\nu_2 *_\sigma \nu_3)(y) \\
&= \int_G \int_G \int_G f(xyz)\sigma(x,yz)d\nu_1(x)\sigma(y,z)d\nu_2(y)d\nu_3)(z) \\
&= \int_G \int_G \int_G f(xyz)\sigma(x,y)\sigma(xy,z)d\nu_1(x)d\nu_2(y)d\nu_3)(z) \\
&= \int_G \int_G f(xz)\sigma(x,z)d(\nu_1 *_\sigma \nu_2)(y)d\nu_3)(z) \\
&= \int_G f(w)d((\nu_1 *_\sigma \nu_2) *_\sigma \nu_3)(w).
\end{aligned}$$

Where we used (2) in Definition 2.17 in the third equality. It follows that twisted convolution is associative and submultiplicative, $M(G,\sigma)$ is thus a Banach algebra.

We can then define the left regular representation of $M(G,\sigma)$ on $L^p(G)$ denoted by λ_p^σ, which is given by

$$\lambda_p^\sigma(\nu)(f)(x) = \int_G \lambda_p^\sigma(y)(f(x))d\nu(y) = \int_G f(y^{-1}x)\sigma(y,y^1x)d\nu(y).$$

for all $\nu \in M(G,\sigma)$ and $f \in L^p(G)$, and $x \in G$. This is a bounded operator by the following, given $\xi \in L^p(G)$, we have

$$\begin{aligned}
\|\mu *_\sigma \xi\|_p = \left(\int \left|\int \lambda_p^\sigma(y)(\xi(x))d\nu(y)\right|^p d\mu(x)\right)^{1/p} &\leq \int \left(\int \left|\lambda_p^\sigma(y)(\xi(x))\right|^p d\mu(x)\right)^{1/p} d\nu(y) \\
= \|\xi\|_p \int_G d\nu \leq \|\xi\|_p \, \|\mu\|_{M^1(G)}
\end{aligned}$$

where the inequality is due to Minkowski's integral inequality.

Let $y \in G$ and let δ_y be the point mass measure associated with y. Note that $\lambda_p^\sigma(\delta_y) = \lambda_p^\sigma(y)$. We denote the L^p-operator algebra generated by the twisted left representation of the the twisted algebra of measures by $M^p_\lambda(G,\sigma) = \overline{\lambda^\sigma_p(M(G,\sigma))}^{\|\cdot\|_p} \subseteq \mathcal{B}(L^p(G))$. Let A be an unital Banach algebra, we define the group of invertable isometries on A by

$$\mathcal{U}(A) \coloneqq \{v \in A \colon v \text{ is invertable and } \|v\| = 1\}$$

which is the Banach-analogue of the unitary group. The first goal of this section is to show that we can algebraically identify G_σ with the invertable isometries on $CV_p(G,\sigma)$, analogous to what Gardella and Thiel showed in [10]. Fix some element $y \in G$, we write $\sigma_y(x) = \overline{\sigma(x,y)}$, and note that $\sigma_y \in \mathcal{U}(L^\infty(\mathcal{A}))$. Recall that right multiplication of y induces a Boolean automorphism $r_y \in \mathrm{Aut}(\mathcal{A})$ and note that $\rho_p(y) = u_{r_{y^{-1}}} \in L^p(\mathcal{A})$. We therefore have that $\rho^{\overline{\sigma}}_p(y) = m_{\sigma_y} u_{r_{y^{-1}}} \in L^p(\mathcal{A})$, which will be important for the next result.

Theorem 3.1. *Let G be locally compact group, let $p \in [1,\infty) \setminus \{2\}$, and let c be a continuous 2-cocycle. Let $v \in \mathcal{U}(CV_p(G,\sigma))$, then there exists unique $\gamma \in \mathbb{T}$ and $g \in G$ such that $v = \gamma\lambda^\sigma_p(g)$.*

Proof. Let $v \in \mathcal{U}(CV_p(G,\sigma))$. Then $v \in \mathrm{Isom}(L^p(\mathcal{A},\bar{\mu}))$, and by Lamperti's Theorem there exits a unique $h \in \mathcal{U}(L^\infty(\mathcal{A}))$ and unique $\varphi \in \mathrm{Aut}(\mathcal{A})$ such that $v = m_h u_\varphi$. Since every $v \in \mathcal{U}(CV_p(G,\sigma))$ commutes with $\rho^{\overline{\sigma}}_p(y)$ for every $y \in G$, we have that $m_{\sigma_y} u_{r_{y^{-1}}} m_h u_\varphi = m_h u_\varphi m_{\sigma_y} u_{r_{y^{-1}}}$. By Lemma 2.8 and 2.7 we can rewrite the equality to

$$m_{\sigma_y(r_{y^{-1}}\circ h)} u_{y_{s^{-1}}\circ\varphi} = m_{h(\varphi\circ\sigma_y)} u_{\varphi\circ r_{s^{-1}}}.$$

This implies that $r_{y^{-1}} \circ \varphi = \varphi \circ r_{y^{-1}}$ and $\sigma_y(r_{y^{-1}} \circ h) = h(\varphi \circ \sigma_u)$. By the first equality it follows by Lemma 2.10 that there exists a unique $g_v \in G$ such that $\varphi = l_{g_v}$. We then have $h(l_{g_v} \circ \sigma_y) = \sigma_y(r_{y^{-1}} \circ h)$. This implies that $h(x)\overline{\sigma(g_v^{-1}x,y)} = \overline{\sigma(x,y)}h(xy)$ for all $x \in G$. Set $x = e$, and we deduce that

$$h(y) = h(e)\overline{\sigma(g_v^{-1},y)} = h(e)\overline{\sigma(g_v,g_v^{-1})}\sigma(g_v,g_v^{-1}y)$$

Now set $\gamma_v = h(e)\overline{\sigma(g_v^{-1},g_v)}$ and note that $\gamma_v \in \mathbb{T}$, then $v(\xi)(x) = \gamma_v\sigma(g_v,g_v^{-1}x)\xi(g_v^{-1}x) = \gamma_v\lambda^\sigma_p(g_v)(\xi)(x)$. □

Let $\mathcal{U}(A)_0$ denote the connect component of $\mathcal{U}(A)$ in the norm topology that contains the unit of A, this is then a normal subgroup of $\mathcal{U}(A)$ and we write

$$\pi_0(\mathcal{U}(A)) \coloneqq \mathcal{U}(A)/\mathcal{U}(A)_0$$

for the quotient.

Proposition 3.2. *Let G be a locally compact group. There is a natural group isomorphism $G_\sigma \cong \mathcal{U}(CV_p(G,\sigma))$ and $G \cong \pi_0(\mathcal{U}(CV_p(G)))$ given by the maps $(\gamma,s) \mapsto \gamma\lambda^\sigma_p(y)$ and $y \mapsto [\lambda^\sigma_p(y)]$ respectively.*

Proof. Define $\Delta \colon \mathcal{U}(CV_p(G,\sigma)) \to G_c$ by $\Delta(v) = (\gamma_v, g_v)$. By Theorem 3.1 the map is injective, and by Lampertis theorem its also surjective. It remains to show that this is a homomorphism. Let $\gamma_1,\gamma_2 \in \mathbb{T}$, and $y_1,y_2 \in G$. Then

$$\begin{aligned}(\gamma_1\lambda^\sigma_p(y_1) \circ \gamma_2\lambda^\sigma_p(y_2))(\xi(x) &= \gamma_1\gamma_2\lambda^\sigma_p(y_1)(\sigma(y_2,y_2^{-1}x)\xi(y_2^{-1}x))\\ &= \gamma_1\gamma_2\sigma(y_1,y_1^{-1}x)\sigma(y_2,(y_1y_2)^{-1}x)\xi((y_1y_2)^{-1}x)\\ &= \gamma_1\gamma_2\sigma(y_1,y_2)\sigma(y_1y_2,(y_1y_2)^{-1}x)\xi((y_1y_2)^{-1}x)\\ &= \gamma_1\gamma_2\sigma(y_1,y_2)\lambda^\sigma_p(y_1y_2)\xi(x)\end{aligned}$$

Where we used (1) in 2.17 with $x_1 = y_1$, $x_2 = y_2$ and $x_3 = (y_1y_2)^{-1}x$. Thus

$$\Delta((\gamma_1\lambda^\sigma_p(y_1) \circ \gamma_2\lambda^\sigma_p(y_2)) = (\gamma_1\gamma_2\sigma(y_1,y_2),y_1,y_2) = (\gamma_1,y_1)(\gamma_2,y_2) = \Delta((\gamma_1\lambda^\sigma_p(y_1))\Delta((\gamma_2\lambda^\sigma_p(y_2))$$

Hence Δ is a homomorphism and the assertion follows. □

Lemma 3.3. *Let G be a locally compact group, let $p \in (1,\infty)$, let c be a continuous 2-cocycle, and let $(f_j)_j$ be a contractive approximate identity in $L^1(G,\sigma)$. Then the net $(\lambda_p(f_j))_j$ converges weak* to 1 in $\mathcal{B}(L^p(G))$.*

Proof. The proof follows Lemma 4.2 in [10]. Let q be the conjugate exponent of p. We can we identify $\mathcal{B}(L^p(G))$ with the dual of $L^p(G)\widehat{\otimes}L^q(G)$. via the dual pairing $(T, f\otimes g) = (T(f), g))$, where $T \in \mathcal{B}(L^p(G))$, $f \in L^p(G), g \in L^q(G)$, see Section 2 in [4]. Now, since the net is bounded, it suffices to show that $(\lambda_p^\sigma(f_j), \omega)) \to (1, \omega)$ where ω is a simple tensor in $L^p(G)\widehat{\otimes}L^q(G)$. Let $\xi \in L^p(G)$ and $\eta \in L^q(G)$. By assumption we have that $(f_j)_j$ is a contractive approximate identity, which means that $\|f_j *_\sigma \xi - \xi\|_p \to 0$, which again implies that

$$(\lambda(f_j), \xi\otimes\eta) = (f_j *_\sigma \xi, \eta) \to (\xi, \eta) = (1, \xi\otimes\eta)$$

and the statement follows. $\square$

Let A be a unital Banach algebra, we write $M_l(A)$ for the left multiplier algebra of A which is defined as $M_l(A) \coloneqq \{S \in \mathcal{B}(A)\colon S(ab) = S(a)b \text{ for all } a, b \in A\}$.

Since $F^p_\lambda(G,\sigma)$ has an approximate identity and $F^p_\lambda(G,\sigma) \subseteq \mathcal{B}(L^p(G))$ is a non degenerate subalgebra, it follows from Theorem 4.1 in [7], that $M_l(F^p_\lambda(G,\sigma))$ has a canonical isometric representation as a unital subalgebra in $\mathcal{B}(L^p(G))$.

Proposition 3.4. *Let G be a locally compact group, let c be a continuous 2-cocycle and let $p \in (1,\infty)$. Then there is a natural isometric inclusion*

$$M^p_\lambda(G,\sigma) \subseteq M_l(F^p_\lambda(G,\sigma)) \subseteq CV_p(G,\sigma).$$

Proof. We will first show the first inclusion. It sufices to show that $\lambda_p^\sigma(M(G,\sigma)) \subseteq M_l(F^p_\lambda(G,\sigma))$. Let $\mu \in M(G,\sigma)$ and let $a \in F^p_\lambda(G,\sigma)$. Then there exists a net (f_j) in $L^1(G,\sigma)$ converging to a. We have that $\mu *_\sigma f_j \in L^1(G)$, and it follows from this that $\lambda_p^\sigma(\mu *_\sigma f_j) \in F^p_\lambda(G,\sigma)$. Thus $\lambda_p^\sigma(\mu *_\sigma f_j) = \lambda_p^\sigma(\mu)\lambda_p^\sigma(f_j) \to \lambda_p^\sigma(\mu)a$, and the inclusion follows.

We will then show that $M_l(F^p_\lambda(G,\sigma)) \subseteq CV_p(G,\sigma)$. Let $S \in M_l(F^p_\lambda(G,\sigma))$, and let $(f_j)_j$ be a contractive approximate identity in $L^1(G,\sigma)$. Note that for every j, $S\lambda_p^\sigma(f_j) \in F^p_\lambda(G,\sigma)$. By Lemma 3.3 $S\lambda_p^\sigma(f_j) \xrightarrow{w^*} S$ and thus $M_l(F^p_\lambda(G))$ is in the weak* closure of $F^p_\lambda(G)$. Since $\lambda_p^\sigma(L^1(G,\sigma)) \subseteq CV_p(G,\sigma)$ and the fact that the commutant algebras are weak operator closed, and hence w^* closed, we deduce that $M_l(F^p_\lambda(G,\sigma)) \subseteq CV_p(G,\sigma)$. $\square$

Proposition 3.5. *Let G be a locally compact group, let $p \in (1,\infty)\setminus\{2\}$, and let c be continuous 2-cocycle. Let A be normed closed subalgebra of $\mathcal{B}(L^p(G))$ such that $M^p_\lambda(G,\sigma) \subseteq A \subseteq CV_p(G,\sigma)$. Then there is a natural group isomorphism*

$$\Delta\colon G_\sigma \xrightarrow{\cong} \mathcal{U}(A) \quad \Delta'\colon G \xrightarrow{\cong} \pi_0(\mathcal{U}(A))$$

Given by $(\gamma, y) \mapsto \gamma\lambda_p^\sigma(y)$, and $y \mapsto [\lambda_p(y)]$.

Proof. $G_\sigma \xrightarrow{\cong} \mathcal{U}(CV_p(G,\sigma))$ follows from Lemma 3.1. We need to prove that $\mathcal{U}(A) = \mathcal{U}(CV_p(G,\sigma))$. Firstly $\mathcal{U}(A) \subseteq \mathcal{U}(CV_p(G,\sigma))$ follow trivially from the assumption. Thus, we only need to show that $\mathcal{U}(CV_p(G,\sigma)) \subseteq \mathcal{U}(A)$. It suffices to show that $\mathcal{U}(CV_p(G,\sigma)) \subseteq A$. let v be any element in $\mathcal{U}(CV_p(G,\sigma))$, then by Lemma 3.1 we get that $v = \gamma\lambda_p^\sigma(y)$ for some $\gamma \in \mathbb{T}$ and $y \in G$. But since $\lambda_p^\sigma(y) = \lambda_p^\sigma(\delta_y)$ we get that $v \in M^p_\lambda(G,\sigma) \subseteq A$, and the statement follows. $\square$

To recover the topology of the group, we need to briefly mention some toplogical concept. Let E be a Banach space and let $(T_j)_j$ be a net in $\mathcal{B}(E)$. We say that $(T_j)_j$ converges to T in the strong operator topology or SOT of $\mathcal{B}(E)$ if and only if $T_j(\xi)$ converges to $T(\xi)$ in the norm of E.

Again let A be a unital Banach algebra. We also need to define the strict operator topology on $M_l(A)$ which is the restriction of the strong operator topology of $\mathcal{B}(A)$ to $M_l(A)$. In other words, for a net $(S_j)_j \subset M_l(A)$, we have that $S_j \xrightarrow{str} S$ for some $S \in M_l(A)$ if and only if $S_j(a) \xrightarrow{\|\cdot\|} S(a)$ for all $a \in A$. We let $\mathcal{U}(M_l((A))_{str}$ denote the group $M_l(\mathcal{U}(A))$ with the strict topology on $M_l((A)$ restricted to the invertable isometries, and we denote $\pi_0(\mathcal{U}(M_l((A)))_{str}$ for the group $\pi_0(\mathcal{U}(M_l((A)))$ endowed with the quotient topology.

Proposition 3.6. *Let G be a locally compact group, let $p \in (1, \infty) \setminus \{2\}$, and let σ be a continuous 2-cocycle. Then there is a natural isomorphism as toplogical groups*

$$\Lambda \colon G_\sigma \xrightarrow{\cong} \mathcal{U}(M_l(F^p_\lambda(G,\sigma)))_{str} \quad \Lambda' \colon G \to \pi_0(\mathcal{U}(M_l(F^p_\lambda(G,\sigma))))_{str}$$

Given by $\Lambda(\gamma, y) = \gamma\lambda^\sigma_p(y)$, $\Lambda'(y) = [\lambda_p(y)]$.

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{T} \times G \arrow[r, "\Lambda"] \arrow[d, "(\gamma,y)\mapsto y"'] & \mathcal{U}(M_l(F^p_\lambda(G,\sigma)))_{str} \arrow[d, "v\mapsto[v]"] \\
G \arrow[r, "\Lambda"] & \pi_0(\mathcal{U}(M_l((F^p_\lambda(G,\sigma)))_{str}.
\end{tikzcd}$$

By Proposition 3.5 and Proposition 3.4 it follows that Λ and Λ' are group isomorphisms. Since the downward maps in the diagram of the last Proposition are toplogical quotient map is suffices to show that Λ is a homeomorphism. Let $(\gamma_j, y_j) \subset G_\sigma$ be a net converging to some element $(\gamma, y) \in G_\sigma$. Given $f \in L^1(G,\sigma)$ we have that $(\delta_{y_j} *_\sigma f)(x) \xrightarrow{\|\cdot\|} (\delta_y *_\sigma f)(x)$ in $L^1(G,\sigma)$. Since λ^σ_p is a contracitve homomorphism, this implies that $\gamma_j\lambda^\sigma_p(y_j)\lambda^\sigma_p(f) = \gamma_j\lambda^\sigma_p(\delta_{y_j} *_\sigma f) \xrightarrow{\|\cdot\|} \gamma\lambda^\sigma_p(\delta_s *_\sigma f) = \gamma\lambda^\sigma_p(y)\lambda^\sigma_p(f)$ in $F^p_\lambda(G,\sigma)$ for every $f \in L^1(G,\sigma)$. Since $F^p_\lambda(G,\sigma)$ is the closure of $\lambda^\sigma_p(f)$ and the net $\left(\gamma_j\lambda^\sigma_p(y_j)\right)_j$ is bounded, we deduce that $\gamma_j\lambda^\sigma_p(y_j)a \xrightarrow{\|\cdot\|} \gamma\lambda^\sigma_p(y)a$ for all $a \in F^p_\lambda(G,\sigma)$, and so we have, by the definition of the strict topology, that $\gamma_j\lambda^\sigma_p(y_j) \xrightarrow{str} \gamma\lambda^\sigma_p(y)$ in $M_l(F^p_\lambda(G,\sigma))$.

Conversely assume that $\gamma_j\lambda^\sigma_p(y_j) \xrightarrow{str} \gamma\lambda^\sigma_p(y)$ in $M_l(F^p_\lambda(G,\sigma))$. We need to show that $(\gamma_j, y_j) \to (\gamma, s)$ in G_σ, which means we have to show that $\gamma_j \to \gamma$ in $\mathbb{T}$ and $y_j \to y$ in G.

We will first show that $y_j \to y$. Let U be a neighbourhood of the unit of G. Choose a neighbourhood V also containing the unit of G such that $V^2V^{-2} \subseteq U$ and $\mu(V) < \infty$. This means that $\chi_V \in L^p(G)$ for every $p \in [1,\infty)$. Note that this implies that $\lambda^\sigma_p(\chi_V) \in F^p_\lambda(G,\sigma)$ and it follows by the definition of the strict topology that $\gamma_j\lambda^\sigma_p(y_j)\lambda^\sigma_p(\chi_V) \xrightarrow{\|\cdot\|} \gamma\lambda^\sigma_p(y)\lambda^\sigma_p(\chi_V)$ in $F^p_\lambda(G,\sigma)$ and so we have $\gamma_j\lambda^\sigma_p(y_j)\lambda^\sigma_p(\chi_V)\chi_V \xrightarrow{\|\cdot\|} \gamma\lambda^\sigma_p(y)\lambda^\sigma_p(\chi_V)\chi_V$ in $L^p(G)$. Observe that

$$\lambda^\sigma_p(\chi_V)(\chi_V)(x) = \int_G \sigma(t, t^{-1}x)\chi(t)\chi_V(t^{-1}x)dt \colon = f(x) \in L^p(G)$$

for all $x \in G$ and note that that $\operatorname{supp}(f) \subseteq V^2$.

