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Homotopy-coherent algebra via Segal conditions

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ABSTRACT

Many homotopy-coherent algebraic structures can be described by Segal-type limit conditions determined by an "algebraic pattern", by which we mean an ∞-category equipped with a factorization system and a collection of "elementary" objects. Examples of structures that occur as such "Segal $\mathcal{O}$-spaces" for an algebraic pattern $\mathcal{O}$ include ∞-categories, (∞, n)-categories, ∞-operads (including symmetric, non-symmetric, cyclic, and modular ones), ∞-properads, and algebras for a (symmetric) ∞-operad in spaces.

In the first part of this paper we set up a general framework for algebraic patterns and their associated Segal objects, including conditions under which the latter are preserved by left and right Kan extensions. In particular, we obtain necessary and sufficient conditions on a pattern $\mathcal{O}$ for free Segal $\mathcal{O}$-spaces to be described by an explicit colimit formula, in which case we say that $\mathcal{O}$ is "extendable".

In the second part of the paper we explore the relationship between extendable algebraic patterns and polynomial monads, by which we mean cartesian monads on presheaf ∞-categories that are accessible and preserve weakly contractible limits. We first show that the free Segal $\mathcal{O}$-space monad for an extendable pattern $\mathcal{O}$ is always polynomial. Next, we prove an ∞-categorical version of Weber's Nerve Theorem for polynomial monads, and use this to define a canonical extendable pattern from any polynomial monad, whose Segal spaces are equivalent to the algebras of the monad. These constructions yield functors between polynomial monads and extendable algebraic patterns, and we show that these exhibit full sub-

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categories of "saturated" algebraic patterns and "complete" polynomial monads as localizations, and moreover restrict to an equivalence between the ∞-categories of saturated patterns and complete polynomial monads.

Contents

1. Introduction

Homotopy-coherent algebraic structures, where identities between operations are replaced by an infinite hierarchy of compatible coherence equivalences, have played an important role in algebraic topology since the 1960s,[1] when they were first introduced in the special case of A_∞-spaces by Stasheff [36], and have since found a variety of applications in many fields of mathematics. From a modern perspective, homotopy-coherent algebraic structures can be considered as the natural algebraic structures in the setting of ∞-categories (which are themselves the homotopy-coherent analogues of categories).

It turns out that many interesting homotopy-coherent algebraic structures can be described by "Segal conditions", i.e. they can be described as functors satisfying a specific type of limit condition. The canonical (and original) example is Segal's [35] description of homotopy-coherently commutative monoids in spaces (or E_∞*-spaces*) as "special Γ-spaces". In ∞-categorical language, these are functors $F\colon \mathbb{F}_* \to \mathcal{S}$, where $\mathbb{F}_*$ is a skeleton of the category of pointed finite sets, with objects $\langle n \rangle := (\{0, 1, \ldots, n\}, 0)$, and $\mathcal{S}$ is the ∞-category of spaces (or ∞-groupoids), which are required to satisfy the following condition:

[1] More general frameworks for homotopy-coherent algebra, such as operads, arose out of work on infinite loop spaces by Boardman–Vogt [8] and May [31] in the early 1970s.

For all n, the map

$$F(\langle n\rangle) \to \prod_{i=1}^{n} F(\langle 1\rangle),$$

induced by the morphisms $\rho_i \colon \langle n\rangle \to \langle 1\rangle$ given by

$$\rho_i(j) = \begin{cases} 0, & j \neq i, \\ 1, & j = i, \end{cases}$$

is an equivalence.

Other key examples of structures described by Segal conditions include:

- associative (or A_∞- or E_1-)monoids, using the simplex category $\mathbf{\Delta}^{\mathrm{op}}$ (in unpublished work of Segal),
- ∞-categories, again using $\mathbf{\Delta}^{\mathrm{op}}$, in the form of Rezk's Segal spaces [33],
- (∞, n)-categories, using Joyal's categories $\mathbf{\Theta}_n^{\mathrm{op}}$, also in work of Rezk [34],
- ∞-operads, using the dendroidal category $\mathbf{\Omega}^{\mathrm{op}}$ of Moerdijk–Weiss [32], in work of Cisinski and Moerdijk [11],
- algebras for an ∞-operad $\mathcal{O}$ (in the sense of [30]) in $\mathcal{S}$, using the "category of operators" $\mathcal{O}$ itself.

Given these and other examples (many of which we will discuss below in §3), we might wonder why so many different algebraic structures can be described by Segal conditions. Our main results in this paper provide an explanation of this situation, by answering the following question:

Question 1.1. *Which homotopy-coherent algebraic structures can be described (in a reasonable way) by Segal conditions, and how canonical is this description?*

Before we describe our answer, we need to formulate a more precise version of this question, by defining the terms that appear. First of all, we will consider algebraic structures on (families of) spaces, which we take to mean algebras for monads on functor ∞-categories $\mathrm{Fun}(\mathcal{I}, \mathcal{S})$ (where $\mathcal{I}$ is any small ∞-category). Next, let us specify what precisely we mean by "Segal conditions". Returning to the example of special Γ-spaces, the category $\mathbb{F}_*$ has the following features that we wish to abstract:

- A morphism $\phi \colon \langle n\rangle \to \langle m\rangle$ is called *inert* if $|\phi^{-1}(j)| = 1$ for $j \neq 0$, and *active* if $\phi^{-1}(0) = \{0\}$. The inert and active morphisms form a factorization system on $\mathbb{F}_*$: every morphism factors as an inert morphism followed by an active morphism, and this decomposition is unique up to isomorphism.

- The morphisms ρ_i are precisely the inert morphisms $\langle n\rangle \to \langle 1\rangle$.
- If $\mathbb{F}_*^{\mathrm{int}}$ denotes the subcategory of $\mathbb{F}_*$ with only inert morphisms, then the special Γ-spaces are precisely the functors $F\colon \mathbb{F}_* \to \mathcal{S}$ such that the restriction $F|_{\mathbb{F}_*^{\mathrm{int}}}$ is a right Kan extension of $F|_{\{\langle 1\rangle\}}$.

These features recur in our other examples, which suggests that the input data for a class of "Segal conditions" should consist of an ∞-category $\mathcal{O}$ equipped with a factorization system (whereby every morphism factors as an "active" morphism followed by an "inert" morphism) and a class of "elementary" objects (or generators). From this data, which we will refer to as an *algebraic pattern*,[2] we obtain the relevant Segal-type limit condition on a functor $F\colon \mathcal{O} \to \mathcal{S}$ by imposing the requirement that for every $O \in \mathcal{O}$ the object $F(O)$ is the limit over all inert morphisms to elementary objects,

$$F(O) \xrightarrow{\sim} \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} F(E);$$

we say that such a functor F is a *Segal $\mathcal{O}$-space*.[3] If $\mathcal{O}$ is any algebraic pattern, and $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{O}, \mathcal{S})$ on the Segal $\mathcal{O}$-spaces, then the restriction functor

$$\mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \to \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S})$$

has a left adjoint. This adjunction is always monadic, and we write $T_{\mathcal{O}}$ for the corresponding monad on $\mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S})$. The monad $T_{\mathcal{O}}$ is then "described by" the algebraic pattern $\mathcal{O}$. In general, however, it is not possible to describe this monad explicitly, because the left adjoint involves an abstract localization. We only want to consider a pattern to be "reasonable" if this localization is unnecessary, in which case $T_{\mathcal{O}}$ is given by a concrete formula, namely as

$$T_{\mathcal{O}}F(E) \simeq \operatorname*{colim}_{X \in \mathrm{Act}_{\mathcal{O}}(E)} \lim_{E' \in \mathcal{O}^{\mathrm{el}}_{X/}} F(E'),$$

where $\mathrm{Act}_{\mathcal{O}}(E)$ is the space of active morphisms to E in $\mathcal{O}$. We call such patterns $\mathcal{O}$ *extendable*, and give explicit necessary and sufficient conditions for a pattern to be extendable in Proposition 8.8.

We can now state the precise version of the previous question that we will address:

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[2] This terminology is inspired by Lurie's *categorical patterns* [30, §B], the key examples of which all arise from algebraic patterns in our sense, and should not be confused with the notion of "pattern" considered by Getzler [17].

[3] Here we write $\mathcal{O}^{\mathrm{int}}$ for the subcategory of $\mathcal{O}$ containing only the inert morphisms, $\mathcal{O}^{\mathrm{el}}$ for the full subcategory of $\mathcal{O}^{\mathrm{int}}$ spanned by the elementary objects, and define $\mathcal{O}^{\mathrm{el}}_{O/} := \mathcal{O}^{\mathrm{el}} \times_{\mathcal{O}^{\mathrm{int}}} \mathcal{O}^{\mathrm{int}}_{O/}$.

Question 1.2. *Which monads on presheaf ∞-categories can be described as the free Segal $\mathcal{O}$-space monad for an extendable algebraic pattern $\mathcal{O}$, and how canonical is this description?*

We will characterize these monads as a certain class of *polynomial*[4] monads, by which we mean the monads on presheaf ∞-categories that are *cartesian*[5] and whose underlying endofunctors are accessible and preserve weakly contractible limits. Our first main result provides functors in both directions between ∞-categories of extendable patterns and of polynomial monads:

Theorem 1.3.

(i) If $\mathcal{O}$ is an extendable algebraic pattern then the free Segal $\mathcal{O}$-space monad $T_{\mathcal{O}}$ is polynomial. This determines a functor $\mathfrak{M}$ from extendable patterns to polynomial monads.

(ii) If T is a polynomial monad on $\mathrm{Fun}(\mathcal{I}, \mathcal{S})$ *then there exists a canonical extendable algebraic pattern $\mathcal{W}(T)$ such that* $\mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S})$ *is equivalent to the ∞-category of T-algebras. This determines a functor $\mathfrak{P}$ from polynomial monads to extendable patterns.*

We prove part (i) in §10 and part (ii) in §13. Part (ii) depends on an ∞-categorical version of Weber's *nerve theorem* [38], which we prove in §11 and use to construct a factorization system on the Kleisli ∞-category of a polynomial monad in §12.

Our second main result characterizes the images of these functors:

Theorem 1.4.

(i) Restricting to slim[6] *extendable patterns, there is a natural transformation $\sigma\colon \mathrm{id} \to \mathfrak{P}\mathfrak{M}$, and the component $\sigma_{\mathcal{O}}$ is an equivalence if and only if the pattern $\mathcal{O}$ is* saturated, *meaning that it is a slim extendable pattern such that the functors*

$$\mathrm{Map}_{\mathcal{O}}(O, –)\colon \mathcal{O} \to \mathcal{S}$$

are Segal $\mathcal{O}$-spaces for $O \in \mathcal{O}$. The pattern $\mathcal{W}(T)$ for a polynomial monad T is always saturated, and the transformation σ exhibits the full subcategory of saturated patterns as a localization of the ∞-category of slim extendable patterns.

[4] The analogous monads on ordinary categories are sometimes called *strongly cartesian* monads.

[5] The cartesian monads are those whose multiplication and unit transformations are *cartesian* natural transformations, which in turn means that their naturality squares are all cartesian, i.e. are pullback squares.

[6] This is a mild technical hypothesis; it is satisfied in almost all examples, and the patterns $\mathcal{W}(T)$ are always slim. Moreover, any extendable pattern can be replaced by a full subcategory that is slim and determines the same monad.

(ii) *There is a natural transformation* $\tau\colon \mathrm{id} \to \mathfrak{M}\mathfrak{P}$, *and* τ_T *is an equivalence for a polynomial monad* T *on* $\mathrm{Fun}(\mathcal{I}, \mathcal{S})$ *if and only if* T *is* complete, *meaning that the essentially surjective functor* $\mathcal{I} \to \mathcal{W}(T)^{\mathrm{el}}$ *is an equivalence. The monad* $T_{\mathcal{O}}$ *for an extendable pattern* $\mathcal{O}$ *is always complete, and the transformation* τ *exhibits the full subcategory of complete polynomial monads as a localization of the* ∞*-category of polynomial monads.*

(iii) *The functors* $\mathfrak{P}$ *and* $\mathfrak{M}$ *restrict to an equivalence between the* ∞*-categories of saturated patterns and complete polynomial monads.*

We will prove part (i) in §14 and parts (ii) and (iii) in §15.

The answer to our question above is thus that the monads of the form $T_{\mathcal{O}}$ for an extendable pattern $\mathcal{O}$ are precisely the *complete* polynomial monads, and there is a *unique* extendable pattern describing this monad that is *saturated*, namely the canonical pattern $\mathcal{W}(T_{\mathcal{O}})$. For example, returning to our initial example of commutative monoids described by an algebraic pattern structure on $\mathbb{F}_*$, this pattern is extendable, with free commutative monoids described by the expected formula

$$X \mapsto \coprod_{n=0}^{\infty} X^{\times n}_{h\Sigma_n},$$

but it is *not* saturated. The corresponding saturated pattern is instead the ∞-category of free commutative monoids on finite sets (i.e. the *Lawvere theory* for commutative monoids), which by work of Cranch [12] can be identified with the (2,1)-category $\mathrm{Span}(\mathbb{F})$ of finite sets with spans (or correspondences) as morphisms; see Example 14.22 for more details.

1.1. Overview

In the first part of the paper we set up a general categorical framework for algebraic patterns and Segal objects. In §2 we introduce these objects more formally and prove some of their basic properties, before we look at examples of algebraic patterns and their Segal objects in §3. We then introduce morphisms of algebraic patterns in §4 and construct an ∞-category of algebraic patterns in §5, where we also prove that this has limits and filtered colimits. Next, we provide conditions under which Segal objects are preserved by right and left Kan extensions in §6 and §7, respectively.

In §8 we apply our work on left Kan extensions to analyze free Segal objects; in particular, we obtain necessary and sufficient conditions for a pattern $\mathcal{O}$ to be extendable, meaning that free Segal $\mathcal{O}$-spaces are described by a colimit formula. In §9 we study (weak) Segal fibrations, which generalize Lurie's definitions of symmetric monoidal ∞-categories and symmetric ∞-operads. We show that any weak Segal fibration over an extendable base is again extendable, and moreover left Kan extension along any morphism of weak Segal fibrations preserves Segal objects; this recovers, for example, the formula of [30] for operadic left Kan extensions of ∞-operad algebras in cartesian monoidal ∞-categories.

In §10 we introduce polynomial monads, and prove that the free Segal $\mathcal{O}$-space monad for any extendable pattern is polynomial. We then prove an ∞-categorical version of Weber's Nerve Theorem for presheaf ∞-categories in §11, and apply this to define a factorization system on the Kleisli ∞-category of a polynomial monad in §12. This gives a canonical algebraic pattern for every polynomial monad, which we study in §13. Next, we study the relationship between an extendable pattern and the canonical pattern of its free Segal space monad; under a mild hypothesis there is a functor between these, and we show that this is an equivalence precisely when the pattern is saturated. Finally, in §15 we study complete polynomial monads, and prove that there is an equivalence between these and saturated patterns.

1.2. Related work

There is an extensive literature on using (finite) limit conditions to describe algebraic structures in category theory, going back at least to Lawvere's thesis [28], where he introduced algebraic theories. Our work is in particular closely related to the "nerve theorem", one version of which *almost* says that a strongly cartesian monad on a presheaf category is described by Segal conditions; this version was first proved in unpublished work of Leinster (though his proof did not use the factorization system), and later extended by Weber [38] to a description of certain *weakly* cartesian monads.[7] We were particularly inspired by the simpler proof given by Berger, Melliès, and Weber [6]. Their work has more recently been extended by Bourke and Garner [9], who study general classes of monads that can be described by some notion of "theories with arities", including in the enriched context.

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[7] Weakly cartesian monads are of interest in the case of ordinary categories, as many "algebraic" monads that involve symmetries, such as the free commutative monoid monad, are not cartesian. This issue disappears if we replace sets by groupoids, and so weakly cartesian monads are not relevant in our ∞-categorical setting.

2. Algebraic patterns and Segal objects

In this section we introduce the basic structures we will study in this paper, namely algebraic patterns and their Segal objects.

Definition 2.1. An *algebraic pattern* $\mathfrak{O}$ is an ∞-category $\mathcal{O}$ equipped with:

- a factorization system $(\mathcal{O}^{\mathrm{int}}, \mathcal{O}^{\mathrm{act}})$, the morphisms in which we refer to as the *inert* and *active* morphisms in $\mathfrak{O}$,
- a full subcategory $\mathcal{O}^{\mathrm{el}} \subseteq \mathcal{O}^{\mathrm{int}}$ whose objects we call the *elementary* objects of $\mathfrak{O}$.

Unless stated otherwise, we will assume by default that algebraic patterns are essentially small.

Remark 2.2. Here a *factorization system* on an ∞-category $\mathcal{C}$ means a pair of subcategories $(\mathcal{C}^L, \mathcal{C}^R)$ such that both contain all objects of $\mathcal{C}$, and for every morphism $f\colon X \to X'$ in $\mathcal{C}$, the space of factorizations

$$\left\{ \begin{tikzcd} & Y \arrow[dr, "r"] & \\ X \arrow[ur, "l"] \arrow[rr, "f"] & & X' \end{tikzcd} \quad : l \in \mathcal{C}^L, r \in \mathcal{C}^R \right\}$$

is contractible.

Remark 2.3. We will often abuse notation and conflate an algebraic pattern with its underlying ∞-category $\mathcal{O}$, i.e. we will simply say that $\mathcal{O}$ is an algebraic pattern.

Notation 2.4. If $\mathcal{O}$ is an algebraic pattern, we will often indicate an inert map between objects O, O' of $\mathcal{O}$ as $O \rightarrowtail O'$ and an active map as $O \rightsquigarrow O'$. These symbols are not meant to suggest any intuition about the nature of inert and active maps.

Notation 2.5. If $\mathcal{O}$ is an algebraic pattern and X is an object of $\mathcal{O}$, then we write $\mathcal{O}^{\mathrm{el}}_{X/}$ for the fibre product of ∞-categories $\mathcal{O}^{\mathrm{el}} \times_{\mathcal{O}^{\mathrm{int}}} \mathcal{O}^{\mathrm{int}}_{X/}$. Thus the objects of $\mathcal{O}^{\mathrm{el}}_{X/}$ are inert morphisms $X \rightarrowtail E$ where E is elementary, and the morphisms are commutative triangles

$$\begin{tikzcd} & X \arrow[dl, tail] \arrow[dr, tail] & \\ E \arrow[rr, tail] & & E' \end{tikzcd}$$

where all morphisms are inert, and E and E' are elementary.

Definition 2.6. Let $\mathcal{O}$ be an algebraic pattern. We say an ∞-category is $\mathcal{O}$-*complete* if it has limits of shape $\mathcal{O}^{\mathrm{el}}_{X/}$ for all $X \in \mathcal{O}$.

Definition 2.7. Let $\mathcal{O}$ be an algebraic pattern and $\mathcal{C}$ an $\mathcal{O}$-complete ∞-category. A *Segal $\mathcal{O}$-object* in $\mathcal{C}$ is a functor $F\colon \mathcal{O} \to \mathcal{C}$ such that for every $X \in \mathcal{O}$ the canonical map

$$F(X) \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} F(E)$$

is an equivalence. We write $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$ spanned by the Segal $\mathcal{O}$-objects.

Notation 2.8. We will often refer to Segal $\mathcal{O}$-objects in the ∞-category $\mathcal{S}$ of spaces as *Segal $\mathcal{O}$-spaces*, and to Segal $\mathcal{O}$-objects in the ∞-category Cat_∞ of ∞-categories as *Segal $\mathcal{O}$-∞-categories.*

Lemma 2.9. *Let $\mathcal{C}$ be an $\mathcal{O}$-complete ∞-category. Then $F\colon \mathcal{O} \to \mathcal{C}$ is a Segal $\mathcal{O}$-object if and only if the restriction $F|_{\mathcal{O}^{\mathrm{int}}}$ is a right Kan extension of $F|_{\mathcal{O}^{\mathrm{el}}}$.*

Proof. Since $\mathcal{C}$ is $\mathcal{O}$-complete, $F|_{\mathcal{O}^{\mathrm{int}}}$ is a right Kan extension of $F|_{\mathcal{O}^{\mathrm{el}}}$ if and only if for all $X \in \mathcal{O}^{\mathrm{int}}$, the natural map

$$F(X) \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} F(E)$$

is an equivalence. $\square$

Definition 2.10. Let $\mathcal{O}$ be an algebraic pattern. For $O \in \mathcal{O}$ we write $y(O)_{\mathrm{Seg}}$ for the colimit $\mathrm{colim}_{E \in (\mathcal{O}^{\mathrm{el}}_{O/})^{\mathrm{op}}} y(E)$ in $\mathrm{Fun}(\mathcal{O}, \mathcal{S})$, where y denotes the Yoneda embedding $\mathcal{O}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{O}, \mathcal{S})$. If $\mathcal{C}$ is a cocomplete ∞-category, and thus is tensored over $\mathcal{S}$, then we can consider $C \otimes y(O)$ and $C \otimes y(O)_{\mathrm{Seg}}$ in $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$ for $C \in \mathcal{C}$.

Lemma 2.11. *Let $\mathcal{O}$ be an algebraic pattern and $\mathcal{C}$ a cocomplete ∞-category.*

(i) $F \in \mathrm{Fun}(\mathcal{O}, \mathcal{C})$ *is a Segal $\mathcal{O}$-object if and only if F is local with respect to the canonical maps $C \otimes y(O)_{\mathrm{Seg}} \to C \otimes y(O)$ for all $O \in \mathcal{O}$.*
(ii) If $\mathcal{C}$ is κ-presentable, then F is a Segal $\mathcal{O}$-object if and only if F is local with respect to these maps where C is κ-compact.
(iii) If $\mathcal{C}$ is presentable, then the full subcategory $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ *is an accessible localization of* $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$.
(iv) If $\mathcal{C}$ is presentable, then so is the ∞-category $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$.

Proof. The object F is local with respect to $C \otimes y(O)_{\mathrm{Seg}} \to C \otimes y(O)$ precisely when the morphism of spaces

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{C})}(C \otimes y(O), F) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{C})}(C \otimes y(O)_{\mathrm{Seg}}, F)$$

is an equivalence. Here we have equivalences

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{C})}(C \otimes y(O), F) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y(O), \mathrm{Map}_{\mathcal{C}}(C, F)) \simeq \mathrm{Map}_{\mathcal{C}}(C, F(O)),$$

using the Yoneda Lemma, and similarly

$$\begin{aligned}\mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{C})}(C \otimes y(O)_{\mathrm{Seg}}, F) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y(O)_{\mathrm{Seg}}, \mathrm{Map}_{\mathcal{C}}(C, F)) \\ &\simeq \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y(E), \mathrm{Map}_{\mathcal{C}}(C, F)) \\ &\simeq \mathrm{Map}_{\mathcal{C}}(C, \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} F(E)).\end{aligned}$$

Thus F is local with respect to this morphism for all C and O if and only if $F(O) \xrightarrow{\sim} \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} F(E)$ for all O, i.e. F is a Segal object. This proves (i). If $\mathcal{C}$ is κ-presentable, then to conclude that the Segal map $F(O) \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} F(E)$ is an equivalence it suffices to consider C in $\mathcal{C}^{\kappa}$, which proves (ii).

It follows that if $\mathcal{C}$ is presentable, then $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ is the full subcategory of objects in $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$ that are local with respect to a *set* of morphisms. Parts (iii) and (iv) then follow from [29, Proposition 5.5.4.15]. □

3. Examples of algebraic patterns

In this section we will briefly describe some examples of algebraic patterns and their associated Segal objects.

Example 3.1. We write $\mathbb{F}_*^{\flat}$ for the algebraic pattern structure on $\mathbb{F}_*$ given by the inert–active factorization system we discussed above in the introduction, with $\mathbb{F}_*^{\flat,\mathrm{el}}$ containing the single object $\langle 1 \rangle$. Then a Segal $\mathbb{F}_*^{\flat}$-space is precisely a commutative monoid, or equivalently a special Γ-space in the sense of [35].

Example 3.2. We can also consider another pattern structure on $\mathbb{F}_*$: We define $\mathbb{F}_*^{\natural}$ by the same factorization system, but now $\mathbb{F}_*^{\natural,\mathrm{el}}$ contains the two objects $\langle 0 \rangle$ and $\langle 1 \rangle$, with the unique inert morphism $\langle 1 \rangle \to \langle 0 \rangle$. Segal $\mathbb{F}_*^{\natural}$-objects are functors $F \colon \mathbb{F}_*^{\natural} \to \mathcal{C}$ such that

$$F(\langle n \rangle) \simeq F(\langle 1 \rangle)^{\times_{F(\langle 0 \rangle)} n},$$

where the right-hand side denotes an iterated fibre product over $F(\langle 0 \rangle)$; this is equivalently a commutative monoid in the slice $\mathcal{C}_{/F(\langle 0 \rangle)}$.

Example 3.3. We write $\boldsymbol{\Delta}$ for the simplex category, i.e. the category of non-empty finite ordered sets $[n] := \{0, \ldots, n\}$ and order-preserving maps between them. A morphism $f \colon [n] \to [m]$ is *inert* if it is the inclusion of a sub-interval, i.e. $f(i) = f(0) + i$ for all i, and *active* if f preserves the end-points, i.e. $f(0) = 0$ and $f(n) = m$. Every morphism in

$\mathbf{\Delta}$ factors uniquely as an active morphism followed by an inert one, so this determines an inert–active factorization system on $\mathbf{\Delta}^{\mathrm{op}}$. Using this factorization system we can define two interesting algebraic pattern structures on $\mathbf{\Delta}^{\mathrm{op}}$:

- $\mathbf{\Delta}^{\mathrm{op},\natural}$ denotes the pattern where $\mathbf{\Delta}^{\mathrm{op},\natural,\mathrm{el}}$ contains the two objects $[0]$ and $[1]$, and the two inert morphisms $[1] \rightrightarrows [0]$,
- $\mathbf{\Delta}^{\mathrm{op},\flat}$ denotes the pattern where $\mathbf{\Delta}^{\mathrm{op},\flat,\mathrm{el}} := \{[1]\}$.

A Segal $\mathbf{\Delta}^{\mathrm{op},\natural}$-object is a functor $F\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ such that

$$F([n]) \xrightarrow{\sim} F([1]) \times_{F([0])} \cdots \times_{F([0])} F([1]).$$

In particular, a Segal $\mathbf{\Delta}^{\mathrm{op},\natural}$-space is precisely a *Segal space* in the sense of Rezk [33], which describes the algebraic structure of an ∞-category. On the other hand, a Segal $\mathbf{\Delta}^{\mathrm{op},\flat}$-object F satisfies

$$F([n]) \simeq F([1])^{\times n},$$

and describes an associative monoid.

Example 3.4. For any integer n the product $\mathbf{\Delta}^{n,\mathrm{op}} := (\mathbf{\Delta}^{\mathrm{op}})^{\times n}$ has a coordinate-wise factorization system (i.e. a morphism is active or inert precisely when all of its components are). Using this we can define two algebraic pattern structures $\mathbf{\Delta}^{n,\mathrm{op},\natural}$ and $\mathbf{\Delta}^{n,\mathrm{op},\flat}$, where

$$\mathbf{\Delta}^{n,\mathrm{op},\natural,\mathrm{el}} := (\mathbf{\Delta}^{\mathrm{op},\natural,\mathrm{el}})^{n}$$

consists of all objects $([i_1], \ldots, [i_n])$ with $i_s = 0$ or 1 for all s, while

$$\mathbf{\Delta}^{n,\mathrm{op},\flat,\mathrm{el}} := \{([1], \ldots, [1])\}.$$

These are both special cases of products of algebraic patterns (Corollary 5.5). Segal $\mathbf{\Delta}^{n,\mathrm{op},\natural}$-spaces are *$n$-uple Segal spaces*, which describe internal ∞-categories in internal ∞-categories in ... in ∞-categories. A special class of these was first introduced by Barwick [1] as a model for (∞, n)-categories. On the other hand, the Dunn–Lurie additivity theorem [30, Theorem 5.1.2.2] implies that Segal $\mathbf{\Delta}^{n,\mathrm{op},\flat}$-objects are equivalent to $\mathbb{E}_n$-algebras, i.e. algebras for the little n-disc operad.

Example 3.5. Let $\mathbf{\Theta}_n$ be defined inductively by $\mathbf{\Theta}_0 := *$ and $\mathbf{\Theta}_n := \mathbf{\Delta} \wr \mathbf{\Theta}_{n-1}$, where for any category $\mathbf{C}$ the wreath product $\mathbf{\Delta} \wr \mathbf{C}$ has objects $[n](C_1, \ldots, C_n)$ with $C_i \in \mathbf{C}$, and morphisms $[n](C_1, \ldots, C_n) \to [m](C'_1, \ldots, C'_m)$ given by morphisms $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}$ together with maps $\psi_{ij}\colon C_i \to C_j$ in $\mathbf{C}$ whenever $\phi(i-1) < j \leq \phi(i)$. (This category was first considered in unpublished work of Joyal; the “wreath product” definition is due to Berger [5].) Then $\mathbf{\Theta}_n$ has an inductively defined factorization system (first defined in [4, Lemma 1.11]): the morphism above is *inert* (or *active*) if ϕ is inert (active) and each ψ_{ij} is inert (active). We can again use this to define two algebraic patterns. To do so we need some notation: We inductively define objects $C_0, \ldots, C_n$ in $\mathbf{\Theta}_n$ by $C_0 := [0]()$ and $C_n := [1](C_{n-1})$, starting with C_0 being the unique object of $\mathbf{\Theta}_0$. Then

- $\mathbf{\Theta}_n^{\mathrm{op},\natural}$ is defined by taking $\mathbf{\Theta}_n^{\mathrm{op},\natural,\mathrm{el}}$ to contain the objects $C_0, \ldots, C_n$; we can depict this category as

$$C_n \rightrightarrows C_{n-1} \rightrightarrows \cdots \rightrightarrows C_0.$$

- $\mathbf{\Theta}_n^{\mathrm{op},\flat}$ is defined by taking $\mathbf{\Theta}_n^{\mathrm{op},\flat,\mathrm{el}}$ to contain the single object C_n.

Segal $\mathbf{\Theta}_n^{\mathrm{op},\natural}$-spaces are then precisely Rezk's $\mathbf{\Theta}_n$*-spaces* [34], which describe the algebraic structure of (∞, n)-categories. On the other hand, Segal $\mathbf{\Theta}_n^{\mathrm{op},\flat}$-objects are again equivalent to $\mathbb{E}_n$-algebras — this follows from [2, Theorem 8.12] together with the Dunn–Lurie additivity theorem.

Example 3.6. All the examples considered so far are special cases of the following construction, due to Barwick: Suppose Φ is a *perfect operator category* in the sense of [2], and let $\Lambda(\Phi)$ be its Leinster category, which is the Kleisli category of a certain monad on Φ. This has an active-inert factorization system by [2, Lemma 7.3], where the active morphisms are the free morphisms on morphisms of Φ. Using this factorization system we can define two natural algebraic patterns:

- $\Lambda(\Phi)^\flat$ is defined by taking $\Lambda(\Phi)^{\flat,\mathrm{el}}$ to consist only of the terminal object $* \in \Phi$,
- $\Lambda(\Phi)^\natural$ is defined by taking $\Lambda(\Phi)^{\natural,\mathrm{el}}$ to contain all objects E such that there is an inert map $* \rightarrowtail E$ in $\Lambda(\Phi)$.

If $\mathbb{O}$ denotes the category of (possibly empty) ordered finite sets then $\Lambda(\mathbb{O}) \simeq \mathbf{\Delta}^{\mathrm{op}}$, while if $\mathbb{F}$ denotes the category of finite sets then $\Lambda(\mathbb{F}) \simeq \mathbb{F}_*$, and these pattern structures agree with those defined above. The same holds for $\mathbf{\Theta}_n^{\mathrm{op}}$, which can be described as the Leinster category of a wreath product $\mathbb{O}^{\wr n}$ of operator categories.

Example 3.7. Let $\mathbf{\Omega}$ be the *dendroidal category* of Moerdijk and Weiss [32, §3]; this can be defined as the category of free operads on trees. This has a natural active-inert factorization system, described for example in [26] (where the inert maps are called "free" and the active ones "boundary-preserving"). Using this we can define an algebraic pattern $\mathbf{\Omega}^{\mathrm{op},\natural}$ where $\mathbf{\Omega}^{\mathrm{op},\natural,\mathrm{el}}$ consists of the *corollas* C_n (i.e. trees with one vertex) and the plain edge η. Segal $\mathbf{\Omega}^{\mathrm{op},\natural}$-spaces are the dendroidal Segal spaces introduced by Cisinski and Moerdijk [11], which describe the algebraic structure of ∞-operads. The Segal condition says that the value of a Segal object at a tree decomposes as a limit over the corollas and edges of the tree. (We can also consider a pattern $\mathbf{\Omega}^{\mathrm{op},\flat}$ where the elementary objects are just the corollas; then Segal $\mathbf{\Omega}^{\mathrm{op},\flat}$-spaces describe ∞-operads with a single object.)

Example 3.8. If Φ is an operator category, let $\mathbf{\Delta}_\Phi$ be the category defined in [2, Definition 2.4]. This has pairs $([m], f\colon [m] \to \Phi)$ as objects, and morphisms $([m], f) \to ([n], g)$ are given by morphisms $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}$ together with certain natural transformations $\eta\colon f \to g \circ \phi$. We define a morphism $(\phi, \eta)\colon ([m], f) \to ([n], g)$ in $\mathbf{\Delta}_\Phi$ to be *inert* if ϕ is inert in $\mathbf{\Delta}$, and *active* if ϕ is active and $\eta_i\colon f(i) \to g(\phi(i))$ is an isomorphism for every $0 \leq i \leq m$. This gives an inert–active factorization system on $\mathbf{\Delta}_\Phi^{\mathrm{op}}$, and we define an algebraic pattern $\mathbf{\Delta}_\Phi^{\mathrm{op},\natural}$ by taking the elementary objects to be $([0], *)$ and $([1], I \to *)$ (where $*$ denotes the terminal object). Then Segal $\mathbf{\Delta}_\Phi^{\mathrm{op},\natural}$-spaces are precisely the Segal Φ-operads of [2, §2], which describe Φ-∞-*operads*. (When Φ is $\mathbb{F}$ these agree with ∞-operads in the sense of Lurie by [2, Theorem 10.16], and with dendroidal Segal spaces by [10, Theorem 1.1].)

Example 3.9. Let $\mathbf{\Gamma}$ be the category of acyclic connected finite directed graphs defined by Hackney, Robertson, and Yau in [19]. Then $\mathbf{\Gamma}^{\mathrm{op}}$ has an inert–active factorization system described in [27, 2.4.14] (where the active maps are called “refinements” and the inert maps are called “convex open inclusions”). Using this we can define an algebraic pattern structure $\mathbf{\Gamma}^{\mathrm{op},\natural}$ by taking the elementary objects to be the elementary graphs with at most one vertex. Segal $\mathbf{\Gamma}^{\mathrm{op},\natural}$-spaces are equivalent to the model of ∞-*properads* as “graphical spaces” satisfying a Segal condition that is briefly discussed in [18]; this is presumably equivalent (after imposing a completeness condition) to the model of ∞-properads as certain presheaves of sets on $\mathbf{\Gamma}$ constructed in [19].

Example 3.10. Let $\mathbf{\Xi}$ denote the category of unrooted trees defined in [20]. Then $\mathbf{\Xi}^{\mathrm{op}}$ has an inert–active factorization system, described in [20, §4], and using this we can give $\mathbf{\Xi}^{\mathrm{op}}$ an algebraic pattern structure $\mathbf{\Xi}^{\mathrm{op},\natural}$ where the elementary objects are the stars and the plain edge. Segal $\mathbf{\Xi}^{\mathrm{op},\natural}$-spaces are then precisely the model for cyclic ∞-operads considered by Hackney, Robertson, and Yau [20].

