Doctoral theses at NTNU, 2022:331

Davood Dadrasajirlou

Hyper-Viscoplastic Modelling of Clay Behaviour

Doctoral thesis

NTNU
Norwegian University of Science and Technology
Thesis for the Degree of
Philosophiae Doctor
Faculty of Engineering
Department of Civil and Environmental
Engineering

NTNU
Norwegian University of
Science and Technology

Davood Dadrasajirlou

Hyper-Viscoplastic Modelling of Clay Behaviour

Thesis for the Degree of Philosophiae Doctor

Trondheim, October 2022

Norwegian University of Science and Technology
Faculty of Engineering
Department of Civil and Environmental Engineering

NTNU
Norwegian University of
Science and Technology

NTNU
Norwegian University of Science and Technology

Thesis for the Degree of Philosophiae Doctor

Faculty of Engineering
Department of Civil and Environmental Engineering

ISBN 978-82-326-5919-7 (printed ver.)
ISBN 978-82-326-6999-8 (electronic ver.)
ISSN 1503-8181 (printed ver.)
ISSN 2703-8084 (online ver.)

Doctoral theses at NTNU, 2022:331

Printed by NTNU Grafisk senter

Preface

The current treatise is a paper-based thesis with a collection of five inter-related research articles enclosed as an appendix.

This thesis is submitted in partial fulfilment of the requirement for the degree of Philosophiae Doctor (PhD) at the Geotechnical Division, Department of Civil and Environmental Engineering, Norwegian University of Science and Technology (NTNU).

The PhD study has been conducted under the supervision of Prof. Gustav Grimstad and Dr. Seyed Ali Ghoreishian Amiri at PoreLab – Centre of Excellence (SFF) and Geotechnical Division, Department of Civil and Environmental Engineering, NTNU, Trondheim, Norway.

The PhD work has been carried out from July 2019 to October 2022. The research council of Norway supported the study through its centre of excellence funding scheme, PoreLab- Porous Media Laboratory, project number 262644.

Members of the evaluation committee of the PhD thesis

Associate Professor Philip James Vardon

Delft University of Technology, Delft, the Netherlands

Assistant Professor Christelle Nadine Abadie

University of Cambridge, Cambridge, UK

Associate Professor Yutao Pan

Norwegian University of Science and Technology, Trondheim, Norway

Professor Gudmund Reidar Eiksund

Norwegian University of Science and Technology, Trondheim, Norway

Supervisors of the PhD study

Professor Gustav Grimstad

Norwegian University of Science and Technology, Trondheim, Norway

Dr Seyed Ali Ghoreishian Amiri

Norwegian University of Science and Technology, Trondheim, Norway

Abstract

The modified cam clay (MCC) model, deduced from the unified and comprehensive behavioural framework of critical state soil mechanics (CSSM), revolutionised the understanding of the mechanical behaviour of soil, particularly clay. Although it was originally energy-based, the MCC model could not stand against the critiques of modern thermodynamics. The laws of thermodynamics (widely considered to be true) summarise the properties of energy and the feasibility of its transformation from one form to another. Constitutive models with no thermodynamic validity cannot confidently be utilised as there is no guarantee against false generation/loss of energy. Soon after its presentation, the MCC model was given a thermodynamic description known today as hyperplasticity. The hyperplastic description lifted the unnecessary normality rule and led to a family of MCC-type models with versatile yield criteria and inelastic flow directions.

In addition to the normality restriction, which results in extreme dilatancy for over-consolidated clays, the original MCC model suffers from several limitations. Perhaps, as pointed out by its founders, the most profound one can be the lack of the concept of time. In developing the MCC model, it is assumed that the state of the material does not spontaneously change with the march of time. Numerous attempts with the overwhelming use of the overstress viscoplastic theory as the vogue approach have been undertaken, but the thermodynamic consistency of most of them is under question. Moreover, no attempts with the hyperplasticty approach have rigorously addressed the issue of time-independency.

This doctoral research contributes to understanding the thermodynamically-based hyperplasticity framework and its application in the constitutive modelling of soil, particularly the viscous behaviour of clay with an orientation toward the CSSM and the isotache viscosity. In this regard, hyperplasticity formalism is laid out after providing a review of the viscous behaviour of clay with a focus on the development of the isotache concept. Next, the MCC model is integrated with the concept of time, resulting in a classical critical state hyper-viscoplastic model with similarities to the soft soil creep model. The profound impact of Ziegler's orthogonality condition, the backbone of the hyperplasticty approach, on the critical state envelope is realised, paving the way for generalising the developed classical hyper-viscoplastic model with isotache viscosity. It is demonstrated that the typical MCC plastic-free energy could not be considered for a rate-dependent system with a single internal variable. Otherwise, by imposing Ziegler's orthogonality condition, the uniqueness of the critical state envelope, which is the useful paradigm of critical state soil mechanics, is lost under different loading rates. A versatile force potential or dissipation rate function is constructed that provides adjustability of the location of the critical state (Spacing Ratio) while securing a unique critical state friction envelope as the useful paradigm of the CSSM to have a unified description of the mechanical behaviour of soil. Non-associated inelastic flow as an essential property of particulate frictional materials is adopted via accommodating an effective pressure-dependent shear dissipative mechanism. This later distinction lifts the unnecessarily restrictive condition of normality invoked in the original overstress and consistency viscoplasticity theories. Moreover, different frictional criteria of Drucker-Prager, Mohr-Coulomb, and Matsuoka-Nakai have been considered, and their features in terms of friction mobilisation and inelastic flow direction are explored. By realising the homothetic functioning of isotache viscosity, an emphasis has been put on the delicate practice of the effective stress ratio tensor (the deviatoric stress tensor normalised by the effective pressure) as an essential state variable of frictional material to achieve all the features mentioned previously such as adjustability of the spacing ratio and non-associativity of inelastic flow with different frictional criteria while securing the uniqueness of critical state envelope. Lastly, the efficacy of the proposed hyper-viscoplastic constitutive model is evaluated by simulating sets of triaxial and true triaxial tests conducted on the Hong Kong Marine Deposit (HKMD) and the Fujinomori clay.

Acknowledgements

I would like to thank my supervisors, Prof. Gustav Grimstad and Dr. Seyed Ali Ghoreishian Amiri, for their trust, support, commitment, and invaluable criticisms. Both are bright and original thinkers with remarkable enthusiasm and the ability to do meaningful research. I have enjoyed and learned very much from our many fascinating discussions throughout my doctoral study. I sincerely appreciate our relationship and look forward to our future collaborations. I also want to warmly thank Ali for being close as a friend and helping me establish myself in Trondheim and overcome the difficulties in the early stages of my PhD journey.

I would like to extend my gratitude to the evaluation committee administered by Assoc. Prof. Yutao Pan, and particularly to my thesis's examiners, Assoc. Prof. Philip James Vardon of Delft University of Technology and Asst. Prof. Christelle Nadine Abadie of the University of Cambridge, who did an excellent job in reviewing my thesis.

I am also grateful to all members of the Geotechnical Division at NTNU. Particularly, I thank Prof. Steinar Nordal and Prof. Gudmund Reidar Eiksund, for their support, professional guidance, discussions, and interest in my research. My sincere gratitude also goes to Arnfinn Emdal for his invaluable support and care and for all the pleasant conversations we had on several occasions. I am also thankful to my friends and colleagues, namely Yeganeh Attari, Herve Vicari, Dr. Emir Ahmet Oguz, Gebray Habtu Alene, Habibollah Sadeghi, Rui Tao, Dr. Quoc Anh Tran, Sigurdur Mar Valsson, and Erik Sørlie, for creating a friendly and cheerful environment.

I would like to give warm thanks to my close friend, Dr. Karim Tarbali, with whom I shared happiness and frustrations since the start of my bachelor's study in 2005 in Iran.

Finally, I am always grateful to my family, particularly my parents and my uncle “Alex”, for their unconditional support and continuous encouragement, without which I would not reach this stage.

Table of content

List of figures

List of tables

List of notations

d	Dissipative power (work) or dissipation function
e	Void ratio
f	Helmholtz free energy potential
G	Shear modulus
g	Dimensionless shear modulus coefficient
I_1, I_2, and I_3	The first, second and third invariants of Cauchy stress tensor $\boldsymbol{\sigma}$
K	Bulk modulus
K_0	Coefficient of earth pressure at rest
k	Dimensionless bulk modulus coefficient
M	Frictional coefficient
n	Rate sensitivity parameter
OCR	Over consolidation ratio
p	Mean effective pressure
p_0	Isotropic pre-consolidation pressure
p_a	Reference pressure (atmospheric pressure) in Helmholtz free energy potential
p_{eq}	Equivalent pressure on isotropic unloading reloading line (IURL)
p_{ref}	Reference pressure for definition of p_0
q	Deviatoric stress invariant equal to the deviatoric stress under triaxial condition
q_i	Heat flux
R	Time resistance in Chapter 2 and Spacing ratio elsewhere
r	Norm of an arbitrary reference volumetric strain rate
S	State variable
s	Entropy
T	Transition function
u	Free energy
w	Flow potential
z	Force potential
$\boldsymbol{\alpha}$	Internal variabel (inelastic strain) tensor
β	Parameter determining the share of stored plastic volumetric work in The conventional rate-independent formulation of hyperplastic MCC
γ	Parameter for the frictional dissipation
$\boldsymbol{\delta}$	Kronecker delta- the substitution tensor
$\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^p$	Total and plastic strain tensor
${}^D\boldsymbol{\varepsilon}$, ${}^D\boldsymbol{\varepsilon}^p$	Total and plastic deviatoric strain tensor
ε_v , ε_v^p	Total and plastic volumetric strain
$\boldsymbol{\eta}$	Stress ratio tensor
θ	Tempreture in Chapter 3 and Lode angle elsewhere
κ	Slope of isotropic unloading reloading line (IURL)
λ	Slope of normal compression line (NCL)
μ	Creep index
ρ	Density

$\boldsymbol{\sigma}$	Cauchy (true) stress tensor
σ_1, σ_2, and σ_3	Principal stresses
υ	Specific volume
ϕ_{cs}	Critical state angle of shearing resistance
$\boldsymbol{\chi}$	Dissipative stress tensor
χ_p, χ_q	Mean and deviatoric invariants of the dissipative stress tensor

List of operations

$\mathrm{tr}(\boldsymbol{x})$	Trace or first invariant of tensor $\boldsymbol{x}$
$\boldsymbol{x}^{-1}$	Inverse of tensor $\boldsymbol{x}$
$\boldsymbol{x}.\boldsymbol{y}$	Product of two tensors: $x_{ik}y_{kj}$
${}^{D}\boldsymbol{x}$	Deviator tensor of $\boldsymbol{x}$: $\boldsymbol{x}-\left(\mathrm{tr}(\boldsymbol{x})/3\right)\boldsymbol{\delta}$
$\dot{\boldsymbol{x}}$	Rate of $\boldsymbol{x}$
∂	Partial differential
$\langle x\rangle$	Macauly bracket: $\left(x+\lvert x\rvert\right)/2$

List of abbreviations

CSSM	Critical state soil mechanics
CSR	Constant strain rate (oedometer test)
DEM	Discrete element method
DKP	Dual kinematic plane
DP	Drucker-Prager frictional criterion
DP-MC	Conventional Mohr-Coulomb criterion as a function of Lode angle
EOP	End of primary consolidation
HKMD	Hong Kong marine deposits
IURL	Isotropic unloading-reloading line on the bi-logarithmic compressional plane
MC	Mohr-Coulomb frictional criterion
MCC	Modified cam clay
MN	Matsuoka-Nakai frictional criterion
NCL	Normal compression line on the bi-logarithmic compressional plane
OCT	Octahedral plane
SMP	Spatially mobilised plane
SRS	Step-changed rate of strain (oedometer test)

Chapter 1 – Introduction

1.1 Background and motivation

The constitutive relation of soil is the core of soil mechanics. It is a mathematical idealisation of the real stress-strain response of soil that plays a vital role in solving boundary value problems via, for instance, the finite element method by providing a link between compatibility and equilibrium.

The study of the mechanics of soils is now over two hundred years old, dating back to 1776 when Coulomb (1776) established the first useful mathematical idealisation for understanding the soil's behaviour. After its generalisation by Mohr (1887) at the end of the 19th century, Coulomb's theory became known as the Mohr-Coulomb criterion. After almost one and a half centuries, Terzaghi, in the first textbook on soil mechanics (Terzaghi 1925), introduced the principle of effective stress, which places a strong emphasis on the particulate nature of soils, i.e., in saturated soil, the solid particle must play a much more important role than the pore water which has no capacity against shear. This milestone in soil mechanics is credited with the birth of modern soil mechanics.

Later, the realisation of soil's frictional character led to some significant modifications to the Mohr-Coulomb criterion. These modifications, manifested by the vogue plasticity theory, resulted in smoother yield surfaces which are attractive from the computational point of view. First, Drucker and Prager (1952) adapted the pressure-independent von Mises criterion (Von Mises 1913) for the pressure-dependent frictional material. Afterwards, two other major modifications, namely the Continuum-Mechanics-based Matsuoka-Nakai criterion (Matsuoka and Nakai 1974) and experimentally-based Lade-Duncan criterion (Lade and Duncan 1975), were introduced to better approximate the shear yield and friction mobilisation of soil by the inclusion of the effect of the intermediate principal stress. These modifications are prime additions to the repertoire of soil mechanics. However, they are exclusively concerned with the shear or friction state of soil and lack a comprehensive description of soil's behaviour.

In the 1950s, Roscoe and his co-workers (Roscoe et al. 1958) presented critical state soil mechanics as a rational behavioural framework for the unified and comprehensive description of soil behaviour. This breakthrough in soil mechanics has been achieved by linking the compressional and shearing behaviour of soil and introducing the concept of the critical state, a state at which a particulate material keeps deforming in shear without volume and stress alteration at the macro-scale. Later, based on the phenomenological framework of critical state soil mechanics and by invoking Drucker's stability postulate (Drucker 1959), the first comprehensive, effective stress constitutive model, namely the Cam Clay model, was developed (Schofield and Wroth 1968). To include the compressional dissipative mechanism, Modified Cam Clay (MCC) model with an elliptical yield surface was subsequently introduced by Roscoe and Burland (1968). Although these two fundamental critical state models are energy-based, they did not satisfy the thermodynamic principals (Collins and Kelly 2002) until Houlsby (1981) provided the thermodynamical expression of the MCC model with some insights and clues on rational enrichment of the model, which still echo today. Houlsby (1981) employed the approach of generalised thermodynamics founded in the book of Ziegler (1977). The systemised version of the approach, termed "hyperplasticity", has been presented more recently by Houlsby and Puzrin (2006).

The MCC model revolutionised our understanding of the behaviour of clay. However, it has several limitations in predicting clay's actual behaviour. Perhaps the most profound limitation is the time independency, as the founders, Roscoe and Burland (1968), pointed out. In the development of the MCC model, it is assumed that the state of the material does not spontaneously change with the march of time (no creep or relaxation). Over the past three decades, overabundant viscoplastic constitutive models have been proposed to address the time-independency along with several other limitations of the MCC model. Almost all developments have been based on the overstress theory of Perzyna (1963), proposed by invoking the stability postulate of Drucker (Olszak and Perzyna 1964), which has been proven to be unnecessarily restrictive for frictional materials. Most of these models cannot stand against thermodynamic scrutiny, making them unreliable for solving boundary value problems. Many employed ad-hoc procedures and assumptions, resulting in complex models (with the same basic assumptions as taken in this research, like small strain, constant temperature, full saturation or full dryness, no bonding effect, etc.) with many material parameters to find excellent fits to certain experimental data. Apart from the daunting task of determining a large number of material parameters, the unjustifiable complication of the models, which appears merely as academic exercises, makes the models non-relocative and dead-end (Kolymbas 2000). In this regard, Gudehus (2011) analogised these models to a "morass of equations" and methaphorised the corresponding literature to "jungle of data".

In the light of the above, the current thesis approaches to incorporate the viscous effect in the MCC model using the hyperplasticity approach. Since the hyperplasticity approach is potential-based and derived from thermodynamic principles, it restricts improper use of mathematics and provides a capacity for developing relocative models (all components of a consitituve model in plasticity theory such as the elastic part, the yield and the plastic potential surfaces, and different hardening rules are specified via only two potential functions) with fewer parameters.

1.2 Aim and objectives

The main goal of the current dissertation is to contribute to understanding and modelling the viscous behaviour of clay. It is one of the first attempts in constructing hyper-viscoplastic models addressing clay's creep and rate-dependent behaviour. More specifically, it extends the hyperplastic version of the MCC model to a hyper-viscoplastic version for viscous modelling of clay that complies with the isotache concept based on which a unique relation between stress, strain, and strain rate exists. To this end, the dissertation endeavours to deliver the following objectives:

1- Development of a further understanding of the hyperplasticity framework for the construction of a thermodynamically consistent constitutive model addressing the viscous behaviour of clay
2- Identification of the conditions imposed by the hyperplasticity framework on the viscous constitutive modelling of clay through consideration of two fundamental concepts of critical state and isotache
3- Development of a generalised hyper-viscoplastic constitutive model for clay considering the essential features such as the viscous behaviour and the friction mobilisation
4- Validation of the developed model against reliable experimental data in the literature with some hints on the evaluation of the model parameters

1.3 Structure of the thesis

This paper-based thesis is organised into five chapters, including the present introduction. The outline of the remaining chapters is as follows.

Chapter 2 is devoted to reviewing the literature on the viscous behaviour of clay and the background to the development of the isotache concept. It also provides a concise review of the existing approaches for constitutive modelling of the viscous behaviour.

Chapter 3 presents a background to the hyperplasticity framework and sets out the formalism and terminology used in the accomplished research articles attached to this thesis.

Chapter 4, which comprises the main body of the thesis, is concerned with the summary of the outcomes and their discussion. It tries to summerise the articles on which this thesis is based on and provide a cogent and coherent account of them. To this end, Chapter 4 is oriented towards the objectives listed.

Chapter 5 concludes the thesis and offers some recommendations for plausible research directions and further relevant developments.

The *Appendix* provides copies of the five research articles disseminated during the three-year research period. These articles are listed in the following section.

1.4 List of publications

- *Paper I*: Grimstad, G., Dadrasajirlou, D., and Amiri, S. a. G. Modelling creep in clay using the framework of hyper-viscoplasticity. *Géotechnique Letters* 10(3):404-408 (2020).
  https://doi.org/10.1680/jgele.20.00004
- *Paper II*: Grimstad, G., Long, M., Dadrasajirlou, D., and Amiri, S. a. G. Investigation of development of the earth pressure coefficient at rest in clay during creep in the framework of hyper-viscoplasticity. *International Journal of Geomechanics* 21(1):04020235 (2021).
  https://doi.org/10.1061/(ASCE)GM.1943-5622.0001883
- *Paper III*: Dadras-Ajirloo, D., Grimstad, G., and Ghoreishian Amiri, S. A. On the isotache viscous modelling of clay behaviour using the hyperplasticity approach. *Géotechnique* Ahead of print (2022).
  https://doi.org/10.1680/jgeot.21.00245
- *Paper IV*: Dadras-Ajirlou, D., Grimstad, G., and Ghoreishian Amiri, S. A. A set of critical state hyper-viscoplastic models with different friction mobilisation criteria. Submitted to *International Journal of Solids and Structures*.
- *Paper V*: Ghoreishian Amiri, S. A., Dadras-Ajirlou, D., and Grimstad, G. On the implementation of hyperplastic models without establishing a yield surface. Submitted to *International Journal for Numerical and Analytical Methods in Geomechanics*.

Chapter 2 – Literature review

2.1 Introduction

This chapter presents a concise review of published literature about clay's viscous (time and rate dependent) behaviour, emphasising the background to the development of the isotache concept. The literature on the viscoplastic constitutive modelling of clay behaviour is also briefly reviewed. A review of clay's inviscid behaviour is beyond this thesis's scope.

2.2 Viscous behaviour of clay

Based on the literature, the viscous behaviour of clay can be examined from two main aspects, i.e., time and rate dependency. The time-dependent feature of clay behaviour has been studied through two dual processes: creep and relaxation. The creep process is defined in the literature as the deformation of a soil element, either volumetric or shear, over time under a constant and persistent effective stress. On the other hand, the relaxation process refers to the decrease of effective stress of the soil element over time under a constant strain condition. Rate dependency relates to the dependence of the stress-strain response of a soil element on the rate of undergone mechanical process (in terms of stress or strain). All these definitions are phenomenological since they are in terms of the continuum quantities of stress and strain. These phenomenological definitions are suited to this thesis's main purpose, constitutive modelling of the time and rate-dependent behaviour of clay using the hyperplasticity approach. In fact, hyperplasticity is based on thermodynamics, a phenomenological approach for the macroscopic description of the change of properties of a system (soil element) without knowing much about the actual states of micro-elements of the system (such as shape of individual particles and their contacts with eachother).

The viscous behaviour of soil, particularly clay, can be affected by several complex time-dependent chemical processes such as bonding, cementation, thixotropy and leaching. At this juncture, the degree of the influence of each of these chemical processes on the time-dependent behaviour of clay is not clearly distinguishable, and their consideration is beyond the scope of this thesis.

2.2.1 Time-dependent behaviour

The consolidation and creep behaviour of clay is usually studied one-dimensionally by the oedometer test (or triaxial test with certain boundary conditions). The pioneering studies in this regard are Buisman (1936) and Gray (1936). They presented the time-dependent behaviour in terms of void ratio and logarithm of time like in Figure 2-1 and qualitatively identified two compressional stages of primary and secondary consolidation based on the change of the effective stress with time. As shown in Figure 2-1, the secondary consolidation phase in the semi-logarithmic compression plan of void ratio (e) versus the logarithm of time (t) is characterised by the coefficient of secondary consolidation or the creep index, C_α:

$$C_\alpha = \frac{\Delta e}{\Delta \log t} = \frac{\Delta e}{\log(t+\Delta t) - \log(t)} \tag{2-1}$$

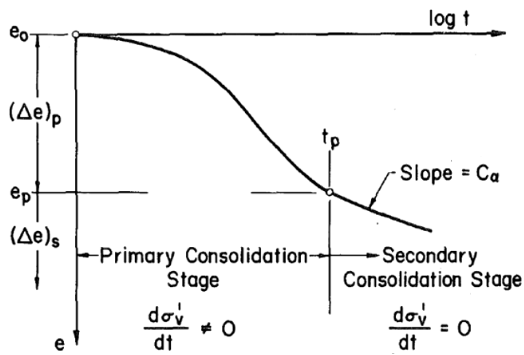


Figure 2-1: Primary and secondary consolidation, after Gray (1936).

The secondary compression behaviour depicted in Figure 2-1 and confirmed by numerous other studies on soft clay might appear as a line (a linear relationship between time in logarithmic scale and strain). This is not generally the case since soil element shows more resistance against further compression as it compresses over time. However, Mesri and Godlewski (1977) and Mesri and Castro (1987), based on a series of tests on various soils, demonstrated that the ratio between C_α and the compression index C_c remains within a narrow range for each soil type. C_c is the slope of the compression line in the classical compression plan of void ratio (e) against the logarithm of the effective vertical stress ($\sigma_v^{'}$):

$$C_c = \frac{\Delta e}{\Delta \log \sigma_v^{'}} = \frac{\Delta e}{\log\left(\sigma_v^{'} + \Delta\sigma_v^{'}\right) - \log\left(\sigma_v^{'}\right)} \tag{2-2}$$

In this thesis, the volumetric part of constitutive models is constructed in the bi-logarithmic compression plane of ln(υ) (υ is the specific volume equal to 1+e) versus ln(p) (p is the mean effective stress), which provides several theoretical and practical benefits (Butterfield 1979, Hashiguchi 1995, Den Haan 1996). Therefore, equations (2-1) and (2-2) can be expressed respectively as:

$$\mu = \frac{\Delta \ln \upsilon}{\Delta \ln t} = \frac{\ln(\upsilon + \Delta\upsilon) - \ln(\upsilon)}{\ln(t - t_0 + \Delta t) - \ln(t - t_0)} = \frac{\Delta \varepsilon_v}{\ln(\tau + \Delta t) - \ln(\tau)} \tag{2-3}$$

$$\lambda = \frac{\Delta \ln \upsilon}{\Delta \ln p} = \frac{\ln(\upsilon + \Delta\upsilon) - \ln(\upsilon)}{\ln(p + \Delta p) - \ln(p)} = \frac{\Delta \varepsilon_v}{\ln(p + \Delta p) - \ln(p)} \tag{2-4}$$

where instead of C_α and C_c, the notations of μ and λ have been used, respectively. Besides using the logarithmic compressional plane for the above expressions, another essential modification is the introduction of a reference time t_0 in equation (2-3) to make the time measure inside the logarithm function an relative measure leading to an objective value for μ independent of unit of time. One must notice that the logarithm of time in equation (2-1) implies that time (t) must be an absolute quantity. However, the measure of absolute time of a certain soil element is out of our grasp because the reference for the absolute time of the element is unknown. By introducing t_0, the meaning of time becomes relative (τ) such that the increment of absolute time (Δt) is equal to the increment of relative time ($\Delta\tau$). Subsequently, it is possible to construct a linear response between strain and the natural logarithm of time while having an objective measure for the creep index. In this regard, according to equation (2-3) to obtain μ, t_0 is required and vice versa. In other words, μ and t_0 must be evaluated consistently to obtain a linear pattern for strain-(relative) time response. More details are provided in the following.

Since the classical work of Buisman (1936) and Gray (1936), the primary and secondary consolidation terminologies started appearing in the literature. The former is associated mainly with the volume change under the dissipation of excess pore pressure, while the latter denotes the deformation under constant effective stress. The primary and secondary terms may imply that these consolidation phases are consecutive (Den Haan 1996), i.e., the secondary compression (under constant effective stress) starts after the end of the primary consolidation (EOP) at time t_p (Figure 2-1). However, it is impossible to distinguish such a consecutiveness for the consolidation process due to the interdependence of the state of soil element (void ratio or effective stress) and time. To clarify the confusion around the classical terminologies of primary and secondary consolidation, Bjerrum (1967) introduced the terms instant and delayed compression. Bjerrum's terminologies are directly related to the state of soil element (effective stress and strain), as depicted in Figure 2-2, in contrast with the classical terms assigned to the consolidation process (the dissipation of pore water pressure). In other words, based on Bjerrum's terminologies, the states of the soil element are mainly related to time since instant and delayed terms carry the concept of time. According to Bjerrum's conceptualisation of compression (Figure 2-2), it can be deduced that creep occurs during the dissipation of pore water.