If the net converges, then every open set of y will contain the net eventually. We will assume that this is not the case, and show that this leads to a contradiction. Since U is any neighbourhood of the unit we assume that $y_j \notin yU$ for all j, then $y_j \notin yV^2V^{-2}$, and thus $y_jV^2 \cap yV^2 = \emptyset$, which implies that $f(y_j^{-1}x)$ and $f(y^{-1}x)$ have disjoint support, which again implies that $c(y_j, y_j^{-1}x)f(y_j^{-1}x)$ and $c(y, y^{-1}x)f(y^{-1}x)$ has disjoint support. Using these facts we have that

$$\left\|\gamma_j c(y_j, y_j^{-1}x)f(y_j^{-1}x) - \gamma c(y, y^{-1}x)f(y^{-1}x)\right\|_p = \left(\int_{y_jV^2} \left|f(y_j^{-1}x)\right|^p dx + \int_{sV^2} \left|f(y^{-1}x)\right|^p dx\right)^{1/p} > 0.$$

for all j. But we already shown that

$$\gamma_j c(y_j, y_j^{-1}x)f(y_j^{-1}x) = \gamma_j\lambda^\sigma_p(y_j)\lambda^\sigma_p(\chi_V)\chi_V(x) \xrightarrow{\|\cdot\|} \gamma\lambda^\sigma_p(y)\lambda^\sigma_p(\chi_V)\chi_V(x) = \gamma c(y, y^{-1}x)f(y^{-1}x),$$

and so we have a contradiction. Thus for every neighbourhood U containing the unit, y_j is eventually in yU for large enough j. Hence $y_j \to y$.

Finally it remains to show that $\gamma_j \to \gamma$. Since $y_j \to y$ in G, we have that $\chi_{y_j V^2} \to \chi_{y V^2}$ in $L^p(G)$. We also have that

$$\left\| \gamma_j \chi_{y_j V^2} - \gamma \chi_{y V^2} \right\|_p \geq \left(\int_{y_j V^2 \cap s V^2} |\gamma_j - \gamma|^p \, dt \right)^{1/p} = |\gamma_j - \gamma| \, \mu(y_j V^2 \cap s V^2)^{1/p}$$

and since the left side converges to zero we deduce that $|\gamma_j - \gamma| \to 0$ since $y_j V^2 \cap y V^2$ is non empty for j large enough. It follows that $\gamma_j \to \gamma$. ☐

Theorem 3.7. *Let G and H be two locally compact groups, let $p \in [1, \infty) \setminus \{2\}$, let σ be a continuous 2-cocycle on G and ρ be a continuous 2-cocycle on H. Then $F_\lambda^p(G, \sigma) \cong F_\lambda^p(H, \rho)$ if and only if $G \cong H$ as toplogical groups and $\sigma \sim \rho$.*

Proof. Assume that that there is an isometric isomorphism $F_\lambda^p(G, \sigma) \to F_\lambda^p(G, \rho)$, this induces an isometric isomorphism

$$\Phi \colon M_l(F_\lambda^p(G, \sigma)) \to M_l(F_\lambda^p(G, \rho))$$

Note that Φ and its inverse are norm continuous and thus strictly continuous, it follows that Φ is a homeomorphsim between the strict topology on $M_l(F_\lambda^p(G, \sigma))$ and $M_l(F_\lambda^p(G, \rho))$. We wll for the rest of the proof write $A = M_l(F_\lambda^p(G, \sigma))$ and $B = M_l(F_\lambda^p(H, \rho))$. This induces an isomorphism of toplogical groups

$$\phi \colon \mathcal{U}(A)_{str} \to \mathcal{U}(B)_{str}$$

and it follows that

$$\phi' \colon \pi_0(\mathcal{U}(A))_{str} \to \pi_0(\mathcal{U}(B))_{str}$$

is an isomorphism of topological groups. By Theorem 3.5 we have the following isomorphisms of topological groups

$$\Lambda_G \colon G_\sigma \xrightarrow{\cong} \mathcal{U}(A) \text{ and } \Lambda_H \colon H_\rho \xrightarrow{\cong} \mathcal{U}(B)$$

We thus have the following commuting diagram

$$\begin{tikzcd}
G_\sigma \arrow[r, "\Lambda_G"] \arrow[d, "(\gamma,g)\mapsto g"'] & \mathcal{U}(A)_{str} \arrow[r, "\phi"] \arrow[d, "v\mapsto[v]"] & \mathcal{U}(B)_{str} \arrow[d, "u\mapsto[u]"] & H_\rho \arrow[l, "\Lambda_H"'] \arrow[d, "(\gamma,h)\mapsto h"] \\
G \arrow[r, "\Lambda'_G"] & \pi_0(\mathcal{U}(A)_{str} \arrow[r, "\phi'"] & \pi_0(\mathcal{U}(B)_{str} & H \arrow[l, "\Lambda'_H"']
\end{tikzcd} \tag{1}$$

where the horizontal maps are isomorphisms of topological groups and the downward maps are quotient maps. It follows that $G \cong H$ as topological groups.

It remains to show $\rho \sim \sigma$. We have shown that $G_\sigma \cong H_\sigma$, denote this isomorphism by $\varphi \colon G_\sigma \to H_\sigma$. Since φ is an isomorphism, for every $g \in G$, we have that $\varphi((1, g) = (\gamma_g, h_g)$ for some unique $\gamma_g \in \mathrm{T}$ and some $h_g \in H$. This induces a continuous map $\gamma \colon G \to \mathbb{T}$ given by $\gamma(g) = \gamma_g$, and a map $h \colon G \to H$ given by $h(g) = h_g$. Observe that h is a continuous injective homomorphism. By the commutative diagram (1), we also have that $\varphi(\gamma, e_G) = (\gamma', e_H)$ for all $\gamma \in \mathbb{T}$ where e_G its the unite of G, e_H is the unit in H and γ' is some element in $\mathbb{T}$. This also induces a map $\pi \colon \mathbb{T} \to \mathbb{T}$ given by $\pi(\gamma) = \gamma'$, this is a continuous invective homomorphism of the unit circle, and since Φ is linear we deduce that $\gamma' = \gamma$. Now let $g_1, g_2 \in G$, then

$$\begin{aligned}
(\gamma(g_1g_2), h(g_1g_2) &= \varphi((1, g_1g_2)) \\
&= \varphi((\overline{\sigma(g_1, g_2)}, e)(1, g_1)(1, g_2)) \\
&= (\overline{\sigma(g_1, g_2)}, e)\varphi((1, g_1))\varphi((1, g_2)) \\
&= ((\gamma(g_1))(\gamma(g_2))\overline{\sigma(g_1, g_2)}, e)(1, h_{g_1})(1, h_{g_2}) \\
&= ((\gamma(g_1)(\gamma(g_2)\overline{\sigma(g_1, g_2)}, e))(\rho(h_{g_1}h_{g_2}), h_{g_1}h_{g_2}) \\
&= ((\gamma(g_1)(\gamma(g_2)\overline{\sigma(g_1, g_2)}\rho(h_{g_1}h_{g_2}), h_{g_1}h_{g_2})
\end{aligned}$$

We thus have that $\gamma(g_1g_2) = \gamma(g_1)\gamma(g_2)\overline{\sigma(g_1, g_2)}\rho(g_1, g_2)$. Without loss of generality we can assume that $G = H$. Rewriting the expression we deduce that $\sigma(g_1, g_2)\overline{\rho(g_1, g_2))} = \gamma(g_1)\gamma(g_2)\overline{\gamma(g_1g_2)}$ which implies that $\sigma \sim \rho$.

Conversely assume that $G \cong H$, we can without loss of generality assume $G = H$ and assume that $\sigma \sim \rho$, then there exists a map $\gamma\colon G \to \mathbb{T}$ such that $\sigma(x_1, x_2)\overline{\rho(x_1, x_2)} = \gamma(x_1)\gamma(x_2)\overline{\gamma(x_1x_2)}$ for every $x_1, x_2 \in G$. Define the map $\phi\colon L^1(G, \sigma) \to L^1(G, \rho)$ by $\phi(f(x)) = \gamma(x)f(x)$. It's easy to see that this is an surjective isometry, and we are going to show that this is also a homomorphsim. First note that $\sigma(y, y^{-1}x)\gamma(x) = \rho(y, y^{-1}x)\gamma(y)\gamma(y^{-1}x)$. We then have that

$$\begin{aligned}
(\phi(f) *_\rho \phi(f))(x) &= (\gamma f *_\rho \gamma g)(x) \\
&= \int_G f(y)\gamma(y)\gamma(y^{-1}x)\rho(y, y^{-1}x)\xi(y^{-1}x)dy \\
&= \gamma(x)(f *_\sigma g)(x) = \phi((f *_\sigma g)(x))
\end{aligned}$$

And so ϕ is an isometric isomorphism. This induces an isometric isomorphism

$$\tilde{\phi}\colon (L^1(G, \sigma), \|\cdot\|_{F^p_\lambda(G,\sigma)}) \to (L^1(G, \rho), \|\cdot\|_{F^p_\lambda(H,\rho)}),$$

which extends to the closures, thus $F^p_\lambda(G, \sigma)$ is isometrically isometric to $F^p_\lambda(H, \sigma)$. □

For $p = 2$ this is well know to not be the case. In fact, $F^2_\lambda(G, \sigma)$ is the reduced group C^*-algebra usually denoted $C^2_\lambda(G, \sigma)$. A well known counterexample is the fact that $C^*_\lambda(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is isometrically isomorphic to $C^*_\lambda(\mathbb{Z}_4)$, but $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$ are not isomorphic.

For the trivial twist Gardella showed that by in [8, Theorem 3.7] that $F^p(G) \cong F^p_\lambda(G)$ if and only if G is amenable. (This was also proved independently by Phillips) using this theorem we will prove that if G is amendable, then $F^p(G, \sigma) \cong F^p_\lambda(G, \sigma)$. To prove this we first need some result form Austad's paper [1].

We say that G is *amenable* if for every $\varepsilon > 0$ and for every compact subset $K \subseteq G$ there exists a compact subset $F \subseteq G$ such that $\mu(FK\Delta F) \leq \varepsilon\mu(F)$.
Example 3.8. Every compact group is amenable.

Let G be a locally compact group and σ a continuous 2-cocycle on G, and E an L^p-space. A *σ-projective isometric representation* of G is a strongly continuous map $\pi\colon G \to \mathcal{U}(\mathcal{B}(E))$ such that

(1) $\pi(x_1)\pi(x_2) = \sigma(x_1, x_2)\pi(x_1x_2)$
(2) $\pi(e) = \mathrm{Id}$.

It is well known that there is a natural bijective correspondence between σ-projective isometric representation of G on E and non degenerate σ-projective representation $L^1(G, \sigma) \to \mathcal{B}(E)$. If π is a σ-projective isometric representation, then the induced non generated σ-twisted representation $\pi\colon L^1(G) \to \mathcal{B}(E)$ is given by

$$\pi(f)(\xi)(x) = \int_G f(y)\pi(y)\xi(x)dy$$

for all $f \in L^1(G)$, and it is called the *integrated form of* π.

Definition 3.9. Let G be a locally compact group, let σ be continuous 2-cocycle in G and let $p \in [1, \infty)$. We define $F^p(G, \sigma)$ the *full σ-twisted group L^p-operator algebra* as the completion of $L^1(G, c)$ in the norm.

$$\|f\|_{F^p} = \sup\{\|\varphi(f)\| : \varphi\colon L^1(G) \to \mathcal{B}(E) \text{ is } \sigma\text{-projective contractivere representation.}\}$$

where E ranges over all L^p spaces.

The argument of Proposition 4.6 in [5] shows that this is in fact an L^p-operator algebra.

Remark 3.10. By the argument of Proposition 2.3 in [11], the full algebra can equivalently be defined as the completion of $L^1(G)$ with respect to non-degenerate σ-projective contractive representation

The σ-projective isometric representation of G induces an isometric representation of G_σ. This is done by sending $\pi\colon G \to \mathcal{U}(\mathcal{B}(L^p(G)))$ to $\pi_\sigma\colon G_\sigma \to \mathcal{U}(\mathcal{B}(L^p(G)))$ where

$$\pi_\sigma(\gamma, x) = \overline{\gamma}\pi(x)$$

We can map $L^p(G)$ isometrically to a subspace of $L^p(G_\sigma)$ with the map $j\colon L^p(G, \sigma) \to L^p(G_\sigma)$ given by

$$j(f)(\gamma, x) = \gamma f(x)$$

By Lemma 3.3 in [1], the embedding is isometric and we thus have that j$(L^p(G))$ is a closed subspace of $L^p(G_\sigma)$ for $p \in [1, \infty]$. For $p = 1$ the embedding is also an homomorphism, and so we have that $j(L^1(G, \sigma))$ is a closed subalgebra of $L^1(G_\sigma)$. By Fourier expanding the function of $L^p(G_\sigma)$ with respect to the second argument we can get an explicit description of the subspace $j(L^p(G, \sigma))$. For any $F \in L^p(G_\sigma)$ and any $x \in G_\sigma$ we have that $\gamma \mapsto F(\gamma, x)$ is a function in $L^p(\mathbb{T}) \subseteq L^p(G_\sigma)$ which means that the Fourier coefficients,

$$F_k(x) = \int_{\mathbb{T}} F(x, \gamma)\overline{\gamma}^n d\gamma$$

are well defined and that the resulting Fourier series

$$F(\gamma, x) = \sum_{k \in \mathbb{Z}} F_k(x)\gamma^k$$

converges in $L^p(\mathbb{T})$. By Lemma 3.4 in [1] we have that

$$j(L^p(G)) = \{F \in L^p(G_\sigma)\colon F_k = 0 \text{ for } k \neq 1\}$$

for $p \in [1, \infty]$. The Following proposition is Proposition 3.5 in [1].

Proposition 3.11. *Let G be a locally compact group and let σ be a continuous 2-cocycle on G. Let $F \in L^1(G_\sigma)$ and let $H \in L^p(G_\sigma)$ Then*

$$(F * H)(\gamma, x) = \sum_{n \in \mathbb{Z}} (F_n *_{c^n} H_n)(x)\gamma^n$$

Let $f \in L^1(G, c)$. From Lemma 3.3 in [1] it follows that

$$\lambda_p(j(f))j(\xi) = j(f) * j(\xi) = j(f *_\sigma g)) = j(\lambda_p^\sigma(f)\xi)$$

for all $\xi \in L^p(\mathcal{G})$, and from the Proposition 3.11 we deduce that $\left\|\lambda_p^\sigma(f)\right\|_{\mathcal{B}(L^p(G)} = \|\lambda_p(j(f)\|_{\mathcal{B}(L^p(G_\sigma)}$. It follows that j map $F_\lambda^p(G, \sigma)$ isometrically to a closed subspace of $F_\lambda^p(G_\sigma)$.

Let π be σ-projective representation of G on E for some L^p space E. Let $f \in L^1(G, c)$, we have that

$$\pi_c(j(f))(\xi) = \int_G \int_{\mathbb{T}} \alpha f(y)\overline{\alpha}\pi(y)\xi d\alpha dy = \pi(f)\xi,$$

which implies that $\|\pi(f)\|_{\mathcal{B}(E)} = \|\pi_c(j(f))\|_{\mathcal{B}(E)}$.

Theorem 3.12. *Let $p \in (1, \infty)$, let G be a locally compact group and let σ be a continuous 2-cocycle. If G is amenable, then there is an isometric isomorphism*

$$F^p(G, c) \cong F^p_\lambda(G, c).$$

Proof. Assume that G is amenable, since $\mathbb{T}$ is compact, this implies that G_σ is amenable, and by Theorem 3.7 in [9] we have that $F^p_\lambda(G_\sigma) \cong F^p(G_\sigma)$. Let $f \in L^1(G, \sigma)$, and let π be any σ-projective representation of G on some L^p-space E. We want to show that $\|\pi(f)\|_{\mathcal{B}(E)} \leq \|\lambda^\sigma_p(f)\|_{\mathcal{B}(L^p(G))}$. Assume that this is not the case, i.e. that $\|\pi(f)\|_{\mathcal{B}(E)} > \|\lambda^\sigma_p(f)\|_{\mathcal{B}(L^p(G))}$. Note that π induces a representation π_σ of G_σ on E and that $\|\pi(f)\|_{\mathcal{B}(E)} = \|\pi_c(j(f))\|_{\mathcal{B}(E)}$. We then have, by the definition, that $\|\pi_c(j(f)\|_{\mathcal{B}(L^p(G_\sigma)} \leq \|j(f)\|_{F^p(G_\sigma}$, but since G is amenable $\|j(f)\|_{F^p(G_\sigma)} = \|\lambda_p(j(f)\|_{\mathcal{B}(L^p(G_\sigma)}$. Thus, we have that

$$\begin{aligned}
\|\pi(f)\|_{\mathcal{B}(E)} &> \|\lambda^c_p(f)\|_{\mathcal{B}(L^p(G))} \\
&= \|\lambda_p(j(f)\|_{\mathcal{B}(L^p(G_\sigma))} \\
&= \|j(f)\|_{F^p(G_\sigma)} \\
&\geq \|\pi_c(j(f)\|_{\mathcal{B}(E)} = \|\pi(f)\|_{\mathcal{B}(E)},
\end{aligned}$$

which is a contradicting. It follows that $\|f\|_{F^p(G,\sigma)} = \|\lambda^\sigma_p(f)\|$ for all $f \in L^1(G, c)$ □

Remark 3.13. Let G be a locally compact group. For $p = 1$ one can easily deduce, as in Proposition 4.9 in [5], that $F^1(G, c) = F^1_\lambda(G) = L^1(G, c)$.

On consequence of this is that the reduced twisted group algebra generated by an amenable group can be characterised in terms of generators and relations.

Example 3.14. Define the function $\sigma \colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$ by $\sigma_\theta((m, n), (p, q)) = e^{2\pi i n p\theta}$ for some irrational number $\theta \in \mathbb{R} \setminus \mathbb{Q}$. This is a continuous 2-cocycle on $\mathbb{Z}^2$ and for $p = 2$, we have that $F^2(\mathbb{Z}^2, \sigma_\theta) = C^*(\mathbb{Z}^2, \sigma_\theta)$, which is the irrational rotation algebra, denoted A_θ. Analogous to this, we will call $F^p(\mathbb{Z}^2, \sigma_\theta)$ *the p-irrational rotation* and denote it by A^p_θ. Write $U_\theta = \delta_{(0,1)}$ and $V_\theta = \delta_{(0,1)}$, these are the generators of $F^p(\mathbb{Z}^2, \sigma_\theta)$ and we have that $U_\theta V_\theta = e^{2\pi i\theta} V_\theta U_\theta$. For $p = 2$ and $\theta = 0$ we have, by the Gelfand transoform, that $F^2(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$, the space of continuous function on the 2-torus. This is why this algebra for is also known as the non commutative tori.

Note that $\mathbb{Z}^2$ is an amenable group and as we have just shown this implies that $F^p(\mathbb{Z}^2, \sigma_\theta) \cong F^p_\lambda(\mathbb{Z}^2, \sigma_\theta)$. Furthermore, let $p \in [1, \infty) \setminus \{2\}$. then, by Theorem 3.7, we have that $F^p_\lambda(\mathbb{Z}^2, \sigma_\theta) \cong F^p_\lambda(\mathbb{Z}^2, \sigma_\phi)$ if and only if $\sigma_\theta \sim \sigma_\phi$, that is, if there exists a continuous map $\gamma \colon \mathbb{Z}^2 \to \mathbb{T}$ such that

$$e^{2\pi i n p\theta} e^{-2\pi i n p\phi} = e^{2\pi i n p(\theta - \phi)} = \gamma(m, n)\gamma(p, q)\overline{\gamma(m + p, n + q)}.$$

Note that $\gamma(1, 0)\gamma(0, 1)\overline{\gamma(1, 1)} = 1$ and $\gamma(0, 1)\gamma(1, 0)\overline{\gamma(1, 1)} = e^{2\pi i(\theta - \phi)}$, and so $e^{2\pi i(\theta - \phi)} = 1$ which means that $\theta - \phi \in \mathbb{Z}$. conversely if $\theta - \phi \in \mathbb{Z}$, then setting $\gamma \equiv 1$ shows that $\sigma_\theta \sim \sigma_\phi$.

4 Étale Groupoids and C^*-cores in L^p operator algebras

4.1 The etalé groupoid

Intuitively a groupoid is a set with partially defined multiplication where the usual group properties hold whenever they make sense, and generalise the notion of groups. We will define groupoids following [14].