Example 3.11. Let $\mathbf{U}$ denote the category of connected graphs defined in [21]. Then $\mathbf{U}^{\mathrm{op}}$ has an inert–active factorization system, described in [21, §2.1], and we can use this to equip $\mathbf{U}^{\mathrm{op}}$ with an algebraic pattern structure $\mathbf{U}^{\mathrm{op},\natural}$ where the elementary objects are the stars and the plain edge. We can also consider an algebraic pattern $\mathbf{U}^{\mathrm{op},\flat}$ where the elementary objects are just the stars; Segal $\mathbf{U}^{\mathrm{op},\flat}$-objects are then the *Segal modular operads* defined by Hackney, Robertson, and Yau [21].

Remark 3.12. Below in §9 we will define (weak) Segal fibrations over an algebraic pattern, which give general classes of examples of algebraic patterns. As a special case, we will see that every ∞-operad $\mathcal{O}$ in the sense of Lurie [30] has an algebraic pattern structure $\mathcal{O}^\flat$ such that a Segal $\mathcal{O}^\flat$-object in an ∞-category $\mathcal{C}$ with finite products is precisely an $\mathcal{O}$-monoid in $\mathcal{C}$.

4. Morphisms of algebraic patterns

In this section we define morphisms of algebraic patterns, and consider when they are compatible with Segal objects. We then discuss some examples of such morphisms.

Definition 4.1. Let $\mathcal{O}$ and $\mathcal{P}$ be algebraic patterns. A *morphism of algebraic patterns* from $\mathcal{O}$ to $\mathcal{P}$ is a functor $f\colon \mathcal{O} \to \mathcal{P}$ such that f preserves both active and inert maps, and takes elementary objects in $\mathcal{O}$ to elementary objects in $\mathcal{P}$.

In general, morphisms of algebraic patterns do not necessarily interact well with Segal objects. We therefore isolate the class of morphisms that preserve Segal objects under restriction:

Definition 4.2. A morphism of algebraic patterns $f\colon \mathcal{O} \to \mathcal{P}$ is called a *Segal morphism* if it satisfies the following condition:

($*$) For all $X \in \mathcal{O}$ the induced functor $\mathcal{O}^{\mathrm{el}}_{X/} \to \mathcal{P}^{\mathrm{el}}_{f(X)/}$ induces an equivalence

$$\lim_{\mathcal{P}^{\mathrm{el}}_{f(X)/}} F \xrightarrow{\sim} \lim_{\mathcal{O}^{\mathrm{el}}_{X/}} F \circ f^{\mathrm{el}}$$

for every Segal $\mathcal{P}$-space $F\colon \mathcal{P} \to \mathcal{S}$.

Remark 4.3. The condition depends only on the restriction of F to $\mathcal{P}^{\mathrm{el}}$, so we could equivalently have considered functors $\mathcal{P}^{\mathrm{el}} \to \mathcal{S}$ that occur as restrictions of Segal $\mathcal{P}$-spaces.

Remark 4.4. In practice, a morphism f is a Segal morphism because the functor $\mathcal{O}^{\mathrm{el}}_{X/} \to \mathcal{P}^{\mathrm{el}}_{f(X)/}$ is coinitial, in which case we say that f is a *strong Segal morphism*. However, the more general definition allows for the following characterization:

Lemma 4.5. *The following are equivalent for a morphism of algebraic patterns $f\colon \mathcal{O} \to \mathcal{P}$:*

(1) f is a Segal morphism.
(2) The functor $f^*\colon \mathrm{Fun}(\mathcal{P}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{S})$ restricts to a functor $\mathrm{Seg}_{\mathcal{P}}(\mathcal{S}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$.
(3) For every ∞-category $\mathcal{C}$, the functor $f^*\colon \mathrm{Fun}(\mathcal{P}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{C})$ restricts to a functor $\mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$.

Proof. It is immediate from the definition that (1) is equivalent to (2) and that (3) implies (2). It remains to check that (2) implies (3). Suppose $F\colon \mathcal{P} \to \mathcal{C}$ is a Segal $\mathcal{P}$-object; we need to show that f^*F is a Segal $\mathcal{O}$-object, i.e. that for all $X \in \mathcal{O}$ the natural map

$$\lim_{\mathcal{P}^{\mathrm{el}}_{f(X)/}} F \to \lim_{\mathcal{O}^{\mathrm{el}}_{X/}} F \circ f^{\mathrm{el}}$$

is an equivalence in $\mathcal{C}$. Equivalently, we must show that for any $C \in \mathcal{C}$, the map of spaces

$$\lim_{\mathcal{P}^{\mathrm{el}}_{f(X)/}} \mathrm{Map}(C, F) \to \lim_{\mathcal{O}^{\mathrm{el}}_{X/}} \mathrm{Map}(C, F) \circ f^{\mathrm{el}}$$

is an equivalence, which is true since $\mathrm{Map}(C, F)$ is a Segal $\mathcal{P}$-space. □

Remark 4.6. One might feel that the Segal property is sufficiently fundamental that it should be included as part of the notion of a morphism of algebraic patterns. However, more general morphisms also turn out to be occasionally useful. For example, the identity functor of $\mathbb{F}_*$ viewed as a functor $\mathbb{F}_*^\flat \to \mathbb{F}_*^\natural$ is a morphism of patterns, but is not a Segal morphism, and we will see later in §6 that it induces a functor from Segal $\mathbb{F}_*^\flat$-objects to Segal $\mathbb{F}_*^\natural$-objects that can be viewed as a right Kan extension along $\mathrm{id}_{\mathbb{F}_*}$.

Proposition 4.7. *Suppose $f\colon \mathcal{O} \to \mathcal{P}$ is a Segal morphism of algebraic patterns, and $\mathcal{C}$ is a presentable ∞-category. Then there is an adjunction*

$$L_{\mathrm{Seg}} f_! \colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \rightleftarrows \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) : f^*$$

where L_{Seg} is the localization functor left adjoint to the inclusion $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{O}, \mathcal{C})$, and $f_!$ is the functor of left Kan extension along f.

Proof. Since f^* restricts to a functor on Segal objects, for $F \in \mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$ and $G \in \mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ we have a natural equivalence

$$\begin{aligned} \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})}(L_{\mathrm{Seg}} f_! F, G) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{C})}(f_! F, G) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{P},\mathcal{C})}(F, f^* G) \\ &\simeq \mathrm{Map}_{\mathrm{Seg}_{\mathcal{P}}(\mathcal{C})}(F, f^* G), \end{aligned}$$

which implies the claim. □

Remark 4.8. Below in §7 we will give conditions on a morphism f such that the left Kan extension functor $f_!$ preserves Segal objects, and so gives a left adjoint to f^* without localizing.

We now consider some examples of morphisms of patterns:

Example 4.9. There is a functor $|{-}|\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbb{F}_*$ which takes an object $[n]$ to $|[n]| := \langle n \rangle$ and a morphism $\alpha\colon [n] \to [m]$ in $\mathbf{\Delta}$ to $|\alpha|\colon |[m]| \to |[n]|$ given by

$$|\alpha|(i) = \begin{cases} j & \text{if } \alpha(j-1) < j \leq \alpha(j) \\ 0 & \text{otherwise} \end{cases}$$

This functor gives a Segal morphism of algebraic patterns $\mathbf{\Delta}^{\mathrm{op},\natural} \to \mathbb{F}_*^\natural$ as well as $\mathbf{\Delta}^{\mathrm{op},\flat} \to \mathbb{F}_*^\flat$.

Example 4.10. There is a functor $\tau_n \colon \mathbf{\Delta}^{n,\mathrm{op}} \to \mathbf{\Theta}_n^{\mathrm{op}}$, defined inductively by setting $\tau_0 :=$ id and

$$\tau_n([i_1], \dots, [i_n]) := [i_1](\tau_{n-1}([i_2], \dots, [i_n]), \dots, \tau_{n-1}([i_2], \dots, [i_n])).$$

This functor gives a Segal morphism of algebraic patterns $\mathbf{\Delta}^{n,\mathrm{op},\natural} \to \mathbf{\Theta}_n^{\mathrm{op},\natural}$ as well as $\mathbf{\Delta}^{n,\mathrm{op},\flat} \to \mathbf{\Theta}_n^{\mathrm{op},\flat}$.

Example 4.11. The previous examples are special cases of the following: Let $f \colon \Phi \to \Psi$ be an *operator morphism* between perfect operator categories, as defined in [2, Definition 1.10]. As discussed in [2, §7] this induces a functor $\Lambda(f) \colon \Lambda(\Phi) \to \Lambda(\Psi)$ between the corresponding Leinster categories, and it is easy to check that this preserves the inert and active morphisms. Since operator morphisms preserve terminal objects by definition, it follows from Example 3.6 that $\Lambda(f)$ preserves elementary objects, and hence gives morphisms of algebraic patterns $\Lambda(\Phi)^\natural \to \Lambda(\Psi)^\natural$ and $\Lambda(\Phi)^\flat \to \Lambda(\Psi)^\flat$. The latter is evidently a Segal morphism, since

$$\Lambda(\Phi)^{\flat,\mathrm{el}}_{I/} \cong \{* \to I\} \cong \{* \to f(I)\} \cong \Lambda(\Psi)^{\flat,\mathrm{el}}_{f(I)/},$$

where the second isomorphism is part of the definition of an operator morphism.

Example 4.12. Every operator category Φ has a unique operator morphism $|{-}| \colon \Phi \to \mathbb{F}$, which gives a Segal morphism $\Lambda(\Phi)^\flat \to \mathbb{F}_*^\flat$. This is also a Segal morphism $\Lambda(\Phi)^\natural \to \mathbb{F}_*^\natural$ provided the category $\Lambda(\Phi)^{\natural,\mathrm{el}}_{I/}$ is weakly contractible for all $I \in \Phi$.

Example 4.13. By [20, Definition 1.20], the category $\mathbf{\Omega}$ of trees can be identified with a subcategory of the category $\mathbf{\Xi}$ of unrooted trees, and [20, Definition 4.2] and [20, Remark 4.3] show that this inclusion gives a morphism of algebraic patterns $\iota \colon \mathbf{\Omega}^{\mathrm{op},\natural} \to \mathbf{\Xi}^{\mathrm{op},\natural}$. The description of morphisms in $\mathbf{\Omega}^{\mathrm{op}}$ in [20, Definition 1.20] implies that for every $X \in \mathbf{\Omega}^{\mathrm{op}}$ and every $\alpha \in \mathbf{\Xi}^{\mathrm{op},\mathrm{el}}_{\iota X/}$, the ∞-category $\mathbf{\Omega}^{\mathrm{op},\mathrm{el}}_{X/} \times_{\mathbf{\Xi}^{\mathrm{op},\mathrm{el}}_{\iota X/}} (\mathbf{\Xi}^{\mathrm{op},\mathrm{el}}_{\iota X/})_{/\alpha}$ has a terminal object. In particular, the functor $\mathbf{\Omega}^{\mathrm{op}}_{X/} \to \mathbf{\Xi}^{\mathrm{op}}_{\iota X/}$ is coinitial, and hence ι is a strong Segal morphism. The resulting functor

$$\iota^* \colon \mathrm{Seg}_{\mathbf{\Xi}^{\mathrm{op},\natural}}(\mathcal{S}) \to \mathrm{Seg}_{\mathbf{\Omega}^{\mathrm{op},\natural}}(\mathcal{S})$$

is the forgetful functor from cyclic ∞-operads to ∞-operads.

5. The ∞-category of algebraic patterns

In this section we construct the ∞-category of algebraic patterns, and describe limits and filtered colimits in this ∞-category. As a first step, we consider the ∞-category of ∞-categories equipped with a factorization system:

Definition 5.1. We define Fact to be the full subcategory of $\mathrm{Fun}(\Lambda^2_2, \mathrm{Cat}_\infty)$ (where Λ^2_2 denotes the category $0 \to 2 \leftarrow 1$) spanned by those cospans

$$\mathcal{C}_L \to \mathcal{C} \leftarrow \mathcal{C}_R$$

that describe factorization systems, i.e. those such that the functors $\mathcal{C}_L, \mathcal{C}_R \to \mathcal{C}$ are essentially surjective subcategory inclusions, and $\mathrm{Fun}_{L,R}(\Delta^2, \mathcal{C}) \to \mathrm{Fun}(\Delta^{\{0,2\}}, \mathcal{C})$ is an equivalence, where the domain is defined as the pullback

$$\begin{tikzcd}
\mathrm{Fun}_{L,R}(\Delta^2, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}(\Delta^1, \mathcal{C}_L) \times \mathrm{Fun}(\Delta^1, \mathcal{C}_R) \arrow[r] & \mathrm{Fun}(\Delta^{\{0,1\}}, \mathcal{C}) \times \mathrm{Fun}(\Delta^{\{1,2\}}, \mathcal{C}).
\end{tikzcd}$$

Proposition 5.2. *The ∞-category* Fact *is closed under limits and filtered colimits in* $\mathrm{Fun}(\Lambda^2_2, \mathrm{Cat}_\infty)$*. In particular, the ∞-category* Fact *has limits and filtered colimits, and the forgetful functor to* Cat_∞ *preserves these.*

This will follow from the following observation:

Lemma 5.3. *In the ∞-category* $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$*, the full subcategories of subcategory inclusions,*[8] *essentially surjective subcategory inclusions, and full subcategory inclusions, are all closed under limits and filtered colimits.*

Proof. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a subcategory inclusion precisely when $\mathcal{C}^\simeq \to \mathcal{D}^\simeq$ is a monomorphism of spaces, and $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(Fx, Fy)$ is a monomorphism of spaces for all $x, y \in \mathcal{C}$. A subcategory inclusion F is essentially surjective if the map $\mathcal{C}^\simeq \to \mathcal{D}^\simeq$ is an equivalence, and a full subcategory inclusion if the maps $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(Fx, Fy)$ are equivalences for all $x, y \in \mathcal{C}$. Since mapping spaces and the underlying space of a limit (or filtered colimit) in Cat_∞ are computed as limits (or filtered colimits) of spaces, it suffices to observe that equivalences and monomorphisms are closed under limits and filtered colimits in $\mathcal{S}$. This is obvious for equivalences, and for monomorphisms it follows from the characterization of these by [29, Lemma 5.5.6.15] as the morphisms

[8] Note that we use "subcategory inclusion" in the equivalence-invariant sense — in other words, a subcategory in our sense must include all equivalences between its objects.

$f: X \to Y$ such that the diagonal $X \to X \times_Y X$ is an equivalence, since filtered colimits commute with finite limits and limits commute. □

Proof of Proposition 5.2. It follows from Lemma 5.3 that cospans of subcategory inclusions are closed under limits and filtered colimits in $\mathrm{Fun}(\Lambda^2_2, \mathrm{Cat}_\infty)$. Since limits commute, the ∞-category $\mathrm{Fun}_{L,R}(\Delta^2, -)$, viewed as a functor $\mathrm{Fun}(\Lambda^2_2, \mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$, preserves limits, which implies that objects such that the natural map $\mathrm{Fun}_{L,R}(\Delta^2, -) \to \mathrm{Fun}(\Delta^{\{0,2\}}, -)$ is an equivalence are also closed under limits. The same holds for filtered colimits, since the objects mapped out of in the definition of $\mathrm{Fun}_{L,R}(\Delta^2, -)$ are compact, and filtered colimits commute with finite limits in Cat_∞. □

Definition 5.4. We now define the ∞-category AlgPatt of algebraic patterns as the full subcategory of the fibre product $\mathrm{Fact} \times_{\mathrm{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ (where the pullback is over $\mathrm{ev}_0 : \mathrm{Fact} \to \mathrm{Cat}_\infty$ and $\mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$) containing the objects

$$\mathcal{C}' \to \mathcal{C}_L \to \mathcal{C} \leftarrow \mathcal{C}_R$$

where $\mathcal{C}' \to \mathcal{C}_L$ is a full subcategory inclusion.

Applying Lemma 5.3 again, now in the case of full subcategory inclusions, we get:

Corollary 5.5. *The full subcategory* AlgPatt *is closed under limits and filtered colimits in*

$$\mathrm{Fun}(\Lambda^2_2, \mathrm{Cat}_\infty) \times_{\mathrm{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty).$$

In particular, AlgPatt *has limits and filtered colimits, and the forgetful functor to* Cat_∞ *preserves these.* □

Remark 5.6. The ∞-category AlgPatt contains *all* morphisms of algebraic patterns; restricting these to Segal morphisms gives a (wide) subcategory $\mathrm{AlgPatt}^{\mathrm{Seg}}$. However, note that Segal morphisms do not seem to be closed under filtered colimits or general pullbacks, though by Lemma 4.5 and the next example they *are* closed under finite products.

Example 5.7. For any pair of algebraic patterns $\mathcal{O}$, $\mathcal{P}$ we have a cartesian product pattern $\mathcal{O} \times \mathcal{P}$. For this we have an equivalence

$$\mathrm{Seg}_{\mathcal{O}\times\mathcal{P}}(\mathcal{C}) \simeq \mathrm{Seg}_{\mathcal{O}}(\mathrm{Seg}_{\mathcal{P}}(\mathcal{C}))$$

for any $\mathcal{O} \times \mathcal{P}$-complete ∞-category $\mathcal{C}$. To see this, observe that a right Kan extension along $\mathcal{O}^{\mathrm{el}} \times \mathcal{P}^{\mathrm{el}} \to \mathcal{O}^{\mathrm{int}} \times \mathcal{P}^{\mathrm{int}}$ can be computed in two stages in two ways, by first doing the right Kan extension to either $\mathcal{O}^{\mathrm{el}} \times \mathcal{P}^{\mathrm{int}}$ or $\mathcal{O}^{\mathrm{int}} \times \mathcal{P}^{\mathrm{el}}$; this shows that $F : \mathcal{O} \times \mathcal{P} \to \mathcal{C}$ is a Segal object if and only if $F(O, -)$ is a $\mathcal{P}$-Segal object for all $O \in \mathcal{O}$ and $F(-, P)$ is an $\mathcal{O}$-Segal object for all $P \in \mathcal{P}$.

Example 5.8. The pattern $\mathbf{\Delta}^{\mathrm{op},\flat}$ can be described as the pullback $\mathbf{\Delta}^{\mathrm{op},\natural} \times_{\mathbb{F}_*^\natural} \mathbb{F}_*^\flat$ using the map $\mathbf{\Delta}^{\mathrm{op},\natural} \to \mathbb{F}_*^\natural$ from Example 4.9 and the identity of $\mathbb{F}_*$ viewed as a morphism of patterns $\mathbb{F}_*^\flat \to \mathbb{F}_*^\natural$. (Similarly, for the other pairs of patterns $\mathcal{O}^\flat, \mathcal{O}^\natural$ mentioned in §3 the pattern $\mathcal{O}^\flat$ is the pullback $\mathcal{O}^\natural \times_{\mathbb{F}_*^\natural} \mathbb{F}_*^\flat$ for a morphism of patterns $\mathcal{O}^\natural \to \mathbb{F}_*^\natural$.)

Example 5.9. Let $\mathbf{\Theta}^{\mathrm{op},\natural}$ be the colimit $\mathrm{colim}_{n\geq 0}\, \mathbf{\Theta}_n^{\mathrm{op},\natural}$ induced by the sequence of natural inclusions $\mathbf{\Theta}_n^{\mathrm{op},\natural} \hookrightarrow \mathbf{\Theta}_{n+1}^{\mathrm{op},\natural}$, $n \geq 0$, where $\mathbf{\Theta}_n^{\mathrm{op},\natural}$ is the algebraic pattern defined in Example 3.5. The underlying category $\mathbf{\Theta}$ is equivalent to that introduced by Joyal [25] to give a definition of weak higher categories. It is easy to see that in this case we have an equivalence

$$\mathrm{Seg}_{\mathbf{\Theta}^{\mathrm{op},\natural}}(\mathcal{S}) \simeq \lim_{n\geq 0} \mathrm{Seg}_{\mathbf{\Theta}_n^{\mathrm{op},\natural}}(\mathcal{S}),$$

so that Segal $\mathbf{\Theta}^{\mathrm{op},\natural}$-spaces model (∞,∞)-categories (in the inductive sense). In particular, the canonical functor $\mathrm{Seg}_{\mathbf{\Theta}^{\mathrm{op},\natural}}(\mathcal{S}) \to \mathrm{Seg}_{\mathbf{\Theta}_n^{\mathrm{op},\natural}}(\mathcal{S})$ gives the underlying (∞,n)-category of an (∞,∞)-category.

6. Right Kan extensions and Segal objects

Our goal in this section is to give a sufficient criterion on a morphism of algebraic patterns $f\colon \mathcal{O} \to \mathcal{P}$ such that right Kan extension along f preserves Segal objects.

Definition 6.1. We say that a morphism $f\colon \mathcal{O} \to \mathcal{P}$ of algebraic patterns has *unique lifting of active morphisms* if for every active morphism $\phi\colon P \to f(O)$ in $\mathcal{P}$, the ∞-groupoid of lifts of ϕ to an active morphism $O' \to O$ in $\mathcal{O}$ is contractible. More precisely, the fibre $(\mathcal{O}^{\mathrm{act}}_{/O})^{\simeq}_{\phi}$ of the morphism

$$(\mathcal{O}^{\mathrm{act}}_{/O})^{\simeq} \to (\mathcal{P}^{\mathrm{act}}_{/f(O)})^{\simeq}$$

at ϕ is contractible. Equivalently, f has unique lifting of active morphisms if this morphism of ∞-groupoids is an equivalence for all $O \in \mathcal{O}$.

Lemma 6.2. *A morphism of algebraic patterns $f\colon \mathcal{O} \to \mathcal{P}$ has unique lifting of active morphisms if and only if it satisfies the following condition:*

($*$) *For all $P \in \mathcal{P}$ the functor*

$$\mathcal{O}^{\mathrm{int}}_{P/} \to \mathcal{O}_{P/}$$

is coinitial.

Proof. By [29, Theorem 4.1.3.1], the functor $\mathcal{O}^{\mathrm{int}}_{P/} \to \mathcal{O}_{P/}$ is coinitial if and only if for every morphism $\phi\colon P \to f(O)$ in $\mathcal{P}$, the ∞-category $(\mathcal{O}^{\mathrm{int}}_{P/})_{/\phi}$ is weakly contractible. This ∞-category has objects pairs

$$\left(O' \xrightarrow{\alpha} O, \quad \begin{tikzcd} & P \arrow[dl, tail, "\iota"'] \arrow[dr, "\phi"] & \\ f(O') \arrow[rr, "f(\alpha)"] & & f(O) \end{tikzcd} \right),$$

where ι is inert. The morphism α has an essentially unique inert–active factorization, and since f is compatible with this factorization we see that the full subcategory of objects where α is active is cofinal. By uniqueness of factorizations a morphism in this subcategory is required to be an equivalence, hence this is an ∞-groupoid, and so $(*)$ is equivalent to this ∞-groupoid being contractible. But an object in this subcategory gives an inert–active factorization of ϕ, and we see that it is equivalent to the ∞-groupoid of lifts of the active part of ϕ to an active morphism in $\mathcal{O}$. $\square$

Proposition 6.3. *Suppose $f\colon \mathcal{O} \to \mathcal{P}$ is a morphism of algebraic patterns that has unique lifting of active morphisms and $\mathcal{C}$ is an $\mathcal{O}$- and $\mathcal{P}$-complete ∞-category such that the pointwise right Kan extension*

$$f_*\colon \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{P}, \mathcal{C})$$

exists. Then f_* restricts to a functor

$$f_*\colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}).$$

Remark 6.4. We emphasize that the condition of unique lifting of active morphisms is far from a *necessary* one. Indeed, the functor f_* will preserve Segal objects if and only if its left adjoint f^* preserves Segal equivalences. In [10] the latter condition was checked for a certain morphism $\tau\colon \mathbf{\Delta}^{1,\mathrm{op}}_{\mathbb{F}} \to \mathbf{\Omega}^{\mathrm{op}}$, which clearly does *not* have unique lifting of active morphisms.

Proof of Proposition 6.3. By Lemma 6.2, the condition that f has unique lifting of active morphisms implies that for any functor $F\colon \mathcal{O} \to \mathcal{C}$, the Beck–Chevalley transformation

$$(f_*F)|_{\mathcal{P}^{\mathrm{int}}} \to f^{\mathrm{int}}_*(F|_{\mathcal{O}^{\mathrm{int}}})$$

is an equivalence. If F is a Segal $\mathcal{O}$-object, then $F|_{\mathcal{O}^{\mathrm{int}}} \simeq i_{\mathcal{O},*}F|_{\mathcal{O}^{\mathrm{el}}}$, where $i_{\mathcal{O}}$ is the inclusion $\mathcal{O}^{\mathrm{el}} \hookrightarrow \mathcal{O}^{\mathrm{int}}$, so in this case we have $(f_*F)|_{\mathcal{P}^{\mathrm{int}}} \simeq f^{\mathrm{int}}_* i_{\mathcal{O},*}F|_{\mathcal{O}^{\mathrm{el}}}$. By naturality of right Kan extensions in the commutative square

$$\begin{tikzcd} \mathcal{O}^{\mathrm{el}} \arrow[r, "f^{\mathrm{el}}"] \arrow[d, "i_{\mathcal{O}}"'] & \mathcal{P}^{\mathrm{el}} \arrow[d, "i_{\mathcal{P}}"] \\ \mathcal{O}^{\mathrm{int}} \arrow[r, "f^{\mathrm{int}}"] & \mathcal{P}^{\mathrm{int}} \end{tikzcd}$$

this can in turn be identified with $i_{\mathcal{P},*}f^{\mathrm{el}}_* F|_{\mathcal{O}^{\mathrm{el}}}$. Moreover, since $\mathcal{P}^{\mathrm{el}}$ is a full subcategory of $\mathcal{P}^{\mathrm{int}}$, we have

$$f^{\mathrm{el}}_* F|_{\mathcal{O}^{\mathrm{el}}} \simeq i^*_{\mathcal{P}} i_{\mathcal{P},*} f^{\mathrm{el}}_* F|_{\mathcal{O}^{\mathrm{el}}} \simeq i^*_{\mathcal{P}} f^{\mathrm{int}}_* i_{\mathcal{O},*} F|_{\mathcal{O}^{\mathrm{el}}} \simeq i^*_{\mathcal{P}} f^{\mathrm{int}}_* F|_{\mathcal{O}^{\mathrm{int}}}.$$

Combining these equivalences, we see that $(f_*F)|_{\mathcal{P}^{\mathrm{int}}} \simeq i_{\mathcal{P},*}(i^*_{\mathcal{P}} f^{\mathrm{int}}_* F|_{\mathcal{O}^{\mathrm{int}}}) \simeq i_{\mathcal{P},*}(f_*F)|_{\mathcal{P}^{\mathrm{el}}}$, where the second equivalence is given by $i^*_{\mathcal{P}} f^{\mathrm{int}}_*(F|_{\mathcal{O}^{\mathrm{int}}}) \simeq i^*_{\mathcal{P}}(f_*F)|_{\mathcal{P}^{\mathrm{int}}} \simeq (f_*F)|_{\mathcal{P}^{\mathrm{el}}}$. Hence f_*F is a Segal $\mathcal{P}$-object. □

Remark 6.5. If f in Proposition 6.3 is moreover a Segal morphism, we get an adjunction

$$f^* : \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) \rightleftarrows \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) : f_*$$

by restricting the adjunction $f^* \dashv f_*$ on functor ∞-categories.

Example 6.6. Suppose we have two categorical patterns $\mathfrak{O}_1$ and $\mathfrak{O}_2$ with the same underlying ∞-category $\mathcal{O}$ and the same inert–active factorization system, and $\mathfrak{O}_1^{\mathrm{el}}$ is a full subcategory of $\mathfrak{O}_2^{\mathrm{el}}$. Then the identity functor of $\mathcal{O}$ gives a morphism of algebraic patterns $\mathfrak{O}_1 \to \mathfrak{O}_2$ for which unique lifting of active morphisms holds trivially. In this case, this just means that the Segal condition for $\mathfrak{O}_1$ is stronger than that for $\mathfrak{O}_2$. For example, this holds for the identity morphism of $\mathbb{F}_*$ viewed as a morphism $\mathbb{F}_*^{\flat} \to \mathbb{F}_*^{\natural}$. On the other hand, the identity functor would typically *not* be a Segal morphism.

Example 6.7. The inclusion $i\colon \{[0]\} \to \boldsymbol{\Delta}^{\mathrm{op},\natural}$ clearly has unique lifting of active morphisms, since the only active morphism to $[0]$ in $\boldsymbol{\Delta}^{\mathrm{op}}$ is the identity. In this case, the right Kan extension functor

$$i_*\colon \mathcal{C} \simeq \mathrm{Fun}(\{[0]\}, \mathcal{C}) \to \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{C})$$

takes an object $C \in \mathcal{C}$ to the simplicial object i_*C given by $(i_*C)_n \simeq \prod_{i=0}^n C$, with face maps corresponding to projections and degeneracies given by diagonal maps. This clearly satisfies the Segal condition. More generally, the inclusion $\boldsymbol{\Theta}_{n-1}^{\mathrm{op},\natural} \hookrightarrow \boldsymbol{\Theta}_n^{\mathrm{op},\natural}$ has unique lifting of active morphisms for all $n \geq 1$.

Example 6.8. Let $\iota\colon \boldsymbol{\Omega}^{\mathrm{op},\natural} \to \boldsymbol{\Xi}^{\mathrm{op},\natural}$ be the Segal morphism of Example 4.13. Since the active morphisms in $\boldsymbol{\Xi}^{\mathrm{op}}$ are the boundary-preserving ones, it is easy to see that ι has unique lifting of active morphisms. Then Proposition 6.3 and Remark 6.5 give an adjunction

$$\iota^* : \mathrm{Seg}_{\boldsymbol{\Xi}^{\mathrm{op}}}(\mathcal{S}) \rightleftarrows \mathrm{Seg}_{\boldsymbol{\Omega}^{\mathrm{op},\natural}}(\mathcal{S}) : \iota_*,$$

where ι_* is a right adjoint to the forgetful functor ι^* from cyclic ∞-operads to ∞-operads. According to [13, §2.15] the analogue of this right adjoint for ordinary cyclic operads was first considered in the unpublished thesis of J. Templeton.

7. Left Kan extensions and Segal objects

In this section we will give conditions under which left Kan extension along a morphism f preserves Segal objects in $\mathcal{C}$. In contrast to the case of right Kan extensions, this requires strong assumptions on both f and the target ∞-category $\mathcal{C}$. Part of the condition is a uniqueness requirement on lifts of inert morphisms, which we consider first:

Definition 7.1. A morphism of algebraic patterns $f : \mathcal{O} \to \mathcal{P}$ is said to have *unique lifting of inert morphisms* if for every inert morphism $f(O) \to P$ the ∞-groupoid of lifts to inert morphisms $O \to O'$ is contractible. More precisely, the fibre $(\mathcal{O}^{\mathrm{int}}_{O/})^{\simeq}_{\phi}$ of the morphism

$$(\mathcal{O}^{\mathrm{int}}_{O/})^{\simeq} \to (\mathcal{P}^{\mathrm{int}}_{f(O)/})^{\simeq}$$

at ϕ is contractible. Equivalently, f has unique lifting of inert morphisms if this morphism of ∞-groupoids is an equivalence for all $O \in \mathcal{O}$.

Lemma 7.2. *A morphism of algebraic patterns $f : \mathcal{O} \to \mathcal{P}$ has unique lifting of inert morphisms if and only if it satisfies the following condition:*

(∗) *For all $P \in \mathcal{P}$ the functor*

$$\mathcal{O}^{\mathrm{act}}_{/P} \to \mathcal{O}_{/P}$$

is cofinal.

Proof. This follows by the same argument as for Lemma 6.2, with the roles of active and inert morphisms reversed. □

Unique lifting of inert morphisms allows us to functorially transport active morphisms along inert morphisms, in the following sense:

Proposition 7.3. *Suppose $f : \mathcal{O} \to \mathcal{P}$ has unique lifting of inert morphisms. Let*

$$\mathcal{X} \subseteq \mathcal{O} \times_{\mathcal{P}} \mathcal{P}^{\Delta^1}$$

be the full subcategory of the fibre product over evaluation at 0, with objects those pairs

$$(O, f(O) \xrightarrow{\phi} P)$$

where ϕ is active. Then the projection $\mathcal{X} \to \mathcal{P}$ given by evaluation at $1 \in \Delta^1$ is a cocartesian fibration, and a morphism

$$\left(\begin{tikzcd} O \arrow[d, "\omega"'] \\ O' \end{tikzcd} , \begin{tikzcd} f(O) \arrow[r, rightsquigarrow] \arrow[d, "f(\omega)"'] & P \arrow[d] \\ f(O') \arrow[r, rightsquigarrow] & P' \end{tikzcd} \right)$$

is cocartesian if and only if ω is inert.

Proof. We first show that such a morphism with ω inert is cocartesian. This means that given a morphism $O \to X$ in $\mathcal{O}$ and a commutative diagram

$$\begin{tikzcd} f(O) \arrow[rr] \arrow[dr, tail, "f(\omega)"'] \arrow[dd, rightsquigarrow] & & f(X) \arrow[dd, rightsquigarrow] \\ & f(O') \arrow[ur, dashed] \arrow[dd, rightsquigarrow] & \\ P \arrow[rr] \arrow[dr] & & Q, \\ & P' \arrow[ur] & \end{tikzcd}$$

there exists a unique lift $O' \to X$ making the diagram commute.

The morphism $O \to X$ has a unique inert–active factorization as $O \rightarrowtail O'' \rightsquigarrow X$. Since f is compatible with the factorization system, we see that the unique inert–active factorization of $f(O) \to Q$ is $f(O) \rightarrowtail f(O'') \rightsquigarrow f(X) \rightsquigarrow Q$.

On the other hand, the inert–active factorization of $f(O') \to Q$ gives another factorization $f(O) \rightarrowtail f(O') \rightarrowtail Q' \rightsquigarrow Q$, where by uniqueness we must have $Q' \simeq f(O'')$. Since f has unique lifts of inert morphisms, the map $f(O') \rightarrowtail f(O'')$ lifts to a unique morphism $O' \rightarrowtail O''$, and moreover by uniqueness the composite $O \rightarrowtail O' \rightarrowtail O''$ must be the inert map $O \rightarrowtail O''$ arising from the factorization of $O \to X$.

Thus, there are unique diagrams

$$\begin{tikzcd} O \arrow[r, tail] \arrow[rrr, bend left] & O' \arrow[r, tail] & O'' \arrow[r, rightsquigarrow] & X, \end{tikzcd}$$

$$\begin{tikzcd} f(O) \arrow[r, tail] \arrow[rrr, bend left] \arrow[d, rightsquigarrow] & f(O') \arrow[r, tail] \arrow[d, rightsquigarrow] & f(O'') \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & f(X) \arrow[dl, rightsquigarrow] \\ P \arrow[r] & P' \arrow[r] & Q, & \end{tikzcd}$$

which give the required unique factorization (since any other factorization through $(O', f(O') \rightsquigarrow P')$ must induce these by uniqueness of inert–active factorizations).

We next check that $\mathcal{X} \to \mathcal{P}$ is a cocartesian fibration. This amounts to showing that cocartesian morphisms exist, and by the first part of the proof it suffices to check that given $(O, f(O) \overset{\phi}{\rightsquigarrow} P)$ with ϕ active and a morphism $P \to P'$, there exists a morphism

$$\left(\begin{tikzcd} O \arrow[d, tail, "\omega"'] \\ O' \end{tikzcd}, \begin{tikzcd} f(O) \arrow[r, squiggly] \arrow[d, tail, "f(\omega)"'] & P \arrow[d] \\ f(O') \arrow[r, squiggly] & P' \end{tikzcd}\right)$$

with ω inert. This again follows from unique lifting of inert morphisms, which ensures that the inert–active factorization of $f(O) \rightsquigarrow P \to P'$ gives such a diagram.

