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A Heuristic Observer Design for an Uncertain Hyperbolic PDE using Distributed Sensing

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Abstract: We design an adaptive observer for semi-linear 2×2 hyperbolic PDEs with parametric uncertainties in both state equations. The proposed method is an extension of a previous result where parametric uncertainties were only allowed in one of the system equation. We utilize partial state measurements of one of the distributed states to estimate the remaining unknown distributed state. The method can be applied to flow rate estimation in fluid flow systems where the pressure is measured.

Keywords: Distributed parameter systems, State estimation, Parameter estimation, Adaptive systems, Fluid flow estimation

1. INTRODUCTION

1.1 Problem formulation

We consider semi-linear 2×2 hyperbolic systems on the form

$$y_t(x,t) + az_x(x,t) = \phi_1^T(y(x,t),x)\theta_1 \tag{1a}$$

$$z_t(x,t) + by_x(x,t) = \phi_2^T(z(x,t),x)\theta_2. \tag{1b}$$

where $a, b \in \mathbb{R}$ and $\phi_1 : \mathbb{R} \times [0,1] \to \mathbb{R}^p, \phi_2 : \mathbb{R} \times [0,1] \to \mathbb{R}^q$ are known and $\theta_1 \in \mathbb{R}^p, \theta_2 \in \mathbb{R}^q$ are unknown constants with $p, q \in \mathbb{N}$. The distributed state $y : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ is assumed measured for all $x \in [0,1]$ while $z : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ is assumed unknown for $x \in (0,1)$, but we assume that both $z(0,t)$ and $z(1,t)$ are measured. In addition, we assume the following.

Assumption 1. System (3) with appropriate boundary and initial conditions has a unique bounded solution $(y(\cdot,t), z(\cdot,t)) \in L_2([0,1])$ for all $t \geq 0$.

Assumption 2. $||y|| \in \mathcal{L}_\infty \Rightarrow ||\phi_1(y,\cdot)|| \in \mathcal{L}_\infty, ||z|| \in \mathcal{L}_\infty \Rightarrow ||\phi_2(z,\cdot)|| \in \mathcal{L}_\infty$, and ϕ_2 satisfies the sector condition

$$(\phi_2^T(z_1,x) - \phi_2^T(z_2,x))\theta_2(z_1 - z_2) \leq 0. \tag{2}$$

for any $z_1, z_2 \in L_2([0,1])$

The goal is to estimate the unknown state $z(x)$ as well as the unknown parameters θ_1, θ_2.

The method presented in this paper can be extended to general hyperbolic systems with $(y(x), z(x)) \in \mathbb{R}^{n+m}$ for $m, n \in \mathbb{N}$ and any coefficient matrix having distinct real eigenvalues. However to simplify the presentation we let $m = n = 1$ and only require $ab > 0$, which implies that (1) is strictly hyperbolic.

1.2 Motivation and previous work

The system (1) can be used to model single-phase fluid flow systems (among others, see Bastin and Coron (2016)) and is derived by considering the mass and momentum balances in an open fluid system. In the following we will therefore refer to (1a) as the *mass balance* and (1b) as the *momentum balance*. An example is oil & gas drilling where a drilling fluid called mud is circulated down the hollow drill-string, through the drilling bit down-hole and up in the annulus surrounding the drill-string all the way to the top of the well. The fluid is used to carry cuttings to the top and provide pressure control in the well. Inadequate pressure control might lead to uncontrolled flows of fluid to or from the surrounding oil or gas reservoir. A reservoir pressure exceeding the well pressure leading to a flow of oil or gas into the well, called a *kick*, might have severe consequences if the reservoir fluids reach the surface. The opposite situation where the well pressure exceeds the reservoir pressure by a sufficiently high margin and the drilling fluid flows into the reservoir, which is called a *loss*, is also undesirable as the integrity of the reservoir might weaken, and the pressure drop caused by a loss might lead to a subsequent kick.

Due to the long length of the well which can be up to $10\,\mathrm{km}$, and even though the sound of speed for a typical drilling fluid can be as high as $1000\,\mathrm{m\,s^{-1}}$, the distributed effects caused by the compressibility of the fluid is sometimes significant and should not be neglected (Berg et al., 2019; Landet et al., 2013). In this paper, we utilize the information in the fast traveling pressure waves to estimate unknown states and parameters in a general PDE model. We assume that part of the sate vector is known and design an adaptive observer to estimate the remaining state. The method developed in this paper is an extension of Holta and Aamo (2019) where we only consider uncertainties in the momentum balance of a 2×2 semi-linear hyperbolic system, and not in the mass balance. The method in Holta and Aamo (2019) is an extension to PDEs of the method developed for non-linear ODEs in Stamnes et al. (2008, 2009) where stability of the observer design is proved by assuming that the non-linearities satisfy a sector condition similar to the condition proposed in Arcak and Kokotovic (1999). Utilizing this special structure avoids the use of canonical transformations (see Marino and Tomei (1992)) which requires that the system is *persistently*

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excited (PE) (Marino and Tomei, 1995). For non-linearities that satisfy Lipschitz conditions, another approach is to use *high-gain* to dominate the non-linearities. See e.g. Besanon et al. (2004) for ODEs and the recent result in Kitsos et al. (2018) for hyperbolic PDEs.