A Groupoid is most naturally defined in the language of category theory, where one define a Groupoid is a *small category with inverses.* So let $\mathcal{G}$ be a small category with inverses, and let $\mathcal{G}^{(0)}$ denote its objects. As the category is small, $\mathcal{G}^{(0)}$ forms a set, the set of *units* of $\mathcal{G}$. We then identify the groupoid with the morphisms where the elements are "arrows" from one object to another object, we denote the source and the range of γ by $s(\gamma)$ and $r(\gamma)$ respectively, we can think of multiplication as composition of the "arrows", and in intuitively multiplication $\alpha\beta$ of elements $\alpha, \beta \in \mathcal{G}$ makes sense if and only if $s(\alpha) = r(\beta)$. Since the groupoid is a category with inverses, every element γ has a inverse γ^{-1}. Every unit can therefor be identified with the identity morphism at the unit. The unit space is under this identification the subset of $\mathcal{G}$ of elements $\gamma\gamma^{-1}$ where γ ranges over $\mathcal{G}$. The range map and source map $r, s \colon \mathcal{G} \to \mathcal{G}^{(0)}$ are thus onto maps given by

$$r(\gamma) = \gamma^{-1}\gamma \qquad \text{and} \qquad s(\gamma) = \gamma\gamma^{-1}. \tag{2}$$

The set of pairs in $\mathcal{G} \times \mathcal{G}$ where multiplication is defined is denoted $\mathcal{G}^{(2)}$, and is called the set of *composable paris*, and is consists of elements $(x, y) \in \mathcal{G} \times \mathcal{G}$ such that $s(x) = r(y)$, in other terms, the morphisms x and y compose.

This is a generalisation of the group in the sense that a group is then a small category with inverses with only one unit, which means that every morphisms trivially compose. i.e multiplication is everywhere defined as we would expect in a group.

In our functional analysis context we want to think of groupoid as we do with groups, as a abstract set equipped certain properties, for instance toplogical properties. We therefore need to define groupoids axiomatic, we will use the definition in [14].

Definition 4.1. A groupoid is a set $\mathcal{G}$ together with a subset $\mathcal{G}^2 \subset \mathcal{G} \times \mathcal{G}$, a product map $\mathcal{G}^2 \to \mathcal{G}$ given by $(\alpha, \beta) \mapsto \alpha\beta$ and a inverse map $\mathcal{G} \to \mathcal{G}$ given by $\alpha \mapsto \alpha^{-1}$ such that

(1) If $(\alpha, \beta), (\beta, \gamma) \in \mathcal{G}^2$, then $(\alpha\beta, \gamma), (\alpha, \beta) \in \mathcal{G}^2$ and

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

(2) $\left(\beta, \beta^{-1}\right) \in \mathcal{G}^2$ and if $(\alpha, \beta) \in \mathcal{G}^2$ then

$$\alpha^{-1}(\alpha\beta) = \beta \qquad\qquad (\alpha\beta)\beta^{-1} = \alpha$$

For each unit $x \in \mathcal{G}^0$ we define

$$\mathcal{G}^x = \{\gamma \in \mathcal{G} \colon r(\gamma) = x\}$$

and

$$\mathcal{G}_x = \{\gamma \in \mathcal{G} \colon s(\gamma) = x\}$$

Given $x \in \mathcal{G}^0$. Note that the set $\mathcal{G}_x^x = \mathcal{G}_x \cap \mathcal{G}^x = \{\gamma \in \mathcal{G} \colon r(\gamma) = s(\gamma) = x\}$ is closed under product and inversion and therefor is a group. We call the group $\mathcal{G}_x^x$ the *isotropy group* at x. One say that x has *tricial isotropy* if $\mathcal{G}_x^x$ only contains x. The set $\mathcal{G}' = \{\gamma \in \mathcal{G} \colon s(\gamma) = r(\gamma)\}$ is called the isotropy bundle. A *topological groupoid* is groupoid $\mathcal{G}$ endowed with a topology with the structure maps continuous. The topology on $\mathcal{G}^{(2)}$ is the relative topology inherited from the product topology $\mathcal{G} \times \mathcal{G}$.

Definition 4.2. A *locally compact groupoid* is a groupoid $\mathcal{G}$ endowed with a topology having a countable basis of Hausdorff open sets with compact closures, such that

(1) multiplication and inversion of arrows are continuous maps

(2) the set of objects $\mathcal{G}^0$, as well as $\mathcal{G}_x$ and $\mathcal{G}^x$ for every $x \in \mathcal{G}^0$, are locally compact Hausdorff spaces in the relative topology inherited from $\mathcal{G}$.

Locally compact groups are the locally compact groupoids with only one object.

Definition 4.3. A locally compact groupoid $\mathcal{G}$ is called *étale* if the range map $r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism.

This implies that the source map is also a local homeomorphism. Note that if $\mathcal{G}$ is étale, then $\mathcal{G}_x$ and $\mathcal{G}^x$ are discrete in the relative topology, which will be important later. We will in the thesis only work with étale groupoid, the study of groupoid algebras in general requires a significant amount of representation-theoretic background, but étale groupoid algebras is sufficient to capture a lot of examples as Renault showed. This is analogue to how discrete groups group algebra setting. The trivial example of étale groupoid is the discrete group

Definition 4.4. Let $\mathcal{G}$ be a locally compact étale groupoid. $\mathcal{G}$ is said to be *topological principal* if the set of points in $\mathcal{G}^0$ with trivial isotropy is dense in $\mathcal{G}^0$. $\mathcal{G}$ is said to be *effective* if the interior $\mathcal{G}'$ is $\mathcal{G}^0$.

Let $\mathcal{G}$ be an étale groupoid. If $\mathcal{G}$ is topologically principal and Hausdorff, then $\mathcal{G}$ is effective, the converse is not necessarily true. The typical example of topological principal groupoid is the transformation groupoid of a topological free action of a discrete group. We will visit the transformation groupoid in section 7.

Open bisections will play a key role in the following sections. A subset $S \subseteq \mathcal{G}$ is called a *bisection* if there exists an open subset $U \subseteq \mathcal{G}$ containing S such that the restriction of the range and source map to U are local homeomorphsim. If S is open then we call S an *open bisection* and the restriction of the range and sours map to S are local homeomorphsims. Note that this is the case if and only if the restriction of the range and sours map to S is injective.

4.2 The C^*-core and the Weyl groupoid

The way we will show the rigidity results is by identifying something we will call the *C^*-core* of the algebras. The results of the section comes from from Section 2 and 3 of the paper [3] by Choi, Gardella and Thiel. We will not treat the details of these results and will refer to [3] for details and proofs, but note that the most important result of the $C^* - core$ which lets us prove the L^p-rigidity results is a consequence of the Lamperti Theorem.

Let A be a unital algebra. An element a of A is called *hermitian* if $\left\|e^{ita}\right\| = 1$ for all $t \in$ R. We denote the set of hermitian elements of A by A_h, which is a closed real linear subspace satisfying $A_h \cap iA_h = \{0\}$.

If A is a unital C^*-algebra, then A consists of the self adjoint elements and $A = A_h + iA_h$. The Vidav–Palmer theorem shows that the converse also hols. So if A is a unital Banach algebra with $A = A_h + iA_h$, then the real-linear involution $x + iy \mapsto x - iy$ is both isometric and an algebra involution that satisfy the C^* identify. This motivates the following definition:

Definition 4.5. Let A be a unital Banach algebra, and let $B \subseteq A$ be a unital closed subalgebra. We say that B is a *unital C^*-subalgebra* of A if $B = B_h + iB_h$.

The following theorem is Theorem 2.9 in [3] and will form the backbone of the main results of the thesis.

Theorem 4.6. *Let $p \in [1, \infty)$, and let A be unital L^p-operator algebra. Set core$(A) = A_h + iA_h$. Then core(A) is the largest unital C^*-subalgebra of A. If $p \neq 2$, then core(A) is commutative.*

The last statement for $p \neq 2$ is a consequence of the much mentioned the Lamperti Theorem, which we gave a detailed exposition in the preliminary specialisation Project. As with the rigidity results in previous section it is this that gives rise to the interesting rigidity results for $p \neq 2$.

Definition 4.7. Let $p \in [1, \infty)$, and let A be a unital L^p-operator algebra. We call the algebra $\mathrm{core}(A)$ for the *C^*-core* of A.

The C^*-core in our case will play the same role as the *maximal abelian subalgebra* does for C^*-algebras, but there two important difference. Firstly the it is C^*-core unique, which will give rise to the rigidity results. Secondly, it is very small, and in many cases it is to small to carry any useful information about the structure of the algebra. The last point will be why we only consider effective groupoids.

Another useful proposition from [3] that will be important is the following

Proposition 4.8. *Let $p, q \in [1, \infty)$, let A be a unital L^p-operator algebra, an let B be a unital L^q-operator algebra. Let $\varphi\colon A \to B$ be a untial contractive linear map. Then $\varphi(core(A)) \subseteq core(B)$ and $\varphi\colon core(A) \to core(B)$ is a $*$-homomorphsim*

Now let A be a unital L^p-operator algebra, for $p \neq 2$, we now know from Theorem 4.6 that the C^*-core is a commutative unital C^*-subalgebra. We will write X_A for its spectrum. Recall that the *spectrum* of an Banach algebra its the set of characters endowed with the w^*-topology. For an unital commutative Banach algebras the spectrum is a compact Hausdorff space, as is the case for X_A, and by the Gelfand Theorem we can identify the core with $C(X_A)$ [13].

We will denote the subset of continuous non-negative real functions by $C(X_A)_+$, this is the set of positive hermitian elements in in A.

Definition 4.9. Let A be a unital L^p-operator algebra. Given open subsets $U, V \subseteq X_A$, and a homeomorphsims $\alpha\colon U \to V$, we say that α is *realisable* (within A) if there exists $a, b \in A$ satisfying the following:

(1) For every $f \in C(X_A)_+$, we have that $bfa, afb \in C(X_A)_+$
(2) $U = \{x \in X_A\colon ba(x) > 0\}$ and $V = \{x \in X_A\colon ab(x) > 0\}$
(3) for $x \in U$, $y \in V$, $f \in C_0(V)$ and $g \in C_0(U)$ we have that

$$f(\alpha(x))ba(x) = bfa(x) \text{ and } g(\alpha^{-1}(y)ab(y) = agb(y)$$

We then say that $n = (a, b)$ is the *admissible pair* that *realises* α, and we write α_n, U_n and U_n for α, U and V respectively.

The realisable pairs will play the role as the L^p-analogue of the normalizers used by Renault in [17] in the context of Cartan pairs of C^*-algebras. i.e the pair (a, b) replaces the pair (a, a^*) where a is a normalizers. In our setting of L^p-operator algebras, there are number of difficulties arising from the lack of canonical involution.

Proposition 4.10. *Let $p \neq 2$, let A be a unital L^p-operator algebra and let $n = (a, b)$ and $m = (c, d)$ be two admissible pairs in A that realises α_n and α_m respectively.*

(1) The inverse of α_n is realised by the reverse of n witch is defined as $n^\sharp = (b, a)$
(2) the product $nm = (ac, db)$ realises the composition

$$\alpha_n \circ \alpha_m \restriction_{U_m \cap \alpha_m^{-1}(U_n)} \colon U_m \cap \alpha_m^{-1}(U_n) \to U_n \cap \alpha_n(V_m)$$

(3) For every $f \in C(X_A)$, the pair $n_f = (f, \overline{f})$ is admissible and realises $\alpha_{n_f} = Id_U$ for $U = U_{supp(f)}$. In particular the identity map on every open set of X_A is realisable.

Proof. (1) follows immediately from the definition in 4.9 were we easily deduce that (b, a) realises the inverse homeomorphsim $U_n \to U_n$ and is thus the inverse of n.

For condition (2), we need to show that $U_{nm} = U_m \cap \alpha_m^{-1}(U_n) = \{x\colon dcba(x) > 0\}$ Noting that $ba \in C(U_n)$ and using condition (3) we have that $dbac(x) = ba(\alpha_m(x))dc(x)$. This is greater then zero if $x \in U_m$ and $\alpha_m(x) \in U_n$. Similarly one can show that $V_{nm} = U_n \cap \alpha_n(V_m)$

Now to show (3), let $f \in C_0(V_{st})$. Using condition (3) for n applied to f and for m applied to bfa, we have the following: $f(\alpha_n(\alpha_m(x)))dbac(x) = f(\alpha_n(\alpha_m(x)))ba(\alpha_m(x))cd(x) = bfa(\alpha_m(x))dc(x) = dbfac(x)$, the statement follows immediately using that $C(X_A)$ is commutative. □

Definition 4.11. A semigroup N is called an *inverse semigroup* if for each $n \in N$, there is a unique $m \in N$ such that

$$nmn = n \qquad\qquad mnm = m \tag{3}$$

we write the element m as $n^\sharp$. The map $n \mapsto n^\sharp$ is called the involution on N

A *partial homeomorphism* of a topological space X is a homeomorphism $U \to V$ between open subsets U and V of X. Let X be a compact Hausdorff space, we denote $\mathrm{Homeo}_{par}(X)$ for the set of partial homeomorphisms of X, which forms a inverse semigroup. Let $\varphi \in \mathrm{Homeo}_{par}(X)$, we denote $\mathrm{dom}(\varphi)$ for the domain of φ in X.

The following corollary follows from Proposition 4.10.

Corollary 4.12. *Let A be a unital L^p-operator algebra, then the set of realisable partial homeomorphism on X_A, denoted $N(A)$ is an inverse subsemigroup of $Homeo_{par}(X_A)$.*

Let X be a compact Hausdorff space, and let N be an inverse subsemigroup of $\mathrm{Homeo}_{par}(X)$. *The groupoid of germs*, $\mathcal{G}(N)$, of N is defined as follows. On the set

$$\{(n, x) \in N \times X : b \in N, x \in \mathrm{dom}(n)\}$$

we define the following equivalence class. $(n, x) \sim (m, y)$ whenever $x = y$ and there exists neighbourhood U of x in X such that $n_{\restriction_U} = m_{\restriction_U}$. We write $[n, x]$ for the equivalence class of (m, x). Then $\mathcal{G}(N)$ has a natural groupoid structure with $r([n, x]) = n(x)$ and $s([n, x]) = x$, multiplication given by

$$[n, m \cdot y][m, y] = [nm, y]$$

and inverse given by

$$[n, x]^{-1} = [n^\sharp, n(x)]$$

The groupoid of germs $\mathcal{G}(N)$ becomes an étale groupoid under the topology given with basic open sets $\mathcal{U}(U, n) = \{[n, x] : x \in U\}$ indexed by elements $n \in N$ and open set $U \subseteq \mathrm{dom}(n)$. The unit space of $\mathcal{G}(B)$ can be canonically defined with X and is thus a compact Hausdorff space. For details see Section 3 of [17].

Definition 4.13. Let A be a unital L^p-operator algebra. We define the *Weyl groupoid* of A denoted $\mathcal{G}_A$ to be the groupoid of germs of $N(A)$, the inverse subsemigroup of realisable partial homeomorphisms of X_A.

The Weyl groupoid is sometimes to small to carry any information about the underlying groupid. For instance when G is a discrete group then $\mathcal{G}_{F^p_\lambda(G)}$ is the trivial groupoid. This is because a discrete group has very large isotropy.(the core is too small). We will therefor be interested in the class of topological principal groupoid, this property ensure that the groupoid has small isotropy, which means that groupoid of germs will be a useful tool.

The following remarsk are from the beginning of Section 3 of [17].

Remark 4.14. Let $\mathcal{G}$ be an étale groupoid, then the open bisection forms an inverse semigroup denoted $\mathcal{S}(\mathcal{G})$. Note that given two subset $U, V \in \mathcal{G}$ composition is given as $UV = \{uv : s(u) = r(v)\}$. It follows that for two open bisection the composition is also an open bisection and that the composition is associative. Given and open bisection B we also have $BB^{-1}B = B$ and $B^{-1}BB^{-1} = B^{-1}$.

Remark 4.15. Note that any open bisection $B \in \mathcal{S}(\mathcal{G})$ defines a homeomorphism $\beta_B : s(B) \to r(B)$ given by $\beta_B(x) = r(Bx)$ for all $x \in s(B)$. Moreover, since the set of partial metamorphism forms an inverse semigroup, the induced map $\beta : S(G) \to \mathrm{Homeo}_{\mathrm{par}}(\mathcal{G}^{(0)})$ is an inverse semigroup homomorphism. We let $\mathcal{P}(\mathcal{G})$ denote the image of β. By Corollary 3.4 in [17], the groupoid of germs of $\mathcal{P}(\mathcal{G})$ is isomorphic to $\mathcal{G}$ if and only if $\mathcal{G}$ is effective. Moreover, if this is the case, then β identifies $\mathcal{S}(\mathcal{G})$ bijectively with $\mathcal{P}(\mathcal{G})$.

5 The reduced σ-twisted groupoid L^p-operator algebra

5.1 The reduced groupoid L^p-operator algebra

In this section we will define the reduced groupoid L^p-operator algebra following Choi, Gardella and Thiel's paper [3].

Let $\mathcal{G}$ be an étale groupoid. If U is an open Hausdorff subset of $\mathcal{G}$, then $C_c(U)$ is the space of compactly supported continuous function on U. We define $C_c(\mathcal{G})$ to be the complex linear span of functions on $C_c(U)$ ranging over all open Hausdorff subsets U of $\mathcal{G}$. In the case when $\mathcal{G}$ is a Hausdorff étale groupoid, this definition coincides with usual definition of $C_c(\mathcal{G})$, namely the compactly supported continuous functions on $\mathcal{G}$.

For the rest of the section $\mathcal{G}$ will denote a locally compact, Hausdorff, étale groupoid.

for $f, g \in C_c(\mathcal{G})$ we define convolution by

$$(f * g)(\gamma) = \sum_{\alpha \in \mathcal{G}_{s(\gamma)}} f(\gamma\alpha^{-1})g(\alpha) = \sum_{\alpha \in \mathcal{G}^{r(\gamma)}} f(\alpha)g(\alpha^{-1}\gamma)$$

for all $\gamma \in \mathcal{G}$. Together with pointwise addition and scalar multiplicaton, this makes $C_c(\mathcal{G})$ a complex unital algebra. [18, proposition 3.2]

For a locally compact étale Hausdorff groupoid $\mathcal{G}$, the relative topology on $\mathcal{G}_x$ and $\mathcal{G}^x$ is discrete for all $x \in \mathcal{G}$, this means that the elements in $C_c(\mathcal{G}_x)$ are finite liner combination of characteristic functions δ_γ for $\gamma \in \mathcal{G}_x$. Using this, we show that one can define convolution between elements in $C_c(\mathcal{G}_x)$ and $l^p(\mathcal{G}_x)$. We first define the I-norm, which can be thought of a a fibrewise 1-norm.

Definition 5.1. Let bbbgfr $f \in C_c(\mathcal{G})$, we define the *I-norm* as follows.

$$\|f\|_I = \max\left\{\sup_{x\in\mathcal{G}} \sum_{\gamma\in\mathcal{G}_x} |f(\gamma)|, \sup_{x\in\mathcal{G}} \sum_{\gamma\in\mathcal{G}^x} |f(\gamma)|\right\}$$

Proposition 5.2. *For $p \in [1, \infty]$ and $x \in \mathcal{G}^0$. Let $f \in C_c(\mathcal{G})$, and let $\xi \in C_c(\mathcal{G}_x)$. Then $f * \xi \in C_c(\mathcal{G}_x)$, and*

$$\|f * \xi\|_p \leq \|f\|_I \|\xi\|_p$$

It follows that there exists a unique contractive representation $\lambda_x : C_c(\mathcal{G}) \to \mathcal{B}(l^p(\mathcal{G}_x))$ satisfying $\lambda_x(f)\xi = f * \xi$ where $f \in C_c(\mathcal{G})$ and $\xi \in C_c(\mathcal{G}_x)$

Proof. The continuity of $f*\xi$ follows from the Dominated convergence theorem and by Proposition 3.2 in [18] we know that $\operatorname{supp}(f*\xi) \subseteq \operatorname{supp}(f)\operatorname{supp}(\xi)$ which implies that $f*\xi$ has compact support on $\mathcal{G}_x$. It remains to show the bound. Using the Minkowski's inequality we have the following inequalities

$$\|f * \xi\|_1 = \sum_{\gamma\in\mathcal{G}_x} \left| \sum_{\alpha\in\mathcal{G}_x} f(\gamma\alpha^{-1})\xi(\alpha) \right| \leq \sum_{\alpha\in\mathcal{G}_\S} |\xi(\alpha)| \sum_{\gamma\in\mathcal{G}_x} |f(\gamma\alpha^{-1})| \leq \|f\|_I \|\xi\|_1$$

$$\|f * \xi\|_\infty = \sup_{\gamma\in\mathcal{G}_x} \left| \sum_{\alpha\in\mathcal{G}_x} f(\gamma\alpha^{-1})\xi(\alpha) \right| \leq \sup_{\gamma\in\mathcal{G}_x} \left(\sum_{\alpha\in\mathcal{G}_x} |f(\gamma\alpha^{-1})| \right) \sup_{\alpha\in\mathcal{G}_x} |\xi(\alpha)| \leq \|f\|_I \|\xi\|_\infty$$

Thus the convolution operator is bounded in the 1- and ∞-norm in $C_c(\mathcal{G}_x)$ with norm atmost $\|f\|_I$. By the Riesz-Thorin interpolation theorem this holds for all p (We don't need to generalise to localizable since $\mathcal{G}_x$ is discrete, and thus σ-finite)

$\square$

We call λ_x the *left regular representation* of $\mathcal{G}$ associated to x. We can now define the reduced groupoid L^p-operator algebra.