It remains to show that ω must be inert for any cocartesian morphism. Since cocartesian morphisms are unique when they exist, this follows from the existence of the cocartesian morphisms we just described. $\square$

Straightening this cocartesian fibration, we get:

Corollary 7.4. *Suppose $f : \mathcal{O} \to \mathcal{P}$ has unique lifting of inert morphisms. Then there is a functor $\mathcal{P} \to \mathrm{Cat}_\infty$ that takes P to $\mathcal{O}^{\mathrm{act}}_{/P}$. The functor $\mathcal{O}^{\mathrm{act}}_{/P} \to \mathcal{O}^{\mathrm{act}}_{/P'}$ assigned to a morphism $P \to P'$ takes a pair $(O, f(O) \rightsquigarrow P)$ to $(O', f(O') \rightsquigarrow P')$ where $f(O) \rightarrowtail f(O') \rightsquigarrow P'$ is the inert–active factorization of $f(O) \rightsquigarrow P \to P'$.* $\square$

Remark 7.5. Let $\mathcal{O}$ be an algebraic pattern, and write $\mathrm{Fun}(\Delta^1, \mathcal{O})_{\mathrm{act}}$ for the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{O})$ spanned by the active morphisms. As a simple special case of the previous result (taking f to be $\mathrm{id}_{\mathcal{O}}$) we see that

$$\mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, \mathcal{O})_{\mathrm{act}} \to \mathcal{O}$$

is a cocartesian fibration. This corresponds to a functor $\mathcal{O} \to \mathrm{Cat}_\infty$ that takes O to $\mathcal{O}^{\mathrm{act}}_{/O}$ and a morphism $\phi : O \to O'$ to a functor $\mathcal{O}^{\mathrm{act}}_{/O} \to \mathcal{O}^{\mathrm{act}}_{/O'}$ that takes $X \rightsquigarrow O$ to $X' \rightsquigarrow O'$, where $X \rightarrowtail X' \rightsquigarrow O'$ is the inert–active factorization of the composite $X \rightsquigarrow O \to O'$.

Remark 7.6. Suppose $f : \mathcal{O} \to \mathcal{P}$ has unique lifting of inert morphisms, and let $\mathcal{X}^{\mathrm{int}} \to \mathcal{P}^{\mathrm{int}}$ be the pullback of the cocartesian fibration $\mathcal{X} \to \mathcal{P}$ of Proposition 7.3 to the subcategory $\mathcal{P}^{\mathrm{int}}$. Then for every active morphism $\phi : f(O) \rightsquigarrow P$ in $\mathcal{P}$ we can define a functor $\mathcal{P}^{\mathrm{int}}_{P/} \to \mathcal{O}^{\mathrm{int}}_{O/}$ as the composite

$$\mathcal{P}^{\mathrm{int}}_{P/} \to \mathcal{X}^{\mathrm{int}}_{(O,\phi)/} \to \mathcal{O}^{\mathrm{int}}_{O/}$$

where the first functor takes $\alpha : P \rightarrowtail P'$ to the cocartesian morphism $(O, \phi) \to (\alpha_! O, \alpha_! \phi)$ for the cocartesian fibration $\mathcal{X}$ (where $\alpha_! \phi$ is the active part of the map $\alpha \circ \phi$), and the second is induced by the forgetful functor $\mathcal{X} \to \mathcal{O}$. In particular, we can restrict to $\mathcal{P}^{\mathrm{el}}_{P/}$ and compose with the functor $\mathcal{O}^{\mathrm{int,op}}_{O/} \to \mathcal{O}^{\mathrm{op}} \xrightarrow{\mathcal{O}^{\mathrm{el}}_{-/}} \mathrm{Cat}_\infty$ to get a functor $\mathcal{P}^{\mathrm{el,op}}_{P/} \to \mathrm{Cat}_\infty$ that takes $\alpha \colon P \to E$ to $\mathcal{O}^{\mathrm{el}}_{\alpha_! O/}$. We write $\mathcal{O}^{\mathrm{el}}(\phi) \to \mathcal{P}^{\mathrm{el}}_{P/}$ for the corresponding cartesian fibration.

Using this functoriality we can now state the conditions we require of a morphism of algebraic patterns:

Definition 7.7. A morphism of algebraic patterns $f \colon \mathcal{O} \to \mathcal{P}$ is *extendable* if the following conditions are satisfied:

(1) The morphism f has unique lifting of inert morphisms.
(2) For $P \in \mathcal{P}$, let $\mathcal{L}_P$ denote the limit of the composite functor $\epsilon_P \colon \mathcal{P}^{\mathrm{el}}_{P/} \to \mathcal{P}^{\mathrm{int}} \to \mathrm{Cat}_\infty$ taking E to $\mathcal{O}^{\mathrm{act}}_{/E}$ (where the second functor is that of Corollary 7.4). Then the canonical functor

$$\mathcal{O}^{\mathrm{act}}_{/P} \to \mathcal{L}_P$$

is cofinal.
(3) For every active morphism $\phi \colon f(O) \rightsquigarrow P$, the canonical functor

$$\mathcal{O}^{\mathrm{el}}(\phi) \to \mathcal{O}^{\mathrm{el}}_{O/}$$

induces an equivalence

$$\lim_{\mathcal{O}^{\mathrm{el}}_{O/}} F \xrightarrow{\sim} \lim_{\mathcal{O}^{\mathrm{el}}(\phi)} F$$

for every functor $F \colon \mathcal{O}^{\mathrm{el}} \to \mathcal{S}$.

Remark 7.8. We have used the limit in condition (2) as this seems the most natural choice in Definition 7.11; we could also have used the lax limit instead, provided the same change is made in Definition 7.11. In the cases of interest the lax limit actually agrees with the usual limit, as it will either be a finite product or a limit of ∞-groupoids, so the distinction turns out not to matter in practice.

Remark 7.9. In practice, condition (3) holds because the map $\mathcal{O}^{\mathrm{el}}(\phi) \to \mathcal{O}^{\mathrm{el}}_{O/}$ is coinitial.

Remark 7.10. Condition (3) implies that for a functor $\Phi \colon \mathcal{O}^{\mathrm{el}} \to \mathcal{C}$, we have an equivalence

$$\lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \Phi(E) \simeq \lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \lim_{E \in \mathcal{O}^{\mathrm{el}}_{\alpha_! O/}} \Phi(E)$$

whenever either limit exists in $\mathcal{C}$. If Φ is a Segal $\mathcal{O}$-object, this implies that the following "relative Segal condition" holds:

$$\Phi(O) \simeq \lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \Phi(\alpha_! O).$$

We now turn to the requirements we must make of our target category, for which we need the following notion:

Definition 7.11. Consider a functor $K\colon \mathcal{I} \to \mathrm{Cat}_\infty$ with corresponding cocartesian fibration $\pi\colon \mathcal{K} \to \mathcal{I}$. Let $\mathcal{L}$ be the limit of K, which we can identify with the ∞-category of cocartesian sections $\mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{I}}(\mathcal{I}, \mathcal{K})$. We then have a functor $p\colon \mathcal{I} \times \mathcal{L} \to \mathcal{K}$ adjoint to the forgetful functor $\mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{I}}(\mathcal{I}, \mathcal{K}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{K})$; the composite $\pi \circ p$ is moreover the projection $\mathcal{L} \times \mathcal{I} \to \mathcal{I}$. This gives a commutative diagram

$$\begin{tikzcd}
\mathcal{L} \times \mathcal{I} \arrow[r, "p"] \arrow[rr, bend left, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{K} \arrow[r, "\pi"] & \mathcal{I} \arrow[d, "\iota"] \\
\mathcal{L} \arrow[rr, "\lambda"] & & *,
\end{tikzcd}$$

which for any ∞-category $\mathcal{C}$ (with appropriate limits and colimits) determines an equivalence of functors between functor ∞-categories

$$p^* \pi^* \iota^* \simeq \mathrm{pr}_1^* \lambda^*.$$

This induces a mate transformation

$$\lambda^* \iota_* \to \mathrm{pr}_{1,*} \mathrm{pr}_2^* \simeq \mathrm{pr}_{1,*} p^* \pi^*,$$

and this is an equivalence: for $\Phi\colon \mathcal{I} \to \mathcal{C}$, $\lambda^* \iota_* \Phi$ is the constant functor with value $\lim_{\mathcal{I}} \Phi$ while the right Kan extension $\mathrm{pr}_{1,*}$ takes limits over $\mathcal{I}$ fibrewise so that $\mathrm{pr}_{1,*}\mathrm{pr}_2^*\Phi$ is also the constant functor with value $\lim_{\mathcal{I}} \Phi$. From this equivalence we in turn obtain, by moving adjoints around, a natural transformation

$$\lambda_! \mathrm{pr}_{1,*} p^* \to \lambda_! \mathrm{pr}_{1,*} p^* \pi^* \pi_! \simeq \lambda_! \lambda^* \iota_* \pi_! \to \iota_* \pi_!.$$

For a functor $F\colon \mathcal{K} \to \mathcal{C}$ we can interpret this as a natural morphism

$$\operatorname*{colim}_{\mathcal{L}} \lim_{\mathcal{I}} p^* F \to \lim_{i \in \mathcal{I}} \operatorname*{colim}_{\mathcal{K}_i} F|_{\mathcal{K}_i}.$$

We say that $\mathcal{I}$*-limits distribute over* K*-colimits* in $\mathcal{C}$ if this morphism is an equivalence for any functor F.

Definition 7.12. Let $f\colon \mathcal{O} \to \mathcal{P}$ be an extendable morphism of algebraic patterns. We say that an ∞-category $\mathcal{C}$ is *f-admissible* if $\mathcal{C}$ is $\mathcal{O}$- and $\mathcal{P}$-complete, the pointwise left Kan extension $f_!\colon \mathrm{Fun}(\mathcal{O},\mathcal{C}) \to \mathrm{Fun}(\mathcal{P},\mathcal{C})$ exists, and $\mathcal{P}^{\mathrm{el}}_{P/}$-limits distribute over ϵ_P-colimits for all $P \in \mathcal{P}$, where ϵ_P is the functor from Definition 7.7(2). In other words, if $\mathcal{C}$ is f-admissible then for every $P \in \mathcal{P}$ and every functor Φ, the natural map

$$\operatorname*{colim}_{(O_E)_{E\in\mathcal{P}^{\mathrm{el}}_{P/}}\in\mathcal{L}_P} \lim_{E\in\mathcal{P}^{\mathrm{el}}_{P/}} \Phi(O_E) \to \lim_{E\in\mathcal{P}^{\mathrm{el}}_{P/}} \operatorname*{colim}_{O_E\in\mathcal{O}^{\mathrm{act}}_{/E}} \Phi(O_E)$$

is an equivalence.

Having made these definitions, we can now state our result on left Kan extensions:

Proposition 7.13. *Suppose $f\colon \mathcal{O} \to \mathcal{P}$ is an extendable morphism of algebraic patterns, and $\mathcal{C}$ is an f-admissible ∞-category. Then left Kan extension along f restricts to a functor*

$$f_!\colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}),$$

given by $f_!\Phi(P) \simeq \mathrm{colim}_{O\in\mathcal{O}^{\mathrm{act}}_{/P}} \Phi(O)$.

Proof. Given $\Phi \in \mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$, we must show that $f_!\Phi$ is a Segal object, i.e. that the natural map

$$(f_!\Phi)(P) \to \lim_{E\in\mathcal{P}^{\mathrm{el}}_{P/}} (f_!\Phi)(E)$$

is an equivalence. We have a sequence of equivalences

$$\begin{aligned}
f_!\Phi(P) &\simeq \mathrm{colim}_{O\in\mathcal{O}_{/P}} \Phi(O) & \\
&\simeq \mathrm{colim}_{O\in\mathcal{O}^{\mathrm{act}}_{/P}} \Phi(O) & \text{(by 7.2)}\\
&\simeq \mathrm{colim}_{O\in\mathcal{O}^{\mathrm{act}}_{/P}} \lim_{E\in\mathcal{P}^{\mathrm{el}}_{P/}} \Phi(O_E) & \text{(by 7.7(3))}\\
&\simeq \mathrm{colim}_{(O_E)_E\in\mathcal{L}_p} \lim_{E\in\mathcal{P}^{\mathrm{el}}_{P/}} \Phi(O_E) & \text{(by 7.7(2))}\\
&\simeq \lim_{E\in\mathcal{P}^{\mathrm{el}}_{P/}} \mathrm{colim}_{O_E\in\mathcal{O}^{\mathrm{act}}_{/E}} \Phi(O_E) & \text{(by 7.12)}\\
&\simeq \lim_{E\in\mathcal{P}^{\mathrm{el}}_{P/}} (f_!\Phi)(E), &
\end{aligned}$$

which completes the proof. □

Having identified conditions under which $f_!$ preserves Segal objects, we now turn to the question of when these conditions hold. For extendability, we will see some general classes of examples below in §9; here, we will discuss two classes of examples where f-admissibility holds. The starting point is the following examples of distributivity of limits over colimits:

Definition 7.14. We say an ∞-category $\mathcal{C}$ is $\times$-admissible if it has finite products and the cartesian product preserves colimits in each variable.

Lemma 7.15. *Suppose $\mathcal{C}$ is $\times$-admissible. Then finite products distribute over all colimits in $\mathcal{C}$.*

Proof. For any functors $F_i \colon \mathcal{I}_i \to \mathcal{C}$ $(i = 1, \dots, n)$ whose colimits exist we have

$$\operatorname*{colim}_{\mathcal{I}_1 \times \cdots \times \mathcal{I}_n} F_1 \times \cdots \times F_n \simeq \operatorname*{colim}_{\mathcal{I}_1} \cdots \operatorname*{colim}_{\mathcal{I}_n} F_1 \times \cdots \times F_n \simeq \operatorname*{colim}_{\mathcal{I}_1} F_1 \times \cdots \times \operatorname*{colim}_{\mathcal{I}_n} F_n. \quad \square$$

Proposition 7.16. *Let $\mathcal{C}$ be a presentable ∞-category and write $t \colon \mathcal{S} \to \mathcal{C}$ for the unique colimit-preserving functor taking $*$ to the terminal object $*_{\mathcal{C}}$ of $\mathcal{C}$. Consider a functor $K \colon \mathcal{I} \to \mathcal{S}$ and suppose the following conditions hold:*

(1) t preserves $\mathcal{I}$-limits.
(2) The functor $\mathcal{C}_{/t(S)} \to \lim_S \mathcal{C} \simeq \operatorname{Fun}(S, \mathcal{C})$ induced by taking pullbacks along $*_{\mathcal{C}} \simeq t(*) \to t(S)$, is an equivalence for $S = \lim_{\mathcal{I}} K(i)$ and $S = K_i$ for all $i \in \mathcal{I}$.

Then $\mathcal{I}$-limits distribute over K-colimits in $\mathcal{C}$.

Proof. Condition (2) implies that we have a commutative diagram of right adjoints

$$\begin{tikzcd} \mathcal{C}_{/t(S)} \arrow[rr, "\sim"] & & \operatorname{Fun}(S, \mathcal{C}) \\ & \mathcal{C} \arrow[ul, "- \times t(S)"] \arrow[ur, "\mathrm{const}"'] & \end{tikzcd}$$

Passing to left adjoints, we get the commutative triangle

$$\begin{tikzcd} \operatorname{Fun}(S, \mathcal{C}) \arrow[rr, "\sim"] \arrow[dr, "\mathrm{colim}"'] & & \mathcal{C}_{/t(S)} \arrow[dl, "\mathrm{src}"] \\ & \mathcal{C}, & \end{tikzcd}$$

from which we see that under the equivalence of (2) the colimit of a diagram $S \to \mathcal{C}$ is given by the source of the corresponding morphism to $t(S)$. Given $F \colon S \to \mathcal{C}$, it follows that we have pullback squares

$$\begin{tikzcd} F(s) \arrow[r] \arrow[d] & \operatorname{colim}_S F \arrow[d] \\ *_{\mathcal{C}} \arrow[r, "t(s)"] & t(S) \end{tikzcd}$$

for $s \in S$.

Now consider a functor $F\colon \mathcal{K} \to \mathcal{C}$, where $\mathcal{K} \to \mathcal{I}$ is the left fibration corresponding to K. We have a commutative square

$$\begin{tikzcd} \operatorname{colim}_L \lim_{\mathcal{I}} F \arrow[r] \arrow[d] & \lim_{i\in\mathcal{I}} \operatorname{colim}_{K_i} F \arrow[d] \\ \operatorname{colim}_L \lim_{\mathcal{I}} *_{\mathcal{C}} \arrow[r] & \lim_{i\in\mathcal{I}} \operatorname{colim}_{K_i} *_{\mathcal{C}}, \end{tikzcd}$$

where $L := \lim_{\mathcal{I}} K(i)$. Here the bottom horizontal map can be identified with the natural map

$$t(L) \simeq \operatorname*{colim}_L *_{\mathcal{C}} \to \lim_{i\in\mathcal{I}} \operatorname*{colim}_{K_i} *_{\mathcal{C}} \simeq \lim_{i\in\mathcal{I}} t(K_i).$$

This is an equivalence by assumption (1). The equivalence of assumption (2) then implies that the top horizontal map is an equivalence if and only if it induces an equivalence on fibres over each map $t(l)\colon *_{\mathcal{C}} \to t(L)$ for $l \in L$. Using the pullback squares above and the fact that limits commute, we see that the map on fibres at $(k_i)_i \in L$ is the identity

$$\lim_{\mathcal{I}} F(k_i) \to \lim_{\mathcal{I}} F(k_i). \quad \square$$

This argument applies to $\mathcal{C}$ being $\mathcal{S}$, or more generally any ∞-topos, giving:

Corollary 7.17. *Given any functor $K\colon \mathcal{I} \to \mathcal{S}$ we have that:*

(i) $\mathcal{I}$-limits distribute over K-colimits in $\mathcal{S}$,
(ii) $\mathcal{I}$-limits distribute over K-colimits in any ∞-topos provided $\mathcal{I}$ is a finite ∞-category.

Proof. Condition (2) of Proposition 7.16 holds in ∞-topoi by descent, [29, Theorem 6.1.3.9], while condition (1) holds for finite limits since t is the left adjoint of a geometric morphism by [29, Proposition 6.3.4.1] and so preserves finite limits. In the case of $\mathcal{S}$ the finiteness condition is unnecessary since t is an equivalence and so preserves all limits. $\square$

Corollary 7.18. *Let $f\colon \mathcal{O} \to \mathcal{P}$ be an extendable morphism of algebraic patterns such that $\mathcal{P}^{\mathrm{el}}_{P/}$ is a finite set for all $P \in \mathcal{P}$. Suppose $\mathcal{C}$ is a $\times$-admissible ∞-category, and assume the pointwise left Kan extension $f_!\colon \mathrm{Fun}(\mathcal{O}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{P}, \mathcal{C})$ exists. Then $\mathcal{C}$ is f-admissible, and the left Kan extension along f restricts to a functor*

$$f_!\colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}). \quad \square$$

Remark 7.19. The assumption of $\times$-admissibility can be slightly weakened: It is enough to assume that the cartesian product in $\mathcal{C}$ preserves colimits of shape $\mathcal{O}^{\mathrm{act}}_{/E}$ in each variable for all $E \in \mathcal{P}^{\mathrm{el}}$.

Corollary 7.20. *Suppose $\mathcal{X}$ is an ∞-topos, and $f\colon \mathcal{O} \to \mathcal{P}$ is an extendable morphism of algebraic patterns such that*

(1) $\mathcal{O}^{\mathrm{act}}_{/E}$ is an ∞-groupoid for all $E \in \mathcal{P}^{\mathrm{el}}$,
(2) the ∞-category $\mathcal{P}^{\mathrm{el}}_{P/}$ is finite for all $P \in \mathcal{P}$ (or arbitrary if $\mathcal{X}$ is the ∞-topos $\mathcal{S}$).

Then $\mathcal{X}$ is f-admissible, and the left Kan extension restricts to a functor

$$f_! \colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{X}) \to \mathrm{Seg}_{\mathcal{P}}(\mathcal{X}). \qquad \square$$

8. Free Segal objects

Suppose $\mathcal{O}$ is an algebraic pattern, and $\mathcal{C}$ an $\mathcal{O}$-complete ∞-category. Restricting Segal objects to the subcategory $\mathcal{O}^{\mathrm{el}}$ gives a functor

$$U_{\mathcal{O}} \colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C}).$$

We think of *free* Segal $\mathcal{O}$-objects as being given by a left adjoint $F_{\mathcal{O}}$ to this functor, when this exists.

The subcategory $\mathcal{O}^{\mathrm{int}}$ has a canonical pattern structure restricted from $\mathcal{O}$ (so only equivalences are active morphisms and the elementary objects are still those of $\mathcal{O}^{\mathrm{el}}$), and using this the inclusion $j_{\mathcal{O}} \colon \mathcal{O}^{\mathrm{int}} \to \mathcal{O}$ is a Segal morphism. The ∞-category $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C})$ is by definition the full subcategory of $\mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{C})$ spanned by the functors that are right Kan extensions along the fully faithful inclusion $i_{\mathcal{O}} \colon \mathcal{O}^{\mathrm{el}} \to \mathcal{O}^{\mathrm{int}}$, which means that the restriction functor $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C})$ is an equivalence. The functor $U_{\mathcal{O}}$ thus factors as the composite

$$\mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{j_{\mathcal{O}}^*} \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \xrightarrow{i_{\mathcal{O}}^*} \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C}),$$

where the second functor is an equivalence with inverse the right Kan extension functor $i_{\mathcal{O},*}$. If $\mathcal{C}$ is presentable, using Proposition 4.7 this means the left adjoint $F_{\mathcal{O}}$ is given by

$$\mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C}) \xrightarrow{i_{\mathcal{O},*}} \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \xrightarrow{L_{\mathrm{Seg}} j_{\mathcal{O},!}} \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}).$$

In this section we will first show that this adjunction is monadic, and then specialize the results of the previous section to $j_{\mathcal{O}}$ to get conditions under which the free Segal objects are described by a formula in terms of limits and colimits.

Monadicity is a special case of the following observation:

Proposition 8.1. *Suppose $f\colon \mathcal{O} \to \mathcal{P}$ is an essentially surjective Segal morphism and $\mathcal{C}$ is a presentable ∞-category. Then:*

(i) A functor $F\colon \mathcal{P} \to \mathcal{C}$ is a Segal object if and only if f^*F is a Segal $\mathcal{O}$-object.

(ii) The adjunction

$$L_{\mathrm{Seg}} f_! \colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \rightleftarrows \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) : f^*$$

is monadic.

Proof. We first prove (i). One direction amounts to f being a Segal morphism, which is true by assumption. To prove the non-trivial direction, observe that for $\Phi \colon \mathcal{O} \to \mathcal{C}$ we have for every $O \in \mathcal{O}$ canonical morphisms

$$\Phi(f(O)) \to \lim_{E \in \mathcal{P}^{\mathrm{el}}_{f(O)/}} \Phi(E) \to \lim_{E' \in \mathcal{O}^{\mathrm{el}}_{O/}} \Phi(f(E')).$$

Here the second morphism is an equivalence since f is a Segal morphism, and if $f^*\Phi$ is a Segal $\mathcal{O}$-object then the composite morphism is an equivalence. Thus the first morphism is an equivalence, and so Φ satisfies the Segal condition at every object of $\mathcal{P}$ in the image of f; since f is essentially surjective this completes the proof.

Using the monadicity theorem for ∞-categories [30, Theorem 4.7.3.5], to prove (ii) it suffices to show that f^* detects equivalences, that $\mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$ has colimits of f^*-split simplicial objects, and these colimits are preserved by f^*. Since f is essentially surjective it is immediate that f^* detects equivalences. Consider an f^*-split simplicial object $p \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$. Let $\overline{p} \colon (\mathbf{\Delta}^{\mathrm{op}})^{\triangleright} \to \mathrm{Fun}(\mathcal{P}, \mathcal{C})$ denote the colimit of p in $\mathrm{Fun}(\mathcal{P}, \mathcal{C})$. Since f^*, viewed as a functor $\mathrm{Fun}(\mathcal{P}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{C})$, is a left adjoint, $f^*\overline{p}$ is the colimit of f^*p in $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$. On the other hand, since f^*p extends to a split simplicial diagram, and all functors preserve colimits of split simplicial diagrams, we see that the colimit of f^*p in $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ is also the colimit in $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$. In particular, $f^*\overline{p}(\infty)$ lies in $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$. By (i) this implies that $\overline{p}(\infty)$ is in $\mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$. This completes the proof, since the colimit of p in $\mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$ is the localization of $\overline{p}(\infty)$, which is already local. □

Applying this to $j_{\mathcal{O}}$, we get:

Corollary 8.2. *Let $\mathcal{O}$ be an algebraic pattern and $\mathcal{C}$ a presentable ∞-category. Then the free-forgetful adjunction*

$$F_{\mathcal{O}} \colon \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C}) \simeq \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \rightleftarrows \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) : U_{\mathcal{O}}$$

is monadic. □

Now we apply the results of the previous section to $j_{\mathcal{O}}$ to understand when the free algebras are simply given by the left Kan extension $j_{\mathcal{O},!}$. It is convenient to first introduce some notation:

Notation 8.3. Let $\mathcal{O}$ be an algebraic pattern. For $O \in \mathcal{O}$ we write $\mathrm{Act}_{\mathcal{O}}(O)$ for the ∞-groupoid of active morphisms to O in $\mathcal{O}$; this is equivalent to $(\mathcal{O}^{\mathrm{int}})^{\mathrm{act}}_{/O}$ since the only active morphisms in $\mathcal{O}^{\mathrm{int}}$ are the equivalences.

Remark 8.4. By Remark 7.5 the ∞-categories $\mathcal{O}^{\mathrm{act}}_{/O}$ are functorial in $O \in \mathcal{O}$. Passing to the underlying ∞-groupoids this means the ∞-groupoids $\mathrm{Act}_{\mathcal{O}}(O)$ are functorial in $O \in \mathcal{O}$, via the factorization system.

Definition 8.5. We say an algebraic pattern $\mathcal{O}$ is *extendable* if the inclusion $j_{\mathcal{O}} \colon \mathcal{O}^{\mathrm{int}} \to \mathcal{O}$ is extendable in the sense of Definition 7.7. This is equivalent to the following pair of conditions:

(1) The morphism

$$\mathrm{Act}_{\mathcal{O}}(O) \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Act}_{\mathcal{O}}(E)$$

is an equivalence for all $O \in \mathcal{O}$. In other words, $\mathrm{Act}_{\mathcal{O}}$ is a Segal $\mathcal{O}$-space.

(2) For every active map $O \xrightarrow{\phi} O'$ in $\mathcal{O}$, the canonical functor $\mathcal{O}^{\mathrm{el}}(\phi) \to \mathcal{O}^{\mathrm{el}}_{O/}$ induces an equivalence on limits

$$\lim_{\mathcal{O}^{\mathrm{el}}_{O/}} F \to \lim_{\mathcal{O}^{\mathrm{el}}(\phi)} F$$

for every functor $F \colon \mathcal{O}^{\mathrm{el}} \to \mathcal{S}$.

Remark 8.6. Condition (2) implies that

$$\lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \Phi(E) \to \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O'/}} \lim_{E \in \mathcal{O}^{\mathrm{el}}_{\alpha_! O/}} \Phi(E)$$

is an equivalence for any functor $\Phi \colon \mathcal{O}^{\mathrm{el}} \to \mathcal{C}$, provided either limit exists, and $O \to \alpha_! O \to E$ is the inert–active factorization of $O \to O' \xrightarrow{\alpha} E$. This in particular implies the following "generalized Segal condition": If Φ is a Segal object, then for any active morphism $\phi \colon O \to O'$, we have

$$\Phi(O) \simeq \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O'/}} \Phi(\alpha_! O).$$

Remark 8.7. In practice, condition (2) holds because the map $\mathcal{O}^{\mathrm{el}}(\phi) \to \mathcal{O}^{\mathrm{el}}_{O/}$ is coinitial. However, with the more general formulation we get the following characterization of the extendable patterns:

Proposition 8.8. *The following are equivalent for an algebraic pattern $\mathcal{O}$:*

(1) $\mathcal{O}$ *is extendable.*
(2) $j_{\mathcal{O},!}\colon \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{S})$ *restricts to a functor* $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$*.*

Proof. Suppose (1) holds. Since $\mathrm{Act}_{\mathcal{O}}(O)$ is an ∞-groupoid for all O, the ∞-category $\mathcal{S}$ is $j_{\mathcal{O}}$-admissible by Corollary 7.20, and so (2) follows from Proposition 7.13.

We now show that (2) implies the two conditions in Definition 8.5. To prove condition (1), consider the terminal object $* \in \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S})$. For this we have

$$j_{\mathcal{O},!} * (O) \simeq \operatorname*{colim}_{\mathrm{Act}_{\mathcal{O}}(O)} * \simeq \mathrm{Act}_{\mathcal{O}}(O),$$

so since $*$ is a Segal $\mathcal{O}^{\mathrm{int}}$-space, assumption (2) implies that $\mathrm{Act}_{\mathcal{O}}(-)$ is a Segal $\mathcal{O}$-space. To prove condition (2), consider $F\colon \mathcal{O}^{\mathrm{el}} \to \mathcal{S}$ and its right Kan extension $F' := i_{\mathcal{O},*}F$, which is a Segal $\mathcal{O}^{\mathrm{int}}$-space. Then $j_{\mathcal{O},!}F'$ is a Segal $\mathcal{O}$-space, which means that in the commutative square

$$\begin{tikzcd}
\operatorname{colim}_{X \in \mathrm{Act}_{\mathcal{O}}(O)} F'(X) \arrow[r] \arrow[d] & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \operatorname{colim}_{Y \in \mathrm{Act}_{\mathcal{O}}(E)} F'(Y) \arrow[d] \\
\mathrm{Act}_{\mathcal{O}}(O) \arrow[r, "\sim"] & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Act}_{\mathcal{O}}(E),
\end{tikzcd}$$

the top horizontal morphism is an equivalence. Hence we get an equivalence on fibres at each active morphism $(\phi\colon X \to O) \in \mathrm{Act}_{\mathcal{O}}(O)$, which we can identify with the natural map

$$F'(X) \xrightarrow{\sim} \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O/}} F'(\alpha_! X).$$

Using the description of F' as a right Kan extension we get

$$\lim_{\mathcal{O}^{\mathrm{el}}_{X/}} F \xrightarrow{\sim} \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O/}} \lim_{\mathcal{O}^{\mathrm{el}}_{\alpha_! X/}} F \simeq \lim_{\mathcal{O}^{\mathrm{el}}(\phi)} F,$$

as required. □

Definition 8.9. We say an ∞-category $\mathcal{C}$ is $\mathcal{O}$-*admissible* if $\mathcal{O}^{\mathrm{el}}_{O/}$-limits distribute over colimits indexed by the functor $\mathcal{O}^{\mathrm{el}}_{O/} \to \mathcal{S}$ taking E to $\mathrm{Act}_{\mathcal{O}}(E)$ for all $O \in \mathcal{O}$.

From Corollaries 7.18 and 7.20 we get:

Example 8.10. Let $\mathcal{O}$ be an extendable algebraic pattern. Then:

(i) $\mathcal{S}$ is $\mathcal{O}$-admissible.
(ii) Any ∞-topos is $\mathcal{O}$-admissible if the ∞-categories $\mathcal{O}^{\mathrm{el}}_{O/}$ are all finite.

(iii) Any $\times$-admissible ∞-category is $\mathcal{O}$-admissible if the ∞-categories $\mathcal{O}^{\mathrm{el}}_{O/}$ are all finite sets.

Corollary 8.11. *Let $\mathcal{O}$ be an extendable algebraic pattern and $\mathcal{C}$ an $\mathcal{O}$-admissible ∞-category. Then left Kan extension along $j_{\mathcal{O}} \colon \mathcal{O}^{\mathrm{int}} \to \mathcal{O}$ restricts to a functor*

$$j_{\mathcal{O},!} \colon \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}),$$

left adjoint to the restriction $j^*_{\mathcal{O}} \colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C})$. This functor is given by

$$j_{\mathcal{O},!}\Phi(O) \simeq \operatorname*{colim}_{O' \in \mathrm{Act}_{\mathcal{O}}(O)} \Phi(O').$$

Combining this with the equivalence $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C})$ given by right Kan extension along $i_{\mathcal{O}}$, we can reformulate this as:

Corollary 8.12. *Let $\mathcal{O}$ be an extendable algebraic pattern and $\mathcal{C}$ an $\mathcal{O}$-admissible ∞-category. Then the restriction*

$$U_{\mathcal{O}} \colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C})$$

has a left adjoint $F_{\mathcal{O}}$, which is given by

$$F_{\mathcal{O}}(\Phi)(O) \simeq j_{\mathcal{O},!} i_{\mathcal{O},*}\Phi(O) \simeq \operatorname*{colim}_{O' \in \mathrm{Act}_{\mathcal{O}}(O)} \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O'/}} \Phi(E).$$

We end this section with some examples of extendable patterns:

Example 8.13. The algebraic patterns $\mathbb{F}_*^\flat$ and $\mathbb{F}_*^\natural$ are extendable. In the former case, we recover the familiar formula for free commutative monoids:

$$U_{\mathbb{F}_*^\flat} F_{\mathbb{F}_*^\flat}(X) \simeq \coprod_{n=0}^{\infty} X^{\times n}_{h\Sigma_n}.$$

In the latter case, we get

$$U_{\mathbb{F}_*^\natural} F_{\mathbb{F}_*^\natural}(X \to Y) \simeq \coprod_{n=0}^{\infty} X^{\times_Y n}_{h\Sigma_n} \to Y,$$

which describes a free commutative monoid on $X \to Y$ in the slice over Y.

Example 8.14. The algebraic patterns $\mathbf{\Delta}^{\mathrm{op},\flat}$ and $\mathbf{\Delta}^{\mathrm{op},\natural}$ are extendable. In the former case, we get the expected formula for free associative monoids:

$$U_{\mathbf{\Delta}^{\mathrm{op},\flat}} F_{\mathbf{\Delta}^{\mathrm{op},\flat}}(X) \simeq \coprod_{n=0}^{\infty} X^{\times n},$$

while in the latter case we get the formula for free ∞-categories:

$$U_{\mathbf{\Delta}^{\mathrm{op},\natural}} F_{\mathbf{\Delta}^{\mathrm{op},\natural}} \left(\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ Y & & Y \end{tikzcd} \right) \simeq \left(\begin{tikzcd} & \coprod_{n=0}^{\infty} X \times_Y \cdots \times_Y X \arrow[dl] \arrow[dr] & \\ Y & & Y \end{tikzcd} \right).$$

Example 8.15. More generally, the algebraic pattern $\mathbf{\Theta}_n^{\mathrm{op},\natural}$ is extendable for every n; the conditions are checked in [22], giving a formula for free (∞, n)-categories. (On the other hand, the pattern $\mathbf{\Theta}_n^{\mathrm{op},\flat}$ is *not* extendable for $n > 1$.)

Example 8.16. The algebraic pattern $\mathbf{\Omega}^{\mathrm{op},\natural}$ is extendable; the conditions are checked in [16, §5.3], giving a formula for free ∞-operads. (On the other hand, the pattern $\mathbf{\Delta}_{\mathbb{F}}^{\mathrm{op},\natural}$ is *not* extendable.)

9. Segal fibrations and weak Segal fibrations

In this section we first consider *Segal fibrations* over an algebraic pattern, which are the cocartesian fibrations corresponding to Segal objects in Cat_∞, and then generalize these to the class of *weak Segal fibrations*; for the pattern $\mathbb{F}_*^\flat$, these objects are respectively symmetric monoidal ∞-categories and symmetric ∞-operads in the sense of [30]. Our main goal is to show that extendability can be lifted from a base pattern to morphisms between (weak) Segal fibrations. Combined with our previous results this allows us, for example, to reproduce (in the cartesian setting) Lurie's formula for operadic Kan extensions along morphisms of symmetric ∞-operads.

