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∞-Operads via symmetric sequences

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Abstract

We construct a generalization of the Day convolution tensor product of presheaves that works for certain double ∞-categories. Using this construction, we obtain an ∞-categorical version of the well-known description of (one-object) operads as associative algebras in symmetric sequences; more generally, we show that (enriched) ∞-operads with varying spaces of objects can be described as associative algebras in a double ∞-category of symmetric collections.

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1 Introduction

The theory of operads is a general framework for encoding and working with algebraic structures, first introduced in the early 70s in order to describe homotopy-coherent algebraic operations on topological spaces [5,46]. Since then, the theory has found many applications in diverse areas of mathematics—aside from algebraic topology, where operads in topological spaces, simplicial sets, and spectra have numerous uses (see for example [5,7,17,46,50], among many others), operads in vector spaces and chain complexes (also known as *linear operads* and *dg-operads*, respectively) are by now a well-studied topic in algebra (see for instance [28,42]), with applications in mathematical physics (cf. [45]) and algebraic geometry (e.g. [41]), while operads in sets have become a standard tool in combinatorics (cf. [29,47]).

Classically, an operad $\mathbf{O}$ in a symmetric monoidal category $\mathbf{C}$ consists of a sequence $\mathbf{O}(n)$ of objects of $\mathbf{C}$, where the symmetric group Σ_n acts on $\mathbf{O}(n)$ (this data is called a *symmetric sequence*) together with a unital and associative composition operation. If $\mathbf{C}$ has colimits indexed by groupoids and the tensor product preserves these, then this data can be conveniently encoded using the *composition product* of symmetric sequences. This is a monoidal structure on symmetric sequences, given by the formula[1]

$$(X \circ Y)(n) \cong \coprod_{k=0}^{\infty} \left(\coprod_{i_1+\cdots+i_k=n} (X(i_1) \otimes \cdots \otimes X(i_k)) \times_{\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}} \Sigma_n \right) \otimes_{\Sigma_k} Y(k);$$

the unit is the symmetric sequence

$$\mathbb{1}(n) = \begin{cases} \emptyset, & n \neq 1 \\ \mathbb{1}, & n = 1 \end{cases},$$

where $\mathbb{1}$ is the unit in $\mathbf{C}$. As first observed by Kelly [40], an operad is then precisely an associative algebra with respect to $\circ$: the multiplication map $\mathbf{O} \circ \mathbf{O} \to \mathbf{O}$ is given by a family of equivariant maps

$$\mathbf{O}(k) \otimes \mathbf{O}(i_1) \otimes \cdots \otimes \mathbf{O}(i_k) \to \mathbf{O}(i_1 + \cdots + i_k),$$

supplying the operadic composition maps, and similarly the unit map $\mathbb{1} \to \mathbf{O}$ corresponds to a unit $\mathbb{1} \to \mathbf{O}(1)$.

In homotopical settings, this classical notion of operads has a number of shortcomings, analogous to those afflicting topological or simplicial categories when we want to work with them only up to homotopy (i.e. consider them as models for ∞-categories). This motivates the introduction of a fully homotopy–coherent version of operads, known as ∞-*operads*. Just as in the case of ∞-categories, there are several useful models for ∞-operads, including those of Lurie [44] (which is currently by far the best-developed), Moerdijk–Weiss [48], Cisinki–Moerdijk [14], and Barwick [3]. These authors only consider ∞-operads in spaces, but the formalism has recently been extended to cover ∞-operads in other symmetric monoidal ∞-categories in [10].

The goal of the present paper is to provide another point of view on (enriched) ∞-operads, by extending to the higher-categorical setting the description of operads as associative algebras in symmetric sequences:

[1] Note that this is the *reverse* ordering of the product compared to many references; this convention corresponds to the one that naturally appears in our ∞-categorical construction.

Theorem *Let $\mathbb{F}^{\simeq}$ be the groupoid of finite sets and bijections. If $\mathcal{V}$ is a symmetric monoidal ∞-category compatible with colimits indexed by ∞-groupoids[2], then there exists a monoidal structure on the ∞-category* $\mathrm{Fun}(\mathbb{F}^{\simeq}, \mathcal{V})$ *of symmetric sequences such that associative algebras are $\mathcal{V}$-enriched ∞-operads. Moreover, the tensor product is described by the same formula as above.*

More precisely, this gives a description of ∞-operads with a single object. It is often convenient to consider the more general notion of operads with many objects (also known as *coloured operads* or *symmetric multicategories*), and the term ∞-operad typically refers to the higher-categorical version of these generalized objects, which also have a description as associative algebras: For a set S, let

$$\mathbb{F}_S^{\simeq} := \coprod_{n=0}^{\infty} S^{\times n}_{h\Sigma_n}$$

denote the groupoid with objects lists $(s_1, \ldots, s_n)$ $(s_i \in S)$ and with a morphism $(s_1, \ldots, s_n) \to (s'_1, \ldots, s'_m)$ given by a bijection $\sigma\colon \{1, \ldots, n\} \xrightarrow{\sim} \{1, \ldots, m\}$ in $\mathbb{F}^{\simeq}$ such that $s_i = s'_{\sigma(i)}$. Then a (symmetric) *S-collection* (or *S-coloured symmetric sequence*) in $\mathbf{V}$ is a functor $\mathbb{F}_S^{\simeq} \times S \to \mathbf{V}$. The category $\mathrm{Fun}(\mathbb{F}_S^{\simeq} \times S, \mathbf{V})$ again has a composition product $\circ$, given by a more complicated version of the formula we gave above, such that an operad with S as its set of objects is precisely an associative algebra for this monoidal structure. Our work also gives an ∞-categorical version of this many-object composition product.

More generally, we can describe operads with varying spaces of objects as associative algebras in a *double category*. We will call a functor $\mathbb{F}_S^{\simeq} \times T \to \mathbf{V}$ an (S, T)-*collection* in $\mathbf{V}$. Then we can define a double category $\mathrm{COLL}(\mathbf{V})$ as follows:

- Objects are sets, and vertical morphisms are maps of sets.
- Horizontal morphisms from S to T are (S, T)-collections.
- Composition of horizontal morphisms is given by a version of the composition product.

An associative algebra in $\mathrm{COLL}(\mathbf{V})$ consists of a set S together with an associative algebra in the category of horizontal endomorphisms of S with composition as monoidal structure, i.e. an associative algebra in S-collections with the composition product. Thus associative algebras are precisely operads, and moreover morphisms of algebras in $\mathrm{COLL}(\mathbf{V})$ are precisely functors of operads. We will produce an ∞-categorical version of this structure:

Theorem *For any symmetric monoidal ∞-category $\mathcal{V}$ compatible with colimits indexed by ∞-groupoids there is a double ∞-category* $\mathrm{COLL}(\mathcal{V})$ *such that* $\mathrm{Alg}(\mathrm{COLL}(\mathcal{V}))$ *is the ∞-category of $\mathcal{V}$-enriched ∞-operads.*[3]

The double ∞-category $\mathrm{COLL}(\mathcal{V})$ admits the same description as its analogue for ordinary categories, except with ∞-groupoids as objects.

In a sequel to this paper [35] we apply this description of ∞-operads to study *algebras* over enriched ∞-operads. In addition, we hope that it can serve as a starting point for a better understanding of *bar-cobar* (or *Koszul*) *duality* for ∞-operads. Over a field of characteristic zero, Koszul duality for dg-operads was introduced by Ginzburg and Kapranov [28], and

2 By this we mean that the underlying ∞-category $\mathcal{V}$ has colimits indexed by ∞-groupoids, and the tensor product preserves such colimits in each variable.

3 In this paper we are considering ∞-operads as *algebraic* objects, i.e. we are not inverting the fully faithful and essentially surjective morphisms or imposing a completeness condition. In the terminology of Ayala–Francis [1], $\mathrm{Alg}(\mathrm{COLL}(\mathcal{V}))$ is the ∞-category of "flagged $\mathcal{V}$-∞-operads".

is by now well understood using model-categorical methods (see e.g. [21,22,27,42,52]). As a first step towards an ∞-categorical approach to Koszul duality, here we construct a bar-cobar adjunction between ∞-operads and ∞-cooperads. Following the approach proposed by Francis and Gaitsgory [20], we obtain this as the bar-cobar adjunction between associative algebras and coassociative coalgebras (constructed in great generality in [44, Sect. 5.2.2]) applied to our monoidal ∞-category of symmetric sequences. This seems likely to agree with existing constructions not only in chain complexes over a field of characteristic 0, but also in other settings such as spectra [8,9], where it is closely related to Goodwillie calculus [7], as well as in $K(n)$-local spectra, where bar-cobar duality has been constructed and applied in work of Heuts [37].

1.1 Overview of results

Let us now give a more detailed overview of the results of this paper. The starting point for our construction is the "coordinate-free" definition of the composition product due to Dwyer and Hess [18, Sect. A.1]. They observe that, if $\mathbb{F}^{[1],\simeq}$ denotes the groupoid of morphisms of finite sets and $\mathbb{F}^{[2],\simeq}$ denotes the groupoid of composable pairs of morphisms of finite sets, then:

- Symmetric sequences in Set are the same thing as symmetric monoidal functors $\mathbb{F}^{[1],\simeq} \to \mathrm{Set}$, with respect to the disjoint union in $\mathbb{F}^{[1],\simeq}$ and the cartesian product of sets.
- Under this identification the composition product of X and Y corresponds (by [18, Lemma A.4]) to the left Kan extension, along the functor $\mathbb{F}^{[2],\simeq} \to \mathbb{F}^{[1],\simeq}$ given by composition, of the restriction of $X \times Y$ from $\mathbb{F}^{[1],\simeq} \times \mathbb{F}^{[1],\simeq}$ to $\mathbb{F}^{[2],\simeq}$. In other words,
$$(X \circ Y)(A \to C) \cong \operatorname*{colim}_{(A \to B \to C) \in \mathbb{F}^{[2],\simeq}_{(A \to C)}} X(A \to B) \times Y(B \to C).$$
If $C \cong *$, then the groupoid $\mathbb{F}^{[2],\simeq}_{(A \to *)}$ of factorizations of $A \to *$ is the groupoid of maps $A \to B$ and isomorphisms $B \xrightarrow{\sim} B'$ under A. An isomorphism class of such objects corresponds to a decomposition $|A| = i_1 + \cdots + i_k$ where $k = |B|$, with a division of A into subsets of size i_j. This can be rewritten to recover the previous formula (with the division of A corresponding to the product with Σ_n for a given partition of $n = |A|$).

After a slight reformulation this description is closely related to Barwick's indexing category $\Delta_{\mathbb{F}}$ for ∞-operads, introduced in [3]. This is the category with objects sequences $S_0 \to S_1 \to \cdots \to S_n$ of morphisms of finite sets, with a map $(S_0 \to \cdots \to S_n) \to (T_0 \to \cdots \to T_m)$ given by a map $\phi\colon [n] \to [m]$ in Δ and injective morphisms $S_i \to T_{\phi(i)}$ such that the squares

$$\begin{tikzcd} S_i \arrow[r] \arrow[d] & S_j \arrow[d] \\ T_{\phi(i)} \arrow[r] & T_{\phi(j)} \end{tikzcd}$$

are cartesian. If $(\Delta_{\mathbb{F}})_{[n]}$ denotes the fibre at $[n]$ of the obvious projection $\Delta_{\mathbb{F}} \to \Delta$, then:

- Symmetric sequences in Set are the same thing as functors $X\colon (\Delta_{\mathbb{F}})^{\mathrm{op}}_{[1]} \to \mathrm{Set}$ such that for every object $S \to T$ the map
$$X(S \to T) \to \prod_{i \in T} X(S_i \to *),$$

induced by the morphisms

$$\begin{tikzcd} S_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \{i\} \arrow[d] \\ S \arrow[r] & T, \end{tikzcd}$$

is an isomorphism.

- Under this identification the composition product of X and Y corresponds to the left Kan extension, along the functor $(\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[2]} \to (\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[1]}$ corresponding to $d_1 \colon [1] \to [2]$, of the restriction of $X \times Y$ along the functor $(\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[2]} \to (\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[1]} \times (\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[1]}$ corresponding to (d_2, d_0). In other words,

$$(X \circ Y)(A \to C) \cong \operatorname*{colim}_{(A' \to B \to C') \in ((\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[2]})_{/(A \to C)}} X(A' \to B) \times Y(B \to C').$$

This is equivalent to the previous description since the inclusion $\mathbb{F}^{[2],\simeq}_{(A \to C)} \to (\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[2],/(A \to C)}$ is cofinal: given an object ξ in the target, which is a diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[d] & & C \arrow[d] \\ A' \arrow[r] & B \arrow[r] & C', \end{tikzcd}$$

where the square is cartesian, the category $(\mathbb{F}^{[2],\simeq}_{(A \to C)})_{\xi/}$ is a contractible groupoid with the single object given by the factorization $A \to B \times_{C'} C \to C$.

The projection $\mathbb{\Delta}_{\mathbb{F}} \to \mathbb{\Delta}$ is a Grothendieck fibration, and the corresponding functor $\Phi \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Cat}$ is a double category, in the sense that it satisfies the Segal condition

$$\Phi_n \xrightarrow{\sim} \Phi_1 \times_{\Phi_0} \cdots \times_{\Phi_0} \Phi_1.$$

We will obtain the composition product by applying to this double category a general construction of monoidal structures on functor categories arising from certain double ∞-categories. In fact, our construction will produce a canonical double ∞-category of which this monoidal ∞-category is a piece, with the full double ∞-category describing ∞-operads with varying spaces of objects.

The construction of this double ∞-category can be seen a variation of the *Day convolution* [16] structure on functor categories: If $\mathbf{C}$ is a small monoidal category and $\mathbf{V}$ is a monoidal category compatible with colimits, then the functor category $\mathrm{Fun}(\mathbf{C}, \mathbf{V})$ has a tensor product, given for functors F and G as the left Kan extension along $\otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ of the composite

$$\mathbf{C} \times \mathbf{C} \xrightarrow{F \times G} \mathbf{V} \times \mathbf{V} \xrightarrow{\otimes} \mathbf{V}.$$

This monoidal structure has the property that an associative algebra in $\mathrm{Fun}(\mathbf{C}, \mathbf{V})$ is the same thing as a lax monoidal functor $\mathbf{C} \to \mathbf{V}$; more generally, the Day convolution has a universal property for algebras over non-symmetric operads.

Day convolution (in the symmetric monoidal setting) was implemented in the ∞-categorical context by Glasman [30].[4] In this paper we extend this to a construction of Day convolution for a class of double ∞-categories:

[4] More recently, Lurie has also given a more general account [44, Sect. 2.2.6].

Theorem 1.1.1 *Suppose $\mathcal{M} \to \mathbb{\Delta}^{\mathrm{op}}$ is a suitable double ∞-category. Then there is a double ∞-category $\widehat{\mathcal{M}}_{\mathcal{S}}$ such that for any generalized non-symmetric ∞-operad $\mathcal{O}$ we have a natural equivalence*[5]

$$\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathcal{M}}_{\mathcal{S}}) \simeq \mathrm{Seg}_{\mathcal{O}\times_{\mathbb{\Delta}^{\mathrm{op}}}\mathcal{M}}(\mathcal{S}).$$

The objects of $\widehat{\mathcal{M}}_{\mathcal{S}}$ are functors $\mathcal{M}_0 \to \mathcal{S}$ and the vertical morphisms are natural transformations of such functors. A horizontal morphism from F to G is a functor $\mathcal{M}_{1,F,G} \to \mathcal{S}$, where $\mathcal{M}_{1,F,G} \to \mathcal{M}_1$ is the left fibration for the composite functor

$$\mathcal{M}_1 \xrightarrow{(d_{1,!},d_{0,!})} \mathcal{M}_0 \times \mathcal{M}_0 \xrightarrow{F\times G} \mathcal{S}.$$

This theorem summarizes the results of Sect. 3: We construct these double ∞-categories in Sect. 3.2 using an unfolding construction introduced in Sect. 3.1, and prove the universal property in Sect. 3.3. Note that the precise meaning of "suitable" we need is quite restrictive. We also show in Sect. 3.4 that we can extract from $\widehat{\mathcal{M}}_{\mathcal{S}}$ a family of monoidal ∞-categories and lax monoidal functors which suffices to describe associative algebras in $\widehat{\mathcal{M}}_{\mathcal{S}}$. Moreover, we consider enriched versions of the theorem, with more general targets than $\mathcal{S}$, in Sect. 3.5.

To obtain our double ∞-categories we use results on ∞-categories of spans due to Barwick [4], and Sect. 2 is devoted to a review of this work, with some slight variations, together with a brief review of non-symmetric ∞-operads and related structures.

In Sect. 4 we apply our results on Day convolution to ∞-operads. In Sect. 4.1 we describe non-enriched ∞-operads as associative algebras in a double ∞-categories of collections in $\mathcal{S}$, and in Sect. 4.2 we extend this to a description of enriched ∞-operads. More precisely, we obtain an equivalence between associative algebras in a double ∞-category of collections and ∞-operads in the sense of Barwick [3], as generalized to enriched ∞-operads in [10]. In Sect. 4.3 we then apply this description of ∞-operads to obtain the bar-cobar adjunction between ∞-operads and ∞-cooperads.

As a warm-up to this description of ∞-operads, in Sect. 3.6 we also consider an additional application of our Day convolution construction, by showing that enriched ∞-categories can be described as associative algebras.

1.2 Related work

There are at least two other approaches to constructing the composition product on symmetric sequences ∞-categorically:

Composition product from free presentably symmetric monoidal categories

An alternative approach to defining the composition product of S-coloured symmetric sequences in Set starts with the observation that $\mathrm{Fun}(\coprod_{n=0}^{\infty} S^n_{h\Sigma_n}, \mathrm{Set})$ is the free presentably symmetric monoidal category generated by S. If $\mathbf{C}$ is a presentably symmetric monoidal category we therefore have a natural equivalence

$$\mathrm{Fun}(S, \mathbf{C}) \simeq \mathrm{Fun}^{L,\otimes}(\mathrm{Fun}(\coprod_{n=0}^{\infty} S^n_{h\Sigma_n}, \mathrm{Set}), \mathbf{C}),$$

[5] The right-hand side is the ∞-category of "Segal $\mathcal{O} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{M}$-spaces", which are functors $\mathcal{O} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{M} \to \mathcal{S}$ satisfying certain Segal-type limit conditions; see Definition 2.1.18 for the precise definition.

where the right-hand side denotes the category of colimit-preserving symmetric monoidal functors. Taking **C** to be

$$\mathrm{Fun}(\mathbb{F}_S^{\simeq}, \mathrm{Set}) \cong \mathrm{Fun}\left(\coprod_{n=0}^{\infty} S^n_{h\Sigma_n}, \mathrm{Set}\right),$$

we get a natural equivalence

$$\mathrm{Fun}(\mathbb{F}_S^{\simeq} \times S, \mathrm{Set}) \simeq \mathrm{Fun}\left(S, \mathrm{Fun}(\mathbb{F}_S^{\simeq}, \mathrm{Set})\right) \simeq \mathrm{Fun}^{L,\otimes}\left(\mathrm{Fun}(\mathbb{F}_S^{\simeq}, \mathrm{Set}), \mathrm{Fun}(\mathbb{F}_S^{\simeq}, \mathrm{Set})\right).$$

Here the right-hand side has an obvious monoidal structure given by composition of functors, and this corresponds under the equivalence to the composition product of S-coloured symmetric sequences. This construction is described in [2, Sect. 2.3]. The one-object variant is much better known; it is attributed to Carboni in the "Author's Note" for [40], and it is also found in Trimble's preprint [51]. There is also an enriched version of this construction, for (coloured) symmetric sequences in a presentably symmetric monoidal category. More recently, this approach has been further developed in [19,23] where it is shown to arise from a 2-categorical construction that produces a 2-category of operads with varying sets of objects (but with *bimodules* of operads as morphisms rather than functors).