Definition 5.3. Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid, and let p$\in [1, \infty)$. we define $F^p_\lambda(\mathcal{G})$, the *reduced groupoid L^p-operator algebra* of $\mathcal{G}$ as the completion of of $C_c(\mathcal{G})$ in the norm

$$\|f\|_\lambda = \sup\{\|\lambda_x(f)\| : x \in \mathcal{G}^0\}$$

Theorem 5.4. *Let $\mathcal{G}$ be a locally compact étale Hausdorff groupoid and let $p \in [1, \infty)$. Then $F^p_\lambda(\mathcal{G})$ is an L^p-operator algebra.*

Proof. By Lemma 1.22 in [15] we have that $E = \bigoplus\limits_{x\in\mathcal{G}^{(0)}} l^p(G_x)$ is an L^p-space. Let $\bigoplus_{x\in\mathcal{G}^0} \lambda^p_x \colon C_c(\mathcal{G}) \to \mathcal{B}(E)$ be the L^p direct sum of the representations as described in Lemma 1.22 in [15]. Given some $f \in C_c(G)$, we have

$$\| \bigoplus_{x\in\mathcal{G}^{(0)}} \lambda_x(f)\| = \sup_{x\in\mathcal{G}} \|\lambda_x(f)\| = \|f\|_\lambda .$$

□

We are most interested in the following result which is Theorem 5.6 in [3]. It implies that a large class of groupoids can be recovered from their reduced groupoid L^p-operator algebras. Our main result of this thesis is an extension of this rigidity results to twisted convolution.

Theorem 5.5. *Let $\mathcal{G}$ be topological principal, Hausdorff, étale groupoid with compact unit space, and let $p \in [1, \infty) \setminus \{2\}$. Then there exists natural identification(of groupoids)*

$$\mathcal{G}_{F^p_\lambda(\mathcal{G}} \cong \mathcal{G}$$

In more familiar terms, this implies the following corollary.

Corollary 5.6. *Let $\mathcal{G}$ and $\mathcal{H}$ be topological principal, Hausdorff, étale groupoid with compact unit space, and let $p \in [1, \infty) \setminus \{2\}$. Then there exists a isometric isomorphism $F^p_\lambda(\mathcal{G}) \cong F^p_\lambda(\mathcal{H})$ if and only if there is a groupoid isomorphism $\mathcal{G} \cong \mathcal{H}$.*

5.2 The reduced σ-twisted groupoid L^p-operator algebra

As with the group we can define twisted convolution for the groupoid algebra using a $2-$*cocycle*. This lets us define the L^p equivalent of the twisted groupoid C^*-algebras as defined in [16], and is a natural generalisation of the groupoid L^p -operator algebra from the last section

Definition 5.7. A *normalised continuous 2-cocycle* for a toplogical groupoid is a continouous map $\sigma \colon \mathcal{G}^2 \to \mathbb{T}$ satisifying the following:

(1) $\sigma(r(\gamma), \gamma) = \sigma(\gamma, s(\gamma)) = 1$ for all $\gamma \in \mathcal{G}$
(2) $\sigma(\alpha, \beta)\sigma(\alpha\beta, \gamma) = \sigma(\beta, \gamma)\sigma(\alpha, \beta\gamma)$ whenever $(\alpha, \beta), (\beta, \gamma) \in \mathcal{G}^2$

Definition 5.8. let σ, ω be two normalised continuous 2-cocycles on $\mathcal{G}$, we say that σ is *cohomologous* to ω, if there exists a continuous function $\gamma \colon \mathcal{G} \to \mathbb{T}$ such that $\sigma(\alpha, \beta)\overline{\omega(\alpha, \beta)} = \gamma(\alpha)\gamma(\beta)\overline{\gamma(\alpha\beta)}$ for all $(\alpha, \beta) \in \mathcal{G}^{(2)}$. γ is called a *coboundary.*

Given a normalised continuous 2-cocycles on $\mathcal{G}$ we will write $[\sigma]$ for the equivalent class of cocycles cohomologous to σ. The collection of equivalent class $H^2(\mathcal{G}, \mathbb{T})$ is know as the *second cohomology group* and forms an Abelian group.

Note that if $\mathcal{G}$ is a locally compact group this definition is reduced to the one for continuous 2-cocycle for groups as in section 1. Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid, and let σ be a normalised 2-cocycle. We can then define the σ-twisted convolution on $C_c(\mathcal{G})$ given by

$$(f *_\sigma g)(\gamma) = \sum_{\alpha\in s(\gamma)} f(\gamma\alpha^{-1})g(\alpha)\sigma(\gamma\alpha^{-1}, \alpha) = \sum_{\alpha\in\mathcal{G}^{r(\gamma)}} f(\alpha)g(\alpha^{-1}\gamma)\sigma(\alpha, \alpha^{-1}\gamma)$$

for $f, g \in C_c(\mathcal{G})$ and all $\gamma \in \mathcal{G}$. We denote the complex algebra formed by σ-twisted convolution by $C_c(\mathcal{G}, \sigma)$. We can also define an involution product on $C_c(\mathcal{G}, \sigma)$ given by

$$f^{*_\sigma}(\gamma) = \overline{\sigma(\gamma^{-1}, \gamma) f(\gamma^{-1})}$$

Just as with Proposition 5.2, for $p \in [1, \infty]$, we define the mapping $\lambda_x^\sigma \colon C_c(G, \sigma) \to l^p(\mathcal{G}_x)$ to be the unique map sucht that $\lambda_x^\sigma(f) = f *_\sigma \xi$. for all $\xi \in C_c(\mathcal{G}_x)$. We call λ_x^σ the *σ-twisted left regular representation* of $\mathcal{G}$ associated to x.

Definition 5.9. Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid, let σ be a 2-cocycle and let $p \in [1, \infty)$. we define $F_\lambda^p(\mathcal{G}, \sigma)$, the *reduced σ-twisted groupoid L^p-operator algebra* of $\mathcal{G}$ as the completion of of $C_c(\mathcal{G}, \sigma)$ in the norm

$$\|f\|_{\lambda^\sigma} = \sup\{\|\lambda_x^\sigma(f)\|_p : x \in \mathcal{G}^0\}$$

As in Proposition 5.4 we have that $\bigoplus_{x \in \mathcal{G}^0} \lambda_x^\sigma$ is an isometric representation of $F_\lambda^p(\mathcal{G}, \sigma)$ on some L^p-space. It follows that $F_\lambda^p(\mathcal{G}, \sigma)$ is an L^p-operator algebra. For the trivial twist this is the reduced L^p-operator algebra from the previous section.

6 The reduced twisted L^p-operator algebra and L^p-rigidity

In section 5 we defined the reduced twisted groupoid algebra from a normalised continuous 2-cocycle. In the following section we will define a more general notion of twisting the groupoid algebra, from something called a *twist*. We will then show an L^p-analogue of Renault reconstruction theorem in [17].

6.1 The reduced twisted groupoid L^p-operator algebra

We will follow Chapter 5 in [18] in defining the twist.

Definition 6.1. Let $\mathcal{G}$ be an étale groupoid. A *twist* over $\mathcal{G}$ is an exact sequence

$$0 \to \mathcal{G}^{(0)} \times \mathbb{T} \xrightarrow{i} \mathcal{E} \xrightarrow{\pi} \mathcal{G} \to 0$$

where $\mathcal{G}^{(0)} \times \mathbb{T}$ is regarded as a trivial group bundle with fibres $\mathbb{T}$, $\mathcal{E}$ is a locally compact Hausdorff groupoid, and i and π are continuous groupoid homomorphisms that restrict to homeomorphisms of unit spaces. The exact sequence need to satisfy the following

(1) i is injective

(2) $\mathcal{E}$ is locally a trivial $\mathcal{G}$-bundle, i.e. every point $\gamma \in \mathcal{G}$ has a bisection neighbourhood U such that there exists a continuous section $S\colon U \to \mathcal{E}$ satisfying $\pi \circ S = \mathrm{id}_U$ and such that the map $(\gamma, z) \mapsto i(r(\gamma), z)S(\gamma)$ is a homeomorphism $U \times \mathbb{T} \to \pi^{-1}(U)$

(3) $i(\mathcal{G}^{(0)} \times \mathbb{T})$ is central in $\mathcal{E}$ i.e. $i(r(\varepsilon), z)\varepsilon = \varepsilon i(s(\varepsilon), z)$ for all $\varepsilon \in \mathcal{E}$ and $z \in \mathbb{T}$.

(4) $\pi^{-1}(\mathcal{G}^{(0)}) = i(\mathcal{G}^{(0)} \times \mathbb{T})$

If $\mathcal{E}$ is a twist over $\mathcal{G}$ for $z \in \mathbb{T}$ and $\varepsilon \in \mathcal{E}$, we write $z \cdot \varepsilon = i(r(\varepsilon), z)\varepsilon$ and $\varepsilon \cdot z = \varepsilon i(s(\varepsilon), z)$, note that $\varepsilon \cdot z = z \cdot \varepsilon$ since $\mathcal{E}$ is central in $\mathcal{G}$. We identify $\mathcal{E}^{(0)}$ with $\mathcal{G}^{(0)}$ via the map $x \mapsto i(x, 1)$. Note that $\mathcal{G}^{(0)}$ is therefor Hausdorff.

Lemma 6.2. *Let $\mathcal{G}$ be an étale groupoid and $\mathcal{E}$ be a twist over $\mathcal{G}$. If two elements $\epsilon, \delta \in \mathcal{E}$ satisfy $\pi(\epsilon) = \pi(\delta)$, then there exists $z \in \mathbb{T}$ such that $z \cdot \epsilon = \delta$*

Proof. $s(\delta) = \pi(\varepsilon^{-1}\delta)$, and so since i is injective there is a unique z such that $i(s(\delta), z) = \varepsilon^{-1}\delta$. $s(\delta) = i(s(\delta), 1) = i(s(\delta), z)i(s(\delta), \overline{z}) = (\varepsilon^{-1}\delta)i(s(\delta), \overline{z}) = (z \cdot \varepsilon)^{-1}\delta$, and so $\delta = z \cdot \varepsilon$. ☐

Remark 6.3. Let $\delta, \varepsilon, \gamma \in \mathcal{E}$ with $\pi(\delta) = \pi(\varepsilon) = \pi(\gamma)$. We will use the notation $\delta = z(\delta, \varepsilon) \cdot \varepsilon$, where $z(\delta, \varepsilon)$ is the element $z \in \mathbb{T}$ such that $\delta = z \cdot \varepsilon$. Note that $\overline{z(\delta, \varepsilon)} = z(\varepsilon, \delta)$, and $z(\varepsilon, \delta)z(\delta, \gamma) = z(\varepsilon, \gamma)$.

Let $\mathcal{G}$ be an locally compact étale Hausdorff groupoid. If σ is a continuous normalised 2-cocycle on $\mathcal{G}$, we can then make $\mathcal{G} \times \mathbb{T}$ in to a groupoid $\mathcal{E}_\sigma$. The unit space, range and source map is given as usual, but multiplication is given by $(\alpha, z)(\beta, w) = (\alpha\beta, wz\sigma(\alpha, \beta))$ and inversion is given by $(\alpha, z)^{-1} = (\alpha, \overline{\sigma(\alpha^{-1}, \alpha)z})$. The groupoid $\mathcal{E}_\sigma$ is analogous to the Mackey group G_σ in section 2. The set inclusion $i\colon \mathcal{G}^{(0)} \times \mathbb{T} \to \mathcal{E}_\sigma$ and the projection $\pi\colon \mathcal{E}_\sigma \to \mathcal{G}$ given by $\pi(\gamma, z) = \gamma$ are groupoid homomorphism. One can then show that $\mathcal{E}_\sigma$ is a twist over $\mathcal{G}$ with respect to i and π.

We can also recover the cohomology class of σ from the twist $\mathcal{E}_\sigma \to \mathcal{G}$. Let S be any continuous section for σ i.e a continuous map $S\colon \mathcal{G} \to \mathcal{E}_\sigma$ such that $\pi \circ S = \mathrm{id}_\mathcal{G}$. For $(\alpha, \beta) \in \mathcal{G}^{(2)}$ we have $\pi\left(S(\alpha)S(\beta)S(\alpha\beta)^{-1}\right) = r(\alpha)$, and so there is a unique element (dependent on α and β) $\omega(\alpha, \beta) \in \mathbb{T}$ such that $S(\alpha)S(\beta)S(\alpha\beta)^{-1} = (r(\alpha), \omega(\alpha, \beta))$ The resulting map $\omega\colon \mathcal{G}^{(2)} \to \mathbb{T}$ is a continuous 2-cocycle. Let S' be another continuous section for σ, and let ω' be defines as ω, but with respect to S'. Let $b\colon \mathcal{G} \to \mathbb{T}$ be the map satisfying the equality $S(\alpha)^{-1}S'(\alpha) = (r(\alpha), b(\alpha))$ for all $\alpha \in \mathcal{G}$, this is in fact a 1-cochain. Then $\omega^{-1}\omega'\colon \mathcal{G}^{(2)} \to \mathbb{T}$ given by $(\omega^{-1}\omega')(\alpha, \beta) = \omega(\alpha, \beta)^{-1}\omega'(\alpha, \beta)$ is equal to the 2-coboundary obtained from the 1-cochain b. Thus the cocycle obtained from different choices of continuous sections S are cohomologous. If we let S be the continuous section given by $S(\gamma) = (\gamma, 1)$ for all $\gamma \in \mathcal{G}$, then $\omega = \sigma$. Thus the cohomology class $[\sigma]$ of σ is equal to that obtained from every continuous section for $\sigma\colon \mathcal{G} \to \mathcal{E}_\sigma$

In general if $\mathcal{E}$ is a twist over $\mathcal{G}$ that admitt a continuous section $S\colon \mathcal{G} \to \mathcal{E}$, then there is a continuous 2-cocycle define by $S(\alpha)S(\beta)S(\alpha\beta)^{-1} = i\,(s(\alpha), \sigma(\alpha, \beta))$ for all $(\alpha, \beta) \in \mathcal{G}^{(2)}$, and thus a isomorphism $\mathcal{E} \cong \mathcal{E}_\sigma$ that is equivariant for i and π. In that case, $\mathcal{E}$ is isomorphic to a twist coming from a continuous 2-cocycle. However it is not clear that every twist admits a continuous section, so the notion of a twist is formally stronger then that of a continuous 2-cocycle.

Definition 6.4. Let $\mathcal{G}$ be an étale groupoid and let $\mathcal{E}$ be a twist over $\mathcal{G}$, we define

$$\Sigma_c(\mathcal{G}, \mathcal{E}) = \{f \in C_c(\mathcal{E})\colon f(z \cdot \varepsilon) = zf(\varepsilon) \forall \varepsilon \in \mathcal{E}, \forall z \in \mathbb{T}\}$$

Let $\gamma \in \mathcal{G}$. For any element $\delta \in \pi^{-1}(\gamma)$ we have a homeomorphism $\mathbb{T} \cong \pi^{-1}(\gamma)$ given by $z \mapsto z \cdot \delta$. We define a measure on $\pi^{-1}(\gamma)$ by pulling back the Haar measure on $\mathbb{T}$. This is independent of the choice of $\delta \in \pi^{-1}(\gamma)$ since the Haar measure on $\mathbb{T}$ is rotation invariant. For every $x \in \mathcal{G}$ we endow $\mathcal{E}_x(\mathcal{E}^x)$ with the measure $\nu_x(\nu^x)$ that agrees with the pulled backed copy of the Haar measure on $\pi^{-1}(\gamma)$ for each $\gamma \in \mathcal{G}_x(\mathcal{G}^x)$. Note that $\pi^{-1}(\gamma)$ has measure 1.

Lemma 6.5. *Let $\mathcal{G}$ be an étale groupoid and let $\mathcal{E}$ be a twist over $\mathcal{G}$. $\Sigma_c(\mathcal{G}, \mathcal{E})$ forms a complex $*$-algebra with product given by convolution*

$$(f * g)(\varepsilon) = \int_{\mathcal{E}^{r(\varepsilon)}} f(\gamma)g(\gamma^{-1}\varepsilon) d\nu^{r(\varepsilon)} = \int_{\mathcal{E}_{d(\varepsilon)}} f(\varepsilon\gamma^{-1})g(\gamma) d\nu_{d(\varepsilon)}$$

and involution given by

$$f^*(\varepsilon) = \overline{f(\varepsilon^{-1})}.$$

Let $\varepsilon \in \mathcal{E}$, for any choice of section (not necessarily continuous) that maps $\alpha \to \tilde{\alpha}$, $\alpha \in \mathcal{G}^{r(\varepsilon)}$, we have

$$(f * g)(\varepsilon) = \sum_{\alpha \in \mathcal{G}^{r(\varepsilon)}} f(\tilde{\alpha})g(\tilde{\alpha}^{-1}\varepsilon) = \sum_{\alpha \in \mathcal{G}^{s(\varepsilon)}} f(\varepsilon\tilde{\alpha}^{-1})g(\tilde{\alpha}) \tag{4}$$

for $f, g \in \Sigma_c(\mathcal{G}, \mathcal{E})$. There is an isomorphism

$$C_c(\mathcal{G}^{(0)}) \cong \{f \in \Sigma_c(\mathcal{G}, \mathcal{E})\colon supp(f) \subseteq i(\mathcal{G}^0 \times \mathbb{T})\} \tag{5}$$

that send $f \in C_c(\mathcal{G}_0)$ to $\tilde{f}$, where $\tilde{f}\colon i(x, z) \mapsto zf(x)$.

Proof. Let $\alpha \in \mathcal{G}^{r(\varepsilon)}$ and $\gamma, \gamma' \in \pi^{-1}(\alpha)$, we then have that $\gamma = z \cdot \gamma'$ for some $z \in \mathbb{T}$

$$f(\gamma)g(\gamma^{-1}\varepsilon) = f(z\gamma')g(\overline{z}\gamma'^{-1}\varepsilon)) = f(\gamma')g(\gamma'^{-1}\varepsilon)$$

So given any section that sends say $\alpha \mapsto \tilde{\alpha}$ we have that $\int_{\pi^{-1}}(\alpha)(f(\gamma)g(\gamma^{-1}\varepsilon)) = f(\tilde{\alpha})g(\tilde{\alpha}\varepsilon)$ since $\pi^{-1}\alpha$ has measure 1. It follows that

$$(f * g)(\varepsilon) = \int_{\mathcal{E}^{r(\varepsilon)}} f(\gamma)g(\gamma^{-1}\varepsilon) d\nu^{r(\varepsilon)} = \sum_{\alpha \in \mathcal{G}^{r(\varepsilon)}} \int_{\gamma \in \pi^{-1}(\alpha)} f(\gamma)g(\gamma^{-1}\varepsilon) = \sum_{\alpha \in \mathcal{G}^{r(\varepsilon)}} f(\tilde{\alpha})g(\tilde{\alpha}^{-1}\varepsilon)$$

From this it follows that $\Sigma_c(\mathcal{G}, \mathcal{E})$ is a $*$-algebra since $C_c(\mathcal{G})$ is a $*$-algebra. Lastly since $x \mapsto i(x, 1)$ is a section, the isomorphism also follows from (4). ☐

Remark 6.6. Each twist $\mathcal{E}$ over $\mathcal{G}$ determines a complex line bundle $\tilde{\mathcal{E}}$ over $\mathcal{G}$. $\tilde{\mathcal{E}} = \mathbb{C} \times \mathcal{E}/\sim$ where $(z\overline{t}, \gamma) \sim (z, t \cdot \gamma)$, we denote the corresponding equivalence class by $[z, \gamma]$, then $\tilde{\mathcal{E}}$ is a line bundle over $\mathcal{G}$ with respect to the fibre map $p\colon \tilde{\mathcal{E}} \to \mathcal{G}$ given by $p([z, \gamma]) \mapsto \pi(\gamma)$. We can regard elements of $\Sigma_c(\mathcal{G}, \mathcal{E})$ as sections of $\tilde{\mathcal{E}}$, where the corresponding section of $f \in \Sigma_c(\mathcal{G}, \mathcal{E})$ is given by

$$\gamma \mapsto [f(\tilde{\gamma}), \tilde{\gamma}]$$

where $\tilde{\gamma} \in \pi^{-1}(\gamma)$, this is well defined due to the equivalence relation.