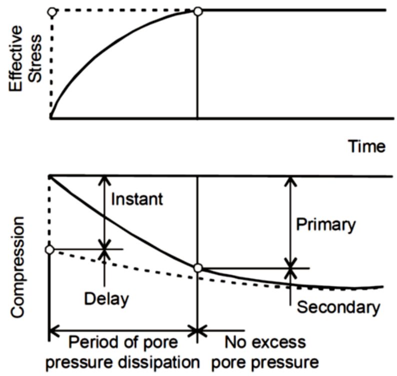


Figure 2-2: Definition of instant and delayed compression, after Bjerrum (1967).

Another concept for resolving the confusion surrounding the classical terms of primary and secondary consolidation is the time resistance proposed by Janbu (Janbu 1969, Janbu 1985). By observing the stepwise loading in the oedometer tests in which each load step is applied instantaneously (equivalent to the instant compression of Bjerrum (1967)) and then kept constant for some time during which the sample gradually compressed (equivalent to the delayed compression of Bjerrum (1967)), Janbu (1969) identified that creep happens in the so-called primary consolidation stage where the dissipation of pore water is the dominant mechanism. To mathematically express the rheological phenomenon of creep, he appealed to the concept of resistance commonly used in physics like thermal or electrical resistance. He defined the property for the soil element called the time resistance (R) as:

$$R = \frac{dt}{d\varepsilon} \tag{2-5}$$

R is the slope of the strain-time response shown in Figure 2-3. As shown in the figure, R generally increases with time, describing the reserve resistance of the soil element against further compression. After transitioning from the primary to the secondary phase (Figure 2-3), due to the gradual dissipation of the excess pore water pressure, the R-t response approaches a straight line with a slope of r_s depicting the pure creep process ($t > t_p$):

$$R_s = r_s\left(t - t_0\right) \tag{2-6}$$

r_s is the creep resistance. t_0 is a constant in time scale (time shift), which can have a negative value. $\tau = t - t_0$, intrinsic time (Den Haan 1996), which corresponds to Bjerrum's equivalent time, is the time appreciation of the soil element after the spike in the excess pore water pressure inside the soil element. Considering dτ = dt, by replacing equation (2-6) in (2-5) after Janbu (1969), Janbu (1985), and doing the integration from t_r, the creep strain can be computed as:

$$\varepsilon_s = \frac{1}{r_s}\int_{t_r-t_0}^{t-t_0}\frac{1}{t-t_0}d\left(t-t_0\right) = \frac{1}{r_s}\int_{\tau_r}^{\tau}\frac{1}{\tau}d\left(\tau\right) = \frac{1}{r_s}\ln\left(\frac{\tau}{\tau_r}\right) \tag{2-7}$$

Based on equation (2-7), one may realise that the creep strain approaches infinity at infinite time, which may appear meaningless. However, does infinite time meaningful? Considering the lifetime of infrastructures, further resistance of soil against further strain (hardening phenomenon), and that the creep strain rate is strictly decreasing with time (derivative of natural logarithm function is strictly decreasing), infinite strain at an infinite time has no practical relevance.

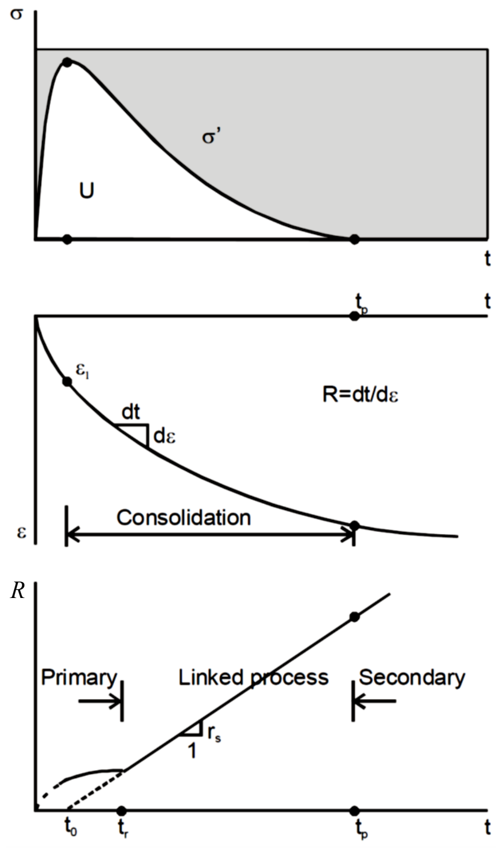


Figure 2-3: Definition of time resistance R and creep resistance r_s for a constant load step in oedometer, after Janbu (1985).

Despite the afore-mentioned reasonings about the overlapping of the primary (dominance of dissipation of pore water pressure) and the secondary (creep) consolidation phases, Mesri and co-workers (Mesri and Godlewski 1977, 1979, Mesri and Choi 1985, Mesri et al. 1994) asserted that the creep strain is independent of the time taken to reach EOP ($t = t_p$), and there is a unique relationship between volumetric strain or void ratio and effective stress at EOP in both laboratory and in-situ with thicker soil layer. In other words, the mechanical behaviour of soil (stress-strain states) during the dissipation of pore water pressure is inviscid and independent of the thickness of the soil element. This led Ladd et al. (1977) to point out the confusion about whether the creep phenomenon consecutively follows the dissipation of pore water or it overlaps the dissipation of pore water, as already considered by Bjerrum (1967) and Janbu (1969). These two cases are introduced as hypotheses A and B, respectively (shown as theories A and B in Figure 2-4 after Hight et al. (1987)) for settlement analysis of thicker soil layers in the field from thin soil specimens in the laboratory.

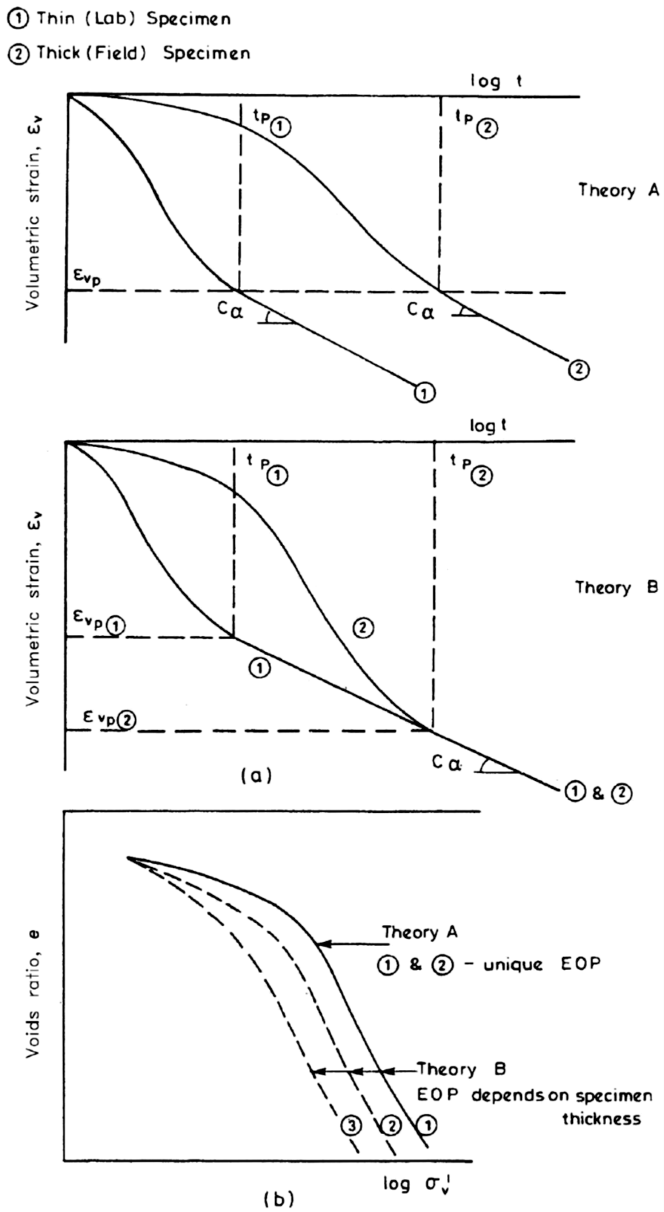


Figure 2-4: Hypothesis A versus Hypothesis B, after Hight et al. (1987).

Degago et al. (2009) and Degago et al. (2011) have attended the preceding controversy and critically examined and re-interpreted the literature data supporting hypothesis A. They demonstrated that by employing a consistent criterion for EOP definition in terms of degree of consolidation (degree of pore water pressure dissipation), the data support hypothesis B and the non-uniqueness of the void ratio at EOP (Figure 2-5).

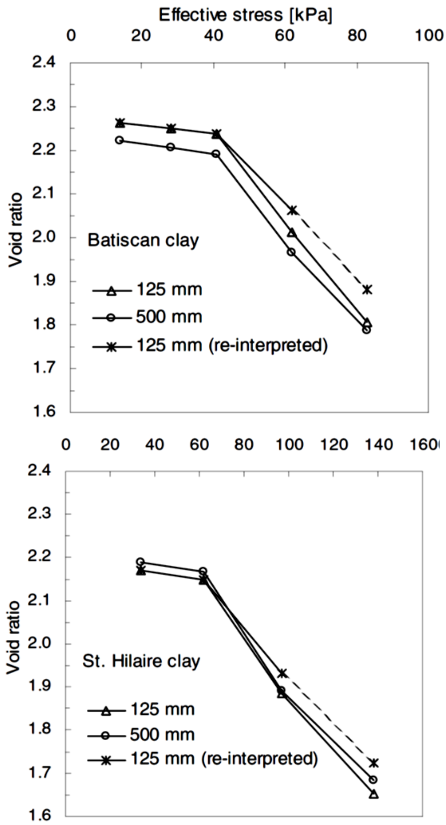


Figure 2-5: Re-interpretation of the data supporting Hypothesis A based on a consistant definition for identification of end of primary consolidation, after Degago et al. (2009).

2.2.2 Strain rate-dependent behaviour

Šuklje (1957) presented the isotache concept based on which a unique relationship between effective stress, the void ratio, and the rate of change of the void ratio can be conceived. Figure 2.6 illustrates the isotache model of Šuklje (1957). An isotache is a contour of the constant rate of change of void ratio shown in the lower graph in Figure 2-6. As shown, the contours are parallel, indicating a unique relationship between effective stress, the void ratio, and the rate of change in the void ratio. The upper graph on the right in Figure 2-6 illustrates the compression behaviour (in terms of effective vertical stress versus void ratio) for samples with thicknesses *n* times the thickness of the reference sample (20 mm). These results support Hypothesis B as the sample size (path of pore water dissipation) influences the compression behaviour.

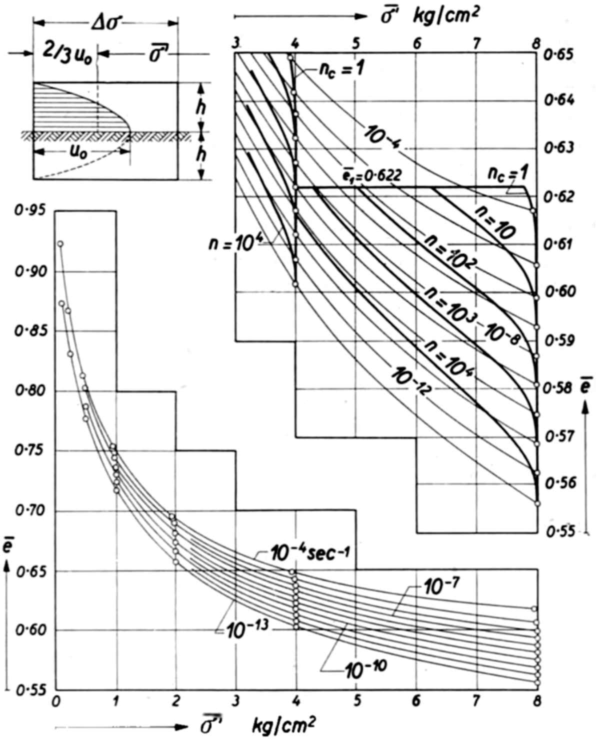


Figure 2-6: Illustration of isotache concept- the contours of isotache is for lacustrine chalk samples- after Šuklje (1957).

Sällfors (1975) confirmed the validity of the isotache concept by conducting several constant strain rate (CSR) oedometer tests on Bäckebol clay. Leroueil et al. (1985) performed a variety of oedometer tests, for instance, CSR and step-changed rate of strain (SRS), on different Champlain Sea clays and advocated the model of Šuklje (1957). Figure 2-8 and Figure 2-7 show the CSR and SRS tests conducted on Batiscan clay, respectively. Figure 2-8 shows that by increasing strain rate, the compression curve generally moves to the right with higher values for the pre-consolidation pressure. However, the compression curve corresponding to the lowest strain rate (1.69×10^{-8}/s) deviates considerably from the isotaches trend. This inconsistency was justified by Leroueil et al. (1985) as the effect of structuration occurring under sufficiently low strain rates.

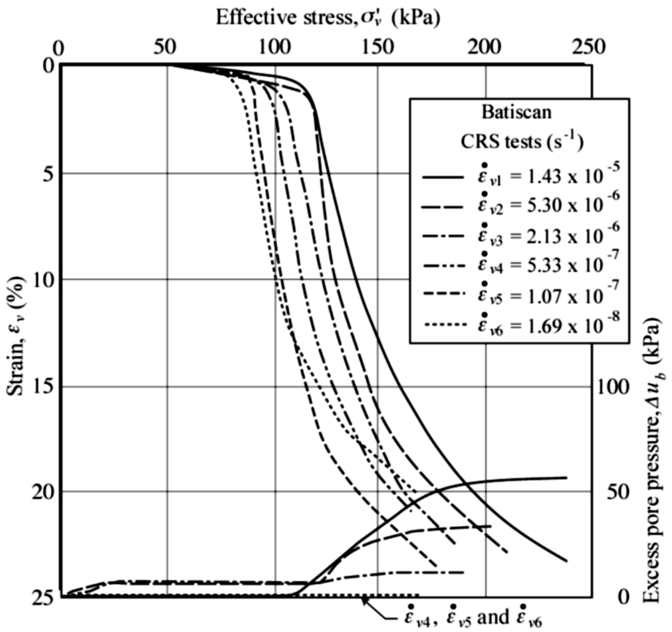


Figure 2-8: Strain-stress response obtained by constant rate of strain (CRS) oedometer tests with different strain rate on Batiscan clay- after Leroueil et al. (1985).

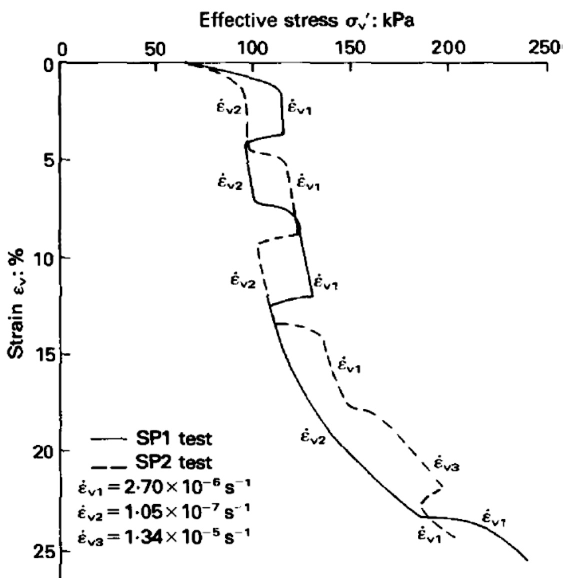


Figure 2-7: Strain-stress response obtained by step-changed rate of strain (SRS) in odeometer test on Batiscan clay- after Leroueil et al. (1985).

Based on the above observations, Leroueil et al. (1985) proposed an isotache model in terms of the total strain rate by deconstructing the stress-strain curves into two (Figure 2-9). The first curve describes the variation of the pre-consolidation pressure ($\sigma_p^{'}$) with the strain rate:

$$\sigma_p^{'} = f\left(\dot{\varepsilon}\right) \tag{2-8}$$

and the other describes the normalised effective stress-strain curve:

$$\frac{\sigma_v^{'}}{\sigma_p^{'}\left(\dot{\varepsilon}\right)} = g\left(\varepsilon_v\right) \tag{2-9}$$

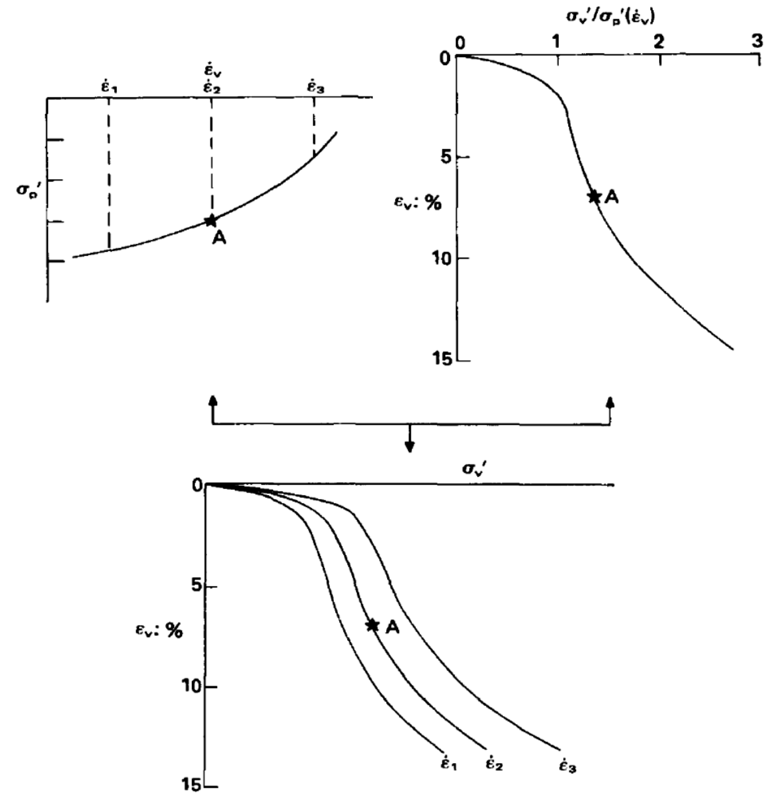


Figure 2-9: Isotache model based on total strain rate- after Leroueil et al. (1985).

Although the isotache model of Leroueil et al. (1985) may exhibit satisfactory performance in predicting creep and strain loading with different loading rates, it was soon realised that it fails to describe the clay behaviour observed in the relaxation process (Yoshikuni et al. 1994). As shown in Figure 2-10, taken from Kim and Leroueil (2001), the excess pore water pressure is generated during the relaxation process in which the total strain is constant. The higher the strain rate at which relaxation begins, the higher the generated excess pore pressure. Since the excess pore water pressure generated under constant total strain (stress relaxation process), it was concluded that the isotache model should be applied to the inelastic strain, not the total strain (Kim and Leroueil 2001). This late conclusion was already practised in the viscoplastic constitutive model proposed by Kutter and Sathialingam (1992) based on the empirical formula of Murakami (1979) regarding the effect of creep on the pre-consolidation pressure.

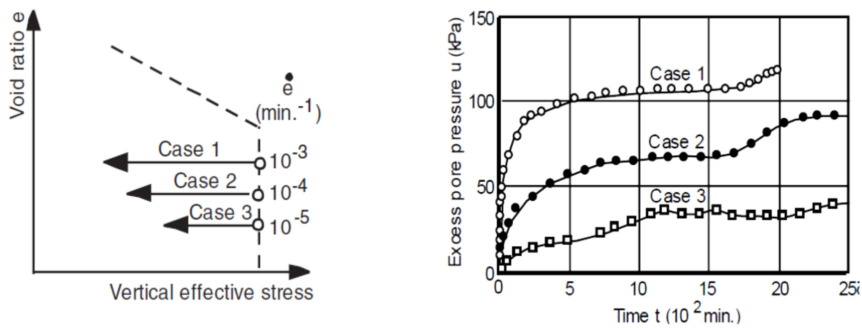


Figure 2-10: Generation of excess pore water pressure during relaxation tests- after Kim and Leroueil (2001) based on the data of Yoshikuni et al. (1994).

The isotache rate-dependent behaviour has also been observed in several triaxial undrained shear loading tests on clay (e.g., Vaid and Campanella (1977), Graham et al. (1983), Tatsuoka et al. (2002)). Figure 2-11 shows the step-changed undrained triaxial compression test on undisturbed Haney clay conducted by Vaid and Campanella (1977). As shown, due to the increase in the axial strain rate, the stress-strain curve moves from the contour associated with the lower strain rate to the upper contour related to the higher strain rate. Graham et al. (1983) extended the work done by Vaid and Campanella (1977) and performed the same type of test but with a relaxation phase before step-changing of the axial strain rate. As shown in Figure 2-12, after the relaxation (denoted by R in the figure) upon a recommencement of the applied strain rate, the path rapidly returns to the contours associated with the applied strain rate, demonstrating a unique stress-strain-strain rate relationship for clay.

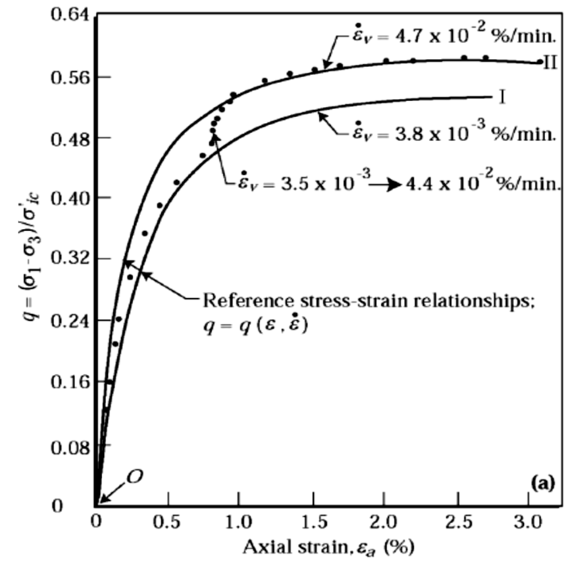


Figure 2-11: Influenec of strain rate on the response of undisturbed Haney clay in undrained triaxial compression test- after Vaid and Campanella (1977).

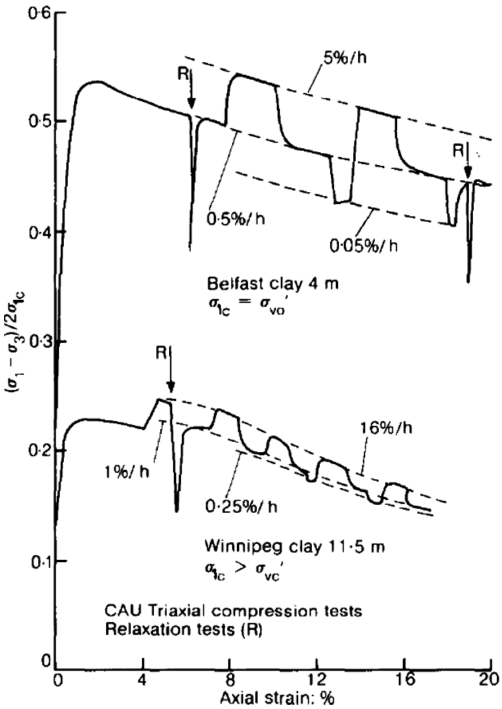


Figure 2-12: Influenec of strain rate after relaxation on the response of undisturbed Haney clay in undrained triaxial compression test- after Graham et al. (1983).

Based on the above-mentioned experimental studies, it can be inferred that the shear strength of clay is rate-dependent, following a unique relationship with strain rate. In this regard, Kulhawy and Mayne (1990), by evaluating the shear rate-dependent response of several worldwide saturated clays, proposed a linear relationship between the undrained shear strength and the logarithm of strain rate (Figure 2-12).

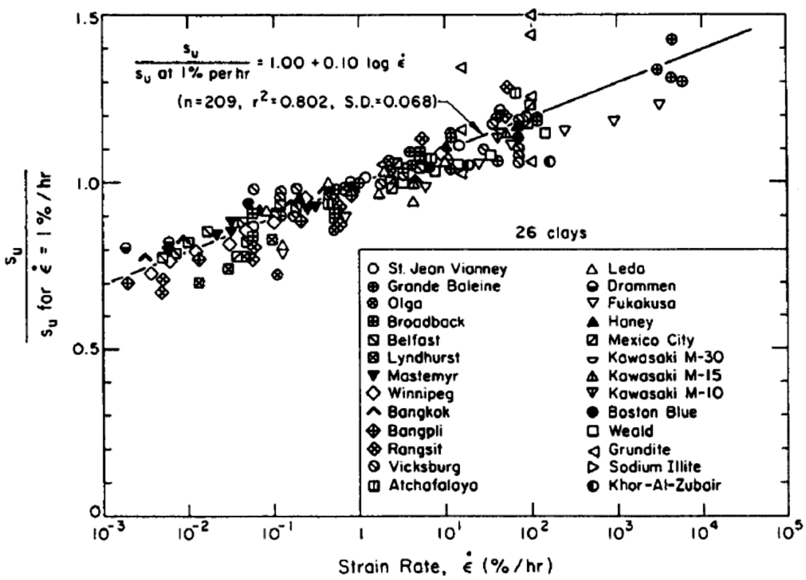


Figure 2-12: Effect of strain rate on undrained shear strength- after Kulhawy and Mayne (1990).

Later based on a more precise evaluation, Qu et al. (2010) advocated the linear relationship between the logarithm of undrained shear strength and the logarithm of strain rate (Figure 2-13). They found out that the slope of this line (denoted by α called rate-sensitivity parameter) confidently is equal to:

$$\alpha = \frac{\mu}{\lambda - \kappa} \tag{2-10}$$

where μ and λ have been previously defined in equations (2-3) and (2-4), respectively, and κ is the index of isotropic reloading (after unloading) in the bi-logarithmic compression plane. Grimstad et al. (2010) have also analytically computed equation (2-10) by linking the time resistance concept and the critical state soil mechanics.