In the drilling system, a kick or loss is by definition an unexpected event caused by inadequate knowledge about system states and properties. In particular, the reservoir pressure and the flow rate at any single point in the well is often unknown. However, using so-called wired-pipe technology where the pressure inside the well is measured, and under a certain excitation criterion, both flow rate and the properties of the reservoir can be estimated . A local inflow of fluid from the reservoir into the well will likely result in an increase in the local frictional pressure drop, and a local loss of fluids from the well into the reservoir will likely lead to a decrease in friction. So by adapting the observer by estimating local frictional coefficients we can both detect and locate kicks or losses. However, a local increase in frictional momentum loss might also be caused by a pack-off of cuttings, or a wash-out between the drill-string and the annulus. To classify an event as a kick or loss we also need to model in- or out-flow of mass from and to the reservoir and acknowledge that parameters governing the mass balance are dependent on the reservoir properties and therefore uncertain. As the method in Holta and Aamo (2019) assumes that all parameters in the mass balance are perfectly known, the method cansnot be applied to distinguish between a in- or out-flow and other incidents. However, in this paper we show that if additional flow rate measurements at the top-side boundary are available, a simple parametric model can be used to estimate uncertainties in the mass balance, thus significantly increasing the applicability of the method first proposed in Holta and Aamo (2019).

Remark 1. Strictly speaking, the system described above with local inflows might require a model where the uncertain parameters are spatially varying. That is,

$$y_t(x) + az_x(x) = \varphi_1(y(x), x)\vartheta_1(x) \tag{3a}$$

$$z_t(x) + by_x(x) = \varphi_2(z(x), x)\vartheta_2(x) \tag{3b}$$

for some uncertain functions $\vartheta_1, \vartheta_2 : [0, 1] \to \mathbb{R}$. However, in most practical applications ϑ_1, ϑ_2 will be piece-wise constant (for example due to a geological fault) and we can define

$$\phi_1^i(y(x), x) = \chi_i(x)\varphi(y(x), x) \tag{4}$$

where ϕ_1^i is an element in ϕ_1 and $\chi_i(x) = 1$ in some subset of $[0, 1]$ and zero otherwise,

$$\theta_i = \chi_i\theta(x_i) \tag{5}$$

for any x_i such that $\chi_i(x_i) = 1$, and similarly for ϕ_2. However, from a mathematical point-of-view the method can straight forwardly be extended to handle spatially varying uncertainties $\theta_2(x)$. That being said, due to implementational concerns regarding robustness and to keep the presentation comparable to Holta and Aamo (2019) we keep the formulation in (1).

1.3 Notation

We avoid arguments in time and write e.g. $y(x)$ for a variable $y : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+$ denotes the set of non-negative real numbers. For $f : \mathbb{R}^+ \to \mathbb{R}$, we use the spaces

$$f \in \mathcal{L}_p \leftrightarrow \left(\int_0^\infty |f(t)|^p dt\right)^{\frac{1}{p}} < \infty \tag{6}$$

for $p \geq 1$ with the particular case $f \in \mathcal{L}_\infty \leftrightarrow \sup_{t\geq 0} |f(t)| < \infty$. A function $u : [0, 1] \to \mathbb{R}$ is said to be in $L_2([0, 1])$ if

$$||u|| := \sqrt{\int_0^1 u^2(x)dx} < \infty. \tag{7}$$

The partial derivative of a function is denoted with a subscript, for example $u_t(x, t) = \frac{\partial}{\partial t}u(x, t)$. For a function of one variable, the derivative is denoted using a prime, that is $f'(x) = \frac{d}{dx}f(x)$. The dot notation is reserved for the derivative of functions of time only; $\dot{f}(t) = \frac{d}{dt}f(t)$.

An operator $\Xi : L_2(0, 1) \to \mathbb{R}$ is called *Fréchet differentiable* at $u \in L_2([0, 1])$ if there exists a bounded linear operator $D_u\Xi : L_2([0, 1]) \to \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{|\Xi[u + h] - \Xi[u] - D_u\Xi[h]|}{||h||} = 0 \tag{8}$$

for $h \in L_2([0, 1])$. If such a bounded linear operator exists, it is unique and we call $D_u\Xi$ the *Fréchet derivative* of Ξ at u.

2. OBSERVER DESIGN

Let $\zeta(x) = ly(x) + z(x)$ for some l such that $\lambda := la > 0$. We have

$$\begin{aligned}\zeta_t(x) + \lambda\zeta_x(x) =&(l^2a - b)y_x(x) \\ &+ l\phi_1^T(y(x), x)\theta_1 + \phi_2^T(z(x), x)\theta_2 \qquad (9)\\ \zeta(0) =&ly(0) + z(0) \qquad (10)\end{aligned}$$