Let $p \in [1,\infty]$. We denote the p-integrable $\mathbb{T}$-equivariant functions on $\mathcal{E}_x$, by $L^p(\mathcal{G}_x,\mathcal{E}_x)$. for $p=\infty$ we denote the $\mathbb{T}$-equivariant supermum bounded functions by $L^\infty(\mathcal{G}_x,\mathcal{E}_x)$.

Lemma 6.7. *Let $p \in [1,\infty]$, let $\mathcal{G}$ be an étale groupoid and $\mathcal{E}$ be a twist over $\mathcal{G}$, and let $x \in \mathcal{G}^{(0)}$, then $L^p(\mathcal{G}_x,\mathcal{E}_x) \cong l^p(\mathcal{G}_x)$.*

Proof. Let $S\colon \mathcal{G}_x \to \mathcal{E}_x$ be a section given by $S(\gamma)=\widetilde{\gamma}$, then for every $\gamma \in \mathcal{G}_x$, then for every $\alpha \in \mathcal{E}_x$ there exists a $z \in \mathbb{T}$ such that $\alpha = \widetilde{\pi(\alpha)}i(x,z)$. Let $f \in l^p(G)$, then define the function $\widetilde{f} \in L^p(\mathcal{G}_x,\mathcal{E}_x)$ as follows: $\widetilde{f}\colon \alpha \mapsto zf(\pi(\alpha))$. where $z \in \mathbb{T}$ is given by $\alpha = \widetilde{\pi(\alpha)}i(x,z)$ (also denoted $z(\alpha,\widetilde{\pi(\alpha)})$. For $p \in [1,\infty)$ we have that

$$\int_{\alpha\in\mathcal{E}_x} |\tilde{f}(\alpha)|d\nu_x = \sum_{\gamma\in\mathcal{G}_x}\int_{\alpha\in\pi^{-1}(\gamma)} |zf(\pi(\alpha))|d\nu_x = \sum_{\gamma\in\mathcal{G}_x}|f(\gamma)|\int_{\alpha\in\pi^{-1}(\gamma)} d\nu_x = \sum_{\gamma\in\mathcal{G}_x}|f(\gamma)|$$

and for $p=\infty$ we have that

$$\sup_{\alpha\in\mathcal{E}_x}|\tilde{f}(\alpha)| = \sup_{\gamma\in\mathcal{G}_x}\sup_{\alpha\in\pi^{-1}(\gamma)}|zf(\gamma)| = \sup_{\gamma\in\mathcal{G}_x}|f(\gamma)|$$

It follows that the map is isometric for all $p \in [1,\infty]$. The map $f \mapsto \widetilde{f}$ is clearly injective. To show surjectivity, let $g \in L^p(\mathcal{G}_x,\mathcal{E}_x)$. Note that there exists a $z \in \mathbb{T}$ such that $\alpha = z\cdot\widetilde{\pi(\alpha)}$, we then have that $g(\alpha) = zg(\widetilde{\pi(\alpha)})$ and so the function $f \in l^p(G_x)$ given by $\gamma \mapsto g(\tilde{\gamma})$ for all $\gamma \in G_x$ is such that $g(\alpha) = \widetilde{f}(\alpha)$ for all $\alpha \in \mathcal{E}_x$. The map is thus an isomorphism and it follows that $L^p(\mathcal{G}_x,\mathcal{E}_x) \cong l^p(\mathcal{G}_x)$. □

We define the left regular representation $\lambda_x\colon \Sigma_c(\mathcal{G},\mathcal{E}) \to \mathcal{B}(L^p(\mathcal{G}_x,\mathcal{E}_x))$ by extension of the convolution formula, i.e the following:

$$\lambda_x(f)\xi(\varepsilon) = \int_{\gamma\in\mathcal{E}_x} f(\varepsilon\gamma^{-1})\xi(\gamma)d\nu_x$$

which is bounded. This can be deduced by the following estimate

$$\|\lambda_x(f)\| \leq \|f\|_I = \max\{\sup_x \int |f|d\nu_x, \sup_x \int |f^*|d\nu_x\}$$

To show this estimate, choose any section $\alpha \mapsto \tilde{\alpha}$. We then have that

$$\lambda_x(f)\xi(\varepsilon) = \int_{\gamma\in\mathcal{E}_x} f(\varepsilon\gamma^{-1})\xi(\gamma)d\nu_x = \sum_{\alpha\in\mathcal{G}_x} f(\varepsilon\tilde{\alpha}^{-1})\xi(\tilde{\alpha})$$

and

$$\int_{\mathcal{E}_x}|f|d\nu_x = \sum_{\alpha\in\mathcal{G}_x}\int_{\gamma\in\pi^{-1}(\alpha)}|f(\alpha)|d\nu_x = \sum_{\alpha\in\mathcal{G}_x}|f(\tilde{\alpha})|$$

and since $|f(\gamma_1)| = |f(\gamma_2)|$ for all $\gamma_1,\gamma_2 \in \mathcal{E}_x$ with $\pi(\gamma_1)=\pi(\gamma_2)$, we have that

$$\sup_{\gamma\in\mathcal{E}_x}|f(\gamma)| = \sup_{\alpha\in\mathcal{G}_x}|f(\tilde{\alpha})|.$$

By Lemma 6.7, The estimate $\|\lambda_x(f)\| \leq \|f\|_I$ then follows as in the proof of Proposition 5.2.

Definition 6.8. Let $p \in [1,\infty)$, and $\mathcal{E}$ be a twist over an étale groupoid $\mathcal{G}$. We define *the reduced twisted L^p-operator algebra* denoted $F^p_\lambda(\mathcal{G};\mathcal{E})$ as the completion of $\Sigma_c(\mathcal{G},\mathcal{E})$ in the norm

$$\|f\|_\lambda = \sup\nolimits_{x\in\mathcal{G}^{(0)}}\|\lambda_x(f)\|$$

Remark 6.9. Let $\mathcal{G}$ be a locally compact étale Hausdorff groupoid. For the trivial twist $\mathcal{E} = \mathcal{G} \times \mathbb{T}$ the continuous map $\alpha \mapsto (\alpha, 1)$ is a section for $\pi\colon \mathcal{G} \times \mathbb{T} \to \mathcal{G}$. The cocycle obtained from this section is the trivial coccyle, and it follows from 4 that $F^p_\lambda(\mathcal{G}; \mathcal{G} \times \mathbb{T}) \cong F^p_\lambda(\mathcal{G})$. Now, let σ be a continuous 2-cocycle on $\mathcal{G}$ and let $\mathcal{E}_\sigma$ be the twist over $\mathcal{G}$ constructed from σ. Then $F^p_\lambda(\mathcal{G}; \mathcal{E}_\sigma) \cong F^p_\lambda(\mathcal{G}, \sigma)$. Let S be the section such that $\gamma \mapsto (\gamma, 1)$ for all $\gamma \in \mathcal{G}$. We then have the isometric isomorphism $\Sigma_c(\mathcal{G}; \mathcal{E}_\sigma) \to C_c(\mathcal{G}, \sigma)$ That sends $f \mapsto \tilde{f}$, where $\tilde{f}(z, \gamma) = zf(\gamma)$. One can also show that $\|f\|_\lambda = \|\tilde{f}\|_\lambda$. which means that the isomorphism extends to the closures $F^p_\lambda(\mathcal{G}; \mathcal{E}_\sigma) \cong F^p_\lambda(\mathcal{G}, \sigma)$.

The fact that $F^p_\lambda(\mathcal{G}; \mathcal{E})$ is an L^p-operator algebra follows as in the proof of Theorem 5.4. Since $L^p(\mathcal{G}_x; \mathcal{E}_x)$ is an L^p-space we have that case then $\bigoplus_{x \in \mathcal{G}^{(0)}} \lambda_x$ is an isometric representation of $F^p_\lambda(\mathcal{G}; \mathcal{E})$ on the L^p-space $\bigoplus_{x \in \mathcal{G}^{(0)}} L^p(\mathcal{G}_x; \mathcal{E}_x)$. Note that $F^p_\lambda(\mathcal{G}; \mathcal{E})$ is unital if and only if $\mathcal{G}^0$ is compact.

The rest of the section follows as section 4 and 5 in [3], and generalises the results there to our setting of the twist.

Remark 6.10. Let $\beta, \varepsilon \in \mathcal{E}$, we define the point mass function on $L^p(\mathcal{G}_x, \mathcal{E}_x)$ as follows: $\delta_\beta(\varepsilon) = z(\beta, \varepsilon)$, if $\pi(\beta) = \pi(\varepsilon)$. If $\pi(\beta) \neq \pi(\varepsilon)$, then $\delta_\beta(\varepsilon) = 0$.

Lemma 6.11. *Let $p \in [1, \infty)$, and let $\mathcal{E}$ be a twist over an étale groupoid $\mathcal{G}$. Let $f \in \Sigma_c(\mathcal{G}; \mathcal{E})$, Then*

$$\|f\|_\infty \leq \|f\|_\lambda \leq \|f\|_I .$$

Furthermore, if U is a bisection, then for every f with supp$(f) \subseteq \pi^{-1}(U)$ we have that $\|f\|_\infty = \|f\|_\lambda$.

Proof. Let $f \in \Sigma_c(\mathcal{G}; \mathcal{E})$. Since $\|\lambda_x(f)\| \leq \|f\|_I$, the second inequality follows immediately. Let $\varepsilon \in \mathcal{E}$ To show the first inequality, we want to show that $|f(\varepsilon)| \leq \|f\|_\lambda$. Set $x = s(\varepsilon)$

$$\|f\|_\lambda \geq \|\lambda_x(f)\| \geq \|\lambda_x(f)\delta_x\| = \| \sum_{\gamma \in \mathcal{G}_x} f(x)\delta_x\|_p \geq |f(\gamma)|$$

If $\mathrm{supp}(f) \subseteq \pi^{-1}(U)$ for a bisection U, then a quick manipulation of terms shows that $\|f\|_I = \|f\|_\infty$, and it follows that $\|f\|_\infty = \|f\|_\lambda$. ☐

Remark 6.12. Note that $\mathcal{G}^0$ is a bisection, and so for all $f \in C_c(G^0)$ the equality $\|f\|_I = \|f\|_\infty$ holds.

The identity map $\Sigma_c(\mathcal{G}, \mathcal{E}) \to C_0(\mathcal{E})$ extends to a linear contractive map $j\colon F^p_\lambda(\mathcal{G}, \mathcal{E}) \to C_0(\mathcal{E})$ by the previous Lemma. Given $a \in F^p_\lambda(\mathcal{G}; \mathcal{E}_x)$ we will write j_a for $j(a) \in C_0(\mathcal{E})$.

Let $p \in [1, \infty]$ and let q be the dual exponents, i.e $\frac{1}{p} + \frac{1}{q} = 1$. For $x \in \mathcal{G}_x$ we identify the dual of $l^p(\mathcal{G}_x)$ with $l^q(\mathcal{G}_x)$. For $\xi \in l^p(\mathcal{G}_x)$ and $\eta \in l^q(\mathcal{G}_x)$ we write $\langle \xi \,, \eta \rangle = \sum_{\mathcal{G}_x} \xi(\tilde{\gamma})\eta(\tilde{\gamma})$ for the dual pairing. For the rest of the section fix any section $\varepsilon \mapsto \tilde{\varepsilon}$ so that $\pi(\tilde{\varepsilon}) = \varepsilon$ for all $\varepsilon \in \mathcal{G}$. Using the the identity in the proof of Proposition 6.7,

Using the identification of Proposition 6.7 we define a operator $\tilde{\lambda}_x\colon l^p(\mathcal{G}_x) \to l^p(\mathcal{G}_x)$ as follows. Let $\xi \in l^p(\mathcal{G}_x)$, we then have $\tilde{\xi} \in L^p(\mathcal{E}_x; \mathcal{G}_x)$ given by $\varepsilon \to z(\varepsilon, \widetilde{\pi(\varepsilon)})\xi(\widetilde{\pi(\varepsilon)})$, we then have that

$$\lambda_x(f)(\tilde{\xi}(\varepsilon)) = \sum_{\alpha \in \mathcal{G}_x} f(\varepsilon\tilde{\alpha}^{-1})\xi(\alpha)$$

and so $\tilde{\lambda}_x$ is given by

$$\tilde{\lambda}_x(f)(\tilde{\xi}(\gamma)) = \sum_{\alpha \in \mathcal{G}_x} f(\tilde{\gamma}\tilde{\alpha}^{-1})\xi(\alpha)$$

The reason we define this operator is that we need the properties of dual action is the following results.

Proposition 6.13. *Let $p \in [1,\infty)$, and let $\mathcal{E}$ be a twist over an étale groupoid $\mathcal{G}$. The map $j\colon F^p_\lambda(\mathcal{G};\mathcal{E}) \to C_0(\mathcal{E})$ is injective and we have*

$$j_a(\gamma) = \langle \tilde{\lambda}_{s(\gamma)}(a)(\delta_{s(\gamma)})\,,\overline{\delta_\gamma}\rangle$$

for all $a \in F^p_\lambda(\mathcal{G};\mathcal{E})$ and all $\gamma \in \mathcal{E}$.

Note that the point mass functions here are the functions as given in Remark 6.12 which we will here see as functions in $l^p(G_x)$ using the identify in the proof of Proposition 6.7. For instance we write $\overline{\delta_\gamma}$ where $\gamma \in \mathcal{E}$ for the function $\alpha \to \overline{\delta_\gamma(\tilde{\alpha})}$ for all $\alpha \in \mathcal{G}$, which is a function in $l^q(G_x)$. This is important as the function $\varepsilon \mapsto \overline{\delta_\gamma}(\varepsilon)$ for $\varepsilon \in \mathcal{E}$ is not $\mathbb{T}$-equivariant, and so is not a function in $L^p(\mathcal{G}_x,\mathcal{E}_x)$.

Proof. For $f \in \Sigma_c(\mathcal{G},\mathcal{E})$ we have that $j_f(\alpha) = f(\alpha)$. We are going to show that

$$\langle \tilde{\lambda}_{s(\alpha)}(f)(\delta_{s(\alpha)})\,,\overline{\delta_\alpha}\rangle = f(\alpha)$$

Let $f \in \Sigma_c(\mathcal{G};\mathcal{E})$, $\alpha \in \mathcal{E}_x$, note that we write δ_x for the function $\alpha \mapsto \delta_x(\tilde{\alpha})$ in $l^p(\mathcal{G}_x)$(not to be confused with the usual point mass function). We then have that

$$\langle \tilde{\lambda}_x(f)\delta_x\,,\overline{\delta_\alpha}\rangle = \sum_{\gamma\in\mathcal{G}_x} \tilde{\lambda}_x(f)\delta_x(\tilde{\gamma})\overline{\delta_\alpha(\tilde{\gamma})}$$

First note that

$$\tilde{\lambda}_x(f)\delta_x(\tilde{\gamma}) = \sum_{\beta\in\mathcal{G}_x} f(\tilde{\gamma}\tilde{\beta}^{-1})\delta_x(\beta) = f(\tilde{\gamma})$$

and that $\alpha = z(\alpha,\widetilde{\pi(\alpha)})\cdot\widetilde{\pi(\alpha)}$. Using these two facts we then have

$$\begin{aligned}\langle \tilde{\lambda}_x(f)\delta_x\,,\delta_\alpha\rangle &= \sum_{\gamma\in\mathcal{G}_x} \tilde{\lambda}_x(f)\delta_x(\tilde{\gamma})\overline{\delta_\alpha(\gamma)} \\ &= f(\widetilde{\pi(\alpha)})\overline{\delta_\alpha(\widetilde{\pi(\alpha)})} = f(\widetilde{\pi(\alpha)})z(\widetilde{\pi(\alpha)},\alpha) = f(\alpha)\end{aligned}$$

as intended. By the continuity of both side with respect to the norm of $F^p_\lambda(\mathcal{G};\mathcal{E}_x)$ this extends to all $a \in F^p_\lambda(\mathcal{G};\mathcal{E}_x)$ ☐

Lemma 6.14. *Let $p \in [1,\infty)$ and let $\mathcal{E}$ be a twist over an étale groupoid $\mathcal{G}$. let $a \in F^p_\lambda(\mathcal{G};\mathcal{E})$, let $x \in \mathcal{G}_0$ and let $\gamma,\alpha \in \mathcal{E}_x$, then*

$$\langle \tilde{\lambda}_x(a)(\delta_\gamma)\,,\overline{\delta_\alpha}\rangle = \langle \tilde{\lambda}_{r(\gamma)}(a)(\delta_{r(\gamma)})\,,\overline{\delta_{\alpha\gamma^{-1}}}\rangle$$

Proof. let $f \in \Sigma_c(\mathcal{G};\mathcal{E})$, then

$$\begin{aligned}\langle \tilde{\lambda}_x(f)(\delta_\gamma)\,,\overline{\delta_\alpha}\rangle &= \sum_{\beta\in\mathcal{G}_x} \tilde{\lambda}_x(f)\delta_\gamma(\beta)\overline{\delta_\alpha(\tilde{\beta})} \\ &= \tilde{\lambda}_x(f)\delta_\gamma(\alpha)z(\alpha,\widetilde{\pi(\alpha)}) \\ &= f(\widetilde{\pi(\alpha)}\widetilde{\pi(\gamma)}^{-1})z(\widetilde{\pi(\gamma)},\gamma)z(\alpha,\widetilde{\pi(\alpha)}) \\ &= f(\alpha\gamma^{-1}) = \langle \tilde{\lambda}_{r(\gamma)}(f)(\delta_{r(\gamma)})\,,\overline{\delta_{\alpha\gamma^{-1}}}\rangle.\end{aligned}$$

By continuity this holds for all $a \in F^p_\lambda(\mathcal{G};\mathcal{E})$. ☐

Given an Banach space E and a operator $T \in \mathcal{B}(E)$. We write T' the *adjoint* of T. The operator $T' \in \mathcal{B}(E^*)$, where E^* is the dual of E, which is given by $\langle x\,,T'x^*\rangle = \langle Tx\,,x^*\rangle$ for all $x \in E$ and $x^* \in E^*$ where $\langle\cdot\,,\cdot\rangle$ represent the dual pairing of E^* and E. Recall that $\|T\| = \|T'\|$.

Lemma 6.15. *Let $\mathcal{E}$ be a twist over $\mathcal{G}$. Let $p \in [1, \infty)$, let q denote its conjugate exponent, and let $x \in \mathcal{G}^{(0)}$. For $a \in F^p_\lambda(\mathcal{G};\mathcal{E})$ we write $\tilde{\lambda}_x(a)' \colon l^q(\mathcal{G}_x) \to l^q(\mathcal{G}_x)$ for the adjoint of $\tilde{\lambda}_x(a)$. We define the contractive linear maps $l_x \colon F^p_\lambda(\mathcal{G};\mathcal{E}) \to l^p(\mathcal{G}_x)$ and $r_x \colon F^p_\lambda(\mathcal{G};\mathcal{E}) \to l^q(\mathcal{G}_x)$ by*

$$r_x(a) = \tilde{\lambda}_x(a)'\overline{\delta_x}, \text{ and } l_x(a) = \tilde{\lambda}_x(a)\delta_x$$

for $a \in F^p_\lambda(\mathcal{G},\mathcal{E})$. Then $r_x(a)(\gamma) = j_a(\gamma^{-1})$ and $l_x(a)(\gamma) = j_a(\gamma)$ for all $\gamma \in \mathcal{E}_x$.