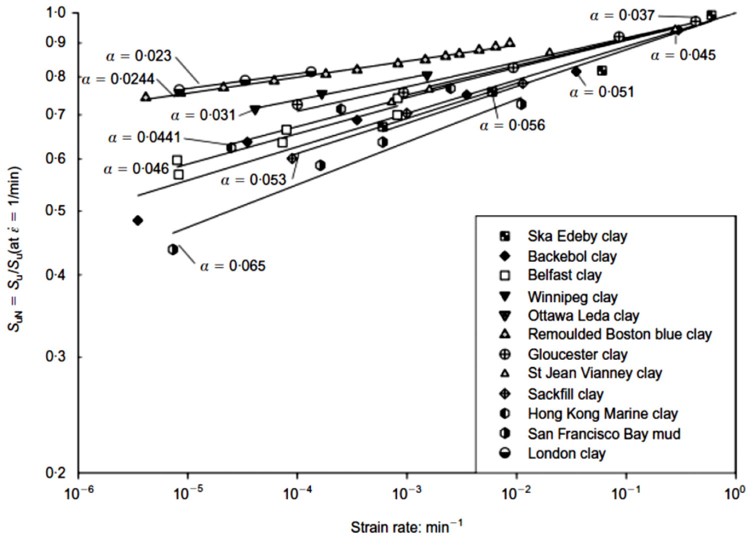


Figure 2-13: Relationship between undrained shear strength and axial strain rate based on undrained triaxial tests on worldwide clays- after Qu et al. (2010).

Although the strain rate affects the stress-strain response and the shear strength, the mobilised friction at the critical state (isochoric shearing) appears to be almost independent of the strain rate. Tests performed at different strain rates on clay samples (e.g., Adachi et al. (1995), Arulanandan et al. (1971), Sheahan et al. (1996), Vaid and Campanella (1977), Zhu (2000), Tafili et al. (2021), e.g.) do not show any significant effect of strain rate on the friction angle at failure. Figure 2-14 from Adachi et al. (1995) shows the stress path results of the undrained triaxial compression and extension tests conducted on the undisturbed normally and lightly overconsolidated samples of Osaka clay. As shown, the higher strain rates give the higher shear strength, but the friction angle at failure is unique. It should be noted that in undrained conditions, the strain rate must remain small enough in order not to generate significant pore pressure gradients inside the specimen. Otherwise, the externally measured stress path response would approach or, in extreme cases, follow the total stress path (Quinn and Brown 2011).

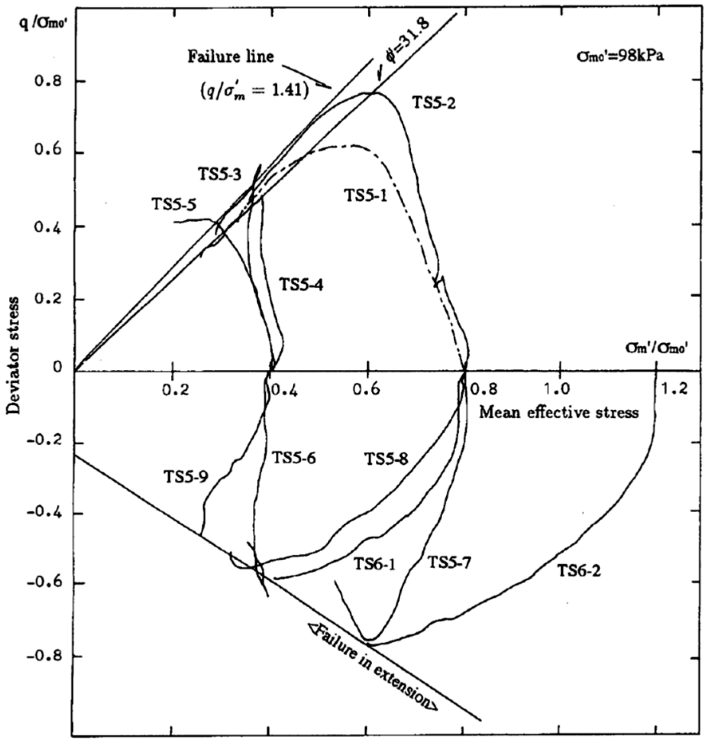


Figure 2-14: Stress path until failue for undrained triaxial tests on Osaka clay- after Adachi et al. (1995).

2.2.3 Evolution of the coefficient of earth pressure at rest (K_0) with time

Estimating the in-situ stress state, particularly the stress ratio between the horizontal and the effective vertical stresses under zero lateral strain conditions (known as K_0), is essential for designing and analysing geotechnical systems. However, the measurement and prediction of K_0 remain difficult. Jacky (1944), theoretically, by analysis of the stress field in a wedge prism of loose granular material, derived the following equation for estimation of K_0:

$$K_0 = \left(1 + \frac{2}{3}\sin\left(\phi_{cs}\right)\right)\left(\frac{1-\sin\left(\phi_{cs}\right)}{1+\sin\left(\phi_{cs}\right)}\right) \tag{2-11}$$

Intriguingly K_0 is related to the angle of shearing resistance (ϕ_{cs}) under isochoric condition (known as the critical state friction) even though the state of stress is away from the failure state (Feda 1984, Michalowski 2005). Equation (2-11), which provides the hydrostatic state for fluids $\left(\phi_{cs} = 0\right)$, implies that the evolution of K_0 can be a frictional mechanism. Recently, Tsegaye (2021) proposed a similar equation for K_0 under the plane strain condition by focusing on the frictional character of the soil and using particular stress- dilatancy relationship (Tsegaye and Benz 2014).

The preloading of frictional granular material is believed to leave residual horizontal stresses locked in, causing an increase in K_0. Schmidt (1966) provided the following well-known empirical expression for the increase of K_0 due to preloading:

$$K_{0,OC} = K_{0,NC} \left(OCR \right)^n \tag{2-12}$$

where OCR stands for the over consolidation ratio representing the degree of preloading and n is a constant for a particular soil. Equation (2-12), or similar ones like the recent proposition by Tsegaye (2021), is only valid for the unloading process (Wroth and Houlsby 1985) since, during the reloading, the value of K_0 rapidly falls due to a slight increase in the effective horizontal stress. However, equation (2-12) gives the same value for both the unloading and reloading processes in the preloading states.

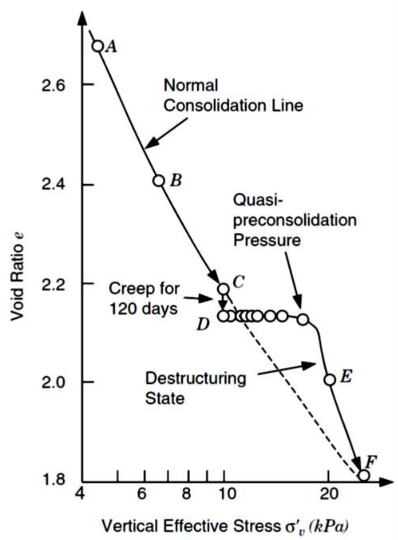


Figure 2-15 Ageing effect on preconsolidation pressure (a measure for preloading) for artificially sedimented Jonquiere clay from Leroueil et al. (1996) .

As shown in Figure 2-15, Creep and ageing could also put the soil's state in the preloading condition. In this regard, Schmertmann (1983) raised the question of whether K_0 varies with time during creep or ageing. However, the literature on this matter remains inconclusive. Some creep analyses of the limited existing experimental data like Kavazanjian and Mitchell (1984) and Mesri and Hayat (1993) support the idea of an increase of K_0 with time. Yet, other studies such as Holtz and Jamiolkowski (1985) and Jamiolkowski et al. (1985) show constant K_0 over time. On the other hand, a decrease of K_0 over time due to the ageing effect (like thixotropy) could be expected (Schmertmann 1983).

2.2.4 Creep and swelling with time after unloading

Despite the limited number of studies, various and confusing terminologies have been employed on this subject. Moreover, most of the studies have been done on expansive soils (such as smectite or bentonite), whose behaviour is not generally representative of the behaviour of fine-grained soils.

Swelling in fine-grained soils is initiated by a reduction of the effective stress due to unloading. When the applied load decreases and soil swells, viscous effects are also observed. Mesri et al. (1978) introduced primary and secondary swelling following the conceptualisation of creep in HypothesisPrimary swelling is associated with pore pressure equalisation and changes in the effective stress. The following swelling over time under constant effective stress is called secondary swelling. Mesri et al. (1978) performed both isotropic and one-dimensional unloading and swelling tests. They proposed a time-dependent swelling model in which the ratio of the primary swelling index (C_s) to the secondary swelling index ($C_{\alpha,s}$) depends on the over-consolidation ratio. This contrasts with the creep model, in which the corresponding ratio is assumed to be constant (Mesri and Godlewski 1977).

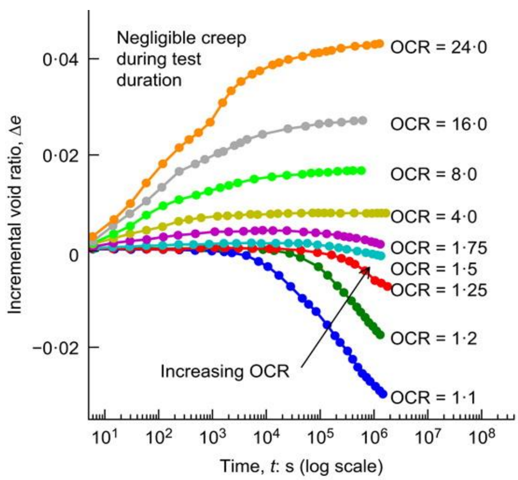


Figure 2-16: Change of void ratio obserced after unloading in oedometer tests on Berthierville clay- after Vergote et al. (2022) based on data of Feng (1991).

Feng (1991)conducted several oedometer tests on Berthierville clay, in which specimens were first loaded in the normally consolidated range and then, at the end-of-primary consolidation, unloaded in one step to various over-consolidation ratios (*OCR*). Then, the specimens were put under constant stress showing the behaviour depicted in Figure 2-16 taken from Vergote et al. (2022). As shown, the recompression occurred similarly to normally consolidated clay after some swelling for the specimens with relatively low OCRs (lower than 1.75). For specimens with higher *OCR*s, the secondary or better to call delayed swelling prevailed over the entire test.

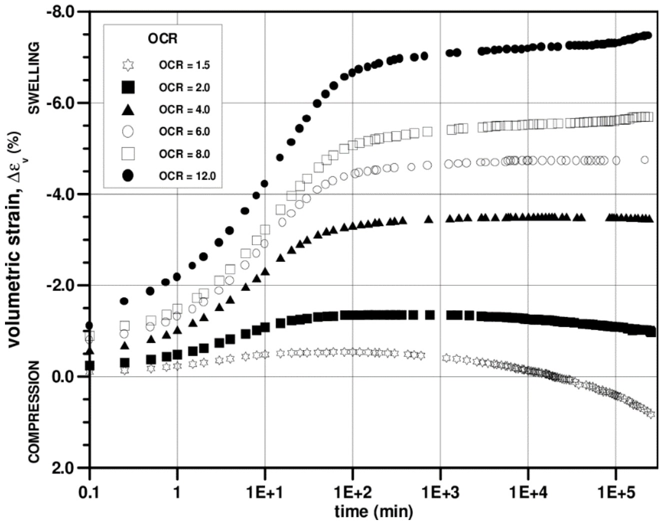


Figure 2-17: Volumetric strains observed after unloading of Sarapuí clay-after Almeida and Marques (2003).

The results shown in Figure 2-17 obtained by Almeida and Marques (2003) for organic Sarapuí clay from Brazil are very similar to those shown in Figure 2-16. Interestingly, the delayed swelling for *OCR* = 4 is insignificant in both studies.

2.2.5 Ageing effect

The effect of ageing is beyond the scope of this thesis. However, because of the importance of the subject in the viscous behaviour of clay, a brief review is provided here.

According to Mitchell (1986), ageing refers to the physicochemical and biological processes over time that result in time-dependent changes in the soil properties (the mechanical properties like strength and stiffness are of interest here). Reduction in the shear strength due to leaching out of salt in pore water in Norwegian clays (Bjerrum 1954) is one example of the ageing effect that causes tremendous shear strength sensitivity in these clays. Cementation and bonding as interparticle chemical forces that are not purely frictional (Lambe and Whitman 1991) are other ageing effects that cause an increase in shear strengths of natural soil over time (Burland 1990). Another ageing effect that causes sensitivity but in a reversible manner is thixotropy. Mitchell and Soga (2005) define thixotropy as an isothermal, reversible, and time-dependent process occurring under conditions of constant composition and volume whereby a material stiffens while at rest and softens or liquefies upon remoulding.

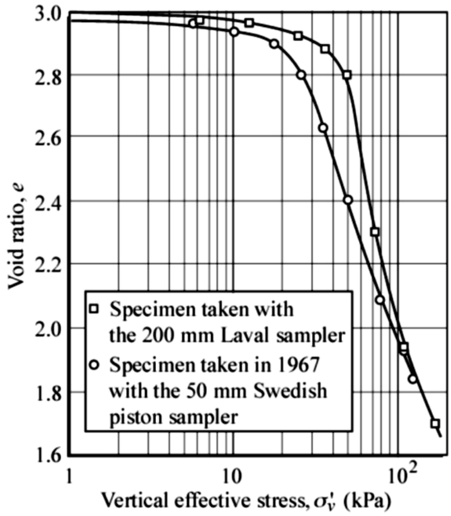


Figure 2-18: Effect of sample disturbance on the stress- strain response of oedometer test on Väsby clay- after Leroueil and Kabbaj (1987).

The rate and time-dependent frameworks in the literature cannot consider the ageing processes. In some viscous modelling approaches (e.g., Grimstad et al. (2010), Nash and Brown (2015)) the bonding effect has been considered as an initial condition for ongoing time-dependent mechanical processes. This modelling consideration calls for high-quality samples and careful experiments to quantify the initial bonding condition. In this case, the importance of the sample quality can be seen in the controversy between proponents of Hypothesis A (Mesri and Choi 1985) and Hypothesis B (Leroueil and Kabbaj 1987) over the case of the Väsby test fill. Figure 2-18 illustrates the effect of sample disturbance on the Väsby clay behaviour obtained by incremental oedometer tests on the samples retrieved by Swedish (Mesri and Choi 1985) and Laval samplers (Leroueil and Kabbaj 1987). Based on the results obtained by the Swedish standard piston sampler, Mesri and Choi (1985) found a relatively good agreement with the field observation using Hypothesis A for the consolidation analysis. However, Leroueil and Kabbaj (1987) questioned the results of Mesri and Choi (1985) by reasoning that the agreement was essentially due to disturbance effects. Leroueil and Kabbaj (1987)'s remark has been confirmed by the numerical analyses of the Väsby test fill using the constitutive models based on each hypothesis (Degago et al. 2011, Degago 2011).

2.3 Viscoplastic modelling approaches

Several theories exist in the literature for the construction of viscoplastic constitutive models. In this section, those commonly used for modelling the viscous behaviour of clay are covered. These theories can generally be categorised into four groups: the overstress theory, the consistency theory, the hypoplasticty, and the hyperplasticty.

2.3.1 Overstress theory

The overstress viscoplasticity theory of Perzyna (1963) is the most popular approach for the construction of a viscoplastic constitutive model of soil. By following the infinitesimal strain hypothesis in this approach, the total strain rate tensor is decomposed into elastic ($\dot{\varepsilon}^{e}$) and inelastic ($\dot{\varepsilon}^{vp}$) parts:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{e} + \dot{\boldsymbol{\varepsilon}}^{vp} \tag{2-13}$$

The elastic strain rate, which is assumed to be inviscid, can be calculated as:

$$\dot{\boldsymbol{\varepsilon}}^{e} = \boldsymbol{D}^{-1} : \dot{\boldsymbol{\sigma}} \tag{2-14}$$

where ':' represents the inner product and $\boldsymbol{D}$ is the fourth order elastic tensor. The inelastic strain rate is time-dependent and captures the viscous effects. It is defined as:

$$\dot{\boldsymbol{\varepsilon}}^{vp} = \left(\frac{\langle \Phi(f) \rangle}{\xi} \right) \frac{\partial g}{\partial \boldsymbol{\sigma}} \tag{2-15}$$

Where '$\langle x \rangle = (x + |x|)/2$' is a ramp function, ξ is the viscosity parameter, and Φ is the overstress function that depends on the so-called dynamic yield surface (*f*) motivated by the notion of yield surface from the classical plasticity theory. According to Simo and Hughes (1998), the overstress function is subjected to the following two requirements:

1) Φ must be at least a C^{1} continuous function
2) $\Phi(f) \geq 0$ and $\Phi(f) = 0$ if and only if $f \leq 0$

g in equation (2-15) is the plastic potential surface whose gradients define the direction of the inelastic flow. In the original overstress theory of Perzyna (1963), by invoking the postulate of

Drucker (1959), it is assumed that $f = g$ which results in the normal or associated plastic flow rule (Olszak and Perzyna 1964). It is well known that this assumption is not tenable for soil as a frictional material. However, surprisingly it has been rarely attempted (e.g., Bodas Freitas et al. (2011)) to include a non-associated flow rule using the overstress viscoplasticity approach.

Numerous viscoplastic constitutive models for clay have been developed using the overstress theory (e.g., Fodil et al. (1997), Hinchberger and Rowe (2005), Sivasithamparam et al. (2015)). Their main difference lies in the definition of the dynamic yield surface and the overstress function. For instance, the two well-accepted viscoplastic models for lightly consolidated clay developed by Adachi and Oka (1982) and Vermeer and Neher (2019) have similar overstress functions employing different dynamic yield surfaces, namely the original cam clay surface (Schofield and Wroth 1968) and the modified cam clay surface (Roscoe and Burland 1968), respectively.

2.3.2 Consistency theory

The consistency theory, proposed by Naghdi and Murch (1963), was developed directly from classical plasticity theory. While the viscous effect in the overstress theory is implicitly accounted for by an overstress function, in the consistency theory, an extra time-dependent internal variable appears in the yield surface of the classical plasticity considering the viscous effect:

$$f\left(\boldsymbol{\sigma}, \boldsymbol{k}, k_t\right) = 0 \tag{2-16}$$

where $\boldsymbol{k}$ hardening variables and k_t is the time-dependent variable characterising the viscous effects.

Partitioning the total strain rates into elastic and inelastic strain rates is also followed in this theory. The inelastic flow rule is expressed as:

$$\dot{\boldsymbol{\varepsilon}}^{vp} = \langle \Lambda \rangle \frac{\partial g}{\partial \boldsymbol{\sigma}} \tag{2-17}$$

where Λ is the non-negative multiplier like in the classical plasticity theory. Viscoplastic loading occurs if the stress state remains on the yield surface, similar to plasticity theory in which plastic loading occurs. This is called the consistency condition applied by the following differential equation:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \boldsymbol{k}} : \dot{\boldsymbol{k}} + \frac{\partial f}{\partial k_t} : \dot{k}_t = 0 \tag{2-18}$$

By employing the decomposition hypothesis for strain rate, the consistency differential equation can be written as:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{D} : \left(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^{vp} \right) + \frac{\partial f}{\partial \boldsymbol{k}} : \dot{\boldsymbol{k}} + \frac{\partial f}{\partial k_t} : \dot{k}_t = 0 \tag{2-19}$$

Now by assuming $\dot{\boldsymbol{k}} = \boldsymbol{B}\Lambda$ and replacing the inelastic flow rule from equation (2-17), Λ can be computed:

$$\Lambda = \frac{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{D} : \dot{\boldsymbol{\varepsilon}}}{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{D} : \frac{\partial g}{\partial \boldsymbol{\sigma}} - \frac{\partial f}{\partial \boldsymbol{k}} : \boldsymbol{B}} + \frac{\left(\frac{\partial f}{\partial k_t} \right) \dot{k}_t}{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{D} : \frac{\partial g}{\partial \boldsymbol{\sigma}} - \frac{\partial f}{\partial \boldsymbol{k}} : \boldsymbol{B}} = \Lambda^{inv} + \Lambda^{vis} \tag{2-20}$$

where Λ^{inv} and Λ^{vis} are the inviscid and viscous plastic multipliers corresponding to the plastic multiplier in the plasticity theory. Similar to the plasticity theory, the loading and unloading processes can be distinguished through the Kuhn–Tucker complementarity conditions:

$$f \le 0, \quad \Lambda \ge 0, \quad \Lambda f = 0 \tag{2-21}$$

The work by Sekiguchi (1984) is one of the earliest applications of the consistency theory in viscoplastic modelling of clay behaviour. However, since then, a limited number of constitutive models have been proposed using the consistency theory. Perhaps, some limitations of the approach pointed out by Liingaard et al. (2004) are the reason for its limited application. According to Liingaard et al. (2004), the consistency theory cannot consider the creep or relaxation processes for the stress states inside the yield surface. Recently, Yao et al. (2015) and Qiao et al. (2016) have formulated viscoplastic constitutive models using the consistency theory that addresses this limitation.

2.3.3 Hypoplasticity

Application of the hypoplasticity theory in viscous modelling of clay is relatively new and limited (e.g., Niemunis (1996, 2003), Gudehus (2004), Niemunis et al. (2009)). The hypoplasticity theory is related to the hypoelasticity. Essentially, it is an expression of stress rate versus strain rate relation in terms of polynomial tensorial functions (H):

$$\dot{\boldsymbol{\sigma}} = \mathrm{H}\left(\boldsymbol{\sigma}, \dot{\boldsymbol{\varepsilon}} \right) \tag{2-22}$$

Wu and Kolymbas (2000) presented the theory's assumptions and requirements, which are allegedly more general or less restrictive than the requirements in the previous approaches. For instance, it is suggested that there is no need for the notions of additive decomposition of strain rates or conceiving of yield and plastic potentials surfaces, which all are essential components of the classical plasticity and viscoplasticity theories.

However, according to the current literature, the viscous part of existing hypoplastic models for clay (e.g., Fuentes et al. (2018), Jerman and Mašín (2020)) are constructed by following the approach of Niemunis (2003) based on the idea of scaling from the overstress theory (considered in the overstress function). This treatment of viscosity necessitates the concept of the dynamic yield surface (probably a convex surface as far as the isotache viscosity is concerned) for the description of the constitutive model in general stress space, which undermines the philosophy of hypoplasticity as an algebraic approach (Wu and Kolymbas 2000) that is being free of geometric constituents.

2.3.4 Hyperplasticity

The hyperplasticity approach (Houlsby and Puzrin 2006) is based on so-called generalised thermodynamics in which a strong emphasis is placed on using internal variables to describe the history or path-dependent behaviour of the material. The First and Second Laws of Thermodynamics, as general nature’s law, are enforced directly so that any model defined within this framework will automatically obey these Laws. In this approach, all elements of a constitutive model, such as the yield and the plastic potential surfaces, hardening rules, and elasticity (in the context of the classical plasticity and viscoplasticity approaches) are derived from the specification of two potentials: the free energy potential and the dissipation or the force potential. The flowchart in Figure 2-19 presented by Collins and Kelly (2002) depicts the general procedure for constructing the hyperplastic constitutive models.

As shown in Figure 2-19, hyperplasticity is a systematic framework with clearly defined steps and rules imposed for the development of models, avoiding the application of arbitrary functions to fit laboratory data which may not obey the laws of thermodynamics. The importance of using potentials in hyperplasticity, which makes it akin to hyperelasticity (Fung 1965), provides possibilities to decrease the number of material parameters with the cost of some amount of mathematics.

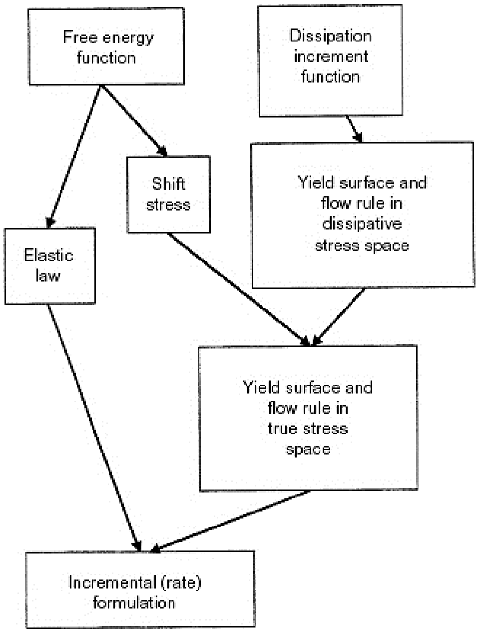


Figure 2-19: Flow chart outlining procedures to develop hyperplastic constitutive laws- after Collins and Kelly (2002).

Compared with the previous theories, hyperplasticity is the most restrictive approach but with unconditional thermodynamics consistency. Models that violate thermodynamics cannot be used with any confidence to void physically unrealistic results, especially for modern constitutive models, which are becoming increasingly complex. Hypoplasticity in the current form is the loosest approach with relatively minor restrictions and a higher possibility of violating the laws of thermodynamics. This is despite the attempt made by Einav (2012) to integrate the hyperplasticity with the hypoplastcity under the same thermodynamics-based framework. The classical viscoplasticity approaches (the overstress and the consistency theories), in general, are sufficiently flexible to have the possibility of disobeying the laws of thermodynamics. Still, with some tedious procedures in a retrospective manner, it is possible to verify their thermodynamic consistency. Overall, with the ideology of hierarchy of elasticity theories borrowed from Houlsby and Puzrin (2006), the previously described theories for viscoplastic constitutive modelling can be classified as shown in Figure 2-20.

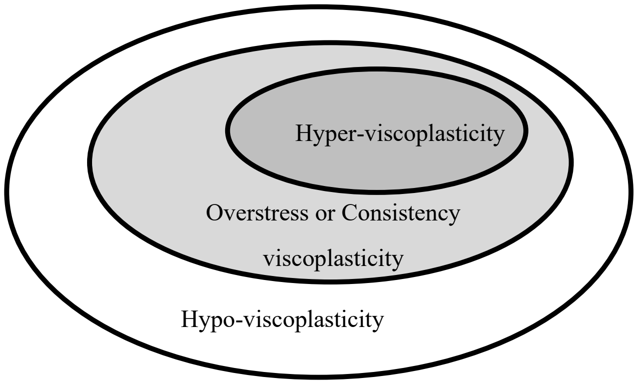


Figure 2-20: Classes of viscoplasticty theory.