To estimate the unknown state ζ, consider the observer

$$\begin{aligned}\hat{\zeta}_t(x) + \lambda\hat{\zeta}_x(x) =&(l^2a - b)y_x(x) \\ &+ l\phi_1^T(y(x), x)\hat{\theta}_1 + \phi_2^T(\hat{z}(x), x)\hat{\theta}_2 \qquad (11a)\\ \hat{\zeta}(0) =&\zeta(0) \qquad (11b)\end{aligned}$$

where $\hat{z}(x) = \hat{\zeta}(x) - ly(x)$ and $\hat{\theta}_1, \hat{\theta}_2$ are estimates of θ_1, θ_2. The error dynamics $\tilde{\zeta}(x) = \zeta(x) - \hat{\zeta}(x)$ then satisfies

$$\begin{aligned}\tilde{\zeta}_t(x) + \lambda\tilde{\zeta}_x(x) =&l\phi_1^T(y(x), x)\tilde{\theta}_1 + \phi_2^T(\hat{z}(x), x)\tilde{\theta}_2 \\ &+ \tilde{\phi}_2^T(z(x), \hat{z}(x), x)\theta_2 \qquad (12a)\\ \tilde{\zeta}(0) =& 0 \qquad (12b)\end{aligned}$$

where $\tilde{\phi}_2(z(x), \hat{z}(x), x) := \phi(z(x), x) - \phi(\hat{z}(x), x)$ and $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$, $i = 1, 2$.

The adaptive law generating $\hat{\theta}_1$ is derived in Section 2.1. Stability of the error system (12), which is the main result (Proposition 2), is proved in Section 2.2 under the assumption of an *ideal* adaptive law for $\hat{\theta}_2$ which at first glance cannot be implemented. Finally, an implementable adaptive law for $\hat{\theta}_2$ is designed in Section 2.3 and shown to asymptotically converge to the ideal adaptive law.

2.1 Estimating the uncertainty in the mass balance

We utilize both the distributed state measurements $y(x)$, $x \in [0, 1]$ and boundary measurements $z(0), z(1)$ and design a swapping-based parameter estimation scheme.

Let the operators Ψ, Ω and Δ be defined as

$$\Psi[y] := \int_0^1 y(x)dx \tag{13a}$$

$$\Omega[y] := \int_0^1 \phi_1(y(x), x)dx \tag{13b}$$

$$\Delta[z] := -a(z(1) - z(0)) \tag{13c}$$

We have

$$\begin{aligned}\dot{\Psi}[y] =& \frac{d}{dt}\int_0^1 y(x)dx = \int_0^1 y_t(x)dx \\ & - a\int_0^1 z_x(x)dx + \int_0^1 \phi_1^T(y(x),x)dx\theta_1 \\ =& \Delta[z] + \Omega^T[y]\theta_1. \end{aligned} \tag{14}$$

Consider the filters

$$\dot{\nu} = -\varsigma\nu + \Omega[y] \tag{15a}$$

$$\dot{\rho} = -\varsigma(\rho - \Psi[y]) + \Delta[z] \tag{15b}$$

for some $\varsigma > 0$ and let

$$\bar{\Psi} := \rho + \nu^T\theta_1. \tag{16}$$

Then the error $e := \Psi[y] - \bar{\Psi}$ satisfies

$$\begin{aligned}\dot{e} =& \Delta[z] + \Omega^T[y]\theta_1 \\ & - (-\varsigma(\rho - \Psi[y]) + \Delta[z] - \varsigma\nu^T\theta_1 + \Omega^T[y]\theta_1) \quad (17)\\ =& -\varsigma e \quad (18)\end{aligned}$$

showing that $e \in \mathcal{L}_2$ and $\bar{\Psi} \to \Psi[y]$ exponentially fast.

Lemma 1. For some $\Gamma_1 = \Gamma_1^T \succ 0$, let

$$\dot{\hat{\theta}}_1 = \Gamma_1\epsilon_1\nu \tag{19}$$

where $\epsilon_1 := \Psi[y] - \rho - \nu\hat{\theta}_1$. Then,

(1) $\epsilon_1, \hat{\theta}_1, \dot{\hat{\theta}}_1 \in \mathcal{L}_\infty$.
(2) $\epsilon_1, \dot{\hat{\theta}}_1 \in \mathcal{L}_2$.
(3) $\Omega^T[y]\tilde{\theta}_1 \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

If in addition $\Omega[y]$ satisfies the PE condition

$$\alpha_0 I \preceq \frac{1}{T}\int_t^{t+T} \Omega[y]\Omega[y]^T d\tau \preceq \alpha_1 I, \tag{20}$$

for some $\alpha_0, \alpha_1, T > 0$, then $\tilde{\theta}_1 \to 0$ exponentially fast.