In the ∞-categorical setting, it is not hard to see that $\mathrm{Fun}(\coprod_{n=0}^{\infty} X^n_{h\Sigma_n}, \mathcal{S})$ is again the free presentably symmetric monoidal ∞-category generated by a space X. One can thus take the same route to obtain a composition product on X-coloured symmetric sequences in the ∞-category of spaces. In the one-object case this approach (including its enriched variant) is worked out in Brantner's thesis [6, Sect. 4.1.2]. However, this approach has not yet been compared to any of the established models for ∞-operads.

Polynomial monads

In [26] we show that ∞-operads with a fixed space of objects X are equivalent to analytic monads on the slice ∞-category $\mathcal{S}_{/X}$. These analytic monads can be viewed as associative algebras under composition in an ∞-category of analytic endofunctors of $\mathcal{S}_{/X}$. The latter can be identified with X-coloured symmetric sequences in $\mathcal{S}$, so this gives an alternative description of ∞-operads as associative algebras for the composition product. Compared to our approach here, this has a number of advantages:

- it makes it clear that an ∞-operad can be recovered from its free algebra monad,
- it clarifies the relation between ∞-operads and trees (because free analytic monads can be described in terms of trees).

It also seems likely that versions of polynomial monads in other ∞-topoi can be used to define operad-like structures that occur in equivariant and motivic homotopy theory. On the other hand, polynomial monads do not seem to extend usefully to give a description of *enriched* ∞-operads.

2 Background on spans and Non-symmetric ∞-operads

In this section we first review non-symmetric ∞-operads and related structures in Sect. 2.1, and then recall some definitions and results regarding spans from [4], with some minor variations to get the generality we need in the next section.

2.1 Review of non-symmetric ∞-operads

For the reader's convenience, we will briefly review some definitions and results related to non-symmetric ∞-operads that we will make frequent use of below. For more details, as well as motivation, we refer the reader to [24,32,44].

Notation 2.1.1 $\mathbb{\Delta}$ denotes the standard simplicial indexing category, i.e. the category of ordered sets $[n] = \{0, 1, \ldots, n\}$ and order-preserving maps. We say a map $\phi\colon [n] \to [m]$ is *inert* if it is the inclusion of a subinterval, i.e. $\phi(i) = \phi(0) + i$ for all i, and *active* if it preserves the end-points, i.e. $\phi(0) = 0$, $\phi(n) = m$. The active and inert maps form a factorization system on $\mathbb{\Delta}$—every morphism factors uniquely as an active map followed by an inert map. We write $\mathbb{\Delta}^{\mathrm{int}}$ for the subcategory of $\mathbb{\Delta}$ containing only the inert maps, and $\mathbb{\Delta}^{\mathrm{el}}$ for the full subcategory of $\mathbb{\Delta}^{\mathrm{int}}$ containing only the objects [0] and [1]; we also use the notation

$$\mathbb{\Delta}^{\mathrm{el}}_{/[n]} := \mathbb{\Delta}^{\mathrm{el}} \times_{\mathbb{\Delta}^{\mathrm{int}}} \mathbb{\Delta}^{\mathrm{int}}_{/[n]}$$

for the category of inert maps to $[n]$ from [0] and [1]

Definition 2.1.2 For $0 \leq i \leq j \leq n$ we write ρ_{ij} for the inclusion $[j-i] \cong \{i, i+1, \ldots, j\} \hookrightarrow [n]$. If $\mathcal{C}$ is an ∞-category with products, then an *associative monoid* in $\mathcal{C}$ is a functor $A\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}$ such that for every n the map $A_n \to \prod_{i=1}^{n} A_1$, induced by the maps $\rho_{(i-1)i}\colon [1] \to [n]$, is an equivalence.

Definition 2.1.3 If $\mathcal{C}$ is an ∞-category with finite limits, then a *category object* in $\mathcal{C}$ is a functor $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}$ such that for every n the map

$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

induced by the maps $\rho_{(i-1)i}$ and ρ_{ii}, is an equivalence.

Remark 2.1.4 A category object in the ∞-category $\mathcal{S}$ of spaces is a *Segal space* in the sense of Rezk [49]. The structure of a Segal space describes precisely the "algebraic" structure of an ∞-category, i.e. a homotopy-coherent composition with identities, but to capture the correct equivalences between ∞-categories one must invert the fully faithful and essentially surjective maps between Segal spaces, or equivalently restrict to the full subcategory of *complete* Segal spaces.

Definition 2.1.5 A *monoidal ∞-category* is a cocartesian fibration $\mathcal{C}^{\otimes} \to \mathbb{\Delta}^{\mathrm{op}}$ such that the corresponding functor $\mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is an associative monoid. Similarly, a *double ∞-category* is a cocartesian fibration $\mathcal{M} \to \mathbb{\Delta}^{\mathrm{op}}$ such that the corresponding functor $\mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is a category object.

Notation 2.1.6 We will use the following terminology to describe double ∞-categories $\mathcal{M} \to \mathbb{\Delta}^{\mathrm{op}}$:

- an object of $\mathcal{M}_0$ is an *object* of the double ∞-category,
- a morphism of $\mathcal{M}_0$ is a *vertical morphism* of the double ∞-category
- an object of $\mathcal{M}_1$ is a *horizontal morphism*,
- a morphism in $\mathcal{M}_1$ is a *square*,
- composition of vertical morphisms is composition in the ∞-category $\mathcal{M}_0$,
- vertical composition of squares is composition in $\mathcal{M}_1$

- composition of horizontal morphisms, as well as horizontal composition of squares, is given by the functor

$$\mathcal{M}_1 \times_{\mathcal{M}_0} \mathcal{M}_1 \xleftarrow{\sim} \mathcal{M}_2 \xrightarrow{d_{1,!}} \mathcal{M}_1.$$

Notation 2.1.7 Given objects $X, Y \in \mathcal{M}_0$, we write $\mathcal{M}(X, Y)$ for the fibre of $\mathcal{M}_1 \xrightarrow{(d_{1,!}, d_{0,!})} \mathcal{M}_0 \times \mathcal{M}_0$ at (X, Y), and call this the ∞-category of horizontal morphisms from X to Y. Given its simplicial origin, it is usually less confusing to write composition of horizontal morphisms in the non-standard order, and we denote it

$$- \odot_Y -\colon \mathcal{M}(X, Y) \times \mathcal{M}(Y, Z) \to \mathcal{M}(X, Z).$$

We will also write $\mathbb{1}_X \in \mathcal{M}(X, X)$ for the horizontal identity.

Definition 2.1.8 A *generalized non-symmetric ∞-operad* is a functor $p\colon \mathcal{O} \to \mathbb{\Delta}^{\mathrm{op}}$ such that

(i) $\mathcal{O}$ has p-cocartesian morphisms over all inert maps in $\mathbb{\Delta}^{\mathrm{op}}$,
(ii) for every n the functor $\mathcal{O}_{[n]} \to \mathcal{O}_{[1]} \times_{\mathcal{O}_{[0]}} \cdots \times_{\mathcal{O}_{[0]}} \mathcal{O}_{[1]}$, induced by the cocartesian morphisms over the maps $\rho_{(i-1)i}$ and ρ_{ii}, is an equivalence,
(iii) for every $X \in \mathcal{O}_{[n]}$, $Y \in \mathcal{O}_{[m]}$ and $\phi\colon [n] \to [m]$ in $\mathbb{\Delta}$, the map

$$\mathrm{Map}^{\phi}_{\mathcal{O}}(Y, X) \to \mathrm{Map}^{\rho_{01}\phi}_{\mathcal{O}}(\rho_{01,!}Y, X) \times_{\mathrm{Map}^{\rho_{11}\phi}(\rho_{11,!}Y, X)} \cdots \times_{\mathrm{Map}^{\rho_{(n-1)(n-1)}\phi}(\rho_{(n-1)(n-1),!}Y, X)} \mathrm{Map}^{\rho_{(n-1)n}\phi}_{\mathcal{O}}(\rho_{(n-1)n,!}Y, X)$$

is an equivalence, where $X \to \rho_{ij,!}X$ is the p-cocartesian morphism over the inert map ρ_{ij} and $\mathrm{Map}^{\phi}_{\mathcal{O}}(Y, X)$ denotes the fibre at ϕ of the map $\mathrm{Map}_{\mathcal{O}}(Y, X) \to \mathrm{Map}_{\mathbb{\Delta}^{\mathrm{op}}}([m], [n])$.

We refer to the cocartesian morphisms over inert morphisms in $\mathbb{\Delta}^{\mathrm{op}}$ as *inert* morphisms in $\mathcal{O}$. A morphism of generalized non-symmetric ∞-operads is a functor over $\mathbb{\Delta}^{\mathrm{op}}$ that preserves inert morphisms; we also refer to a morphism of generalized non-symmetric ∞-operads $\mathcal{O} \to \mathcal{P}$ as an $\mathcal{O}$-*algebra* in $\mathcal{P}$ and write $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ for the ∞-category of these. More generally, if $\mathcal{O}$ and $\mathcal{P}$ are generalized non-symmetric ∞-operads over $\mathcal{Q}$ we write $\mathrm{Alg}_{\mathcal{O}/\mathcal{Q}}(\mathcal{P})$ for the analogous ∞-category of commutative triangles of morphisms of generalized non-symmetric ∞-operads

$$\begin{tikzcd} \mathcal{O} \arrow[rr] \arrow[dr] & & \mathcal{P} \arrow[dl] \\ & \mathcal{Q}. & \end{tikzcd}$$

Definition 2.1.9 A *non-symmetric ∞-operad* is a generalized non-symmetric ∞-operad $\mathcal{O}$ such that $\mathcal{O}_{[0]} \simeq *$.

Notation 2.1.10 If $\mathcal{O}$ is a generalized non-symmetric ∞-operad and x is an object of $\mathcal{O}_n$, we will often write $x \to x_{ij}$ for the cocartesian morphism over ρ_{ij} for $0 \leq i \leq j \leq n$.

Lemma 2.1.11 *Suppose $\mathcal{O}$ is a generalized non-symmetric ∞-operad. Let $\mathcal{O}'_0$ be a full subcategory of $\mathcal{O}_0$ and $\mathcal{O}'_1$ be a full subcategory of $\mathcal{O}_1$ such that for $x \in \mathcal{O}'_1$ the objects x_{00} and x_{11} are in $\mathcal{O}'_0$. If $\mathcal{O}'$ denotes the full subcategory of $\mathcal{O}$ spanned by objects x such that $x_{ii} \in \mathcal{O}'_0$ and $x_{(i-1)i}$ is in $\mathcal{O}'_1$ for all i, then*

(i) *$\mathcal{O}'$ is also a generalized non-symmetric ∞-operad,*
(ii) *the inclusion $j\colon \mathcal{O}' \hookrightarrow \mathcal{O}$ preserves inert morphisms,*

(iii) *for any generalized non-symmetric ∞-operad $\mathcal{P}$ the functor*

$$j_* \colon \mathrm{Alg}_{\mathcal{P}}(\mathcal{O}') \to \mathrm{Alg}_{\mathcal{P}}(\mathcal{O})$$

given by composition with j is fully faithful, with image the algebras $\mathcal{P} \to \mathcal{O}$ such that the restrictions $\mathcal{P}_i \to \mathcal{O}_i$ factor through $\mathcal{O}'_i$ for $i = 0, 1$.

Proof By definition, we have pullback squares

$$\begin{tikzcd} \mathcal{O}'_n \arrow[r] \arrow[d] & \mathcal{O}_n \arrow[d, "\sim"] \\ \mathcal{O}'_1 \times_{\mathcal{O}'_0} \cdots \times_{\mathcal{O}'_0} \mathcal{O}'_1 \arrow[r] & \mathcal{O}_1 \times_{\mathcal{O}_0} \cdots \times_{\mathcal{O}_0} \mathcal{O}_1, \end{tikzcd}$$

so that the left vertical map is an equivalence. Condition (iii) in Definition 2.1.8 is also immediate from $\mathcal{O}'$ being a full subcategory. If x is in $\mathcal{O}'$ and $x \to y$ is an inert morphism in $\mathcal{O}$, then by the definition of $\mathcal{O}'$ the object y is also in $\mathcal{O}'$, so $\mathcal{O}'$ inherits cocartesian morphisms over inert morphisms from $\mathcal{O}$. This proves (i) and (ii), and (iii) is immediate from the definition of $\mathrm{Alg}_{\mathcal{P}}(\mathcal{O}')$ as a full subcategory of $\mathrm{Fun}_{/\Delta^{\mathrm{op}}}(\mathcal{P}, \mathcal{O}')$. □

Definition 2.1.12 If $\mathcal{C}$ is an ∞-category with finite products and $\mathcal{O}$ is a generalized non-symmetric ∞-operad, then an $\mathcal{O}$-*monoid* in $\mathcal{C}$ is a functor $M \colon \mathcal{O} \to \mathcal{C}$ such that for every $x \in \mathcal{O}_{[n]}$, the map $M(x) \to \prod_{i=1}^m M(x_{(i-1)i})$ induced by the cocartesian morphisms $x \to x_{(i-1)i}$ over $\rho_{(i-1)i}$, is an equivalence. We write $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ for the ∞-category of $\mathcal{O}$-monoids in $\mathcal{C}$, a full subcategory of $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$.

Definition 2.1.13 Let $\mathcal{O}$ be a generalized non-symmetric ∞-operad. An $\mathcal{O}$-*monoidal ∞-category* is a cocartesian fibration $\mathcal{U}^{\otimes} \to \mathcal{O}$ such that the corresponding functor $\mathcal{O} \to \mathrm{Cat}_{\infty}$ is an $\mathcal{O}$-monoid; for $X \in \mathcal{O}_{[1]}$ we often write $\mathcal{U}_X$ for the fibre of $\mathcal{U}^{\otimes}$ at X. Note that the composite $\mathcal{U}^{\otimes} \to \mathcal{O} \to \Delta^{\mathrm{op}}$ is again a generalized non-symmetric ∞-operad (and a double ∞-category if $\mathcal{O}$ is one). We call a morphism of generalized non-symmetric ∞-operads over $\mathcal{O}$ between $\mathcal{O}$-monoidal ∞-categories a *lax $\mathcal{O}$-monoidal functor*, and say that it is $\mathcal{O}$-*monoidal* if it preserves all cocartesian morphisms over $\mathcal{O}$.

Definition 2.1.14 If $\mathcal{V}^{\otimes} \to \mathcal{O}$ is an $\mathcal{O}$-monoidal ∞-category, we write $\mathcal{V}_{\otimes} \to \mathcal{O}^{\mathrm{op}}$ for the corresponding cartesian fibration. Then $\mathcal{V}^{\mathrm{op},\otimes} := (\mathcal{V}_{\otimes})^{\mathrm{op}} \to \mathcal{O}$ is again an $\mathcal{O}$-monoidal ∞-category; this describes the $\mathcal{O}$-monoidal structure on $\mathcal{V}_X^{\mathrm{op}}$ ($X \in \mathcal{O}_{[1]}$) given by the same operations as those on $\mathcal{V}_X$.

Proposition 2.1.15 *If $\mathcal{M}$ is a generalized non-symmetric ∞-operad and $\mathcal{C}$ is an ∞-category with products, then there is a natural equivalence*

$$\mathrm{Alg}_{\mathcal{M}}(\mathcal{C}) \simeq \mathrm{Mon}_{\mathcal{M}}(\mathcal{C}).$$

Proof This is a special case of [11, Proposition 5.1] (which generalizes the version for symmetric ∞-operads, [44, Proposition 2.4.1.7]). □

The ∞-categorical analogue of Day convolution was first constructed by Glasman [30] for symmetric monoidal ∞-categories. It was generalized by Lurie [44, Sect. 2.2.6] to $\mathcal{O}$-monoidal ∞-categories where $\mathcal{O}$ is a (symmetric) ∞-operad and further extended by Hinich to *flat* ∞-operads [39]. The following is a special case of another generalization, proved in [11]:

Proposition 2.1.16 *Let $\mathcal{O}$ be a generalized non-symmetric ∞-operad and $\mathcal{U}^{\otimes} \to \mathcal{O}$ an $\mathcal{O}$-monoidal ∞-category. There exists an $\mathcal{O}$-monoidal ∞-category $\mathcal{U}^{\otimes}_{\mathcal{S}} \to \mathcal{O}$, natural with respect to $\mathcal{O}$-monoidal functors, such that for $X \in \mathcal{O}_{[1]}$ we have $(\mathcal{U}^{\otimes}_{\mathcal{S}})_X \simeq \mathrm{Fun}(\mathcal{U}^{\otimes}_X, \mathcal{S})$ and with the universal property that for every generalized non-symmetric ∞-operad $\mathcal{P}$ over $\mathcal{O}$ we have a natural equivalence*

$$\mathrm{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{U}^{\otimes}_{\mathcal{S}}) \simeq \mathrm{Alg}_{\mathcal{P}\times_{\mathcal{O}}\mathcal{U}^{\otimes}}(\mathcal{S}) \simeq \mathrm{Mon}_{\mathcal{P}\times_{\mathcal{O}}\mathcal{U}^{\otimes}}(\mathcal{S}).$$

Moreover, if $\mathcal{P}(\mathcal{U})^{\otimes} := \mathcal{U}^{\mathrm{op},\otimes}_{\mathcal{S}}$ then there is a fully faithful $\mathcal{O}$-monoidal functor

$$\mathcal{U}^{\otimes} \hookrightarrow \mathcal{P}(\mathcal{U})^{\otimes},$$

given over $X \in \mathcal{O}_{[1]}$ by the Yoneda embedding $\mathcal{U}^{\otimes}_X \hookrightarrow \mathcal{P}(\mathcal{U}^{\otimes}_X)$.

Proof This is a special case of [11, Proposition 6.16 and Corollary 6.21]. □

Remark 2.1.17 The functor $\mathcal{U}^{\otimes}_1 \to \mathcal{O}_1$ is a cocartesian fibration. Let $U : \mathcal{O}_1 \to \mathrm{Cat}_{\infty}$ denote the corresponding functor. Then $\mathcal{U}^{\otimes}_{\mathcal{S},1} \to \mathcal{O}_1$ is the cartesian fibration for the functor $\mathrm{Fun}(U, \mathcal{S}) : \mathcal{O}_1^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$ defined by composition, or equivalently the cocartesian fibration for the induced functor given by the left adjoints, i.e. left Kan extensions along the functors $U(f)$ for f in $\mathcal{U}^{\otimes}_1$.

Definition 2.1.18 Let $\mathcal{C}$ be an ∞-category with pullbacks and $\mathcal{O}$ a generalized non-symmetric ∞-operad. A *Segal $\mathcal{O}$-object* in $\mathcal{C}$ is a functor $F : \mathcal{O} \to \mathcal{C}$ such that for every object $X \in \mathcal{O}$ over $[n] \in \mathbb{\Delta}$, the morphism

$$F(X) \to F(X_{01}) \times_{F(X_{11})} \cdots \times_{F(X_{(n-1)(n-1)})} F(X_{(n-1)n})$$

is an equivalence. We write $\mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$ spanned by the Segal $\mathcal{O}$-objects.

Proposition 2.1.19 *Let $\mathcal{O}$ be a generalized non-symmetric ∞-operad. The restriction functor*

$$\mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \to \mathrm{Fun}(\mathcal{O}_0, \mathcal{S})$$

is a cartesian fibration, and the fibre at $\Xi : \mathcal{O}_0 \to \mathcal{S}$ is equivalent to $\mathrm{Mon}_{\mathcal{O}_{\Xi}}(\mathcal{S})$ where $\mathcal{O}_{\Xi} \to \mathcal{O}$ is the left fibration for the functor $\mathcal{O} \to \mathcal{S}$ obtained as the right Kan extension of Ξ along the inclusion $\mathcal{O}_0 \hookrightarrow \mathcal{O}$.