The application of the hyperplasticity approach in constitutive modelling of clay behaviour is limited (perhaps because of its restrictions which bring cumbersome mathematics), and it has chiefly focused on the inviscid aspects (e.g., Einav and Puzrin (2003, 2004); Yan and Li (2011); Zhang et al. (2018); Rollo and Amorosi (2020)). The work done by Houlsby and Puzrin (2002) and Puzrin and Houlsby (2003) are pioneering attempts in viscous modelling of clay behaviour using the hyperplastic approach. In addition to establishing a rigorous procedure for constructing the hyper-viscoplastic models, they formulated a viscous model based on the rate process theory (Mitchell et al. 1968) as an example to demonstrate the approach's potential. Despite the excellent agreement between the measurements and the simulations (Puzrin and Houlsby 2003) of undrained triaxial test on natural Haney clay (Vaid and Campanella 1977), the proposed model can only describe the shear behaviour.

More recently, Aung et al. (2019) employed the hyperplasticity framework to formulate a constitutive model for the soil’s creep and rate-dependent behaviour. However, some severe theoretical flaws are inherent in their application of the framework that led to violation of the first law of thermodynamics and the principle of maximal rate of dissipation (Ziegler 1977), which is the cornerstone of the hyperplasticity framework. For instance, the method of the Lagrange multiplier has been wrongly practised (an ad-hoc value for the Lagrange multiplier has been assumed). The dimension of the dissipation rate is not of energy type (power per volume). Moreover, it appears that the effect of the plastic part of free energy or the shift stress (back stress) on the inelastic flow rule has been neglected. In addition, the proposed formulation, which reportedly follows the critical state soil mechanics, does not secure a unique friction envelope at the critical state under different loading rates.

Similarly, Jacquey and Regenauer-Lieb (2021) extended the rate-independent family of critical state models of Collins and Hilder (2002) to include the rate dependency with the non-associated flow rule. However, in addition to the questionable dimension they used for viscosity (viscosity has units of $\mathrm{Pa}^{-1/n}\mathrm{s}^{1/n}$ with n being introduced as a "generic degree"), it has not been realised that the choice of using the specific form of the rate-independent hyperplastic critical state model with plastic-free energy (Collins and Hilder 2002) comes at the cost of non-uniqueness of the friction mobilisation at the critical state under different loading rates. Numerous experimental studies (e.g., Adachi et al. (1995), Arulanandan et al. (1971), Sheahan et al. (1996), Vaid and Campanella (1977), Zhu (2000), and Tafili et al. (2021)) have demonstrated that the mobilised friction at critical state does not significantly depend on the loading rate. This experimental finding supports the fact that the Coulomb sliding friction (macro-scale) is approximately independent of the rate of mechanical processes. This can be ideally equivalent to the uniqueness of the critical friction envelope under different loading rates.

Chapter 3 – Hyperplasticity formalism

3.1 Introduction

A plethora of constitutive models for the description of reversible (conservative) and irreversible (dissipative) behaviour of geomaterials have been proposed without thermodynamic consideration. While some of these models, especially in certain circumstances, can be thermodynamically consistent, the unconditional thermodynamic consistency of most of them is under question. Thermodynamically inconsistent constitutive models cannot be confidently employed for solving boundary value problems (BVP), although they may give excellent fits to experimental data under certain loading conditions. This is because in solving a BVP there could be a series of loading for which energy is spuriously dissipated or generated that is physically meaningless. As a rule, fulfilling the laws of thermodynamics (almost universally considered true) is a must. Still, it might not necessarily be sufficient for constructing mathematically rigorous and robust constitutive models with a minimum number of parameters.

Generally, there are two ways to verify the consistency of a constitutive model with the second law of thermodynamics that determines the direction and feasibility of the irreversible processes. One is to specify the model freely without constraints and then ensure post hoc that energy always dissipates due to any irreversible processes. However, despite subscription to the second law (for example, by invoking the stability postulate of Drucker (1959)), the model might disobey the first law of thermodynamics (conservation of energy) if the reversible behaviour of the system is not described by a state function. Specifically, for a reversible process, the system does not find its initial state due to a false generation or dissipation of energy (e.g., Zytynski et al. (1978)). Many existing constitutive models for geomaterials are prone to the latest flaw. Despite fulfilling the thermodynamic requirements, the post hoc approach is unsystematic and mathematically not rigorous. It might easily and unintentionally arrive at ad hoc assumptions and procedures, especially when different interacting mechanisms are involved. This may lead to a substantial number of so-called material parameters that must be evaluated.

The alternative approach is to commence directly with the thermodynamic laws and then specify a model with predefined constraints. One such approach is the hyperplasticity framework (Houlsby and Puzrin 2006). Hyperplasticity is based on generalised thermodynamics with internal variables to describe the material's history (Maugin 1999). It is founded on the thermodynamics of Ziegler (1977, 1983), based on which the linear theory of Onsager (1931) is generalised to nonlinear phenomenological processes (Ziegler 1958). It provides a rigorous and systematic procedure to establish a hierarchy of thermodynamically consistent constitutive models without the unnecessary and restrictive postulate of Drucker (1959). In this approach, by applying the orthogonality postulate of Ziegler (1977), all elements of a constitutive model can be defined with the specification of two potentials: the free energy potential and the force potential (or dissipation rate function). As a result, the derived formulations would be compact and portable, opening possibilities for reducing the number of material parameters. This distinction of being a completely potential-based approach makes the hyperplasticty framework mathematically rigorous although significant amount of mathematics might be required.

In the following, the thermodynamic preliminaries for the hyperplasticty framework are overviewed. Then, the essential constituents of the hyperplasticity framework are presented. For a more detailed account of the framework, refer to the books of Ziegler (1983) and Houlsby and Puzrin (2006).

3.2 Thermodynamic preliminaries

Thermodynamics is essentially a phenomenological approach to describing the equilibrium states and properties of macroscopic systems by constructing a consistent mathematical framework. The thermodynamical system refers to the part of the universe that is of interest, and the rest of the universe is called the surrounding (the universe is the union of the system and its surrounding). Based on the notion of equilibrium (mechanical or thermal), each law of thermodynamics introduces crucially important macroscopic properties for any system. These properties include temperature, internal energy, and entropy deduced from the zeroth, first and the second laws of thermodynamics, respectively. General accounts of thermodynamics and its application in continuum mechanics can be found in the books by Ziegler (1983), De Groot and Mazur (1984), and Maugin (1999).

This thesis adopts the infinitesimal strain hypothesis in the Cartesian coordinate system, which is reasonable for most geotechnical processes. Therefore, all extensive variables (like energy and entropy) are considered as specific quantities and expressed per unit volume instead of per unit mass by avoiding a ubiquitous factor of the density ρ in the ensuing mathematics (Houlsby and Puzrin 2006).

To construct the mathematical framework of the hyperplastcity, it suffices to begin with the first and second laws. According to the first law of thermodynamics (the law of conservation of energy), the universe's energy is constant. It can only change from one form to another by transferring between the system and its surroundings. Generally, there are three mechanisms for transferring energy into and out of a system: heat transfer, work transfer, and mass flow. The system considered in the current thesis is a closed system, meaning no mass exchange with the surrounding can occur. Therefore, the first law can be mathematically expressed in its local form as:

$$\dot{u} = \sigma_{ij}\dot{\varepsilon}_{ij} - q_{i,i} \tag{3-23}$$

where the internal energy (u) is generally changed by two forms of power input, the mechanical power (the first term) and the heat supply per unit volume (the second term is written as the divergence of the heat flux vector q_i). It should be noted that the heat transfer is a non-convective process since the considered system is thermodynamically closed. Mechanical power generally represents the transfer of energy by uniform motion, whereas heat denotes the transfer of energy by non-uniform or random motions. Due to the transfer of mechanical and thermal power or energy, the states of a system, which can be quantified by deformation and temperature, can alter, subsequently leading to a change in internal energy. Internal energy can be interpreted as a measure of a system's capacity at a particular state to do work.

While the first law deals with the quantification and feasibility of the transfer of energy in thermal and mechanical forms, the second law introduces entropy (s) as another property of the system to identify the direction that energy tends to disperse naturally. For instance, according to Clausius's statement of the second law, no process is possible whose sole result is the transfer of heat from a cold to a hot body. Equivalently, according to Kelvin's statement of the second law, no process is possible whose sole result is the complete conversion of heat to mechanical power. Mathematically, the useful form for expressing the second law is the Clausius-Duhem inequality (Houlsby and Puzrin 2006) in the local form:

$$\dot{s} \geq -\left(\frac{q_i}{\theta}\right)_{,i} = -\frac{q_{i,i}}{\theta} + \left(\frac{\varsigma_i}{\theta}\right)\theta_{,i} \tag{3-24}$$

where θ and q_i/θ are temperature and the entropy flux (ς_i), respectively. Equation (3-24), which states that the entropy increment of the universe is non-negative for any process, can be extended as:

$$d = \theta\dot{s} + q_{i,i} - \varsigma_i\theta_{,i} \geq 0 \tag{3-25}$$

where d is the total dissipative power in the sense that the last term on the left side of the inequality is known as the thermal dissipation, which is always non-negative since the heat flux is always in the opposite direction of the temperature gradient. The rest of the total dissipation can then be considered mechanical dissipation. The thermal dissipation is insignificant compared to the mechanical dissipation for slow processes, so it can be argued that the mechanical dissipation must be non-negative (Houlsby and Puzrin 2006). Moreover, the temperature gradient is zero for the isothermal condition (constant temperature) adopted in this thesis. Therefore, by combining equations (3-23) and (3-25) we have:

$$\dot{u} + d = \sigma_{ij}\dot{\varepsilon}_{ij} + \theta\dot{s} \tag{3-26}$$

which is subjected to the imposed condition by the second law, $d \geq 0$. This equation is the starting point for developing the hyperplasticity formalism in the following section. But before that, Legendre transform is introduced, which provides a different and simplified perspective to equation (3-26) under certain conditions, for example, isothermal condition.

3.3 Legendre transform

Legendre transform is essentially a method of transferring between potentials by interchanging the role of independent and dependent variables. This valuable transformation, which is extensively used in theoretical mechanics, provides different and more simplified (under certain circumstances) perspectives to the considered system's behaviour while pertaining all information and properties of the physical theory. The detailed account of this elegant mathematical tool is beyond this thesis. Interested readers could refer to Callen (1960), Sewell (1969), and Houlsby and Puzrin (2006). The following briefly introduces some important properties of the Legendre transform.

Consider the potential function $\mathrm{X} = \mathrm{X}(x_1, \alpha)$. Only two independent variables, x_1 and α, are considered for simplicity. Since the function X acts as a potential, it can be written:

$$y_1 = \frac{\partial \mathrm{X}(x_1, \alpha)}{\partial x_1} \tag{3-27}$$

where y_1 is the dependent variable conjugated to x_1. Here, we are interested in transforming X to the other form (Y, for example) in which the roles of x_1 and y_1 are interchanged; y_1 acts as an independent variable conjugated to x_1. Therefore, throughout the transformation, α serves as a passive variable. The $\mathrm{Y} = \mathrm{Y}(y_1, \alpha)$ can be defined via the Legendre transform:

$$\mathrm{Y}(y_1, \alpha) = \pm\left[\mathrm{X}(x_1, \alpha) - x_1 y_1\right] \tag{3-28}$$

where the preference of sign is the matter of the particular application. It can be deduced that this transformation has the properties of:

$$x_1 = \mp\frac{\partial \mathrm{Y}}{\partial y_1} \tag{3-29}$$

$$\frac{\partial Y}{\partial \alpha} = \pm \frac{\partial X}{\partial \alpha} \tag{3-30}$$

Another important property of the Legendre transform is related to homogenous functions. Suppose X is a homogeneous function of the independent variable x_1 with the homogeneity order of m. By imposing Euler's theory for homogenous functions, it follows that Y is a homogeneous function of its independent variable y_1 (conjugated to x_1) with the homogeneity order of n such that:

$$\frac{1}{n} = \pm\left(\frac{1}{m} - 1\right) \tag{3-31}$$

3.4 Hyperplasticity formalism

In the approach of thermodynamics with internal variables, the state of a material element (a thermodynamic system) is characterised by a set of variables called internal variables to record the effect of the history, which is manifested microscopically and hidden to the macro scale (external observer who can only thermodynamically see the material element as a black box). Internal variables are in addition to standard independent macroscopic state variables like strain. For the purpose of this thesis, a single kinematic internal variable in tensorial form is chosen as the plastic strain. It is worth noting that there can generally be more internal variables in different forms depending on the (irreversible) mechanism they are associated with. Following Houlsby and Puzrin (2006), the internal variable is denoted by α_{ij} herein. The meaning for notions of independent and dependent variables will be seen in the following.

Since the current state of the material quantifies the internal energy to represent the path-independent property of the material (the capacity), it is conceived as a function of independent state and internal variables, i.e., $u(\varepsilon_{ij}, \alpha_{ij}, s)$. Therefore, the rate of change of the internal energy can be given as:

$$\dot{u} = \left(\frac{\partial u}{\partial \varepsilon_{ij}}\right)\dot{\varepsilon}_{ij} + \left(\frac{\partial u}{\partial \alpha_{ij}}\right)\dot{\alpha}_{ij} + \left(\frac{\partial u}{\partial s}\right)\dot{s} \tag{3-32}$$

On the other hand, to describe the dissipative processes, Ziegler (1977) postulated that instead of treating the dissipation d in equation (3-26) as secondary importance just as the checkpoint to follow the second law (non-negative dissipation) like in the conventional methods, d must be a primary function describing the dissipative power. Since dissipation is essentially path and history-dependent (it matters how the current state is reached), the dissipation function (d) must be a function of rates of the internal variable (representing incremental path-dependent behaviour associated with the micro-mechanisms hidden to the macro scale), in addition to the state and internal variables. Since dissipative power is always non-negative based on the second law, any function for d must retain its sign irrespective of the undertaken dissipative process represented by the rate of the internal variable. This means d is not a state function of its primary variable, the rate of internal variables. Further, Ziegler (1977) postulated that the dissipation function d must be a positively homogenous or pseudo-homogeneous function of rates of internal variables. Based on Euler's theorem, for a positively homogeneous dissipation function of order n, there is:

$$d = \frac{1}{n}\left(\frac{\partial d}{\partial \dot{\alpha}_{ij}}\right)\dot{\alpha}_{ij} = \left(\frac{\partial z}{\partial \dot{\alpha}_{ij}}\right)\dot{\alpha}_{ij} \tag{3-33}$$

where based on Houlsby and Puzrin (2002), z, which is called force potential (more detail provided in the following section), is defined as the dissipation function divided by the homogeneity order n. For the particular case of rate-independent dissipative behaviour, n is one, and d equals z. Throughout this chapter, the general form of homogeneity for d is employed since the rate-independency is not the prime subject of this thesis. The homogeneity or pseudo-homogeneity of d is a powerful postulate that reciprocally is linked to the fundamental definition of power (here is a dissipative power) if it is accepted that d exists as a primary function of the rate of internal variables. This importance leads not only to the determination of the thermodynamic dissipative stresses conjugated to the rate of internal variables but also to maximising the dissipation rate. More mathematical details regarding the maximum dissipation are provided in the next section.

It is worth noting that the dissipation or the force potential can also be a function of state and internal variables. However, the role of these variables is passive. The most important consequence of the state dependency of the dissipation function for geomaterials, which appears naturally because of the frictional dissipative mechanism, is the non-associativity of the flow rule (Collins and Houlsby 1997). This feature of frictional material is rejected by the unnecessarily restrictive postulate of Drucker (1959).

Having established the rates of change of the internal energy and dissipation, equation (3-26) can now be extended by replacing equations (3-32) and (3-33) and grouping each rate terms:

$$\left(\frac{\partial u}{\partial \varepsilon_{ij}} - \sigma_{ij}\right)\dot{\varepsilon}_{ij} + \left(\frac{\partial u}{\partial s} - \theta\right)\dot{s} + \left(\frac{\partial u}{\partial \alpha_{ij}} + \frac{\partial z}{\partial \dot{\alpha}_{ij}}\right)\dot{\alpha}_{ij} = 0 \tag{3-34}$$

Now since entropy and strain are state variables and subsequently their rates are arbitrary at any thermodynamic state (free variation), for a pure reversible heating process in which $\dot{\varepsilon}_{ij}$ and $\dot{\alpha}_{ij}$ are both zero, one can deduce that:

$$\frac{\partial u}{\partial s}=\theta \tag{3-35}$$

Equation (3-35) demonstrates an example for the notion of dependent and independent variables. As the equation indicates, the temperature is a dependent variable since it can be derived by differentiating the potential function u in terms of the conjugated, independent variable entropy. So, one can apply the Legendre transform to interchange the role of these variables and to obtain the other form of the free energy, the Helmholtz free energy (f):

$$f\left(\varepsilon_{ij},\alpha_{ij},\theta\right)=u\left(\varepsilon_{ij},\alpha_{ij},s\right)-\theta s \tag{3-36}$$

Now by practising the property of the Legendre transform for passive variables of ε_{ij} and α_{ij}, it can be obtained:

$$\frac{\partial f}{\partial \varepsilon_{ij}}=\frac{\partial u}{\partial \varepsilon_{ij}} \tag{3-37}$$

$$\frac{\partial f}{\partial \alpha_{ij}}=\frac{\partial u}{\partial \alpha_{ij}} \tag{3-38}$$

By replacing equations (3-35), (3-37), and (3-38) in equation (3-34), for an isothermal condition, we have:

$$\left(\frac{\partial f}{\partial \varepsilon_{ij}}-\sigma_{ij}\right)\dot{\varepsilon}_{ij}+\left(\frac{\partial f}{\partial \alpha_{ij}}+\frac{\partial z}{\partial \dot{\alpha}_{ij}}\right)\dot{\alpha}_{ij}=0 \tag{3-39}$$

where the Helmholtz free energy is a potential function in terms of ε_{ij} and α_{ij} under the isothermal condition. Again, since the strain is a state variable, and f is state property, the strain rate is subsequently arbitrary at any thermodynamic state. Therefore, the following relation between stress-strain can be deduced:

$$\frac{\partial f}{\partial \varepsilon_{ij}}=\sigma_{ij} \tag{3-40}$$

However, the same conclusion as the above cannot be deduced for the second differential term in equation (3-39) because z is not a state function and its partial derivative with respect to $\dot{\alpha}_{ij}$ generally depends on $\dot{\alpha}_{ij}$. The only conclusion is that the internal variable has no exclusive contribution to the free energy increase but only contributes to the increase of the dissipation (entropy production). To satisfy the last differential term, generally, two possibilities emerge:

1) $\frac{\partial f}{\partial \alpha_{ij}}+\frac{\partial z}{\partial \dot{\alpha}_{ij}}=0$ or equivalently $\chi_{ij}-\bar{\chi}_{ij}=0$ (3-41)

2) $\left(\frac{\partial f}{\partial \alpha_{ij}}+\frac{\partial z}{\partial \dot{\alpha}_{ij}}\right)\dot{\alpha}_{ij}=0$ or equivalently $\left(\chi_{ij}-\bar{\chi}_{ij}\right)\dot{\alpha}_{ij}=\tilde{\chi}_{ij}\dot{\alpha}_{ij}=0$ (3-42)

where after Houlsby and Puzrin (2006), $\bar{\chi}_{ij}$ and χ_{ij} are called the generalised and dissipative stresses, respectively, to conveniently distinguish the non-dissipative and dissipative internal stresses that are both work-conjugated to the rate of the internal variable. $\bar{\chi}_{ij}$ is the conservative stress since it is derived from the internal energy potential. $\tilde{\chi}_{ij}$ is called gyroscopic stress (Ziegler 1983) since $\tilde{\chi}_{ij}$ is orthogonal to $\dot{\alpha}_{ij}$ resulting in zero power.

Ideally, each of two possibilities can be employed to define the dissipative power. In the second possibility, the dissipative stress can be obtained by defining the gyroscopic stress, provided that the non-dissipative power is derived from the internal energy potential. Here, one gyroscopic stress needs to be defined for one internal variable. Compare this with the general situation with *n* internal variables. In that case, *n* gyroscopic stresses must be specified. Then, the corresponding dissipative stresses can be obtained and subsequently, the dissipation rate can be examined to be non-negative for all conditions. One must notice that in this possibility, the dissipation function or force potential are not needed to be specified since the dissipative power is specified passively at the cost of determining *n* gyroscopic stresses for *n* internal variables.

However, Ziegler (1983) argues that phenomenologically (no microscopic considerations), the gyroscopic stresses or forces (like electromagnetic effects) are not present, particularly in the mechanics of continuum media. In support of his argument, he refers to Onsager's widely used theory for linear irreversible processes in which magnetic fields, as a source for gyroscopic forces, must be excluded to hold the reciprocal relation. Therefore, Ziegler (1958, 1977) postulated the first alternative, the orthogonality condition, to generalise Onsager's linear theory to nonlinear irreversible processes. Applying Ziegler's orthogonality condition makes the number of unknown variables (internal and dependent state variables) equal to the number of equations supplied by the two potentials without any side conditions. This means the system of equations is closed and ready for analytical or numerical treatments to find the acceptable solution to the unknowns. The system of equations that needs to be solved can be casted as:

$$\begin{aligned}\frac{\partial f}{\partial \varepsilon_{ij}}&=\sigma_{ij}\\ \frac{\partial z}{\partial \dot{\alpha}_{ij}}&=-\frac{\partial f}{\partial \alpha_{ij}}\end{aligned}\qquad(3\text{-}43)$$

subjected to the homogeneity or pseudo-homogeneity requirement for z, which must always be non-negative for any $\dot{\alpha}_{ij}$.

Ziegler's powerful postulates makes the hyperplasticity framework a straightforward and useful approach for describing a system since all information related to the conservative and the dissipative behaviour of the system can be extracted from two potentials. However, one must be cautious about the consequences of applying Ziegler's orthogonality postulate on the system or the material's behaviour under different processes and conditions.

3.5 Ziegler's orthogonality condition and maximum dissipation

This section, following Ziegler (1983), demonstrates why Ziegler's postulate is an orthogonality condition leading to the maximum dissipation rate, which makes the postulate a stronger statement than the second law. To do so, no prerequisite assumptions and conditions regarding the form of the dissipation function, like homogeneity condition, are made. The only prerequisites are the second law that is non-negativity of dissipation rate, and the fundamental postulate of power (or known as energy or work per time increment), here a dissipative power:

$$d\left(\dot{\alpha}_{ij}, r\right) = \chi_{ij}\dot{\alpha}_{ij} \tag{3-44}$$

in which d is a primary invariant entity, i.e., a primary function of the rate of internal variable as the primary variable. Note that χ_{ij} is not yet defined but only considered as a variable conjugated to $\dot{\alpha}_{ij}$. For generality, r is defined as a set of state and internal variables with the passive role. The above construction is essentially phenomenological and free from microscopic complexities. The fundamental postulate of power determines the dissipative stress (χ_{ij}) as the space of linear mappings taking elements of $\dot{\alpha}_{ij}$ into scalars of dissipative power, which imposes χ_{ij} to be so-called dual space of $\dot{\alpha}_{ij}$.

Considering the above construction, now the task is to determine χ_{ij} in a way that maximises the dissipation rate (d) subjected to the constraint of the fundamental postulate of power. To do so, the following Lagrangian can be conceived:

$$L = d + \Lambda\left(d - \chi_{ij}\dot{\alpha}_{ij}\right) \tag{3-45}$$

where Λ is the Lagrange multiplier. The extrema of d can be found by:

$$\frac{\partial L}{\partial \dot{\alpha}_{ij}} = \frac{\partial d}{\partial \dot{\alpha}_{ij}} + \Lambda\left(\frac{\partial d}{\partial \dot{\alpha}_{ij}} - \chi_{ij}\right) = 0 \tag{3-46}$$

which leads to the definition of χ_{ij} as:

$$\chi_{ij} = \left(\frac{1+\Lambda}{\Lambda}\right)\frac{\partial d}{\partial \dot{\alpha}_{ij}} \tag{3-47}$$

This definition for χ_{ij} indicates that the dissipative stress is in the direction of the gradient of the dissipation function, like the deduction made from the homogeneity condition of *d*. By replacing this definition for χ_{ij} in equation (3-44), the dissipative power can be expressed as:

$$d\left(\dot{\alpha}_{ij}, r\right) = \chi_{ij}\dot{\alpha}_{ij} = \left(\frac{1+\Lambda}{\Lambda}\right)\frac{\partial d}{\partial \dot{\alpha}_{ij}}\dot{\alpha}_{ij} \geq 0 \tag{3-48}$$

Considering that based on the second law, *d* must be positive definite and contain the origin $\left(\dot{\alpha}_{ij} = 0\right)$, it can be deduced that *d* must be a differentiable convex surface of $\dot{\alpha}_{ij}$, that is maximised by the definition of χ_{ij} in equation (3-47) obtained based on the extremum principle. Since χ_{ij} is orthogonal to the level set of *d*, this result is known as Ziegler's orthogonality postulate. By combining equations (3-47) and (3-48), the dissipative stress can be expressed solely based on the dissipation function:

$$\chi_{ij} = \left(\frac{d}{\frac{\partial d}{\partial \dot{\alpha}_{ij}}\dot{\alpha}_{ij}}\right)\frac{\partial d}{\partial \dot{\alpha}_{ij}} \tag{3-49}$$

which in combination with the postulate of Ziegler (equation (3-41)), leads to the derivation of the complete response of a system, conservative or dissipative, just by the specification of two functions of *u* (or other dual forms of free energy like *f,* which is of interest here) and *d*, while subscribing to the principle of maximum dissipation (or entropy production).