Proof. Since ϕ_1 and therefore ν is bounded by assumption, Property (1) and (2) follow from Ioannou and Sun (2012, Th. 4.3.2). For Property (3), we have $\epsilon_1 = \nu\tilde{\theta}_1 + e$, so that

$$\begin{aligned}\dot{\epsilon}_1 =& \dot{\nu}\tilde{\theta}_1 - \nu\dot{\hat{\theta}}_1 + \dot{e} \\ =& -c\epsilon_1 + \Omega[y]\tilde{\theta}_1 - \epsilon_1\gamma_1\nu\nu^T - \varsigma e \\ =& -\epsilon_1(c + \gamma_1\nu\nu^T) + \Omega[y]\tilde{\theta}_1 - \varsigma e, \end{aligned} \tag{21}$$

so that

$$\begin{aligned}(\Omega[y]\tilde{\theta}_1)^2 \le& 2\dot{\epsilon}_1^2 + 2\epsilon_1^2(c + \gamma_1\nu\nu^T)^2 \\ \le& 2\frac{d}{dt}(\epsilon_1\dot{\epsilon}_1) - 2\epsilon_1\frac{d^2\epsilon_1}{dt^2} + 4\epsilon_1^2(c + \gamma_1\nu\nu^T)^2 \\ & + 4\varsigma^2 e^2 \end{aligned} \tag{22}$$

Therefore,

$$\begin{aligned}\int_0^t (\Omega[y]\tilde{\theta}_1)^2 d\tau \le& 2\int_0^t \frac{d}{d\tau}(\epsilon_1\dot{\epsilon}_1)d\tau + 2\bar{\epsilon}_1\int_0^t \frac{d^2\epsilon_1}{d\tau^2}d\tau \\ & + 2(c + \gamma_1\bar{\nu}^2)^2\int_0^t \epsilon_1^2 d\tau + 4\varsigma^2\int_0^t e^2 d\tau \\ \le& 4\bar{\epsilon}_1(\dot{\epsilon}_1(t) - \dot{\epsilon}_1(0)) + 2(c + \gamma_1\bar{\nu}^2)^2\int_0^t \epsilon_1^2 d\tau \\ & + 4\varsigma^2\int_0^t e^2 d\tau \end{aligned} \tag{23}$$

where $\bar{\epsilon}_1 = \sup_{t\ge 0}|\epsilon_1|$ and $\bar{\nu} = \sup_{t\ge 0}||\nu||$ (the latter exists by assumption). Letting $t \to \infty$ on both sides of the above inequality and using the fact that $\epsilon_1, e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{\epsilon}_1 \in \mathcal{L}_\infty$ shows that the left hand side is bounded which concludes the proof of Property (3). If $\Omega[y]$ is PE, it trivially follows that ν is PE (Ioannou and Sun, 2012, Lemma 4.8.3 (ii)) and exponential convergence of $\hat{\theta}_1$ to θ_1 follows again from Ioannou and Sun (2012, Th. 4.3.2). □

2.2 Main result and stability proof

To study the stability of (12) consider the Lyapunov function candidate

$$V_0 = \frac{1}{2}\int_0^1 W(x)\tilde{\zeta}^2(x)dx \tag{24}$$

for some $W(x) > 0$ satisfying $W'(x) \le -cW(x)$ for some $c > 0$, e.g. $W(x) = e^{-cx}$. Differentiating (24) with respect to time and inserting the dynamics (12a) yield

$$\begin{aligned}\dot{V}_0 =& -\lambda\int_0^1 W(x)\tilde{\zeta}(x)\tilde{\zeta}_x(x)dx \\ & + \int_0^1 W(x)\tilde{\zeta}(x)l\phi_1^T(y(x),x)\tilde{\theta}_1 dx \quad (25)\\ & + \int_0^1 W(x)\tilde{\zeta}(x)\phi_2^T(\hat{z}(x),x)\tilde{\theta}_2 dx \\ & + \int_0^1 W(x)\tilde{\zeta}(x)\tilde{\phi}_2^T(z(x),\hat{z}(x),x)\theta_2 dx. \quad (26)\end{aligned}$$

Using integration by parts, splitting the second term using Young's inequality and applying the sector condition in Assumption 2 to the last term, keeping in mind that $\tilde{\zeta}(x) = \tilde{z}(x)$, yield

$$\begin{aligned}\dot{V}_0 \le& -(c\lambda - 1)V_0 + \frac{1}{2}\int_0^1 W(x)(l\phi_1^T(y(x),x)\tilde{\theta}_1)^2 dx \quad (27)\\ & + \int_0^1 W(x)\tilde{\zeta}(x)\phi_2^T(\hat{z}(x),x)\tilde{\theta}_2 dx. \quad (28)\end{aligned}$$

To deal with the parametric uncertainties we augment the function (24) as follows.

$$V = V_0 + a_1V_1 + V_2. \tag{29}$$

where

$$V_i = \frac{1}{2}\tilde{\theta}_i^T\Gamma_i^{-1}\tilde{\theta}_i \tag{30}$$

for $i = 1, 2$, $a_1 > 0$ and $\Gamma_2 = \Gamma_2^T \succ 0$. Differentiating with respect to time and inserting (19) and (27) yield

$$\dot{V} \le -(c\lambda - 1)V_0 + \tilde{\theta}_2^T\Gamma_2^{-1}\left(\dot{\hat{\theta}}_2^* - \dot{\hat{\theta}}_2\right) - \tilde{\theta}_1^T H_1\tilde{\theta}_1 \tag{31}$$

where

$$\begin{aligned}H_1 :=& a_1(\nu\nu^T) \\ & - \frac{l^2}{2}\int_0^1 W(x)\phi_1(y(x),x)\phi_1^T(y(x),x)dx \end{aligned} \tag{32}$$

and

$$\dot{\hat{\theta}}_2^* := \Gamma_2\int_0^1 W(x)\phi_2(\hat{z}(x),x)\tilde{\zeta}(x)dx. \tag{33}$$

We conclude the above discussion by stating the main result.