Definition 9.1. Let $\mathcal{O}$ be an algebraic pattern. A *Segal $\mathcal{O}$-fibration* is a cocartesian fibration $\mathcal{E} \to \mathcal{O}$ whose corresponding functor $\mathcal{O} \to \mathrm{Cat}_\infty$ is a Segal $\mathcal{O}$-∞-category.

Examples 9.2.

(i) A Segal $\mathbb{F}_*^\flat$-fibration is a symmetric monoidal ∞-category.
(ii) A Segal $\mathbf{\Delta}^{\mathrm{op},\flat}$-fibration is a monoidal ∞-category, and a Segal $\mathbf{\Delta}^{\mathrm{op},\natural}$-fibration is a double ∞-category.
(iii) Segal $\mathbf{\Delta}^{n,\mathrm{op},\flat}$-fibrations and Segal $\mathbf{\Theta}_n^{\mathrm{op},\flat}$-fibrations both describe $\mathbb{E}_n$-monoidal ∞-categories.

Definition 9.3. Suppose $\mathcal{O}$ is an algebraic pattern, and $\pi \colon \mathcal{E} \to \mathcal{O}$ is a Segal $\mathcal{O}$-fibration. We say a morphism in $\mathcal{E}$ is *inert* if it is cocartesian and lies over an inert morphism in

$\mathcal{O}$, and *active* if it lies over an active morphism in $\mathcal{O}$; moreover, we say an object of $\mathcal{E}$ is *elementary* if it lies over an elementary object of $\mathcal{O}$.

Lemma 9.4. *Equipped with this data, $\mathcal{E}$ is an algebraic pattern, and $\pi\colon \mathcal{E} \to \mathcal{O}$ is a Segal morphism.*

Proof. The inert and active morphisms form a factorization system by [30, Proposition 2.1.2.5], so we have defined an algebraic pattern structure on $\mathcal{E}$. To see that π is a Segal morphism it suffices to show that for $\overline{X} \in \mathcal{E}_X$ the induced functor

$$\mathcal{E}^{\mathrm{el}}_{\overline{X}/} \to \mathcal{O}^{\mathrm{el}}_{X/}$$

is coinitial. But this functor is an equivalence since for each inert morphism $X \to E$ with E elementary there is a unique cocartesian morphism with source $\overline{X}$ lying over it. $\square$

We now show that we can lift extendability along Segal fibrations:

Proposition 9.5. *Consider a commutative square*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "F"] \arrow[d, "p"'] & \mathcal{F} \arrow[d, "q"] \\ \mathcal{O} \arrow[r, "f"] & \mathcal{P}, \end{tikzcd}$$

where f is an extendable morphism of algebraic patterns, $p\colon \mathcal{E} \to \mathcal{O}$ and $q\colon \mathcal{F} \to \mathcal{P}$ are Segal fibrations, and F preserves cocartesian morphisms. Then F is extendable. Moreover, if $\mathcal{C}$ is f-admissible and either

(i) $\mathcal{P}^{\mathrm{el}}_{P/}$-limits distribute over η-colimits in $\mathcal{C}$ for all functors $\eta\colon \mathcal{P}^{\mathrm{el}}_{P/} \to \mathrm{Cat}_\infty$ and all $P \in \mathcal{P}$, or

(ii) p and q are left fibrations, and $\mathcal{P}^{\mathrm{el}}_{/P}$-limits distribute over η-colimits in $\mathcal{C}$ for all functors $\eta\colon \mathcal{P}^{\mathrm{el}}_{/P} \to \mathcal{S}$ and all $P \in \mathcal{P}$,

then $\mathcal{C}$ is F-admissible.

Proof. It is immediate from the definitions that F preserves inert and active morphisms. We now observe that F has unique lifting of inert morphisms. Given $\overline{O} \in \mathcal{E}$ lying over $O \in \mathcal{O}$, and an inert morphism $\overline{\epsilon}\colon F(\overline{O}) \rightarrowtail \overline{P}$ in $\mathcal{F}$, lying over $\epsilon\colon f(O) \rightarrowtail P$ in $\mathcal{P}$, there exists a unique inert morphism $\gamma\colon O \rightarrowtail O'$ such that $f(\gamma) \simeq \epsilon$, since f is extendable. Since inert morphisms in $\mathcal{E}$ are cocartesian, there exists a unique inert morphism $\overline{\gamma}\colon \overline{O} \rightarrowtail \overline{O}'$ lying over γ. Moreover, as F preserves cocartesian morphisms, the morphism $F(\overline{\gamma})$ is the unique inert morphism over ϵ with source $F(\overline{O})$, i.e. $F(\overline{\gamma}) \simeq \overline{\epsilon}$, and since cocartesian morphisms are unique, $\overline{\gamma}$ is the unique inert morphism that maps to $\overline{\epsilon}$.

For every active morphism $\overline{\phi}\colon F(\overline{O}) \rightsquigarrow \overline{P}$ lying over $\phi\colon f(O) \rightsquigarrow P$, equivalences of the type $\mathcal{E}^{\mathrm{el}}_{\overline{X}/} \simeq \mathcal{O}^{\mathrm{el}}_{X/}$ imply that $\mathcal{E}^{\mathrm{el}}(\overline{\phi}) \to \mathcal{E}^{\mathrm{el}}_{\overline{O}/}$ is equivalent to $\mathcal{O}^{\mathrm{el}}(\phi) \to \mathcal{O}^{\mathrm{el}}_{O/}$, hence condition (3) in Definition 7.7 follows immediately from f being extendable. It remains to prove condition (2). For $\overline{P} \in \mathcal{F}$ lying over $P \in \mathcal{P}$ and $\overline{\epsilon}\colon \overline{P} \rightarrowtail \overline{P}'$ an inert morphism in $\mathcal{F}$ lying over $\epsilon\colon P \rightarrowtail P'$, we have a functor

$$\mathcal{E}^{\mathrm{act}}_{/\overline{P}} \to \mathcal{E}^{\mathrm{act}}_{/\overline{P}'},$$

which fits in a commutative square

$$\begin{tikzcd}
\mathcal{E}^{\mathrm{act}}_{/\overline{P}} \arrow[r] \arrow[d] & \mathcal{E}^{\mathrm{act}}_{/\overline{P}'} \arrow[d] \\
\mathcal{O}^{\mathrm{act}}_{/P} \arrow[r] & \mathcal{O}^{\mathrm{act}}_{/P'}.
\end{tikzcd}$$

We claim that here the vertical functors are cocartesian fibrations, and the top horizontal functor preserves cocartesian morphisms. The functor

$$\mathcal{E}_{/\overline{P}} := \mathcal{E} \times_{\mathcal{F}} \mathcal{F}_{/\overline{P}} \to \mathcal{O} \times_{\mathcal{P}} \mathcal{P}_{/P} =: \mathcal{O}_{/P}$$

is a fibre product of cocartesian fibrations along functors that preserve cocartesian morphisms, hence it is again a cocartesian fibration. We can write $\mathcal{E}^{\mathrm{act}}_{/\overline{P}}$ as a pullback $\mathcal{E}_{/\overline{P}} \times_{\mathcal{O}_{/P}} \mathcal{O}^{\mathrm{act}}_{/P}$, hence $\mathcal{E}^{\mathrm{act}}_{/\overline{P}} \to \mathcal{O}^{\mathrm{act}}_{/P}$ is a pullback of a cocartesian fibration and so is itself cocartesian. Moreover, a morphism in $\mathcal{E}^{\mathrm{act}}_{/\overline{P}}$ is cocartesian if and only if its image in $\mathcal{E}$ is cocartesian (since the functor $\mathcal{F}_{/\overline{P}} \to \mathcal{F}$ detects cocartesian morphisms, by [29, Proposition 2.4.3.2]). Since inert morphisms are cocartesian, this implies that the top horizontal functor preserves cocartesian morphisms by the 3-for-2 property of cocartesian morphisms ([29, Proposition 2.4.1.7]).

For $\overline{P} \in \mathcal{F}$ we therefore have a commutative square

$$\begin{tikzcd}
\mathcal{E}^{\mathrm{act}}_{/\overline{P}} \arrow[r] \arrow[d] & \lim_{\alpha\colon P \rightarrowtail E \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{E}^{\mathrm{act}}_{/\alpha_!\overline{P}} \arrow[d] \\
\mathcal{O}^{\mathrm{act}}_{/P} \arrow[r] & \lim_{\alpha\colon P \rightarrowtail E \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{O}^{\mathrm{act}}_{/E},
\end{tikzcd}$$

where the vertical functors are cocartesian fibrations, the top horizontal functor preserves cocartesian morphisms, and the bottom horizontal functor is cofinal, since f is extendable. Our goal is to show that the top horizontal functor is cofinal. Since pullbacks of cofinal functors along cocartesian fibrations are cofinal by [29, Proposition 4.1.2.15], it suffices to show that the square is cartesian, which in this situation is equivalent to the functor on fibres being an equivalence.

Since the fibration $\mathcal{E}^{\mathrm{act}}_{/\overline{P}} \to \mathcal{O}^{\mathrm{act}}_{/P}$ is a fibre product, its fibre at $(O, \psi\colon f(O) \rightsquigarrow P)$ is the fibre product $\mathcal{E}_O \times_{\mathcal{F}_{f(O)}} (\mathcal{F}^{\mathrm{act}}_{/\overline{P}})_\psi$; since $\mathcal{F}$ is cocartesian over $\mathcal{P}$, we can use the cocartesian pushforward over ψ to identify this with a fibre product $\mathcal{E}_O \times_{\mathcal{F}_P} \mathcal{F}_{P/\overline{P}}$ over the composite functor $\mathcal{E}_O \to \mathcal{F}_{f(O)} \xrightarrow{\psi_!} \mathcal{F}_P$.

If ψ is active, then as f is extendable and $\mathcal{E} \to \mathcal{O}$ is a Segal fibration we have an equivalence

$$\mathcal{E}_O \xrightarrow{\sim} \lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{E}_{\alpha_! O}$$

by Remark 7.10. Putting this together with the equivalence $\mathcal{F}_{P/\overline{P}} \xrightarrow{\sim} \lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{F}_{E/\alpha_!\overline{P}}$ (and similarly for $\mathcal{F}_P$) we get

$$(\mathcal{E}^{\mathrm{act}}_{/\overline{P}})_{(O,\psi)} \xrightarrow{\sim} \lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} (\mathcal{E}^{\mathrm{act}}_{/\alpha_!\overline{P}})_{(E,\psi_\alpha)},$$

i.e. the functor we get on fibres is indeed an equivalence, which completes the proof that F is extendable.

For admissibility, observe that since $\lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{E}^{\mathrm{act}}_{/\alpha_!\overline{P}} \to \lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{O}^{\mathrm{act}}_{/E}$ is a cocartesian fibration, if we compute the colimit of a functor Φ over its source in two stages using the left Kan extension along this functor, we get

$$\operatorname*{colim}_{\lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{E}^{\mathrm{act}}_{/\alpha_!\overline{P}}} \Phi \simeq \operatorname*{colim}_{(\omega_\alpha) \in \lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{O}^{\mathrm{act}}_{/E}} \operatorname*{colim}_{\lim_{\alpha \in \mathcal{P}^{\mathrm{el}}_{P/}} (\mathcal{E}^{\mathrm{act}}_{/\alpha_!\overline{P}})_{\omega_\alpha}} \Phi,$$

from which we see that F-admissibility follows from f-admissibility plus either (i) or (ii). □

Definition 9.6. Let $\mathcal{O}$ be an algebraic pattern. A *weak Segal* $\mathcal{O}$*-fibration* is a functor $p\colon \mathcal{E} \to \mathcal{O}$ such that:

(1) For every object $\overline{X}$ in $\mathcal{E}$ lying over $X \in \mathcal{O}$ and every inert morphism $i\colon X \to Y$ in $\mathcal{O}$ there exists a p-cocartesian morphism $\bar{\imath}\colon \overline{X} \to \overline{Y}$ lying over i.

(2) For every object $X \in \mathcal{O}$, the functor

$$\mathcal{E}_X \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} \mathcal{E}_E,$$

induced by the cocartesian morphisms over inert maps, is an equivalence.

(3) Given $\overline{X}$ in $\mathcal{E}_X$, choose a cocartesian lift $\xi\colon (\mathcal{O}^{\mathrm{el}}_{X/})^{\triangleleft} \to \mathcal{E}$ of the diagram of inert morphisms from X in $\mathcal{O}$, taking $-\infty$ to $\overline{X}$. Then for any $Y \in \mathcal{O}$ and $\overline{Y} \in \mathcal{E}_Y$, the commutative square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}}(\overline{Y},\overline{X}) \arrow[r] \arrow[d] & \lim_{E\in\mathcal{O}^{\mathrm{el}}_{X/}} \mathrm{Map}_{\mathcal{E}}(\overline{Y},\xi(E)) \arrow[d] \\
\mathrm{Map}_{\mathcal{O}}(Y,X) \arrow[r] & \lim_{E\in\mathcal{O}^{\mathrm{el}}_{X/}} \mathrm{Map}_{\mathcal{O}}(Y,E)
\end{tikzcd}$$

is cartesian.

Remark 9.7. Condition (3) in the definition can be rephrased as: For every map $\phi\colon Y \to X$ in $\mathcal{O}$, the natural map

$$\mathrm{Map}^{\phi}_{\mathcal{E}}(\overline{Y},\overline{X}) \to \lim_{\alpha\colon X\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{X/}} \mathrm{Map}^{\alpha\phi}_{\mathcal{E}}(\overline{Y},\alpha_!\overline{X})$$

is an equivalence, where $\mathrm{Map}^{\phi}_{\mathcal{E}}(\overline{Y},\overline{X})$ denotes the fibre at ϕ of $\mathrm{Map}_{\mathcal{E}}(\overline{Y},\overline{X}) \to \mathrm{Map}_{\mathcal{O}}(Y,X)$. If ϕ is active, let $Y \overset{\alpha_Y}{\rightarrowtail} Y_\alpha \overset{\phi_\alpha}{\rightsquigarrow} E$ denote the inert–active factorization of $Y \overset{\phi}{\rightsquigarrow} X \overset{\alpha}{\rightarrowtail} E$, then combining this equivalence with the cocartesian morphisms $\overline{Y} \rightarrowtail \alpha_{Y,!}\overline{Y}$ over α_Y we obtain an equivalence

$$\mathrm{Map}^{\phi}_{\mathcal{E}}(\overline{Y},\overline{X}) \simeq \lim_{\alpha\colon X\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{X/}} \mathrm{Map}^{\phi_\alpha}_{\mathcal{E}}(\alpha_{Y,!}\overline{Y},\alpha_!\overline{X}).$$

Examples 9.8.

(i) A weak Segal $\mathbb{F}_*^\flat$-fibration is a symmetric ∞-operad, and a weak Segal $\mathbb{F}_*^\natural$-fibration is a generalized ∞-operad, in the sense of [30].

(ii) A weak Segal $\mathbf{\Delta}^{\mathrm{op},\flat}$-fibration is a non-symmetric ∞-operad, and a weak Segal $\mathbf{\Delta}^{\mathrm{op},\natural}$-fibration is a generalized non-symmetric ∞-operad, as considered in [15].

(iii) If Φ is a perfect operator category and $\Lambda(\Phi)$ is its Leinster category, then a weak Segal $\Lambda(\Phi)^\flat$-fibration is a Φ-∞-operad, in the sense of [2], and weak Segal $\Lambda(\Phi)^\natural$-fibrations are the natural extension of generalized ∞-operads to *generalized* Φ-∞-*operads*.

(iv) Weak Segal $\mathbf{\Theta}_n^{\mathrm{op},\natural}$-fibrations can be viewed as an ∞-categorical analogue of the *n-operads* of Batanin [3].

Definition 9.9. Suppose $\mathcal{O}$ is an algebraic pattern, and $\pi\colon \mathcal{E} \to \mathcal{O}$ is a weak Segal $\mathcal{O}$-fibration. We say a morphism in $\mathcal{E}$ is *inert* if it is cocartesian and lies over an inert morphism in $\mathcal{O}$, and *active* if it lies over an active morphism in $\mathcal{O}$; moreover, we say an object of $\mathcal{E}$ is *elementary* if it lies over an elementary object of $\mathcal{O}$.

Lemma 9.10. *Equipped with this data, $\mathcal{E}$ is an algebraic pattern, and $\pi\colon \mathcal{E} \to \mathcal{O}$ is a Segal morphism.*

Proof. As Lemma 9.4. □

Remark 9.11. A cocartesian fibration $\mathcal{E} \to \mathcal{O}$ is a Segal fibration if and only if it is a weak Segal fibration.

Remark 9.12. Suppose $\mathcal{E} \to \mathcal{O}$ and $\mathcal{F} \to \mathcal{O}$ are weak Segal fibrations. Then a morphism $\mathcal{E} \to \mathcal{F}$ over $\mathcal{O}$ is a Segal morphism if and only if it preserves inert morphisms.

Remark 9.13. Let $\mathrm{Cat}^{\mathrm{WSF}}_{\infty/\mathcal{O}}$ denote the subcategory of $\mathrm{Cat}_{\infty/\mathcal{O}}$ whose objects are the weak Segal fibrations and whose morphisms are those that preserve inert morphisms. This ∞-category is described by a *categorical pattern* in the sense of [30, §B], and so arises from a combinatorial model category by [30, Theorem B.0.20]. It follows that $\mathrm{Cat}^{\mathrm{WSF}}_{\infty/\mathcal{O}}$ is a presentable ∞-category.

For weak Segal fibrations we can prove a weaker version of Proposition 9.5; for this we need the following consequence of extendability, which we learned from Roman Kositsyn:

Lemma 9.14. *Let $\mathcal{O}$ be an extendable pattern. Then the functor $\mathcal{O} \to \mathrm{Cat}_\infty$ taking O to $\mathcal{O}^{\mathrm{act}}_{/O}$ from Remark 7.5 is a Segal $\mathcal{O}$-∞-category. In particular, for any active maps $\phi\colon X \rightsquigarrow O$, $\psi\colon Y \rightsquigarrow O$ in $\mathcal{O}$, the morphism of mapping spaces*

$$\mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/O}}(\phi, \psi) \to \lim_{\alpha\colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/E}}(\phi_\alpha, \psi_\alpha)$$

is an equivalence.

Proof. We must show that for any $O \in \mathcal{O}$, the functor

$$\mathcal{O}^{\mathrm{act}}_{/O} \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{O}^{\mathrm{act}}_{/E}$$

is an equivalence; to see this it suffices to check that it is an equivalence on underlying ∞-groupoids and is fully faithful. The map on underlying ∞-groupoids is the map $\mathrm{Act}_{\mathcal{O}}(O) \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Act}_{\mathcal{O}}(E)$, which is an equivalence by assumption since $\mathcal{O}$ is extendable. Given active maps $\phi\colon X \rightsquigarrow O$, $\psi\colon Y \rightsquigarrow O$, the morphism of mapping spaces

$$\mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/O}}(\phi, \psi) \to \lim_{\alpha\colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/E}}(\phi_\alpha, \psi_\alpha)$$

fits in a commutative cube

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/O}}(\phi,\psi) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{Act}_{\mathcal{O}}(Y) \arrow[dr] \arrow[dd] & \\
& \lim_{\alpha\colon O\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/E}}(\phi_\alpha,\psi_\alpha) \arrow[rr] \arrow[dd] & & \lim_{\alpha\colon O\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Act}_{\mathcal{O}}(\alpha_!Y) \arrow[dd] \\
\{\phi\} \arrow[rr] \arrow[dr] & & \mathrm{Act}_{\mathcal{O}}(X) \arrow[dr] & \\
& \lim_{\alpha\colon O\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{O/}} \{\phi_\alpha\} \arrow[rr] & & \lim_{\alpha\colon O\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{O/}} \mathrm{Act}_{\mathcal{O}}(\alpha_!X),
\end{tikzcd}$$

where the back and front faces are cartesian. Since $\mathcal{O}$ is extendable, we can apply the "extended Segal condition" of Remark 7.10 to $\mathrm{Act}_{\mathcal{O}}(-)$ and conclude the horizontal morphisms in the right-hand square are equivalences. It follows that the map on fibres in the left square is also an equivalence, as required. □

Using this we can prove the following key observation:

Proposition 9.15. *Suppose $\mathcal{O}$ is an extendable algebraic pattern. Consider a commutative triangle*

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{F} \arrow[dl, "q"] \\
& \mathcal{O}, &
\end{tikzcd}$$

where p and q are weak Segal fibrations and f preserves inert morphisms. Then for any $F \in \mathcal{F}$ the functor

$$\mathcal{E}^{\mathrm{act}}_{/F} \to \lim_{\alpha\in\mathcal{O}^{\mathrm{el}}_{q(F)/}} \mathcal{E}^{\mathrm{act}}_{/\alpha_!F}$$

is an equivalence.

Proof. For any active morphisms $\phi\colon Y \rightsquigarrow X$, $\psi\colon X \rightsquigarrow q(F)$ in $\mathcal{O}$ and $\alpha \in \mathcal{O}^{\mathrm{el}}_{q(F)/}$ the inert-active factorization gives a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, squiggly, "\phi"] \arrow[d, tail, "\alpha_Y"'] & X \arrow[r, squiggly] \arrow[d, tail, "\alpha_X"] & q(F) \arrow[d, tail, "\alpha"] \\
Y_\alpha \arrow[r, squiggly, "\phi_\alpha"] & X_\alpha \arrow[r, squiggly] & E.
\end{tikzcd}$$

By combining Remark 7.10 (the "generalized Segal condition") with the argument of Remark 9.7 we then get an equivalence

$$\mathrm{Map}^{\phi}_{\mathcal{E}}(\overline{Y},\overline{X}) \xrightarrow{\sim} \lim_{\alpha\in\mathcal{O}^{\mathrm{el}}_{q(F)/}} \mathrm{Map}^{\phi_\alpha}_{\mathcal{E}}(\alpha_{Y,!}\overline{Y},\alpha_{X,!}\overline{X}).$$

Thus in the commutative square

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{E}^{\mathrm{act}}_{/F}}(\overline{Y}, \overline{X}) \arrow[r] \arrow[d] & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{q(F)/}} \mathrm{Map}_{\mathcal{E}^{\mathrm{act}}_{/\alpha_! F}}(\alpha_{Y,!}\overline{Y}, \alpha_{X,!}\overline{X}) \arrow[d] \\ \mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/q(F)}}(Y, X) \arrow[r] & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{q(F)/}} \mathrm{Map}_{\mathcal{O}^{\mathrm{act}}_{/E}}(Y_\alpha, X_\alpha), \end{tikzcd}$$

the map on fibres is an equivalence for all $\phi\colon Y \rightsquigarrow X$, which means the square is cartesian. The bottom horizontal morphism is an equivalence by Lemma 9.14 since $\mathcal{O}$ is extendable. Hence we see that the functor $\mathcal{E}^{\mathrm{act}}_{/F} \to \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{q(F)/}} \mathcal{E}^{\mathrm{act}}_{/\alpha_! F}$ induces equivalences on mapping spaces, and so is fully faithful. To see that this functor is also essentially surjective, consider the commutative square of ∞-groupoids

$$\begin{tikzcd} (\mathcal{E}^{\mathrm{act}}_{/F})^{\simeq} \arrow[r] \arrow[d] & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{q(F)/}} (\mathcal{E}^{\mathrm{act}}_{/\alpha_! F})^{\simeq} \arrow[d] \\ (\mathcal{O}^{\mathrm{act}}_{/q(F)})^{\simeq} \arrow[r] & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{q(F)/}} (\mathcal{O}^{\mathrm{act}}_{/E})^{\simeq}; \end{tikzcd}$$

we want to show that the top horizontal morphism is an equivalence. The bottom horizontal morphism is an equivalence by assumption, since $\mathcal{O}$ is extendable; it therefore suffices to show the map on fibres over $\phi\colon O \to q(F)$ is an equivalence. The fibre $(\mathcal{E}^{\mathrm{act}}_{/F})^{\simeq}_{\phi}$ we can identify with $\mathcal{E}^{\simeq}_{O} \times_{\mathcal{F}^{\simeq}_{O}} (\mathcal{F}_{/F})^{\simeq}_{\phi}$. By condition (2) in Definition 9.6 we have an equivalence $\mathcal{E}^{\simeq}_{O} \simeq \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{E}^{\simeq}_{E}$, and similarly for $\mathcal{F}$. Moreover, condition (3) implies that in the commutative square

$$\begin{tikzcd} (\mathcal{F}_{/F})^{\simeq}_{\phi} \arrow[r] \arrow[d] & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O/}} (\mathcal{F}_{/\alpha_! F})^{\simeq}_{\phi_\alpha} \arrow[d] \\ \mathcal{F}^{\simeq}_{O} \arrow[r, "\sim"] & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{F}^{\simeq}_{E}, \end{tikzcd}$$

the map on fibres over each object of $\mathcal{F}^{\simeq}_{O}$ is an equivalence, hence the top horizontal morphism is an equivalence. Since limits commute, it follows that we have an equivalence

$$(\mathcal{E}^{\mathrm{act}}_{/F})^{\simeq}_{\phi} \to \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O/}} (\mathcal{E}^{\mathrm{act}}_{/\alpha_! F})^{\simeq}_{\phi_\alpha},$$

which completes the proof. □

Corollary 9.16. *Suppose $\mathcal{O}$ is an extendable algebraic pattern. Then any morphism between weak Segal fibrations over $\mathcal{O}$ that preserves inert morphisms is extendable.*

Proof. Suppose $\mathcal{E}$ and $\mathcal{F}$ are weak Segal fibrations over $\mathcal{O}$. Then any morphism of algebraic patterns $f\colon \mathcal{E} \to \mathcal{F}$ over $\mathcal{O}$ has unique lifting of inert morphisms, as an inert morphism is uniquely determined by its source and its image in $\mathcal{O}$. Moreover, f satisfies condition (2) in Definition 7.7 by Proposition 9.15, and condition (3) reduces to the extendability of $\mathcal{O}$. □

Corollary 9.17. *Suppose $\mathcal{O}$ is an extendable algebraic pattern, and $\mathcal{E} \to \mathcal{O}$ is a weak Segal fibration. Then $\mathcal{E}$ is extendable.*

Proof. The restriction $\mathcal{E}^{\mathrm{int}} \to \mathcal{O}$ is also a weak Segal fibration, hence we can apply Corollary 9.16 to the inclusion $\mathcal{E}^{\mathrm{int}} \to \mathcal{E}$. □

Example 9.18. The pattern $\mathbb{F}_*^\flat$ is extendable. Our previous results therefore specialize to tell us that any morphism $f \colon \mathcal{O} \to \mathcal{P}$ of symmetric ∞-operads is extendable. If $\mathcal{C}$ is a cocomplete $\times$-admissible ∞-category, we conclude that left Kan extension along f restricts to a functor $f_! \colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$, given by the formula

$$(f_!F)(P) \simeq \operatorname*{colim}_{O \in \mathcal{O}^{\mathrm{act}}_{/P}} F(O).$$

Note that this agrees with the formula for operadic left Kan extensions from [30, §3.1.2], though in our case the target must be a cartesian symmetric monoidal ∞-category.

Example 9.19. Let us spell out the description of free Segal $\mathcal{O}$-objects for a symmetric ∞-operad $\mathcal{O} \to \mathbb{F}_*$ in a bit more detail. We can identify $\mathcal{O}^{\mathrm{el}}$ with the ∞-groupoid $\mathcal{O}^{\simeq}_{\langle 1\rangle}$, and for $X \in \mathcal{O}_{\langle 1\rangle}$ the space $\mathrm{Act}_{\mathcal{O}}(X)$ decomposes as $\coprod_{n=0}^{\infty} \mathrm{Act}_{\mathcal{O}}(X)_n$, where $\mathrm{Act}_{\mathcal{O}}(X)_n$ is the space of morphisms to X in $\mathcal{O}$ lying over the unique active morphism $\langle n\rangle \to \langle 1\rangle$ in $\mathbb{F}_*$. If $\mathcal{C}$ is a cocomplete $\times$-admissible ∞-category, then for $F \in \mathrm{Fun}(\mathcal{O}^{\simeq}_{\langle 1\rangle}, \mathcal{C})$ our formula for the free Segal $\mathcal{O}$-object monad $T_{\mathcal{O}}$ gives:

$$(T_{\mathcal{O}}F)(X) \simeq \coprod_{n=0}^{\infty} \operatorname*{colim}_{(Y_1,\ldots,Y_n) \in \mathrm{Act}_{\mathcal{O}}(X)_n} F(Y_1) \times \cdots \times F(Y_n).$$

If $\mathcal{O}^{\simeq}_{\langle 1\rangle}$ is contractible, we can identify the space $\mathcal{O}(n)$ of n-ary operations with the fibre of $\mathrm{Act}_{\mathcal{O}}(X) \to \mathrm{Act}_{\mathbb{F}_*}(\langle 1\rangle) \simeq B\Sigma_n$, and so rewrite this as the familiar formula

$$T_{\mathcal{O}}C \simeq \coprod_{n=0}^{\infty} \operatorname*{colim}_{B\Sigma_n} \operatorname*{colim}_{\mathcal{O}(n)} C \times \cdots \times C \simeq \coprod_{n=0}^{\infty} \left(\mathcal{O}(n) \times C^{\times n}\right)_{h\Sigma_n}$$

for $C \in \mathcal{C} \simeq \mathrm{Fun}(\mathcal{O}^{\simeq}_{\langle 1\rangle}, \mathcal{C})$.

Remark 9.20. Our description of free algebras differs from what Lurie calls "free algebras" in [30, Section 3.1.3], because Lurie defines these to be given by operadic Kan extension along the inclusion $\mathcal{O} \times_{\mathbb{F}_*} \mathbb{F}_*^{\mathrm{int}} \to \mathcal{O}$ where the source is the subcategory containing *all* morphisms in $\mathcal{O}$ lying over inert morphisms in $\mathbb{F}_*$, not just the cocartesian ones. Lurie's construction amounts to specifying the unary operations in advance and freely adding the n-ary operations for $n > 1$, while our version adds all the operations freely.

Example 9.21. The pattern $\mathbf{\Delta}^{\mathrm{op},\flat}$ is also extendable. The analogues of Examples 9.18 and 9.19 hence also hold for non-symmetric ∞-operads.

Example 9.22. The patterns $\mathbb{F}_*^{\natural}$ and $\mathbf{\Delta}^{\mathrm{op},\natural}$ are also extendable. Hence any morphism of *generalized* symmetric or non-symmetric ∞-operads is extendable.

Remark 9.23. Suppose

$$\begin{tikzcd} \mathcal{O} \arrow[rr, "f"] \arrow[dr] & & \mathcal{P} \arrow[dl] \\ & \mathbb{F}_* & \end{tikzcd}$$

is a morphism of generalized symmetric ∞-operads. Then the previous example does *not* say that we can compute free Segal $\mathcal{P}^\flat$-objects on Segal $\mathcal{O}^\flat$-objects, as $f_!$ generally will not restrict to a functor between these. In the definition of extendability, condition (1) is still automatic (as the inert morphisms in $\mathbb{F}_*^{\natural}$ and $\mathbb{F}_*^{\flat}$ are the same), while condition (3) reduces to $\mathbb{F}_*^{\flat}$ being extendable. Thus the morphism $f^\flat \colon \mathcal{O}^\flat \to \mathcal{P}^\flat$ is extendable if and only if for all P over $\langle n \rangle$ in $\mathbb{F}_*$ the functor

$$\mathcal{O}^{\mathrm{act}}_{/P} \to \prod_{i=1}^{n} \mathcal{O}^{\mathrm{act}}_{/\rho_{i,!}P}$$

is cofinal, where $\rho_i \colon \langle n \rangle \to \langle 1 \rangle$ is as in the introduction.

10. Polynomial monads from patterns

In this section we introduce the notion of *polynomial monad* on an ∞-category of presheaves, and prove that the free Segal $\mathcal{O}$-space monad for an extendable pattern $\mathcal{O}$ is polynomial. Moreover, we show that this is compatible with Segal morphisms of algebraic patterns, yielding a functor

$$\mathfrak{M} \colon \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}} \to \mathrm{PolyMnd}$$

between the subcategory of AlgPatt consisting of extendable patterns and Segal morphisms, and an ∞-category of polynomial monads. We start by introducing some terminology:

Definition 10.1. A natural transformation $\phi\colon F \to G$ is *cartesian* if the naturality squares

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\phi_x"'] & F(y) \arrow[d, "\phi_y"] \\ G(x) \arrow[r, "G(f)"'] & G(y) \end{tikzcd}$$

are all cartesian.

Definition 10.2. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a *local right adjoint* if for every $c \in \mathcal{C}$ the induced functor $\mathcal{C}_{/c} \to \mathcal{D}_{/Fc}$ is a right adjoint.

Lemma 10.3. *If $\mathcal{C}$ and $\mathcal{D}$ are presentable ∞-categories, then the following are equivalent for a functor $F\colon \mathcal{C} \to \mathcal{D}$:*

(1) F is accessible and preserves weakly contractible limits.
(2) F is a local right adjoint.
(3) The functor $F_{/*}\colon \mathcal{C} \to \mathcal{D}_{/F(*)}$ has a left adjoint.

Proof. The equivalence of (1) and (2) was proved as [16, Proposition 2.2.8]. Since (3) is a special case of (2), it remains to prove that (3) implies (1). By the adjoint functor theorem [29, Corollary 5.5.2.9], it follows from (3) that $F_{/*}$ is accessible and preserves limits. The forgetful functor $\mathcal{D}_{/F(*)} \to \mathcal{D}$ preserves and creates all colimits, as well as weakly contractible limits, by [16, Lemma 2.2.7], so this implies that F itself is accessible and preserves weakly contractible limits. $\square$

Definition 10.4. A monad T is *cartesian* if its multiplication and unit are cartesian natural transformations, and is *polynomial* if it is cartesian and the underlying endofunctor is a local right adjoint.

Remark 10.5. For ordinary categories, our notion of polynomial monads is the same as the *strongly cartesian* monads considered in [6]. For monads on ∞-categories of the form $\mathcal{S}_{/X}$ for $X \in \mathcal{S}$, we recover the polynomial monads studied in [16] (see Theorem 2.2.3 there), which is our reason for adopting this terminology.

Proposition 10.6. *If $\mathcal{O}$ is an extendable algebraic pattern, then the free Segal $\mathcal{O}$-space monad $T_{\mathcal{O}}$ on $\mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S})$ is a polynomial monad.*

Proof. Since $\mathrm{Seg}_{\mathcal{O}^{(\mathrm{int})}}(\mathcal{S})$ is an accessible localization of $\mathrm{Fun}(\mathcal{O}^{(\mathrm{int})}, \mathcal{S})$, the inclusions $\mathrm{Seg}_{\mathcal{O}^{(\mathrm{int})}}(\mathcal{S}) \hookrightarrow \mathrm{Fun}(\mathcal{O}^{(\mathrm{int})}, \mathcal{S})$ are accessible and preserve limits. The endofunctor $T_{\mathcal{O}}$ of $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})$ factors as a composite

$$\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S}) \hookrightarrow \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S}) \xrightarrow{j_{\mathcal{O},!}} \mathrm{Fun}(\mathcal{O}, \mathcal{S}) \xrightarrow{j_{\mathcal{O}}^{*}} \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S}),$$

where the composite lands in the subcategory $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})$. To see that $T_{\mathcal{O}}$ is a local right adjoint it suffices to show that the three functors in this composition are accessible and preserve weakly contractible limits. All three functors are clearly accessible and except for $j_{\mathcal{O},!}$ they preserve limits. It therefore remains to show that $j_{\mathcal{O},!}$ preserves weakly contractible limits. By Lemma 7.2 for $O \in \mathcal{O}$ and $F \in \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S})$, the value of $j_{\mathcal{O},!}F$ at O is $\mathrm{colim}_{X \in \mathrm{Act}_{\mathcal{O}}(O)} F(X)$. Since $\mathrm{Act}_{\mathcal{O}}(O) = (\mathcal{O}^{\mathrm{int}})^{\mathrm{act}}_{/O}$ is an ∞-groupoid, this factors through the forgetful functor $\mathcal{S}_{/\mathrm{Act}_{\mathcal{O}}(O)} \to \mathcal{S}$, which detects weakly contractible limits by [16, Lemma 2.2.7]. It therefore suffices to show that the functor $\mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S}) \to \mathcal{S}_{/\mathrm{Act}_{\mathcal{O}}(O)}$ taking F to $\mathrm{colim}_{X \in \mathrm{Act}_{\mathcal{O}}(O)} F(X) \to \mathrm{Act}_{\mathcal{O}}(O)$ preserves weakly contractible limits. But this factors as restriction along $\mathrm{Act}_{\mathcal{O}}(O) \to \mathcal{O}^{\mathrm{int}}$, which certainly preserves limits, followed by the colimit functor $\mathrm{Fun}(\mathrm{Act}_{\mathcal{O}}(O), \mathcal{S}) \to \mathcal{S}_{/\mathrm{Act}_{\mathcal{O}}(O)}$, which is an equivalence.