3.6 Force and flow potential

It has been shown previously that the dissipative stress can be derived by derivation of the dissipation function with respect to the internal variable rate. As can be seen, based on Euler's theorem for homogenous functions of order *n*, the scaling factor inside the parenthesis in equation (3-49) is equal to 1/*n*. For rate-independent material, *n* is unity, and as a result, the dissipative stress can be purely derived from the derivation of the dissipation function:

$$\chi_{ij} = \frac{\partial d}{\partial \dot{\alpha}_{ij}} \tag{3-50}$$

in this case, *d* serves as a true potential for the dissipative stress and provides a derivation relationship between independent (rate of the internal variable) and dependent variables (dissipative stress). However, for the rate-dependent material, which is of interest, with a homogenous dissipation function of order *n*, the dissipative stress can be derived as:

$$\chi_{ij} = \left(\frac{1}{n}\right)\frac{\partial d}{\partial \dot{\alpha}_{ij}} \tag{3-51}$$

This definition for χ_{ij} indicates that the dissipation function does not act as a true potential but rather as a pseudopotential meaning that the partial derivative of *d* provides only the direction, but not the magnitude of χ_{ij} (Houlsby and Puzrin 2002). By introducing the force potential as $z = d/n$, the dissipative stress can then be obtained by:

$$\chi_{ij} = \frac{\partial z}{\partial \dot{\alpha}_{ij}} \tag{3-52}$$

which indicates that *z* is a potential for dissipative stresses. Therefore, *z* was called the force potential previously. Now, a Legendre transformation can be made to interchange the role of the dependent and independent variables and identify the flow potential *w*:

$$w\left(\chi_{ij},\sigma_{ij},\alpha_{ij}\right) = \chi_{ij}\dot{\alpha}_{ij} - z\left(\dot{\alpha}_{ij},\sigma_{ij},\alpha_{ij}\right) = d\left(\dot{\alpha}_{ij},\sigma_{ij},\alpha_{ij}\right) - z\left(\dot{\alpha}_{ij},\sigma_{ij},\alpha_{ij}\right) \tag{3-53}$$

where it can be deduced:

$$\dot{\alpha}_{ij} = \frac{\partial w}{\partial \chi_{ij}} \tag{3-54}$$

which means that *w* is a potential for the rate of the internal variable with a resemblance to the dynamic yield surface in the overstress viscoplastic theory of Perzyna (1963). The flow potential is a useful form for a dissipative potential for the conventional numerical implementation of the derived hyper-viscoplastic constitutive models as it can readily be employed in the strain-based incremental formulation. Based on the property of Legendre transform, since *z* is a homogenous function of order *n* in terms of $\dot{\alpha}_{ij}$, *w* must be a homogenous function of order *m* in terms of χ_{ij}:

$$\frac{1}{m} + \frac{1}{n} = 1 \tag{3-55}$$

On the other hand, if *z* is a first order homogenous function of $\dot{\alpha}_{ij}$, then $z = d$ and $w = 0$. In this case, *w* is similar to the yield function. To obtain *w* from *d* in this case, the degenerate form of Legendre transform (Collins and Houlsby 1997) or the convex analysis (Srinivasa 2010) must be employed. This case has not been practised in this thesis.

Chapter 4 – Summary, results, and discussions

4.1 Introduction

This chapter summarises and discusses the outcomes and significance of this PhD project. The PhD work and its outcomes are disseminated in five research articles attached to the current thesis. The following narration and accounts of results are oriented towards the number of objectives defined in the first chapter, which are as follows:

1. Development of a further understanding of the hyperplasticity framework for the construction of a thermodynamically consistent constitutive model addressing the viscous behaviour of clay- *Paper I* and *Paper V*
2. Identification of the conditions imposed by the hyperplasticity framework on the viscous constitutive modelling of clay through consideration of two fundamental concepts of critical state and isotache- *Paper I*, *Paper II*, *Paper III*, and *Paper IV*
3. Development of a generalised hyper-viscoplastic constitutive model for clay considering the essential features such as the viscous behaviour and the friction mobilisation- *Paper III* and *Paper IV*
4. Validation of the developed model against reliable experimental data in the literature with some hints on the evaluation of the model parameters- *Paper III* and *Paper IV*

4.2 Summary, results, and discussions

The following accounts of results and key contributions of the five papars are devided into four parts, each of which directly addresses the corresponding number of objectives provided in the previous section.

1. The first objective regarding understanding the hyperplasticity framework and its thermodynamic and mathematic backgrounds is addressed in the previous chapter, in addition to *Paper I* and *Paper V*. The general idea of the hyperplastic framework is that all the constituents of a constitutive model, such as elastic moduli, inelastic flow rule and hardening rules, are encompassed in two scalar functions of the free energy and the dissipation, and they do not need to be imposed as separate features. This compact formulation of the hyperplasicity framework, which obeys the principle of maximum dissipation (or entropy production), stems from Ziegler's orthogonality postulate (Ziegler 1983). As presented in the previous chapter and *Paper V*, the conservative (path-independent) and dissipative response (path-dependent) of a

system have resulted directly from thermodynamic principles, and they are linked by Ziegler's orthogonality postulate. As a result of this powerful postulate, all information regarding incremental stress-strain response can be extracted by solving a closed form and generally non-linear system of equations. This realization resulted in a rather novel technique for implementation of hyperplastic models exempted from establishing an explicit form for the yield surface or the inelastic flow potential using the cumbersome Legendre transform (degenerate form for the rate-independent case). Conventionally, establishing an analytical expression for the yield or inelastic flow potential is a prerequisite to the numerical implementation of hyperplastic constitutive models. However, according to the proposed technique, the hyperplastic constitutive models can be implemented directly from the free energy and the dissipation functions through the linearisation of the closed form nonlinear system of equations using the Newton-Raphson method. This is a useful approach since, occasionally, conducting Legendre transform is not trivial. Note that the role of the Legendre transform that is central to thermodynamics and mechanics cannot be undermined. Here, it is only underlined that this straightforward approach can pave the road for more efficient application of the hyperplastic constitutive models in non-linear finite element analyses. In this regard, the proposed approach is successfully practised for a broad spectrum of standard rate-independent hyperplastic constitutive models, specifically, a family of critical state models with the Matsuoka-Nakai (MN) frictional criterion and a family of cone models with different frictional criteria, dilatancy rules, and frictional hardenings. The yield surface for these models in the meridian and deviatoric planes is numerically constructed. The models are also applied in the simulation of the conventional triaxial tests. An obvious step forward would be utilising the proposed approach in solving boundary value problems and examining its numerical efficiency.

Along the first objective, through undrestanding of the mathematical formulation of the hyperplasticity approach, a force potential with a similar structure to the Modified Cam Clay (MCC) dissipation function is proposed in *Paper I* (Grimstad et al. 2020) to accommodate the isotache viscosity (Šuklje 1957). The proposed force potential is:

$$z=\frac{rp_0}{n}\left(\frac{\dot{\varepsilon}_v^p+\sqrt{\left(\dot{\varepsilon}_v^p\right)^2+\left(M\dot{\varepsilon}_s^p\right)^2}}{2r}\right)^n \tag{4-56}$$

where p_0 is the consolidation pressure, M is the dimensionless frictional coefficient at the critical state, n is the rate sensitivity parameter, r is the norm of an arbitrary reference volumetric strain rate, $\dot{\varepsilon}_v^p$ and $\dot{\varepsilon}_s^p$ are the volumetric and shear components of the rate of the inelastic strain (internal variable). *Paper I* (Grimstad et al. 2020) is one of few attempts to integrate the two essential and widely used phenomenological descriptions of clay's behaviour, namely the critical state soil mechanics (CSSM) (Schofield and Wroth 1968) and isotache concept (Šuklje 1957), using the thermodynamically consistent hyperplasticity approach (Houlsby and Puzrin 2006). The derived model is the hyper-viscoplastic expression for the classical Soft-Soil-Creep model proposed by Vermeer and Neher (2019) based on the overstress theory of Perzyna (1963) and the time resistance concept of Janbu (1969), Janbu (1985). This evolvement also led to the recognition of an imposed condition by Ziegler's orthogonality on the modelling of clay's behaviour using the isotache and CSSM, which brings us to address the second objective in the following.

2. In addressing the second objective, it is realised that imposing Ziegler's orthogonality condition has an unappealing consequence for a rate-dependent system with a single internal variable. Before exposing this restriction, it would be worth glancing over the history of evolvement of the hyperplastic MCC model (rate-independent).

   Based on Houlsby (1981), who originally presented the hyperplastic description of the MCC model, the inelastic power can be expressed as:

$$W^p = \frac{1}{2} p_0 \dot{\varepsilon}_v^p + \frac{1}{2} p_0 \sqrt{\left(\dot{\varepsilon}_v^p\right)^2 + \left(M \dot{\varepsilon}_s^p\right)^2} \tag{4-57}$$

   The first term is considered by Houlsby (1981) as stored inelastic power and is transferred to free energy. Subsequently, the remaining inelastic power in equation (4-57) is assigned to the dissipative function. Alternately, Collins and Houlsby (1997) and Houlsby (2000) presented another hyperplastic form for the MCC model with the same performance by considering the whole inelastic power in equation (4-57) as the dissipation function. This non-unique expression of the MCC model brought Houlsby (2000) to the conclusion that free energy and dissipation are generally "unobservable". In other words, it is impossible to prove which form of description is "right".

   On the other hand, Collins and his co-workers, on numerous occasions (Collins and Hilder 2002, Collins and Kelly 2002, Collins 2003, 2005a, 2005b), proposed the first alternative as a correct choice. Collins and Hilder (2002) and Collins (2005b) reasoned that for a process with a closed loop for inelastic strain rate, the first term in equation (4-57) has no contribution to the dissipation and can be identified as a recoverable inelastic power. In a further extension, Collins and Hilder (2002) proposed a family of rate-independent, critical state models with versatile yield surfaces and non-associated inelastic flow rules for a system with a single internal variable. By introducing two coefficients, *A* and *B,* with stress dimension, the Helmholtz free energy (*f*) and dissipation function for this case can be respectively expressed as:

$$f = f^e + \frac{1}{2}(\lambda-\kappa)\beta p_{ref}\exp\left(\frac{\varepsilon_v^p}{\lambda-\kappa}\right) = f^e + \frac{1}{2}(\lambda-\kappa)\beta p_0 \tag{4-58}$$

$$d = \sqrt{\left(A\dot{\varepsilon}_v^p\right)^2 + \left(BM\dot{\varepsilon}_s^p\right)^2} \tag{4-59}$$

$$A = (1-\beta)p + \frac{1}{2}\beta p_0 \tag{4-60}$$

$$B = (1-\gamma)p + \frac{1}{2}\beta\gamma p_0 \tag{4-61}$$

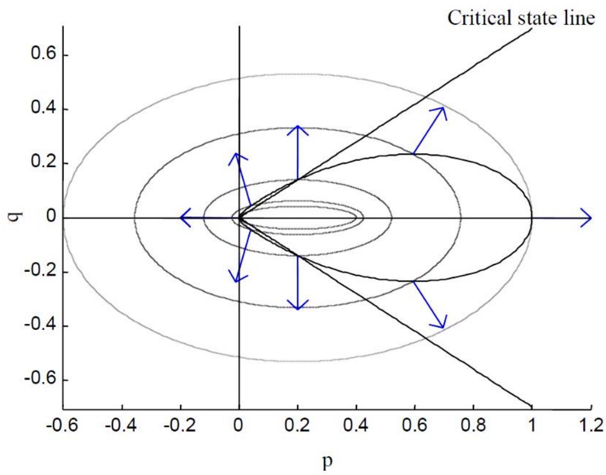


Figure 4-1: Yield surface in dissipative and true stress space together with inelastic flow directions based on Collins's proposition for β = 0.4 and γ = 0.3- after Collins (2005a).

In the above equations, p is the mean effective stress (pressure). λ and κ are slopes of isotropic unloading and normal compression lines, respectively, on the bi-logarithmic compressional plane of $\ln\upsilon - \ln p$ where υ is the specific volume. In Collin's proposition, by imposing Ziegler's postulate, the location of the critical state on the mean effective stress axis is adjustable due to the adjustability of the dissipative and stored shares of the plastic volumetric power via parameter β, while simultaneously, the critical state friction envelope remains unique. Figure 4-1 illustrates this distinction.

Considering the above distinctions, it is tempting to employ the approach of Collins and Hilder (2002) to construct a family of the critical state model for an equivalent rate-dependent system like in Aung et al. (2019) and, more recently, Jacquey and Regenauer-Lieb (2021). However, *Paper I* (Grimstad et al. 2020), *Paper II* (Grimstad et al. 2021), *Paper III* (Dadras-Ajirloo et al. 2022), and *Paper IV* have demonstrated that for such a rate-dependent system whose phenomenological description of its behaviour should follow the CSSM, the approach of Collins cannot be employed. Otherwise, there would be no unique mobilised friction at the critical state (no plastic volume change). This is not acceptable since a unique envelope for the mobilised friction at the critical state is an essential paradigm in the CSSM for a unified description of the behaviour of soils. Moreover, several experimental studies (e.g., Adachi et al. (1995), Arulanandan et al. (1971), Sheahan et al. (1996), Vaid and Campanella (1977), Zhu (2000), Tafili et al. (2021)) have implied that the mobilised friction at the critical state does not significantly depend on the loading rate. This experimental finding supports the fact that the Coulomb sliding friction (macro-scale) is approximately independent of the rate of mechanical processes.

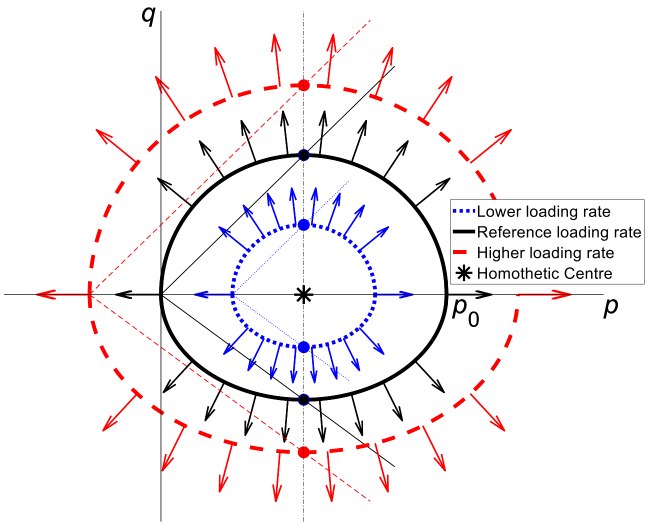


Figure 4-2: Dynamic yield surfaces together with inelastic flow direction in the true stress space for the conventional Mohr-Coulomb (DP-MC) friction criterion showing the non-uniqueness of the critical state friction under different loading rates due to movement of the homothetic centre as the consequence of consideration of plastic free energy for the rate-dependent system with a single internal variable.

Two sources for the non-uniqueness of friction at the critical state from applying Collins's approach for the rate-dependent case have been diagnosed. As the first source, *Paper I* (Grimstad et al. 2020) and *Paper IV* showed that pure plastic free energy should not be included. Figure 4-2 and Figure 4-3 from *Paper IV* illustrate the non-uniqueness of the critical state friction at different rates due to the presence of the pure plastic free energy for the Mohr-Coulomb (MC) and the Matsuoka-Nakai frictional criteria (Matsuoka and Nakai 1974), respectively.

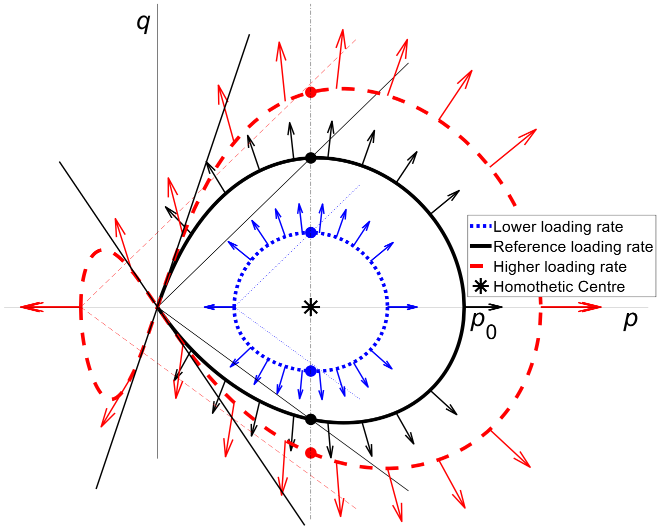


Figure 4-3: Dynamic yield surfaces together with inelastic flow direction in the true stress space for the conventional Mohr-Coulomb (DP-MC) friction criterion showing the non-uniqueness of the critical state friction under different loading rates due to movement of the homothetic centre as the consequence of consideration of plastic free energy for the rate-dependent system with a single internal variable.

Excluding the plastic free energy may not necessarily guarantee a unique critical state friction envelope for rate dependency if the force potential (dissipation function) is coupled with the internal variable such that affects the homothety of viscous scaling (e.g., via coefficient *B* introduced by Collins and Hilder (2002)). This is the second source exploited in *Paper II* (Grimstad et al. 2021) to consider the development of the earth Pressure coefficient at rest in clay during creep. Figure 4-4 from *Paper IV* depicts the effect of this source considering the Matsuoka-Nakai frictional criterion.

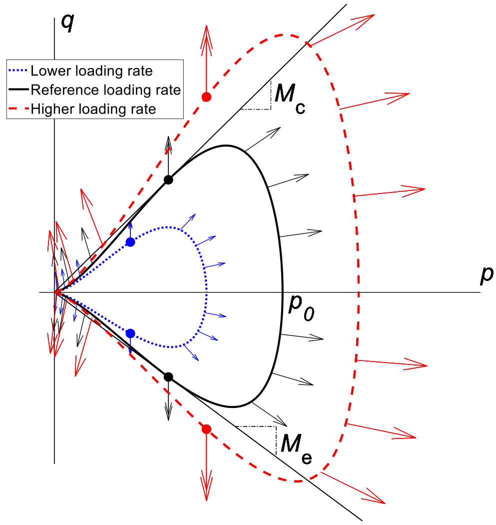


Figure 4-4: Dynamic surfaces together with inelastic flow direction in the true stress space for Matsuoka-Nakai (MN) friction criteria showing the non-uniqueness of the critical state friction under different loading rates due to the non-homotheticity of the viscous scaling as the consequence of mapping of frictional coefficient.

3. As the third objective, through *Paper III* and *Paper IV*, a hyper-viscoplastic constitutive model is developed to address the issues identified previously. The proposed constitutive model is completely potential-based, with only seven dimensionless material parameters. It complies with the CSSM and the isotache concept. The most distinctive feature of the model regarding the existing hyper-viscoplastic models is the securement of a unique friction envelope at the critical state while adopting versatile dynamic yield surfaces (in the overstress viscoplastic terminology) and non-associated inelastic flow rules. This distinction has been achieved through use of the stress ratio tensor (the deviatoric stress tensor normalised by the mean effective stress) in a novel force potential as an essential state variable of frictional material. The proposed force potential could equivalently represent the dissipation function for the rate-independent case, which can further highlight the remark of Houlsby (2000) on the dissipation's unobservability. In the following, some novel features of the model are briefly introduced.

   First, the versatile Helmholtz form of the free energy proposed by Houlsby et al. (2005) is specialised for the isotache concept to enable an objective and reference-independent measurement for the isotache viscous properties. The Helmholtz free energy can be expressed as:

$$f = \left(\frac{p_{ref}}{k} \right) \exp\left(k\left(\varepsilon_v - \varepsilon_v^p \right) + kg\,\mathrm{tr}\left(\left({}^D\boldsymbol{\varepsilon} - {}^D\boldsymbol{\varepsilon}^p \right)^2 \right) \right) \tag{4-62}$$

where k and g are dimensionless material parameters. ${}^D\boldsymbol{\varepsilon}$ and ${}^D\boldsymbol{\varepsilon}^p$ are total and inelastic deviatoric tensors, respectively. The operator “tr(.)” denotes the trace of a tensor. This free energy results in the same bulk modulus as the MCC model, which linearly depends on the mean effective stress. The mean effective stress (pressure) dependency can also be obtained for the shear modulus from equation (4-62). The shear modulus in the MCC model, adopted by almost all viscoplastic models in the literature, is also linearly mean effective stress-dependent via incorporation of the Poisson ratio. However, the traditional MCC elastic moduli are thermodynamically inconsistent and can create energy in a closed-loop loading (Zytynski et al. 1978). The by-product of having pressure dependency for the elastic shear modulus with the thermodynamic consistency is another feature called ‘stress-induced anisotropy’ (Houlsby et al. 2005). This anisotropy is an imposed condition by the first law of thermodynamics and is not related to the fundamental structure of the material (Houlsby et al. 2005). Note also that the compliance with the concepts of critical state under the isotache viscosity rejected the pure plastic free energy from the Helmholtz energy of the system with a single internal variable (the realised limitation in the preceding objective).

Another feature of the proposed model is having a versatile dynamic yield surface and non-associated flow rule. This has been achieved in *Paper III* (Dadras-Ajirloo et al. 2022) and *Paper IV* by generalising the force potential developed in *Paper I* (Grimstad et al. 2020) (equation (4-56)) and considering the impact of Ziegler’s orthogonality postulate. The generalized force potential is:

$$z = \frac{rp_0}{n} \left(\frac{\dot{\varepsilon}_v^p + \sqrt{\left(\bar{T}\dot{\varepsilon}_v^p \right)^2 + \left(\bar{M}\dot{\bar{\varepsilon}}_s^p \right)^2}}{Rr} \right)^n \tag{4-63}$$

where except R and $\bar{T}$, all other parameters are the same as equation (4-56). $\dot{\bar{\varepsilon}}_s^p$ and $\bar{M}$ are a measure of the inelastic shear strain rate and the corresponding frictional coefficient defined based on a certain frictional criterion. R is called spacing ratio, specifying the relative location of the critical state on the mean effective stress axis. Chen and Yang (2017) reported that R could vary between 1.5 to 4.0 for clays. The dimensionless coefficient $\bar{T}$ in equation (4-63) is a variant of the Logistic function, which due to its performance called the transition function of the state variable $\bar{S}$ defined as:

$$\bar{T} = \frac{R}{2} + \left(\frac{R-2}{2}\right)\tanh\left(\bar{S}\right) \tag{4-64}$$

where $\bar{S}$ is:

$$\bar{S} = \left(\frac{\bar{M}}{\bar{\eta}}\right)^2 - \left(\frac{\bar{\eta}}{\bar{M}}\right)^2 \tag{4-65}$$

in which $\bar{\eta}$ is a measure of the stress ratio indicating the mobilised friction associated with a certain frictional criterion. According to the previous chapter, the Legendre transform of the force potential is the flow potential (w) which defines the evolution of the internal variable (inelastic strain). The flow potential can be found as follows:

$$w = rp_0\left(\frac{n-1}{n}\right)\left(\frac{\bar{p}_{eq}}{p_0}\right)^{\left(\frac{n}{n-1}\right)} \tag{4-66}$$

where $\bar{p}_{eq}$ is known as the size of the dynamic yield surface (Perzyna 1963), whose division by p_0 represents the relative rate of the ongoing process. It is defined as:

$$\bar{p}_{eq} = \frac{R\chi_p\sqrt{\left(\left(\chi_p\bar{M}\right)^2 - \left(\bar{\chi}_q\right)^2\right)^2 + \left(\bar{\chi}_q\left(\bar{T}\chi_p\bar{M} + \sqrt{\left(\chi_p\bar{M}\right)^2 + \left(\bar{T}^2-1\right)\left(\bar{\chi}_q\right)^2}\right)\right)^2}}{\left(\begin{array}{l}\bar{T}\left(\left(\chi_p\bar{M}\right)^2 - \left(\bar{\chi}_q\right)^2\right) + \\ \sqrt{\left(\left(\chi_p\bar{M}\right)^2 - \left(\bar{\chi}_q\right)^2\right)^2 + \left(\bar{\chi}_q\left(\bar{T}\chi_p\bar{M} + \sqrt{\left(\chi_p\bar{M}\right)^2 + \left(\bar{T}^2-1\right)\left(\bar{\chi}_q\right)^2}\right)\right)^2}\end{array}\right)} \tag{4-67}$$

in which χ_p and $\bar{\chi}_q$ are the mean and shear the dissipative stresses conjugated to $\dot{\varepsilon}_v^p$ and $\bar{\dot{\varepsilon}}_s^p$, respectively. As expected from the property of dual homogeneous functions presented in the previous chapter, the degree of homogeneity of w with respect to dissipative stress is $n/(n\text{-}1)$ since the degree of homogeneity of z is n.