Proposition 2. Consider the state estimation error system (12) with $\hat{\theta}_1$ generated by the adaptive law in Lemma 1 and $\hat{\theta}_2$ satisfying

$$\dot{\hat{\theta}}_2 = \dot{\hat{\theta}}_2^* \tag{34}$$

with any initial estimates

$$\hat{\theta}_2(0) = \hat{\theta}_2^*(0). \tag{35}$$

Then, the state estimation error $||\tilde{\zeta}||$ is bounded. Moreover, if PE condition (20) is satisfied, then

$$||\tilde{\zeta}|| \to 0. \tag{36}$$

Proof. Selecting c such that $c_0 := c\lambda - 1 > 0$ and inserting (34) into (31) yield

$$\dot{V} \leq -c_0 V_0 - \tilde{\theta}_1^T H_1 \tilde{\theta}_1. \tag{37}$$

For any $c_2 > 0$ and all

$$V_0 \geq c_0^{-1}(c_2 - \tilde{\theta}_1^T H_1 \tilde{\theta}_1) \tag{38}$$

we have $V \leq -c_2 V_0$. By (Khalil, 1996, Th 4.18), V and consequently $||\tilde{\zeta}||, \hat{\theta}_1, \hat{\theta}_2$ are bounded. Furthermore, if the PE condition (20) holds, then it can be shown that (Ioannou and Sun, 2012, Sec. 4.8.3)

$$\int_t^{t+T} \tilde{\theta}_1^T(\nu\nu^T)\tilde{\theta}_1 d\tau \geq h_1 V_1 \geq 0 \tag{39}$$

for the same $T > 0$ specifying (20) and some $h_1 > 0$. Since ϕ_1 is bounded by assumption, we can lower bound the second term in (32) which together with (39) for sufficiently large $a_1 > 0$ give the lower bound

$$\int_t^{t+T} \tilde{\theta}_1^T H_1 \tilde{\theta}_1 d\tau \geq \int_t^{t+T} (a_1 h_1 - h_2) V_1 d\tau > 0. \tag{40}$$

for some $h_2 > 0$. Selecting $a_1 > h_2 h_1^{-1}$ in (29) yields

$$\dot{V} \leq -c_0 V_0 \tag{41}$$

so that

$$c_0 \int_0^\infty V_0 d\tau \leq V(0) - V(\infty) \tag{42}$$

which since the right hand side is bounded and $V_0 \geq 0$, implies $V_0, ||\tilde{\zeta}||^2 \in \mathcal{L}_1$. By (Liu and Krstic, 2001, Lemma 3.1) it then follows that $V_0 \to 0$ and consequently (36). □

2.3 Estimating the uncertainty in the momentum balance

The *ideal* adaptive law $\hat{\theta}_2^*$ defined by (33) is not implementable as $\tilde{\zeta}$ is a-priori unknown. Instead, we heuristically seek an adaptive law *resembling* the non-implementable law in the sense

$$\dot{\hat{\theta}}_2 \to \dot{\hat{\theta}}_2^* \tag{43}$$

as $t \to \infty$. Simplifying the notation by defining

$$\Phi[\hat{z}](x) := -\int_x^1 W(\xi)\phi_2(\hat{z}(\xi), \xi)d\xi, \tag{44}$$

we have

$$\dot{\hat{\theta}}_2^* = \Gamma_2 \int_0^1 \Phi'[\hat{z}](x)\tilde{\zeta}(x)dx. \tag{45}$$

Utilizing that $\Phi[\hat{z}](1) = \tilde{\zeta}(0) = 0$ and using integration by parts, we equivalently have

$$\dot{\hat{\theta}}_2^* = \Gamma_2 \int_0^1 \Phi[\hat{z}](x)\tilde{\zeta}_x(x)dx. \tag{46}$$

Proposition 3. Consider the signal $\hat{\sigma} : \mathbb{R}^{\to}\mathbb{R}^q$ defined by

$$\dot{\hat{\sigma}} = \int_0^1 \eta[\hat{z}](x)\left(-by_x + \phi_2^T(\hat{z}(x), x)\hat{\theta}_2\right)dx \tag{47a}$$

$$\hat{\sigma}(0) = \hat{\theta}_2^*(0) \tag{47b}$$

where

$$\eta[\hat{z}] = \lambda^{-1}\Gamma_2\Phi[\hat{z}], \tag{48}$$

the operator $\Xi : L_2([0,1]) \to \mathbb{R}^q$ satisfying

$$D_{\hat{z}}\Xi[h] = \int_0^1 \eta[\hat{z}](x)h(x)dx \tag{49a}$$

$$\Xi[\hat{z}(\cdot, 0)] = 0, \tag{49b}$$

and let

$$\hat{\theta}_2 = \hat{\sigma} - \Xi[\hat{z}]. \tag{50}$$

If the PE condition (20) in Lemma 1 is satisfied and $||\eta[\hat{z}]|| \in \mathcal{L}_\infty$, then

$$\dot{\hat{\theta}}_2 \to \dot{\hat{\theta}}_2^* \tag{51}$$

exponentially fast.