Proof As [31, Theorem 7.5]. □

Definition 2.1.20 Let $\mathcal{O}$ be a generalized non-symmetric ∞-operad and let $\mathcal{U}^{\otimes}$ be an $\mathcal{O}$-monoidal ∞-category. We write

$$\mathrm{Algd}_{\mathcal{O}}(\mathcal{U}) \to \mathrm{Fun}(\mathcal{O}_0, \mathcal{S})$$

for the cartesian fibration corresponding to the functor $X \mapsto \mathrm{Alg}_{\mathcal{O}_X/\mathcal{O}}(\mathcal{U})$ and refer to its objects as $\mathcal{O}$-*algebroids* in $\mathcal{U}$.

Example 2.1.21 $\mathbb{\Delta}^{\mathrm{op}}$-algebroids in a monoidal ∞-category $\mathcal{V}$ are algebras in $\mathcal{V}$ for the family $\mathbb{\Delta}^{\mathrm{op}}_X$ $(X \in \mathcal{S})$ of generalized non-symmetric ∞-operads. These were called *categorical algebras* in [24], where they were used to model ∞-categories enriched in $\mathcal{V}$.

Remark 2.1.22 Propositions 2.1.19 and 2.1.15 identify $\mathrm{Algd}_{\mathcal{O}}(\mathcal{S})$ with $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$. If $\mathcal{U}^{\otimes}$ is a small $\mathcal{O}$-monoidal ∞-category then the natural equivalence of Proposition 2.1.16 gives an equivalence

$$\mathrm{Alg}_{\mathcal{O}_{\Xi}/\mathcal{O}}(\mathcal{U}^{\otimes}_{\mathcal{S}}) \simeq \mathrm{Alg}_{\mathcal{U}^{\otimes}_{\Xi}}(\mathcal{S}),$$

natural in Ξ, and so an equivalence

$$\mathrm{Algd}_{\mathcal{O}}(\mathcal{U}^{\otimes}_{\mathcal{S}}) \simeq \mathrm{Algd}_{\mathcal{U}^{\otimes}}(\mathcal{S}) \simeq \mathrm{Seg}_{\mathcal{U}^{\otimes}}(\mathcal{S}).$$

Combined with the $\mathcal{O}$-monoidal Yoneda embedding, we get:

Corollary 2.1.23 *Let $\mathcal{O}$ be a generalized non-symmetric ∞-operad and $\mathcal{U}^{\otimes}$ a small $\mathcal{O}$-monoidal ∞-category. Then there is a fully faithful functor*

$$\mathrm{Algd}_{\mathcal{O}}(\mathcal{U}) \hookrightarrow \mathrm{Seg}_{\mathcal{U}^{\mathrm{op},\otimes}}(\mathcal{S})$$

with image those Segal $\mathcal{U}^{\mathrm{op},\otimes}$-spaces Φ such that for every $x \in \mathcal{O}_{[1]}$, $p \in \Phi(x_{00})$, and $q \in \Phi(x_{11})$ the presheaf

$$\Phi_{x,p,q} \colon (\mathcal{U}^{\otimes}_{x})^{\mathrm{op}} \simeq \mathcal{U}^{\mathrm{op},\otimes}_{x} \xrightarrow{\Phi} \mathcal{S}_{/\Phi(x_{00})\times\Phi(x_{11})} \xrightarrow{(-)_{(p,q)}} \mathcal{S},$$

obtained by taking fibres at (p, q), is representable.

Definition 2.1.24 Let K be a collection of ∞-categories. Following [44, Definition 3.1.1.18] we say that an $\mathcal{O}$-monoidal ∞-category $\mathcal{V}^{\otimes}$ is *compatible with K-colimits* if

- the ∞-category $\mathcal{V}_X$ has K-colimits for every object $X \in \mathcal{O}_1$,
- for every active morphism $f \colon X \to Y$ in $\mathcal{O}$ with $X \in \mathcal{O}_n$ and $Y \in \mathcal{O}_1$, the functor

$$\prod_{i=1}^{n} \mathcal{V}_{X_{(i-1)i}} \simeq \mathcal{V}^{\otimes}_{X} \xrightarrow{f_!} \mathcal{V}_Y,$$

  induced by the cocartesian morphisms over f, preserves K-colimits in each variable.

Lemma 2.1.25 *Let $\pi \colon \mathcal{O} \to \mathbb{\Delta}^{\mathrm{op}}$ be a generalized non-symmetric ∞-operad.*

(i) *If for every active morphism $\phi \colon [1] \to [n]$ in $\mathbb{\Delta}$ and every $X \in \mathcal{O}_n$, there is a locally π-cocartesian morphism $X \to \phi_! X$ in $\mathcal{O}$, then π is a locally cocartesian fibration.*
(ii) *If in addition for every active map $\phi \colon [2] \to [n]$ and $X \in \mathcal{O}_n$, the canonical map*

$$(\phi d_1)_! X \to d_{1,!}\phi_! X$$

is an equivalence, then π is a cocartesian fibration.

Proof We first prove that $\mathcal{O}$ has locally cocartesian morphisms over any active map $\alpha \colon [n] \to [m]$ in $\mathbb{\Delta}$. Given $x \in \mathcal{O}_m$ and $y \in \mathcal{O}_n$, we have

$$\mathrm{Map}^{\alpha}_{\mathcal{O}}(x, y) \simeq \lim_{\rho_{ij}\in\mathbb{\Delta}^{\mathrm{el,op}}_{/[n]}} \mathrm{Map}^{\alpha_{ij}}_{\mathcal{O}}(x_{\alpha(i)\alpha(j)}, y_{ij})$$

where α_{ij} is the active part of $\alpha \circ \rho_{ij}$. By assumption we have locally cocartesian morphisms $x_{\alpha(i)\alpha(j)} \to \alpha_{ij,!}x_{\alpha(i)\alpha(j)}$ (if $i = j$ this is just the identity), so we can rewrite this as

$$\lim_{\rho_{ij}\in\mathbb{\Delta}^{\mathrm{el,op}}_{/[n]}} \mathrm{Map}_{\mathcal{O}_{j-i}}(\alpha_{ij,!}x_{\alpha(i)\alpha(j)}, y_{ij}) \simeq \mathrm{Map}_{\mathcal{O}_n}(\alpha_! x, y),$$

where $\alpha_! x$ is the object of $\mathcal{O}_n \simeq \lim_{\rho_{ij} \in \Delta^{\mathrm{el,op}}_{/[n]}} \mathcal{O}_{j-i}$ corresponding to the family of objects $\alpha_{ij,!} x_{\alpha(i)\alpha(j)}$. Thus we have a locally cocartesian morphism $x \to \alpha_! x$.

Next, suppose $\phi\colon [n] \to [m]$ is an arbitrary map in Δ, and let $[n] \xrightarrow{\alpha} [k] \xrightarrow{\iota} [m]$ be its active-inert factorization. Then for $x \in \mathcal{O}_m$ the composite $x \to \iota_! x \to \alpha_! \iota_! x$ is locally cocartesian over ϕ, where the first map is cocartesian over ι and the second is locally cocartesian over α: for $y \in \mathcal{O}_n$ we have an equivalence

$$\mathrm{Map}^{\phi}_{\mathcal{O}}(x, y) \simeq \mathrm{Map}^{\alpha}_{\mathcal{O}}(\iota_! x, y) \simeq \mathrm{Map}_{\mathcal{O}_n}(\alpha_! \iota_! x, y),$$

since $x \to \iota_! x$ is cocartesian. This shows that $\mathcal{O} \to \Delta^{\mathrm{op}}$ is a locally cocartesian fibration.

Before we prove part (ii), we make a further observation in the general case: Suppose $\alpha\colon [n] \to [m]$ is active, $\iota\colon [l] \to [n]$ is inert, x is an object of $\mathcal{O}_m$, $x \to \alpha_! x$ is locally cocartesian, and $\alpha_! x \to \iota_! \alpha_! x$ is cocartesian. Then it follows from the decomposition above of $\alpha_!$ in terms of locally cocartesian morphisms over the unique active maps $[1] \to [n]$ that $x \to \iota_! \alpha_! x$ is locally cocartesian over $\phi := \alpha\iota$.

It remains to prove (ii), for which we have to check that the assumption implies that locally cocartesian morphisms over active maps compose, i.e. for active morphisms

$$[m] \xrightarrow{\alpha} [n] \xrightarrow{\beta} [k]$$

the natural map $(\beta\alpha)_! X \to \alpha_! \beta_! X$ is an equivalence for $X \in \mathcal{O}_k$. Using the decomposition of locally cocartesian morphisms above we can immediately reduce to the case where $m = 1$. Now if α is surjective, we must have $n = 0$ or 1; if $n = 0$ then $\beta = \mathrm{id}_{[0]}$, while if $n = 1$ then $\alpha = \mathrm{id}_{[1]}$ — in either case the claim is trivially true. We can therefore assume that α is not surjective, in which case we can find a factorization of α as

$$[1] \xrightarrow{d_1} [2] \xrightarrow{\alpha'} [n]$$

where $\alpha'(1) \neq \alpha'(0), \alpha'(2)$; using this factorization we get for $X \in \mathcal{O}_k$ a commutative square

$$\begin{array}{ccc} (\beta\alpha' d_1)_! X & \longrightarrow & d_{1,!}(\beta\alpha')_! X \\ \downarrow & & \downarrow \\ (\alpha' d_1)_! \beta_! X & \longrightarrow & d_{1,!}\alpha'_! \beta_! X. \end{array}$$

Here our assumption guarantees the horizontal maps are equivalences, and we want to show the left vertical map is an equivalence. It thus suffices to show the right vertical map is an equivalence, for which it's enough to prove $(\beta\alpha')_! X \to \alpha'_! \beta_! X$ is an equivalence since $d_{1,!}$ is a functor. Our assumption on $\alpha'(1)$ means this decomposes as a pair of maps

$$[1] = \{0, 1\} \to \{\alpha'(0), \ldots, \alpha'(1)\} \to \{\beta\alpha'(0), \ldots, \beta\alpha'(1)\}$$

and similarly with $\{1, 2\}$, where

$$\{\alpha'(0), \ldots, \alpha'(1)\}, \{\alpha'(1), \ldots, \alpha'(2)\} < n.$$

This means we can reduce to our assumption by inducting on n. Combined with our previous observations we have then shown that locally cocartesian morphisms compose in general, since it holds for all combinations of active and inert maps. Thus π is a cocartesian fibration, as required. □

2.2 ∞-Categories of spans

For compatibility with [4] we will work with ∞-categories as *quasicategories*, i.e. simplicial sets satisfying the horn-filling condition for inner horns, in this and the next subsections.

Definition 2.2.1 Let $\epsilon\colon \mathbb{\Delta} \to \mathbb{\Delta}$ be the functor $[n] \mapsto [n]\star[n]^{\mathrm{op}}$. This induces a functor $\epsilon^*\colon \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ given by composition with ϵ; this functor is the *edgewise subdivision* of simplicial sets. If $\mathcal{C}$ is an ∞-category, we will write $\mathrm{Tw}^r\,\mathcal{C} := \epsilon^*\mathcal{C}$ and refer to this as the *twisted arrow ∞-category* of $\mathcal{C}$.

Remark 2.2.2 By [44, Proposition 5.2.1.3] the simplicial set $\mathrm{Tw}^r\,\mathcal{C}$ is an ∞-category if $\mathcal{C}$ is one, and the projection $\mathrm{Tw}^r\,\mathcal{C} \to \mathcal{C}\times\mathcal{C}^{\mathrm{op}}$ (induced by the inclusions $[n], [n]^{\mathrm{op}} \to [n]\star[n]^{\mathrm{op}}$) is a right fibration.

Remark 2.2.3 If $\mathbf{C}$ is an ordinary category, then it is easy to see that $\mathrm{Tw}^r\,\mathbf{C}$ can be identified with the *twisted arrow category* of $\mathbf{C}$. This has morphisms $c \to d$ in $\mathbf{C}$ as objects, and diagrams

$$\begin{tikzcd} c \arrow[r] \arrow[d] & d \\ c' \arrow[r] & d' \arrow[u] \end{tikzcd}$$

as morphisms from $c \to d$ to $c' \to d'$, with composition induced from composition in $\mathbf{C}$. Unwinding the definition of $\mathrm{Tw}^r\,\mathcal{C}$ for $\mathcal{C}$ an ∞-category, we see that its objects and morphisms admit the same description in terms of $\mathcal{C}$.

Example 2.2.4 The twisted arrow category $\mathrm{Tw}^r(\Delta^n)$ is the poset of pairs (i, j) with $0 \leq i \leq j \leq n$ where $(i, j) \leq (i', j')$ if $i \leq i'$, $j' \leq j$.

Warning 2.2.5 There are two possible conventions for the definition of $\mathrm{Tw}^r\,\mathcal{C}$: Instead of the definition we have given we could instead consider $[n] \mapsto [n]^{\mathrm{op}}\star[n]$; let us call the resulting simplicial set $\mathrm{Tw}^\ell\,\mathcal{C}$ — this is the definition of the twisted arrow ∞-category used in [4] (there called $\widetilde{\mathcal{O}}(\mathcal{C})$). We clearly have $\mathrm{Tw}^r\,\mathcal{C} \cong (\mathrm{Tw}^\ell\,\mathcal{C})^{\mathrm{op}}$, which explains why op's appear in different places here compared to [4].

Definition 2.2.6 The functor ϵ^* has a right adjoint $\epsilon_*\colon \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$, given by right Kan extension. Explicitly, $\epsilon_* X$ is determined by $\mathrm{Hom}(\Delta^n, \epsilon_* X) \cong \mathrm{Hom}(\mathrm{Tw}^r(\Delta^n), X)$. If $\mathcal{C}$ is an ∞-category, we write $\overline{\mathrm{Span}}(\mathcal{C})$ for the simplicial set $\epsilon_*\mathcal{C}$.

Definition 2.2.7 Let $\mathrm{Tw}^r(\Delta^n)_0$ denote the full subcategory of $\mathrm{Tw}^r(\Delta^n)$ spanned by the objects (i, j) where $j - i \leq 1$. We say a simplex $\Delta^n \to \overline{\mathrm{Span}}(\mathcal{C})$ is *cartesian* if the corresponding functor $F\colon \mathrm{Tw}^r(\Delta^n) \to \mathcal{C}$ is the right Kan extension of its restriction to $\mathrm{Tw}^r(\Delta^n)_0$, or equivalently if for all integers $0 \leq i \leq k \leq l \leq j \leq n$, the square

$$\begin{tikzcd} F(i, j) \arrow[r] \arrow[d] & F(k, j) \arrow[d] \\ F(i, l) \arrow[r] & F(k, l) \end{tikzcd}$$

is cartesian. We write $\mathrm{Span}(\mathcal{C})$ for the simplicial subset of $\overline{\mathrm{Span}}(\mathcal{C})$ containing only the cartesian simplices.

Remark 2.2.8 A morphism $\mathcal{I} \to \overline{\mathrm{Span}}(\mathcal{C})$ corresponds to a functor $F\colon \mathrm{Tw}^r(\mathcal{I}) \to \mathcal{C}$. Unwinding the definitions, we see that the map to $\overline{\mathrm{Span}}(\mathcal{C})$ takes $i \in \mathcal{I}$ to $F(i \xrightarrow{\mathrm{id}} i)$ and a morphism $f\colon i \to j$ to the value of F at the span

$$\begin{tikzcd} i \arrow[r, equal] \arrow[d, equal] & i \arrow[r, "f"] \arrow[d, "f"] & j \arrow[d, equal] \\ i \arrow[r, "f"] & j \arrow[r, equal] & j. \end{tikzcd}$$

A functor $\mathcal{I} \to \mathrm{Span}(\mathcal{C})$ then corresponds to a functor $\mathrm{Tw}^r(\mathcal{I}) \to \mathcal{C}$ such that for all composable morphisms $f\colon i \to j$, $g\colon j \to k$, the value of F at the commutative square

$$\begin{tikzcd} & i \arrow[dl, equal] \arrow[drr] \arrow[dd] & & \\ i \arrow[drr] \arrow[dd] & & & j \arrow[dl, equal] \arrow[dd] \\ & k \arrow[drr, equal] & j \arrow[dd, equal] & \\ j \arrow[ur] \arrow[drr, equal] & & & k \\ & & j \arrow[ur] & \end{tikzcd}$$

in $\mathrm{Tw}^r(\mathcal{I})$ is a cartesian square in $\mathcal{C}$.

Proposition 2.2.9 (Barwick, [4, Proposition 3.4]) *If $\mathcal{C}$ is an ∞-category with pullbacks, then* $\mathrm{Span}(\mathcal{C})$ *is an ∞-category.* $\square$

Definition 2.2.10 Following Barwick [4], we say a *triple* is a list $(\mathcal{C}, \mathcal{C}^F, \mathcal{C}^B)$ where $\mathcal{C}$ is an ∞-category and $\mathcal{C}^B$ and $\mathcal{C}^F$ are both subcategories of $\mathcal{C}$ containing all the equivalences. We will call the morphisms in $\mathcal{C}^B$ the *backwards* morphisms and the morphisms in $\mathcal{C}^F$ the *forwards* morphisms in the triple. We say a triple is *adequate* if for every morphism $f\colon x \to y$ in $\mathcal{C}^F$ and $g\colon z \to y$ in $\mathcal{C}^B$, there is a pullback square

$$\begin{tikzcd} w \arrow[r, "f'"] \arrow[d, "g'"'] & z \arrow[d, "g"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

where f' is in $\mathcal{C}^F$ and g' is in $\mathcal{C}^B$.

Example 2.2.11 If $\mathcal{C}$ is any ∞-category, we have the triple $(\mathcal{C}, \mathcal{C}, \mathcal{C})$ where all morphisms are both forwards and backwards morphisms. We call this the *maximal* triple on $\mathcal{C}$; it is adequate if and only if $\mathcal{C}$ has pullbacks.

Remark 2.2.12 In [4], the forwards morphisms are called *ingressive* and the backwards morphisms are called *egressive*.

Definition 2.2.13 Given a triple $(\mathcal{C}, \mathcal{C}^F, \mathcal{C}^B)$ we define $\overline{\mathrm{Span}}_{B,F}(\mathcal{C})$ to be the simplicial subset of $\overline{\mathrm{Span}}(\mathcal{C})$ containing only those simplices that correspond to maps $\sigma\colon \mathrm{Tw}^r(\Delta^n) \to \mathcal{C}$ such that for all i, j the map $\sigma(i, j) \to \sigma(i+1, j)$ lies in $\mathcal{C}^F$ and the map $\sigma(i, j) \to \sigma(i, j-1)$ lies in $\mathcal{C}^B$. We write $\mathrm{Span}_{B,F}(\mathcal{C})$ for the simplicial subset of $\overline{\mathrm{Span}}_{B,F}(\mathcal{C})$ containing the cartesian simplices with this property.

Proposition 2.2.14 (Barwick, [4, Proposition 5.6]) *If $(\mathcal{C}, \mathcal{C}^F, \mathcal{C}^B)$ is an adequate triple, then $\mathrm{Span}_{B,F}(\mathcal{C})$ is an ∞-category.* □

2.3 Spans and fibrations

Definition 2.3.1 Given an adequate triple $(\mathcal{B}, \mathcal{B}^F, \mathcal{B}^B)$ and an inner fibration $p\colon \mathcal{E} \to \mathcal{B}$ such that $\mathcal{E}$ has p-cartesian morphisms over morphisms in $\mathcal{B}^B$, we define a triple $(\mathcal{E}, \mathcal{E}^F, \mathcal{E}^B)$ by taking $\mathcal{E}^B$ to consist of cartesian morphisms over morphisms in $\mathcal{B}^B$ and $\mathcal{E}^F$ to consist of all morphisms lying over morphisms in $\mathcal{B}^F$.