In the following, some definitions for $\bar{\dot{\varepsilon}}_s^p$, $\bar{M}$ and subsequently $\bar{\chi}_q$ are provided to specialise the previous generalized formulation to specific friction criteria. For the Drucker-Prager (DP) friction criterion, it suffices to replace the following definition for the measure of shear strain rate on the octahedral plane with the unit norm characteristic tensor of $\sqrt{\boldsymbol{\delta}/\mathrm{tr}(\boldsymbol{\delta})}$:

$$\bar{\dot{\varepsilon}}_s^p = \dot{\varepsilon}_{s,oct}^p = \sqrt{2}\sqrt{\frac{\mathrm{tr}\left(\boldsymbol{\delta}.\left({}^D\dot{\boldsymbol{\varepsilon}}^p\right)^2\right)}{\mathrm{tr}\left(\boldsymbol{\delta}\right)}} \tag{4-68}$$

where $\boldsymbol{\delta}$ is the Kronecker delta, tr(.) is the trace of a tensor, and ${}^{D}\dot{\boldsymbol{\varepsilon}}^{p}$ is the inelastic deviatoric strain rate:

$$ {}^{D}\dot{\boldsymbol{\varepsilon}}^{p} = \dot{\boldsymbol{\varepsilon}}^{p} - \mathrm{tr}\left(\left(\sqrt{\frac{\boldsymbol{\delta}}{\mathrm{tr}(\boldsymbol{\delta})}} \right)^{2} . \dot{\boldsymbol{\varepsilon}}^{p} \right) \boldsymbol{\delta} = \dot{\boldsymbol{\varepsilon}}^{p} - \left(\frac{\dot{\varepsilon}_{v}^{p}}{3} \right) \boldsymbol{\delta} \tag{4-69} $$

The dissipative shear stress conjugated to $\dot{\varepsilon}_{s,oct}^{p}$ can be computed as:

$$ \bar{\chi}_{q} = \sqrt{\frac{3}{2} \mathrm{tr}\left({}^{D}\boldsymbol{\chi}^{2} \right)} \tag{4-70} $$

in which ${}^{D}\boldsymbol{\chi}$ is the deviatoric dissipative stress tensor. Based on equation (4-70), after applying Ziegler orthogonality condition $(\boldsymbol{\chi} = \boldsymbol{\sigma})$, the conjugated dissipative shear stress is obtained on the octahedral plane the same as $\dot{\varepsilon}_{s,oct}^{p}$. $\bar{M}$ conventionally could be assumed to be equal to the critical friction under triaxial compression (M_c):

$$ \bar{M} = M_{c} = \frac{6\sin(\phi_{cs})}{3 - \sin(\phi_{cs})} \tag{4-71} $$

or the critical friction under triaxial extension (M_e):

$$ \bar{M} = M_{e} = \frac{6\sin(\phi_{cs})}{3 + \sin(\phi_{cs})} \tag{4-72} $$

where ϕ_{cs} is the critical state angle of shearing resistance of the particulate system in the macroscale. For the DP criterion, the equivalent isotropic stress measure $\left(p_{eq}^{DP}\right)$ or the size of the dynamic surface in the true stress space can be computed as:

$$ \bar{p}_{eq} = p_{eq}^{DP} = \frac{Rp\sqrt{\left((M_c)^2 - (\eta_{oct})^2\right)^2 + \left(\eta_{oct}\left(T_{oct}M_c + \sqrt{(M_c)^2 + \left(T_{oct}{}^2 - 1\right)(\eta_{oct})^2}\right)\right)^2}}{\left(\begin{array}{l} T_{oct}\left((M_c)^2 - (\eta_{oct})^2\right) + \\ \sqrt{\left((M_c)^2 - (\eta_{oct})^2\right)^2 + \left(\eta_{oct}\left(T_{oct}M_c + \sqrt{(M_c)^2 + \left(T_{oct}{}^2 - 1\right)(\eta_{oct})^2}\right)\right)^2} \end{array} \right)} \tag{4-73} $$

where the transition function defined in equation (4-64) is specialised for the DP criterion $\left(T_{oct}\right)$ with the state function of:

$$ \bar{S} = S_{oct} = \left(\frac{M_c}{\eta_{oct}} \right)^2 - \left(\frac{\eta_{oct}}{M_c} \right)^2 \tag{4-74} $$

In equations (4-73) and (4-74), η_{oct} is the measure of stress ratio on the octahedral plane:

$$\eta_{oct} = \sqrt{\frac{3}{2}\mathrm{tr}\left(\boldsymbol{\eta}^2\right)} \tag{4-75}$$

where $\boldsymbol{\eta} = {}^{D}\boldsymbol{\sigma}/p$ is the stress ratio tensor defined as the deviatoric stress tensor normalised by the mean effective stress (p). Since the octahedral plane is stationary and the assumed frictional coefficient is constant (independent of state), the mobilised friction in the DP criterion is isotropic, i.e., it does not vary with the change of shear loading (Lode angle independent). However, numerous experiments conducted by the true triaxial test (e.g., Yong and Mckyes (1971), Lade and Musante (1978), Nakai et al. (1986), Kirkgard and Lade (1993), Prashant and Penumadu (2004), Prashant and Penumadu (2005), and Ye et al. (2014)) have demonstrated that the shear strength of soil is essentially anisotropic and varies according to the orientation of shear loading. A conventional method to address this limitation of the DP friction criterion is the incorporation of some shape function of the Lode angle (e.g., Van Eekelen (1980) and Panteghini and Lagioia (2018)). One of the widely used shape functions is obtained through Mohr-Coulomb (MC) criterion:

$$M_\theta = \frac{3\sin\left(\phi_{cs}\right)}{\sqrt{3}\cos\left(\theta\right) + \sin\left(\theta\right)\sin\left(\phi_{cs}\right)} \tag{4-76}$$

where θ is Lode angle defined as:

$$\begin{aligned} \theta &= -\frac{1}{3}\sin^{-1}\left(\sqrt{6}\mathrm{tr}\left(\boldsymbol{e}_\theta{}^3\right)\right) \\ \boldsymbol{e}_\theta &= \frac{\boldsymbol{\eta}}{\sqrt{\mathrm{tr}\left(\boldsymbol{\eta}^2\right)}} \end{aligned} \tag{4-77}$$

To consider this recent case of the MC criterion, the only modification to equations (4-73) and (4-74) is the replacement of M_c with M_θ provided in equation (4-76).

Another considered friction criterion is the Matsuoka-Nakai (MN) criterion (Matsuoka and Nakai 1974) which, based on several real and virtual experiments done by the Discrete element method (DEM), is physically more appealing. The MN criterion is derived from the continuum-mechanics-based concept of "Spatial Mobilised Plane (SMP)" (Matsuoka 1976) that can be characterised by the unit norm tensor of $\sqrt{\boldsymbol{\sigma}^{-1}/\mathrm{tr}\left(\boldsymbol{\sigma}^{-1}\right)}$. According to Collins (2003), the MN criterion can be obtained by finding the shear strain rate on the "Dual Kinematic Plane" (DKP), which is the dual form of the SMP plane characterised by the unit norm tensor of $\sqrt{\boldsymbol{\sigma}/\mathrm{tr}\left(\boldsymbol{\sigma}\right)}$.

This dual treatment of friction and shear strain rates guarantees the work-conjugacy of the shear dissipative stress and inelastic shear strain rate. However, according to Nixon (1999), there appears to be a redundant and unnecessarily complex term in the shear strain rate on the DKP defined by Collins (2003). To exempt from this redundant term and to obtain the MN friction criterion, the shear rate on the DKP can be aptly cast as:

$$\bar{\dot{\varepsilon}}_s^p = \dot{\varepsilon}_{s,DKP}^p = \sqrt{\frac{\mathrm{tr}\left(\boldsymbol{\sigma}.\left({}^D\dot{\boldsymbol{\varepsilon}}^p\right)^2\right)}{\mathrm{tr}\left(\boldsymbol{\sigma}\right)}} \tag{4-78}$$

which leads to the following definition for the conjugated dissipative shear stress:

$$\bar{\chi}_q = \sqrt{p\,\mathrm{tr}\left(\boldsymbol{\sigma}^{-1}.{}^D\boldsymbol{\chi}^2\right)} \tag{4-79}$$

which after applying Ziegler orthogonality condition $\left(\boldsymbol{\chi} = \boldsymbol{\sigma}\right)$, the conjugated dissipative shear stress is obtained on the SMP. Now, by assuming:

$$\bar{M} = M_{SMP} = 2\sqrt{\frac{2}{3}}\tan\left(\phi_{cs}\right) \tag{4-80}$$

and replacing equations (4-78) and (4-80) in the force potential, the p_{eq}^{MN} in the true stress pace that is specialized for the MN friction can be computed as:

$$\bar{p}_{eq} = p_{eq}^{MN} = \frac{Rp\sqrt{\left(\left(M_{SMP}\right)^2 - \left(\eta_{SMP}\right)^2\right)^2 + \left(\eta_{SMP}\left(T_{SMP}M_{SMP} + \sqrt{\left(M_{SMP}\right)^2 + \left(T_{SMP}{}^2 - 1\right)\left(\eta_{SMP}\right)^2}\right)\right)^2}}{\left(\begin{array}{l} T_{SMP}\left(\left(M_{SMP}\right)^2 - \left(\eta_{SMP}\right)^2\right) + \\ \sqrt{\left(\left(M_{SMP}\right)^2 - \left(\eta_{SMP}\right)^2\right)^2 + \left(\eta_{SMP}\left(T_{SMP}M_{SMP} + \sqrt{\left(M_{SMP}\right)^2 + \left(T_{SMP}{}^2 - 1\right)\left(\eta_{SMP}\right)^2}\right)\right)^2} \end{array}\right)} \tag{4-81}$$

where the transition function T_{SMP} is specialised for the MN criterion with the state function of:

$$\bar{S} = S_{SMP} = \left(\frac{M_{SMP}}{\eta_{SMP}}\right)^2 - \left(\frac{\eta_{SMP}}{M_{SMP}}\right)^2 \tag{4-82}$$

η_{SMP} in above equations represents the mobilised friction or the shear stress ratio on the SMP:

$$\eta_{SMP} = \sqrt{p\,\mathrm{tr}\left(\boldsymbol{\sigma}^{-1}\right) - 3} = \sqrt{\frac{I_1 I_2}{3I_3} - 3} \tag{4-83}$$

in which I_1, I_2 and I_3 are the first, second and third stress invariants.

It is of prime importance to realise that the shear dissipation on the stationary octahedral plane led to the friction mobilisation (η_{oct}) on the very same plane for the DP and the MC. The only difference is that in the MC case, the frictional coefficient varies with the Lode angle due to the directional change of $\boldsymbol{\eta}$. This is suggestive of the MC being renamed the DP-MC. On the other hand, for the MN criterion, the shear dissipation on the evolving DKP resulted in the friction mobilisation on the evolving SMP (η_{SMP}). In fact, the dissipative shear strain rate on the DKP (equation (4-78)) can be re-expressed as:

$$\dot{\varepsilon}^{p}_{s,DKP} = \sqrt{\left(\dot{\varepsilon}^{p}_{s,oct}\right)^2 + \mathrm{tr}\left(\boldsymbol{\eta}.\left({}^{D}\dot{\boldsymbol{\varepsilon}}^{p}\right)^2\right)} \tag{4-84}$$

where $\dot{\varepsilon}^{p}_{s,oct}$ and $\dot{\varepsilon}^{p}_{s,DKP}$ are the shear strain rates on the octahedral plane and the DKP defined in equations (4-68) and (4-78), respectively. DEM studies (e.g., Radjai et al. (1998), Alonso-Marroquín et al. (2005), Antony and Kruyt (2009), and Radjai and Azéma (2009)) imply that the stress ratio $\boldsymbol{\eta}$ could represent the degree of the bimodal dissipative behavioural feature of particulate systems (development of the strong and weak force chains). It has been numerously demonstrated that the strong force chains carry almost the entire deviatoric stress while the weak force chains contribute only to the mean stress. Based on equation (4-84), as the stress ratio increases and the bimodal behaviour intensifies, the dissipative shear strain rate deviates from the dissipative strain rate on the fixed octahedral plane. The consequences of this deviation (stress-induced anisotropy) in terms of friction mobilisation and the corresponding inelastic flow are explored in detail in *Paper IV*.

As an example, Figure 4-5 compares the dynamics yield surfaces computed in equations (4-73) and (4-81) associated with the friction criteria of the DP, the DP-MC, and the MN. The visualisations have been done in the normalised triaxial $p/\bar{p}_{eq} - q/\bar{p}_{eq}$ plane with $q = \sigma_1 - \sigma_3$ and $p = \left(\sigma_1 + 2\sigma_3\right)/3$ for the simplest case of $R = 2$ and subsequently $\bar{T} = T_{SMP} = T_{oct} = 1$. σ_1 and σ_3 are the maximum and minimum principal stresses.

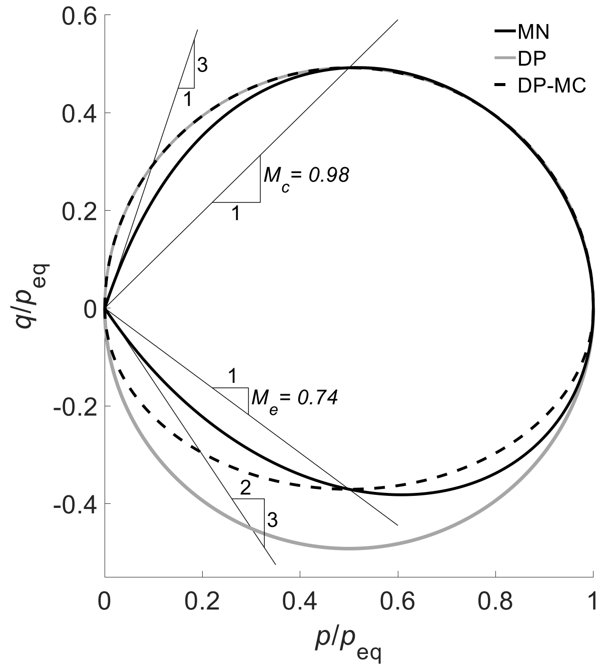


Figure 4-5- Dynamic yield surface associated with different frictional criteria of Matsuoka-Nakai (MN), Drucker-Prager (DP), and the conventional Mohr-Coulomb constructed via a shape function of Lode angle (DP-MC).

As shown, on the left side of the critical state line for the triaxial compression and extension, where the soil state is relatively denser, the dynamic surface with the MN friction considerably deviates from the ones associated with the DP-MC and the DP and becomes tangential to the lines called tension cut-off. The tension cut-off lines with a slope of 1:3 for compression and 2:3 for extension separate the stress states with negative values of the principal stresses under axisymmetric conditions. As illustrated, the principal stresses in the DP and DP-MC dynamic yield surfaces can become negative, which is meaningless for particulate material without tensile capacity. In contrast, for the MN dynamic yield surface, negative principal stresses are not attainable. On the extensional looser side, the MN friction mobilises towards the critical friction in such a way that deviates from the DP and approaches the DP-MC at the critical state.

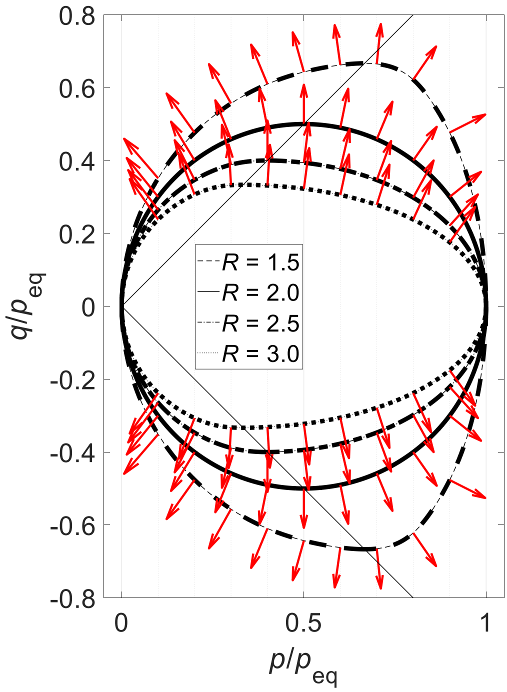


Figure 4-6: Dynamic surfaces associated with the Drucker-Prager (DP) frictional criterion together with the corresponding inelastic flow directions in the normalised true stress space for different values of spacing ratio (R) and $\phi_{cs} = 25^{\circ}$.

Figure 4-6 and Figure 4-7 show the effect of spacing ratio R on the dynamic yield surfaces and inelastic flows associated with the MN and the DP criteria, respectively. As shown, the dilatancy (the ratio of volumetric to shear strain rate) is controlled by the ratio of the mobilised friction to the critical friction and the spacing ratio. This is the fundamental premise of the CSSM employed through the sophisticated and comprehensive force potential defined in equation (4-63) which guarantees the uniqueness of the critical friction envelope.

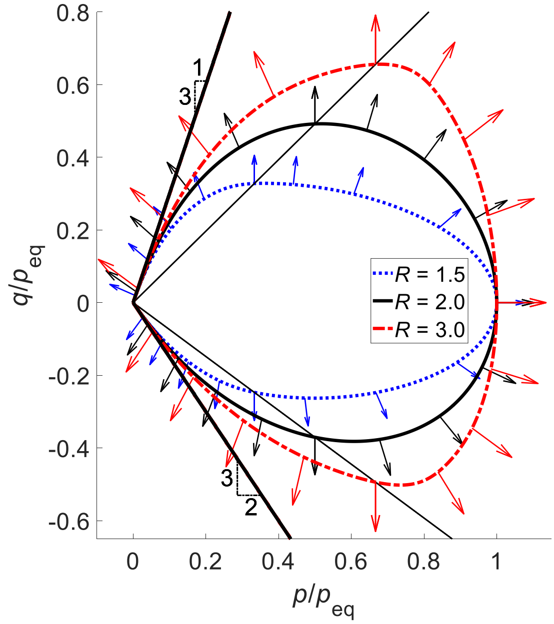


Figure 4-7: Dynamic surfaces associated with the Matsuoka-Nakai (MN) frictional criterion together with the corresponding inelastic flow directions in the normalised true stress space for different values of spacing ratio (R) and $\phi_{cs} = 25^{\circ}$.

It is still possible to increase the versatility of the force potential by incorporating the "frictional dissipation", i.e., the pressure sensitivity of a dissipative shear process attributed to the mobilisation of internal friction at grains/aggregates contacts. To do so, by preserving the dimension of the force potential, $\bar{M}$ in equation (4-63) can be mapped by the ratio of mean effective stress p to $\tilde{p}_{eq}$:

$$z = \frac{rp_0}{n}\left(\frac{\dot{\varepsilon}_v^p + \sqrt{\left(\bar{T}\dot{\varepsilon}_v^p\right)^2 + \left(\tilde{M}\dot{\bar{\varepsilon}}_s^p\right)^2}}{Rr}\right)^n$$
$$\tilde{M} = \bar{M}\sqrt{1-\gamma+\gamma\left(\frac{Rp}{\tilde{p}_{eq}}\right)} \tag{4-85}$$

where $\tilde{p}_{eq}$ is the dynamic equivalent pressure associated with a particular frictional criterion in the true stress space, which, based on the foregoing development, can be defined as:

$$\tilde{p}_{eq} = \frac{Rp\sqrt{\left(\left(\bar{M}\right)^2-\left(\bar{\eta}\right)^2\right)^2+\left(\bar{\eta}\left(\bar{T}\bar{M}+\sqrt{\left(\bar{M}\right)^2+\left(\bar{T}^2-1\right)\left(\bar{\eta}\right)^2}\right)\right)^2}}{\bar{T}\left(\left(\bar{M}\right)^2-\left(\bar{\eta}\right)^2\right)+\sqrt{\left(\left(\bar{M}\right)^2-\left(\bar{\eta}\right)^2\right)^2+\left(\bar{\eta}\left(\bar{T}\bar{M}+\sqrt{\left(\bar{M}\right)^2+\left(\bar{T}^2-1\right)\left(\bar{\eta}\right)^2}\right)\right)^2}} \tag{4-86}$$

with this definition for $\tilde{p}_{eq}$, $\tilde{M}$ can be extended as follows:

$$\tilde{M} = \bar{M}\sqrt{1+\gamma\left(\frac{\bar{T}\left(\left(\bar{M}\right)^2-\left(\bar{\eta}\right)^2\right)}{\sqrt{\left(\left(\bar{M}\right)^2-\left(\bar{\eta}\right)^2\right)^2+\left(\bar{\eta}\left(\bar{T}\bar{M}+\sqrt{\left(\bar{M}\right)^2+\left(\bar{T}^2-1\right)\left(\bar{\eta}\right)^2}\right)\right)^2}}\right)} \tag{4-87}$$

which the previous definition for the transition function $\bar{T}$ still holds as:

$$\begin{aligned}\bar{T} &= \frac{R}{2}+\left(\frac{R-2}{2}\right)\tanh\left(\bar{S}\right)\\ \bar{S} &= \left(\frac{\bar{M}}{\bar{\eta}}\right)^2-\left(\frac{\bar{\eta}}{\bar{M}}\right)^2\end{aligned} \tag{4-88}$$

by considering the new frictional coefficient $\tilde{M}$ defined in equations (4-85) and (4-87), the flow potential has the same form as equation (4-66) with $\bar{p}_{eq}$ modified as:

$$\bar{p}_{eq} = \frac{R\chi_p\sqrt{\left(\left(\chi_p\tilde{M}\right)^2-\left(\bar{\chi}_q\right)^2\right)^2+\left(\bar{\chi}_q\left(\bar{T}\chi_p\tilde{M}+\sqrt{\left(\chi_p\tilde{M}\right)^2+\left(\bar{T}^2-1\right)\left(\bar{\chi}_q\right)^2}\right)\right)^2}}{\left(\begin{aligned}&\bar{T}\left(\left(\chi_p\tilde{M}\right)^2-\left(\bar{\chi}_q\right)^2\right)+\\&\sqrt{\left(\left(\chi_p\tilde{M}\right)^2-\left(\bar{\chi}_q\right)^2\right)^2+\left(\bar{\chi}_q\left(\bar{T}\chi_p\tilde{M}+\sqrt{\left(\chi_p\tilde{M}\right)^2+\left(\bar{T}^2-1\right)\left(\bar{\chi}_q\right)^2}\right)\right)^2}\end{aligned}\right)} \tag{4-89}$$

which after imposing the Ziegler orthogonality condition (no shift stress as the pure plastic energy is neglected to secure a unique critical state envelope), the dynamic yield surface in the true stress space can be expressed as:

$$\bar{p}_{eq} = \frac{Rp\sqrt{\left(\left(\tilde{M}\right)^2-\left(\bar{\eta}\right)^2\right)^2+\left(\bar{\eta}\left(\bar{T}\tilde{M}+\sqrt{\left(\tilde{M}\right)^2+\left(\bar{T}^2-1\right)\left(\bar{\eta}\right)^2}\right)\right)^2}}{\bar{T}\left(\left(\tilde{M}\right)^2-\left(\bar{\eta}\right)^2\right)+\sqrt{\left(\left(\tilde{M}\right)^2-\left(\bar{\eta}\right)^2\right)^2+\left(\bar{\eta}\left(\bar{T}\tilde{M}+\sqrt{\left(\tilde{M}\right)^2+\left(\bar{T}^2-1\right)\left(\bar{\eta}\right)^2}\right)\right)^2}} \tag{4-90}$$

which is similar to equation (4-86) but with a different frictional coefficient $\left(\tilde{M}\right)$. Note that in recent development from equation (4-85) to equation (4-90), $\bar{M}$ and $\bar{\eta}$ are defined based on a certain frictional criterion. For instance, $\bar{M}$ and $\bar{\eta}$ are defined for the DP criterion in equations (4-71) and (4-75), and for the MN criterion, in equations (4-80) and (4-83), respectively.

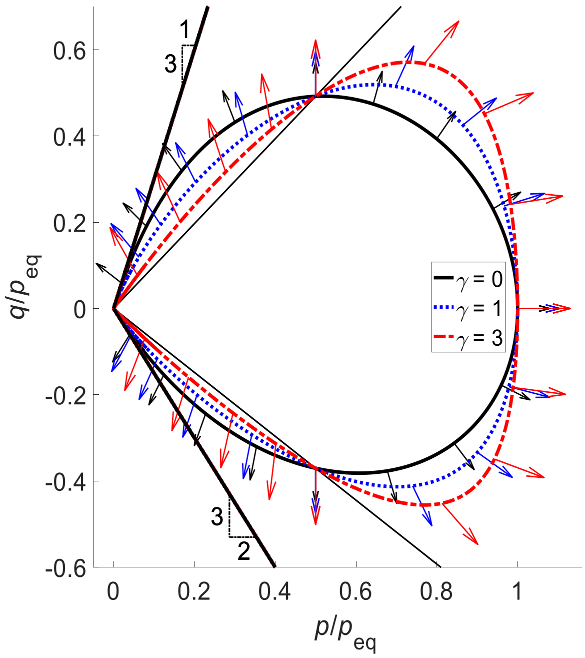


Figure 4-8: Dynamic surfaces associated with the Matsuoka-Nakai (MN) friction criterion together with the corresponding inelastic flow directions in the normalised true stress space for different values of parameter γ incorporating the frictional dissipation while $\phi_{cs} = 25^{\circ}$ and $R = 2$.

According to equation (4-85), the parameter γ regulates the intensity of the pressure dependency of the shear dissipation. Figure 4-8 illustrates the effect of the frictional dissipation on the dynamic convex loci and their corresponding inelastic flow directions considering the MN criterion. As shown, with the increase of γ, the convex dynamic loci get more teardrop shapes while conforming to the tension cut-off due to the engagement of the MN friction. Moreover, incorporating the frictional dissipation mitigates the dilation on the denser side of the critical state. In contrast, it causes an increase in the contraction (negative dilatancy) on the looser side.

4. As the last objective, the efficacy of the developed hyper-viscoplastic model with DP-MC and MN friction criteria is evaluated by simulating the conventional and the true triaxial tests

conducted on the Hong Kong Marine Deposits (HKMD) (Yin and Zhu 1999, Zhu 2000) and the Fujinomori clay (Nakai and Matsuoka 1986, Nakai et al. 1986). The model requires seven dimensionless parameters, as displayed in Table 4-1. The rate sensitivity parameter *n* according to *Paper I* (Grimstad et al. 2020) and Grimstad et al. (2010) equals:

$$n = 1 + \frac{\mu}{\lambda - \kappa} \tag{4-91}$$

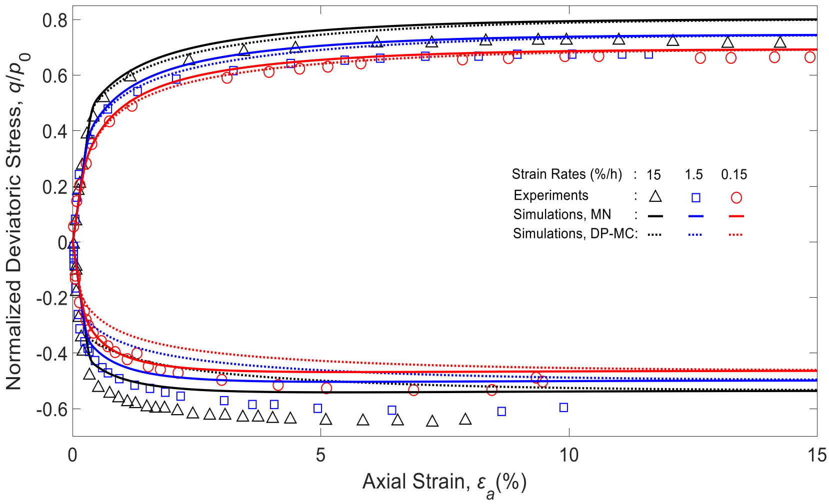
where μ is the creep index. The parameter γ controls the non-associated flow direction and teardrop shape for the dynamic yield surface.