Proof. Consider the auxiliary signal $\sigma : \mathbb{R}^+ \to \mathbb{R}^q$ defined as

$$\sigma := \theta_2 + \Xi[\hat{z}]. \tag{52}$$

It is evident that $\sigma - \hat{\sigma} =: \tilde{\sigma} = \tilde{\theta}_2$ and therefore $\dot{\hat{\sigma}} = \dot{\hat{\theta}}_2$. We thus need to show that $\dot{\hat{\sigma}} \to \dot{\hat{\theta}}_2^*$. Differentiating (52) with respect to time yields

$$\dot{\sigma} = \frac{d}{dt}\Xi[\hat{z}] = D_{\hat{z}}\Xi[\hat{z}_t]. \tag{53}$$

Inserting (49) gives

$$\begin{aligned}\dot{\sigma} &= \int_0^1 \eta[\hat{z}](x)\hat{z}_t(x)dx \\ &= \int_0^1 \eta[\hat{z}](x)\left(\hat{\zeta}_t(x) + \lambda z_x(x) - l\phi_1^T(y(x), x)\theta_1\right)dx.\end{aligned} \tag{54}$$

Subtracting (47a) from (54) yields

$$\dot{\hat{\theta}}_2 = \dot{\hat{\sigma}} = \int_0^1 \eta[\hat{z}](x)\left(\lambda\tilde{z}_x(x) - l\phi_1^T(y(x), x)\tilde{\theta}_1\right)dx \tag{55}$$

which in view of (46) and (48) and the fact that $\tilde{\zeta}(x) = \tilde{z}(x)$, is equivalent to

$$\dot{\hat{\sigma}} = \dot{\hat{\theta}}_2^* - \int_0^1 \eta[\hat{z}](x)\left(l\phi_1^T(y(x), x)\tilde{\theta}_1\right)dx. \tag{56}$$

If and $||\eta[\hat{z}]|| \in \mathcal{L}_\infty$ and the PE condition (20) is satisfied, $||\eta[\hat{z}]|| ||l\phi_1^T(y, \cdot)\tilde{\theta}_1|| \to 0$ and we obtain the desired result (51). From (47b) and (49b) we have that (35) is satisfied. □

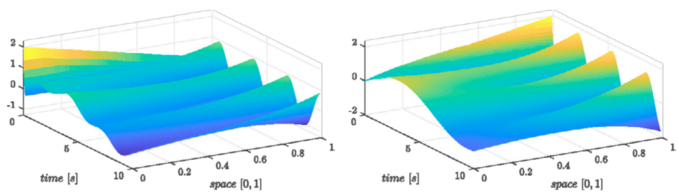
To implement the adaptive law (50) at any single time t_1, we need to evaluate Ξ at $\hat{z}(\cdot, t_1)$. We proceed by computing the incremental value $\Xi[\hat{z}(\cdot, t_1)] - \Xi[\hat{z}(\cdot, t_0)]$ for any $t_1 > t_0 > 0$. Let $S(\gamma) = \hat{z}(\cdot, t_0) + \gamma[\hat{z}(\cdot, t_1) - \hat{z}(\cdot, t_0)]$. Evaluating Ξ at $S(\gamma)$ and differentiating with respect to γ yield

$$\begin{aligned}\frac{d}{d\gamma}\Xi[S(\gamma)] &= D_{S(\gamma)}\Xi[S'(\gamma)] \\ &= D_{S(\gamma)}\Xi[\hat{z}(\cdot, t_1) - \hat{z}(\cdot, t_0)] \\ &= \int_0^1 \eta[S(\gamma)](x)\left(\hat{z}(x, t_1) - \hat{z}(x, t_0)\right)dx\end{aligned} \tag{57}$$

Integrating both sides from $\gamma = 0$ to $\gamma = 1$ and using $S(1) = \hat{z}(\cdot, t_1)$ and $S(0) = \hat{z}(\cdot, t_0)$ yield

$$\begin{aligned}\Xi[\hat{z}(\cdot, t_1)] = &\int_0^1\int_0^1 \eta[S(\gamma)](x)\left(\hat{z}(x, t_1) - \hat{z}(x, t_0)\right)dxd\gamma \\ &+ \Xi[\hat{z}(\cdot, t_0)]\end{aligned} \tag{58}$$

Remark 2. Using the adaptive law suggested in Proposition 3, the condition (34) in Proposition 2 is only satisfied asymptotically. Consequently, we have not formally established the boundedness and convergence properties of $||\tilde{\zeta}||$ and $\tilde{\theta}_2$. However,



(a) State $y(x,t)$. (b) State $z(x,t)$.

Fig. 1. Case 1. System states.

the two intermediate results in Proposition 2 and 3 suggest an adaptive observer for system (1) that can be tested in simulations or with experimental data. In the next section, the observer is tested in two case simulations.