Proposition 2.3.2 *In the situation of Definition 2.3.1, the triple $(\mathcal{E}, \mathcal{E}^F, \mathcal{E}^B)$ is adequate. Moreover, we have a pullback square of simplicial sets*

$$\begin{tikzcd} \mathrm{Span}_{B,F}(\mathcal{E}) \arrow[r] \arrow[d] & \overline{\mathrm{Span}}_{B,F}(\mathcal{E}) \arrow[d] \\ \mathrm{Span}_{B,F}(\mathcal{B}) \arrow[r] & \overline{\mathrm{Span}}_{B,F}(\mathcal{B}). \end{tikzcd}$$

This is a consequence of the following simple observation:

Lemma 2.3.3 *Let $p\colon \mathcal{E} \to \mathcal{B}$ be an inner fibration, and suppose we have a pullback square*

$$\begin{tikzcd} a' \arrow[r, "f'"] \arrow[d, "\alpha"'] & b' \arrow[d, "\beta"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$$

in $\mathcal{B}$. If $\bar{b}' \to \bar{b}$ is a morphism in $\mathcal{E}$ over β and there exist p-cartesian morphisms $\bar{a} \to \bar{b}$ over f and $\bar{a}' \to \bar{b}'$ over f', then the commutative square

$$\begin{tikzcd} \bar{a}' \arrow[r] \arrow[d] & \bar{b}' \arrow[d] \\ \bar{a} \arrow[r] & \bar{b} \end{tikzcd}$$

(where the left vertical morphism is induced by the universal property of $\bar{a} \to \bar{b}$) is cartesian.

Proof For any $\bar{x}$ in $\mathcal{E}$ over $x \in \mathcal{B}$ we have a commutative cube

$$\begin{tikzcd} & \mathrm{Map}_{\mathcal{E}}(\bar{x}, \bar{a}') \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Map}_{\mathcal{E}}(\bar{x}, \bar{b}') \arrow[dl] \arrow[dd] \\ \mathrm{Map}_{\mathcal{E}}(\bar{x}, \bar{a}) \arrow[rr] \arrow[dd] & & \mathrm{Map}_{\mathcal{E}}(\bar{x}, \bar{b}) \arrow[dd] & \\ & \mathrm{Map}_{\mathcal{B}}(x, a') \arrow[rr] \arrow[dl] & & \mathrm{Map}_{\mathcal{B}}(x, b') \arrow[dl] \\ \mathrm{Map}_{\mathcal{B}}(x, a) \arrow[rr] & & \mathrm{Map}_{\mathcal{B}}(x, b) & \end{tikzcd}$$

in the ∞-category of spaces. Here the bottom face is cartesian since a' is a pullback, and the front and back faces are cartesian since the morphisms $\bar{a} \to \bar{b}$ and $\bar{a}' \to \bar{b}'$ are p-cartesian. Therefore the top face is also cartesian. Since this holds for all $\bar{x} \in \mathcal{E}$ this means $\bar{a}'$ is the pullback $\bar{a} \times_{\bar{b}} \bar{b}'$, as required. □

Proof of Proposition 2.3.2 Adequacy follows immediately from Lemma 2.3.3. Moreover, this lemma also shows that an n-simplex of $\overline{\mathrm{Span}}_{B,F}(\mathcal{E})$ lies in $\mathrm{Span}_{B,F}(\mathcal{E})$ if and only if it maps to an n-simplex of $\mathrm{Span}_{B,F}(\mathcal{B})$, giving the pullback square. □

Definition 2.3.4 For K a simplicial set, let $\mathrm{Tw}^r_B(K)$ denote the marked simplicial set $(\mathrm{Tw}^r(K), B)$ where B is the set of "backwards" maps, i.e. those lying in the image of $K^{\mathrm{op}} \to \mathrm{Tw}^r(K)$.

In the remaining part of this subsection we give a reformulation of the results of [4, Sect. 12] that will be convenient for us.

Proposition 2.3.5 *For $0 < k < n$, the map $\mathrm{Tw}^r_B(\Lambda^n_k)^{\mathrm{op}} \to \mathrm{Tw}^r_B(\Delta^n)^{\mathrm{op}}$ is marked anodyne in the sense of [43, Definition 3.1.1.1].*

Proof This follows from the filtration defined in [4, Sect. 12], using [4, Proposition 12.14]. □

Corollary 2.3.6 *If $\mathcal{E} \to \mathcal{B}$ is as in Definition 2.3.1, then*

(i) *$\overline{\mathrm{Span}}_{B,F}(\mathcal{E}) \to \overline{\mathrm{Span}}_{B,F}(\mathcal{B})$ is an inner fibration.*
(ii) *$\mathrm{Span}_{B,F}(\mathcal{E}) \to \mathrm{Span}_{B,F}(\mathcal{B})$ is an inner fibration.*

Proof To prove (i) we must show that there exists a lift in every commutative square

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \overline{\mathrm{Span}}_{B,F}(\mathcal{E}) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \overline{\mathrm{Span}}_{B,F}(\mathcal{B}) \end{tikzcd}$$

with $0 < k < n$. This is equivalent to giving a lift in the corresponding commutative square

$$\begin{tikzcd} \mathrm{Tw}^r\,\Lambda^n_k \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathrm{Tw}^r\,\Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{B}. \end{tikzcd}$$

Here the lift exists by Proposition 2.3.5, since by definition the backwards maps go to cartesian morphisms in $\mathcal{E}$. Now (ii) follows from the pullback square in Proposition 2.3.2. □

Proposition 2.3.7 *Let $p\colon \mathcal{E} \to \mathcal{B}$ be as in Definition 2.3.1, and assume that in addition $\mathcal{E}$ has locally p-cocartesian edges over morphisms in $\mathcal{B}^F$. Then:*

(i) *$\overline{\mathrm{Span}}_{B,F}(\mathcal{E}) \to \overline{\mathrm{Span}}_{B,F}(\mathcal{B})$ is a locally cocartesian fibration,*
(ii) *$\mathrm{Span}_{B,F}(\mathcal{E}) \to \mathrm{Span}_{B,F}(\mathcal{B})$ is a locally cocartesian fibration,*

A span $X \xleftarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ is locally p-cocartesian if and only if g is a locally p-cocartesian morphism in $\mathcal{E}$.

Proof We first prove (i). Consider a 1-simplex ϕ of $\overline{\mathrm{Span}}_{B,F}(\mathcal{B})$, which corresponds to a span $b \xleftarrow{f} b' \xrightarrow{g} b''$ in $\mathcal{B}$. We wish to show that the pullback $\phi^*\overline{\mathrm{Span}}_{B,F}(\mathcal{E}) \to \Delta^1$ is a cocartesian fibration. Pick an object e of $\mathcal{E}$ lying over b. Then a 1-simplex of $\overline{\mathrm{Span}}_{B,F}(\mathcal{E})$ with source e lying over ϕ is a span $e \xleftarrow{\bar{f}} e' \xrightarrow{\bar{g}} e''$ where $\bar{f}$ is a cartesian morphism over f and $\bar{g}$ is any morphism over g. The space of maps from e to e'' in $\phi^*\overline{\mathrm{Span}}_{B,F}(\mathcal{E})$ can therefore be identified with the space $\mathrm{Map}_{\mathcal{E}}(e', e'')_g$ of maps in $\mathcal{E}$ lying over g. From this it follows immediately that if $\bar{g}\colon e' \to e''$ is a locally cocartesian morphism from e' over g then the span $e \xleftarrow{\bar{f}} e' \xrightarrow{\bar{g}} e''$ is locally cocartesian, as required. This proves (i), from which (ii) follows by the pullback square of Proposition 2.3.2. □

Corollary 2.3.8 *Let $p\colon \mathcal{E} \to \mathcal{B}$ be as in Definition 2.3.1, and assume in addition:*

(1) *$\mathcal{E}$ has p-cocartesian edges over morphisms in $\mathcal{B}^F$.*
(2) *Consider a pullback square*

$$\begin{tikzcd} a' \arrow[r, "f'"] \arrow[d, "\alpha"'] & b' \arrow[d, "\beta"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$$

in $\mathcal{B}$ with α, β in $\mathcal{B}^F$ and f', f in $\mathcal{B}^B$. Let $\bar{b}'$ be an object of $\mathcal{E}$ over b', and suppose $\bar{b}' \xrightarrow{\bar{\beta}} \bar{b}$ is a p-cocartesian morphism over β and $\bar{a} \xrightarrow{\bar{f}} \bar{b}$ and $\bar{a}' \xrightarrow{\bar{f}'} \bar{b}'$ are p-cartesian morphisms over f and f'. Then in the commutative square

$$\begin{tikzcd} \bar{a}' \arrow[r, "\bar{f}'"] \arrow[d, "\bar{\alpha}"'] & \bar{b}' \arrow[d, "\bar{\beta}"] \\ \bar{a} \arrow[r, "\bar{f}"'] & \bar{b} \end{tikzcd}$$

induced by the universal property of $\bar{f}$, the morphism $\bar{\alpha}$ is again p-cocartesian.

Then $\mathrm{Span}_{B,F}(\mathcal{E}) \to \mathrm{Span}_{B,F}(\mathcal{B})$ *is a cocartesian fibration.*

Proof We know from Proposition 2.3.7 that $\mathrm{Span}_{B,F}(\mathcal{E}) \to \mathrm{Span}_{B,F}(\mathcal{B})$ is a locally cocartesian fibration. By [43, Proposition 2.4.2.8] it therefore suffices to show that the locally cocartesian morphisms are closed under composition. Lemma 2.3.3 shows that this is indeed the case under the given assumptions. □

3 Day convolution for double ∞-categories

In this section we carry out the main technical construction of this paper: We show that for a certain class of double ∞-categories $\mathcal{M}$, there exists a *Day convolution* double ∞-category $\widehat{\mathcal{M}}_{\mathcal{S}}$ such that for any non-symmetric ∞-operad $\mathcal{O}$ we have a natural equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathcal{M}}_{\mathcal{S}}) \simeq \mathrm{Seg}_{\mathcal{O}\times_{\Delta^{\mathrm{op}}}\mathcal{M}}(\mathcal{S}).$$

In Sect. 3.1 we introduce an "unfolding" construction that we use to define $\widehat{\mathcal{M}}_{\mathcal{S}}$ in Sect. 3.2; we then establish the universal property in Sect. 3.3. Next we prove in Sect. 3.4 that we may view associative algebras in $\widehat{\mathcal{M}}$ as algebras in a family of monoidal ∞-categories. We also consider enriched variants of the Day convolution construction in Sect. 3.5, and in Sect. 3.6 we illustrate the theory by discussing the example of enriched ∞-categories.

3.1 An unfolding construction

Suppose we have a cocartesian fibration $p\colon \mathcal{E} \to \mathcal{U}$ and a cartesian fibration $q\colon \mathcal{U} \to \mathcal{B}$. Our goal in this subsection is to construct for every cocomplete ∞-category $\mathcal{X}$ a cocartesian fibration $\widetilde{\mathcal{E}}_{\mathcal{X}} \to \mathcal{B}$ with the universal property that for any functor $\mathcal{C} \to \mathcal{B}$ there is a natural equivalence

$$\mathrm{Map}_{/\mathcal{B}}(\mathcal{C}, \widetilde{\mathcal{E}}_{\mathcal{X}}) \xrightarrow{\sim} \mathrm{Map}(\mathcal{C} \times_{\mathcal{B}} \mathcal{E}, \mathcal{X}).$$

Remark 3.1.1 Recall that a functor of ∞-categories f is called an *exponentiable*, *flat*, or *Conduché fibration* if the functor f^* given by pullback along f has a right adjoint f_*. Both cartesian and cocartesian fibrations are examples of exponentiable fibrations, hence the composite $qp\colon \mathcal{E} \to \mathcal{B}$ is an exponentiable fibration. The universal property of $\widetilde{\mathcal{E}}_{\mathcal{X}}$ is that of $(qp)_*(\mathcal{X} \times \mathcal{E})$, but it is not clear from the latter that $\widetilde{\mathcal{E}}_{\mathcal{X}}$ will be a cocartesian fibration if $\mathcal{X}$ is cocomplete.

To define $\widetilde{\mathcal{E}}_{\mathcal{X}}$ we first introduce an "unfolding construction" that uses p and q to construct a functor $\mathcal{B} \to \mathrm{Span}(\mathrm{Cat}_\infty)$ that takes $b \in \mathcal{B}$ to the fibre $\mathcal{E}_b$ of the composite $\mathcal{E} \to \mathcal{B}$, and takes a morphism $f\colon b \to b'$ to the top row in the diagram

$$\begin{tikzcd}
\mathcal{E}_b \arrow[d, "p_b"] \arrow[dr, phantom, "\llcorner"] & f^*\mathcal{E}_b \arrow[l] \arrow[r, "f_!"] \arrow[d] & \mathcal{E}_{b'} \arrow[d, "p_{b'}"] \\
\mathcal{U}_b & \mathcal{U}_{b'} \arrow[l, "f^*"] \arrow[r, equal] & \mathcal{U}_{b'},
\end{tikzcd}$$

where $f^*\colon \mathcal{U}_{b'} \to \mathcal{U}_b$ is the functor given by the cartesian morphisms over f and the left square is a pullback; an object of $f^*\mathcal{E}_b$ then corresponds to a pair $(x \in \mathcal{E}_b, u \in \mathcal{U}_{b'})$ such that $p(x) \simeq f^*u$ in $\mathcal{U}_b$, and the top right morphism takes (x, u) to the cocartesian pushforward $\bar{f}_!x$ where $\bar{f}\colon f^*u \to u$ is the q-cartesian morphism over f.

Construction 3.1.2 Let $q^\vee\colon \mathcal{U}^\vee \to \mathcal{B}^{\mathrm{op}}$ be the cocartesian fibration dual to $q\colon \mathcal{U} \to \mathcal{B}$ (i.e. the cocartesian fibration corresponding to the same functor as q). By [25, Theorem 4.5], the *free* cocartesian fibration on $q^\vee$ is $\mathcal{U}^\vee \times_{\mathcal{B}^{\mathrm{op}}} (\mathcal{B}^{\mathrm{op}})^{\Delta^1} \to \mathcal{B}^{\mathrm{op}}$, where the fibre product uses $q^\vee$ and evaluation at 0 and the functor to $\mathcal{B}^{\mathrm{op}}$ uses evaluation at 1. Since $q^\vee$ is a cocartesian fibration, the identity induces a functor

$$\mathcal{U}^\vee \times_{\mathcal{B}^{\mathrm{op}}} (\mathcal{B}^{\mathrm{op}})^{\Delta^1} \to \mathcal{U}^\vee$$

over $\mathcal{B}^{\mathrm{op}}$ that preserves cocartesian morphisms. Dualizing again, we obtain a morphism of cartesian fibrations

$$(\mathcal{U}^\vee \times_{\mathcal{B}^{\mathrm{op}}} (\mathcal{B}^{\mathrm{op}})^{\Delta^1})^\vee \to \mathcal{U}$$

that preserves cartesian morphisms. The following lemma identifies the source of this functor with $\mathcal{U}^\vee \times_{\mathcal{B}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{B}^{\mathrm{op}})$:

Lemma 3.1.3 *For any functor $f\colon \mathcal{C} \to \mathcal{B}$, the cartesian fibration $(\mathcal{C} \times_{\mathcal{B}} \mathcal{B}^{\Delta^1})^\vee \to \mathcal{B}^{\mathrm{op}}$ dual to the free cocartesian fibration on f is equivalent to*

$$\mathcal{C} \times_{\mathcal{B}} \mathrm{Tw}^r(\mathcal{B}) \to \mathcal{B}^{\mathrm{op}}.$$

Proof We can write the cocartesian fibration $\mathcal{C} \times_{\mathcal{B}} \mathcal{B}^{\Delta^1} \to \mathcal{B}$ as the fibre product

$$(\mathcal{C} \times \mathcal{B}) \times_{(\mathcal{B} \times \mathcal{B})} \mathcal{B}^{\Delta^1}$$

of cocartesian fibrations over $\mathcal{B}$. Since dualization of fibrations is an equivalence of ∞-categories, it preserves fibre products, hence we obtain an equivalence

$$(\mathcal{C} \times_{\mathcal{B}} \mathcal{B}^{\Delta^1})^{\vee} \simeq (\mathcal{C} \times \mathcal{B}^{\mathrm{op}}) \times_{(\mathcal{B} \times \mathcal{B}^{\mathrm{op}})} (\mathcal{B}^{\Delta^1})^{\vee} \simeq \mathcal{C} \times_{\mathcal{B}} (\mathcal{B}^{\Delta^1})^{\vee}.$$

By [36, Proposition A.2.4] the dual of $\mathcal{B}^{\Delta^1} \to \mathcal{B}$ is $\mathrm{Tw}^r(\mathcal{B}) \to \mathcal{B}^{\mathrm{op}}$, which completes the proof. □

Definition 3.1.4 Given a cartesian fibration $\mathcal{U} \to \mathcal{B}$, we have constructed a canonical functor

$$\mathfrak{c}_{\mathcal{U}} \colon \mathcal{U}^{\vee} \times_{\mathcal{B}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{B}) \to \mathcal{U}.$$

For $\mathcal{E} \to \mathcal{U}$ a cocartesian fibration, we define the *unfolding* $\mathrm{Unf}(\mathcal{E})$ as the fibre product $(\mathcal{U}^{\vee} \times_{\mathcal{B}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{B})) \times_{\mathcal{U}} \mathcal{E}$, using $\mathfrak{c}_{\mathcal{U}}$, and write $\bar{\mathfrak{c}}_{\mathcal{U}}$ for the induced map $\mathrm{Unf}(\mathcal{E}) \to \mathcal{E}$ over $\mathfrak{c}_{\mathcal{U}}$. The projection $\mathrm{Unf}(\mathcal{E}) \to \mathrm{Tw}^r(\mathcal{B})$ is then a cocartesian fibration, since it decomposes as a composite

$$(\mathcal{U}^{\vee} \times_{\mathcal{B}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{B})) \times_{\mathcal{U}} \mathcal{E} \to \mathcal{U}^{\vee} \times_{\mathcal{B}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{B}) \to \mathrm{Tw}^r(\mathcal{B}),$$

where the first map is a pullback of the cocartesian fibration $\mathcal{E} \to \mathcal{U}$ and the second is a pullback of the cocartesian fibration $\mathcal{U}^{\vee} \to \mathcal{B}^{\mathrm{op}}$.

Remark 3.1.5 Given a functor $\mathcal{C} \to \mathcal{B}$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Unf}(\mathcal{C} \times_{\mathcal{B}} \mathcal{E}) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{C} \times_{\mathcal{B}} \mathcal{E} \arrow[dr] \arrow[dd] & \\
& \mathrm{Unf}(\mathcal{E}) \arrow[rr] \arrow[dd] & & \mathcal{E} \arrow[dd] \\
(\mathcal{C} \times_{\mathcal{B}} \mathcal{U})^{\vee} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{C}) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{C} \times_{\mathcal{B}} \mathcal{U} \arrow[dr] \arrow[dd] & \\
& \mathcal{U}^{\vee} \times_{\mathcal{B}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{B}) \arrow[rr] \arrow[dd] & & \mathcal{U} \arrow[dd] \\
\mathrm{Tw}^r(\mathcal{C}) \arrow[rr] \arrow[dr] & & \mathcal{C} \arrow[dr] & \\
& \mathrm{Tw}^r(\mathcal{B}) \arrow[rr] & & \mathcal{B}.
\end{tikzcd}$$

In the top cube, the back and front faces are cartesian by definition of unfolding, and the right face is cartesian since the bottom right and right composite squares are cartesian. This implies that the left square in the top cube is cartesian. Moreover, since dualization of fibrations is compatible with pullbacks we have $(\mathcal{C} \times_{\mathcal{B}} \mathcal{U})^{\vee} \simeq \mathcal{U}^{\vee} \times_{\mathcal{B}^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}}$, and hence

$$(\mathcal{C} \times_{\mathcal{B}} \mathcal{U})^{\vee} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{C}) \simeq \mathcal{U}^{\vee} \times_{\mathcal{B}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{C}) \simeq \left(\mathcal{U}^{\vee} \times_{\mathcal{B}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{B})\right) \times_{\mathrm{Tw}^r(\mathcal{B})} \mathrm{Tw}^r(\mathcal{C}).$$

The bottom left face in the diagram is therefore cartesian, and so the left composite square is a pullback. Thus unfolding is compatible with base change, in the sense that we have a natural equivalence

$$\mathrm{Unf}(\mathcal{C} \times_{\mathcal{B}} \mathcal{E}) \xrightarrow{\sim} \mathrm{Tw}^r(\mathcal{C}) \times_{\mathrm{Tw}^r(\mathcal{B})} \mathrm{Unf}(\mathcal{E}).$$

Lemma 3.1.6 *The cocartesian fibration* $\mathrm{Unf}(\mathcal{E}) \to \mathrm{Tw}^r(\mathcal{B})$ *corresponds to a functor* $\mathfrak{U}_{\mathcal{E}} : \mathcal{B} \to \mathrm{Span}(\mathrm{Cat}_\infty)$.