Table 4-1: Parameters of the hyperviscoplastic model and their values for HKMD and Fujinomori clay

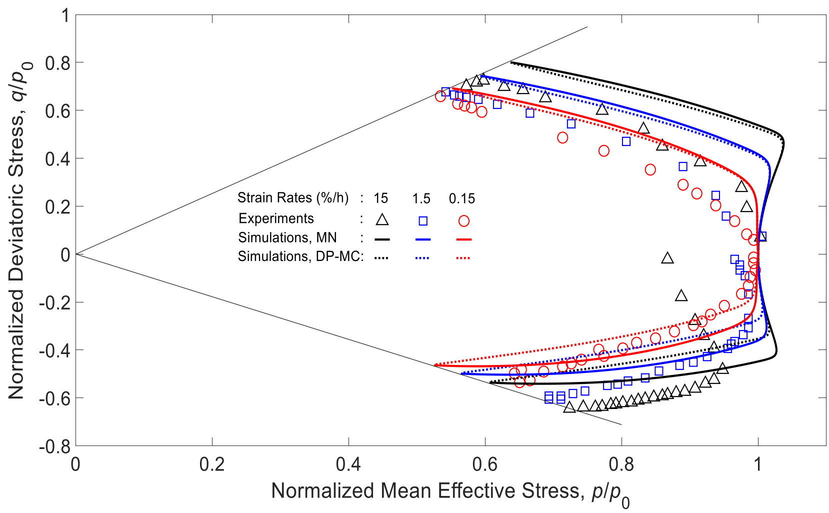
Parameter	**Description**	**HKMD**	**Fujinomori clay**
κ	slope of isotropic unloading line on bi-logarithmic compression plane	0.018	0.0112
λ	slope of normal compression line on bi-logarithmic compression plane	0.0793	0.0508
g	shear modulus coefficient	42	88.2
ϕ_{cs}	angle of shearing resistance at critical state	31.5	33.7
R	spacing ratio	2.6	2.2
γ	parameter for frictional dissipation	0	0
μ	creep index	0.0025	0.003

Figure 4-9 illustrates the simulated and experimental results of undrained triaxial tests conducted at constant strain loading rates of 0.15, 1.5, and 15%/h on the HKMD specimens. Before shearing, each sample was normally consolidated to isotropic mean effective stress of 400 kPa. As shown, the model's response with DP-MC and MN friction criteria are almost identical for the undrained compression tests. However, for the undrained extension tests, as expected from Figure 4-5, the MN criterion gives higher deviatoric stress throughout all tests.

To assess the model's performance for the denser (over-consolidated) states, the undrained triaxial tests with the strain rate of 1.5%/h on the specimens of HKMD with different over-consolidation ratios $(OCR = p/p_0)$ are also simulated. According to Figure 4-10, the MN criterion results in a lower deviatoric and stress ratio than the DP-MC criterion for high *OCR*s $(OCR = 4{,}8)$, supported by the observation in Figure 4-5.



(a)



(b)

Figure 4-9: Comparison between experimental and simulated results of undrained triaxial tests under different strain rates conducted on normally-consolidated and reconstituted HKMD in terms of (a) stress-strain and (b) stress path responses. The numerical simulations have been done with frictional criteria of Matsuoka-Nakai (MN) and the conventional Mohr-Coulomb constructed via a shape function of Lode angle (DP-MC).

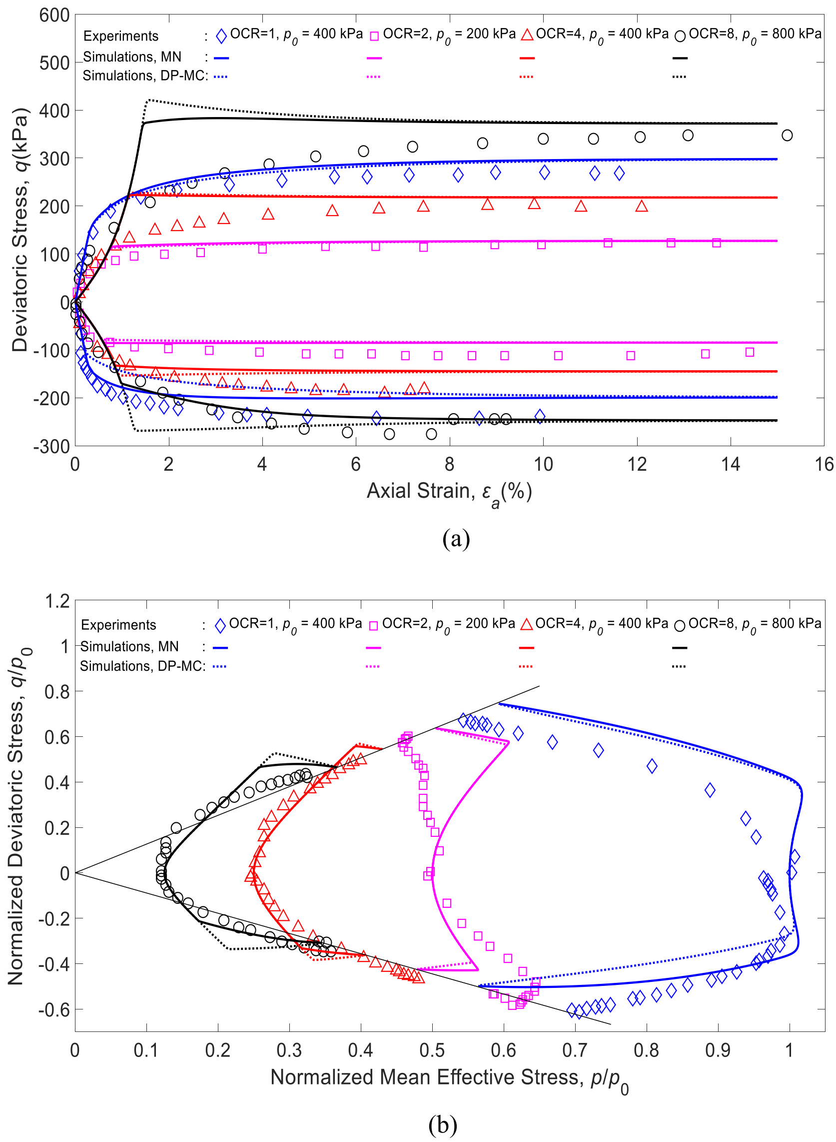
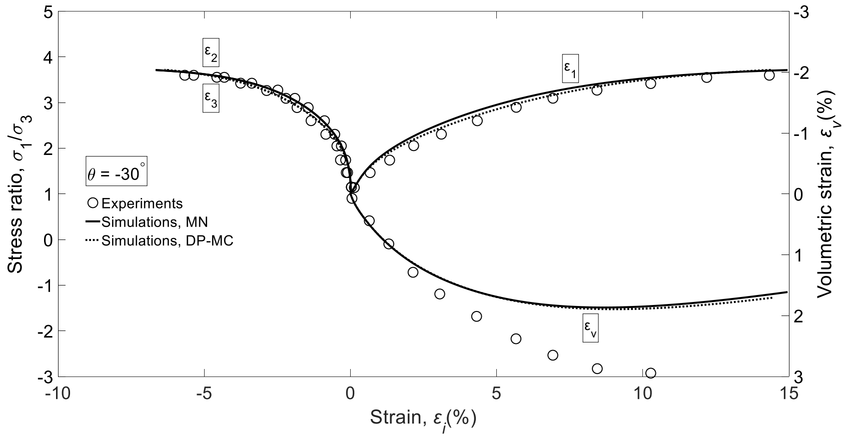
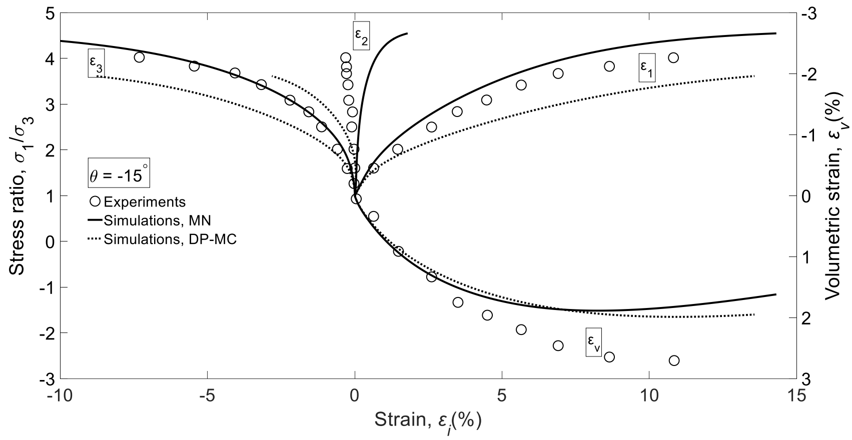


Figure 4-10: Comparison between experimental and simulated undrained triaxial compression tests under constant axial strain rate of 1.5%/h on reconstituted HKMD with different over-consolidation ratio (*OCR*) in terms of (a) stress-strain and (b) stress path responses. The numerical simulations have been done with frictional criteria of Matsuoka-Nakai (MN) and the conventional Mohr-Coulomb constructed via a shape function of Lode angle (DP-MC).

For further evaluation of the model performance, particularly the stress-induced anisotropic friction mobilization (according to equation (4-84)), the drained true triaxial tests with shear loadings under constant mean stress and different Lode angles on normally consolidated Fujinomori clay are simulated. The results are illustrated in Figure 4-11, which indicates that the hyper-visoplastic model with the MN friction criterion generally provides a better prediction.

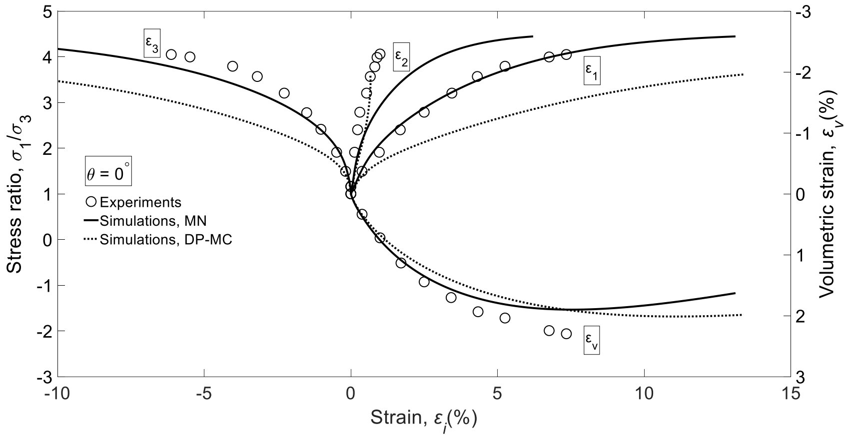


(a)

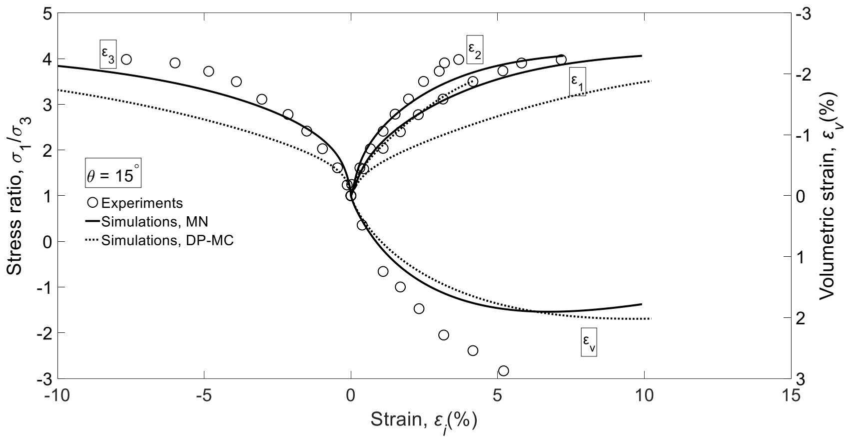


(b)

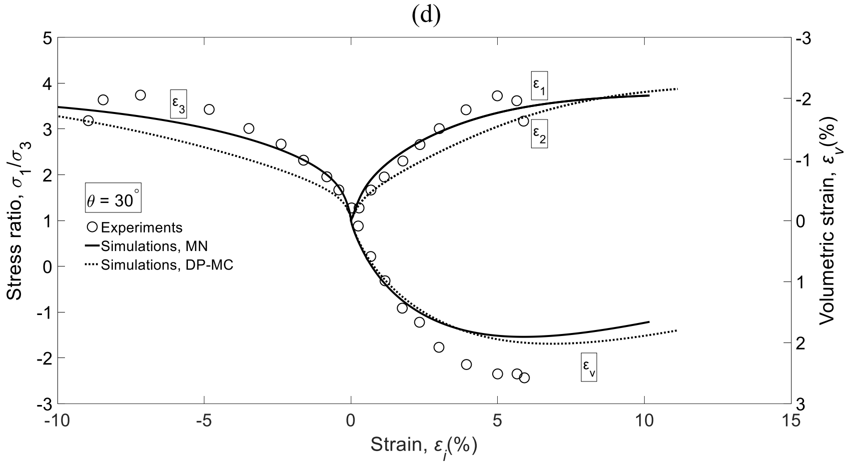
Figure 4-11: Comparison between experimental and simulated drained true triaxial tests on normally-consolidated and reconstituted Fujinomori clay under different Lode angles of (a) -30°, (b) -15°, (c) 0°, (d) 15°, and (e) 30°. The numerical simulations have been done with frictional criteria of Matsuoka-Nakai (MN) and the conventional Mohr-Coulomb constructed via a shape function of Lode angle (DP-MC).



(c)



(d)



(e)

Figure 4-11: Continued

Lastly, a drained triaxial extension test at an arbitrary high *OCR* is numerically conducted to illustrate the capability of the MN friction criterion in attaining the stress states with negative principal stresses. As shown in Figure 4-12, in the simulation with the MC friction, σ_3 becomes significantly negative, whereas, in the simulation with the MN friction, σ_3 is restricted to be strictly positive throughout the test.

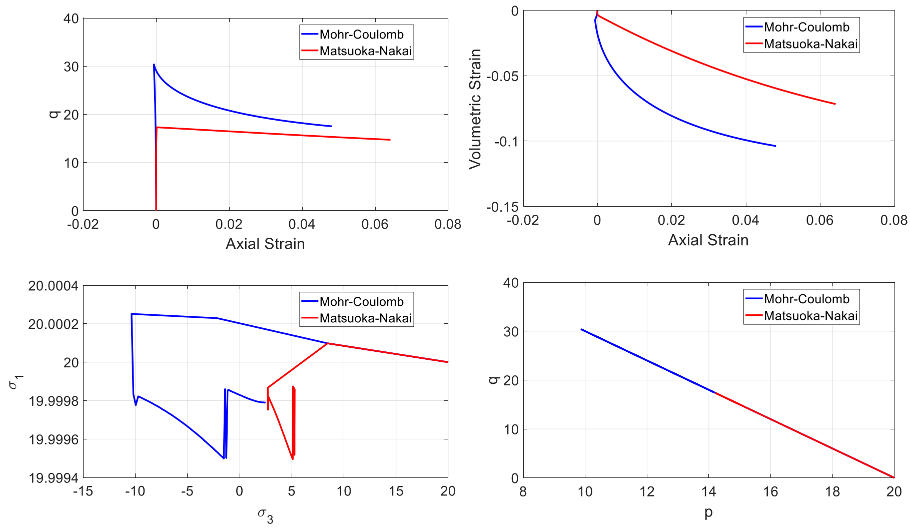


Figure 4-12: Comparison between performance of the proposed hyperplastic model with frictional criteria of the Matsuoka-Nakai (MN) and the conventional Mohr-Coulomb constructed via a shape function of Lode angle (DP-MC) in resctricting tensile principal stresses during a drained triaxial extension test.

Chapter 5 – Conclusions and recommendations

5.1 Introduction

Some general conclusions drawn from this doctoral research are presented in the following. This chapter ends with outlining some limitations of the developed hyper-viscoplastic model and some suggestions for future developments.

5.2 Conclusions

- Hyperplasticity has been previously demonstrated as a promising thermodynamically consistent approach for constitutive modelling of soil behaviour. As a further highlight, another useful feature of the hyperplasticity approach has been explored in this research (*Paper V*). Specifically, an integration algorithm directly based on the two potentials of free energy and dissipation (it can be pseudo-potential) has been proposed. Ziegler's orthogonality postulate is the kernel of this development as it makes hyperplasticity a potential-based approach confirming the maximal dissipation principle. The algorithm is computationally attractive and convenient since there is no need to establish an explicit expression for the yield or the plastic potential surfaces, which are essential elements in the plasticity theory. Notably, the idea of "plastic shooting" has been proposed that might help with numerical instabilities associated with the particular form of yield or plastic potential surfaces. For instance, as shown in *Paper IV*, the Matsuoka-Nakai friction criterion (Matsuoka and Nakai 1974) prevents tensile principal stresses, which are inconsistent with granular material without tensile capacities. However, according to the conventional implicit stress return algorithm, the trial stress state resulting from the "elastic shooting" can be in the tensile region where the Matsuoka-Nakai friction criterion is undefined, causing numerical instability.
- According to the current literature, the efficacy of the hyperplasticity theory for modelling the inviscid behaviour of clay has been explored extensively. Therefore, this doctoral study has been concerned with applying the hyperplasticity approach to accommodate the viscous effects of the clay behaviour. In particular, the hyperplastic version of the MCC model has been extended to the hyper-viscoplastic version by employing the isotache viscosity (*Paper I* and *III*). The resulting model is equivalent to the soft soil creep model proposed by Vermeer and Neher (2019). Along these lines, an important consequence of the orthogonality postulate of Ziegler (1977) is identified (*Paper I*, *IV*). It has been demonstrated that compliance with the uniqueness of the critical state friction envelope (the paradigm of the critical state soil

mechanics) rejects the purely plastic part of the free energy for a system with a single internal variable. Consequently, the formalised dissipation function (or force potential) is of the same structure as one of two alternative forms for MCC dissipation functions presented by Houlsby (2000). It should be noted that the non-unique hyperplastic expression of the MCC model brought Houlsby (2000) to conclude that energy functions are not objectively observable quantities.

- Inspired by the works of Collins and co-workers (Collins 2002, Collins and Hilder 2002, Collins and Kelly 2002), who repeatedly demonstrated the thermodynamically supported practice of the non-associated flow rule for the MCC model under the non-viscous condition, this doctoral research extends this particular property of frictional material to the viscous condition (*Paper III*). This has been achieved by recognising the homothetic functioning of the isotache viscosity and by the delicate practice of the stress ratio tensor (the deviatoric stress tensor per the mean or spherical effective stress measure) as an essential state variable of frictional material in the dissipation or the force potential. In this regard, it should be emphasised that the overstress theory of Perzyna (1963), which has been widely used for the viscoplastic modelling of soil behaviour, is based on Drucker's stability postulate (Olszak and Perzyna 1964) that restricts the inelastic viscous flow to be associative (normal to the dynamic yield surface). Moreover, the proposed generalisation of the dissipation or the force potential guarantees the uniqueness of the friction mobilisation at the critical state, which is the essential paradigm in the critical state soil mechanics for a unified description of the general mechanical behaviour of soil.
- Along the lines of the above, another condition for losing a unique critical state envelope is identified (*Papers II* and *IV*). This condition results from the direct use of the frictional mechanism proposed by Collins and Kelly (2002) in the force potential to accommodate the non-associated inelastic viscous flow. Grimstad et al. (2021) (*Papers II*) have exploited this condition to shed light on the controversial matter of evolution of the coefficient of earth pressure at rest during the creep. It is demonstrated that the coefficient of earth pressure at rest can be expressed by an increasing function of the creep strain in such a way that as the frictional dissipative mechanism intensifies (this comes with distortion of homothetic viscosity), the change of the coefficient of earth pressure at rest with creep increases.
- Lastly, different friction criteria of the Drucker-Prager, the Mohr-Coulomb, and the Matsuoka-Nakai have been integrated with the hyper-viscoplastic MCC model, and their effects on the friction mobilisation and the inelastic flow direction have been explored (*Paper IV*). It has been demonstrated that the model with the Matsuoka-Nakai criterion exhibits the stress ratio-induced anisotropy in both the friction mobilisation and inelastic flow direction. This has been

considered by an apt form for the measure of the dissipative shear strain rate that deviates from the stationary octahedral plane. This deviation is controlled by the stress ratio tensor ($\boldsymbol{\eta}$), which, based on several DEM studies (e.g., Radjai et al. (1998), Alonso-Marroquín et al. (2005), Antony and Kruyt (2009), and Radjai and Azéma (2009)), is an essential state variable of a particulate frictional system that represents the degree of the bimodal stress transmission through the development of the weak and the strong force chains. It has been numerously demonstrated that the strong force chains carry almost the entire deviatoric stress while the weak force chains contribute only to the mean stress. Thus, it can be deduced that as the stress ratio increases and the bimodal behaviour intensifies.

5.3 Recommendations

The proposed hyper-viscoplastic MCC model is basic (only the concept of time is integrated with hyperplastic MCC model). As a result, the potential for further development of the model is significant. Some of the directions for further development are as follows:

- An obvious direction for future research is evaluating the proposed integrative approach for hyper-(visco)plastic models. This evaluation can also be conducted on the efficacy of the proposed hyper-viscoplastic model in solving boundary value problems with viscous dominant effects.
- The isothermal condition is adopted in this research. However, it is well known that the viscous behaviour of soil, particularly clay, significantly depends on the temperature change. Considering the temperature effect on viscous behaviour would be a worthy study. Golchin et al. (2022) recently proposed a thermo-mechanical constitutive model for the inviscid behaviour of fine-grained soils using the hyperplasticity approach. Interestingly, pure plastic free energy is also rejected in this work.
- The system or the material in this research is assumed to be fully saturated or fully dry. However, in some cases, the actual condition is not even close to this idealisation. Most of the viscoplastic models in the literature based on different viscoplastic theories are also limited to this assumption. Therefore, consideration of the viscous effects for unsaturated soil would be a valuable contribution.
- In natural soil, predominantly clayey types, there is frequently a certain amount of bonding between the particles and peds. Moreover, mixing soil with materials such as cement, lime, gypsum, fly ash, etc., is an approach to improve the soil's mechanical properties. The natural bonding and the artificial cementation are sensitive and might degrade due to loading and

mobilised mechanisms. The effect typically manifests as softening forms in the stress-strain response, particularly for natural soils. A possible way to incorporate this effect by the hyperplasticty approach has been proposed by Yan and Li (2011). In this regard, one must be critical of the definition of the Mohr-Coulomb criterion and the meaning of different bonding strength components considered by Yan and Li (2011). Another possible method is using the framework proposed by Einav et al. (2007), which includes the damage mechanics.

- Several experimental studies on the effects of non-isotropic consolidation on the behaviour of clayey soils imply that the gross yield and plastic potential envelops distort. This effect is neglected in this work. The most straightforward choice for accommodating this effect can be adapting the dissipation function proposed by Dafalias (1986). However, there is no consensus regarding the evolution of the distortion (known as rotational hardening) due to the change of loading direction and approaching the critical state.
- The proposed basic model can be used to analyse quasi-instantaneous and delayed strain localisation-related problems. The application of the potential-based model in this area appears appealing as it covers the non-associativeness of the flow and the viscous effects, which are significant factors for strain localisation problems.
- This research is concerned with a rate-dependent system with a single internal variable. Due to the impact of Ziegler's orthogonality postulate on the uniqueness of the critical state envelope, the pure plastic free energy was rejected. However, the plastic free energy with multiple internal variables is promising in addressing the loading history and cyclic loading effects. To the author's knowledge, no viscoplastic model can cover these challenging aspects of soil behaviour, particularly clay, leaving aside the thermodynamic consistency. Puzrin and Houlsby (2001, 2003)'s generalisation of multiple internal variables through the incorporation of internal functions, which are practised by Houlsby (2000) and Einav and Puzrin (2003, 2004) for the rate-independent case, appears to be a viable way forward. However, the matter here is not as simple as it looks for the rate-independent case. In this regard, one should consider the meaning of creep, storage and release of the internal plastic energy, and impact of loading rate on the critical state if its uniqueness is believed to be useful.

References

Adachi, T. and F. Oka (1982). Constitutive equations for normally consolidated clay based on elasto-viscoplasticity. *Soils and Foundations* 22(4): 57-70.

Adachi, T., F. Oka, T. Hirata, T. Hashimoto, J. Nagaya, M. Mimura and T. B. S. Pradhan (1995). Stress-strain behavior and yielding characteristics of eastern osaka clay. *Soils and Foundations* 35(3): 1-13.

Almeida, M. and M. Marques (2003). The behaviour of sarapuí soft clay. *Characterization and engineering properties of natural soils*, Swets & Zeitlinger, Lisse, the Netherlands.

Alonso-Marroquín, F., S. Luding, H. J. Herrmann and I. Vardoulakis (2005). Role of anisotropy in the elastoplastic response of a polygonal packing. *Physical Review E* 71(5): 051304.

Antony, S. J. and N. P. Kruyt (2009). Role of interparticle friction and particle-scale elasticity in the shear-strength mechanism of three-dimensional granular media. *Physical Review E* 79(3): 031308.

Arulanandan, K., C. K. Shen and R. B. Young (1971). Undrained creep behaviour of a coastal organic silty clay. *Géotechnique* 21(4): 359-375.

Aung, Y., H. Khabbaz and B. Fatahi (2019). Mixed hardening hyper-viscoplasticity model for soils incorporating non-linear creep rate – h-creep model. *International Journal of Plasticity* 120: 88-114.

Bjerrum, L. (1954). Geotechnical properties of norwegian marine clays. *Géotechnique* 4(2): 49-69.

Bjerrum, L. (1967). Engineering geology of norwegian normally-consolidated marine clays as related to settlements of buildings. *Géotechnique* 17(2): 83-118.

Bodas Freitas, T. M., D. M. Potts and L. Zdravkovic (2011). A time dependent constitutive model for soils with isotach viscosity. *Computers and Geotechnics* 38(6): 809-820.