Next, we show that the multiplication transformation $T_{\mathcal{O}}^2 \to T_{\mathcal{O}}$ is cartesian. For $O \in \mathcal{O}$, we have an equivalence

$$(T_{\mathcal{O}}^2 F)(O) \simeq \operatorname*{colim}_{X \in \mathrm{Act}_{\mathcal{O}}(O)} (T_{\mathcal{O}} F)(O) \simeq \operatorname*{colim}_{X \in \mathrm{Act}_{\mathcal{O}}(O)} \operatorname*{colim}_{Y \in \mathrm{Act}_{\mathcal{O}}(X)} F(Y) \simeq \operatorname*{colim}_{(Y \rightsquigarrow X \rightsquigarrow O) \in \mathrm{Act}^2_{\mathcal{O}}(O)} F(Y),$$

where $\mathrm{Act}^2_{\mathcal{O}}(O) \to \mathrm{Act}_{\mathcal{O}}(O)$ is the left fibration for the functor taking $X \rightsquigarrow O$ to $\mathrm{Act}_{\mathcal{O}}(X)$. We then have an identification

$$\mathrm{Act}^2_{\mathcal{O}}(O) \simeq \{Y \overset{g}{\rightsquigarrow} X \overset{f}{\rightsquigarrow} O : f, g \text{ active}\}$$

under which the multiplication transformation $T_{\mathcal{O}}^2 F(X) \to T_{\mathcal{O}} F(X)$ is the morphism induced on colimits by the map $\mathrm{Act}^2_{\mathcal{O}}(O) \to \mathrm{Act}_{\mathcal{O}}(O)$ given by composition of active morphisms. Given $F \to G$, we want to show that the square

$$\begin{tikzcd}
\mathrm{colim}_{(Y \rightsquigarrow X \rightsquigarrow O) \in \mathrm{Act}^2_{\mathcal{O}}(O)} F(Y) \arrow[r] \arrow[d] & \mathrm{colim}_{(Y \rightsquigarrow X \rightsquigarrow O) \in \mathrm{Act}^2_{\mathcal{O}}(O)} G(Y) \arrow[d] \\
\mathrm{colim}_{(Y \rightsquigarrow O) \in \mathrm{Act}_{\mathcal{O}}(O)} F(Y) \arrow[r] & \mathrm{colim}_{(Y \rightsquigarrow O) \in \mathrm{Act}_{\mathcal{O}}(O)} G(Y)
\end{tikzcd}$$

is cartesian. To see this it suffices to show that the square on fibres over $(Y \overset{f}{\rightsquigarrow} O) \in \mathrm{Act}_{\mathcal{O}}(O)$ is cartesian. The fibre $(T_{\mathcal{O}}^2 F(X))_f$ we can identify with the colimit over the fibre

$$\mathrm{Act}^2_{\mathcal{O}}(O)_f \simeq \left\{ \begin{tikzcd} & X \arrow[dr] & \\ Y \arrow[ur] \arrow[rr, "f"] & & O \end{tikzcd} \right\}$$

of the *constant* functor with value $F(Y)$. The square of fibres is therefore

$$\begin{tikzcd}
\mathrm{Act}^2_{\mathcal{O}}(O)_f \times F(Y) \arrow[r] \arrow[d] & \mathrm{Act}^2_{\mathcal{O}}(O)_f \times G(Y) \arrow[d] \\
F(Y) \arrow[r] & G(Y),
\end{tikzcd}$$

which is indeed cartesian.

The value of the unit transformation $F(O) \to T_{\mathcal{O}}F(O)$ is similarly induced by the map $\{\mathrm{id}_O\} \to \mathrm{Act}_{\mathcal{O}}(O)$. To see that the unit transformation is cartesian we must show that for $F \to G$ the square

$$\begin{tikzcd}
F(O) \arrow[r] \arrow[d] & G(O) \arrow[d] \\
\mathrm{colim}_{(Y \rightsquigarrow O) \in \mathrm{Act}_{\mathcal{O}}(O)} F(Y) \arrow[r] & \mathrm{colim}_{(Y \rightsquigarrow O) \in \mathrm{Act}_{\mathcal{O}}(O)} G(Y)
\end{tikzcd}$$

is cartesian. It again suffices to consider the square of fibres over $(X \overset{f}{\rightsquigarrow} O) \in \mathrm{Act}_{\mathcal{O}}(O)$. The fibre of $\{\mathrm{id}_O\} \to \mathrm{Act}_{\mathcal{O}}(O)$ at f is the space

$$P_f := \mathrm{Map}_{\mathrm{Act}_{\mathcal{O}}(O)}(\mathrm{id}_O, f)$$

of paths from $\mathrm{id}_{\mathcal{O}}$ to f in $\mathrm{Act}_{\mathcal{O}}(O)$ (which is empty if id_O and f are not equivalent), and the square of fibres is

$$\begin{tikzcd}
P_f \times F(O) \arrow[r] \arrow[d] & P_f \times G(O) \arrow[d] \\
F(X) \arrow[r] & G(X),
\end{tikzcd}$$

which is cartesian as required. □

Remark 10.7. We can regard polynomial monads as being the monads in an $(\infty, 2)$-category whose objects are presheaf ∞-categories, whose morphisms are local right adjoints, and whose 2-morphisms are cartesian transformations. The natural morphisms between polynomial monads are then the lax morphisms of monads in this $(\infty, 2)$-category. If T is a polynomial monad on $\mathcal{S}^{\mathcal{I}}$ and S is a polynomial monad on $\mathcal{S}^{\mathcal{J}}$, then by the results of [23] these correspond to commutative squares

$$\begin{tikzcd}
\mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}) \arrow[r, "\Phi"] \arrow[d, "U_S"'] & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \arrow[d, "U_T"] \\
\mathcal{S}^{\mathcal{J}} \arrow[r, "f^*"] & \mathcal{S}^{\mathcal{I}},
\end{tikzcd}$$

for some functor $f\colon \mathcal{I} \to \mathcal{J}$, such that the mate transformation

$$F_T f^* \to \Phi F_S$$

is cartesian. Noting the contravariance here, this motivates the following definition of an ∞-category of polynomial monads:

Definition 10.8. Consider the pullback

$$\mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty) \times_{\widehat{\mathrm{Cat}}_\infty} \mathrm{Cat}_\infty$$

along $\mathrm{ev}_1\colon \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty) \to \widehat{\mathrm{Cat}}_\infty$ and $\mathcal{S}^{(-)}\colon \mathrm{Cat}_\infty^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$. We write $\mathrm{PolyMnd}^{\mathrm{op}}$ for the subcategory of this pullback whose objects are the monadic right adjoints of polynomial monads, and whose morphisms are commutative squares whose mate transformations are cartesian.

Remark 10.9. Note that since U_T detects pullbacks, the mate transformation above is cartesian if and only if the transformation

$$Tf^* \to f^* S$$

obtained by composing with U_T is cartesian.

Next, we observe that any Segal morphism between extendable patterns gives a morphism of polynomial monads:

Proposition 10.10. *Suppose $f\colon \mathcal{O} \to \mathcal{P}$ is a Segal morphism between extendable patterns. Then the mate transformation*

$$j_{\mathcal{O},!} f^{\mathrm{int},*} \to f^* j_{\mathcal{P},!}$$

of functors $\mathrm{Seg}_{\mathcal{P}^{\mathrm{int}}}(\mathcal{S}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$ is cartesian.

Proof. We have to show that for every morphism $\Phi \to \Psi$ the commutative square

$$\begin{tikzcd}
j_{\mathcal{O},!} f^{\mathrm{int},*}\Phi \arrow[r] \arrow[d] & f^* j_{\mathcal{P},!}\Phi \arrow[d] \\
j_{\mathcal{O},!} f^{\mathrm{int},*}\Psi \arrow[r] & f^* j_{\mathcal{P},!}\Psi
\end{tikzcd}$$

is cartesian in $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$. Since $\mathrm{Seg}_{\mathcal{P}^{\mathrm{int}}}(\mathcal{S})$ has a terminal object it suffices to consider $\Psi \simeq *$, in which case we obtain the commutative square

$$\begin{tikzcd}
\operatorname{colim}_{X\in \mathrm{Act}_{\mathcal{O}}(E)} \Phi(fX) \arrow[r] \arrow[d] & \operatorname{colim}_{Y\in \mathrm{Act}_{\mathcal{P}}(f(E))} \Phi(Y) \arrow[d] \\
\mathrm{Act}_{\mathcal{O}}(E) \arrow[r] & \mathrm{Act}_{\mathcal{P}}(f(E))
\end{tikzcd}$$

after evaluating at an object $E \in \mathcal{O}^{\mathrm{el}}$. To show that this square is cartesian, it now suffices to observe that for every point $(X \to E) \in \mathrm{Act}_{\mathcal{O}}(E)$, the map on fibres is the identity $\Phi(fX) \to \Phi(fX)$. $\square$

Definition 10.11. We let $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}}$ denote the subcategory of AlgPatt whose objects are the extendable patterns and whose morphisms are the Segal morphisms.

Corollary 10.12. *The functor* $\mathrm{AlgPatt}^{\mathrm{Seg}} \to \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)^{\mathrm{op}}$ *taking a pattern* $\mathcal{O}$ *to the monadic right adjoint* $U_{\mathcal{O}}\colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \to \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S})$ *restricts to a functor* $\mathfrak{M}\colon \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}} \to \mathrm{PolyMnd}$. $\square$

11. Generic morphisms and the nerve theorem

In the previous section we saw that the free Segal space monad for any extendable pattern was a polynomial monad. Our next goal is to extract an extendable pattern from any polynomial monad. As a first step towards this, in this section we prove an ∞-categorical version of Weber's nerve theorem [38]; our proof was particularly inspired by that of Berger, Melliès, and Weber [6].

We begin by defining *generic morphisms* with respect to a local right adjoint functor, and extend some basic observations about them from [37] to the ∞-categorical setting.

Definition 11.1. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a local right adjoint functor between presentable ∞-categories. Let $L_*\colon \mathcal{D}_{/F(*)} \to \mathcal{C}$ be the left adjoint to $F_{/*}\colon \mathcal{C} \to \mathcal{D}_{/F(*)}$; we will abusively write L_*D for the value of L_* at an object $D \to F(*)$. For any morphism $D \xrightarrow{\phi} F(C)$ in $\mathcal{D}$, we can view ϕ as a morphism in $\mathcal{D}_{/F(*)}$ via the map $F(q)\colon F(C) \to F(*)$, where q is the unique morphism $C \to *$. We say ϕ is F-*generic* (or just *generic* if F is clear from context) if the adjoint morphism

$$L_*D \simeq L_*(F(q) \circ \phi) \to C$$

is an equivalence. (In other words, the generic morphisms are precisely the unit morphisms $D \to F_{/*}L_*D$.)

Remark 11.2. Using the universal property of the left adjoint, we can rephrase this definition purely in terms of F as follows: $\phi\colon D \to F(B)$ is F-generic if for every commutative square

$$\begin{tikzcd} D \arrow[r, "\psi"] \arrow[d, "\phi"'] & F(A) \arrow[d, "F(\alpha)"] \\ F(B) \arrow[r, "F(\beta)"] \arrow[ur, dotted] & F(*) \end{tikzcd}$$

there exists a unique morphism $\gamma \colon B \to A$ such that $F(\gamma) \circ \phi \simeq \psi$ and the equivalence in the square arises by combining this with the canonical equivalence $F(\alpha) \circ F(\gamma) \simeq F(\alpha\gamma) \simeq F(\beta)$ induced by $*$ being terminal. This is the version of the definition considered in [37].

Lemma 11.3. *Let $\phi \colon D \to F(B)$ be an F-generic morphism. Then given a commutative square*

$$\begin{tikzcd} D \arrow[r, "\psi"] \arrow[d, "\phi"'] & F(A) \arrow[d, "F(\alpha)"] \\ F(B) \arrow[r, "F(\beta)"] \arrow[ur, dotted] & F(X), \end{tikzcd}$$

there exists a unique commutative triangle

$$\begin{tikzcd} A \arrow[rr, "\gamma"] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & X & \end{tikzcd}$$

such that $F(\gamma) \circ \phi \simeq \psi$ and the equivalence in the square arises by combining this with the equivalence $F(\alpha) \circ F(\gamma) \simeq F(\alpha\gamma) \simeq F(\beta)$ given by applying F to the triangle.

Proof. The existence of a unique filler in the original square is equivalent to the existence of such a filler in the adjoint square

$$\begin{tikzcd} L_X D \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"] & X. \end{tikzcd}$$

Since F preserves pullbacks, if ξ denotes the unique morphism $X \to *$ we have a commutative square of right adjoints

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F_{/*}"] \arrow[d, "\xi^*"'] & \mathcal{D}_{/F(*)} \arrow[d, "F(\xi)^*"] \\ \mathcal{C}_{/X} \arrow[r, "F_{/X}"] & \mathcal{D}_{/F(X)}. \end{tikzcd}$$

This induces a corresponding square of left adjoints

$$\begin{tikzcd} \mathcal{D}_{/F(X)} \arrow[r, "L_X"] \arrow[d, "F(\xi)_!"'] & \mathcal{C}_{/X} \arrow[d, "\xi_!"] \\ \mathcal{D}_{/F(*)} \arrow[r, "L_*"] & \mathcal{C}. \end{tikzcd}$$

Thus $\xi_! L_X \simeq L_* F(\xi)_!$; since $\xi_!$ detects equivalences, we see that for $D \xrightarrow{\phi} F(B) \xrightarrow{F(\beta)} F(X)$ the adjoint morphism $L_X D \to B$ over X is equivalent to $L_* X \to B$ computed using the morphism $F(B) \to F(*)$ that is the image of $B \to *$, as this is the composite $F(B) \xrightarrow{F(\beta)} F(X) \xrightarrow{F(\xi)} F(*)$. Since ϕ is generic, it therefore follows that the map $L_X D \to B$ is also an equivalence, hence the unique filler arises from the composite $B \simeq L_X D \to A$. $\square$

Remark 11.4. For any morphism $\phi\colon D \to F(C)$, if $\psi\colon L_* D \to C$ is the adjoint morphism, we can write ϕ as a composite

$$D \xrightarrow{\eta_D} F(L_* D) \xrightarrow{F(\psi)} F(C),$$

where η_D is the unit of the adjunction $L_* \dashv F_{/*}$. This is the *unique* factorization of ϕ as a generic morphism followed by a morphism in the image of F; we will often refer to this as the *generic–free factorization* of ϕ.

Lemma 11.5 *(Cf. [37, Proposition 5.10]). Suppose $F, G\colon \mathcal{C} \to \mathcal{D}$ are local right adjoint functors between presentable ∞-categories and $\phi\colon F \to G$ is a cartesian natural transformation. Then a morphism $f\colon D \to F(C)$ is F-generic if and only if the composite $D \to F(C) \to G(C)$ is G-generic.*

Proof. Since ϕ is a cartesian transformation, we have natural cartesian squares

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & G(X) \arrow[d] \\ F(*) \arrow[r, "\phi(*)"] & G(*). \end{tikzcd}$$

This means we can write $F_{/*}$ as the composite

$$\mathcal{C} \xrightarrow{G_{/*}} \mathcal{D}_{/G(*)} \xrightarrow{\phi(*)^*} \mathcal{D}_{/F(*)}.$$

But then the left adjoint $L_{*,F}$ of $F_{/*}$ is the composite

$$\mathcal{D}_{/F(*)} \xrightarrow{\phi(*)_!} \mathcal{D}_{/G(*)} \xrightarrow{L_{*,G}} \mathcal{C},$$

where $L_{*,G}$ denotes the left adjoint to $G_{/*}$. Given $f\colon D \to F(C)$, this means the adjoint morphism $L_{*,F}D \to C$ is the same as the adjoint morphism $L_{*,G}D \to C$ for the composite $D \to F(C) \to G(C)$. $\square$

Lemma 11.6 *(Cf. [37, Lemma 5.14]). Suppose $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ are local right adjoint functors between presentable ∞-categories. If $f\colon D \to F(C)$ is F-generic and $g\colon E \to G(D)$ is G-generic, then the composite*

$$E \xrightarrow{g} G(D) \xrightarrow{G(f)} GF(C)$$

is GF-generic.

Proof. The functor $(GF)_{/*}$ factors in two steps as

$$\mathcal{C} \xrightarrow{F_{/*}} \mathcal{D}_{/F(*)} \xrightarrow{G_{/F(*)}} \mathcal{E}_{/GF(*)}.$$

The left adjoint is therefore also computed in two steps; to find the morphism adjoint to $G(f)g$ we first get the commutative diagram

$$\begin{tikzcd} L_{*,G}E \arrow[r, "\sim"] \arrow[dr] & D \arrow[r, "f"] \arrow[d] & F(C) \arrow[dl] \\ & F(*), & \end{tikzcd}$$

and then $L_{*,F}L_{*,G}E \xrightarrow{\sim} L_{*,F}D \xrightarrow{\sim} C$, which is an equivalence as required. $\square$

Definition 11.7. Suppose $\mathcal{I}$ is a small ∞-category and T is a polynomial monad on the functor ∞-category $\mathcal{S}^{\mathcal{I}}$. We define $\mathcal{U}(T)^{\mathrm{op}}$ to be the full subcategory of $\mathcal{S}^{\mathcal{I}}$ spanned by the objects X that admit a generic morphism $I \to TX$ with $I \in \mathcal{I}^{\mathrm{op}}$ (regarded as an object of $\mathcal{S}^{\mathcal{I}}$ through the Yoneda embedding). We write $\mathcal{W}(T)^{\mathrm{op}}$ for the full subcategory of $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}})$ spanned by the free T-algebras on the objects of $\mathcal{U}(T)$.

Remark 11.8. From the definition of generic morphisms it follows that we can equivalently describe the objects of $\mathcal{U}(T)^{\mathrm{op}}$ as those of the form L_*I for some $I \in \mathcal{I}^{\mathrm{op}}$ and some morphism $I \to T*$ in $\mathcal{S}^{\mathcal{I}}$.

Lemma 11.9. *Let T be a polynomial monad on $\mathcal{S}^{\mathcal{I}}$.*

(i) For any object $X \in \mathcal{S}^{\mathcal{I}}$, the unit map $X \to T(X)$ is generic.
(ii) If $X \xrightarrow{\phi} T(Y)$ and $Y \xrightarrow{\psi} T(Z)$ are generic morphisms, then the composite

$$X \xrightarrow{\phi} TY \xrightarrow{T\psi} T^2Z \xrightarrow{\mu_Z} TZ$$

is generic, where μ denotes the multiplication transformation of the monad.

Proof. Since T is a polynomial monad, the unit transformation $\mathrm{id} \to T$ is cartesian and so by Lemma 11.5 the unit map $X \to TX$ is generic for all X (since an id-generic map is precisely an equivalence).

The composite $X \xrightarrow{\phi} TY \xrightarrow{T\psi} T^2Z$ is T^2-generic by Lemma 11.6, and as the multiplication μ is a cartesian transformation this implies the composite of this with $\mu_Z \colon T^2Z \to TZ$ is T-generic by Lemma 11.5. □

Proposition 11.10. *Let T be a polynomial monad on $\mathcal{S}^{\mathcal{I}}$.*

(i) The full subcategory $\mathcal{U}(T)^{\mathrm{op}}$ contains $\mathcal{I}^{\mathrm{op}}$.
(ii) For any generic morphism $X \to TY$ with $X \in \mathcal{U}(T)^{\mathrm{op}}$, the object Y also lies in $\mathcal{U}(T)^{\mathrm{op}}$.

Proof. The unit map $I \to TI$ is generic by Lemma 11.9(i). Hence $I \to TI \to T*$ is a generic–free factorization, where the second map is the image under T of the unique map $I \to *$. This shows that I is in $\mathcal{U}(T)^{\mathrm{op}}$, which proves (i).

To prove (ii), observe that since X is in $\mathcal{U}(T)^{\mathrm{op}}$, we have a generic morphism $I \to TX$ with I in $\mathcal{I}^{\mathrm{op}}$. Then by Lemma 11.9(ii) the composite

$$I \to TX \to T^2Y \xrightarrow{\mu_Y} TY$$

is also generic, which means that Y is also in $\mathcal{U}(T)^{\mathrm{op}}$. □

Remark 11.11. Note that the functor $\mathcal{U}(T) \to \mathcal{W}(T)$ need not exhibit $\mathcal{U}(T)$ as a subcategory of $\mathcal{W}(T)$.

Our goal is now to show that the algebras for the polynomial monad T can be described in terms of the ∞-categories $\mathcal{U}(T)$ and $\mathcal{W}(T)$ — this is the content of the nerve theorem. The next proposition gives the key input needed to prove this.

Notation 11.12. Given a functor $j \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{S}^{\mathcal{I}}$, we let

$$\nu_{\mathcal{A}} \colon \mathcal{S}^{\mathcal{I}} \to \mathrm{Fun}((\mathcal{S}^{\mathcal{I}})^{\mathrm{op}}, \mathcal{S}) \xrightarrow{j^*} \mathcal{S}^{\mathcal{A}}$$

denote the composition of the Yoneda embedding and j^*. Thus $\nu_{\mathcal{A}}$ takes $\Phi \colon \mathcal{I} \to \mathcal{S}$ to $\mathrm{Map}_{\mathcal{S}^{\mathcal{I}}}(j(-), \Phi)$.

Proposition 11.13. *Let T be a polynomial monad on $\mathcal{S}^{\mathcal{I}}$.*

(i) The functor $\nu_{\mathcal{U}(T)} \colon \mathcal{S}^{\mathcal{I}} \to \mathcal{S}^{\mathcal{U}(T)}$ is fully faithful, and given by right Kan extension along the inclusion $\mathcal{I} \hookrightarrow \mathcal{U}(T)$.

(ii) For every $\Phi \in \mathcal{S}^{\mathcal{I}}$, the diagram

$$(\mathcal{U}(T)^{\mathrm{op}})^{\triangleright}_{/\Phi} \to \mathcal{S}^{\mathcal{I}}$$

is a colimit diagram.

(iii) For every Φ in $\mathcal{S}^{\mathcal{I}}$ the composite diagram

$$(\mathcal{U}(T)^{\mathrm{op}})^{\triangleright}_{/\Phi} \to \mathcal{S}^{\mathcal{I}} \xrightarrow{T} \mathcal{S}^{\mathcal{I}} \xrightarrow{\nu_{\mathcal{U}(T)}} \mathcal{S}^{\mathcal{U}(T)}$$

is a colimit diagram. (In other words, the colimit diagram in (ii) is preserved by the functor $\nu_{\mathcal{U}(T)}T$.)

The proof uses the following technical observation:

Lemma 11.14. *Suppose $j\colon \mathcal{A}^{\mathrm{op}} \hookrightarrow \mathcal{S}^{\mathcal{I}}$ is a full subcategory of a presheaf ∞-category $\mathcal{S}^{\mathcal{I}}$ such that $\mathcal{I}^{\mathrm{op}}$ (viewed as a full subcategory of $\mathcal{S}^{\mathcal{I}}$ via the Yoneda embedding) is contained in $\mathcal{A}^{\mathrm{op}}$, so that we have a fully faithful functor $i\colon \mathcal{I} \to \mathcal{A}$. Then:*

(i) $\nu_{\mathcal{A}}$ is equivalent to the functor $i_*\colon \mathcal{S}^{\mathcal{I}} \to \mathcal{S}^{\mathcal{A}}$ given by right Kan extension along i.

(ii) $\nu_{\mathcal{A}}$ is fully faithful.

(iii) For every Φ in $\mathcal{S}^{\mathcal{I}}$, the diagram

$$(\mathcal{A}^{\mathrm{op}})^{\triangleright}_{/\Phi} \to \mathcal{S}^{\mathcal{I}}$$

is a colimit diagram, and this colimit is preserved by $\nu_{\mathcal{A}}$.

Proof. For any $\Phi \in \mathcal{S}^{\mathcal{I}}$, the diagram $(\mathcal{I}^{\mathrm{op}})^{\triangleright}_{/\Phi} \to \mathcal{S}^{\mathcal{I}}$ is a colimit, so we have a natural equivalence

$$\nu_{\mathcal{A}}\Phi(a) \simeq \mathrm{Map}(j(a), \Phi) \simeq \mathrm{Map}(\operatorname*{colim}_{x \in (\mathcal{I}^{\mathrm{op}})_{/j(a)}} y(x), \Phi) \simeq \lim_{x \in \mathcal{I}_{a/}} \Phi(x) \simeq (i_*\Phi)(a).$$

This proves (i). Since $i\colon \mathcal{I} \to \mathcal{A}$ is fully faithful, it follows that i_* is also fully faithful, which proves (ii). To prove (iii), since $\nu_{\mathcal{A}}$ is fully faithful it suffices to show that the composite

$$(\mathcal{A}^{\mathrm{op}})^{\triangleright}_{/\Phi} \to \mathcal{S}^{\mathcal{I}} \xrightarrow{\nu_{\mathcal{A}}} \mathcal{S}^{\mathcal{A}}$$

is a colimit diagram. But this is now a Yoneda cocone for $\mathcal{A}^{\mathrm{op}}$, which is always a colimit in $\mathcal{S}^{\mathcal{A}}$. $\square$

Proof of Proposition 11.13. (i) and (ii) follow from Proposition 11.10(i) and Lemma 11.14. To prove (iii), since colimits in functor categories are computed objectwise, it suffices to show that for every $X \in \mathcal{U}(T)$ and $\Phi \in \mathcal{S}^{\mathcal{I}}$, the morphism

$$\operatorname*{colim}_{Y \in (\mathcal{U}(T)^{\mathrm{op}})_{/\Phi}} \mathrm{Map}_{\mathcal{S}^{\mathcal{I}}}(X, TY) \to \mathrm{Map}_{\mathcal{S}^{\mathcal{I}}}(X, T\Phi)$$

is an equivalence. Let $\mathcal{E} \to (\mathcal{U}(T)^{\mathrm{op}})_{/\Phi}$ be the left fibration for the functor $(\mathcal{U}(T)^{\mathrm{op}})_{/\Phi} \to \mathcal{S}$ taking Y to $\mathrm{Map}_{\mathcal{S}^{\mathcal{I}}}(X, TY)$; then we have a pullback square

$$\begin{tikzcd}
\mathcal{E} \arrow[rr] \arrow[d] & & \mathcal{S}^{\mathcal{I}}_{X/} \arrow[d] \\
(\mathcal{U}(T)^{\mathrm{op}})_{/\Phi} \arrow[r] & \mathcal{S}^{\mathcal{I}} \arrow[r, "T"] & \mathcal{S}^{\mathcal{I}},
\end{tikzcd}$$

so that an object of $\mathcal{E}$ is a pair $(Y \to \Phi, X \to TY)$. By [29, Proposition 3.3.4.5], the space $\mathrm{colim}_{Y \in (\mathcal{U}(T)^{\mathrm{op}})_{/\Phi}} \mathrm{Map}_{\mathcal{S}^{\mathcal{I}}}(X, TY)$ is equivalent to the space $\|\mathcal{E}\|$ obtained by inverting all morphisms in $\mathcal{E}$, and the morphism we are interested in is the map of spaces induced by the functor of ∞-categories $\mathcal{E} \to \mathrm{Map}_{\mathcal{S}^{\mathcal{I}}}(X, T\Phi)$ taking $(Y \xrightarrow{\alpha} \Phi, X \to TY)$ to the composite $X \to TY \xrightarrow{T\alpha} T\Phi$. By [29, Proposition 4.1.1.3] a morphism of spaces that arises from a cofinal functor of ∞-categories is an equivalence, so it suffices to show that the functor $\mathcal{E} \to \mathrm{Map}_{\mathcal{S}^{\mathcal{I}}}(X, T\Phi)$ is cofinal. Since every functor to an ∞-groupoid is a cartesian fibration, to prove this we may apply [29, Lemma 4.1.3.2], which says that a cartesian fibration with weakly contractible fibres is cofinal. It thus suffices to check that the fibres $\mathcal{E}_\phi$ at $\phi\colon X \to T\Phi$ are weakly contractible. But the fibre $\mathcal{E}_\phi$ is the ∞-category of factorizations of ϕ of the form $X \to TY \xrightarrow{T\alpha} T\Phi$. Since T is a local right adjoint, this ∞-category has an initial object, corresponding to the generic-free factorization $X \to TY \to T\Phi$, as Y also lies in $\mathcal{U}(T)$ by Proposition 11.10(ii); hence $\mathcal{E}_\phi$ is indeed weakly contractible, as required. □

Theorem 11.15 *(Nerve Theorem). Suppose T is a polynomial monad on $\mathcal{S}^{\mathcal{I}}$, and let j_T denote the restriction of F_T^{op} to a functor $\mathcal{U}(T) \to \mathcal{W}(T)$. Then the commutative square*

$$\begin{tikzcd}
\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \arrow[r, "\nu_{\mathcal{W}(T)}"] \arrow[d, "U_T"'] & \mathrm{Fun}(\mathcal{W}(T), \mathcal{S}) \arrow[d, "j_T^*"] \\
\mathcal{S}^{\mathcal{I}} \arrow[r, "\nu_{\mathcal{U}(T)}"] & \mathrm{Fun}(\mathcal{U}(T), \mathcal{S})
\end{tikzcd}$$

is cartesian, and the mate transformation

$$j_{T,!}\nu_{\mathcal{U}(T)} \to \nu_{\mathcal{W}(T)} F_T$$

is an equivalence. In particular, $\nu_{\mathcal{W}(T)}\colon \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \to \mathrm{Fun}(\mathcal{W}(T), \mathcal{S})$ is fully faithful, and the left adjoint $j_{T,!}$ restricts to F_T.

Proof. We want to apply [16, Proposition 5.3.5] to conclude that the square is cartesian. All the requirements for this are clearly satisfied, with one exception: We must show that the mate transformation

$$j_{T,!}\nu_{\mathcal{U}(T)} \to \nu_{\mathcal{W}(T)}F_T$$

is an equivalence, i.e. is given by an equivalence when evaluated at every object $\Phi \in \mathcal{S}^{\mathcal{I}}$. We first consider the case of $X \in \mathcal{U}(T)^{\mathrm{op}} \subseteq \mathcal{S}^{\mathcal{I}}$. Then $\nu_{\mathcal{U}(T)}X$ is the presheaf on $\mathcal{U}(T)$ represented by X, hence $j_{T,!}\nu_{\mathcal{U}(T)}X$ is represented by $j_TX \simeq F_TX$, and so $j_{T,!}\nu_{\mathcal{U}(T)}X \xrightarrow{\sim} \nu_{\mathcal{W}(T)}F_TX$, as required.

Now let $\Phi \in \mathcal{S}^{\mathcal{I}}$ be a general object. Since j_T^* detects equivalences, it suffices to show that the evaluation of the transformation

$$j_T^*j_{T,!}\nu_{\mathcal{U}(T)} \to j_T^*\nu_{\mathcal{W}(T)}F_T \simeq \nu_{\mathcal{U}(T)}T$$

at Φ is an equivalence. We know from Lemma 11.14(iii) and Proposition 11.13(iii) that Φ is the colimit of the diagram $(\mathcal{U}(T)^{\mathrm{op}})_{/\Phi} \to \mathcal{S}^{\mathcal{I}}$ taking $X \to \Phi$ to X, and this colimit is preserved by the functors $\nu_{\mathcal{U}(T)}$ and $\nu_{\mathcal{U}(T)}T$. Since $j_T^*j_{T,!}$ preserves colimits (being itself a left adjoint), we have a commutative square

$$\begin{tikzcd}
\operatorname{colim}_{X\in(\mathcal{U}(T)^{\mathrm{op}})_{/\Phi}} j_T^*j_{T,!}\nu_{\mathcal{U}(T)}X \arrow[r] \arrow[d, "\wr"'] & \operatorname{colim}_{X\in(\mathcal{U}(T)^{\mathrm{op}})_{/\Phi}} \nu_{\mathcal{U}(T)}TX \arrow[d, "\wr"] \\
j_T^*j_{T,!}\nu_{\mathcal{U}(T)}\Phi \arrow[r] & \nu_{\mathcal{U}(T)}T\Phi,
\end{tikzcd}$$

where the vertical morphisms are equivalences. Moreover, the top horizontal morphism is an equivalence, since it is the colimit of equivalences $j_T^*j_{T,!}\nu_{\mathcal{U}(T)}X \xrightarrow{\sim} \nu_{\mathcal{U}(T)}TX$ for $X \in \mathcal{U}(T)^{\mathrm{op}}$. The bottom horizontal morphism is therefore also an equivalence, which completes the proof. $\square$

Corollary 11.16. $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}})$ *is equivalent to the full subcategory of* $\mathrm{Fun}(\mathcal{W}(T), \mathcal{S})$ *spanned by functors that are local with respect to the morphisms*

$$j_{T,!}(\operatorname*{colim}_{I\in(\mathcal{I}_{X/})^{\mathrm{op}}} y(I)) \to j_{T,!}y(X)$$

for $X \in \mathcal{U}(T)$*. In particular* $\mathrm{Alg}_T(\mathcal{S})^{\mathcal{I}}$ *is an accessible localization of* $Fun(\mathcal{W}(T), \mathcal{S})$ *and so a presentable* ∞*-category.* $\square$

We now want to show that the ∞-categories $\mathcal{U}(T)$ and $\mathcal{W}(T)$ are compatible with morphisms of polynomial monads.

Proposition 11.17. *Let* T *be a polynomial monad on* $\mathcal{S}^{\mathcal{I}}$ *and* S *a polynomial monad on* $\mathcal{S}^{\mathcal{J}}$*, and suppose we have a commutative square*

$$\begin{tikzcd}
\mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}) \arrow[r, "\Phi"] \arrow[d, "U_S"'] & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \arrow[d, "U_T"] \\
\mathcal{S}^{\mathcal{J}} \arrow[r, "f^*"] & \mathcal{S}^{\mathcal{I}},
\end{tikzcd}$$

such that the mate transformation $F_T f^* \to \Phi F_S$ is cartesian.