Proof Given morphisms $a \xrightarrow{f} b \xrightarrow{g} c$ in $\mathcal{B}$, we have the commutative diagram of ∞-categories

$$\begin{tikzcd}
 & & g^*f^*\mathcal{E}_a \arrow[dl] \arrow[dd] \arrow[drr] & & & \\
 & f^*\mathcal{E}_a \arrow[dl] \arrow[dd] \arrow[drr] & & & g^*\mathcal{E}_b \arrow[dl] \arrow[dd] \arrow[dr] & \\
\mathcal{E}_a \arrow[dd] & & \mathcal{U}_c \arrow[dl] \arrow[drr, equal] & \mathcal{E}_b \arrow[dd] & & \mathcal{E}_c \arrow[dd] \\
 & \mathcal{U}_b \arrow[dl] \arrow[drr, equal] & & & \mathcal{U}_c \arrow[dl] \arrow[dr, equal] & \\
\mathcal{U}_a & & & \mathcal{U}_b & & \mathcal{U}_c
\end{tikzcd}$$

and by Remark 2.2.8 we must show that the commutative square in the top level is cartesian. But in the commutative cube the bottom, back left, and front right faces are cartesian, hence so is the top face. $\square$

We can now define $\widetilde{\mathcal{E}}_{\mathcal{X}}$ using the following construction:

Definition 3.1.7 For any ∞-category $\mathcal{X}$, let $p_{\mathcal{X}} : \mathcal{F}_{\mathcal{X}} \to \mathrm{Cat}_\infty$ denote the cartesian fibration correspoding to the functor $\mathrm{Fun}(\text{–}, \mathcal{X}) : \mathrm{Cat}_\infty^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$. If $\mathcal{X}$ is cocomplete, then this is also a cocartesian fibration (with cocartesian morphisms given by left Kan extensions). We then have a locally cocartesian fibration $\mathrm{Span}_{B,F}(\mathcal{F}_{\mathcal{X}}) \to \mathrm{Span}(\mathrm{Cat}_\infty)$ by Proposition 2.3.7, where $\mathcal{F}_{\mathcal{X}}$ is equipped with the triple structure from Definition 2.3.1.

Definition 3.1.8 Given a cocartesian fibration $\mathcal{E} \to \mathcal{U}$ and a cartesian fibration $\mathcal{U} \to \mathcal{B}$, we let $\widetilde{\mathcal{E}}_{\mathcal{X}}$ for a cocomplete ∞-category $\mathcal{X}$ be defined by the pullback

$$\begin{tikzcd}
\widetilde{\mathcal{E}}_{\mathcal{X}} \arrow[r] \arrow[d] & \mathrm{Span}_{B,F}(\mathcal{F}_{\mathcal{X}}) \arrow[d, "p_{\mathcal{X}}"] \\
\mathcal{B} \arrow[r, "\mathfrak{U}_{\mathcal{E}}"] & \mathrm{Span}(\mathrm{Cat}_\infty).
\end{tikzcd}$$

Then $\widetilde{\mathcal{E}}_{\mathcal{X}} \to \mathcal{B}$ is a locally cocartesian fibration.

Lemma 3.1.9 *Let $\mathcal{X}$ be a cocomplete ∞-category. The locally cocartesian fibration $\widetilde{\mathcal{E}}_{\mathcal{X}} \to \mathcal{B}$ is a cocartesian fibration.*

Proof We must show that the locally cocartesian morphisms are closed under composition. For morphisms $a \xrightarrow{f} b \xrightarrow{g} c$ in $\mathcal{B}$, we have the cartesian square

$$\begin{tikzcd}
g^*f^*\mathcal{E}_a \arrow[r, "F'"] \arrow[d, "G'"] & g^*\mathcal{E}_b \arrow[d, "G"] \\
f^*\mathcal{E}_a \arrow[r, "F"] & \mathcal{E}_b
\end{tikzcd}$$

as above, and we must show that the mate transformation

$$F'_! G'^* \to G^* F_!$$

of functors $\mathrm{Fun}(f^*\mathcal{E}_a, \mathcal{X}) \to \mathrm{Fun}(g^*\mathcal{E}_b, \mathcal{X})$ is an equivalence. At $\phi \in \mathrm{Fun}(f^*\mathcal{E}_a, \mathcal{X})$ and $x \in g^*\mathcal{E}_b$, the mate transformation evaluates to the natural map of colimits

$$\operatorname*{colim}_{y \in (g^* f^* \mathcal{E}_a)_{/x}} \phi(G'y) \to \operatorname*{colim}_{z \in (f^*\mathcal{E}_a)_{/Gx}} \phi(z)$$

arising from the functor $(g^* f^* \mathcal{E}_a)_{/x} \to (f^*\mathcal{E}_a)_{/Gx}$ induced by G. It thus suffices to show that this functor is cofinal.

By definition, $(g^* f^* \mathcal{E}_a)_{/x}$ is the pullback $g^* f^* \mathcal{E}_a \times_{g^*\mathcal{E}_b} (g^*\mathcal{E}_b)_{/x}$. Since $g^*\mathcal{E}_b$ and $g^* f^* \mathcal{E}_a$ are pulled back along $\mathcal{U}_c \to \mathcal{U}_b$, we can rewrite this to see that there is a natural pullback square

$$\begin{tikzcd} (g^* f^* \mathcal{E}_a)_{/x} \arrow[r] \arrow[d] & (f^*\mathcal{E}_a)_{/Gx} \arrow[d] \\ (\mathcal{U}_c)_{/\pi_c x} \arrow[r] & (\mathcal{U}_b)_{/\pi_b G x}, \end{tikzcd}$$

where π_t denotes the projection $\mathcal{E}_t \to \mathcal{U}_t$. In this square the right vertical functor is a cocartesian fibration, and the bottom horizontal functor is cofinal since both $(\mathcal{U}_c)_{/\pi_c x}$ and $(\mathcal{U}_b)_{/\pi_b G x}$ have a terminal object, which is preserved by this functor. It follows by [43, Proposition 4.1.2.15] that the top horizontal functor is also cofinal, as required. □

Remark 3.1.10 The cocartesian fibration $\widetilde{\mathcal{E}}_{\mathcal{X}} \to \mathcal{B}$ corresponds to a functor $\mathcal{B} \to \mathrm{Cat}_\infty$. This takes $b \in \mathcal{B}$ to $\mathrm{Fun}(\mathcal{E}_b, \mathcal{X})$ and a morphism $f: b \to b'$ to the composite functor

$$\mathrm{Fun}(\mathcal{E}_b, \mathcal{X}) \to \mathrm{Fun}(f^*\mathcal{E}_b, \mathcal{X}) \to \mathrm{Fun}(\mathcal{E}_{b'}, \mathcal{X})$$

where the first functor is given by composition with $f^*\mathcal{E}_b \to \mathcal{E}_b$ and the second by left Kan extension along $f^*\mathcal{E}_b \to \mathcal{E}_{b'}$. Both $f^*\mathcal{E}_b$ and $\mathcal{E}_{b'}$ are cocartesian fibrations over $\mathcal{U}_{b'}$, and the functor $f_!: f^*\mathcal{E}_b \to \mathcal{E}_{b'}$ preserves cocartesian morphisms. The following lemma therefore implies that the left Kan extension along $f_!$ can be computed fibrewise, i.e. for $\Phi: f^*\mathcal{E}_b \to \mathcal{X}$ and $x \in \mathcal{E}_{b'}$ over $u \in \mathcal{U}_{b'}$ we have

$$(f_!)_!\Phi(x) \simeq \operatorname*{colim}_{(\mathcal{E}_{b,f^*u})_{/x}} \Phi,$$

where we have used the equivalence $(f^*\mathcal{E}_b)_u \simeq \mathcal{E}_{b,f^*u}$, and the slice

$$(\mathcal{E}_{b,f^*u})_{/x} := \mathcal{E}_{b,f^*u} \times_{\mathcal{E}_{b',u}} (\mathcal{E}_{b',u})_{/x}$$

is defined using the functor $\mathcal{E}_{b,f^*u} \to \mathcal{E}_{b',u}$ given by cocartesian pushforward along the cartesian morphism $\bar{f}: f^*u \to u$. We obtain the following description of the functor $\mathrm{Fun}(\mathcal{E}_b, \mathcal{X}) \to \mathrm{Fun}(\mathcal{E}_{b'}, \mathcal{X})$ arising from the cocartesian fibration $\widetilde{\mathcal{E}}_{\mathcal{X}} \to \mathcal{B}$: For $\Psi: \mathcal{E}_b \to \mathcal{X}$, its image is the functor $\mathcal{E}_{b'} \to \mathcal{X}$ that to $x \in \mathcal{E}_{b',u}$ assigns

$$\operatorname*{colim}_{(y, \bar{f}_! y \to x) \in (\mathcal{E}_{b,f^*u})_{/x}} \Psi(y).$$

Lemma 3.1.11 *Consider a commutative triangle of ∞-categories*

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{F} \arrow[dl, "q"] \\ & \mathcal{B}, & \end{tikzcd}$$

where p and q are cocartesian fibrations and f preserves cocartesian morphisms. Then for $x \in \mathcal{F}_b$ the inclusion

$$\mathcal{E}_{b/x} := \mathcal{E}_b \times_{\mathcal{F}_b} \mathcal{F}_{b/x} \to \mathcal{E} \times_{\mathcal{F}} \mathcal{F}_{/x} =: \mathcal{E}_{/x}$$

is cofinal. In particular, if $\mathcal{C}$ is a cocomplete ∞-category then the left Kan extension $f_!F$ along f of any functor $F\colon \mathcal{E} \to \mathcal{C}$ can be computed fibrewise over $\mathcal{B}$, i.e. for $x \in \mathcal{F}_b$ we have

$$f_!F(x) \simeq \operatorname*{colim}_{y \in \mathcal{E}_{b/x}} F(y).$$

Proof By [43, Theorem 4.1.3.1] it suffices to check that for every object $\eta = (y, f(y) \xrightarrow{\phi} x)$ in $\mathcal{E}_{/x}$, the ∞-category $(\mathcal{E}_{b/x})_{\eta/}$ is weakly contractible. This ∞-category has as objects maps $y \to y'$ over $q(\phi)$ together with commutative triangles

$$\begin{tikzcd} f(y) \arrow[d] \arrow[dr, "\phi"] & \\ f(y') \arrow[r, "\phi'"'] & x \end{tikzcd}$$

where ϕ' lies over id_b. It therefore has an initial object, given by the cocartesian morphism $\psi\colon y \to y'$ over $q(\phi)$ together with the canonical factorization $f(y) \xrightarrow{f(\psi)} f(y') \to x$ that exists since $f(\psi)$ is again a cocartesian morphism. □

Remark 3.1.12 We can identify sections of the cocartesian fibration $\widetilde{\mathcal{E}}_{\mathcal{X}}$ as follows: For any functor $\phi\colon \mathcal{C} \to \mathcal{B}$, we have

$$\mathrm{Map}_{/\mathcal{B}}(\mathcal{C}, \widetilde{\mathcal{E}}_{\mathcal{X}}) \simeq \left\{ \begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "\phi"] & \mathrm{Span}_{B,F}(\mathcal{F}_{\mathcal{X}}) \arrow[d, "\mathrm{Span}(p_{\mathcal{X}})"] \\ \mathcal{B} \arrow[r, "\mathfrak{U}_{\mathcal{E}}"] & \mathrm{Span}(\mathrm{Cat}_\infty) \end{tikzcd} \right\} \subseteq \left\{ \begin{tikzcd} \mathrm{Tw}^r(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{Tw}^r(\phi)"] & \mathcal{F}_{\mathcal{X}} \arrow[d, "p_{\mathcal{X}}"] \\ \mathrm{Tw}^r(\mathcal{B}) \arrow[r] & \mathrm{Cat}_\infty \end{tikzcd} \right\}.$$

By the pullback square in Proposition 2.3.2, the only condition for a point of the right-hand ∞-groupoid to lie in the image of $\mathrm{Map}_{/\mathcal{B}}(\mathcal{C}, \widetilde{\mathcal{E}}_{\mathcal{X}})$ is that the functor $\mathrm{Tw}^r(\mathcal{C}) \xrightarrow{\Phi} \mathcal{F}_{\mathcal{X}}$ takes morphisms in $\mathrm{Tw}^r(\mathcal{C})$ of the form

$$\begin{tikzcd} c \arrow[r, equal] \arrow[d, "f"] & c \arrow[d, equal] \\ d & c \arrow[l, "f"] \end{tikzcd}$$

to cartesian morphisms in $\mathcal{F}_{\mathcal{X}}$. This amounts to the natural transformation

$$\begin{tikzcd} \phi(f)^*\mathcal{E}_{\phi(c)} \arrow[d] \arrow[dr, "\Phi(f)"] \arrow[d, phantom, shift left=4ex, "\swarrow"] & \\ \mathcal{E}_{\phi(c)} \arrow[r, "\Phi(\mathrm{id}_c)"'] & \mathcal{X} \end{tikzcd}$$

being an equivalence. Since $\mathcal{F}_{\mathcal{X}}$ is by definition the cartesian fibration for the functor $\mathrm{Fun}(–, \mathcal{X})$ we can use [25, Proposition 7.3] and Remark 3.1.5 to obtain an equivalence

$$\left\{\begin{tikzcd} \mathrm{Tw}^r(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{Tw}^r(\phi)"] & \mathcal{F}_{\mathcal{X}} \arrow[d, "p_{\mathcal{X}}"] \\ \mathrm{Tw}^r(\mathcal{B}) \arrow[r] & \mathrm{Cat}_\infty \end{tikzcd}\right\} \simeq \mathrm{Map}(\mathrm{Tw}^r(\mathcal{C}) \times_{\mathrm{Tw}^r(\mathcal{B})} \mathrm{Unf}(\mathcal{E}), \mathcal{X}) \simeq \mathrm{Map}(\mathrm{Unf}(\mathcal{C} \times_{\mathcal{B}} \mathcal{E}), \mathcal{X}),$$

natural in $\mathcal{C}$, under which $\mathrm{Map}_{/\mathcal{B}}(\mathcal{C}, \widetilde{\mathcal{E}}_{\mathcal{X}})$ is identified with the functors $\mathrm{Tw}^r(\mathcal{C}) \times_{\mathrm{Tw}^r(\mathcal{B})} \mathrm{Unf}(\mathcal{E}) \to \mathcal{X}$ that take morphisms

$$(f : c \to d, e \in \mathcal{E}_{\phi(c)}, u \in \mathcal{U}_{\phi(d)}, \pi_{\phi(c)}e \simeq \phi(f)^*u) \to (\mathrm{id}_c, e, \phi(f)^*u, \pi_{\phi(c)}e \simeq \phi(f)^*u),$$

over the morphism in $\mathrm{Tw}^r(\mathcal{C})$ above, to equivalences in $\mathcal{X}$.

Notation 3.1.13 Let $W_{\mathcal{U}/\mathcal{B}}$ denote the class of morphisms in $\mathcal{U}^\vee \times_{\mathcal{B}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{B})$ (or $\mathrm{Unf}(\mathcal{U})$) of the form

$$(u \in \mathcal{U}_b, f : a \to b) \to (f^*u, \mathrm{id}_a),$$

corresponding to the cocartesian morphism $u \to f^*u$ in $\mathcal{U}^\vee$ and the morphism

$$\begin{tikzcd} a \arrow[r, equal] \arrow[d, "f"] & a \arrow[d, equal] \\ b & a \arrow[l, "f"] \end{tikzcd}$$

in $\mathrm{Tw}^r(\mathcal{B})$; this is the union over $b \in \mathcal{B}$ of the classes $W_{\mathcal{U}/\mathcal{B},b}$ of such morphisms that lie over b. Then let $\overline{W}_{\mathcal{E}/\mathcal{B}}$ denote the class of morphisms in $\mathrm{Unf}(\mathcal{E})$ consisting of cocartesian morphisms lying over the morphisms in $W_{\mathcal{U}}$.

Remark 3.1.14 Using this notation, Remark 3.1.12 identifies the space $\mathrm{Map}_{/\mathcal{B}}(\mathcal{C}, \widetilde{\mathcal{E}}_{\mathcal{X}})$ of sections with the space of functors $\mathrm{Unf}(\mathcal{C} \times_{\mathcal{B}} \mathcal{E}) \to \mathcal{X}$ that take the morphisms in $\overline{W}_{\mathcal{C} \times_{\mathcal{B}} \mathcal{E}/\mathcal{C}}$ to equivalences in $\mathcal{X}$, or equivalently the space

$$\mathrm{Map}(\mathrm{Unf}(\mathcal{C} \times_{\mathcal{B}} \mathcal{E})[\overline{W}^{-1}_{\mathcal{C} \times_{\mathcal{B}} \mathcal{E}/\mathcal{C}}], \mathcal{X})$$

of functors from the localization at $\overline{W}_{\mathcal{C} \times_{\mathcal{B}} \mathcal{E}/\mathcal{C}}$. Our next goal is to identify this localization with $\mathcal{C} \times_{\mathcal{B}} \mathcal{E}$.