Proof. Let $x \in \mathcal{G}^{(0)}$, and let $\gamma \in \mathcal{G}_x$. For $a \in F^p_\lambda(\mathcal{G},\mathcal{E})$ we have by Proposition 6.14 that

$$l_x(a)(\gamma) = \langle \tilde{\lambda}_x(a)(\delta_x)\,, \overline{\delta_{\tilde{\gamma}}} \rangle = j_a(\tilde{\gamma})$$

Similarly

$$r_x(a)(\gamma) = \langle \delta_{\tilde{\gamma}}\,, \tilde{\lambda}_x(a)'\overline{\delta_x} \rangle = \langle \tilde{\lambda}_x(a)(\delta_{\tilde{\gamma}})\,, \overline{\delta_x} \rangle = \langle \tilde{\lambda}_{s(\gamma^{-1})}(a)(\delta_{s(\gamma^{-1})})\,, \overline{\delta_{\tilde{\gamma}^{-1}}} \rangle = j_a(\tilde{\gamma}^{-1})$$

for all $\gamma \in \mathcal{G}_x$ where the second equality is by definition of the Banach space adjoint, the this equality follows from the previous lemma. It remains to show that the maps are contractive. We have. $\|l_x\| = \sup\{\|l_x(a)\|_p : \|a\|_\lambda \le 1\}$. Let $f \in \Sigma_c(\mathcal{G},\mathcal{E})$. Since

$$\|l_x(f)\|_p = \|\tilde{\lambda}_x(f)(\delta_x)\|_p = \|\lambda_x(f)(\delta_x)\|_p = \|f * \delta_x\|_p\,,$$

it follows that $\|l_x\| \le 1$. Similarly it follows that r_x is contractive. □

Proposition 6.16. *Let $\mathcal{G}$ be an étale groupoid and let $\mathcal{E}$ be a twist over $\mathcal{G}$. Let $a, b \in F^p_\lambda(\mathcal{G};\mathcal{E})$, $s \in \mathcal{G}$. Then*

$$j_{a*b}(\gamma) = \int_{\alpha \in \mathcal{E}_{s(\gamma)}} j_a(\gamma\alpha^{-1}) j_b(\alpha) d\nu_{s(\gamma)}$$

Proof. Let We first want to show that show that $j_a(\gamma\tilde{\alpha}^{-1}) = r_x(\delta_{\gamma^{-1}} * a)(\alpha)$ for all $\alpha \in \mathcal{G}_x$. Let $f \in \Sigma_c(\mathcal{G};\mathcal{E})$. We then have that

$$\begin{aligned} r_x(\delta_{\gamma^{-1}} * f)(\alpha) &= \langle \delta_{\tilde{\alpha}}\,, \tilde{\lambda}_x(\delta_{\gamma^{-1}} * f)'(\overline{\delta}_x) \rangle = \langle \tilde{\lambda}_x(\delta_{\gamma^{-1}} * f)(\delta_{\tilde{\alpha}})\,, \overline{\delta_x} \rangle \\ &= \sum_{s \in \mathcal{G}_x} (\delta_{\gamma^{-1}} * f)(\tilde{s}\tilde{\alpha}^{-1})\overline{\delta_x(s)} = (\delta_{\gamma^{-1}} * f)(\tilde{\alpha}^{-1}) = f(\gamma\tilde{\alpha}^{-1}) = j_f(\gamma\tilde{\alpha}^{-1}) \end{aligned}$$

Let $\gamma \in \mathcal{E}_x$, we now want to show that $\tilde{\lambda}_x(a)'\overline{\delta}_\gamma(\alpha) = j_a(\gamma\tilde{\alpha}^{-1})$ for all $\alpha \in \mathcal{G}_x$. As usual let $f \in \Sigma_c(\mathcal{G};\mathcal{E})$. Using Lemma 6.14 in the third sted, we have that

$$\tilde{\lambda}_x(a)'\overline{\delta}_\gamma(\alpha) = \langle \delta_{\tilde{\alpha}}\,, \tilde{\lambda}_x(a)'(\overline{\delta}_\gamma) \rangle = \langle \lambda_x(a)(\delta_{\tilde{\alpha}})\,, \overline{\delta}_\gamma \rangle = \langle \lambda_{r(\alpha)}(a)(\delta_{r(\alpha)})\,, \overline{\delta}_{\gamma\tilde{\alpha}^{-1}} \rangle = j_a(\gamma\tilde{\alpha}^{-1})$$

combining the two statements we finally have

$$\begin{aligned} j_{a*b}(\gamma) &= \langle \lambda_x(a * b)(\delta_x)\,, \delta_\gamma \rangle = \langle \lambda_x(a)\left(\lambda_x(b)(\delta_x)\right)\,, \delta_\gamma \rangle \\ &= \langle \lambda_x(a)'\delta_\gamma\,, \lambda_x(b)(\delta_x) \rangle = \langle r_x(\delta_{\alpha^{-1}} * a)\,, l_x(b) \rangle \\ &= \sum_{\alpha \in \mathcal{G}_x} j_a(\gamma\tilde{\alpha}^{-1}) j_b(\tilde{\alpha}) = \int_{\alpha \in \mathcal{E}_x} j_a(\gamma\alpha^{-1}) j_b(\alpha) d\nu_x \end{aligned}$$

which is well defined since the sum is absolute convergent as it is given by the dual paring of elements in $l^p(\mathcal{G}_x)$ and $l^q(\mathcal{G}_x)$, the last equality follows from (4). □

We can finally identify the C^*-core of $F^p_\lambda(\mathcal{G};\mathcal{E})$

Theorem 6.17. *Let $p \in [1, \infty) \setminus \{2\}$, and let $\mathcal{E}$ be a twist over an étale groupoid $\mathcal{G}$ with compact unit space. Then core($F^p_\lambda(\mathcal{G};\mathcal{E}) = C(\mathcal{G}^0)$*

Proof. Let $a \in F^p_\lambda(\mathcal{G};\mathcal{E})$. With the map given in Proposition 6.13 we want to to show that $\mathrm{supp}(j_a) \subseteq i(\mathcal{G}^0 \times \mathbb{T})$. Fix $x \in \mathcal{G}^0$, then $\lambda_x\colon F^p_\lambda(\mathcal{G};\mathcal{E}) \to \mathcal{B}(L^p(\mathcal{G}_x,\mathcal{E}_x))$ is a contractive representation and it follows from Proposition 2.13 in [3] that $\lambda_x(\mathrm{core}(F^p_\lambda(\mathcal{G};\mathcal{E}))) \subseteq \mathrm{core}(\mathcal{B}(L^p(\mathcal{G}_x,\mathcal{E}_x)))$ and so by Example 2.11 [3] we have that $\lambda_x(a)$ is a multiplication operator in $\mathcal{B}(L^p(\mathcal{G}_x,\mathcal{E}_x)$, let $\gamma \in \mathcal{E}$ and set $x = s(\gamma)$, since $\lambda_x(a)$ is a multiplcation operator we have that $\lambda_x(a)\delta_x = c\delta_x$ for some constant $c \in \mathbb{T}$. We therefore have the following

$$\begin{aligned} j_a(\gamma) &= \langle \lambda_x(a)\delta_x \,, \delta_\gamma\rangle = \langle c\delta_x \,, \overline{\delta_\gamma}\rangle = c\sum_{\beta\in\mathcal{G}_x} \delta_x(\tilde{\beta})\overline{\delta_\gamma(\tilde{\beta})} \\ &= c\overline{\delta_\gamma(x).} \end{aligned}$$

(Note that we identify x with $i(x,1) \in \mathcal{E}$) So if $\gamma \notin i(\mathcal{G}^{(0)} \times \mathbb{T})$ we have that $j_a(\gamma) = 0$. It follows that $\mathrm{supp}(j_a) \subseteq i(\mathcal{G}^{(0)} \times \mathbb{T})$. By Proposition 6.16 we have that j is an homomorphism, by Proposition 6.11 it is also isometric, and by Proposition 6.13 it is injective. Using the identity 5 we have that j induces a *-isomorphism $\mathrm{core}(F^p_\lambda(\mathcal{G};\mathcal{E}) \cong C(\mathcal{G}^0)$. ☐

We will for the rest of the thesis omit the the map j from notation and just write $a(\gamma)$ for $j_a(\gamma)$, and when the result is independent of the choice of section we will omit the the mention of the section and just write $\tilde{\gamma}$. for $S(\gamma)$, where S is a section and $\gamma \in \mathcal{G}$.

Next we will show the relationship between two classes of partial homeomorphisms on $\mathcal{G}^{(0)}$, the ones induced by open bisection and the ones realised by admissible pairs of $F^p_\lambda(\mathcal{G},\sigma)$. Given a topological space X, let $U \subseteq X$ be an open set, we say U is the (complex) cozero set if there exists a continuous function $f\colon X \to (\mathbb{C})\mathbb{R}$ such that $U = \mathrm{supp}(f)$

Proposition 6.18. *Let $\mathcal{E}$ a twist over an étale groupoid $\mathcal{G}$ with compact unit space. Let $p \in [1,\infty)\setminus\{2\}$, and let B be an open bisection of $\mathcal{G}$ with associated partial homeomorphism $\beta\colon s(B) \to r(B)$. Let $U \subseteq \mathcal{G}^{(0)}$ be a cozero set. Then the restriction of β_B to U is realisable by an admissible pair in $F^p_\lambda(\mathcal{G};\Sigma)$*

Proof. We can without loss of generality replace B with $\{\gamma \in B\colon s(\gamma) \in U\}$ and assume $s(B) = U$. We then need to prove that β_B, given as in Remark 4.15, is realisable by an admissible pair in $F^p_\lambda(\mathcal{G};\mathcal{E})$. Let $h \in C(\mathcal{G}^{(0)})$ be any function such that $\mathrm{supp}(h) = s(B)$. Choose a non-vanishing continuous $\mathbb{T}$-equivariant function $u\colon \pi^{-1}(B) \to \mathbb{C}$. Replace $u(\gamma)$ with $u(\gamma)/|u(\gamma)|$ and we may assume that $|u(\gamma)| = 1$. Define the functions $a, b\colon \mathcal{E} \to \mathbb{C}$ as the following

$$a(\alpha) = \begin{cases} u(\alpha)h(s(\alpha)) & \text{if } \pi(\alpha) \in B \\ 0 & else \end{cases} \qquad b(\alpha) = \begin{cases} \overline{u(\alpha^{-1})h(r(\alpha))} & \text{if } \pi(\alpha^{-1}) \in B \\ 0 & else \end{cases}$$

for all $\alpha \in \mathcal{E}$, then a and b are $\mathbb{T}$-equivariant. Since both function a, b have support on $\pi^{-1}(B)$ it means the I-norm is equal to ∞-norm for a and b by Lemma 6.11. It follows that $a, b \in F^p_\lambda(\mathcal{G},\mathcal{E})$ since they are I-norm limits of elements of elements in $C_c(\pi^{-1}(B))$. We then want to show that $s = (a, b)$ is the admissible pair that realises β_B. First we will prove that s satisfy the axioms in Definition 4.9. Let $f \in C(\mathcal{G}^{(0)})_+$, we then have

$$bfa(\gamma) = \int_{\alpha\in\mathcal{E}_{s(\gamma)}} b(\gamma\alpha^{-1})f(r(\alpha))a(\alpha)d\nu_s(\gamma).$$

Now if $bfa(\gamma) \neq 0$, then there exists an element $\alpha \in \mathcal{E}_{s(\gamma)}$ such that $b(\gamma\alpha^{-1})f(r(\alpha))a(\alpha) \neq 0$, then $\pi((\gamma\alpha^{-1})^{-1}) = \pi(\alpha\gamma^{-1}) \in B$ and $\pi(\alpha) \in B$. Since B is an open bisection, the range map restricted to B is injective. This then implies that $\gamma^{-1} = s(\alpha) \in \mathcal{G}^{(0)}$. Thus it follows that $\mathrm{supp}(bfa) \subseteq i(\mathcal{G}^{(0)} \times \mathbb{T})$ which means that $bfa \in C(\mathcal{G}^{(0)})$. We then need to show that bfa is positive. Since

$$bfa(x) = \int_{\alpha\in\mathcal{E}_x} b(\alpha^{-1})f(x)a(\alpha)d\nu_x = \int_{\alpha\in\mathcal{E}_x} |h(x)|^2 f(x)d\nu_x,$$

it follows that $bfa(x) > 0$ for all $x \in \mathcal{G}^0$, hence by Remark **??**, we have that $bfa \in C(\mathcal{G}^{(0)})_{\mathbb{R}_+}$. Similarly one can show that $afb \in C(\mathcal{G}^{(0)})_{\mathbb{R}_+}$. Thus the first condition of Definition 4.9 holds.

Now, to show the second condition note that from the first condition, it follows that $ab, ba \in C(\mathcal{G}^{(0)})_+$.Let $x \in d(B)$. Since B is an open bicection, let $\alpha_0 \in B$ be the unique element such that $d(\alpha_0) = x$ and let $\tilde{\alpha_0} \in \pi^{-1}(\alpha_0)$ then

$$ba(x) = \int_{\alpha \in \mathcal{E}_x} b(\alpha^{-1})a(\alpha) d\nu_s = b(\tilde{\alpha_0}^{-1})a(\tilde{\alpha_0}) = |h(x)|^2$$

It follows that $ba(x) = |h(x)|^2$. On the other hand if $x \in \mathcal{G}^{(0)} \setminus s(B)$, then

$$ba(x) = \int_{\alpha \in \mathcal{E}_x} b(\alpha^{-1})a(\alpha) = 0$$

since $b(\alpha^{-1}), a(\alpha) = 0$ for all $\alpha \in \mathcal{G}_x$. Thus $ba = |h|^2$, and it follows that $s(B) = \operatorname{supp}(h) = \operatorname{supp}(ba)$. Similarly let $x \in d(B)$. For $x \in \mathcal{G}^{(0)} \setminus s(B)$, $ab(\beta_B(x)) = 0$, and so $|h|^2 = |ab \circ \beta_B|$. Thus $r(B) = \beta_B(s(B)) = \operatorname{supp}(ab)$.

Finally to show the third condition let $x \in s(B)$ and let $f \in C_0(r(B))$, then again since B in an open bisection, there is a unique element $\alpha_0 \in B$ such that $s(\alpha_0) = x$, then we have

$$\begin{aligned} bfa(x) &= \int_{\alpha \in \mathcal{E}_x} b(\alpha^{-1}) f(r(\alpha)) a(\alpha) d\nu_x = \sum_{\alpha \in \mathcal{E}_x} b(\tilde{\alpha}^{-1}) f(r(\alpha)) a(\tilde{\alpha}) \\ &= b(\tilde{\alpha_0}^{-1}) f(r(\alpha_0)) a(\tilde{\alpha_0}) f(\beta_B(x)) h(x)^2 = f(\beta_B(x)) ba(x) \end{aligned}$$

Similarly for $y \in r(B)$ and $g \in C_0(s(B))$ it follows that $g(\beta_B^{-1}(y))ab(y) = agb(y)$. Hence $s = (a, b)$ is the admissible pair that realises β_B, and the Proposition follows. □

Remark 6.19. If $\mathcal{G}^{(0)}$ is metrizable, then every open set is a cozero set. In general we have that $\mathcal{G}^0$ is compact Hausdorff(Since $\mathcal{G}^{(0)}$ is a subspace of $\mathcal{E}$)space so we have that for $x \in \mathcal{G}^{(0)}$ there exists an open cozero neighbourhood U of x.

The next proposition makes the "converse" relation. We find an open bisection given a admissible pair.

Proposition 6.20. *Let $\mathcal{E}$ a twist over an étale groupoid $\mathcal{G}$ with compact unit space. Let $p \in [1, \infty) \setminus \{2\}$, and let $n = (a, b)$ be and admissible pair in $F^p_\lambda(\mathcal{G}; \Sigma)$. Set*

$$B_n \cong \{\pi(\gamma) \in \mathcal{G} \colon a(\gamma), b(\gamma) \neq 0 \text{ for any } \gamma \in \mathcal{E}\}.$$

Then B_n is an open bisection of $\mathcal{G}$ and $\alpha_n = \beta_{B_n}$.

We will call B_n *the open bisection induced by n.*

Proof. First note that if $a(\gamma) = 0$, then $a(\varepsilon) = 0$ for all $\epsilon \in \mathcal{E}$ with $\pi(\gamma) = \pi(\varepsilon)$

First choose $f \in C(\mathcal{G}^0)$ with $f(\gamma) = 1$ for all $\gamma \in \mathcal{G}^0$. Then since (a, b) is realizable pair we have that $ba(\gamma) > 0$, that is as a function in $\Sigma_c(\mathcal{G}; \mathcal{E})$ we have that $ba(x) > 0$ and $\operatorname{supp}(ba) \subseteq i(\mathcal{G}' \times \mathbb{T}))$. Let $\delta \in \mathcal{G}$ Note that $b(\gamma^{-1})a(\gamma) = b(\alpha^{-1})a(\alpha)$ for any $\alpha, \gamma \in \pi^{-1}(\delta)$, in that case we will write $b(\delta^{-1})a(\delta)$.

$$ba(x) = \sum_{\gamma \in \mathcal{G}_x} b(\gamma^{-1})a(\gamma) > 0.$$

Which means that $b(\gamma^{-1})a(\gamma) \in \mathbb{R}$. We want to show that for any $\gamma \in \mathcal{G}$, $b(\gamma^{-1})a(\gamma) \geq 0$. Assume that $b(\gamma^{-1})a(\gamma) < 0$. Since a and b are continuous there is an open neighbourhood U of γ such that $a(\alpha)b(\alpha^{-1}) < 0$ for all $\alpha \in U$. Set $V = s(U)$ which is an open subset of $\mathcal{G}^{(0)}$. Sicne $\mathcal{G}$ is topological principle there is an $x_0 \in V$ with trivial isotropy. Fix $\alpha_0 \in U$ such that $s(\alpha_0) = x_0$, set

$y = r(\alpha_0)$, then, since x_0 has trivial isotropy, α_0 is the unique element in $\mathcal{E}_{x_0}$ with y as it range. Since

$$ba(x_0) = \int_{\alpha \in \mathcal{E}_{x_0}} b(\alpha^{-1})a(\alpha) d\nu_{x_0} = \sum_{\alpha \in \mathcal{G}_{x_0}} b(\tilde{\alpha}^{-1})a(\tilde{\alpha})$$

converges absolutely, the set $\{\alpha \in \mathcal{G}_{x_0} : b(\alpha^{-1})a(\alpha)\}$ is at most countable. Set $t = b(\alpha_0^{-1})a(\alpha_0) < 0$, and choose a neighbourhood W of y such that

$$\sum_{\alpha \in \mathcal{G}_{x_0}, r(\alpha) \in W \setminus \{y\}} |b(\alpha^{-1})a(\alpha)| < |t| = -t$$

Choose $f \in C_0(\mathcal{G}^{(0)})$ with $0 \leq f \leq 1$, $f(y) = 1$ and $\mathrm{supp}(f) \subseteq W$. Then

$$\begin{aligned} bfa(x_0) &= \int_{\alpha \in \mathcal{E}_{x_0}} b(\alpha^{-1})f(r(\alpha))a(\alpha) d\nu_x = \sum_{\alpha \in \mathcal{G}_{x_0}} b(\alpha^{-1})f(r(\alpha))a(\alpha) \\ &= \sum_{\alpha \in \mathcal{G}_{x_0}, r(\alpha) \in W \setminus \{y\}} b(\alpha^{-1})f(r(\alpha))a(\alpha) + b(\alpha_0^{-1})a(\alpha_0) < 0 \end{aligned}$$

which contradicts condition (1) of Definition 4.9, and we thus have that $b(\gamma^{-1})a(\gamma) \geq 0$ for all $\gamma \in \mathcal{G}$ (This is independent of the choice of section).

Let $\gamma \in B$, we want to prove that $s(\gamma) \in U_n$ and $r(\gamma) \in V_n$. Set $x = s(\gamma)$ we then have that

$$ba(x) = \sum_{\alpha \in B_x} b(\alpha^{-1})a(\alpha) > b(\gamma^{-1})a(\gamma) > 0$$

which by (2) in Defintion 4.9 implies that $x \in U_n$. Similarly it follows that $y = r(\gamma) \in V_n$.