3. SIMULATION

We simulate two cases. Case 1 where the PE condition (20) is satisfied and Case 2 where it is not. The system (1), observer (11), adaptive law (19), filter (47) and operator (58) are simulated in MATLAB for 10 seconds. The PDEs are discretized using 100 spatial discretization points and solved using the *method of lines* by first transforming the hyperbolic system (1) to a Riemann invariant form. In both cases, the system parameters where

$$a = b = 4 \tag{59a}$$

$$\phi_1^T(y(x),x) = \begin{cases} \left[-5y(x)\ \ 0\right], & x < 0 \\ \left[0\ \ -5y(x)\right], & x \geq 0 \end{cases} \tag{59b}$$

$$\phi_2^T(z(x),x) = \begin{cases} \left[-z(x)\ \ 0\right], & x < 0 \\ \left[0\ \ -z(x)\right], & x \geq 0 \end{cases} \quad \theta_1 = [2\ \ 4] \tag{59c}$$

$$\theta_2 = [5\ \ 7]. \tag{59d}$$

Observe that the chosen ϕ_2 satisfies Assumption 2. For case 1, we used the boundary conditions

$$z(0,t) = \sin(\frac{t}{2}) \tag{60a}$$

$$y(1,t) = \sin(2t) \tag{60b}$$

while for case 2, we used

$$z(0,t) = 0 \tag{61a}$$

$$y(1,t) = \sin(2t). \tag{61b}$$

Compatible initial conditions were selected as

$$y(x,0) = y(1,0) + 2(1-x) \tag{62a}$$

$$z(x,0) = 2x + z(0,0). \tag{62b}$$

Finally, the design parameters are

$$L = 0.8 \tag{63a}$$

$$\hat{\theta}_1(0) = \hat{\theta}_2(0) = 0 \tag{63b}$$

$$\Gamma_1 = \Gamma_2 = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \tag{63c}$$

$$W(x) = 2 - x. \tag{63d}$$

For case 1, with states shown in Figure 1, the PE condition (20) is satisfied and $\tilde{\theta}_1 \to 0$ as can be seen in Figure 4. The conditions of Proposition 3 is satisfied and for $\dot{\hat{\theta}}_2 = \dot{\hat{\theta}}_2^*$ it follows from Proposition 2 that $||\tilde{\zeta}|| \to 0$ which can be seen in Figures 2 and 3. The parameter estimates $\hat{\theta}_2$ also converge to their true value as can be seen in Figure 5.

For case 2, with states shown in Figure 6, the PE condition (20) is not satisfied and $\hat{\theta}_1$ converge to a constant $\bar{\theta}_1 \neq \theta_1$ as can be seen in Figure 9. Still, as shown in Figures 7 and 8 the estimation error $||\tilde{\zeta}||$ is bounded and converge to a set close

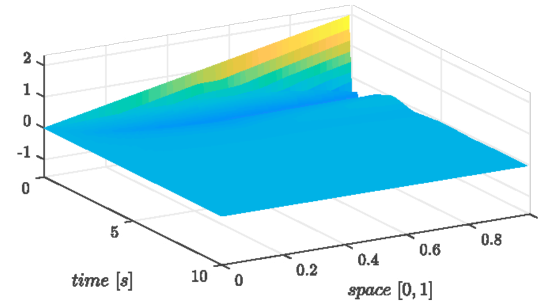


Fig. 2. Case 1. State estimation error $\tilde{z}(x,t)$.

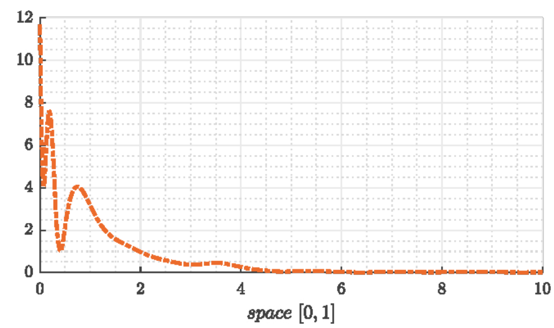
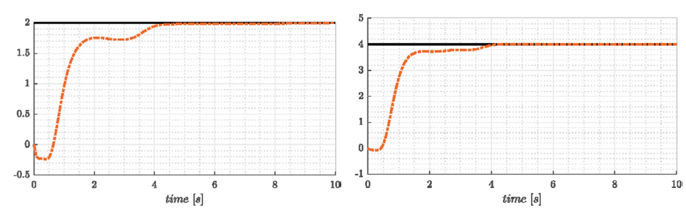
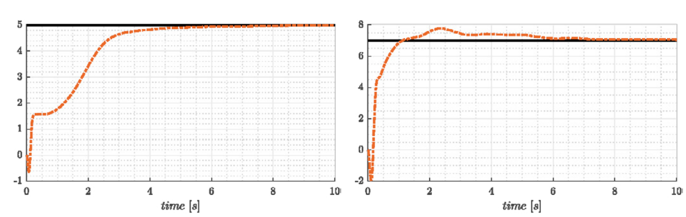


Fig. 3. Case 1. State estimation error $||\tilde{z}||$.