Buisman, A. (1936). Results of long duration settlement tests. *Proc. 1st ICSMFE*, Cambridge, Massachusetts, USA, Graduate School of Engineering, Harvard University.

Burland, J. B. (1990). On the compressibility and shear strength of natural clays. *Géotechnique* 40(3): 329-378.

Butterfield, R. (1979). A natural compression law for soils (an advance on e–log p′). *Géotechnique* 29(4): 469-480.

Callen, H. B. (1960). *Thermodynamics and an introduction to thermostatistics*.

Chen, Y. N. and Z. X. Yang (2017). A family of improved yield surfaces and their application in modeling of isotropically over-consolidated clays. *Computers and Geotechnics* 90: 133-143.

Collins, I. F. (2002). Associated and non-associated aspects of the constitutive laws for coupled elastic/plastic materials. *International Journal of Geomechanics* 2(2): 259-267.

Collins, I. F. (2003). A systematic procedure for constructing critical state models in three dimensions. *International Journal of Solids and Structures* 40(17): 4379-4397.

Collins, I. F. (2005a). Elastic/plastic models for soils and sands. *International Journal of Mechanical Sciences* 47(4): 493-508.

Collins, I. F. (2005b). The concept of stored plastic work or frozen elastic energy in soil mechanics. *Géotechnique* 55(5): 373-382.

Collins, I. F. and T. Hilder (2002). A theoretical framework for constructing elastic/plastic constitutive models of triaxial tests. *International Journal for Numerical and Analytical Methods in Geomechanics* 26(13): 1313-1347.

Collins, I. F. and G. T. Houlsby (1997). Application of thermomechanical principles to the modelling of geotechnical materials. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 453(1964): 1975-2001.

Collins, I. F. and P. A. Kelly (2002). A thermomechanical analysis of a family of soil models. *Géotechnique* 52(7): 507-518.

Coulomb, C. A. (1776). Essai sur une application des r`egles des maximis et minimis `a quelques problemes de statique relatifs `a l'architecture. *Mem. Acad. Roy. Pres. divers Sav* 5(7).

Dadras-Ajirloo, D., G. Grimstad and S. A. Ghoreishianamiri (2022). On the isotache viscous modelling of clay behaviour using the hyperplasticity approach. *Géotechnique* 0(0): 1-43.

Dafalias, Y. F. (1986). An anisotropic critical state soil plasticity model. *Mechanics Research Communications* 13(6): 341-347.

De Groot, S. and P. Mazur (1984). *Non-equilibrium thermodynamics*. New York, Dover.
Degago, S., G. Grimstad, H. Jostad and S. Nordal (2009). The non-uniqueness of the end-of-primary (eop) void ratio-effective stress relationship. *Proceedings of the 17th International Conference on Soil Mechanics and Geotechnical Engineering*, IOS Press.
Degago, S., S. Nordal, G. Grimstad and H. Jostad (2011). Analyses of väsby test fill according to creep hypothesis a and b. *13th international conference of the international association for computer methods and advances in geomechanics*, Melbourne, Australia.
Degago, S. A. (2011). *On creep during primary consolidation of clays*. PhD Thesis, Norwegian University of Science and Technology, Trondheim, Norway.
Degago, S. A., G. Grimstad, H. P. Jostad, S. Nordal and M. Olsson (2011). Use and misuse of the isotache concept with respect to creep hypotheses a and b. *Géotechnique* 61(10): 897-908.
Den Haan, E. J. (1996). A compression model for non-brittle soft clays and peat. *Géotechnique* 46(1): 1-16.
Drucker, D. C. (1959). A definition of stable inelastic material. *Journal of Applied Mechanics* 26(1): 101-106.
Drucker, D. C. and W. Prager (1952). Soil mechanics and plastic analysis or limit design. *Quarterly of applied mathematics* 10(2): 157-165.
Einav, I. (2012). The unification of hypo-plastic and elasto-plastic theories. *International Journal of Solids and Structures* 49(11): 1305-1315.
Einav, I., G. T. Houlsby and G. D. Nguyen (2007). Coupled damage and plasticity models derived from energy and dissipation potentials. *International Journal of Solids and Structures* 44(7): 2487-2508.
Einav, I. and A. M. Puzrin (2003). Evaluation of continuous hyperplastic critical state (chcs) model. *Géotechnique* 53(10): 901-913.
Einav, I. and A. M. Puzrin (2004). Continuous hyperplastic critical state (chcs) model: Derivation. *International Journal of Solids and Structures* 41(1): 199-226.
Feda, J. (1984). Ko-coefficient of sand in triaxial apparatus. *Journal of Geotechnical Engineering* 110(4): 519-524.
Feng, T.-W. (1991). *Compressibility and permeability of natural soft clays and surcharging to reduce settlements*. PhD Thesis, University of Illinois at Urbana-Champaign, Champaign, IL, USA.
Fodil, A., W. Aloulou and P. Y. Hicher (1997). Viscoplastic behaviour of soft clay??? *Géotechnique* 47(3): 581-591.
Fuentes, W., M. Tafili and T. Triantafyllidis (2018). An isa-plasticity-based model for viscous and non-viscous clays. *Acta Geotechnica* 13(2): 367-386.
Fung, Y. C. (1965). *Foundations of solid mechanics.* , Englewood Cliffs, NJ, Prentice-Hall.
Golchin, A., P. J. Vardon and M. A. Hicks (2022). A thermo-mechanical constitutive model for fine-grained soils based on thermodynamics. *International Journal of Engineering Science* 174: 103579.
Graham, J., J. H. A. Crooks and A. L. Bell (1983). Time effects on the stress-strain behaviour of natural soft clays. *Géotechnique* 33(3): 327-340.
Gray, H. (1936). Progress report on research on the consolidation of fine-grained soils. *Proceedings, First International Conference on Soil Mechanics and Foundation Engineering*, Cambridge Massachusetts, USA, Graduate School of Engineering, Harvard University.
Grimstad, G., D. Dadrasajirlou and S. a. G. Amiri (2020). Modelling creep in clay using the framework of hyper-viscoplasticity. *Géotechnique Letters* 10(3): 404-408.
Grimstad, G., S. A. Degago, S. Nordal and M. Karstunen (2010). Modeling creep and rate effects in structured anisotropic soft clays. *Acta Geotechnica* 5(1): 69-81.
Grimstad, G., M. Long, D. Dadrasajirlou and S. a. G. Amiri (2021). Investigation of development of the earth pressure coefficient at rest in clay during creep in the framework of hyper-viscoplasticity. *International Journal of Geomechanics* 21(1): 04020235.
Gudehus, G. (2004). A visco-hypoplastic constitutive relation for soft soils. *Soils and Foundations* 44(4): 11-25.
Gudehus, G. (2011). *Physical soil mechanics*, Springer Berlin, Heidelberg.
Hashiguchi, K. (1995). On the linear relations of v–ln p and ln v–ln p for isotropic consolidation of soils. *International Journal for Numerical and Analytical Methods in Geomechanics* 19(5): 367-376.

Hight, D. W., R. J. Jardine and A. Gens (1987). *The behaviour of soft clays. Embankments on soft clays*. Athens, Public Works Research Centre**:** 33-158.
Hinchberger, S. D. and R. K. Rowe (2005). Evaluation of the predictive ability of two elastic-viscoplastic constitutive models. *Canadian Geotechnical Journal* 42(6): 1675-1694.
Holtz, R. D. and M. B. Jamiolkowski (1985). Discussion of time dependence of lateral earth pressure. *Journal of Geotechnical Engineering* 111(10): 1239-1242.
Houlsby, G. (2000). Critical state models and small-strain stiffness. *Developments in Theoretical Geomechanics. Proceedings of the Booker Memorial Symposium*, Sydney, N.S.W., Rotterdam, Netherlands ; Brookfiels, VT : A.A. Balkema.
Houlsby, G. and A. Puzrin (2006). Principles of hyperplasticity: An approach to plasticity theory based on thermodynamic principles. London, *Springer Verlag***:** 375.
Houlsby, G. T. (1981). *Study of plasticity theories and their applicability to soils*. PhD Thesis, University of Cambridge, Cambridge, UK.
Houlsby, G. T., A. Amorosi and E. Rojas (2005). Elastic moduli of soils dependent on pressure: A hyperelastic formulation. *Géotechnique* 55(5): 383-392.
Houlsby, G. T. and A. M. Puzrin (2002). Rate-dependent plasticity models derived from potential functions. *Journal of Rheology* 46(1): 113-126.
Jacky, J. (1944). The coefficient of earth pressure at rest. In hungarian (a nyugalmi nyomas tenyezoje). *J. of the Society of Hungarian Architects and Engineers (Magyar Mernok es Epitesz-Egylet Kozlonye)* 78: 355-358.
Jacquey, A. B. and K. Regenauer-Lieb (2021). Thermomechanics for geological, civil engineering and geodynamic applications: Rate-dependent critical state line models. *Rock Mechanics and Rock Engineering* 54(10): 5355-5373.
Jamiolkowski, M., C. Ladd, J. Germaine and R. Lancellotta (1985). New developments in field and laboratory testing of soils. *International conference on soil mechanics and foundation engineering. 11*.
Janbu, N. (1969). The resistance concept applied to deformations of soils. *Proceedings of the 7th international conference on soil mechanics and foundation engineering, Mexico city*.
Janbu, N. (1985). Soil models in offshore engineering. *Géotechnique* 35(3): 241-281.
Jerman, J. and D. Mašín (2020). Hypoplastic and viscohypoplastic models for soft clays with strength anisotropy. *International Journal for Numerical and Analytical Methods in Geomechanics* 44(10): 1396-1416.
Kavazanjian, E. and J. K. Mitchell (1984). Time dependence of lateral earth pressure. *Journal of Geotechnical Engineering* 110(4): 530-533.
Kim, Y. T. and S. Leroueil (2001). Modeling the viscoplastic behaviour of clays during consolidation: Application to berthierville clay in both laboratory and field conditions. *Canadian Geotechnical Journal* 38(3): 484-497.
Kirkgard, M. M. and P. V. Lade (1993). Anisotropic three-dimensional behavior of a normally consolidated clay. *Canadian Geotechnical Journal* 30(5): 848-858.
Kolymbas, D. (2000). *The misery of constitutive modelling*. Constitutive modelling of granular materials. D. Kolymbas. Berlin, Heidelberg, Springer Berlin Heidelberg**:** 11-24.
Kulhawy, F. H. and P. W. Mayne (1990). Manual on estimating soil properties for foundation design, *Electric Power Research Inst., Palo Alto, CA (USA); Cornell Univ., Ithaca ...*.
Kutter, B. L. and N. Sathialingam (1992). Elastic-viscoplastic modelling of the rate-dependent behaviour of clays. *Géotechnique* 42(3): 427-441.
Ladd, C., R. Foot, K. Ishihara, F. Schlosser and H. Poulos (1977). Stress-deformation and strength characteristics. State of the art report. Proc. 9th Int. Conf. Soil Mech. Found. Engng Tokyo. 2**:** 421-494.
Lade, P. V. and J. M. Duncan (1975). Elastoplastic stress-strain theory for cohesionless soil. *Journal of the Geotechnical Engineering Division* 101(10): 1037-1053.
Lade, P. V. and H. M. Musante (1978). Three-dimensional behavior of remolded clay. *Journal of the Geotechnical Engineering Division* 104(2): 193-209.
Lambe, T. W. and R. V. Whitman (1991). *Soil mechanics*, John Wiley & Sons.
Leroueil, S. and M. Kabbaj (1987). Discussion of settlement analysis of embankments on soft clays. *Journal of Geotechnical Engineering* 113(9): 1067-1070.

Leroueil, S., M. Kabbaj, F. Tavenas and R. Bouchard (1985). Stress–strain–strain rate relation for the compressibility of sensitive natural clays. *Géotechnique* 35(2): 159-180.
Leroueil, S., D. Perret and J. Locat (1996). Strain rate and structuring effects on the compressibility of a young clay. *Measuring and modeling time dependent soil behavior*, ASCE.
Liingaard, M., A. Augustesen and P. V. Lade (2004). Characterization of models for time-dependent behavior of soils. *International Journal of Geomechanics* 4(3): 157-177.
Matsuoka, H. (1976). On the significance of the "spatial mobilized plane&quot. *Soils and Foundations* 16(1): 91-100.
Matsuoka, H. and T. Nakai (1974). Stress-deformation and strength characteristics of soil under three different principal stresses. *Proceedings of the Japan Society of Civil Engineers* 1974(232): 59-70.
Maugin, G. A. (1999). *The thermomechanics of nonlinear irreversible behaviors*, World Science Series on Nonlinear Science.
Mesri, G. and A. Castro (1987). Cα/cc concept and k0 during secondary compression. *Journal of Geotechnical Engineering* 113(3): 230-247.
Mesri, G. and Y. Choi (1985). The uniqueness of the end of primary void ratio- effective stress relationship. . Proceedings of the 11th ICSMFE, San Francisco, 12-16 AUGUST 1985. 2**:** 587-590.
Mesri, G. and Y. K. Choi (1985). Settlement analysis of embankments on soft clays. *Journal of Geotechnical Engineering* 111(4): 441-464.
Mesri, G. and P. M. Godlewski (1977). Time- and stress-compressibility interrelationship. *Journal of the Geotechnical Engineering Division* 103(5): 417-430.
Mesri, G. and P. M. Godlewski (1979). Closure to time and stress-compressibility interrelationship. *Journal of the Geotechnical Engineering Division* 105(1): 106-113.
Mesri, G. and T. M. Hayat (1993). The coefficient of earth pressure at rest. *Canadian Geotechnical Journal* 30(4): 647-666.
Mesri, G., D. O. K. Lo and T. W. Feng (1994). Settlement of embankments on soft clays. *Proceedings of the Conference on Vertical and Horizontal Deformations of Foundations and Embankments. Part 2 (of 2)*, Publ by ASCE.
Mesri, G., C. R. Ullrich and Y. K. Choi (1978). The rate of swelling of overconsolidated clays subjected to unloading. *Géotechnique* 28(3): 281-307.
Michalowski, R. L. (2005). Coefficient of earth pressure at rest. *Journal of Geotechnical and Geoenvironmental Engineering* 131(11): 1429-1433.
Mitchell, J. K., R. G. Campanella and A. Singh (1968). Soil creep as a rate process. *Journal of the Soil Mechanics and Foundations Division* 94(1): 231-253.
Mitchell, J. K. and K. Soga (2005). *Fundamentals of soil behavior*, John Wiley & Sons New York.
Mohr, O. (1887). Ueber die bestimmung und die graphische darstellung von trägheitsmomenten ebener flächen. *Zivilingenieur* 33: 43-68.
Murakami, Y. (1979). Excess pore-water pressure and preconsolidation effect developed in normally consolidated clays of some age. *Soils and Foundations* 19(4): 17-29.
Naghdi, P. M. and S. A. Murch (1963). On the mechanical behavior of viscoelastic/plastic solids. *Journal of Applied Mechanics* 30(3): 321-328.
Nakai, T. and H. Matsuoka (1986). A generalized elastoplastic constitutive model for clay in three-dimensional stresses. *Soils and Foundations* 26(3): 81-98.
Nakai, T., H. Matsuoka, N. Okuno and K. Tsuzuki (1986). True triaxial tests on normally consolidated clay and analysis of the observed shear behavior using elastoplastic constitutive models. *Soils and Foundations* 26(4): 67-78.
Nash, D. and M. Brown (2015). Influence of destructuration of soft clay on time-dependent settlements: Comparison of some elastic viscoplastic models. *International Journal of Geomechanics* 15(5): A4014004.
Niemunis, A. (1996). A visco-plastic model for clay and its fe-implementation. *Resultats Recents en Mechanique des Sols et des Roches*: 151-162.
Niemunis, A. (2003). *Extended hypoplastic models for soils*, Inst. für Grundbau und Bodenmechanik Vienna.

Niemunis, A., C. E. Grandas-Tavera and L. F. Prada-Sarmiento (2009). Anisotropic visco-hypoplasticity. *Acta Geotechnica* 4(4): 293.
Nixon, S. A. (1999). *Yield functions suitable for granular materials*. University of Aberdeen (UK)
Olszak, W. and P. Perzyna (1964). On elastic/visco-plastic soils. *IUTAM Symp."Rheology and Soil Mechanics*, Grenoble, Springer-Verlag.
Onsager, L. (1931). Reciprocal relations in irreversible processes. Ii. *Physical Review* 38(12): 2265-2279.
Panteghini, A. and R. Lagioia (2018). An approach for providing quasi-convexity to yield functions and a generalized implicit integration scheme for isotropic constitutive models based on 2 unknowns. *International Journal for Numerical and Analytical Methods in Geomechanics* 42(6): 829-855.
Perzyna, P. (1963). The constitutive equations for rate sensitive plastic materials. *Quarterly of applied mathematics* 20(4): 321-332.
Prashant, A. and D. Penumadu (2004). Effect of intermediate principal stress on overconsolidated kaolin clay. *Journal of Geotechnical and Geoenvironmental Engineering* 130(3): 284-292.
Prashant, A. and D. Penumadu (2005). A laboratory study of normally consolidated kaolin clay. *Canadian Geotechnical Journal* 42(1): 27-37.
Puzrin, A. M. and G. T. Houlsby (2001). A thermomechanical framework for rate-independent dissipative materials with internal functions. *International Journal of Plasticity* 17(8): 1147-1165.
Puzrin, A. M. and G. T. Houlsby (2003). Rate-dependent hyperplasticity with internal functions. *Journal of Engineering Mechanics* 129(3): 252-263.
Qiao, Y., A. Ferrari, L. Laloui and W. Ding (2016). Nonstationary flow surface theory for modeling the viscoplastic behaviors of soils. *Computers and Geotechnics* 76: 105-119.
Qu, G., S. D. Hinchberger and K. Y. Lo (2010). Evaluation of the viscous behaviour of clay using generalised overstress viscoplastic theory. *Géotechnique* 60(10): 777-789.
Quinn, T. and M. Brown (2011). Effect of strain rate on isotropically consolidated kaolin over a wide range of strain rates in the triaxial apparatus. *Fifth International Symposium on Deformation Characteristics of Geomaterials*, IOS Press.
Radjai, F. and E. Azéma (2009). Shear strength of granular materials. *European Journal of Environmental and Civil Engineering* 13(2): 203-218.
Radjai, F., D. E. Wolf, M. Jean and J.-J. Moreau (1998). Bimodal character of stress transmission in granular packings. *Physical Review Letters* 80(1): 61-64.
Rollo, F. and A. Amorosi (2020). Saniclay-t: Simple thermodynamic-based anisotropic plasticity model for clays. *Computers and Geotechnics* 127: 103770.
Roscoe, K. H. and J. Burland (1968). On the generalized stress-strain behaviour of wet clay. *Engineering Plasticity*, Cambridge, Cambridge University Press, New York.
Roscoe, K. H., A. N. Schofield and C. P. Wroth (1958). On the yielding of soils. *Géotechnique* 8(1): 22-53.
Sällfors, G. (1975). *Preconsolidation pressure of soft, high plasticity clays*. PhD Thesis, Chalmers University of Technology, Göteberg, Sweden.
Schmertmann, J. H. (1983). A simple question about consolidation. *Journal of Geotechnical Engineering* 109(1): 119-122.
Schmidt, B. (1966). Earth pressures at rest related to stress history. *Canadian Geotechnical Journal* 3(4): 239-242.
Schofield, A. N. and P. Wroth (1968). *Critical state soil mechanics*. New York, McGraw-Hill
Sekiguchi, H. (1984). Theory of undrained creep rupture of normally consolidated clay based on elasto-viscoplasticity. *Soils and Foundations* 24(1): 129-147.
Sewell, M. J. (1969). On dual approximation principles and optimization in continuum mechanics. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* 265(1162): 319-351.
Sheahan, T. C., C. C. Ladd and J. T. Germaine (1996). Rate-dependent undrained shear behavior of saturated clay. *Journal of Geotechnical Engineering* 122(2): 99-108.
Simo, J. C. and T. J. Hughes (1998). *Computational inelasticity*, Springer Science & Business Media.

Sivasithamparam, N., M. Karstunen and P. Bonnier (2015). Modelling creep behaviour of anisotropic soft soils. *Computers and Geotechnics* 69: 46-57.
Srinivasa, A. R. (2010). Application of the maximum rate of dissipation criterion to dilatant, pressure dependent plasticity models. *International Journal of Engineering Science* 48(11): 1590-1603.
Šuklje, L. (1957). The analysis of the consolidation process by the isotache method. *Proceedings of the 4th International Conference on Soil Mechanics and Foundation Engineering*, London, UK.
Tafili, M., T. Wichtmann and T. Triantafyllidis (2021). Experimental investigation and constitutive modeling of the behaviour of highly plastic lower rhine clay under monotonic and cyclic loading. *Canadian Geotechnical Journal* 58(9): 1396-1410.
Tatsuoka, F., M. Ishihara, B. Herve Di and R. Kuwano (2002). Time-dependent shear deformation characteristics of geomaterials and their simulation. *Soils and Foundations* 42(2): 103-129.
Terzaghi, K. (1925). *Erdbaumechanik auf bodenphysikalischer grundlage*. Vienna, Franz Deuticke.
Tsegaye, A. B. (2021). The coefficient of earth pressure at rest for soils--a theoretical insight. *arXiv preprint arXiv:2111.02956*.
Tsegaye, A. B. and T. Benz (2014). Plastic flow and state-dilatancy for geomaterials. *Acta Geotechnica* 9(2): 329-342.
Vaid, Y. P. and R. G. Campanella (1977). Time-dependent behavior of undisturbed clay. *Journal of the Geotechnical Engineering Division* 103(7): 693-709.
Van Eekelen, H. a. M. (1980). Isotropic yield surfaces in three dimensions for use in soil mechanics. *International Journal for Numerical and Analytical Methods in Geomechanics* 4(1): 89-101.
Vergote, T. A., C. F. Leung and S. C. Chian (2022). Modelling creep and swelling after unloading under constant load and relaxation with bayesian updating. *Géotechnique* 72(6): 496-509.
Vermeer, P. and H. Neher (2019). *A soft soil model that accounts for creep*. Beyond 2000 in computational geotechnics. R. B. J. Brinkgreve, Routledge**:** 249-261.
Von Mises, R. (1913). Mechanik der festen körper im plastisch-deformablen zustand. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1913: 582-592.
Wu, W. and D. Kolymbas (2000). *Hypoplasticity then and now*. Constitutive modelling of granular materials. D. Kolymbas. Berlin, Heidelberg, Springer Berlin Heidelberg**:** 57-105.
Yan, W. M. and X. S. Li (2011). A model for natural soil with bonds. *Géotechnique* 61(2): 95-106.
Yao, Y.-P., L.-M. Kong, A.-N. Zhou and J.-H. Yin (2015). Time-dependent unified hardening model: Three-dimensional elastoviscoplastic constitutive model for clays. *Journal of Engineering Mechanics* 141(6): 04014162.
Ye, G.-L., B. Ye and F. Zhang (2014). Strength and dilatancy of overconsolidated clays in drained true triaxial tests. *Journal of Geotechnical and Geoenvironmental Engineering* 140(4): 06013006.
Yin, J.-H. and J.-G. Zhu (1999). Measured and predicted time-dependent stress-strain behaviour of hong kong marine deposits. *Canadian Geotechnical Journal* 36(4): 760-766.
Yong, R. N. and E. Mckyes (1971). Yield and failure of a clay under triaxial stresses. *Journal of the Soil Mechanics and Foundations Division* 97(1): 159-176.
Yoshikuni, H., H. Nishiumi, S. Ikegami and K. Seto (1994). The creep and effective-stress-relaxation behavior on one-dimensional consolidation. *Proceedings of the 29th Japan National Conference on Soil Mechanics and Foundation Engineering*.
Zhang, Z., Y. Chen and Z. Huang (2018). A novel constitutive model for geomaterials in hyperplasticity. *Computers and Geotechnics* 98: 102-113.
Zhu, J.-G. (2000). *Experimental study and elastic visco-plastic modelling of the time-dependent stress-strain behaviour of hong kong marine deposits*. PhD Thesis, Hong Kong Polytechnic University, Hong Kong.
Ziegler, H. (1958). An attempt to generalize onsager's principle, and its significance for rheological problems. *Zeitschrift für angewandte Mathematik und Physik ZAMP* 9(5): 748-763.
Ziegler, H. (1977). *An introduction to thermodynamics*. Amsterdam, North-Holland
Ziegler, H. (1983). *An introduction to thermomechanics*. Amsterdam, North-Holland.
Zytynski, M., M. F. Randolph, R. Nova and C. P. Wroth (1978). On modelling the unloading-reloading behaviour of soils. *International Journal for Numerical and Analytical Methods in Geomechanics* 2(1): 87-93.

Appendix- Collection of journal articles

Modelling creep in clay using the framework of hyper-viscoplasticity

published in *Géotechnique Letters*

Paper I

This paper is not included due to copyright restrictions available at publisher site
https://doi.org/10.1680/jgele.20.00004 and
NTNU Open
https://hdl.handle.net/11250/2725862

Investigation of Development of the Earth Pressure Coefficient at Rest in Clay During Creep in the Framework of Hyper-Viscoplasticity

published in *International Journal of Geomechanics*

Paper II

On the isotache viscous modelling of clay behaviour using the hyperplasticity approach

published in *Géotechnique*

Paper III

This paper is not included due to copyright restrictions available at publisher site
https://doi.org/10.1680/jgeot.21.00245

On the isotache viscous modelling of clay behaviour using the hyperplasticity approach

Submitted to *International Journal of Solids and Structures*

PaperIV

This paper is awaiting publication and is not included in NTNU Open

On the implementation of hyperplastic models without establishing a yield surface

Submitted to *International Journal for Numerical and Analytical Methods in Geomechanics*

PaperV

This paper is awaiting publication and is not included in NTNU Open

ISBN 978-82-326-5919-7 (printed ver.)
ISBN 978-82-326-6999-8 (electronic ver.)
ISSN 1503-8181 (printed ver.)
ISSN 2703-8084 (online ver.)

NTNU

Norwegian University of
Science and Technology