Proposition 3.1.15 *The functor $\mathfrak{c}_{\mathcal{U}} : \mathcal{U}^\vee \times_{\mathcal{B}^{\mathrm{op}}} \mathrm{Tw}^r(\mathcal{B}) \to \mathcal{U}$ exhibits $\mathcal{U}$ as the localization at the class $W_{\mathcal{U}/\mathcal{B}}$.*

Proof The functor $\mathfrak{c}_{\mathcal{U}}$ is by construction a map of cartesian fibrations over $\mathcal{B}$ that preserves cartesian morphisms. On fibres over $b \in \mathcal{B}$, the functor

$$\mathfrak{c}_{\mathcal{U},b} : \mathcal{U}^\vee \times_{\mathcal{B}^{\mathrm{op}}} (\mathcal{B}_{b/})^{\mathrm{op}} \to \mathcal{U}_b$$

takes $(u \in \mathcal{U}_a, f : b \to a)$ to $f^*u \in \mathcal{U}_b$. This has a canonical section s_b, taking $u \in \mathcal{U}_b$ to (u, id_b). Moreover, the cocartesian morphisms in $\mathcal{U}^\vee$ determine a natural transformation $\mathrm{id} \to s_b\mathfrak{c}_{\mathcal{U},b}$, given for $(u, f : b \to a)$ by the cocartesian morphism $u \to f^*u$ and the triangle

$$\begin{tikzcd} & b \arrow[dl, "f"'] \arrow[dr, equal] & \\ a & & b. \arrow[ll, "f"] \end{tikzcd}$$

It follows that $\mathfrak{c}_{\mathcal{U},b}$ exhibits $\mathcal{U}_b$ as the localization of $\mathcal{U}^\vee \times_{\mathcal{B}^{\mathrm{op}}} (\mathcal{B}_{b/})^{\mathrm{op}}$ at the class $W_{\mathcal{U}/\mathcal{B},b}$. The same argument also shows that $\mathcal{U}_b$ is the localization at the larger class $W'_{\mathcal{U}/\mathcal{B},b}$ of morphisms

$$(u, f: b \to a) \to (u', f': b \to a')$$

over

$$\begin{tikzcd} & b \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ a & & a' \arrow[ll, "\alpha"] \end{tikzcd}.$$

and $\alpha^* u \to u'$, such that the induced morphism $f^*u \simeq f'^*\alpha^*u \to f'^*u'$ is an equivalence in $\mathcal{U}_b$. This class is compatible with cartesian pullback, in the sense that $\beta^* W'_{\mathcal{U}/\mathcal{B},b} \subseteq W'_{\mathcal{U}/\mathcal{B},b'}$ for $\beta: b' \to b$, and so [38, Proposition 2.1.4] implies that $\mathfrak{c}_{\mathcal{U}}$ exhibits $\mathcal{U}$ as the localization of $\mathcal{U}^\vee \times^{\mathrm{op}}_{\mathcal{B}} \mathrm{Tw}^r(\mathcal{B})$ at $W'_{\mathcal{U}/\mathcal{B}} := \bigcup_b W'_{\mathcal{U}/\mathcal{B},b}$. It thus only remains to see that localizing at $W'_{\mathcal{U}/\mathcal{B}}$ is the same as localizing at $W_{\mathcal{U}/\mathcal{B}}$, which follows from applying the 2-for-3 property of localizations using the diagram

$$\begin{tikzcd} b \arrow[r, equal] \arrow[d, "f"] & b \arrow[r, equal] \arrow[d, "f'"] & b \arrow[d, equal] \\ a & a' \arrow[l, "\alpha"] & b \arrow[l, "f'"] \end{tikzcd}$$

in $\mathrm{Tw}^r(\mathcal{B})$. $\square$

Corollary 3.1.16 *The functor* $\bar{\mathfrak{c}}_{\mathcal{U}}: \mathrm{Unf}(\mathcal{E}) \to \mathcal{E}$ *exhibits* $\mathcal{E}$ *as the localization of* $\mathrm{Unf}(\mathcal{E})$ *at the class* $\overline{W}_{\mathcal{E}/\mathcal{B}}$.

Proof It follows from [38, Proposition 2.1.4] and its proof that if $\mathcal{X} \to \mathcal{Y}$ is a cocartesian fibration and $\eta: \mathcal{Y}' \to \mathcal{Y}$ exhibits $\mathcal{Y}$ as the localization of $\mathcal{Y}'$ at the morphisms in W, then the canonical functor $\mathcal{X}' \to \mathcal{X}$ from the pullback $\mathcal{X}'$ of $\mathcal{X}$ along η exhibits $\mathcal{X}$ as the localization of $\mathcal{X}'$ at the cocartesian morphisms over W. $\square$

From this the universal property of $\widetilde{\mathcal{E}}_{\mathcal{X}}$ now follows using Remark 3.1.14:

Corollary 3.1.17 *For any ∞-category $\mathcal{X}$, there is a natural equivalence*

$$\mathrm{Map}_{/\mathcal{B}}(\mathcal{C}, \widetilde{\mathcal{E}}_{\mathcal{X}}) \simeq \mathrm{Map}(\mathcal{C} \times_{\mathcal{B}} \mathcal{E}, \mathcal{X}),$$

natural in $\mathcal{C} \in \mathrm{Cat}_{\infty/\mathcal{B}}$. $\square$

Remark 3.1.18 We remark briefly on the naturality of the construction in the cocartesian fibration. Suppose then that we have a commutative triangle

$$\begin{tikzcd} \mathcal{E}_0 \arrow[rr, "\phi"] \arrow[dr, "p_0"'] & & \mathcal{E}_1 \arrow[dl, "p_1"] \\ & \mathcal{U}, & \end{tikzcd}$$

and a cartesian fibration $q: \mathcal{U} \to \mathcal{B}$. We can replace the triangle by a cocartesian fibration $p: \mathcal{E} \to \mathcal{U} \times \Delta^1$ and then apply the construction to p and $q \times \Delta^1: \mathcal{U} \times \Delta^1 \to \mathcal{B} \times \Delta^1$ to obtain for any cocomplete ∞-category $\mathcal{X}$ a cocartesian fibration $\widetilde{\mathcal{E}}_{\mathcal{X}} \to \mathcal{B} \times \Delta^1$. By naturality of the construction the fibres at $i = 0, 1$ identify with $\widetilde{\mathcal{E}}_{i,\mathcal{X}} \to \mathcal{B}$ and so this cocartesian fibration corresponds to a commutative triangle

$$\begin{tikzcd} \widetilde{\mathcal{E}}_{0,\mathcal{X}} \arrow[rr] \arrow[dr] & & \widetilde{\mathcal{E}}_{1,\mathcal{X}} \arrow[dl] \\ & \mathcal{B}, & \end{tikzcd}$$

where the horizontal functor preserves cocartesian morphisms. On the fibre over $b \in \mathcal{B}$, this is the functor $\mathrm{Fun}(\mathcal{E}_{0,b}, \mathcal{X}) \to \mathrm{Fun}(\mathcal{E}_{1,b}, \mathcal{X})$ given by left Kan extension along $\phi_b \colon \mathcal{E}_{0,b} \to \mathcal{E}_{1,b}$. Making the same construction with Δ^1 replaced by Δ^n for all n it is easy to see that we get a functor $\widetilde{(-)}_{\mathcal{X}}$ from (small) cocartesian fibrations over $\mathcal{B}$ to (large) cocartesian fibrations over $\mathcal{B}$.

3.2 The day convolution double ∞-category

We now apply the construction of the previous subsection to obtain the Day convolution for a double ∞-category. First we need some notation:

Definition 3.2.1 Let $\mathbb{\Sigma}^n$ denote the partially ordered set of pairs of integers (i, j), $0 \leq i \leq j \leq n$, with $(i, j) \leq (i', j')$ if $i \leq i' \leq j' \leq j$. This determines a functor $\mathbb{\Sigma}^\bullet \colon \mathbb{\Delta} \to \mathrm{Cat}$ by taking $\phi \colon [n] \to [m]$ to the functor $\mathbb{\Sigma}^n \to \mathbb{\Sigma}^m$ that sends (i, j) to $(\phi(i), \phi(j))$; we write $\widehat{\mathbb{\Sigma}} \to \mathbb{\Delta}^{\mathrm{op}}$ for the cartesian fibration corresponding to this functor. We can also define a functor $\Pi \colon \widehat{\mathbb{\Sigma}} \to \mathbb{\Delta}^{\mathrm{op}}$ by sending $([n], (i, j))$ to $[j - i]$, with a map $([n], (i, j)) \to ([m], (i', j'))$, which corresponds to a map $\phi \colon [m] \to [n]$ in $\mathbb{\Delta}$ such that $(i, j) \leq (\phi(i'), \phi(j'))$ in $\mathbb{\Sigma}^m$, to the map $[j' - i'] \to [j - i]$ obtained by restricting ϕ to a map $\{i', i' + 1, \ldots, j'\} \to \{i, i + 1, \ldots, j\}$.

Remark 3.2.2 We can identify $\mathbb{\Sigma}^n$ with $\mathrm{Tw}^r(\Delta^n)$ and with $\mathbb{\Delta}^{\mathrm{int,op}}_{/[n]}$.

Definition 3.2.3 Let $\mathcal{M} \to \mathbb{\Delta}^{\mathrm{op}}$ be a double ∞-category. Then the base change

$$\mathbb{\Sigma}\mathcal{M} := \mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \widehat{\mathbb{\Sigma}} \to \widehat{\mathbb{\Sigma}}$$

along the functor Π is a cocartesian fibration. Applying the unfolding construction of the previous subsection to this together with the cartesian fibration $\widehat{\mathbb{\Sigma}} \to \mathbb{\Delta}^{\mathrm{op}}$, we get a new cocartesian fibration $\mathrm{Unf}(\mathbb{\Sigma}\mathcal{M}) \to \mathrm{Tw}^r(\mathbb{\Delta}^{\mathrm{op}})$, corresponding to a functor $\mathfrak{U}_{\mathbb{\Sigma}\mathcal{M}} \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Span}(\mathrm{Cat}_\infty)$, from which we obtain another cocartesian fibration

$$\widehat{\mathcal{M}}^+_{\mathcal{X}} := \widetilde{\mathbb{\Sigma}\mathcal{M}}_{\mathcal{X}} \to \mathbb{\Delta}^{\mathrm{op}},$$

where $\mathcal{X}$ is any cocomplete ∞-category.

Remark 3.2.4 The cocartesian fibration $\widehat{\mathcal{M}}^+_{\mathcal{X}} \to \mathbb{\Delta}^{\mathrm{op}}$ corresponds to a functor $\mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that takes $[n]$ to $\mathrm{Fun}(\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^n, \mathcal{X})$ and a morphism $\phi \colon [m] \to [n]$ in $\mathbb{\Delta}$ to

$$\mathrm{Fun}(\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^n, \mathcal{X}) \to \mathrm{Fun}(\phi^*(\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^n), \mathcal{X}) \to \mathrm{Fun}(\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^m, \mathcal{X})$$

where $\phi^*(\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^n)$ is equivalently the pullback $\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^m$ along the composite $\mathbb{\Sigma}^m \xrightarrow{\phi} \mathbb{\Sigma}^n \to \mathbb{\Delta}^{\mathrm{op}}$, the first functor is given by composition with the induced functor $\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^n \to \phi^*(\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^n)$, and the second by left Kan extension along $\phi^*(\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^n) \to \mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^m$. Since both $\phi^*(\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^n)$ and $\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^m$ are cocartesian fibrations over $\mathbb{\Sigma}^m$, and the functor preserves cocartesian morphisms, Lemma 3.1.11 implies that this left Kan extension is given at ($x \in \mathcal{M}_{j-i}$, $(i, j) \in \mathbb{\Sigma}^m$) by the colimit over

$$\mathcal{M}_{\phi(j)-\phi(i)/x} := \mathcal{M}_{\phi(j)-\phi(i)} \times_{\mathcal{M}_{j-i}} \mathcal{M}_{j-i/x},$$

where the functor $\mathcal{M}_{\phi(j)-\phi(i)} \to \mathcal{M}_{j-i}$ arises from the cocartesian morphisms over the restriction of ϕ to an (active) morphism $[j-i] \cong \{i, i+1, \ldots, j\} \to \{\phi(i), \phi(i)+1, \ldots, \phi(j)\} \cong [\phi(j)-\phi(i)]$. Note in particular that this is the identity if ϕ is an inert morphism.

Remark 3.2.5 Continuing from Remark 3.2.4, let us spell out the description a little further in the case of $d_1 \colon [1] \to [2]$: A functor $F \colon \mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^1 \to \mathcal{X}$ assigns to every $x \in \mathcal{M}_1$ a span

$$F(d_{1,!}x) \leftarrow F(x) \rightarrow F(d_{0,!}x)$$

in $\mathcal{X}$, while a functor $G \colon \mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^2 \to \mathcal{X}$ assigns to $(x, y) \in \mathcal{M}_2 \simeq \mathcal{M}_1 \times_{\mathcal{M}_0} \mathcal{M}_1$ a diagram

$$\begin{tikzcd}[column sep=small] & & G(x,y) \arrow[dl] \arrow[dr] & & \\ & G(x) \arrow[dl] \arrow[dr] & & G(y) \arrow[dl] \arrow[dr] & \\ G(d_{0,!}x) & & G(d_{1,!}x) \simeq G(d_{0,!}y) & & G(d_{1,!}y). \end{tikzcd}$$

In the first step, this is taken to the functor $d_1^*(\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^2) \to \mathcal{X}$ that to (x, y) assigns the span

$$G(d_{0,!}x) \leftarrow G(x, y) \rightarrow G(d_{1,!}y),$$

which then in the second step is taken to the functor $\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^1 \to \mathcal{X}$ that to $x \in \mathcal{M}_1$ assigns

$$G(d_{0,!}x) \leftarrow \operatorname*{colim}_{(x',y') \in \mathcal{M}_{2/x}} G(x', y') \rightarrow G(d_{1,!}x),$$

where the colimit is over the fibre product $\mathcal{M}_{2/x} := \mathcal{M}_2 \times_{\mathcal{M}_1} \mathcal{M}_{1/x}$ defined using $d_{1,!} \colon \mathcal{M}_2 \to \mathcal{M}_1$.

Remark 3.2.6 Note that if $\phi \colon [m] \to [n]$ is an inert morphism in $\mathbb{\Delta}$, then $\phi^*(\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^n) \to \mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^m$ is an equivalence, and so the functor

$$\mathrm{Fun}(\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^n, \mathcal{X}) \to \mathrm{Fun}(\mathcal{M} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathbb{\Sigma}^m, \mathcal{X})$$

is simply given by restriction.

We now define a subobject of $\widehat{\mathcal{M}}^+_{\mathcal{X}}$ that, in good cases, will be a double ∞-category:

Definition 3.2.7 Let $\mathbb{\Lambda}^n$ be the full subcategory of $\mathbb{\Sigma}^n$ on the objects (i, j) such that $j - i \leq 1$. We define $\mathbb{\Lambda}\mathcal{M}_n$ to be the pullback

$$\begin{tikzcd} \mathbb{\Lambda}\mathcal{M}_n \arrow[r] \arrow[d] & \mathbb{\Sigma}\mathcal{M}_n \arrow[d] \\ \mathbb{\Lambda}^n \arrow[r] & \mathbb{\Sigma}^n, \end{tikzcd}$$

and write $\widehat{\mathcal{M}}_{\mathcal{X}}$ for the full subcategory of $\widehat{\mathcal{M}}^+_{\mathcal{X}}$ spanned by the functors $\mathbb{\Sigma}\mathcal{M}_n \to \mathcal{X}$ that are right Kan extensions of their restrictions to $\mathbb{\Lambda}\mathcal{M}_n$, for all n.

Lemma 3.2.8 *A functor $F\colon \Sigma\mathcal{M}_n \to \mathcal{X}$ is a right Kan extension of its restriction to $\mathbb{A}\mathcal{M}_n$ if and only if for every object $((i,j) \in \Sigma^n, x_{ij} \in \mathcal{M}_{j-i})$ and every integer k, $i \leq k \leq j$, the commutative square*

$$\begin{array}{ccc} F(x_{ij}) & \longrightarrow & F(x_{ik}) \\ \downarrow & & \downarrow \\ F(x_{kj}) & \longrightarrow & F(x_{kk}) \end{array}$$

is cartesian, where $x_{ij} \to x_{i'j'}$ denotes the cocartesian morphism over $(i,j) \to (i',j')$.

Proof For $\xi = ((i,j) \in \Sigma^n, x_{ij} \in \mathcal{M}_{j-i})$, let $\mathbb{A}\mathcal{M}'_{n,\xi/}$ denote the full subcategory of $\mathbb{A}\mathcal{M}_{n,\xi/} := \mathbb{A}\mathcal{M}_n \times_{\Sigma\mathcal{M}_n} \Sigma\mathcal{M}_{n/\xi}$ spanned by the cocartesian morphisms. Then $\mathbb{A}\mathcal{M}'_{n,\xi/} \simeq \mathbb{A}^n_{(i,j)/} \simeq \mathbb{A}^{j-i}$ and the inclusion $\mathbb{A}\mathcal{M}'_{n,\xi/} \hookrightarrow \mathbb{A}\mathcal{M}_{n,\xi/}$ is coinitial. It follows that F is a right Kan extension of its restriction to $\mathbb{A}\mathcal{M}_n$ if and only if $F(x_{ij})$ is the limit over $F(x_{i'j'})$ with $(i',j') \in \mathbb{A}^n_{(i,j)/}$. Depicting the category $\mathbb{A}^n_{(i,j)/}$ as

$$\begin{array}{ccccccccc} & (i,i+1) & & & \cdots & & & (j-1,j) & \\ \swarrow & & \searrow & \swarrow & & \searrow & \swarrow & & \searrow \\ (i,i) & & & (i+1,i+1) & & (j-1,j-1) & & & (j,j), \end{array}$$

we see that the condition is that the map

$$F(x_{ij}) \to F(x_{i,i+1}) \times_{F(x_{(i+1)(i+1)})} \cdots \times_{F(x_{(j-1)(j-1)})} F(x_{(j-1)j})$$

must be an equivalence. Inducting on $j-i$ and using that limits commute, we see that this condition holds for all (i,j) if and only if the given commutative squares are all cartesian. □

Definition 3.2.9 Let $\mathcal{M}$ be a double ∞-category and $\mathcal{X}$ a cocomplete ∞-category with pullbacks. Fix an active morphism $\phi\colon [m] \to [n]$ in $\mathbb{A}$, an object $x_{0m} \in \mathcal{M}_m$, and an integer $0 \leq k \leq m$. Suppose given functors

$$\begin{aligned} F_{0k}&\colon \mathcal{M}_{\phi(k)/x_{0k}} \to \mathcal{X}, \\ F_{kk}&\colon \mathcal{M}_{0/x_{kk}} \to \mathcal{X}, \\ F_{km}&\colon \mathcal{M}_{n-\phi(k)/x_{km}} \to \mathcal{X}, \end{aligned}$$

where $x_{0m} \to x_{ij}$ is the cocartesian morphism over $\{i,\ldots,j\} \hookrightarrow [m]$ and $\mathcal{M}_{\phi(j)-\phi(i)/x_{ij}}$ is defined using the restriction of ϕ to an active morphism

$$[j-i] \cong \{i, i+1, \ldots, j\} \to \{\phi(i), \phi(i)+1, \ldots, \phi(j)\} \cong [\phi(i) - \phi(i)],$$

together with natural transformations $F_{0k} \to F_{kk}|_{\mathcal{M}_{\phi(k)/x_{0k}}}$, $F_{km} \to F_{kk}|_{\mathcal{M}_{n-\phi(k)/x_{km}}}$. Then define $F_{0m}\colon \mathcal{M}_{n/x_{0m}} \to \mathcal{X}$ as the fibre product $F_{0k} \times_{F_{kk}} F_{km}$ using these natural transformations and the equivalence $\mathcal{M}_{n/x_{0m}} \simeq \mathcal{M}_{\phi(k)/x_{0k}} \times_{\mathcal{M}_{0/x_{kk}}} \mathcal{M}_{n-\phi(k)/x_{km}}$. We then have an equivalence

$$\operatorname*{colim}_{\mathcal{M}_{n/x_{0m}}} F_{0m} \simeq \operatorname*{colim}_{\mathcal{M}_{\phi(k)/x_{0k}} \times_{\mathcal{M}_{0/x_{kk}}} \mathcal{M}_{n-\phi(k)/x_{km}}} F_{0k} \times_{F_{kk}} F_{km}.$$

This induces a canonical distributivity morphism

$$\operatorname*{colim}_{\mathcal{M}_{n/x_{0m}}} F|_{0m} \to \left(\operatorname*{colim}_{\mathcal{M}_{\phi(k)/x_{0k}}} F_{0k}\right) \times_{\left(\operatorname{colim}_{\mathcal{M}_{0/x_{kk}}} F_{kk}\right)} \left(\operatorname*{colim}_{\mathcal{M}_{n-\phi(k)/x_{km}}} F_{km}\right).$$

We say $\mathcal{M}$ is $\mathcal{X}$-*admissible* if this morphism is always an equivalence, i.e. if colimits over these slices of $\mathcal{M}$ distribute over pullbacks.