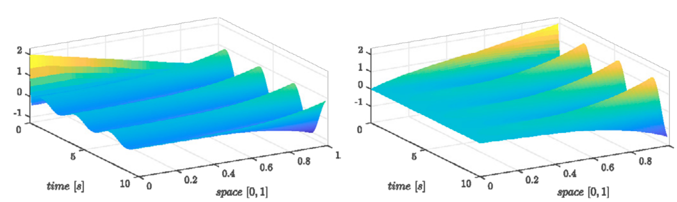
(a) $\hat{\theta}_{1,1}$ vs. $\theta_{1,1}$. (b) $\hat{\theta}_{1,2}$ vs. $\theta_{1,2}$.

Fig. 4. Case 1. Parameter estimates $\hat{\theta}_1$ (red dotted) vs. true parameters θ_1 (solid black).



(a) $\hat{\theta}_{2,1}$ vs. $\theta_{2,1}$. (b) $\hat{\theta}_{2,2}$ vs. $\theta_{2,2}$.

Fig. 5. Case 1. Parameter estimates $\hat{\theta}_2$ (red dotted) vs. true parameters θ_2 (solid black).



(a) State $y(x,t)$. (b) State $z(x,t)$.

Fig. 6. Case 2. System states.

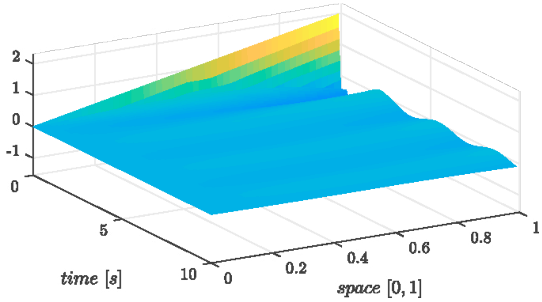


Fig. 7. Case 2. State estimation error $\tilde{z}(x,t)$.

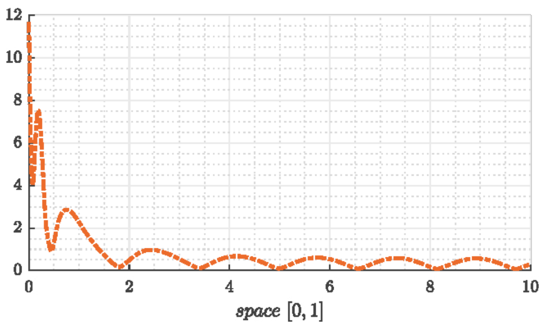
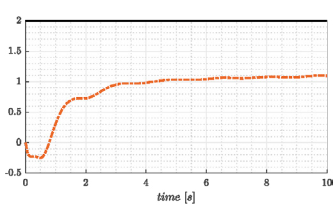
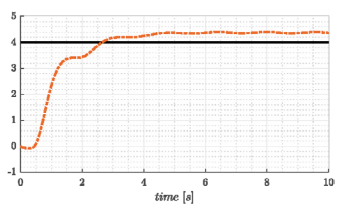


Fig. 8. Case 2. State estimation error $||\tilde{z}||$.

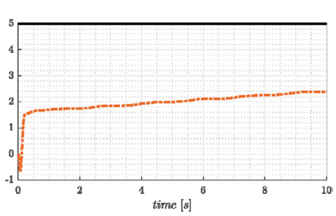


(a) $\hat{\theta}_{1,1}$ vs. $\theta_{1,1}$.

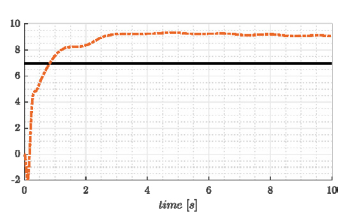


(b) $\hat{\theta}_{1,2}$ vs. $\theta_{1,2}$.

Fig. 9. Case 2. Parameter estimates $\hat{\theta}_1$ (red dotted) vs. true parameters θ_1 (solid black).



(a) $\hat{\theta}_{2,1}$ vs. $\theta_{2,1}$.



(b) $\hat{\theta}_{2,2}$ vs. $\theta_{2,2}$.

Fig. 10. Case 2. Parameter estimates $\hat{\theta}_2$ (red dotted) vs. true parameters θ_2 (solid black).

to zero, as guaranteed by Proposition 2. Figure 10 shows that parameter convergence is not achieved or at least is very slow.

4. CONCLUDING REMARKS

We have designed an adaptive observer estimating the distributed state of a semi-linear 2×2 hyperbolic system and uncertainties appearing in both state equations by relying on partial distributed state measurement and boundary measurements. The scheme can be used to estimate the flow rate in a single-phase fluid flow system where the in-/out-flow of mass and momentum gain/loss are parametrically uncertain by relying on distributed pressure measurements. With no mass in- or out-flux, any uncertain local gain or loss of momentum can be estimated using pressure measurements only. With mass in or out-flux, we use boundary measurements of the flow rate in addition to the pressure measurements to estimate net gain or loss of mass. Remark that for any single point in time, only the aggregate net in-/out-flow can be estimated. Situations with flow-loops, that is inflow in one region and an outflow of equal size in another region, is not detectable using boundary measurements only. However, if the local inflow varies in a certain way and is sufficiently distinct compared to other regions, it is possible to also estimate local in- or out-flow phenomena, and not only the aggregate net flow. These conditions are all covered by the persistence of excitation criterion guaranteeing convergence of the parametric uncertainties in the mass balance.

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