Let $\gamma \in B$, set $x = s(\gamma)$, We want to show that $r(\gamma) = \alpha_s(x)$. Assume that $r(\gamma) \neq \alpha_n(x)$. Choose $f \in C(V_s)_+$, with $f(\alpha_n(x)) = 0$ and $f(r(\gamma) = 1$.

$$\begin{aligned} f(\alpha_n(x)) = \frac{bfa(x)}{ba(x)} &= \int_{\alpha \in \mathcal{E}_{s(\gamma)}} \frac{b(\alpha^{-1})f(r(\alpha))a(\alpha)}{ba(x)} d\nu_s \\ &= \sum_{\alpha \in \mathcal{E}_{s(\gamma)}} \frac{b(\alpha^{-1})f(r(\alpha))a(\alpha)}{ba(x)} > \frac{b(\gamma^{-1})a(\gamma)}{ba(x)} > 0 \end{aligned}$$

Define the set $T = \{\gamma \in G \colon s(\gamma) \in U_n, r(\gamma) = \alpha_n \circ s(\gamma)\}$. We have already shown that $B \subseteq T$, which implies that $BB^{-1} \subseteq TT^{-1} \subseteq \mathcal{G}' = \{\gamma \colon s(\gamma) = r(\gamma)\}$. We then need to show that BB^{-1} is open. let $i \colon \mathcal{E} \to \mathcal{E}$ denote the inversion map. Since a and $b \circ i$ are continuous functions on $\mathcal{E}$, we know that their supports are open subsets of $\mathcal{E}$. we then have that $B = \pi(\mathrm{supp}(a) \cap \mathrm{supp} b \circ i)$ since π is an open map. Since $\mathcal{G}$ is an effect groupoid we have that BB^{-1} is an open set contained in $\mathcal{G}^{(0)}$. Similarly on can show that $B^{-1}B$ is an open set contained in $\mathcal{G}^{(0)}$. It follows that B is an open bijection.

Now let $x \in U_n$. We want to show that there exists a $\gamma \in B$ with $s(\gamma) = x$. For any $x \in U_n$ we have that

$$0 < ba(x) = \sum_{\alpha \in \mathcal{G}_x} b(\tilde{\alpha})a(\tilde{\alpha})$$

This implies that there exists $\gamma \in \mathcal{G}_x$ with $b(\gamma)a(\gamma) > 0$, which means that $\gamma \in B$ and $s(\gamma) = x$.

We thus have that $s(B) = U_n$. It remains to show that $\alpha_n = \beta_B$. Let $x \in U_n$, and let $\gamma \in B$ be the elemnt such that $s(\gamma) = x$. Then $\beta_B(x) = r(\gamma) = \alpha_n(x)$, and the proposition follows. ☐

Theorem 6.21. *Let $\mathcal{E}$ a twist over an étale groupoid $\mathcal{G}$ with compact unit space and let $p \in [1, \infty) \setminus \{2\}$. Then there is a natural identification of groupoids*

$$\mathcal{G}_{F^p_\lambda(\mathcal{G};\mathcal{E})} \cong \mathcal{G}.$$

Proof. By Theorem 6.17 we identify the core of $F^p_\lambda(\mathcal{G};\mathcal{E})$ with $C(\mathcal{G}^{(0)})$. Let $\mathcal{A}$ be the set of all partial homeomorphisms realised by admissible pairs in $F^p_\lambda(\mathcal{G};\mathcal{E})$ and let $\mathcal{B}$ be the family of partial homeomorphsims on on $\mathcal{G}^{(0)}$ induces by open bisection of $\mathcal{G}$. By Proposition 6.18 we have that $\mathcal{A} \subseteq \mathcal{B}$. The converse holds if every open subsets of $\mathcal{G}^{(0)}$ is a cozero set, which is not the case in general, but it holds locally, i.e. for every $x \in \mathcal{G}^{(0)}$, there exists a cozero neighbourhood U of x. Using this fact we have by Proposition 6.20 that for every $\beta \in \mathcal{B}$ and $x \in \mathcal{G}^{(0)}$, there exists an open neighbourhood of x such that $\beta_{\restriction U} \in \mathcal{A}$. It follows that the groupoids of germs of $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. The groupoids of germs of $\mathcal{A}$ is by definition $\mathcal{G}_{F^p_\lambda(\mathcal{G};\mathcal{E})}$. By Corollary 3.3 in [17], the groupoid of germs of $\mathcal{G}$ is isomorphic to $\mathcal{B}$ since $\mathcal{G}$ is effective □

Remark 6.22. Explicitly the isomorphism $\mathcal{G}_{F^p_\lambda(\mathcal{G};\mathcal{E})} \to \mathcal{G}$ is given as follows. For $[n,x] \in \mathcal{G}_{F^p_\lambda(\mathcal{G};\mathcal{E})}$ we let B_n denote the open bisection of the admissible pair n as in Proposition 6.20. Then the isomorphism is given by $[n,x] \mapsto B_n x$. See the proof of Corollary 3.4 in [17] for details.

We observe that the groupoid of germs does not "recover" the twist, and we only "recover" the groupiod as in the non twisted case.

6.2 The Weyl twist and L^p-rigidity

In the previous section we "recovered" the groupoid from the algebra, in this section we will see how to recover the twist $\mathcal{E}$ over $\mathcal{G}$ from the algebra. We will follow Renault's line of thought in [17], where he recovers the twist in the C^*-algebra setting.

Let A be a unital L^p-operator algebra, $p \neq 2$ and let $N(A)$ denote inverse subsemigroup of realisable partial homeomorphisms of X_A. On the set

$$\{(n,x) \in N(A) \times X_A \colon n = (a,b) \in N, ba(x) > 0\},$$

define a equivalence relation as follows: $(n,x) \approx (m,y)$ whenever $x = y$ and there exists $f,g \in C(X_A)$ with $f(x), g(x) > 0$ such that $n_f n = n_g m$. The quotient $\mathcal{E}_A$ by this equivalent relation has a natural groupoid structure, we denote $[\![n,x]\!]$ for the equivalence class of the element (n,x). The range and source map is given as the following $r([\![n,x]\!]) = \alpha_n(x)$ and $s([\![n,x]\!]) = x$. multiplication is given by $[\![n,\alpha_n(x)]\!][\![m,x]\!] = [\![nm,x]\!]$, and inversion by $[\![n,x]\!]^{-1} = [\![n^\sharp, \alpha_n(x)]\!]$. The unit space is canonically identified with X_A. The quotient becomes an étale groupoid under the topology with basic open sets $\mathcal{U}(U,n,V) = \{[\![n,x]\!] \colon x \in U\}$ indexed over $n \in N(A)$ and open set $U \subseteq U_n$, where U_n is given as in Definition 4.9. We call $\mathcal{E}_A$ the *Weyl twist* of A, analogous to the Renault in [17].

Proof. It is straight forward to show that the Weyl twist is a groupoid using Definition 4.1, the only issue is to show that multiplication is well defined, which we will do; let $(n,x) \approx (n',x)$ and $(m,\alpha_n(x)) \approx (m',\alpha_n(x))$, where $n = (a,b), n' = (a',b'), m = (c,d)$ and $m' = (c',d')$, then there exists $f, f', g, g' \in C(X_A)$ such that $nn_f = n'n_{f'}$ and $mn_g = m'n_{g'}$. We want to show that $(nm,x) \approx (n'm',x)$. Note that n_f realises the identity map on some set $U_{n_f} \subseteq \mathcal{G}^{(0)}$.

Since $fa(\gamma) = a(\gamma)f(r(\gamma)) = a(f \circ \alpha_m)(\gamma)$ we have that $n_f m = mn_{f\circ\alpha_m}$. Using this fact we have the following:

$$nn_f mn_g = nmn_{f\circ\alpha_m} n_g = nmn_{(f\circ\alpha_m)g}$$

and so

$$nmn_{(f\circ\alpha_m)g} = n'm'n_{(f'\circ\alpha_m)g'}$$

It thus follows that $(nm,x) \approx (n'm',x)$ and thus, multiplication is well defined. To show that the groupoid is étale let $U \subseteq \mathcal{G}^{(0)}$, then

$$s^{-1}(U) = \bigcup_{n \in N(A) \colon U \cap U_n \neq \emptyset} \{[\![n,x]\!] \colon x \in U \cap U_n\}$$

which is union of basic open sets and thus open. It follows that the source map is continuous. □

Proposition 6.23. *Let A be an unital L^p-operator algebra for some $p \in [1, \infty) \setminus \{2\}$. There is an injective continuous groupoid homomorphism $i_A\colon X_A \times \mathbb{T} \to \mathcal{E}_A$ given by $i_A((x, z)) = [\![n_f, x]\!]$ for $f \in C(X_A)$ such that $f(x) = z$, and there is a continuous surjective groupoid homomorphism $\pi_A\colon \mathcal{E}_A \to \mathcal{G}_A$ given by $\pi_A([\![n, x]\!]) = [n, x]$.*

Proof. Let $x \in X_A$, $z \in \mathbb{T}$, and let $f, g \in C(X_A)$ with $f(x) = g(x) = z$. We then have that $(n_f, x) \approx (n_g, x)$ since $\overline{z}f(x), \overline{z}g(x) > 0$ and $f\overline{z}g = g\overline{z}f$, and so $n_f n_{\overline{z}g} = n_g n_{\overline{z}f}$, i_A is therefor well defined.

Let $x, y \in X_A$, and let $z, w \in \mathbb{T}$, and let $f, g \in C(X_A)$ with $f(x) = z$ and $g(y) = w$. If $(x, n_f) \approx (y, n_g)$, then $x = y$, and there exists $f', g' \in C(X_A)$ with $f(x), g(x) > 0$ such that $n_f n_{f'} = n_g n_{g'}$ and so $ff' = gg'$ which implies that $zf'(x) = wg'(x)$. But since $f'(x), g'(x) > 0$, this implies that $z = w$, and so i_A is injective.

Let $(x_n, z_n)_n$ be an net in $\mathcal{G}^{(0)} \times \mathbb{T}$ converging to some $(x, z) \in \mathcal{G}^{(0)} \times \mathbb{T}$. Let $i(x, z) = [\![n_f, x]\!]$ for some $f \in C(\mathcal{G}^0)$, and note that every basic open set containing $[\![n_f, x]\!]$ is of the form $\{[\![n_f, x]\!]\colon x \in V\}$ for some open neighbourhood of $U \subseteq \operatorname{supp}(f)$. Let U be an arbitrary small set of this form, then for every $m \in \mathbb{N}$ there exists a neighbourhood U of x containing x_n for $n > m$, then $\{[\![n_f, x_n]\!]\colon x \in V\}$ contains $i(x_n, z_n) = [\![n_f, x_n]\!]$ for every $n > m$. i_A is thus continuous.

Now let $(x, n) \approx (x, m)$, we want to show that $(x, n) \sim (x, m)$. since $(x, n) \approx (x, m)$ we have that there exists $f, f' \in C(X_A)$ with $f(x), f'(x) > 0$ such that $nn_f = mn_{f'}$. By multiplying with some positive $h \in C(X_A)$ with $h(x) = 1$, we can assume that $\operatorname{supp}(f) = \operatorname{supp}(f')$. Set $U = \operatorname{supp}(f)$. $\alpha_{n\restriction_U} = \alpha_{nn_f} = \alpha_{mn'_f} = \alpha_{m\restriction_U}$. Thus π_A is surjective and well defined.

Fix $n \in N(A)$ and $U \subseteq U_n$ then $\{[n, x]\colon x \in U\}$ is a basic open set of $\mathcal{G}_A$ and its is clear the the preimage is a union of basic opens sets in $\mathcal{E}_A$, π_A is thus continuous. ☐

Proposition 6.24. *Let A be a unital L^p-operator algebra, then the sequence $X_a \times \mathbb{T} \xrightarrow{i_A} \mathcal{E}_A \xrightarrow{\pi_A} \mathcal{G}_A$ is exact.*

Proof. Need to show that $\operatorname{Im}(i_A) = \operatorname{Ker}(\pi_A)$. For any $f, g \in C(X_A)$, since the partial homeomorphism realised by n_f, α_{n_f} is the identity map on the support of f. by continuity of there is a neighbourhood, U of x such that f and g are positive on U. It follows that α_{n_f} is equal to α_{n_g} on U. and so since the 0 function is continuous we have that $\operatorname{Ker}(\pi_A)$ is all elements in $\mathcal{E}_A$ of the form $[\![n_f, x]\!]$ where $f \in C(X_A)$, $x \in X_A$. Now let $(n_f, x) \sim (n_g, x)$, but $(n_f, x) \approx (n_g, x)$ if and only if $f(x)/|f(x)| = g(x)/|g(x)|$, and so the image of i_A is precisely the element of the form $[\![n_f, x]\!]$ where $f \in C(X_A)$ and $x \in X_A$. ☐

Theorem 6.25. *Let $p \in [1, \infty) \setminus \{3\}$, let $\mathcal{G}$ be an effective étale groupoid and let $\mathcal{E}$ be a twist over $\mathcal{G}$. Then there is an isomorphism $\varphi\colon \mathcal{E}_{F^p_\lambda(\mathcal{G};\mathcal{E})} \to E$ such that the following diagram*

$$\begin{tikzcd}
\mathcal{G}^{(0)} \times \mathbb{T} \arrow[r, "i_{F^p_\lambda(\mathcal{G};\mathcal{E})}"] \arrow[d, "="] & \mathcal{E}_{F^p_\lambda(\mathcal{G};\mathcal{E})} \arrow[r, "\pi_{F^p_\lambda(\mathcal{G};\mathcal{E})}"] \arrow[d, "\varphi"] & \mathcal{G}_{F^p_\lambda(\mathcal{G};\mathcal{E})} \arrow[d, "\theta"] \\
\mathcal{G}^{(0)} \times \mathbb{T} \arrow[r, "i"] & \mathcal{E} \arrow[r, "\pi"] & \mathcal{G}
\end{tikzcd} \tag{6}$$

commutes.

Proof. By Theorem 6.21 we have that right vertical arrows is an isomorphism, it therefore suffices to define φ and show that that it is a groupoid homomorphism which commutes in the diagram.

Let $(n, x) \approx (m, x)$, where $n = (a, b)$ and $m = (c, d)$ then there exists $f, g \in C(\mathcal{G}^{(0)})$ with $f(x), g(x) > 0$ such that $nn_f = mn_g$, i.e $(af, \overline{f}b) = (cg, \overline{g}c)$ which written out gives us the equalities $a(\gamma)f(s(\gamma)) = c(\gamma)g(s(\gamma))$ and $b(\gamma^{-1})\overline{f(r(\gamma))} = d(\gamma^{-1})\overline{g(r(\gamma))}$ for all $\gamma \in \mathcal{G}$. We denote B_n the open bisection in $\mathcal{G}$ induced by n given in Proposition 6.20. Note that $B_{nn_f} = B_{mn_g}$. It follows from the equities above that there exists a neighbourhood of $B_n x$ where B_m and B_n

agree. In particular $B_m x = B_n x$, and we therefore have that $a(\widetilde{B_n x})f(x) = c(\widetilde{B_n x})g(x)$ (similarly $b(\widetilde{B_n x}^{-1})f(x) = d(\widetilde{B_n x}^{-1})g(x)$). rewriting this, we have $a(\widetilde{B_n x}) = c(\widetilde{B_n x}) \cdot g(x)/f(x)$ where $g(x)/f(x)$ is a positive constant. This implies that $a(\widetilde{B_n x})/|a(\widetilde{B_n x}| = c(\widetilde{B_n x})/|a(\widetilde{B_n x})|$.(Similarly it follows that $b((B_n x)^{-1})/|b((B_n x)^{-1})| = d((B_n x)^{-1})/|d((B_n x)^{-1})|$)

Note that $ba(x) > 0$ and since B_n is an open bisection there is a unique element γ such that $s(\gamma) = x$ and $b(\gamma^{-1})a(\gamma) \neq 0$. Note that $\gamma = B_n x$. Thus $ba(x) = b(\widetilde{B_n x}^{-1})a(\widetilde{B_n x}) > 0$. This means that $\arg(a(\widetilde{B_n x})) = -\arg(b((\widetilde{B_n x})^{-1}))$ and so $a(\widetilde{B_n x})/|a(\widetilde{B_n x})| = \overline{b((\widetilde{B_n x})^{-1})/|b((\widetilde{B_n x})^{-1})|}$. The map is therefore independent of the choice of a or b, and thus well defined.

Note that we can identify elements in $\mathcal{E}$ with elements in $\tilde{\mathcal{E}}$ with the map $\mathcal{E} \to \tilde{\mathcal{E}}$ given by $\varepsilon \mapsto [1, \varepsilon]$. We define the map $\varphi \colon \mathcal{E}_{F^p_\lambda(\mathcal{G};\mathcal{E})} \to \tilde{\mathcal{E}}$ by

$$[\![n, x]\!] \mapsto \left[\overline{\frac{a(\widetilde{B_n x})}{|a(\widetilde{B_n x})|}}, \widetilde{B_n x}\right]$$

We then have $\left[\overline{\frac{a(B\widetilde{B_n x}))}{|a(\widetilde{B_n x})|}}, \widetilde{B_n x}\right] = \left[1, \frac{a(\widetilde{B_n x})}{|a(\widetilde{B_n x})|} \cdot \widetilde{B_n x}\right]$ and it follows that $\frac{a(\widetilde{B_n x})}{|a(\widetilde{B_n x})|} \cdot \widetilde{B_n x} \in \mathcal{E}$. From the previous paragraph we have that this is well defined.

We then need to show that this is a homomorphism. Let $n = (a, b)$ and $m = (b, c)$ be two admissible pairs. Pick $x, y \in \mathcal{G}^{(0)}$ such that $x = \alpha_m(y)$. Then $[\![n, x]\!][\![m, y]\!] = [\![nm, y]\!]$, where $nm = (ac, bd)$. We first need to show that $B_n x B_m y = B_{nm} y$ Set $\gamma = B_n x B_m y$. Want to show that $\gamma \in B_{nm}$ and $s(\gamma) = y$. First note that $s(\gamma) = s(B_n x B_m y) = s(B_m y) = y$, we then have that $ac(\tilde{\gamma}) = \sum_{\alpha \in \mathcal{G}_y} a(\tilde{\gamma}\tilde{\alpha}^{-1})c(\tilde{\alpha}) = a(\widetilde{B_n x})c(\widetilde{B_m y}) \neq 0$. Similarly $bd(\tilde{\gamma}^{-1}) \neq 0$. It follows that $B_n x B_m y = B_{nm} y$ and using this we have that

$$\frac{ac(\widetilde{B_{nm} y})}{|ac(\widetilde{B_{nm} y})|} = \frac{a(\widetilde{B_n x})}{|a(\widetilde{B_n x})|} \frac{c(\widetilde{B_m y})}{|c(\widetilde{B_m y})|}$$

which means that φ is a groupoid homomorphsim.

Finally we need to show that the diagram commutes, let $x \in \mathcal{G}^{(0)}$, $z \in \mathbb{T}$ and let $f \in C(\mathcal{G}^{(0)})$ be any function such that $f(x) = z$. Note that $[\![n_f, x]\!] = i_{F^p_\lambda(\mathcal{G};\mathcal{E})}(x, z)$. We then have that $\varphi([\![n_f, x]\!]) = [\overline{f(x)}, x] = [1, f(x) \cdot x] = [1, i(x, z)]$ as expected. Now let $[\![n, x]\!] \in \mathcal{E}_{F^p_\lambda(\mathcal{G};\mathcal{E})}$, we then have that

$$\pi(\varphi([\![n, x]\!])) = \pi\left(\frac{a(\widetilde{B_n x})}{|a(\widetilde{B_n x})|} \cdot \widetilde{B_n x}\right) = B_n x = \theta([n, x]),$$

where θ is given as in Remark 6.22. Hence the diagram commutes, and it follows that φ is a groupoid isomorphism. $\square$

Similarly to the cohomology group for cohomologous cocycles, one can construct a group of isomorphic twists. We will Follow Section 5.2 in [18] in doing so.

Let $\mathcal{G}$ be an étale groupoid and let $\mathcal{E}$ and $\mathcal{F}$ be two twists over $\mathcal{G}$. Two twists are said to be *properly isomorphic* if there exists a groupoid isomorphism ϕ such that the following diagram

$$\begin{tikzcd} \mathcal{G}^{(0)} \times \mathbb{T} \arrow[r] \arrow[d, "="] & \mathcal{E} \arrow[r] \arrow[d, "\phi"] & \mathcal{G} \arrow[d, "="] \\ \mathcal{G}^{(0)} \times \mathbb{T} \arrow[r, "i"] & \mathcal{F} \arrow[r] & \mathcal{G} \end{tikzcd} \tag{7}$$

commutes. We write $[\mathcal{E}]$ for the equivalence class of the twist $\mathcal{E}$ over $\mathcal{G}$ containing all twists over $\mathcal{G}$ properly isomorphic to $\mathcal{E}$. The collection of equivalence classes of proper isomorphic twists on $\mathcal{G}$ is denoted $\mathrm{Tw}(\mathcal{G})$.