(i) If $X \to TY$ is T-generic, then the composite $f_!X \to f_!TY \to Sf_!Y$ is S-generic, where $f_! \colon \mathcal{S}^{\mathcal{I}} \to \mathcal{S}^{\mathcal{J}}$ is the left adjoint to f^*, given by left Kan extension along f, and the natural transformation $f_!T \to Sf_!$ is obtained from the mate by applying U_T and moving adjoints around.
(ii) The functor $f_!$ restricts to a functor $\mathcal{U}(T)^{\mathrm{op}} \to \mathcal{U}(S)^{\mathrm{op}}$.
(iii) The functor $\Phi \colon \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \to \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}})$ has a left adjoint Ψ.
(iv) The functor Ψ restricts to a functor $\mathcal{W}(S)^{\mathrm{op}} \to \mathcal{W}(T)^{\mathrm{op}}$, and we have a commutative square

$$\begin{tikzcd} \mathcal{U}(T) \arrow[r, "f_!^{\mathrm{op}}"] \arrow[d, "F_T^{\mathrm{op}}"'] & \mathcal{U}(S) \arrow[d, "F_S^{\mathrm{op}}"] \\ \mathcal{W}(T) \arrow[r, "\Psi^{\mathrm{op}}"] & \mathcal{W}(S). \end{tikzcd}$$

Proof. We first prove (i). Let u denote the map $T* \simeq Tf^** \to f^*S*$. Since the transformation $T_{/*}f^* \simeq Tf^* \to f^*S$ is cartesian, the functor $(Tf^*)_{/*} \colon \mathcal{S}^{\mathcal{J}} \to \mathcal{S}^{\mathcal{I}}_{/T*}$ is equivalent to the composite

$$\mathcal{S}^{\mathcal{J}} \xrightarrow{S_{/*}} \mathcal{S}^{\mathcal{J}}_{/S*} \xrightarrow{f^*} \mathcal{S}^{\mathcal{J}}_{/f^*S*} \xrightarrow{u^*} \mathcal{S}^{\mathcal{I}}_{/T*}.$$

This means we have a corresponding equivalence of left adjoints

$$f_!L_*^T \simeq L_*^S f_! u_!.$$

Since $X \to TY$ is T-generic, the adjoint map $L_*^T X \to Y$ is an equivalence, hence so is $f_!L_*^T X \to f_!Y$. But under the equivalence of left adjoints this map $L_*^S f_! u_! X \xrightarrow{\sim} f_!Y$ is adjoint to $f_!X \to f_!TY \to Sf_!Y$, as required.

To prove (ii), we must show that if X is in $\mathcal{U}(T)^{\mathrm{op}}$, so that there is a generic morphism $I \to TX$ with $I \in \mathcal{I}^{\mathrm{op}}$, then $f_!X$ is in $\mathcal{U}(S)^{\mathrm{op}}$. By (i), the composite $f(I) \simeq f_!I \to f_!TX \to Sf_!X$ is T-generic. Since $f(I)$ is in $\mathcal{J}$, this implies that $f_!X$ is in $\mathcal{U}(T)^{\mathrm{op}}$.

To show part (iii), note that by Corollary 11.16 the ∞-categories $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}})$ and $\mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}})$ are presentable. Since U_S detects equivalences, preserves limits, and is accessible, and f^* preserves both limits and colimits, it follows that Φ is accessible and preserves limits. By the adjoint functor theorem this implies that Φ has a left adjoint Ψ, as required.

From our commutative square of right adjoints we now get an equivalence $\Psi F_T \simeq F_S f_!$. By definition the ∞-categories $\mathcal{W}(T)^{\mathrm{op}}$ and $\mathcal{W}(S)^{\mathrm{op}}$ consist of free algebras on objects of $\mathcal{U}(T)^{\mathrm{op}}$ and $\mathcal{U}(S)^{\mathrm{op}}$, respectively, so it follows from (ii) that Ψ takes $\mathcal{W}(T)^{\mathrm{op}}$ to $\mathcal{W}(S)^{\mathrm{op}}$, and gives the required commutative square. $\square$

Lemma 11.18. *The functor $\Psi\colon \mathcal{W}(T)^{\mathrm{op}} \to \mathcal{W}(S)^{\mathrm{op}}$ of the previous proposition preserves free maps and takes morphisms which are adjoint to T-generic maps to morphisms which are adjoint to S-generic maps.*

Proof. The commutativity of the square of Proposition 11.17.(iv) shows that Ψ preserves free maps. Suppose $\alpha\colon F_T X \to F_T Y$ is a morphism in $\mathcal{W}(T)^{\mathrm{op}}$ which is adjoint to a T-generic map $X \to TY$, we want to see that $\Psi\alpha$ is adjoint to an S-generic morphism. By the equivalence $\Psi F_T \simeq F_S f_!$ of Proposition 11.17 and the construction of the generic–free factorization the map $\Psi\alpha$ is adjoint to the composite

$$f_!X \xrightarrow{\eta^S f_!} Sf_!X \to Sf_!Y,$$

where η^S is the unit of the monad S. We claim that there is a commutative diagram

$$\begin{tikzcd} & f_!TX \arrow[r] \arrow[d] & f_!TY \arrow[d] \\ f_!X \arrow[ur, "f_!\eta^T"] \arrow[r, "\eta^S f_!"'] & Sf_!X \arrow[r] & Sf_!Y \end{tikzcd}$$

where η^T is the unit of T, the right horizontal maps are induced by α and the vertical maps are induced by the equivalence $Sf_! \simeq U_S \Psi F_T$ together with the natural transformation $\tau\colon f_!U_T \to U_S\Psi$ adjoint to the unit $\eta^S_{f_!}\colon f_! \to Sf_!$. The square in the diagram commutes by naturality. To see that the triangle commutes we first observe that τ is also adjoint to the counit map $\epsilon_\Psi\colon F_S U_S \Psi \to \Psi$. Using this it is easy to see that left triangle is adjoint to a triangle

$$\begin{tikzcd} F_S f_! X \arrow[r] \arrow[dr, "\mathrm{id}"'] & F_S f_! TX \arrow[d] \\ & F_S f_! X \end{tikzcd}$$

which is equivalent to the commutative triangle

$$\begin{tikzcd} \Psi F_T X \arrow[r] \arrow[dr, "\mathrm{id}"'] & \Psi F_T U_T F_T X \arrow[d, "\Psi\epsilon_{F_T X}"] \\ & \Psi F_T X \end{tikzcd}$$

obtained from the adjunction identities. This shows that the diagram above commutes, and hence $\Psi\alpha$ is adjoint to the composite $f_!X \to f_!TX \to f_!TY \to Sf_!Y$, which is S-generic by Proposition 11.17(i). $\square$

Combining the preceding results, we get the following:

Corollary 11.19. *In the situation of Proposition 11.17, we have a commutative cube*

$$\begin{tikzcd}
\mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}) \arrow[rr, hook] \arrow[dr] \arrow[dd] & & \mathrm{Fun}(\mathcal{W}(S), \mathcal{S}) \arrow[dr] \arrow[dd] & \\
& \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \arrow[rr, hook] \arrow[dd] & & \mathrm{Fun}(\mathcal{W}(T), \mathcal{S}) \arrow[dd] \\
\mathcal{S}^{\mathcal{J}} \arrow[rr, hook] \arrow[dr] & & \mathrm{Fun}(\mathcal{U}(S), \mathcal{S}) \arrow[dr] & \\
& \mathcal{S}^{\mathcal{I}} \arrow[rr, hook] & & \mathrm{Fun}(\mathcal{U}(T), \mathcal{S}),
\end{tikzcd}$$

which exhibits the morphism of polynomial monads $T \to S$ as arising from the commutative square in Proposition 11.17(iv). $\square$

Proof. Taking left adjoints, the morphism of polynomial monads $T \to S$ gives a commutative square

$$\begin{tikzcd}
\mathcal{S}^{\mathcal{I}} \arrow[r, "f_!"] \arrow[d, "F_T"'] & \mathcal{S}^{\mathcal{J}} \arrow[d, "F_S"] \\
\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \arrow[r, "\Psi"] & \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}),
\end{tikzcd}$$

and we have shown that this restricts to a commutative square

$$\begin{tikzcd}
\mathcal{U}(T)^{\mathrm{op}} \arrow[r] \arrow[d] & \mathcal{U}(S)^{\mathrm{op}} \arrow[d] \\
\mathcal{W}(T)^{\mathrm{op}} \arrow[r] & \mathcal{W}(S)^{\mathrm{op}}
\end{tikzcd}$$

relating these full subcategories. Thus we have a commutative cube

$$\begin{tikzcd}
\mathcal{U}(T)^{\mathrm{op}} \arrow[rr, hook] \arrow[dr] \arrow[dd] & & \mathcal{S}^{\mathcal{I}} \arrow[dr] \arrow[dd] & \\
& \mathcal{U}(S)^{\mathrm{op}} \arrow[rr, hook] \arrow[dd] & & \mathcal{S}^{\mathcal{J}} \arrow[dd] \\
\mathcal{W}(T)^{\mathrm{op}} \arrow[rr, hook] \arrow[dr] & & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \arrow[dr] & \\
& \mathcal{W}(S)^{\mathrm{op}} \arrow[rr, hook] & & \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}).
\end{tikzcd}$$

The right-hand square consists of cocomplete ∞-categories and colimit-preserving functors, so this canonically extends to presheaves on the left-hand square, giving a commutative cube

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{U}(T),\mathcal{S}) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{S}^{\mathcal{I}} \arrow[dr] \arrow[dd] & \\
& \mathrm{Fun}(\mathcal{U}(S),\mathcal{S}) \arrow[rr] \arrow[dd] & & \mathcal{S}^{\mathcal{J}} \arrow[dd] \\
\mathrm{Fun}(\mathcal{W}(T),\mathcal{S}) \arrow[rr] \arrow[dr] & & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \arrow[dr] & \\
& \mathrm{Fun}(\mathcal{W}(S),\mathcal{S}) \arrow[rr] & & \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}})
\end{tikzcd}$$

This consists entirely of left adjoints, and passing to right adjoints we get the cube we want. □

12. Factorization systems from polynomial monads

Suppose T is a polynomial monad on $\mathcal{S}^{\mathcal{I}}$. Then a morphism $F_T X \to F_T Y$ in the Kleisli ∞-category $\mathcal{K}(T)$ has a canonical factorization of the form

$$F_T X \to F_T L_* X \to F_T Y$$

adjoint to the generic-free factorization of $X \to TY$ as $X \to TL_* X \to TY$ through the unit of the local left adjoint L_*. Our first goal in this section is to show that this canonical factorization is well-defined, in the sense that if we have equivalences $F_T X \simeq F_T X'$, $F_T Y \simeq F_T Y'$ in $\mathcal{K}(T)$ (which need not come from morphisms in $\mathcal{S}^{\mathcal{I}}$), then there is a commutative diagram

$$\begin{tikzcd}
F_T X \arrow[r] \arrow[d, "\wr"] & F_T L_* X \arrow[r] \arrow[d, "\wr"] & F_T Y \arrow[d, "\wr"] \\
F_T X' \arrow[r] & F_T L_* X' \arrow[r] & F_T Y'
\end{tikzcd}$$

where the middle vertical map is again an equivalence. We can then say that a morphism $\phi\colon F_T X \to F_T Y$ is

- *inert* if the map $F_T X \to F_T L_* X$ in the canonical factorization is an equivalence,
- *active* if the map $F_T L_* X \to F_T Y$ in the canonical factorization is an equivalence,

as this does not depend on the choice of the objects X and Y. We will see that the inert morphisms are obtained by closing the free morphisms under equivalences (which need not all be free), while the active morphisms are precisely those that are adjoint to generic morphisms. Our main goal in this section is to prove that these classes give a factorization system:

Theorem 12.1. *Let T be a polynomial monad on $\mathcal{S}^{\mathcal{I}}$. Then the active and inert morphisms give a factorization system on $\mathcal{K}(T)$, whereby every morphism factors as an active* *morphism followed by an inert morphism; this factorization is precisely the canonical factorization, up to equivalence.*

This factorization system restricts to the full subcategory $\mathcal{W}(T)^{\mathrm{op}}$, which induces a canonical pattern structure on $\mathcal{W}(T)$; in the next section we will discuss how this relates to the original monad T.

We start with some observations relating the local left adjoint of T to the Kleisli ∞-category:

Notation 12.2. For $X \in \mathcal{S}^{\mathcal{I}}$, we write $L_X : \mathcal{S}^{\mathcal{I}}_{/T(X)} \to \mathcal{S}^{\mathcal{I}}_{/X}$ for the left adjoint of the functor $T_X : \mathcal{S}^{\mathcal{I}}_{/X} \to \mathcal{S}^{\mathcal{I}}_{/T(X)}$ induced by T.

Proposition 12.3. *Let $\mathcal{K}(T)$ denote the Kleisli ∞-category of T, i.e. the full subcategory of $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}})$ spanned by the free algebras. For $\phi\colon FY \to FX$ in $\mathcal{K}(T)$, the ∞-category*

$$(\mathcal{S}^{\mathcal{I}}_{/X})_{\phi/} := \mathcal{S}^{\mathcal{I}}_{/X} \times_{\mathcal{K}(T)_{/FX}} (\mathcal{K}(T)_{/FX})_{\phi/}$$

has an initial object.

Proof. An object in this ∞-category is a morphism $f\colon Z \to X$ together with a commutative triangle

$$\begin{tikzcd} FY \arrow[rr] \arrow[dr, "\phi"'] & & FZ \arrow[dl, "F(f)"] \\ & FX. & \end{tikzcd}$$

This corresponds to a commutative triangle

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr, "\phi'"'] & & TZ \arrow[dl, "T(f)"] \\ & TX, & \end{tikzcd}$$

which in turn corresponds to

$$\begin{tikzcd} L_XY \arrow[rr] \arrow[dr, "\phi''"'] & & Z \arrow[dl, "f"] \\ & X. & \end{tikzcd}$$

Thus $L_XY \xrightarrow{\phi''} X$ gives an initial object, as required. □

Corollary 12.4. *The functor $F_X\colon \mathcal{S}^{\mathcal{I}}_{/X} \to \mathcal{K}(T)_{/FX}$ given by F has a left adjoint $\mathcal{L}_X$, which takes $\phi\colon FY \to FX$ to the corresponding map $\phi''\colon L_XY \to X$.*

Proof. By a standard argument the functor F_X is a right adjoint if and only if $(\mathcal{S}^{\mathcal{I}}_{/X})_{\phi/}$ has an initial object, which is the statement of Proposition 12.3. □

Remark 12.5. For $f\colon Y \to X$, the counit map $\mathcal{L}_X F_X(f) \to f$ is given by the commutative triangle

$$\begin{tikzcd} L_X Y \arrow[rr, "\sim"] \arrow[dr] & & Y \arrow[dl, "f"] \\ & X, & \end{tikzcd}$$

where the map $L_X Y \to Y$ is the map adjoint to the unit $Y \to TY$ which is an equivalence by Lemma 11.9. It follows that F_X is fully faithful, and so $\mathcal{L}_X$ exhibits $\mathcal{S}^{\mathcal{I}}_{/X}$ as a localization of $\mathcal{K}(T)_{/FX}$.

Remark 12.6. The functor $F_X\colon \mathcal{S}^{\mathcal{I}}_{/X} \to \mathcal{K}(T)_{/FX}$ also has a right adjoint U_X, which takes $\phi\colon FY \to FX$ to the morphism obtained as the pullback of $U\phi\colon TY \to TX$ along the unit map $\epsilon_X\colon X \to TX$. Note that the unit $\mathrm{id} \to U_X F_X$ is an equivalence since ϵ is a cartesian transformation, which also implies that F_X is fully faithful.

Remark 12.7. For $\phi\colon FY \to FX$, the unit map $\phi \to F_X\mathcal{L}_X(\phi)$ is the commutative triangle

$$\begin{tikzcd} FY \arrow[rr] \arrow[dr, "\phi"'] & & FL_X Y \arrow[dl] \\ & FX, & \end{tikzcd}$$

i.e. the canonical factorization of ϕ. By naturality, this means we can extend any commutative triangle

$$\begin{tikzcd} FY \arrow[rr, "\psi"] \arrow[dr] & & FY' \arrow[dl] \\ & FX & \end{tikzcd}$$

to a commutative diagram

$$\begin{tikzcd} FY \arrow[rr, "\psi"] \arrow[d] & & FY' \arrow[d] \\ FL_X Y \arrow[rr, "F_X\mathcal{L}_X\psi"] \arrow[dr] & & FL_X Y' \arrow[dl] \\ & FX & \end{tikzcd}$$

relating the canonical factorizations of the two maps to FX. The next observations will allow us to prove that the canonical factorization is also natural when we vary FX.

Proposition 12.8. *For every object $C \in \mathcal{S}^{\mathcal{I}}$ we have a commutative diagram*

$$\begin{tikzcd} \mathcal{S}^{\mathcal{I}}_{/T^2C} \arrow[r, "\mu_{C,!}"] \arrow[d, "L_{TC}"'] & \mathcal{S}^{\mathcal{I}}_{/TC} \arrow[d, "L_C"] \\ \mathcal{S}^{\mathcal{I}}_{/TC} \arrow[r, "L_C"] & \mathcal{S}^{\mathcal{I}}_{/C}, \end{tikzcd}$$

where the top horizontal map is given by composition with the C-component of the multiplication $\mu\colon T^2 \to T$.

Proof. It suffices to show that the diagram

$$\begin{tikzcd} \mathcal{S}^{\mathcal{I}}_{/T^2C} & \mathcal{S}^{\mathcal{I}}_{/TC} \arrow[l, "\mu_C^*"] \\ \mathcal{S}^{\mathcal{I}}_{/TC} \arrow[u, "T_{TC}"] & \mathcal{S}^{\mathcal{I}}_{/C} \arrow[l, "T_C"] \arrow[u, "T_C"'] \end{tikzcd}$$

of the corresponding right adjoints commutes. Given an object $\alpha\colon B \to C$ in $\mathcal{S}^{\mathcal{I}}_{/C}$, its image under the composite of the right vertical and the upper horizontal map is the left vertical map of the pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & TB \arrow[d, "T\alpha"] \\ T^2C \arrow[r, "\mu_C"] & TC. \end{tikzcd}$$

Since the multiplication μ of the polynomial monad T is a cartesian natural transformation, the map $A \to T^2C$ can be identified with $T^2\alpha\colon T^2B \to T^2C$ which is the same as the image of α under the composite of T_C and T_{TC}. $\square$

Corollary 12.9. *Given morphisms $\alpha\colon A \to TB$ and $\beta\colon B \to TC$, we have adjoint morphisms $L_BA \to B$ and $L_CB \to C$; we also have the composites $A \xrightarrow{\alpha} TB \xrightarrow{T\beta} T^2C \xrightarrow{\mu} TC$ and $L_BA \to B \to TC$ with adjoints $L_CA \to C$ and $L_CL_BA \to C$. These are equivalent, i.e. $L_CA \simeq L_CL_BA$.*

Proof. Consider the diagram

$$\begin{tikzcd} \mathcal{S}^{\mathcal{I}}_{/TB} \arrow[r, "(T\beta)_!"] \arrow[d, "L_B"'] & \mathcal{S}^{\mathcal{I}}_{/T^2C} \arrow[r, "\mu_{C,!}"] \arrow[d, "L_{TC}"] & \mathcal{S}^{\mathcal{I}}_{/TC} \arrow[d, "L_C"] \\ \mathcal{S}^{\mathcal{I}}_{/B} \arrow[r, "\beta_!"] & \mathcal{S}^{\mathcal{I}}_{/TC} \arrow[r, "L_C"] & \mathcal{S}^{\mathcal{I}}_{/C}. \end{tikzcd}$$

Here the left square commutes since it is the square of left adjoints corresponding to the square

$$\begin{tikzcd} \mathcal{S}^{\mathcal{I}}_{/TC} \arrow[r, "\beta^*"] \arrow[d, "T_{TC}"] & \mathcal{S}^{\mathcal{I}}_{/B} \arrow[d, "T_B"] \\ \mathcal{S}^{\mathcal{I}}_{/T^2C} \arrow[r, "(T\beta)^*"] & \mathcal{S}^{\mathcal{I}}_{/TB}, \end{tikzcd}$$

which commutes since T preserves pullbacks, and the right square commutes by Proposition 12.8. By construction the morphisms $L_C A \to C$ and $L_C L_B A \to C$ are given by $L_C \mu_{C,!} T\beta_!(\alpha)$ and $L_C \beta_! L_B(\alpha)$, and so are equivalent by the commutativity of the outer square. □

Proposition 12.10. *Given a commutative square*

$$\begin{tikzcd} F_T A \arrow[r] \arrow[d] & F_T B \arrow[d] \\ F_T C \arrow[r] & F_T D, \end{tikzcd}$$

in $\mathcal{K}(T)$, there exists a canonical commutative diagram

$$\begin{tikzcd} F_T A \arrow[r] \arrow[d] & F_T L_B A \arrow[r] \arrow[d] & F_T B \arrow[d] \\ F_T C \arrow[r] & F_T L_D C \arrow[r] & F_T D. \end{tikzcd}$$

Here the diagram associated to the degenerate square

$$\begin{tikzcd} F_T A \arrow[r] \arrow[d, equal] & F_T B \arrow[d, equal] \\ F_T A \arrow[r] & F_T B \end{tikzcd}$$

is the degenerate diagram

$$\begin{tikzcd} F_T A \arrow[r] \arrow[d, equal] & F_T L_B A \arrow[r] \arrow[d, equal] & F_T B \arrow[d, equal] \\ F_T A \arrow[r] & F_T L_B A \arrow[r] & F_T B. \end{tikzcd}$$

Moreover, we have compatibility with composition, in the sense that if we have a commutative diagram

$$\begin{tikzcd} F_T A \arrow[r] \arrow[d] & F_T B \arrow[d] \\ F_T C \arrow[r] \arrow[d] & F_T D \arrow[d] \\ F_T X \arrow[r] & F_T Y, \end{tikzcd}$$

then the vertical composite of the associated diagrams

$$\begin{tikzcd} F_T A \arrow[r] \arrow[d] & F_T L_B A \arrow[r] \arrow[d] & F_T B \arrow[d] \\ F_T C \arrow[r] \arrow[d] & F_T L_D C \arrow[r] \arrow[d] & F_T D \arrow[d] \\ F_T X \arrow[r] & F_T L_Y X \arrow[r] & F_T Y, \end{tikzcd}$$

is the diagram associated to the composite square

$$\begin{tikzcd} F_T A \arrow[r] \arrow[d] & F_T B \arrow[d] \\ F_T X \arrow[r] & F_T Y. \end{tikzcd}$$

Proof. We can view the original square as a pair of morphisms

$$F_T C \longleftarrow F_T A \longrightarrow F_T B$$

in $\mathcal{K}(T)_{/F_T D}$. Adding the canonical factorization of the arrow $F_T A \to F_T B$, the naturality of the unit for the adjunction $\mathcal{L}_D \dashv F_D$ gives a commutative diagram

$$\begin{tikzcd} F_T C \arrow[d] & F_T A \arrow[l] \arrow[r] \arrow[d] & F_T L_B A \arrow[r] \arrow[d] & F_T B \arrow[d] \\ F_T L_D C & F_T L_D A \arrow[l] \arrow[r, "\sim"] & F_T L_D L_B A \arrow[r] & F_T L_D B \end{tikzcd}$$

over $F_T D$, where the second arrow in the bottom row is an equivalence by Corollary 12.9. If we invert this equivalence we can contract the diagram to

$$\begin{tikzcd}
F_T A \arrow[r] \arrow[d] & F_T L_B A \arrow[r] \arrow[d] & F_T B \\
F_T C \arrow[r] & F_T L_D C &
\end{tikzcd}$$

over $F_T D$; adding $F_T D$ back in now gives the desired diagram.

From the degenerate square we can make the extended diagram

$$\begin{tikzcd}
F_T A \arrow[r, equal] \arrow[d] & F_T A \arrow[r, "\eta_{F_T A}"] \arrow[d, "\eta_{F_T A}"'] & F_T L_B A \arrow[r] \arrow[d, "\eta_{F_T L_B A}"'] \arrow[dd, bend left, equal] & F_T B \arrow[d] \arrow[dd, bend left, equal] \\
F_T L_B A \arrow[r, equal] & F_T L_B A \arrow[r, "F_B \mathcal{L}_B \eta_{F_T A}", "\sim"'] \arrow[dr, bend right, equal] & F_T L_B L_B A \arrow[r] \arrow[d, "F_T \epsilon_{L_B A}"'] & F_T L_B B \arrow[d, "F_T \epsilon_{L_B}"'] \\
 & & F_T L_B A \arrow[r] & F_T B,
\end{tikzcd}$$

all over $F_T B$. Here the adjunction identities for $\mathcal{L}_B \dashv F_B$ imply that the two composites $F_T L_B A \to F_T L_B A$ are identities, as indicated; this means the general definition indeed specializes to give the degenerate diagram in this case.

To see we have compatibility with composition, consider the diagram

$$\begin{tikzcd}
F_T A \arrow[r] \arrow[d] & F_T L_B A \arrow[r] \arrow[d] & F_T B \arrow[d] \\
F_T C \arrow[r] \arrow[d] & F_T L_D C \arrow[r] & F_T D \\
F_T X & &
\end{tikzcd}$$

in $\mathcal{K}(T)_{/F_T Y}$. Using the unit for the adjunction $\mathcal{L}_Y \dashv F_Y$ this extends to a commutative diagram

$$\begin{tikzcd}
F_T A \arrow[rrr] \arrow[dr] \arrow[ddd] & & & F_T L_B A \arrow[rrr] \arrow[dr] \arrow[ddd] & & & F_T B \arrow[dr] \arrow[ddd] & \\
& F_T C \arrow[rrr] \arrow[dr] \arrow[ddd] & & & F_T L_D C \arrow[rrr] \arrow[ddd] & & & F_T D \arrow[ddd] \\
& & F_T X \arrow[ddd] & & & & & \\
F_T L_Y A \arrow[rrr, "\sim"] \arrow[dr] & & & F_T L_Y L_B A \arrow[rrr] \arrow[dr] & & & F_T L_Y B \arrow[dr] & \\
& F_T L_Y C \arrow[rrr, "\sim"] \arrow[dr] & & & F_T L_Y L_D C \arrow[rrr] & & & F_T L_Y D \\
& & F_T L_Y X & & & & &
\end{tikzcd}$$

over $F_T Y$, where the two indicated morphisms are equivalences by Corollary 12.9. Inverting these, we can contract the diagram to

$$\begin{tikzcd}
F_TA \arrow[rr] \arrow[d] & & F_TL_BA \arrow[r] \arrow[d] \arrow[ddl] & F_TB \arrow[d] \\
F_TC \arrow[rr, crossing over] \arrow[dd] & & F_TL_DC \arrow[r] \arrow[d] & F_TD \arrow[dd] \\
& F_TL_YA \arrow[r] \arrow[dr] & F_TL_YC \arrow[d] & \\
F_TX \arrow[rr] & & F_TL_YX \arrow[r] & F_TY
\end{tikzcd}$$

where we see both the composite of the diagrams for the two squares and the diagram for the composite square, as required. □

Corollary 12.11. *A commutative square*

$$\begin{tikzcd}
F_TA \arrow[r] \arrow[d, "\wr"'] & F_TB \arrow[d, "\wr"] \\
F_TC \arrow[r] & F_TD
\end{tikzcd}$$

in $\mathcal{K}(T)$, where the vertical morphisms are equivalences, can be extended to a commutative diagram

$$\begin{tikzcd}
F_TA \arrow[r] \arrow[d, "\wr"'] & F_TL_BA \arrow[r] \arrow[d, "\wr"] & F_TB \arrow[d, "\wr"] \\
F_TC \arrow[r] & F_TL_DC \arrow[r] & F_TD
\end{tikzcd}$$

where the middle vertical map is also an equivalence.

Proof. This follows immediately from the compatibility of the diagrams in Proposition 12.10 with composition and identities. □

In other words, the canonical factorizations in $\mathcal{K}(T)$ are invariant under equivalences. This means the following conditions on morphisms are well-defined:

Definition 12.12. We say a morphism $\phi\colon F_TA \to F_TB$ in $\mathcal{K}(T)$ with canonical factorization

$$F_TA \to F_TL_BA \to F_TB$$

is *inert* if the morphism $F_TA \to F_TL_BA$ in the canonical factorization is an equivalence, and *active* if the morphism $F_TL_BA \to F_TB$ in the canonical factorization is an equivalence.

Our goal in the rest of this section is to show that the active and inert morphisms form a factorization system on $\mathcal{K}(T)$. We start with some observations about equivalences in $\mathcal{K}(T)$ that will lead to a simpler characterization of the active maps.

Lemma 12.13. *If T is a polynomial monad then the free functor F_T is conservative: if $F_T(\phi)$ is an equivalence for some $\phi\colon X \to Y$ in $\mathcal{S}^{\mathcal{I}}$ then ϕ is an equivalence.*

Proof. The functor $F_Y : \mathcal{S}^{\mathcal{I}}_{/Y} \to \mathcal{K}(T)_{/F_TY}$ is fully faithful by Remark 12.5. The inverse of $F_T\phi$ gives a morphism

$$\begin{tikzcd} F_TY \arrow[rr, "(F_T\phi)^{-1}"] \arrow[dr, equal] & & F_TX \arrow[dl, "F_T\phi"] \\ & F_TY & \end{tikzcd}$$

in $\mathcal{K}(T)_{/F_TY}$ between objects in the image of F_Y, hence it is also in the image of F_Y and lifts to an equivalence in $\mathcal{S}^{\mathcal{I}}_{/Y}$ by faithfulness. $\square$

Lemma 12.14. *Given a commutative triangle*

$$\begin{tikzcd} FA \arrow[rr, "\phi"] \arrow[dr, "\alpha"'] & & FB \arrow[dl, "\beta"] \\ & FC, & \end{tikzcd}$$

and morphisms $a\colon A \to C$ and $b\colon B \to C$ with equivalences $\alpha \simeq F(a)$ and $\beta \simeq F(b)$, then the triangle lifts to a unique commutative triangle

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "a"'] & & B \arrow[dl, "b"] \\ & C. & \end{tikzcd}$$

Proof. The functor $F_C : \mathcal{S}^{\mathcal{I}}_{/C} \to \mathcal{K}(T)_{/F_TC}$ is fully faithful by Remark 12.5. This immediately implies the result, since the first triangle is precisely a morphism in $\mathcal{K}(T)_{/F_TC}$ between objects in the image of F_C. $\square$

Lemma 12.15. *Every equivalence $\phi\colon F_TX \xrightarrow{\sim} F_TY$ is adjoint to a generic map.*

Proof. Regarding ϕ as a morphism in $\mathcal{K}(T)_{/F_TY}$, we apply $\mathcal{L}_Y$ to get a commutative triangle

$$\begin{tikzcd} L_Y X \arrow[rr, "\mathcal{L}_Y \phi", "\sim"'] \arrow[dr] & & L_Y Y \arrow[dl, "\sim"] \\ & Y, & \end{tikzcd}$$

where the right diagonal map is an equivalence by Lemma 11.9(i) and the horizontal map is an equivalence by the functoriality of $\mathcal{L}_Y$. Hence the left diagonal map is also an equivalence, which is precisely the condition for the map $X \to TY$ adjoint to ϕ to be generic. □

Lemma 12.16. *A morphism $\phi\colon F_T X \to F_T Y$ is active if and only if the adjoint morphism $\phi'\colon X \to TY$ is generic.*

Proof. Let $\phi''\colon L_Y X \to Y$ be the map adjoint to ϕ'. By definition, ϕ is active if and only if in the canonical factorization

$$F_T X \xrightarrow{\phi_a} F_T L_Y X \xrightarrow{\phi_i} F_T Y,$$

the map $\phi_i = F_T(\phi'')$ is an equivalence. By Lemma 12.13 this happens if and only if ϕ'' is an equivalence, which is precisely the condition for ϕ' to be generic. □

Lemma 12.17. *For any morphism $\phi\colon X \to Y$ in $\mathcal{S}^{\mathcal{I}}$, the free morphism $F_T(\phi)\colon F_T X \to F_T Y$ is inert.*

Proof. The commutative triangle

$$\begin{tikzcd} F_T X \arrow[d, equal] \arrow[dr, "F_T \phi"] & \\ F_T X \arrow[r, "F_T \phi"'] & F_T Y \end{tikzcd}$$

is adjoint to

$$\begin{tikzcd} X \arrow[d, "\eta_X"'] \arrow[dr] & \\ TX \arrow[r, "T\phi"] & TY \end{tikzcd}$$

which is in turn adjoint to

$$\begin{tikzcd} L_* X \arrow[d, "\lambda"', "\wr"] \arrow[dr] & \\ X \arrow[r, "\phi"] & Y, \end{tikzcd}$$

where λ is an equivalence since the unit map η_X is generic by Lemma 11.9(i). By adjointness we see that F_T takes the last triangle to the right triangle in the commutative diagram

$$\begin{tikzcd} F_T X \arrow[r] \arrow[dr, equal] & F_T L_* X \arrow[d, "F_T\lambda"', "\wr"] \arrow[dr] & \\ & F_T X \arrow[r, "F_T\phi"] & F_T Y, \end{tikzcd}$$

where the upper horizontal map is adjoint to the unit $X \to TL_*X$; this is an equivalence as $F_T\lambda$ is one. By definition the upper horizontal map and the right diagonal map give the canonical factorization of $F_T\phi$, hence $F_T\phi$ is inert. □

Warning 12.18. Note, however, that it is *not* necessarily true that every inert map is of the form $F_T(\phi)$ for ϕ a morphism in $\mathcal{S}^{\mathcal{I}}$: The equivalences in $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}})$ need not all be in the image of $\mathcal{S}^{\mathcal{I}}$.

Remark 12.19. By Lemmas 12.16 and 12.17, the canonical factorization of a morphism $\phi\colon F_TA \to F_TB$ as $F_TA \to F_TL_*A \to F_TB$ is a factorization of ϕ as an active morphism followed by an inert morphism.

Lemma 12.20. *Given a commutative square*

$$\begin{tikzcd} F_T A \arrow[r, "\phi"] \arrow[d, "\psi"'] & F_T B \arrow[d, "F_T\beta"] \\ F_T C \arrow[r, "F_T\gamma"] \arrow[ur, dashed] & F_T D \end{tikzcd}$$

where ψ is active, there exists a unique diagonal filler, which is of the form $F_T(\alpha)$ for a unique commutative triangle

$$\begin{tikzcd} C \arrow[rr, "\alpha"] \arrow[dr, "\gamma"'] & & B \arrow[dl, "\beta"] \\ & D. & \end{tikzcd}$$

Proof. As in Lemma 11.3, the square is adjoint to

$$\begin{tikzcd} A \arrow[r, "\phi'"] \arrow[d, "\psi'"'] & TB \arrow[d, "T\beta"] \\ TC \arrow[r, "T\gamma"] & TD, \end{tikzcd}$$

which in turn is adjoint to

$$\begin{tikzcd}
L_*A \arrow[r, "\phi''"] \arrow[d, "\wr"'] & B \arrow[d, "\beta"] \\
C \arrow[r, "\gamma"] & D,
\end{tikzcd}$$

where the left vertical morphism is an equivalence since ψ is active and this implies that ψ' is generic by Lemma 12.16. This square has a unique filler, which in turn corresponds to a unique filler in the original square, since we saw in Lemma 12.14 that all fillers are uniquely of this form. $\square$

Proof of Theorem 12.1. We check the requirements of [29, Definition 5.2.8.8] (which are equivalent to our previous definition of a factorization system by [29, Proposition 5.2.8.17]). We must thus check:

(1) The classes of inert and active maps are closed under retracts.
(2) The active maps are left orthogonal to the inert maps, i.e. for every commutative square

$$\begin{tikzcd}
F_T A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & F_T B \arrow[d, "\gamma"] \\
F_T C \arrow[r, "\delta"] \arrow[ur, dashed] & F_T D
\end{tikzcd}$$

where β is active and γ is inert, there exists a unique filler.
(3) Every morphism can be factored as an active map followed by an inert map.

Condition (3) is by now clear, since by Remark 12.19 the canonical factorization gives an active-inert factorization. For condition (1), suppose we have a retract diagram

$$\begin{tikzcd}
F_T A \arrow[r] \arrow[d, "\phi"'] \arrow[rr, bend left, equal] & F_T A' \arrow[r] \arrow[d, "\psi"] & F_T A \arrow[d, "\phi"] \\
F_T B \arrow[r] \arrow[rr, bend right, equal] & F_T B' \arrow[r] & F_T B.
\end{tikzcd}$$

By applying Proposition 12.10 to the two squares, we obtain a commutative diagram

$$
\begin{tikzcd}
F_T A \arrow[r] \arrow[d, "\phi_a"'] \arrow[rr, bend left, equal] & F_T A' \arrow[r] \arrow[d, "\psi_a"] & F_T A \arrow[d, "\phi_a"] \\
F_T L_* A \arrow[r, "f"] \arrow[d, "\phi_i"'] & F_T L_* A' \arrow[r, "g"] \arrow[d, "\psi_i"] & F_T L_* A \arrow[d, "\phi_i"] \\
F_T B \arrow[r] \arrow[rr, bend right, equal] & F_T B' \arrow[r] & F_T B,
\end{tikzcd}
$$

relating the canonical factorizations of ϕ and ψ, where the compatibility with composition and identities in Proposition 12.10 implies that $gf \simeq \mathrm{id}$. If ψ is active then by definition the map labelled ψ_i is an equivalence, and so ϕ_i is a retract of an equivalence; hence ϕ_i is also an equivalence, which means ϕ is active. The same argument shows that inert morphisms are also closed under retracts.