Remark 3.2.10 Given a functor $F\colon \Sigma\mathcal{M}_n \to \mathcal{X}$ that is right Kan extended from $\mathbb{A}\mathcal{M}_n$, the condition of $\mathcal{X}$-admissibility implies an equivalence

$$\operatorname*{colim}_{\mathcal{M}_{n/x_{0m}}} F|_{\mathcal{M}_{n/x_{0m}}} \to \left(\operatorname*{colim}_{\mathcal{M}_{\phi(k)/x_{0k}}} F|_{\mathcal{M}_{\phi(k)/x_{0k}}}\right) \times_{\left(\operatorname{colim}_{\mathcal{M}_{0/x_{kk}}} F|_{\mathcal{M}_{0/x_{kk}}}\right)} \left(\operatorname*{colim}_{\mathcal{M}_{n-\phi(k)/x_{km}}} F|_{\mathcal{M}_{n-\phi(k)/x_{km}}}\right).$$

Example 3.2.11 If $\mathcal{M}_0$ is an ∞-groupoid (in which case we may view $\mathcal{M}$ as an $(\infty,2)$-category, in the sense of a (not necessarily complete) 2-fold Segal space), then $(\mathcal{M}_0)_{/x_{kk}} \simeq *$, so $\mathcal{M}$ is $\mathcal{X}$-admissible provided pullbacks in $\mathcal{X}$ preserve colimits, i.e. colimits in $\mathcal{X}$ are universal. In particular, $\mathcal{M}$ is $\mathcal{X}$-admissible for any ∞-topos $\mathcal{X}$.

Proposition 3.2.12 *Suppose $\mathcal{M}$ is an $\mathcal{X}$-admissible double ∞-category. Then $\widehat{\mathcal{M}}_{\mathcal{X}}$ is a double ∞-category.*

Proof Using the description of $\widehat{\mathcal{M}}_{\mathcal{X}}$ in Lemma 3.2.8 we see that the condition of $\mathcal{X}$-admissibility is precisely set up so that for $F \in \widehat{\mathcal{M}}_{\mathcal{X},n}$ and $\phi\colon [m] \to [n]$ in Δ, the cocartesian morphism $F \to \phi_! F$ in $\widehat{\mathcal{M}}^{+}_{\mathcal{X}}$ lies in $\widehat{\mathcal{M}}_{\mathcal{X},m}$. Hence $\widehat{\mathcal{M}}_{\mathcal{X}} \to \Delta^{\mathrm{op}}$ is a cocartesian fibration. It remains to check that $\widehat{\mathcal{M}}_{\mathcal{X}}$ is a double ∞-category, i.e. that the functor

$$\widehat{\mathcal{M}}_{\mathcal{X},n} \to \widehat{\mathcal{M}}_{\mathcal{X},1} \times_{\widehat{\mathcal{M}}_{\mathcal{X},0}} \cdots \times_{\widehat{\mathcal{M}}_{\mathcal{X},0}} \widehat{\mathcal{M}}_{\mathcal{X},1}$$

is an equivalence. From the definition of $\widehat{\mathcal{M}}_{\mathcal{X}}$ it is immediate that we may identify $\widehat{\mathcal{M}}_{\mathcal{X},n}$ with $\operatorname{Fun}(\mathbb{A}\mathcal{M}_n, \mathcal{X})$, where $\mathbb{A}\mathcal{M}_n$ is equivalent to the pullback $\mathcal{M} \times_{\Delta^{\mathrm{op}}} \mathbb{A}^n$. Under this equivalence our functor is given by composition with

$$\mathbb{A}\mathcal{M}_1 \amalg_{\mathbb{A}\mathcal{M}_0} \cdots \amalg_{\mathbb{A}\mathcal{M}_0} \mathbb{A}\mathcal{M}_1 \to \mathbb{A}\mathcal{M}_n.$$

Since $\mathcal{M} \to \Delta^{\mathrm{op}}$ is a cocartesian fibration, pullback along it preserves colimits, hence this functor is equivalent to

$$\mathcal{M} \times_{\Delta^{\mathrm{op}}} (\mathbb{A}^1 \amalg_{\mathbb{A}^0} \cdots \amalg_{\mathbb{A}^0} \mathbb{A}^1) \to \mathcal{M} \times_{\Delta^{\mathrm{op}}} \mathbb{A}^n,$$

which is an equivalence by [33, Proposition 5.13]. $\square$

In the case where $\mathcal{X}$ is $\mathcal{S}$, we can give a more explicit description of the ∞-category $\widehat{\mathcal{M}}_{\mathcal{S}}(F,G)$ of horizontal morphisms from F to G, i.e. the fibre of $\widehat{\mathcal{M}}_{\mathcal{S},1} \to \widehat{\mathcal{M}}_{\mathcal{S},0}^{\times 2}$ at (F,G):

Notation 3.2.13 Given $F, G\colon \mathcal{M}_0 \to \mathcal{S}$ with corresponding left fibrations $\mathcal{F}, \mathcal{G} \to \mathcal{M}_0$, let $\mathcal{M}_{1,F,G} \to \mathcal{M}_1$ be the left fibration defined by the pullback

$$\begin{tikzcd}
\mathcal{M}_{1,F,G} \arrow[r] \arrow[d] & \mathcal{F} \times \mathcal{G} \arrow[d] \\
\mathcal{M}_1 \arrow[r] & \mathcal{M}_0 \times \mathcal{M}_0.
\end{tikzcd}$$

This left fibration corresponds to the functor

$$\mathcal{M}_1 \to \mathcal{M}_0^{\times 2} \xrightarrow{F \times G} \mathcal{S}.$$

Lemma 3.2.14 *The ∞-category $\Sigma\mathcal{M}_1$ is equivalent to the pushout* $(\mathcal{M}_1 \times \Sigma^1) \amalg_{\mathcal{M}_1 \amalg \mathcal{M}_1} (\mathcal{M}_0 \amalg \mathcal{M}_0)$.

Proof By definition $\Sigma\mathcal{M}_1$ is the pullback $\Sigma^1 \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{M}$. The category Σ^1 can be written as a pushout $\Delta^1 \amalg_{\{0\}} \Delta^1$, and since pullbacks along cocartesian fibrations preserve colimits we get a decomposition $\Sigma\mathcal{M}_1 \simeq (\Delta^1 \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{M}) \amalg_{\mathcal{M}_1} (\Delta^1 \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{M})$. By [25, Lemma 3.8] (which summarizes results of [43, Sect. 3.2.2]) for any cocartesian fibration $\mathcal{E} \to \Delta^1$ there is a pushout $\mathcal{E} \simeq \mathcal{E}_0 \times \Delta^1 \amalg_{\mathcal{E}_0 \times \{1\}} \mathcal{E}_1$. Applying this we get an equivalence

$$\Sigma\mathcal{M}_1 \simeq (\mathcal{M}_1 \times \Delta^1 \amalg_{\mathcal{M}_1 \times \{1\}} \mathcal{M}_0) \amalg_{\mathcal{M}_1 \times \{0\}} (\mathcal{M}_1 \times \Delta^1 \amalg_{\mathcal{M}_1 \times \{1\}} \mathcal{M}_0),$$

which we can rewrite as the desired expression. □

Proposition 3.2.15 *Given $F, G \colon \mathcal{M}_0 \to \mathcal{S}$, the ∞-category $\widehat{\mathcal{M}}_{\mathcal{S}}(F, G)$ is equivalent to the functor ∞-category* $\mathrm{Fun}(\mathcal{M}_{1,F,G}, \mathcal{S})$.

Proof By Lemma 3.2.14 we have a pullback square

$$\begin{array}{ccc} \mathrm{Fun}(\Sigma\mathcal{M}_1, \mathcal{S}) & \longrightarrow & \mathrm{Fun}(\mathcal{M}_1 \times \Sigma^1, \mathcal{S}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{M}_0, \mathcal{S})^{\times 2} & \longrightarrow & \mathrm{Fun}(\mathcal{M}_1, \mathcal{S})^{\times 2}. \end{array}$$

Now since $\Sigma^1 \simeq \{0, 1\}^{\triangleleft}$, for any ∞-category $\mathcal{C}$ we can identify the fibre of

$$\mathrm{Fun}(\Sigma^1, \mathcal{C}) \to \mathcal{C}^{\times 2}$$

at x, y with $\mathcal{C}_{/x,y} := \mathcal{C}_{/p}$ for the diagram $p \colon \{0, 1\} \to \mathcal{C}$ picking out x and y, and if $\mathcal{C}$ has products then $\mathcal{C}_{/x,y} \simeq \mathcal{C}_{/x\times y}$ by the universal property of the limit. We therefore have an equivalence between the fibre of $\mathrm{Fun}(\mathcal{M}_1 \times \Sigma^1, \mathcal{S}) \to \mathrm{Fun}(\mathcal{M}_1, \mathcal{S})^{\times 2}$ at (α, β) and $\mathrm{Fun}(\mathcal{M}_1, \mathcal{S})_{/\alpha\times\beta}$. Now [25, Proposition 9.7] describes this as $\mathrm{Fun}(\mathcal{E}, \mathcal{S})$ where $\mathcal{E} \to \mathcal{M}_1$ is the left fibration for the functor $\alpha \times \beta$. Together with the pullback square above, this identifies the fibre of $\mathrm{Fun}(\Sigma\mathcal{M}_1, \mathcal{S}) \to \mathrm{Fun}(\mathcal{M}_0, \mathcal{S})^{\times 2}$ at F, G with $\mathrm{Fun}(\mathcal{M}_{1,F,G}, \mathcal{S})$, as required. □

Remark 3.2.16 Let us reformulate the description of the horizontal composition from Remark 3.2.5 in terms of our description of horizontal morphisms: Suppose Φ is a horizontal morphism from F to G and Ψ is a horizontal morphism from G to H, so that we may view Φ as a functor $\mathcal{M}_{1,F,G} \to \mathcal{S}$ and Ψ as a functor $\mathcal{M}_{1,G,H} \to \mathcal{S}$, then their composite is the functor $\mathcal{M}_{1,F,H} \to \mathcal{S}$ given by

$$(x, p \in F(x_{00}), q \in H(x_{11})) \mapsto \operatorname*{colim}_{y\in\mathcal{M}_{2/x}} \operatorname*{colim}_{p'\in F(y_{00})_p} \operatorname*{colim}_{q'\in H(y_{22})_q} \operatorname*{colim}_{r\in H(y_{11})} \Phi(y_{01}, p', r) \times \Psi(y_{12}, r, q'),$$

where y_{ij} and x_{ij} denote the cocartesian pushforwards of y and x along the inert inclusion of $[j - i]$ as $\{i, i + 1, \dots, j\}$.

Note that if the ∞-category $\mathcal{M}_0$ is an ∞-groupoid, the formula above simplifies to

$$(x, p \in F(x_{00}), q \in H(x_{11})) \mapsto \operatorname*{colim}_{y\in\mathcal{M}_{2/x}} \operatorname*{colim}_{r\in H(y_{11})} \Phi(y_{01}, p, r) \times \Psi(y_{12}, r, q).$$

Remark 3.2.17 More generally, we can write the composition of n horizontal morphisms

$$\Phi_i \colon \mathcal{M}_{1,F_{i-1},F_i} \to \mathcal{S}, \qquad i = 1, \dots, n,$$

as the functor $\mathcal{M}_{1,F_0,F_n} \to \mathcal{S}$ given by

$$(x, p \in F_0(x_{00}), q \in F_n(x_{11})) \mapsto \operatorname*{colim}_{y\in\mathcal{M}_{n/x}} \operatorname*{colim}_{(t_0,\dots,t_n)\in F(y)_{p,q}} \Phi_1(y_{01}, t_0, t_1) \times \cdots \times \Phi_n(y_{(n-1)n}, t_{n-1}, t_n),$$

where $F(y)_{p,q} := F(y_{00})_p \times F(y_{11}) \times \cdots \times F(y_{(n-1)(n-1)}) \times F(y_{nn})_q$.

Remark 3.2.18 Let $\widehat{\mathcal{M}}^{\otimes}_{\mathcal{X},*}$ denote the full subcategory of $\widehat{\mathcal{M}}_{\mathcal{X}}$ spanned by functors $F\colon \Sigma\mathcal{M}_n \to \mathcal{X}$ such that for all maps $[0] \to [n]$ in $\mathbb{\Delta}$ the composite $\Sigma\mathcal{M}_0 \to \Sigma\mathcal{M}_n \to \mathcal{X}$ is constant at the terminal object of $\mathcal{X}$. For such F we have an equivalence

$$\operatorname*{colim}_{\mathcal{M}_{n/x_{0m}}} F|_{\mathcal{M}_{n/x_{0m}}} \simeq \operatorname*{colim}_{\mathcal{M}_{\phi(k)/x_{0k}} \times_{\mathcal{M}_{0/x_{kk}}} \mathcal{M}_{n-\phi(k)/x_{km}}} F|_{\mathcal{M}_{\phi(k)/x_{0k}}} \times F|_{\mathcal{M}_{n-\phi(k)/x_{km}}},$$

giving a canonical morphism

$$\operatorname*{colim}_{\mathcal{M}_{n/x_{0m}}} F|_{\mathcal{M}_{n/x_{0m}}} \to \left(\operatorname*{colim}_{\mathcal{M}_{\phi(k)/x_{0k}}} F|_{\mathcal{M}_{\phi(k)/x_{0k}}}\right) \times \left(\operatorname*{colim}_{\mathcal{M}_{n-\phi(k)/x_{km}}} F|_{\mathcal{M}_{n-\phi(k)/x_{km}}}\right).$$

$\widehat{\mathcal{M}}^{\otimes}_{\mathcal{X},*}$ is a monoidal ∞-category under the weaker hypothesis that this morphism is an equivalence. This holds, in particular, if $\mathcal{X}$ has finite products that commute with colimits in each variable, and the functors

$$\mathcal{M}_{n/x_{0m}} \to \mathcal{M}_{\phi(k)/x_{0k}} \times \mathcal{M}_{n-\phi(k)/x_{km}}$$

are cofinal.

Remark 3.2.19 From Remark 3.1.18 we see that any morphism of double ∞-categories $\mu\colon \mathcal{N} \to \mathcal{M}$ (i.e. a functor over $\mathbb{\Delta}^{\mathrm{op}}$ that preserves cocartesian morphisms) induces a functor $\widehat{\mu}\colon \widehat{\mathcal{N}}^+_{\mathcal{X}} \to \widehat{\mathcal{M}}^+_{\mathcal{X}}$ (given by taking left Kan extensions). However, even if both $\mathcal{N}$ and $\mathcal{M}$ are $\mathcal{X}$-admissible this does not necessarily restrict to a functor $\widehat{\mathcal{N}}_{\mathcal{X}} \to \widehat{\mathcal{M}}_{\mathcal{X}}$. Using Lemma 3.1.11 we see that this happens precisely when for every $x \in \mathcal{M}_2 \simeq \mathcal{M}_1 \times_{\mathcal{M}_0} \mathcal{M}_1$ the natural distributivity morphism

$$\operatorname*{colim}_{\mathcal{N}_{1/x_{01}} \times_{\mathcal{N}_{0/x_{11}}} \mathcal{N}_{1/x_{12}}} F_{01} \times_{F_{11}} F_{12} \to \operatorname*{colim}_{\mathcal{N}_{1/x_{01}}} F_{01} \times_{\operatorname{colim}_{\mathcal{N}_{0/x_{11}}} F_{11}} \operatorname*{colim}_{\mathcal{N}_{1/x_{12}}} F_{12}$$

is an equivalence for all functors $F\colon \mathcal{N}_2 \to \mathcal{S}$ in $\widehat{\mathcal{N}}_{\mathcal{X}}$. This happens, for instance, if $\mathcal{N}_{0/x_{11}}$ is a contractible ∞-groupoid and colimits in $\mathcal{X}$ are universal, or if $\mathcal{X}$ is $\mathcal{S}$ and all ∞-categories of the form $\mathcal{N}_{0/x_{11}}$ and $\mathcal{N}_{1/x_{01}}$ admit cofinal functors from ∞-groupoids (since colimits indexed by spaces distribute over limits by [12, Corollary 7.17]).

3.3 The universal property

Suppose $\mathcal{M}$ is an $\mathcal{X}$-admissible double ∞-category and $\mathcal{O}$ is a generalized non-symmetric ∞-operad. Our goal in this subsection is to show that there is a natural equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathcal{M}}_{\mathcal{X}}) \simeq \mathrm{Seg}_{\mathcal{O} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{M}}(\mathcal{X}).$$

Remark 3.3.1 We already know from Corollary 3.1.17 that $\widehat{\mathcal{M}}^+_{\mathcal{X}} \to \mathbb{\Delta}^{\mathrm{op}}$ has the universal property that there is a natural equivalence

$$\mathrm{Map}_{/\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{I}, \widehat{\mathcal{M}}^+_{\mathcal{X}}) \simeq \mathrm{Map}(\mathcal{I} \times_{\mathbb{\Delta}^{\mathrm{op}}} \Sigma\mathcal{M}, \mathcal{X}).$$

Our first goal is to reduce the right-hand side to functors from $\mathcal{I} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{M}$ by a further localization.

Remark 3.3.2 For $\mathcal{M} := \mathbb{\Delta}^{\mathrm{op}}$, the universal property we want was proved as [34, Corollary 3.11], and our proof will build on the constructions made there.

Notation 3.3.3 We recall some notation from [34]:

- The functor $\Pi\colon \widehat{\Sigma} \to \mathbb{\Delta}^{\mathrm{op}}$ has a section $\Psi\colon \mathbb{\Delta}^{\mathrm{op}} \to \widehat{\Sigma}$, taking $[n]$ to $([n],(0,n))$; there is also a natural transformation $\eta\colon \mathrm{id}_{\widehat{\Sigma}} \to \Psi\Pi$ given at $([n],(i,j))$ by the map $([n],(i,j)) \to ([j-i],(0,j-i))$ lying over the inert map $\rho_{ij}\colon [j-i] \to [n]$. Note that $\Pi\eta \simeq \mathrm{id}_{\Pi}$.
- We also have $p\Psi \cong \mathrm{id}_{\mathbb{\Delta}^{\mathrm{op}}}$, where p is the Cartesian fibration $\widehat{\Sigma} \to \mathbb{\Delta}^{\mathrm{op}}$, and a natural transformation $\epsilon\colon \Psi p \to \mathrm{id}_{\widehat{\Sigma}}$ given at $([n],(i,j))$ by the natural maps $([n],(0,n)) \to ([n],(i,j))$.
- We let I denote the set of cartesian morphisms in $\widehat{\Sigma}$ that lie over inert morphisms in $\mathbb{\Delta}^{\mathrm{op}}$.

The functor Π exhibits $\mathbb{\Delta}^{\mathrm{op}}$ as the localization of $\widehat{\Sigma}$ at I by [34, Proposition 3.8]. We can extend this as follows:

Proposition 3.3.4 *The projection $\Pi_{\mathcal{M}}\colon \Sigma\mathcal{M} \to \mathcal{M}$ exhibits $\mathcal{M}$ as the localization of $\Sigma\mathcal{M}$ at the set $I_{\mathcal{M}}$ of cocartesian morphisms that lie over I.*

Proof Let W be the class of morphisms in $\widehat{\Sigma}$ that are mapped to isomorphisms (i.e. identities) by Π and let $W_{\mathcal{M}}$ be the class of morphisms in $\Sigma\mathcal{M}$ that are mapped to equivalences by $\Pi_{\mathcal{M}}$; then $W_{\mathcal{M}}$ is precisely the class of cocartesian morphisms over W.