Let $\mathcal{E}$ and $\mathcal{E}'$ be two twists over $\mathcal{G}$. On the set

$$\mathcal{E} \times_{\pi}^{\pi'} \mathcal{E}' := \{(\varepsilon, \varepsilon') \in \mathcal{E} \times \mathcal{E}' : \pi(\varepsilon) = \pi'(\varepsilon')\},$$

define the following equivalence relation: $(\varepsilon, \varepsilon') \sim (\delta, \delta')$ if and only if there exists $z \in \mathbb{T}$ such that $z \cdot \varepsilon = \delta$ and $\overline{z} \cdot \varepsilon' = \delta'$. The quotient $\mathcal{E} * \mathcal{E}' := \mathcal{E} \times_{\pi}^{\pi'} \mathcal{E}' / \sim$ is in fact a twist over $\mathcal{G}$ given by

$$\mathcal{G}^{(0)} \times \mathbb{T} \xrightarrow{i * i'} \mathcal{E} * \mathcal{E}' \xrightarrow{\pi * \pi'} \mathcal{G}$$

where $(i * i)(x, z) = [i(x, z), i'(1, x)]$ and $(\pi * \pi')([\varepsilon, \varepsilon']) = \pi(\varepsilon)$. The collection $\mathrm{Tw}(\mathcal{G})$ forms an abelian group under the group operation given by $\mathcal{E} + \mathcal{E}' = [\mathcal{E} * \mathcal{E}]$ and identity given by the class of the trivial twist. By Proposition 2.13 in [3], an isometric isomorphism of two L^p-operator algebra induces a injective C^*-homomorphsim of the C^*-cores of the algebras. We therefor have the following corollary from Theorem 6.25.

Corollary 6.26. *Let $\mathcal{G}$ and $\mathcal{H}$ be effective étale groupoids with compact unit spaces, let $p \in [1, \infty) \backslash \{2\}$, $\mathcal{E}$ be a be twist over $\mathcal{G}$ and $\mathcal{F}$ be a twist over $\mathcal{H}$. Then there is an isometric isomorphism $F_\lambda^p(\mathcal{E}; \mathcal{G}) \cong F_\lambda^p(\mathcal{E}; \mathcal{H})$ if and only if there is an isomorphism of groupoids $\mathcal{G} \cong \mathcal{H}$, and $[\mathcal{E}] = [\mathcal{F}]$ in $Tw(\mathcal{G})$.*

There is in an group isomorphism between $H^2(\mathcal{G}, \mathbb{T})$ and the subset of $\mathrm{Tw}(\mathcal{G})$ consisting of twists by continuous sections [12, Section 4]. We therefore have the the following corollary, analogous to Theorem 3.7, the main result in Section 2.

Corollary 6.27. *Let $\mathcal{G}$ and $\mathcal{H}$ be topologically principal, Hausdorff, étale groupoids with compact unit spaces, let $p \in [1, \infty) \setminus \{2\}$, let σ be a 2-cocycle on $\mathcal{G}$ and ρ be a 2-cocycle on $\mathcal{H}$. Then there is an isometric isomorphism $F_\lambda^p(\mathcal{G}, \sigma) \cong F_\lambda^p(\mathcal{H}, \rho)$ if and only if there is an isomorphism of groupoids $\mathcal{G} \cong \mathcal{H}$ and σ and ρ are cohomologous.*

But this is not a generalisation of Theorem 3.7 as locally compact groups are neither étale nor topological principle, and even thou discrete groups are étale, they are not topological principle. In fact, the only group that can be topological principle is the trivial group.

7 The twisted crossed product

Definition 7.1. Let G be a toplogical group and X a toplogical space, we define the *action* of G on X as a function $G \times X \to X$, given by $(g, x) \mapsto g \cdot x$, satisfying the following

(1) $e \cdot x = x$

(2) $g \cdot (h \cdot x) = (gh) \cdot x$.

We will denote the action of G on X by $G \curvearrowright X$. We will follow [2] and [5] in defining the twisted crossed product.

Definition 7.2. Let G be a discrete group and let A be a unital L^p-operator algebra. A *twisted pair* (α, σ) for A and G are two maps $\alpha\colon G \to \operatorname{Aut}(A)$ and $\sigma\colon G \times G \to \mathcal{U}(A)$ satisfying the following:

(1) $\alpha_x(\sigma(y, z))\sigma(x, yz) = \sigma(x, y)\sigma(xy, z)$

(2) $\alpha_x \circ \alpha_y = \operatorname{Ad}(\sigma(x, y)) \circ \alpha_{xy}$

(3) $\sigma(x, e) = \sigma(e, x) = 1$

Where for $v \in \mathcal{U}(A)$, $\operatorname{Ad}(v) \in \operatorname{Aut}(A)$ is the inner automorphism of A given by $\operatorname{Ad}(v)(a) = vav^{-1}$ for all $a \in A$. We call the quadruple (G, A, α, σ) a *twisted L^p-operator algebra dynamical system.*

We call σ the *normalised 2-cocycle of the twisted pair.* Whenever $A = \mathbb{C}$ we have that α is the trivial map and σ is the normalised 2-cocycle as in section 3. If σ is trivial, then (G, A, α) is the L^p-operator algebra dynamical system as described in section 7 of [5].

Note that if A is a commutative algebra, then $\operatorname{Ad}(v)(a) = a$, and thus $\alpha_g^{-1} = \alpha_{g^{-1}}$

Definition 7.3. Let G be a discrete group, let A be a unital L^p-operator algebra and let (α, σ) and (α, c) be two twisted pairs for A and G. We say that σ and c are cohomologous if there exists a map $\phi\colon G \to \mathcal{U}(A)$ such that

$$\sigma(g, h)c(g, h)^{-1} = \alpha_g(\phi(h))\phi(g)\phi(gh)^{-1}$$

We will write $C_c(G, A)$ for continuous functions $G \to A$ with finite support. $C_c(G, A)$ becomes a algebra denoted $C_c(G, A, \alpha, \sigma)$ with product given by

$$(f * g)(x) = \sum_G f(y)\alpha_y(g(y^{-1}x))\sigma(y, y^{-1}x)$$

for all $f, g \in C_c(G, A)$.

Definition 7.4. Let G be a discrete group, let A be a unital L^p-operator algebra and let (α, σ) be a twisted pair for A and G. A *twisted covariant representation* of (G, A, α, σ) on an L^p space E is a pair (φ, u) where $\varphi\colon A \to \mathcal{B}(E)$ is representation and $u\colon G \to \operatorname{Isom}(E)$ is a map such that

$$u_g\varphi(a)u_g^{-1} = \varphi(\alpha_g(a))$$

and

$$u_g u_h = \varphi(\sigma(g, h))u_{gh}$$

for all $g, h \in G$ and all $a \in A$. We define the associated *integral representation* $\varphi \rtimes u\colon C_c(G, A, \alpha, \sigma) \to \mathcal{B}(E)$ by

$$(\varphi \rtimes u)(f)(\xi) = \sum_{s \in G} \varphi(f(s))(u_s(\xi))$$

for all $f \in C_c(G, A, \alpha, \sigma)$ and all $\xi \in E$.

Definition 7.5. Let G be a discrete group, let A be a unital L^p-operator algebra and let (α, σ) be a twisted pair for A and G. Let $\varphi_0 \colon A \to \mathcal{B}(E_0)$ be any representation. Write $E = L^p(G, E_0)$, we define the associated *twisted regular covariant pair* (φ, u) by

$$\varphi(a)(\xi)(t) = \varphi_0(\alpha_t^{-1}(a))\xi(t)$$

and

$$u_s(\xi)(t) = \varphi_0(\alpha_t^{-1}(\sigma(s, s^{-1}t)))\xi(s^{-1}t)$$

for all $a \in A$, all $s, t \in G$ and all $\xi \in E$.

Definition 7.6. Let A be a unital L^p-operator algebra, let G be a locally compact group and let $p \in (1, \infty)$. We define *the twisted reduced crossed product* $F^p_\lambda(G, A, \alpha, \sigma)$ to be the completion of $C_c(G, A, \alpha, \sigma)$ in the norm

$$\|f\| = \sup\{\|(\varphi \rtimes u)(f)\| : (\varphi, u) \text{ is a regular covariant pair}\}.$$

Let X be a compact Hausdorff space, and let A be of the form $C(X)$ so that α comes from an action of G on X via homeomorphisms, let σ be a 2-cocycle such that (α, σ) is a twisted pair for A and G. For the rest of the section we will only work with the twisted reduced crossed product of this form. We will just write $F^p_\lambda(G, C(X), \sigma)$ to mean this algebra.

Example 7.7. Let $A = \mathbb{C}$, and σ be a continuous 2-cocycle on G, then $(G, \mathbb{T}, \text{trivial}, \sigma)$ is twisted dynamical system and we have $F^p_\lambda(G, \mathbb{C}, \sigma) \cong F^p_\lambda(G, \sigma)$.

Definition 7.8. Let G be discrete group and X a compact space, given a group action $G \to \mathrm{Homeo}(X)$ of G on X, we define the *transformation groupoid* $G \ltimes X$, as the set $G \times X$ with groupoid operations given as

$$(g, h \cdot x))(h, x) = (gh, x) \qquad \text{and} \qquad (g, x)^{-1} = (g^{-1}, \alpha_g(x))$$

Since G is discrete, the groupoid $G \ltimes X$ is étale equipped with the natural product topology of $G \times X$. The unit space is $e \times X$, which we identify with X. This means that we can identify $C(X)$ with a subalgebra of $C_c(G \ltimes X, \sigma)$ where σ is some normalised continuous 2-cocycle on $G \ltimes X$. Let $\alpha \colon G \to \mathrm{Aut}(C(X))$ be given by $\alpha_g(f)(x) = f(g^{-1} \cdot x)$.

Let $\sigma \colon G \ltimes X^{(2)} \to \mathbb{T}$ be a normalised continuous 2-cocycle. Note that every pair in $G \ltimes X^{(2)}$ is of the form $((g, h \cdot x), (h, x))$ for some $g, h \in G$ and $x \in X$. Given $g, h \in G$, we then define the continuous function $\tilde{\sigma}(g, h)(x) = \sigma((g, g^{-1} \cdot x), (h, h^{-1}g^{-1}x)$. In fact $\tilde{\sigma} \in C(X, \mathbb{T}) = \mathcal{U}(C(X))$ and we can then show that $(\alpha, \tilde{\sigma})$ is a twisted pair for $C(X)$ and G, as defined in Definition 7.2. Since $C(X)$ is commutative, the second and third condition follows trivially. We need to show the first condition. Let $g, h, k \in G$ and $x \in X$, we then have the following:

$$\begin{aligned}\alpha_g(\tilde{\sigma}(h, k))(x)\tilde{\sigma}(g, hk)(x) &= \sigma\left((h, (gh)^{-1} \cdot x), (k, (ghk)^{-1} \cdot x)\right) \sigma\left((g, g^{-1} \cdot x), (hk, (ghk)^{-1} \cdot x\right) \\ &= \sigma((g, g^{-1} \cdot x), (h, (gh)^{-1})\sigma(gh, (gh)^{-1} \cdot x)(k, (ghk)^{-1} \cdot x)) \\ &= \tilde{\sigma}(g, h)(x)\tilde{\sigma}(gh, k)(x)\end{aligned}$$

where we used (2) in Definition 5.7 in the second equality with $\alpha = (g, g^{-1} \cdot x)$, $\beta = (h, (gh)^{-1} \cdot x)$ and $\gamma = (k, (ghk)^{-1} \cdot x)$. $(\alpha, \tilde{\sigma})$ is thus a twisted pair for $C(X)$ and G.

We now want to show the converse relation. Let $\tilde{\sigma}$ be any map such that $(\alpha, \tilde{\sigma})$ is a twisted pair for $C(X)$ and G. Define $\sigma \colon G \ltimes X^{(2)} \to \mathbb{T}$ as follows:

$$\sigma((g, h \cdot x), (h, x)) = \tilde{\sigma}(g, h)(gh \cdot x) \tag{8}$$

for all $(g, h \cdot x)(h, x) \in G \ltimes X^{(2)}$. We need to show that this is a 2-cocycle on $G \ltimes X$ as defined in 5.7. The first condition follows immediately, we need to show the second condition. First note that every composable triple in (α, β, γ) in $G \ltimes X$ is of the form $\alpha = (g, h \cdot x), \beta = (h, x), \gamma = (k, k^{-1} \cdot x)$

for some $g, h, k \in G$, and $x \in X$. . Now, given such a composable triple, we have the following

$$\begin{aligned}\sigma((g, h \cdot x), (h, x))\sigma((gh, x), (k, k^{-1} \cdot x)) &= \tilde{\sigma}_{g,h}(gh \cdot x))\tilde{\sigma}_{gh,k}(gh \cdot x)) \\ &= \tilde{\sigma}_{g,h}\tilde{\sigma}_{gh,k}(gh \cdot x)) \\ &= \alpha_g(\tilde{\sigma}_{h,k})\tilde{\sigma}_{g,hk}(gh \cdot x) \\ &= \tilde{\sigma}_{h,k}(h \cdot x))\tilde{\sigma}_{g,hk}(gh \cdot x) \\ &= \tilde{\sigma}_{h,k}(hk(k^{-1} \cdot x))\tilde{\sigma}_{g,hk}(ghk(k^{-1} \cdot x)) \\ &= \sigma((h, x), (k, k^{-1}x))\sigma((g, h \cdot x), (hk, k^{-1} \cdot x))\end{aligned}$$

where we used (1) of Definition 7.2 in the third equality. This means σ is a 2-cocycle. It follows that we have a one to one relation between continuous normalised 2-cocycles on $G \ltimes X$ and twisted pair for G and $C(X)$. Let σ, c be two cohomologous normalised 2-cocycles on $G \ltimes X$, we now want to show that this implies that $\tilde{\sigma}$ and $\tilde{c}$ are cohomologous. If $\sigma \sim c$, then there exists a map $\phi\colon \to \mathbb{T}$ such that

$$\sigma((g, h \cdot x), (h, x))\overline{c((g, h \cdot x), (h, x))} = \phi(g, h \cdot x)\phi(h, x)\overline{\phi(gh, x)}$$

for every composable pair $((g, h \cdot x), (h, x)) \in G \ltimes X^{(2)}$. We then define the function $\tilde{\phi}\colon G \to C(X, \mathbb{T})$ given by $\tilde{\phi}(g)(x) = \phi(g, g^{-1}x)$ for all $g \in G$. Write $\tilde{\phi}_g(x)$ for $\tilde{\phi}(g)(x)$ and note that $\tilde{\phi}_g^{-1}(x) = \overline{\phi(g, g^{-1}x)}$. We then have the following:

$$\begin{aligned}\tilde{\sigma}_{g,h}(x)\tilde{c}_{g,h}(x)^{-1} &= \sigma((g, g^{-1} \cdot x), (h, h^{-1}g^{-1}x)\overline{c((g, g^{-1} \cdot x), (h, h^{-1}g^{-1}x)} \\ &= \phi(g, g^{-1} \cdot x)\phi(h, h^{-1}g^{-1}x)\overline{\phi(gh, (gh)^{-1}x)} \\ &= \tilde{\phi}_g(x)\alpha_g(\tilde{\phi}_h)(x)\tilde{\phi}_{gh}^{-1}(x)\end{aligned}$$

It follows that $\tilde{\sigma}$ and $\tilde{c}$ are cohomologous as defined in Definition 7.3. Similarly assume that $\tilde{\sigma}$ and $\tilde{c}$ are cohomologous with coboundary $\tilde{\phi}$. Define the map $\phi\colon G \ltimes X \to \mathbb{T}$ given by $\phi(g, x) = \tilde{\phi}(g^{-1} \cdot x)$. We then have that

$$\begin{aligned}\sigma((g, h \cdot x), (h, x))\overline{c((g, h \cdot x), (h, x))} &= \tilde{\sigma}_{g,h}(gh \cdot x)\tilde{c}_{g,h}^{-1}(gh \cdot x) \\ &= \tilde{\phi}_g(gh \cdot x)\alpha_g(\tilde{\phi}_h)(gh \cdot x)\tilde{\phi}_{gh}^{-1}(gh \cdot x) \\ &= \phi(g, h \cdot x)\phi(h, x)\overline{\phi(gh, x)}\end{aligned}$$

and thus, σ and c are cohomologous. Let $\varphi\colon C_c(G, C(X), \sigma) \to C_c(G \ltimes X, \tilde{\sigma})$ be given by

$$\varphi(f\delta_g)(s, x) = \begin{cases} f(sx) & \text{if}\, s = g \\ 0 & else \end{cases} \tag{9}$$

For $f \in C(X)$ and $g \in G$. $f\delta_g \in C_c(G, C(X))$ denotes the characteristic function that maps $g \in G$ to $f \in C(x)$ and everything else to 0. Since every function in $C_c(G, C(X)$ has finite support, it follows that every element can be written as a unique linear combination of characteristic functions. Let u_g denote the characteristic function on $g \times X$ in $C_c(G \ltimes X, \sigma)$ and note that $\varphi(f\delta_g) = fu_g$, and since we can identify $C(X)$ with a subalgebra of $C_c(G \ltimes X, \sigma)$ and the fact that G is discrete, every function on $C_c(G \ltimes X)$ can be written as a linear combinations of fu_g for $f \in C(X)$ and $g \in G$. It remains to show that this is a homomorphism. Let $f_1, f_2 \in C(X)$, and let $g_1, g_2 \in G$, we then have that

$$\begin{aligned}\varphi(f_1\delta_{g_1} * f_2\delta_{g_2})(s, x) &= \varphi(f_1\alpha_{g_1}(f_2)\tilde{\sigma}(g_1, g_2)u_{g_1g_2})(s, x) \\ &= f_1(g_1g_2x)f_2(g_2x)\sigma(g_1, g_2, x)(g_2, x)\end{aligned}$$

which means that

$$\begin{aligned}(\varphi(f_1\delta_{g_1}) *_\sigma \varphi(f_2\delta_{g_2}))(s, x) &= \varphi(fu_{g_1})(sg_2^{-1}, g_2 \cdot x)f_2(g_2 \cdot x)\sigma((sg_2^{-1}, g_2 \cdot x), (g_2, x)) \\ &= f_1(g_1g_2 \cdot x)f_2(g_2 \cdot x)\sigma(g_1, g_2 \cdot x)(g_2, x) \\ &= \varphi(f_1\delta_{g_1} * f_2\delta_{g_2})(s, x).\end{aligned}$$

It follows that the map φ is an isomorphism. We summaries the results in the following lemma.

Lemma 7.9. *Let $G \curvearrowright X$ be an action of discrete group on a compact Hausdorff space, let $(\alpha, \tilde{\sigma})$ be a twisted pair for G and $C(X)$. Then there is an normalised continuous 2-cocycle σ on $G \ltimes X$ given as in (8) such that there is an isomorphism $C_c(G, C(X), \tilde{\sigma}) \cong C_c(G \ltimes X, \sigma)$ given as in (9). Furthermore if there are two twisted pairs $(\alpha, \tilde{\sigma})$ and $(\alpha, \tilde{\rho})$ for G and $C(X)$, then $\tilde{\sigma}$ and $\tilde{\rho}$ are cohomologous if and only if σ and ρ are cohomologous.*

In section 6 in [3] it is shown that this isomorphism can be extended isometrically to the closures for the trivial 2-cocycle, i.e. that $F^p_\lambda(G \ltimes X) \cong F^p_\lambda(G, C(X))$. This lets one apply the L^p-rigidity result to L^p-crossed products by topologically free actions [3, Theorem 6.7]. To generalise this result we need to define the the reduced twisted L^p-operator algebra analogous to how Gardella and Lupini define the reduced L^p-operator algebra in [6], and show that this definition is equivalent to Definition 5.9. The definition in [6] is based on the formulation of the reduced C^*-algebra in Section 3.1 of [14], and to generalise the result we need to extend the results in [6] to σ-projective representations. This can be done following [16], where some of the results are shown for σ-projective unitary representations. This is unfortunately outside the time scope of this thesis.

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