It remains to prove (2). Consider a commutative square

$$
\begin{tikzcd}
F_T A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & F_T B \arrow[d, "\gamma"] \\
F_T C \arrow[r, "\delta"] & F_T D
\end{tikzcd}
$$

with β active and γ inert. Including the canonical factorizations of γ and δ, we get a diagram

$$
\begin{tikzcd}
F_T A \arrow[rr, "\alpha"] \arrow[dd, "\beta"'] & & F_T B \arrow[d, "\wr"] \\
 & & F_T L_* B \arrow[d] \\
F_T C \arrow[r] \arrow[urr, dashed] & F_T L_* C \arrow[r] \arrow[ur, dashed] & F_T D,
\end{tikzcd}
$$

and since the map $F_T B \to F_T L_* B$ is an equivalence (since γ is inert), a lift in the original square corresponds to a lift $F_T C \to F_T L_* B$ here. Applying Lemma 12.20 to the square

$$
\begin{tikzcd}
F_T A \arrow[r] \arrow[d] & F_T L_* B \arrow[d] \\
F_T L_* C \arrow[r] \arrow[ur, dashed] & F_T D,
\end{tikzcd}
$$

we see that there is a unique diagonal filler $F_T L_* C \to F_T L_* B$, which comes from a unique commutative triangle

$$\begin{tikzcd} L_*C \arrow[rr] \arrow[dr] & & L_*B \arrow[dl] \\ & D. & \end{tikzcd}$$

This gives in particular a lift in the original square, but now applying Lemma 12.20 to a square

$$\begin{tikzcd} F_T C \arrow[r] \arrow[d] & F_T L_* B \arrow[d] \\ F_T L_* C \arrow[r] \arrow[ur, dashed] & F_T D, \end{tikzcd}$$

we see that any lift $F_T C \to F_T L_* B$ must factor through $F_T L_* C$ and so must be the lift we just constructed. □

Let us say that a morphism in $\mathcal{W}(T)$ is inert or active if it corresponds to an inert or active morphism in $\mathcal{K}(T)$ under the inclusion $\mathcal{W}(T)^{\mathrm{op}} \hookrightarrow \mathcal{K}(T)$. Then the factorization system we constructed restricts to one on $\mathcal{W}(T)$:

Corollary 12.21. *The inert and active morphisms restrict to a factorization system on $\mathcal{W}(T)$.*

Proof. It is enough to show that for a morphism $F_T I \to F_T J$ in $\mathcal{W}(T)^{\mathrm{op}}$, if $F_T I \to F_T X \to F_T J$ is its active-inert factorization in $\mathcal{K}(T)$, then $F_T X$ also lies in $\mathcal{W}(T)^{\mathrm{op}}$. Since the canonical factorization is an active-inert factorization, this follows from Proposition 11.10(ii). □

13. Patterns from polynomial monads

Suppose T is a polynomial monad on $\mathcal{S}^{\mathcal{I}}$. In the previous section we saw that the ∞-category $\mathcal{W}(T)$ has a canonical inert–active factorization system. Using this we can define a natural algebraic pattern structure on $\mathcal{W}(T)$ by taking the elementary objects to be those of the form $F_T(I)$ with $I \in \mathcal{S}^{\mathcal{I}}$ in the image of $\mathcal{I}^{\mathrm{op}}$ under the Yoneda embedding.

In this section we will study these algebraic patterns. We will see that $\mathcal{W}(T)$ is always an extendable pattern, and that the free Segal $\mathcal{W}(T)$-space monad is closely related to the original monad T: there is a canonical morphism $T \to T_{\mathcal{W}(T)}$ in PolyMnd, which induces an equivalence on ∞-categories of algebras. Moreover, the patterns $\mathcal{W}(T)$ are natural in T, and so determine a functor

$$\mathfrak{P}\colon \mathrm{PolyMnd} \to \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}};$$

the morphisms $T \to T_{\mathcal{W}(T)}$ then give a natural transformation $\mathrm{id} \to \mathfrak{M}\mathfrak{P}$.

Notation 13.1. In the first part of this section we fix a polynomial monad T on $\mathcal{S}^{\mathcal{I}}$. From our work in the previous two sections we then have the following commutative diagram, where it will be convenient to name the various functors as indicated:

$$\begin{tikzcd} \mathcal{I} \arrow[r, "e"] \arrow[d, hook, "i"'] & \mathcal{W}(T)^{\mathrm{el}} \arrow[d, hook, "i'"] \\ \mathcal{U}(T) \arrow[r, "u"] \arrow[dr, "j"'] & \mathcal{W}(T)^{\mathrm{int}} \arrow[d, "j'"] \\ & \mathcal{W}(T). \end{tikzcd}$$

Proposition 13.2. *Let $\mathcal{K}(T)^{\mathrm{int}}$ denote the subcategory of $\mathcal{K}(T)$ containing only the inert morphisms. Then the slice $\mathcal{K}(T)^{\mathrm{int}}_{/F_TX}$ is equivalent to the full subcategory of $\mathcal{K}(T)_{/F_TX}$ spanned by the inert morphisms to FX. The functor $F_X \colon \mathcal{S}^{\mathcal{I}}_{/X} \to \mathcal{K}(T)_{/F_TX}$ restricts to an equivalence*

$$\mathcal{S}^{\mathcal{I}}_{/X} \xrightarrow{\sim} \mathcal{K}(T)^{\mathrm{int}}_{/F_TX}$$

with inverse $\mathcal{L}_X$.

Proof. It follows from the existence of the active–inert factorization system on $\mathcal{K}(T)$ that if we have a commutative triangle

$$\begin{tikzcd} F_TA \arrow[rr] \arrow[dr] & & F_TB \arrow[dl] \\ & F_TX & \end{tikzcd}$$

where the two diagonal morphisms are inert, then the horizontal morphism is also inert. This implies that $\mathcal{K}(T)^{\mathrm{int}}_{/F_TX}$ is the full subcategory of $\mathcal{K}(T)_{/F_TX}$ spanned by the inert morphisms. Moreover, every inert morphism to F_TX is equivalent to a free morphism in $\mathcal{K}(T)_{/F_TX}$, so this full subcategory consists precisely of the objects in the image of F_X. Since F_X is fully faithful by Remark 12.5, it follows that the adjunction $\mathcal{L}_X \dashv F_X$ restricts to an equivalence between $\mathcal{S}^{\mathcal{I}}_{/X}$ and $\mathcal{K}(T)^{\mathrm{int}}_{/F_TX}$. □

Restricting this equivalence, we get the following:

Corollary 13.3. *The functor F_X^{op} restricts to an equivalence*

$$\mathcal{I}_{X/} \xrightarrow{\sim} \mathcal{W}(T)^{\mathrm{el}}_{F_TX/}$$

for every $X \in \mathcal{U}(T)$. □

Corollary 13.4. *The top commutative square in Notation 13.1 induces a commutative square of functors to $\mathcal{S}$. Taking the mate of this square gives a commutative square*

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}},\mathcal{S}) \arrow[r, "i'_*"] \arrow[d, "e^*"'] & \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}},\mathcal{S}) \arrow[d, "u^*"] \\ \mathrm{Fun}(\mathcal{I},\mathcal{S}) \arrow[r, "i_*"] & \mathrm{Fun}(\mathcal{U}(T),\mathcal{S}), \end{tikzcd}$$

and this is moreover cartesian.

Proof. To see that there is such a commutative square amounts to checking that the mate transformation

$$u^* i'_* \Phi \to i_* e^* \Phi$$

is an equivalence for $\Phi \colon \mathcal{W}(T)^{\mathrm{el}} \to \mathcal{S}$. Evaluated at $X \in \mathcal{U}(T)$, this is the map on limits

$$\lim_{\mathcal{W}(T)^{\mathrm{el}}_{F_T X/}} \Phi \to \lim_{\mathcal{I}_{X/}} \Phi e,$$

induced by the functor $\mathcal{I}_{X/} \to \mathcal{W}(T)^{\mathrm{el}}_{F_T X/}$. Since this functor is an equivalence by Corollary 13.3, the mate transformation is indeed an equivalence. The functors i_* and i'_* are fully faithful, since they are given by right Kan extensions along the fully faithful functors i and i'. To see that the square is cartesian it therefore suffices to check that an object $\Phi \in \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}},\mathcal{S})$ is in the image of i'_* if and only if $u^*\Phi$ is in the image of i_*. Here Φ is in the image of i'_* if and only if the unit map $\Phi \to i'_* i'^* \Phi$ is an equivalence. The functor u^* is conservative, because u is essentially surjective, and so this holds if and only if $u^*\Phi \to u^* i'_* i'^* \Phi$ is an equivalence. We can identify the composite

$$u^*\Phi \to u^* i'_* i'^* \Phi \xrightarrow{\sim} i_* e^* i'^* \Phi \simeq i_* i^* u^* \Phi$$

with the unit map for $i^* \dashv i_*$, and since the mate transformation is an equivalence this means that the latter is an equivalence if and only if Φ is in the image of i'_*. As i_* is also fully faithful, this condition holds precisely when $u^*\Phi$ is in the image of i_*, as required. $\square$

Corollary 13.5. *We have a commutative diagram*

$$\begin{tikzcd} \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \arrow[r, hook, "\nu_{\mathcal{W}(T)}"] \arrow[d] \arrow[dd, bend right=60, "U_T"'] & \mathrm{Fun}(\mathcal{W}(T),\mathcal{S}) \arrow[d, "j'^*"] \arrow[dd, bend left=60, "j^*"] \\ \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}},\mathcal{S}) \arrow[r, hook, "i'_*"] \arrow[d, "e^*"'] & \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}},\mathcal{S}) \arrow[d, "u^*"] \\ \mathrm{Fun}(\mathcal{I},\mathcal{S}) \arrow[r, hook, "i_*"] & \mathrm{Fun}(\mathcal{U}(T),\mathcal{S}), \end{tikzcd}$$

where both squares are cartesian.

Proof. By the Nerve Theorem 11.15, we have a cartesian square

$$\begin{tikzcd} \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \arrow[r, "\nu_{\mathcal{W}(T)}"] \arrow[d, "U_T"'] & \mathrm{Fun}(\mathcal{W}(T), \mathcal{S}) \arrow[d, "j^*"] \\ \mathrm{Fun}(\mathcal{I}, \mathcal{S}) \arrow[r, hook, "i_*"] & \mathrm{Fun}(\mathcal{U}(T), \mathcal{S}). \end{tikzcd}$$

Here the right vertical functor j^* factors as $\mathrm{Fun}(\mathcal{W}(T), \mathcal{S}) \xrightarrow{j'^*} \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}}, \mathcal{S}) \xrightarrow{u^*} \mathrm{Fun}(\mathcal{U}(T), \mathcal{S})$. The left vertical functor therefore factors uniquely through the pullback of i_* along u^*, which we can identify with $\mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}}, \mathcal{S})$ by Corollary 13.4. This gives the desired commutative diagram. Here the bottom and outer squares are cartesian, and so the top square is also cartesian. □

Corollary 13.6. *We have a commutative square*

$$\begin{tikzcd} \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \arrow[r, "\sim"] \arrow[d] & \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S}) \arrow[d] \\ \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}}, \mathcal{S}) \arrow[r, "\sim"] & \mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S}) \end{tikzcd}$$

where the horizontal functors are equivalences.

Proof. By definition, $\mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S})$ is the essential image of the fully faithful functor i'_* in $\mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}}, \mathcal{S})$, and $\mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S})$ is the full subcategory of $\mathrm{Fun}(\mathcal{W}(T), \mathcal{S})$ spanned by the functors whose restriction along j' lies in this full subcategory; we thus have a pullback square

$$\begin{tikzcd} \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S}) \arrow[r, hook] \arrow[d] & \mathrm{Fun}(\mathcal{W}(T), \mathcal{S}) \arrow[d, "j'^*"] \\ \mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S}) \arrow[r, hook] & \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}}, \mathcal{S}). \end{tikzcd}$$

The top cartesian square in the diagram of Corollary 13.5 factors through this, giving a commutative diagram

$$\begin{tikzcd} \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \arrow[r] \arrow[d] & \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S}) \arrow[r, hook] \arrow[d] & \mathrm{Fun}(\mathcal{W}(T), \mathcal{S}) \arrow[d, "j'^*"] \\ \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}}, \mathcal{S}) \arrow[r, "i_*", "\sim"'] & \mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S}) \arrow[r, hook] & \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}}, \mathcal{S}). \end{tikzcd}$$

Here the left-hand square is cartesian, since the outer and right-hand squares are cartesian, and so the induced functor $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \to \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S})$ is indeed an equivalence. □

Proposition 13.7. *For $X \in \mathcal{U}(T)$, the functor*

$$\mathcal{U}(T)_{X/} \to \mathcal{U}(T)_{F_T X/} := \mathcal{U}(T) \times_{\mathcal{W}(T)^{\mathrm{int}}} \mathcal{W}(T)^{\mathrm{int}}_{F_T X/}$$

is coinitial.

Proof. By [29, Theorem 4.1.3.1] it suffices to check that for $Y, \phi\colon FY \to FX$, the slice ∞-category $(\mathcal{U}(T)_{X/})_{/\phi}$ is weakly contractible. Here the canonical factorization of ϕ determines a terminal object, as in the proof of Proposition 12.3. □

Corollary 13.8. *There are natural equivalences of functors*

$$\begin{aligned} \mathrm{id} &\xrightarrow{\sim} u_* u^*, \\ u_! u^* &\xrightarrow{\sim} \mathrm{id}, \\ j_! u^* &\xrightarrow{\sim} j'_!. \end{aligned}$$

Proof. For $\Phi\colon \mathcal{W}(T)^{\mathrm{int}} \to \mathcal{S}$ the unit map $\Phi \to u_* u^* \Phi$ evaluates at $F_T X \in \mathcal{W}(T)^{\mathrm{int}}$ as

$$\Phi(F_T X) \to \lim_{\mathcal{U}(T)_{F_T X/}} \Phi \circ u,$$

which is an equivalence by Corollary 13.7. This gives the first equivalence, which implies the second by passing to left adjoints. Applying $j'_!$ this gives the third equivalence, since $j'_! u_! \simeq (j'u)_! \simeq j_!$. □

Corollary 13.9. *The algebraic pattern $\mathcal{W}(T)$ is extendable.*

Proof. We must show that $j'_!$ restricts to a functor $\mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S}) \to \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S})$. Thus for $\Phi \in \mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S})$ we must show that $j'_!\Phi$ is a Segal object. By Corollary 13.8 the functor $j'_!\Phi$ is equivalent to $j_! u^* \Phi$. But since Φ is by assumption in $\mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S})$, we know by Corollary 13.4 that $u^*\Phi$ is right Kan extended from $\mathcal{I}$. Hence $j_! u^* \Phi$ is in $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \simeq \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S})$ by Theorem 11.15, as required. □

Corollary 13.10. *Inverting the equivalence of Corollary 13.6, we have a commutative square*

$$\begin{tikzcd} \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S}) \arrow[r, "\phi", "\sim"'] \arrow[d, "U_{\mathcal{W}(T)}"'] & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \arrow[d, "U_T"] \\ \mathcal{S}^{\mathcal{W}(T)^{\mathrm{el}}} \arrow[r, "e^*"] & \mathcal{S}^{\mathcal{I}}. \end{tikzcd}$$

This square is a morphism of polynomial monads $T \to T_{\mathcal{W}(T)}$.

Proof. Since $\mathcal{W}(T)$ is extendable, we know that the free Segal $\mathcal{W}(T)$-space monad $T_{\mathcal{W}(T)}$ is polynomial by Proposition 10.6. For the square to be a morphism of polynomial monads, it remains to show that the mate transformation $F_T e^* \to \phi F_{\mathcal{W}(T)}$ is cartesian. The equivalence $U_T\phi \simeq e^* U_{\mathcal{W}(T)}$ gives an equivalence of left adjoints $\phi^{-1}F_T \simeq F_{\mathcal{W}(T)}e_!$ under which the mate transformation corresponds to the transformation

$$\phi F_{\mathcal{W}(T)} e_! e^* \to \phi F_{\mathcal{W}(T)}$$

induced by the counit $e_! e^* \to \mathrm{id}$. This counit is easily seen to be cartesian (as in [16, Lemma 2.1.5]), and since $U_{\mathcal{W}(T)}$ is conservative and preserves limits, it suffices to check this implies the transformation

$$T_{\mathcal{W}(T)} e_! e^* \to T_{\mathcal{W}(T)}$$

is cartesian, which is true since $T_{\mathcal{W}(T)}$ preserves pullbacks. □

We now show that the pattern $\mathcal{W}(T)$ is natural with respect to morphisms of polynomial monads:

Theorem 13.11. *There is a functor*

$$\mathfrak{P} \colon \mathrm{PolyMnd} \to \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}}$$

that takes a polynomial monad T on $\mathcal{S}^{\mathcal{I}}$ to the algebraic pattern $\mathcal{W}(T)$, and a natural transformation

$$\tau \colon \mathrm{id} \to \mathfrak{M}\mathfrak{P},$$

given by the morphism $T \to \mathfrak{M}(\mathcal{W}(T))$ from Corollary 13.10, where $\mathfrak{M}$ is the functor from Corollary 10.12 that takes an extendable pattern $\mathcal{O}$ to the free Segal $\mathcal{O}$-space monad.

Proof. Suppose we have a morphism of polynomial monads $T \to S$, given by a functor $f \colon \mathcal{I} \to \mathcal{J}$ and a commutative square

$$\begin{tikzcd} \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}) \arrow[r, "\Phi"] \arrow[d, "U_S"'] & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \arrow[d, "U_T"] \\ \mathcal{S}^{\mathcal{J}} \arrow[r, "f^*"] & \mathcal{S}^{\mathcal{I}}. \end{tikzcd}$$

By Proposition 11.17, the functor Φ has a left adjoint Ψ which restricts to a functor $\Psi^{\mathrm{op}} \colon \mathcal{W}(T) \to \mathcal{W}(S)$. Lemma 11.18 implies that this functor preserves active and inert morphisms, since the active morphisms are precisely those that are adjoint to generic morphisms by Lemma 12.16, while the inert morphisms are the composites of free morphisms and equivalences. The commutative square from Proposition 11.17(iv) restricts to a commutative square

$$\begin{tikzcd} \mathcal{I} \arrow[r, "f"] \arrow[d] & \mathcal{J} \arrow[d] \\ \mathcal{W}(T) \arrow[r, "\Psi^{\mathrm{op}}"] & \mathcal{W}(S), \end{tikzcd}$$

and so Ψ^{op} also preserves elementary objects. Thus Ψ^{op} is a morphism of algebraic patterns.

It follows from Corollary 13.6 and Corollary 11.19 that $\Phi\colon \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}) \to \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}})$ can be identified with the restriction of $(\Psi^{\mathrm{op}})^*$ to Segal objects, thus Ψ^{op} is a Segal morphism by Lemma 4.5.

Since this construction is obviously compatible with composition we obtain a functor

$$\mathfrak{P}\colon \mathrm{PolyMnd} \to \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}}.$$

Using Corollary 13.4 the commutative cube in Corollary 11.19 extends to a commutative diagram

$$\begin{tikzcd} \mathrm{Alg}_S(\mathcal{S}^{\mathcal{J}}) \arrow[rr, hook] \arrow[dr] \arrow[dd] & & \mathrm{Fun}(\mathcal{W}(S), \mathcal{S}) \arrow[dr] \arrow[dd] & \\ & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \arrow[rr, hook, crossing over] & & \mathrm{Fun}(\mathcal{W}(T), \mathcal{S}) \arrow[dd] \\ \mathrm{Fun}(\mathcal{W}(S)^{\mathrm{el}}, \mathcal{S}) \arrow[rr, hook] \arrow[dr] \arrow[dd] & & \mathrm{Fun}(\mathcal{W}(S)^{\mathrm{int}}, \mathcal{S}) \arrow[dr] \arrow[dd] & \\ & \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}}, \mathcal{S}) \arrow[from=uu, crossing over] \arrow[rr, hook, crossing over] & & \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}}, \mathcal{S}) \arrow[dd] \\ \mathcal{S}^{\mathcal{J}} \arrow[rr, hook] \arrow[dr] & & \mathrm{Fun}(\mathcal{U}(S), \mathcal{S}) \arrow[dr] & \\ & \mathcal{S}^{\mathcal{I}} \arrow[from=uu, crossing over] \arrow[rr, hook] & & \mathrm{Fun}(\mathcal{U}(T), \mathcal{S}), \end{tikzcd}$$

where the left side gives the naturality square

$$\begin{tikzcd} S \arrow[r] \arrow[d] & T_{\mathcal{W}(S)} \arrow[d] \\ T \arrow[r] & T_{\mathcal{W}(T)}. \end{tikzcd}$$

Since this construction is again compatible with composition, it gives a natural transformation $\mathrm{id} \to \mathfrak{M}\mathfrak{P}$. $\square$

Variant 13.12. Let us say that a *flagged algebraic pattern* is a pair $(\mathcal{O}, \mathcal{I} \to \mathcal{O}^{\mathrm{el}})$ where $\mathcal{O}$ is an algebraic pattern and $\mathcal{I} \to \mathcal{O}^{\mathrm{el}}$ is an essentially surjective functor of ∞-categories.

We write FlAlgPatt for the full subcategory of $\mathrm{AlgPatt} \times_{\mathrm{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ spanned by the flagged algebraic patterns, and $\mathrm{FlAlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}}$ for the subcategory consisting of flagged algebraic patterns whose underlying patterns are extendable, with morphisms those such that the underlying morphisms of patterns are Segal morphisms. As a variant of the construction of $\mathfrak{P}$ above, we can define a functor

$$\mathfrak{P}' \colon \mathrm{PolyMnd} \to \mathrm{FlAlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}}$$

that takes a polynomial monad T on $\mathcal{S}^{\mathcal{I}}$ to the flagged algebraic pattern $(\mathcal{W}(T), \mathcal{I} \xrightarrow{e} \mathcal{W}(T)^{\mathrm{el}})$. Note that we can recover the monad T from this flagged pattern, since U_T is equivalent to the composite

$$\mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S}) \xrightarrow{U_{\mathcal{W}(T)}} \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}}, \mathcal{S}) \xrightarrow{e^*} \mathrm{Fun}(\mathcal{I}, \mathcal{S}).$$

For any flagged extendable pattern $(\mathcal{O}, f \colon \mathcal{I} \to \mathcal{O}^{\mathrm{el}})$ the composite

$$\mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \xrightarrow{U_{\mathcal{O}}} \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S}) \xrightarrow{f^*} \mathrm{Fun}(\mathcal{I}, \mathcal{S})$$

is a monadic right adjoint (since f^* preserves all limits and colimits and is conservative when f is essentially surjective), but we do not know under what conditions on f the corresponding monad is polynomial. This means that we do not have a satisfactory flagged version of the functor $\mathfrak{M}$ in general. However, if we restrict to patterns $\mathcal{O}$ such that $\mathcal{O}^{\mathrm{el}}$ is an ∞-groupoid, then this construction *does* give a polynomial monad for any essentially surjective morphism f of ∞-groupoids, since in this case the left adjoint $f_!$ preserves weakly contractible limits by [16, Lemma 2.2.10] and the unit and counit for the adjunction $f_! \dashv f^*$ are cartesian transformations by [16, Lemma 2.1.5].

14. Saturation and canonical patterns

Suppose $\mathcal{O}$ is an extendable algebraic pattern. Then the free Segal $\mathcal{O}$-space monad $T_{\mathcal{O}}$ is polynomial, and our results in the previous section associate to this another algebraic pattern $\overline{\mathcal{O}} := \mathcal{W}(T_{\mathcal{O}})$ such that there is an equivalence[9]

$$\mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \simeq \mathrm{Seg}_{\overline{\mathcal{O}}}(\mathcal{S}).$$

In this section we will explore the relationship between the patterns $\mathcal{O}$ and $\overline{\mathcal{O}}$. We will show that under a mild hypothesis on $\mathcal{O}$ (which can always be enforced by passing to a full subcategory without changing the monad) there is a canonical morphism of patterns $\mathcal{O} \to \overline{\mathcal{O}}$, which gives a natural transformation

[9] In the next section, we will see that furthermore the patterns $\mathcal{O}$ and $\overline{\mathcal{O}}$ determine the same polynomial monad.

$$\mathrm{id} \to \mathfrak{PM}.$$

We will also give an explicit necessary and sufficient condition on $\mathcal{O}$ for the map $\mathcal{O} \to \overline{\mathcal{O}}$ to be an equivalence, and discuss some examples where this holds.

Notation 14.1. In the first part of this section we fix an extendable pattern $\mathcal{O}$, and use the notations

$$\mathcal{O}^{\mathrm{el}} \xrightarrow{i} \mathcal{O}^{\mathrm{int}} \xrightarrow{j} \mathcal{O}$$

for the standard inclusions.

We begin by studying the localized Yoneda embedding

$$\mathcal{O}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{O}, \mathcal{S}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$$

for a pattern $\mathcal{O}$, which will give the canonical map to $\overline{\mathcal{O}}$.

Notation 14.2. Let $\Lambda_{\mathcal{O}}^{(\mathrm{int})} \colon \mathcal{O}^{(\mathrm{int}),\mathrm{op}} \to \mathrm{Seg}_{\mathcal{O}^{(\mathrm{int})}}(\mathcal{S})$ denote the composite of the Yoneda embedding $y_{\mathcal{O}}^{(\mathrm{int})} \colon \mathcal{O}^{(\mathrm{int}),\mathrm{op}} \to \mathrm{Fun}(\mathcal{O}^{(\mathrm{int})}, \mathcal{S})$ with the localization $\mathrm{Fun}(\mathcal{O}^{(\mathrm{int})}, \mathcal{S}) \to \mathrm{Seg}_{\mathcal{O}^{(\mathrm{int})}}(\mathcal{S})$.

Lemma 14.3. *For $X \in \mathcal{O}$, there is an equivalence*

$$\Lambda_{\mathcal{O}} X \simeq F_{\mathcal{O}} \Lambda_{\mathcal{O}}^{\mathrm{int}} X$$

in $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$. This equivalence is natural with respect to inert morphisms, i.e. we have a commutative square

$$\begin{tikzcd}
\mathcal{O}^{\mathrm{op}} \arrow[r, "\Lambda_{\mathcal{O}}"] & \mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \\
\mathcal{O}^{\mathrm{int},\mathrm{op}} \arrow[u, "j^{\mathrm{op}}"] \arrow[r, "\Lambda_{\mathcal{O}}^{\mathrm{int}}"] & \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S}). \arrow[u, "F_{\mathcal{O}}"']
\end{tikzcd}$$

Proof. For $\Phi \in \mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$, we have natural equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}} X, \Phi) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y_{\mathcal{O}} X, \Phi) \\
&\simeq \Phi(X), \\
\mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(F_{\mathcal{O}} \Lambda_{\mathcal{O}}^{\mathrm{int}} X, \Phi) &\simeq \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})}(\Lambda_{\mathcal{O}}^{\mathrm{int}} X, U_{\mathcal{O}} \Phi) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{O}^{\mathrm{int}},\mathcal{S})}(y_{\mathcal{O}}^{\mathrm{int}} X, U_{\mathcal{O}} \Phi) \\
&\simeq U_{\mathcal{O}} \Phi(X) \\
&\simeq \Phi(X).
\end{aligned}$$

The objects $\Lambda_{\mathcal{O}}X$ and $F_{\mathcal{O}}\Lambda_{\mathcal{O}}^{\mathrm{int}}X$ therefore corepresent the same copresheaf on $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$ and hence are equivalent. Moreover, this equivalence is by construction natural in $\mathcal{O}^{\mathrm{int}}$. □

Lemma 14.4. *The map*

$$\mathrm{Map}_{\mathcal{O}}(X,Y) \to \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}Y, \Lambda_{\mathcal{O}}X)$$

given by the functor $\Lambda_{\mathcal{O}}$ fits in a commutative square

$$\begin{tikzcd}
\operatorname{colim}_{O\to Y\in \mathrm{Act}_{\mathcal{O}}(Y)} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,O) \arrow[r] \arrow[d, "\wr"'] & \operatorname{colim}_{O\to Y\in \mathrm{Act}_{\mathcal{O}}(Y)} \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})}(\Lambda_{\mathcal{O}}^{\mathrm{int}}O, \Lambda_{\mathcal{O}}^{\mathrm{int}}X) \arrow[d, "\wr"] \\
\mathrm{Map}_{\mathcal{O}}(X,Y) \arrow[r] & \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}Y, \Lambda_{\mathcal{O}}X),
\end{tikzcd}$$

where the vertical maps are equivalences and the top horizontal map comes from the functor $\Lambda_{\mathcal{O}}^{\mathrm{int}}$.

Proof. From the commutative square of functors in Lemma 14.3 we get for all $O \in \mathcal{O}$ a commutative square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,O) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})}(\Lambda_{\mathcal{O}}^{\mathrm{int}}O, \Lambda_{\mathcal{O}}^{\mathrm{int}}X) \arrow[d] \\
\mathrm{Map}_{\mathcal{O}}(X,O) \arrow[r] & \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}O, \Lambda_{\mathcal{O}}X),
\end{tikzcd}$$

where the right-hand map can be identified with $\Lambda_{\mathcal{O}}^{\mathrm{int}}X(O) \to \operatorname{colim}_{O'\in \mathrm{Act}_{\mathcal{O}}(O)} \Lambda_{\mathcal{O}}^{\mathrm{int}}X(O')$ which is the canonical map to the colimit from the component at id_O. On the other hand, for any active morphism $O \to Y$ we have a natural commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{O}}(X,O) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}O, \Lambda_{\mathcal{O}}X) \arrow[r, "\sim"] \arrow[d] & T_{\mathcal{O}}\Lambda_{\mathcal{O}}^{\mathrm{int}}X(O) \arrow[r, "\sim"] \arrow[d] & \operatorname{colim}_{O'\in \mathrm{Act}_{\mathcal{O}}(O)} \Lambda_{\mathcal{O}}^{\mathrm{int}}X(O') \arrow[d] \\
\mathrm{Map}_{\mathcal{O}}(X,Y) \arrow[r] & \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}X, \Lambda_{\mathcal{O}}Y) \arrow[r, "\sim"] & T_{\mathcal{O}}\Lambda_{\mathcal{O}}^{\mathrm{int}}X(Y) \arrow[r, "\sim"] & \operatorname{colim}_{O''\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda_{\mathcal{O}}^{\mathrm{int}}X(O''),
\end{tikzcd}$$

where the description of $F_{\mathcal{O}}\Lambda_{\mathcal{O}}^{\mathrm{int}}X$ as a left Kan extension implies that the right-hand map is given on the component $\Lambda_{\mathcal{O}}^{\mathrm{int}}X(O')$ for $O' \to O$ by the canonical map $\Lambda_{\mathcal{O}}^{\mathrm{int}}X(O') \to \operatorname{colim}_{O''\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda_{\mathcal{O}}^{\mathrm{int}}X(O'')$ for the component at $O' \to O \to Y$. Putting these two diagrams together we therefore obtain natural commutative squares

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,O) \arrow[r] \arrow[d] & \Lambda_{\mathcal{O}}^{\mathrm{int}}X(O) \arrow[d] \\
\mathrm{Map}_{\mathcal{O}}(X,Y) \arrow[r] & \operatorname{colim}_{O'\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda_{\mathcal{O}}^{\mathrm{int}}X(O'),
\end{tikzcd}$$

for every active morphism $\phi\colon O \to Y$, where the right vertical map is the canonical one from the component of the colimit at ϕ. Taking colimits over $\mathrm{Act}_{\mathcal{O}}(Y)$ we therefore get a commutative square

$$\begin{tikzcd} \mathrm{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,O) \arrow[r] \arrow[d] & \mathrm{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}} X(O) \arrow[d, equal] \\ \mathrm{Map}_{\mathcal{O}}(X,Y) \arrow[r] & \mathrm{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}} X(O). \end{tikzcd}$$

Here the inert–active factorization system on $\mathcal{O}$ implies that the left vertical map is an equivalence, since its fibre at a morphism $\psi\colon X \to Y$ can be identified with the space of inert–active factorizations of ψ, and this completes the proof. □

Remark 14.5. For $Y \in \mathcal{O}$, we have a natural equivalence

$$\mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})}(\Lambda^{\mathrm{int}}_{\mathcal{O}} Y, \Phi) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{O}^{\mathrm{int}},\mathcal{S})}(y^{\mathrm{int}}_{\mathcal{O}} Y, \Phi) \simeq \Phi(Y).$$

In particular,

$$\mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})}(\Lambda^{\mathrm{int}}_{\mathcal{O}} Y, T_{\mathcal{O}}*) \simeq \mathrm{Act}_{\mathcal{O}}(Y),$$

and so a morphism $\Lambda^{\mathrm{int}}_{\mathcal{O}} Y \to T_{\mathcal{O}}*$ corresponds to an active morphism $X \to Y$ in $\mathcal{O}$.

We will now show that this equivalence identifies active morphisms in $\mathcal{O}$ with generic morphisms in $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})$:

Proposition 14.6. *Suppose $\Lambda^{\mathrm{int}}_{\mathcal{O}} Y \xrightarrow{\eta} T_{\mathcal{O}}*$ corresponds to the active morphism $X \overset{\phi}{\rightsquigarrow} Y$ in $\mathrm{Act}_{\mathcal{O}}(Y)$ under the equivalence of Remark 14.5. Then the generic–free factorization of η is*

$$\Lambda^{\mathrm{int}}_{\mathcal{O}} Y \xrightarrow{\hat{\phi}} T_{\mathcal{O}}\Lambda^{\mathrm{int}}_{\mathcal{O}} X \to T_{\mathcal{O}}*,$$

where the first morphism is adjoint to $\Lambda_{\mathcal{O}}(\phi)\colon \Lambda_{\mathcal{O}} Y \to \Lambda_{\mathcal{O}} X$.

Proof. We first check that this factorization exists. By Lemma 14.4 the morphism $\hat{\phi}$ adjoint to $\Lambda(\phi)$ corresponds to the point in $T_{\mathcal{O}}\Lambda^{\mathrm{int}}_{\mathcal{O}} X(Y) \simeq \mathrm{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}} X(O)$ given by the composite

$$\{\mathrm{id}_X\} \to \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,X) \to \underset{O\in \mathrm{Act}_{\mathcal{O}}(Y)}{\mathrm{colim}} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,O) \to \underset{O\in \mathrm{Act}_{\mathcal{O}}(Y)}{\mathrm{colim}} \Lambda^{\mathrm{int}}_{\mathcal{O}} X(O),$$

where the second morphism is the canonical one from the component of the colimit at ϕ. We therefore have a commutative diagram

$$\begin{tikzcd}
 & & * \arrow[dll] \arrow[d] \\
\operatorname{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,O) \arrow[d] \arrow[rr] & & \mathrm{Act}_{\mathcal{O}}(Y), \\
\operatorname{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}}X(O) \arrow[rru] & &
\end{tikzcd}$$

where the outer triangle corresponds to the desired factorization

$$\begin{tikzcd}
 & \Lambda^{\mathrm{int}}_{\mathcal{O}}Y \arrow[dl, "\hat{\phi}"'] \arrow[dr, "\eta"] & \\
T_{\mathcal{O}}\Lambda^{\mathrm{int}}_{\mathcal{O}}X \arrow[rr] & & T_{\mathcal{O}}(*).
\end{tikzcd}$$

Now we must show that $\hat{\phi}$ is generic, so suppose we have a commutative square

$$\begin{tikzcd}
\Lambda^{\mathrm{int}}Y \arrow[r, "\theta"] \arrow[d, "\hat{\phi}"'] & T_{\mathcal{O}}\Phi \arrow[d] \\
T_{\mathcal{O}}\Lambda^{\mathrm{int}}X \arrow[r] & T_{\mathcal{O}}*,
\end{tikzcd}$$

where the top horizontal map corresponds to a point p in the fibre $\Phi(X)$ of $T_{\mathcal{O}}\Phi(Y) \simeq \operatorname{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Phi(O)$ at ϕ. Suppose we have a commutative triangle of the form

$$\begin{tikzcd}
 & \Lambda^{\mathrm{int}}_{\mathcal{O}}Y \arrow[dl, "\hat{\phi}"'] \arrow[dr, "\theta"] & \\
T_{\mathcal{O}}\Lambda^{\mathrm{int}}X \arrow[rr, "T\psi"] & & T_{\mathcal{O}}\Phi.
\end{tikzcd}$$

This amounts to an equivalence between p and the image

$$\begin{aligned}
* \xrightarrow{\mathrm{id}_X} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,X) &\to \operatorname*{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,O) \to \operatorname*{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}}X(O) \\
&\to \operatorname*{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Phi(O).
\end{aligned}$$

But since the last map arises from $T\psi$, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,X) \arrow[r] \arrow[d] & \Lambda^{\mathrm{int}}_{\mathcal{O}}X(X) \arrow[r] \arrow[d] & \Phi(X) \arrow[d] \\
\operatorname{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X,O) \arrow[r] & \operatorname{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Lambda^{\mathrm{int}}_{\mathcal{O}}X(O) \arrow[r] & \operatorname{colim}_{O\in \mathrm{Act}_{\mathcal{O}}(Y)} \Phi(O),
\end{tikzcd}$$

which tells us that ψ must be the morphism $\Lambda^{\mathrm{int}}_{\mathcal{O}}X \to \Phi$ obtained by localizing the unique natural transformation $y^{\mathrm{int}}_{\mathcal{O}}X \to \Phi$ that takes id_X to the point p. Thus $\hat{\phi}$ satisfies the universal property of generic morphisms described in Remark 11.2. By the uniqueness of generic–free factorizations, this completes the proof. □

This proposition allows us to identify the objects of $\mathcal{U}(T_{\mathcal{O}})$:

Definition 14.7. We say an object $O \in \mathcal{O}$ is *necessary* if it admits an active morphism $O \to E$ for some $E \in \mathcal{O}^{\mathrm{el}}$, and denote by $\mathcal{O}^{\circ}$ the full subcategory of $\mathcal{O}$ spanned by the necessary objects. We say the pattern $\mathcal{O}$ is *slim* if all objects are necessary, and write $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{slim,ext}}$ for the full subcategory of $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{ext}}$ spanned by the slim extendable patterns.

Corollary 14.8. *Let $\mathcal{O}$ be an extendable algebraic pattern, and let $\overline{\mathcal{O}} := \mathcal{W}(T_{\mathcal{O}})$ denote the corresponding canonical pattern. Then:*

(i) The objects of $\mathcal{U}(T_{\mathcal{O}})$ are the objects of $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})$ of the form $\Lambda^{\mathrm{int}}_{\mathcal{O}}X$ with $X \in \mathcal{O}^{\circ}$. Thus $\Lambda^{\mathrm{int}}_{\mathcal{O}}$ induces an essentially surjective functor $\mathcal{O}^{\circ,\mathrm{int}} \to \mathcal{U}(T_{\mathcal{O}})$.
(ii) The objects of $\overline{\mathcal{O}}$ are the objects of $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$ of the form $\Lambda_{\mathcal{O}}X$ with $X \in \mathcal{O}^{\circ}$. Thus $\Lambda_{\mathcal{O}}$ induces an essentially surjective functor $\mathcal{O}^{\circ} \to \overline{\mathcal{O}}$.
(iii) A morphism $\Lambda_{\mathcal{O}}X \to \Lambda_{\mathcal{O}}Y$ is active if and only if it is a composite of an equivalence and the image of an active morphism $X \to Y$ in $\mathcal{O}^{\circ}$. In particular, the functor $\mathcal{O}^{\circ} \to \overline{\mathcal{O}}$ preserves inert and active morphisms.

Proof. By definition, the objects of $\mathcal{U}(T_{\mathcal{O}})$ are the objects Φ of $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})$ that admit a generic morphism $\Lambda^{\mathrm{int}}_{\mathcal{O}}E \to T_{\mathcal{O}}\Phi$ with $E \in \mathcal{O}^{\mathrm{el}}$. Such a generic morphism is determined by a morphism $\Lambda^{\mathrm{int}}_{\mathcal{O}}E \to T_{\mathcal{O}}*$, and from Proposition 14.6 we see that the generic–free factorizations of such morphisms yield precisely the objects of $\mathcal{O}^{\circ}$. This proves (i), from which (ii) follows using Lemma 14.3. Finally, as active morphisms in $\overline{\mathcal{O}}$ are those morphisms which are adjoint to generic maps by Lemma 12.16, the first part of (iii) follows from the identification of such generic morphisms with active morphisms in $\mathcal{O}$ in Proposition 14.6. This shows that $\Lambda_{\mathcal{O}}$ preserves active morphisms, while the commutative square of Lemma 14.3 implies that it preserves inert morphisms, since free morphisms in $\overline{\mathcal{O}}$ are in particular inert. □

Remark 14.9. If $\mathcal{O}$ is a slim extendable pattern, then Corollary 14.8 says that $\overline{\mathcal{O}}$ has the same objects as $\mathcal{O}$, and the active morphisms are obtained by combining active morphisms from $\mathcal{O}$ with equivalences (which may not all come from $\mathcal{O}$).

Remark 14.10. If O is necessary and $O' \to O$ is an active morphism, then O' is also necessary. This implies that the inert–active factorization system in $\mathcal{O}$ restricts to $\mathcal{O}^{\circ}$, and that $\mathrm{Act}_{\mathcal{O}}(O) \simeq \mathrm{Act}_{\mathcal{O}^{\circ}}(O)$ for $O \in \mathcal{O}^{\circ}$. It follows that $\mathcal{O}^{\circ}$ is extendable when $\mathcal{O}$ is. In this case we therefore have a commutative diagram

$$\begin{tikzcd}
\mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \arrow[rr] \arrow[d] & & \mathrm{Seg}_{\mathcal{O}^{\circ}}(\mathcal{S}) \arrow[d] \\
\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S}) \arrow[rr] \arrow[dr, "\sim"'] & & \mathrm{Seg}_{\mathcal{O}^{\circ,\mathrm{int}}}(\mathcal{S}) \arrow[dl, "\sim"] \\
& \mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S}), &
\end{tikzcd}$$

where the vertical maps are monadic right adjoints. The lower horizontal map is an equivalence since the diagonal maps are equivalences. Since the two monads on $\mathrm{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{S})$ are the same (by definition $\mathrm{Act}_{\mathcal{O}}(E) \simeq \mathrm{Act}_{\mathcal{O}^{\circ}}(E)$ for $E \in \mathcal{O}^{\mathrm{el}}$), the top horizontal morphism is also an equivalence. Thus the patterns $\mathcal{O}$ and $\mathcal{O}^{\circ}$ describe the same monad, and so the objects of $\mathcal{O}$ that do not lie in $\mathcal{O}^{\circ}$ are in this sense *unnecessary*.

Examples 14.11. The examples of patterns discussed in §3 are all slim, with the exception of the pattern $\mathbf{\Delta}_{\Phi}^{\mathrm{op},\natural}$ of Example 3.8. The corresponding slim pattern $\mathbf{\Delta}_{\Phi}^{\mathrm{op},\natural,\circ}$ is the full subcategory spanned by objects $([m], f)$ such that $f(m) \cong *$. Another non-slim example is the extension of the dendroidal category $\mathbf{\Omega}^{\mathrm{op},\natural}$ to a category of forests considered in [24], which has $\mathbf{\Omega}^{\mathrm{op},\natural}$ as its slim subpattern.

Remark 14.12. If T is a polynomial monad on $\mathcal{S}^{\mathcal{I}}$ then the algebraic pattern $\mathcal{W}(T)$ is slim. This follows from the fact that objects in $\mathcal{W}(T)$ can be identified with objects in $\mathcal{U}(T)$, i.e. objects X admitting a generic map $I \to TX$ with $I \in \mathcal{I}$. Since $\mathcal{I}$ has the same objects as $\mathcal{W}(T)^{\mathrm{el}}$ and every generic map is adjoint to an active morphism in $\mathcal{W}(T)$, the algebraic pattern $\mathcal{W}(T)$ is indeed slim. We can thus regard $\mathfrak{P}$ as a functor

$$\mathrm{PolyMnd} \to \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{slim,ext}}.$$

Remark 14.13. Suppose $f\colon \mathcal{O} \to \mathcal{P}$ is a Segal morphism between slim extendable patterns. Then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}^{\mathrm{op}} \arrow[r, "f^{\mathrm{op}}"] \arrow[d] & \mathcal{P}^{\mathrm{op}} \arrow[d] \\
\mathrm{Fun}(\mathcal{O}, \mathcal{S}) \arrow[r, "f_!"] \arrow[d] & \mathrm{Fun}(\mathcal{P}, \mathcal{S}) \arrow[d] \\
\mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \arrow[r] & \mathrm{Seg}_{\mathcal{P}}(\mathcal{S}).
\end{tikzcd}$$

In other words, we have a commutative square

$$\begin{tikzcd}
\mathcal{O}^{(\mathrm{int}),\mathrm{op}} \arrow[r, "f^{(\mathrm{int}),\mathrm{op}}"] \arrow[d, "\Lambda_{\mathcal{O}}"'] & \mathcal{P}^{(\mathrm{int}),\mathrm{op}} \arrow[d, "\Lambda_{\mathcal{P}}"] \\
\mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \arrow[r] & \mathrm{Seg}_{\mathcal{P}}(\mathcal{S}),
\end{tikzcd}$$

which restricts to a commutative square

$$\begin{tikzcd} \mathcal{O} \arrow[r, "f"] \arrow[d] & \mathcal{P} \arrow[d] \\ \overline{\mathcal{O}} \arrow[r] & \overline{\mathcal{P}}, \end{tikzcd}$$

where all the morphisms are Segal morphism of algebraic patterns. Thus we have proved:

Proposition 14.14. *There is a natural transformation* $\sigma\colon \mathrm{id} \to \mathfrak{PM}$ *of functors* $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{slim,ext}} \to \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{slim,ext}}$.

Our next goal is to identify when the map $\sigma_{\mathcal{O}}$ is an equivalence, which turns out to correspond to the following condition:

Definition 14.15. If $\mathcal{O}$ is a slim extendable pattern, we say that $\mathcal{O}$ is *saturated* if for every object $O \in \mathcal{O}$ the copresheaf

$$\mathrm{Map}_{\mathcal{O}}(O, –)\colon \mathcal{O} \to \mathcal{S}$$

is a Segal $\mathcal{O}$-space. We write $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{sat}}$ for the full subcategory of $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{slim,ext}}$ spanned by the saturated patterns.

Proposition 14.16. *The following conditions are equivalent for a slim extendable pattern* $\mathcal{O}$*:*

(1) $\mathcal{O}$ *is saturated.*
(2) For every $X \in \mathcal{O}$*, the canonical functor* $\mathcal{O}^{\mathrm{int},\triangleleft}_{X/} \to \mathcal{O}$ *is a limit diagram.*
(3) The Yoneda embedding $\mathcal{O}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{O}, \mathcal{S})$ *factors through* $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$*.*
(4) The functor $\Lambda_{\mathcal{O}}\colon \mathcal{O}^{\mathrm{op}} \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$ *is fully faithful.*

Proof. The equivalence of (1), (2), and (3) is clear, and it is also clear that (3) implies (4). We prove the remaining implication from (4) to (3) by showing that (4) implies that for every $X \in \mathcal{O}$ there is an equivalence $y_{\mathcal{O}}X \simeq \Lambda_{\mathcal{O}}X$ in $\mathrm{Fun}(\mathcal{O}, \mathcal{S})$. We have

$$\begin{aligned} \mathrm{Map}_{\mathcal{O}}(X, Y) &\xrightarrow{\sim} \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y_{\mathcal{O}}Y, y_{\mathcal{O}}X) \to \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}Y, \Lambda_{\mathcal{O}}X) \\ &\to \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y_{\mathcal{O}}Y, \Lambda_{\mathcal{O}}X), \end{aligned}$$

where the first map is the Yoneda embedding. Since the composition of the first two morphisms is an equivalence by (4), the second map is an equivalence. The last map is an equivalence because $\Lambda_{\mathcal{O}}X$ is a local object and $y_{\mathcal{O}}Y \to \Lambda_{\mathcal{O}}Y$ is a local equivalence. Hence, we have

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y_{\mathcal{O}}Y, y_{\mathcal{O}}X) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{Fun}(\mathcal{O},\mathcal{S})}(y_{\mathcal{O}}Y, \Lambda_{\mathcal{O}}X)$$

for every object $Y \in \mathcal{O}$, which then implies that $y_{\mathcal{O}}X \simeq \Lambda_{\mathcal{O}}X$ in $\mathrm{Fun}(\mathcal{O},\mathcal{S})$ by the Yoneda Lemma. □

Lemma 14.17. *Suppose T is a polynomial monad. Then the pattern $\mathcal{W}(T)$ is saturated.*

Proof. We already know the pattern $\mathcal{W}(T)$ is extendable (by Corollary 13.9) and slim (by Remark 14.12). By definition, $\mathcal{W}(T)^{\mathrm{op}}$ is a full subcategory of $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}})$, and the Nerve Theorem 11.15 implies that the restricted Yoneda functor $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \to \mathrm{Fun}(\mathcal{W}(T),\mathcal{S})$ is fully faithful with image $\mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S})$. This implies in particular that the Yoneda embedding of $\mathcal{W}(T)$ takes values in Segal $\mathcal{W}(T)$-spaces, which implies that $\mathcal{W}(T)$ is saturated by Proposition 14.16. □

Lemma 14.17 implies in particular that the pattern $\overline{\mathcal{O}}$ is always saturated, which gives the following:

Corollary 14.18. *The morphism $\sigma_{\mathcal{O}}\colon \mathcal{O} \to \overline{\mathcal{O}}$ is an equivalence if and only if $\mathcal{O}$ is saturated.* □

Corollary 14.19. *The natural transformation σ exhibits the full subcategory $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{sat}}$ as a localization of $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{slim,ext}}$.*

Proof. Let $L := \mathfrak{P}\mathfrak{M}$; then the essential image of L is precisely $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{sat}}$: by Corollary 14.18 the image of L contains all saturated patterns, while all patterns in the image of L are saturated by Lemma 14.17. To see that L and σ exhibit $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{sat}}$ as a localization, we apply [29, Proposition 5.2.7.4]. It suffices to verify condition (3) of this result, namely that the two morphisms

$$\sigma_{L\mathcal{O}}, L(\sigma_{\mathcal{O}})\colon L\mathcal{O} \to LL\mathcal{O}$$

are both equivalences for all $\mathcal{O}$ in $\mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{slim,ext}}$. For $\sigma_{L\mathcal{O}}$ this holds by Corollary 14.18, since $L\mathcal{O}$ is saturated, and for $L(\sigma_{\mathcal{O}})$ it holds since $\sigma_{\mathcal{O}}$ induces an equivalence

$$\sigma^*_{\mathcal{O}}\colon \mathrm{Seg}_{L\mathcal{O}}(\mathcal{S}) \xrightarrow{\sim} \mathrm{Seg}_{\mathcal{O}}(\mathcal{S}),$$

and $L\sigma_{\mathcal{O}}$ is obtained by restricting the inverse of this equivalence. □

The following proposition shows that we can equivalently characterize saturated patterns in terms of their subcategories of inert morphisms:

Proposition 14.20. *The following conditions are equivalent for a slim extendable pattern $\mathcal{O}$:*

(1) $\mathcal{O}$ is saturated.
(2) For every $X \in \mathcal{O}$, the functor

$$\mathrm{Map}_{\mathcal{O}^{\mathrm{int}}}(X, \text{–})\colon \mathcal{O}^{\mathrm{int}} \to \mathcal{S}$$

is a Segal $\mathcal{O}^{\mathrm{int}}$-space.
(3) For every X in $\mathcal{O}$, the diagram $\mathcal{O}^{\mathrm{el},\triangleleft}_{X/} \to \mathcal{O}^{\mathrm{int}}$ is a limit diagram.
(4) The Yoneda embedding $\mathcal{O}^{\mathrm{int,op}} \to \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S})$ factors through $\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})$.
(5) The functor $\Lambda^{\mathrm{int}}_{\mathcal{O}}\colon \mathcal{O}^{\mathrm{int,op}} \to \mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})$ is fully faithful.

Proof. The equivalence of conditions (2)–(5) follows exactly as in the proof of Proposition 14.16. It remains to show that these conditions are equivalent to $\mathcal{O}$ being saturated.

Since $\mathcal{O}$ is by assumption extendable, by Proposition 8.8 we have a commutative square

$$\begin{tikzcd}
\mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \arrow[r] & \mathrm{Fun}(\mathcal{O}, \mathcal{S}) \\
\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S}) \arrow[r] \arrow[u, "F_{\mathcal{O}}"] & \mathrm{Fun}(\mathcal{O}^{\mathrm{int}}, \mathcal{S}). \arrow[u, "j_{\mathcal{O},!}"']
\end{tikzcd}$$

Omitting notation for the horizontal inclusions, we have equivalences

$$\Lambda_{\mathcal{O}} X \simeq F_{\mathcal{O}} \Lambda^{\mathrm{int}}_{\mathcal{O}} X \simeq j_{\mathcal{O},!} \Lambda^{\mathrm{int}}_{\mathcal{O}} X.$$

If condition (4) holds, then $\Lambda^{\mathrm{int}}_{\mathcal{O}} X$ is the representable presheaf $y^{\mathrm{int}}_{\mathcal{O}} X$, hence

$$j_{\mathcal{O},!} \Lambda^{\mathrm{int}}_{\mathcal{O}} X \simeq j_{\mathcal{O},!} y^{\mathrm{int}}_{\mathcal{O}} X \simeq y_{\mathcal{O}} j_{\mathcal{O}}(X).$$

In other words, $\Lambda_{\mathcal{O}} X$ is precisely the presheaf represented by $X \in \mathcal{O}$, which implies that $\mathcal{O}$ is saturated by Proposition 14.16.

Conversely, suppose $\mathcal{O}$ is saturated. By Proposition 14.16 this means that for every $X \in \mathcal{O}$, the diagram $\mathcal{O}^{\mathrm{el},\triangleleft}_{X/} \to \mathcal{O}$ is a limit diagram. To show that this diagram is then also a limit in the subcategory $\mathcal{O}^{\mathrm{int}}$ (and hence verify condition (3)), it is enough to show that a morphism $\phi\colon Y \to X$ is inert if the composites $Y \to X \rightarrowtail E$ are all inert. Using the inert–active factorization system, we see that it suffices to consider the case where ϕ is active and prove that it is an equivalence. Recall that we have a morphism

$$\mathrm{Act}_{\mathcal{O}}(X) \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} \mathrm{Act}_{\mathcal{O}}(E),$$

which takes $\phi\colon Y \rightsquigarrow X$ to the active parts of the inert–active decompositions of the composites $Y \rightsquigarrow X \rightarrowtail E$. Since these composites are inert, the image of ϕ is given by id_E for all $E \in \mathcal{O}^{\mathrm{el}}_{X/}$, so that ϕ has the same image as id_X. But since $\mathcal{O}$ is extendable, this map of ∞-groupoids is an equivalence, and hence ϕ is equivalent to id_X in $\mathrm{Act}_{\mathcal{O}}(X)$, which means precisely that ϕ is an equivalence. □

We end this section by looking at some examples of saturated and non-saturated patterns.

Examples 14.21. The patterns $\mathbf{\Delta}^{n,\mathrm{op},\natural}$, $\mathbf{\Theta}_n^{\mathrm{op},\natural}$, and $\mathbf{\Omega}^{\mathrm{op},\natural}$ (described in Examples 3.3, 3.5, and 3.7, respectively) are all saturated. In the case of $\mathbf{\Delta}^{\mathrm{op},\natural}$, for example, this amounts to the observation that the object $[n] \in \mathbf{\Delta}^{\mathrm{int}}$ is a colimit,

$$[1] \amalg_{[0]} \cdots \amalg_{[0]} [1] \simeq [n],$$

while for $\mathbf{\Omega}^{\mathrm{op},\natural}$ the required colimit in $\mathbf{\Omega}^{\mathrm{int}}$ amounts to a decomposition of a tree as a colimit of its nodes and edges, and follows from [26, Proposition 1.1.19].

Example 14.22. The pattern $\mathbb{F}_*^\flat$ from Example 3.1 is *not* saturated: The functor $\Lambda^{\mathrm{int}}_{\mathbb{F}_*^\flat} \colon \mathbb{F}_*^{\flat,\mathrm{int},\mathrm{op}} \to \mathcal{S}$ takes $\langle n\rangle$ to a finite set $\mathbf{n}$ with n elements, and an inert morphism $\langle n\rangle \to \langle m\rangle$ to the map $\mathbf{m} \to \mathbf{n}$ that takes $i \in \mathbf{m}$ to its unique preimage $\phi^{-1}(i) \in \mathbf{n}$. Thus inert morphisms correspond bijectively to *injective* morphisms of finite sets, and the functor is not fully faithful. The canonical pattern $\overline{\mathbb{F}}_*^\flat \subseteq \mathrm{Seg}_{\mathbb{F}_*^\flat}(\mathcal{S})^{\mathrm{op}}$ consists of the free commutative monoids on finite sets. By work of Cranch [12] this can be identified with the (2,1)-category $\mathrm{Span}(\mathbb{F})$ whose objects are finite sets and whose morphisms are *spans* of finite sets, with $\mathbb{F}_* \to \overline{\mathbb{F}}_*$ identifying $\mathbb{F}_*$ with the subcategory where the morphisms from I to J are spans $I \leftarrow K \to J$ with the backward map *injective*.

Example 14.23. More generally, for any ∞-operad $\mathcal{O}$ (in the sense of [30]) the canonical pattern $\overline{\mathcal{O}}$ can be identified with the opposite of the ∞-category of finitely generated free $\mathcal{O}$-monoids in $\mathcal{S}$, i.e. the *Lawvere theory* for $\mathcal{O}$-monoids.

Remark 14.24. See [14,7] for more on Lawvere theories in the ∞-categorical context. Note that the monads corresponding to Lawvere theories always preserve sifted colimits, so the (coloured) Lawvere theories that fit into our theory are precisely the monads on $\mathcal{S}^X$ for an ∞-groupoid X that preserve both sifted colimits and weakly contractible limits. These are precisely the *analytic* monads studied in [16], where they are identified with ∞-operads in the sense of dendroidal Segal spaces.

Example 14.25. The pattern $\mathbf{\Gamma}^{\mathrm{op},\natural}$ of Example 3.9 is not saturated. We expect that its saturation is the $(2,1)$-category of graphs implicitly defined by Kock in [27, §3.3].

15. Completion of polynomial monads

In this section we will study a class of polynomial monads that is particularly closely related to algebraic patterns, namely the *complete* ones in the following sense:

Definition 15.1. Let T be a polynomial monad on $\mathcal{S}^{\mathcal{I}}$. We say that T is *complete* if the functor $\mathcal{I} \to \mathcal{W}(T)^{\mathrm{el}}$ underlying $\tau_T : T \to \mathfrak{M}\mathfrak{P}T$ is an equivalence. We write cPolyMnd for the full subcategory of PolyMnd spanned by the complete polynomial monads.

We will see that the polynomial monad corresponding to an extendable pattern is *always* complete, so that the functor $\mathfrak{M}$ takes values in cPolyMnd. Moreover, we will show that the transformation $\tau\colon \mathrm{id} \to \mathfrak{M}\mathfrak{P}$ exhibits cPolyMnd as a localization of PolyMnd, and the functors $\mathfrak{M}$ and $\mathfrak{P}$ restrict to an *equivalence*

$$\mathrm{cPolyMnd} \simeq \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{sat}}$$

between complete polynomial monads and saturated patterns.

Remark 15.2. The term *complete* is inspired by the equivalence of [16] between dendroidal Segal spaces and *analytic* monads, which are the polynomial monads on presheaves over ∞-groupoids that preserve sifted colimits. Under this equivalence, the complete dendroidal Segal spaces (meaning those whose underlying Segal spaces are complete in the sense of Rezk [33]) are precisely those analytic monads that are complete in our sense.

We begin by giving some alternative descriptions of the complete polynomial monads:

Proposition 15.3. *Let T be a polynomial monad on $\mathcal{S}^{\mathcal{I}}$. The following are equivalent:*

(1) T is complete.
(2) The morphism $\tau_T : T \to \mathfrak{M}\mathfrak{P}T$ is an equivalence.
(3) The functor $u\colon \mathcal{U}(T) \to \mathcal{W}(T)^{\mathrm{int}}$ is an equivalence.
(4) The functor $j\colon \mathcal{U}(T) \to \mathcal{W}(T)$ is a subcategory inclusion, i.e. it is faithful and induces an equivalence $\mathcal{U}(T)^{\simeq} \xrightarrow{\sim} \mathcal{W}(T)^{\simeq}$ on underlying ∞-groupoids.
(5) The functor j is faithful and every equivalence is in its image.

Proof. To see that (1) is equivalent to (2), observe that the morphism τ_T in PolyMnd is given by the morphism $e\colon \mathcal{I} \to \mathcal{W}(T)^{\mathrm{el}}$ together with the commutative square

$$\begin{array}{ccc} \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S}) & \xrightarrow{\sim} & \mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}}, \mathcal{S}) & \xrightarrow{e^*} & \mathrm{Fun}(\mathcal{I}, \mathcal{S}), \end{array}$$

and so τ_T is an equivalence if and only if e is an equivalence.

It is clear that (3) implies (1), since e is obtained from u by restricting to a full subcategory. Conversely, if e is an equivalence, then the commutative square of Corollary 13.4 gives a commutative square

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{W}(T)^{\mathrm{el}}, \mathcal{S}) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S}) \arrow[d, "u^*"] \\
\mathrm{Fun}(\mathcal{I}, \mathcal{S}) \arrow[r, "\sim"] & \mathrm{Seg}_{\mathcal{U}(T)}(\mathcal{S}),
\end{tikzcd}$$

where $\mathrm{Seg}_{\mathcal{U}(T)}(\mathcal{S})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{U}(T), \mathcal{S})$ of functors right Kan extended from $\mathcal{I}$; the functor

$$u^* \colon \mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S}) \to \mathrm{Seg}_{\mathcal{U}(T)}(\mathcal{S})$$

is therefore an equivalence. Here $\mathcal{W}(T)^{\mathrm{int,op}}$ is a full subcategory of $\mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S})$ via the Yoneda embedding by Proposition 14.20, since $\mathcal{W}(T)$ is saturated by Lemma 14.17. Moreover, $\mathcal{U}(T)^{\mathrm{op}}$ is a full subcategory of $\mathrm{Seg}_{\mathcal{U}(T)}(\mathcal{S})$ by Proposition 11.13. The inverse of u^* is given by left Kan extension $u_!$ followed by localization from $\mathrm{Fun}(\mathcal{W}(T)^{\mathrm{int}}, \mathcal{S})$ to $\mathrm{Seg}_{\mathcal{U}(T)}(\mathcal{S})$, which restricts to just u on $\mathcal{U}(T)^{\mathrm{op}}$ since $\mathcal{W}(T)^{\mathrm{int,op}}$ is already in $\mathrm{Seg}_{\mathcal{W}(T)^{\mathrm{int}}}(\mathcal{S})$. Hence u is the restriction of the equivalence $(u^*)^{-1}$ to a full subcategory, which implies that u is indeed an equivalence.

Since $\mathcal{W}(T)^{\mathrm{int}}$ is by definition a subcategory of $\mathcal{W}(T)$, (3) immediately implies (4). On the other hand, (4) implies (3) since the inert morphisms in $\mathcal{W}(T)$ are precisely those that are composites of morphisms in the image of u and equivalences in $\mathcal{W}(T)$.

Finally, (4) trivially implies (5), while given (5) we know that

$$\mathrm{Map}_{\mathcal{U}(T)}(X, Y) \to \mathrm{Map}_{\mathcal{W}(T)}(jX, jY)$$

is a monomorphism whose image contains the components that correspond to equivalences in $\mathcal{W}(T)$. Since j is conservative by Lemma 12.13, the components that map to these are precisely those that correspond to equivalences in $\mathcal{U}(T)$, so that j restricts to an equivalence $\mathcal{U}(T)^{\simeq} \to \mathcal{W}(T)^{\simeq}$. □

Proposition 15.4. *Suppose $\mathcal{O}$ is a slim extendable pattern. Then $T_{\mathcal{O}}$ is a complete polynomial monad.*

For the proof we need the following observation:

Lemma 15.5. *Suppose $\phi \colon X \to Y$ is an active morphism such that $\Lambda_{\mathcal{O}}\phi$ is an equivalence in $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$. Then ϕ is an equivalence in $\mathcal{O}$.*

Proof. Suppose $\alpha \colon \Lambda_{\mathcal{O}}X \to \Lambda_{\mathcal{O}}Y$ is the inverse of $\Lambda_{\mathcal{O}}\phi$. By Proposition 14.6 we can factor α as $\Lambda_{\mathcal{O}}X \xrightarrow{\Lambda_{\mathcal{O}}\psi} \Lambda_{\mathcal{O}}Y' \xrightarrow{\alpha'} \Lambda_{\mathcal{O}}Y$ where α' is free and ψ is an active morphism determined up to equivalence in $\mathcal{O}$ (and both $\Lambda_{\mathcal{O}}\psi$ and α' are equivalences since this is an active–inert factorization). Now the composite $\alpha\Lambda_{\mathcal{O}}\phi$ is the identity, so by Proposition 14.6 the composite $\phi\psi$ lies in the same component of $\mathrm{Act}_{\mathcal{O}}(Y)$ as id_Y, i.e. $\phi\psi$ must be an equivalence. Applying the same argument to ψ, we see that ψ has inverses on both sides in $\mathcal{O}$ and so is an equivalence, hence ϕ is also an equivalence. □

Proof of Proposition 15.4. By Proposition 15.3 the polynomial monad $T_{\mathcal{O}}$ is complete if and only if $j\colon \mathcal{U}(T_{\mathcal{O}}) \to \mathcal{W}(T_{\mathcal{O}})$ is faithful and all equivalences are in its image.

Since $\mathcal{O}$ is slim, the objects of $\mathcal{U}(T_{\mathcal{O}})$ are precisely the objects $\Lambda^{\mathrm{int}}_{\mathcal{O}}X$ for $X \in \mathcal{O}^{\mathrm{int}}$, by Corollary 14.8. To show that j is faithful, we must check that for all $X, Y \in \mathcal{O}^{\mathrm{int}}$, the map

$$\mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}^{\mathrm{int}}}(\mathcal{S})}(\Lambda^{\mathrm{int}}_{\mathcal{O}}X, \Lambda^{\mathrm{int}}_{\mathcal{O}}Y) \to \mathrm{Map}_{\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})}(\Lambda_{\mathcal{O}}X, \Lambda_{\mathcal{O}}Y)$$

is a monomorphism. Lemma 14.4 and Remark 14.5 imply that this map can be identified with the map

$$(\Lambda^{\mathrm{int}}_{\mathcal{O}}X)(Y) \to \operatorname*{colim}_{O \in \mathrm{Act}_{\mathcal{O}}(Y)} (\Lambda^{\mathrm{int}}_{\mathcal{O}}X)(O),$$

given by taking $(\Lambda^{\mathrm{int}}_{\mathcal{O}}X)(Y)$ to the component in the colimit corresponding to $\mathrm{id}_Y \in \mathrm{Act}_{\mathcal{O}}(Y)$. This component is of the form $(\mathcal{O}^{\simeq})_{/Y}$ and so is contractible, which means that the colimit decomposes as a disjoint union of $(\Lambda^{\mathrm{int}}_{\mathcal{O}}X)(Y)$ and the colimit over the other components of $\mathrm{Act}_{\mathcal{O}}(Y)$. This means j is indeed faithful.

Now suppose $\alpha\colon \Lambda_{\mathcal{O}}X \to \Lambda_{\mathcal{O}}X'$ is an equivalence in $\mathcal{W}(T_{\mathcal{O}})$. Then by Proposition 14.6 we can factor α as

$$\Lambda_{\mathcal{O}}X \xrightarrow{\Lambda_{\mathcal{O}}\phi} \Lambda_{\mathcal{O}}Y \xrightarrow{j\psi} \Lambda_{\mathcal{O}}X',$$

where ϕ is active and both $\Lambda_{\mathcal{O}}\phi$ and $j\psi$ are equivalences (since this is in particular an active–inert factorization). Then Lemma 15.5 implies that ϕ is an equivalence in $\mathcal{O}$; but then ϕ is also inert, and so the commutative square in Lemma 14.3 implies that $\Lambda_{\mathcal{O}}\phi$ is $j(\Lambda^{\mathrm{int}}_{\mathcal{O}}\phi)$. Thus α is in the image of j, as required. □

Remark 15.6. It follows from Proposition 15.3 that for any slim extendable pattern $\mathcal{O}$, the morphism $\tau_{T_{\mathcal{O}}}\colon T_{\mathcal{O}} \to T_{\overline{\mathcal{O}}}$ is an equivalence, i.e. the extendable patterns $\mathcal{O}$ and $\overline{\mathcal{O}}$ correspond to the same monad. The saturated pattern $\overline{\mathcal{O}}$ is thus a canonical pattern associated to the free Segal $\mathcal{O}$-space monad $T_{\mathcal{O}}$.

Corollary 15.7. *The natural transformation $\tau\colon \mathrm{id} \to \mathfrak{MP}$ exhibits the full subcategory* cPolyMnd *as a localization of* PolyMnd*.*

Proof. Let $L := \mathfrak{MP}$; then the essential image of L is precisely cPolyMnd: by Proposition 15.3 the image of L contains all complete polynomial monads, while all monads in the image of L are complete by Proposition 15.4.

To see that L and τ exhibit cPolyMnd as a localization, we again apply the criterion of [29, Proposition 5.2.7.4](3). We must thus show that the two morphisms

$$\tau_{LT}, L(\tau_T)\colon LT \to LLT$$

are both equivalences for all T in PolyMnd. For τ_{LT} this holds by Proposition 15.3, since LT is complete, while for $L(\tau_T)$ it holds since $\mathfrak{P}(\tau_T)$ is an equivalence (given by restricting the equivalence $\mathrm{Alg}_T(\mathcal{S}^{\mathcal{I}}) \xrightarrow{\sim} \mathrm{Seg}_{\mathcal{W}(T)}(\mathcal{S})$ to a full subcategory). □

Theorem 15.8. *The functors $\mathfrak{M}$ and $\mathfrak{P}$ restrict to give an equivalence*

$$\mathrm{cPolyMnd} \simeq \mathrm{AlgPatt}^{\mathrm{Seg}}_{\mathrm{sat}}.$$

Proof. We have shown that $\mathfrak{M}\mathcal{O}$ is always complete and $\mathfrak{P}T$ is always saturated, so the functors do restrict to these full subcategories. Moreover, we know that $\sigma_{\mathcal{O}}$ is an equivalence if and only if $\mathcal{O}$ is saturated, and τ_T is an equivalence if and only if T is complete. These natural transformations therefore restrict to natural equivalences on the full subcategories of saturated patterns and complete polynomial monads, and hence exhibit the restrictions of $\mathfrak{P}$ and $\mathfrak{M}$ as inverse equivalences. □

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