The section Ψ pulls back to a section $\Psi_{\mathcal{M}}\colon \mathcal{M} \to \Sigma\mathcal{M}$, and since $\Pi\eta \simeq \mathrm{id}_{\Pi}$ the transformation η pulls back to a natural transformation $\eta_{\mathcal{M}}\colon \mathrm{id}_{\Sigma\mathcal{M}} \to \Psi_{\mathcal{M}}\Pi_{\mathcal{M}}$. Then $\eta_{\mathcal{M}}$ is componentwise given by cocartesian morphisms in $\Sigma\mathcal{M}$ that lie over morphisms in W, and so this data becomes an equivalence of ∞-categories after localizing at W. In particular, $\Pi_{\mathcal{M}}$ exhibits $\mathcal{M}$ as the localization of $\Sigma\mathcal{M}$ at $W_{\mathcal{M}}$. It thus only remains to see that the localization at $I_{\mathcal{M}}$ is the same as the localization at $W_{\mathcal{M}}$. Since the morphisms involved are cocartesian, this follows from the same 2-of-3 argument as in the proof of [34, Proposition 3.8]. □

Proposition 3.3.5 *Suppose $f\colon \mathcal{I} \to \mathbb{\Delta}^{\mathrm{op}}$ is any functor such that $\mathcal{I}$ has f-cocartesian morphisms over inert maps in $\mathbb{\Delta}^{\mathrm{op}}$. Then there is a functor $\overline{\Pi}\colon \mathcal{I} \times_{\mathbb{\Delta}^{\mathrm{op}}} \Sigma\mathcal{M} \to \mathcal{I} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{M}$ lying over Π, which exhibits $\mathcal{I} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{M}$ as the localization at the set $I_{\mathcal{I},\mathcal{M}}$ of morphisms whose image in $\mathbb{\Delta}^{\mathrm{op}}$ is inert, whose image in $\mathcal{I}$ is cocartesian, and whose image in $\widehat{\Sigma}$ is cartesian (and whose image in $\mathcal{M}$ is therefore an equivalence).*

Proof As the proof of [34, Proposition 3.9], using the lifts defined in the proof of the previous proposition. □

Corollary 3.3.6 *Suppose $f\colon \mathcal{I} \to \mathbb{\Delta}^{\mathrm{op}}$ is any functor such that $\mathcal{I}$ has f-cocartesian morphisms over inert maps in $\mathbb{\Delta}^{\mathrm{op}}$. Then there is an equivalence*

$$\mathrm{Map}^{\mathrm{int}}_{/\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{I}, \widehat{\mathcal{M}}^{+}_{\mathcal{X}}) \simeq \mathrm{Map}(\mathcal{I} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{M}, \mathcal{X}),$$

natural in $\mathcal{I}$, where $\mathrm{Map}^{\mathrm{int}}_{/\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{I}, \widehat{\mathcal{M}}^{+}_{\mathcal{X}})$ denotes the subspace of $\mathrm{Map}_{/\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{I}, \widehat{\mathcal{M}}^{+}_{\mathcal{X}})$ consisting of functors that preserve the cocartesian morphisms over inert maps in $\mathbb{\Delta}^{\mathrm{op}}$.

Proof Using Proposition 3.3.5 we may identify $\mathrm{Map}(\mathcal{I} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{M}, \mathcal{X})$ with the subspace of $\mathrm{Map}(\mathcal{I} \times_{\mathbb{\Delta}^{\mathrm{op}}} \Sigma\mathcal{M}, \mathcal{X}) \simeq \mathrm{Map}_{/\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{I}, \widehat{\mathcal{M}}^{+}_{\mathcal{X}})$ consisting of functors that take morphisms in $I_{\mathcal{I},\mathcal{M}}$ to equivalences. Unwinding the definitions, we see that (as a cocartesian morphism in $\widehat{\mathcal{M}}^{+}_{\mathcal{X}}$ over an inert morphism does not involve a left Kan extension) these precisely correspond to the functors that preserve cocartesian morphisms over inert morphisms in $\mathbb{\Delta}^{\mathrm{op}}$. □

Replacing $\mathcal{I}$ by $\mathcal{I} \times \Delta^{\bullet}$ this induces an equivalence

$$\mathrm{Fun}^{\mathrm{int}}_{/\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{I}, \widehat{\mathcal{M}}^{+}_{\mathcal{X}}) \simeq \mathrm{Fun}(\mathcal{I} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{M}, \mathcal{X}).$$

Restricting further to functors with value in $\widehat{\mathcal{M}}_{\mathcal{X}}$, we get:

Corollary 3.3.7 *Suppose $\mathcal{O}$ is a generalized non-symmetric ∞-operad. Then there is an equivalence*

$$\mathrm{Fun}^{\mathrm{int}}_{/\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{O}, \widehat{\mathcal{M}}_{\mathcal{X}}) \simeq \mathrm{Seg}_{\mathcal{O} \times_{\mathbb{\Delta}^{\mathrm{op}}} \mathcal{M}}(\mathcal{X}),$$

natural in $\mathcal{O}$. □

3.4 Day convolution monoidal structures

Our goal in this subsection is to show that $\widehat{\mathcal{M}}_{\mathcal{X}}$ induces a family of monoidal ∞-categories, and in some cases (including for associative algebras) algebras in $\widehat{\mathcal{M}}_{\mathcal{X}}$ are algebras in these monoidal ∞-categories. This will follow from a general observation about *framed* double ∞-categories, which we will consider after some simple observations about algebras in double ∞-categories:

Definition 3.4.1 For any ∞-category $\mathcal{C}$, let $\mathbb{\Delta}^{\mathrm{op}}_{\mathcal{C}} \to \mathbb{\Delta}^{\mathrm{op}}$ be the terminal double ∞-category with $\mathcal{C}$ as its fibre at $[0]$. This is defined as the cocartesian fibration for the functor $\mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$ obtained as the right Kan extension along $\{[0]\} \hookrightarrow \mathbb{\Delta}^{\mathrm{op}}$ of the functor $\{[0]\} \to \mathrm{Cat}_{\infty}$ with value $\mathcal{C}$. Thus $(\mathbb{\Delta}^{\mathrm{op}}_{\mathcal{C}})_n \simeq \mathcal{C}^{\times(n+1)}$ with cocartesian morphisms over face maps given by projections and those over degeneracies given by diagonals. Note that there is in particular a canonical functor $\mathcal{C} \times \mathbb{\Delta}^{\mathrm{op}} \to \mathbb{\Delta}^{\mathrm{op}}_{\mathcal{C}}$, given fibrewise by the diagonal $\mathcal{C} \to \mathcal{C}^{\times(n+1)}$.

Definition 3.4.2 Given a double ∞-category $\mathcal{M}$, define $\mathcal{M}^{\otimes}$ as the pullback

$$\begin{tikzcd}
\mathcal{M}^{\otimes} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\
\mathcal{M}_0 \times \mathbb{\Delta}^{\mathrm{op}} \arrow[r] & \mathbb{\Delta}^{\mathrm{op}}_{\mathcal{M}_0}.
\end{tikzcd}$$

This is a pullback of cocartesian fibrations over $\mathbb{\Delta}^{\mathrm{op}}$ along functors that preserve cocartesian morphisms, hence $\mathcal{M}^{\otimes} \to \mathbb{\Delta}^{\mathrm{op}}$ is again a cocartesian fibration.

Proposition 3.4.3 *Suppose $\mathcal{O}$ is a generalized non-symmetric ∞-operad such that the inclusion $\mathcal{O}_0 \to \mathcal{O}$ induces an equivalence $\mathcal{O}_0 \xrightarrow{\sim} \mathcal{O}[I^{-1}]$ where I is the class of inert morphisms in $\mathcal{O}$. Let $\mathcal{M}$ be a double ∞-category. Then*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}^{\otimes}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{M})$$

is an equivalence.

Proof The double ∞-category $\mathcal{M}^{\otimes}$ is defined by a pullback square of generalized non-symmetric ∞-operads, so we have a pullback square

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}^{\otimes}) \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{M}) \arrow[d] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}_0 \times \mathbb{\Delta}^{\mathrm{op}}) \arrow[r] & \mathrm{Alg}_{\mathcal{O}}(\mathbb{\Delta}^{\mathrm{op}}_{\mathcal{M}_0}).
\end{tikzcd}$$

It therefore suffices to show that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}_0 \times \mathbb{\Delta}^{\mathrm{op}}) \to \mathrm{Alg}_{\mathcal{O}}(\mathbb{\Delta}^{\mathrm{op}}_{\mathcal{M}_0})$ is an equivalence if $\mathcal{O}$ satisfies the assumptions.

By definition $\mathrm{Alg}_{\mathcal{O}}(\mathbb{\Delta}^{\mathrm{op}}_{\mathcal{M}_0})$ is equivalent to $\mathrm{Fun}(\mathcal{O}_0, \mathcal{M}_0)$, while we may identify $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}_0 \times \mathbb{\Delta}^{\mathrm{op}})$ with the ∞-category of functors $\mathcal{O} \to \mathcal{M}_0$ that take inert morphisms to equivalences. We therefore have an equivalence if $\mathcal{O}_0 \to \mathcal{O}[I^{-1}]$ is an equivalence. □

Corollary 3.4.4 *Let $\mathcal{M}$ be a double ∞-category and $\mathcal{O}$ a non-symmetric ∞-operad such that the ∞-category $\mathcal{O}$ is weakly contractible. Then*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}^{\otimes}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{M})$$

is an equivalence.

Proof We must show that $\mathcal{O}[I^{-1}]$ is contractible, where I is the class of inert morphisms. Since $\mathcal{O}$ is weakly contractible, it suffices to check that inverting the inert morphisms in $\mathcal{O}$ amounts to inverting all morphisms. To see this we first observe that for any map $\phi\colon [n] \to [m]$ in $\mathbb{\Delta}$, we have a commutative triangle

$$\begin{tikzcd} & {[0]} \arrow[dl] \arrow[dr] & \\ {[n]} \arrow[rr, "\phi"] & & {[m]} \end{tikzcd}$$

where the maps from [0] are inert. Given a morphism $X \to Y$ in $\mathcal{O}$ we therefore have a commutative triangle

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & (), & \end{tikzcd}$$

where () denotes the unique object of $\mathcal{O}_0$ and the diagonal morphisms are cocartesian morphisms over the (inert) morphisms from [0] in the first triangle. □

Since $\mathbb{\Delta}^{\mathrm{op}}$ is weakly contractible, as a special case we have:

Corollary 3.4.5 *Let $\mathcal{M}$ be a double ∞-category. Then*

$$\mathrm{Alg}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{M}^{\otimes}) \to \mathrm{Alg}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{M})$$

is an equivalence. □

Definition 3.4.6 A double ∞-category $\mathcal{M}$ is *framed* if the functor $(d_{1,!}, d_{0,!})\colon \mathcal{M}_1 \to \mathcal{M}_0^{\times 2}$ is a cartesian fibration.

Remark 3.4.7 By [26, Proposition A.4.4], $(d_{1,!}, d_{0,!})$ is a cartesian fibration if and only if it is a cocartesian fibration, and this is also equivalent to the existence of "companions and conjoints" in (the homotopy double category of) $\mathcal{M}$.

Proposition 3.4.8 *Suppose $\mathcal{M}$ is a framed double ∞-category. Then $\mathcal{M}^{\otimes} \to \mathcal{M}_0$ is a cartesian fibration, and corresponds to a functor from $\mathcal{M}_0$ to monoidal ∞-categories and lax monoidal functors.*

Proof We apply the dual of [36, Lemma A.1.10] to $\mathcal{M}^{\otimes} \to \mathcal{M}_0 \times \mathbb{\Delta}^{\mathrm{op}}$ to conclude that $\mathcal{M}^{\otimes} \to \mathcal{M}_0$ is a cartesian fibration. We know that $\mathcal{M}^{\otimes} \to \mathbb{\Delta}^{\mathrm{op}}$ is a cocartesian fibration, so it remain to check that for $[n] \in \mathbb{\Delta}^{\mathrm{op}}$ the functor $\mathcal{M}^{\otimes}_n \to \mathcal{M}_0$ is a cartesian fibration. This functor lives in a pullback square

$$\begin{tikzcd}
\mathcal{M}^{\otimes}_n \arrow[r] \arrow[d] & \mathcal{M}_n \arrow[d] \\
\mathcal{M}_0 \arrow[r] & \mathcal{M}_0^{\times(n+1)}.
\end{tikzcd}$$

Here the right vertical map is equivalent to the iterated fibre product

$$\mathcal{M}_1 \times_{\mathcal{M}_0} \cdots \times_{\mathcal{M}_0} \mathcal{M}_1 \to (\mathcal{M}_0^{\times 2}) \times_{\mathcal{M}_0} \cdots \times_{\mathcal{M}_0} (\mathcal{M}_0^{\times 2}),$$

and so is a cartesian fibration since $\mathcal{M}$ is framed. It follows that $\mathcal{M}^{\otimes}_n \to \mathcal{M}_0$ is a cartesian fibration, and hence we can conclude that $\mathcal{M}^{\otimes} \to \mathcal{M}_0$ is a cartesian fibration and that cartesian morphisms lie over equivalences in $\mathbb{\Delta}^{\mathrm{op}}$.

The fibre $\mathcal{M}^{\otimes}_X \to \mathbb{\Delta}^{\mathrm{op}}$ at $X \in \mathcal{M}_0$ is a monoidal ∞-category, so it remains to check that the functor $f^*\colon \mathcal{M}^{\otimes}_Y \to \mathcal{M}^{\otimes}_X$ corresponding to the cartesian morphisms over $f\colon X \to Y$ in $\mathcal{M}_0$ is lax monoidal, i.e. preserves those cocartesian morphisms that lie over inert morphisms in $\mathbb{\Delta}^{\mathrm{op}}$.

Since the cartesian morphisms lie over identities in $\mathbb{\Delta}^{\mathrm{op}}$, it is equivalent to check that for every inert morphism $\phi\colon [m] \to [n]$, the functor $\mathcal{M}^{\otimes}_n \to \mathcal{M}^{\otimes}_m$, given by cocartesian pushforward along ϕ, preserves cartesian morphisms over $\mathcal{M}_0$. To see this it suffices to consider the outer face maps $[n-1] \to [n]$ (as any inert morphism is a composite of these). In this case we have a commutative diagram

$$\begin{tikzcd}[row sep=small, column sep=small]
\mathcal{M}^{\otimes}_n \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{M}_n \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{M}_1 \arrow[dr] \arrow[dd] & \\
& \mathcal{M}^{\otimes}_{n-1} \arrow[rr, crossing over] & & \mathcal{M}_{n-1} \arrow[rr, crossing over] & & \mathcal{M}_0 \arrow[dd, equal] \\
\mathcal{M}_0 \arrow[rr] \arrow[dr, equal] & & \mathcal{M}_0^{n+1} \arrow[rr] \arrow[dr] & & \mathcal{M}_0^2 \arrow[dr] & \\
& \mathcal{M}_0 \arrow[rr] \arrow[from=uu, crossing over] & & \mathcal{M}_0^n \arrow[rr] \arrow[from=uu, crossing over] & & \mathcal{M}_0.
\end{tikzcd}$$

Here all the vertical morphisms are cartesian fibrations, and in the left-hand cube the front and back faces are cartesian. It therefore suffices to show that $\mathcal{M}_n \to \mathcal{M}_{n-1}$ takes cartesian morphisms over $\mathcal{M}_0^{n+1}$ to cartesian morphisms over $\mathcal{M}_0^n$.

In the right-hand cube the top and bottom faces are cartesian, i.e. $\mathcal{M}_n \to \mathcal{M}_0^{n+1}$ is a fibre product of cartesian fibrations, and both morphisms to $\mathcal{M}_0$ preserve cartesian morphisms (since all morphisms are cartesian for the identity functor). A morphism in $\mathcal{M}_n$ is hence cartesian if and only if its images in both $\mathcal{M}_{n-1}$ and $\mathcal{M}_1$ are cartesian, and in particular the functor to $\mathcal{M}_{n-1}$ preserves cartesian morphisms, as required. □

Remark 3.4.9 The underlying ∞-category of the monoidal ∞-category $\mathcal{M}^{\otimes}_X$ is the ∞-category $\mathcal{M}(X, X)$ of horizontal endomorphisms of X, and the monoidal structure is given by horizontal composition. Since $\mathcal{M}$ is framed, a vertical morphism $f\colon X \to Y$ gives rise to two horizontal morphisms, say $f^{\triangleright} := (f, \mathrm{id}_Y)^*\mathbb{1}_Y$ from X to Y and $f^{\triangleleft} := (\mathrm{id}_Y, f)^*\mathbb{1}_Y$

from Y to X, and $f^{\triangleright}$ is left adjoint to $f^{\triangleleft}$. The underlying functor of the lax monoidal functor $\mathcal{M}_Y^{\otimes} \to \mathcal{M}_X^{\otimes}$ is the functor $\mathcal{M}(Y, Y) \to \mathcal{M}(X, X)$ given by $\Phi \mapsto f^{\triangleright} \odot_Y \Phi \odot_Y f^{\triangleleft}$, where we use $\odot_Y$ for horizontal composition over Y as in Notation 2.1.7. The lax monoidal structure comes from the unit transformation (note the non-standard order of composition)

$$f^{\triangleright} \odot_Y \Phi \odot_Y f^{\triangleleft} \odot_X f^{\triangleright} \odot_Y \Psi \odot_Y f^{\triangleleft} \to f^{\triangleright} \odot_Y \Phi \odot_Y \Psi \odot_Y f^{\triangleleft}.$$

Corollary 3.4.10 *Suppose $\mathcal{M}$ is a framed double ∞-category and $\mathcal{O}$ is a non-symmetric ∞-operad such that the ∞-category $\mathcal{O}$ is weakly contractible. Then the restriction functor*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}) \to \mathrm{Fun}(\mathcal{O}_0, \mathcal{M}_0) \simeq \mathcal{M}_0$$

is a cartesian fibration, corresponding to the functor $\mathcal{M}_0^{\mathrm{op}} \to \mathrm{Cat}_\infty$ taking $X \in \mathcal{M}_0$ to $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}_X^{\otimes})$.

Proof Observe that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}_{/\Delta^{\mathrm{op}}}(\mathcal{O}, \mathcal{M}^{\otimes}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{O}, \mathcal{M}^{\otimes}) \arrow[d] \\
\mathrm{Fun}(\mathcal{O}, \mathcal{M}_0) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{O}, \mathcal{M}_0 \times \Delta^{\mathrm{op}}) \arrow[d] \\
* \arrow[r] & \mathrm{Fun}(\mathcal{O}, \Delta^{\mathrm{op}}),
\end{tikzcd}$$

where the bottom and outer squares are clearly cartesian. Hence the top square is also cartesian, and here [43, Proposition 3.1.2.1] implies that all but the top left vertex are cartesian fibrations over $\mathrm{Fun}(\mathcal{O}, \mathcal{M}_0)$ and the morphisms to $\mathrm{Fun}(\mathcal{O}, \mathcal{M}_0 \times \Delta^{\mathrm{op}})$ preserve cartesian morphisms. Hence $\mathrm{Fun}_{/\Delta^{\mathrm{op}}}(\mathcal{O}, \mathcal{M}^{\otimes}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{M}_0)$ is a cartesian fibration.

Since $\mathcal{O}$ is weakly contractible, the inclusion $\mathcal{M}_0 \to \mathrm{Fun}(\mathcal{O}, \mathcal{M}_0)$ of the constant functors is a full subcategory. Let $\mathrm{Fun}'_{/\Delta^{\mathrm{op}}}(\mathcal{O}, \mathcal{M}^{\otimes}) \to \mathcal{M}_0$ denote the pullback of $\mathrm{Fun}_{/\Delta^{\mathrm{op}}}(\mathcal{O}, \mathcal{M}^{\otimes})$ along this inclusion; then $\mathrm{Fun}'_{/\Delta^{\mathrm{op}}}(\mathcal{O}, \mathcal{M}^{\otimes})$ is a full subcategory of $\mathrm{Fun}_{/\Delta^{\mathrm{op}}}(\mathcal{O}, \mathcal{M}^{\otimes})$ that contains $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}^{\otimes})$. The projection $\mathrm{Fun}'_{/\Delta^{\mathrm{op}}}(\mathcal{O}, \mathcal{M}^{\otimes}) \to \mathcal{M}_0$ is a cartesian fibration, so to show that $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}^{\otimes}) \to \mathcal{M}_0$ is a cartesian fibration it suffices to check that for every cartesian morphism in $\mathrm{Fun}'_{/\Delta^{\mathrm{op}}}(\mathcal{O}, \mathcal{M}^{\otimes})$ whose target is an $\mathcal{O}$-algebra, its source is also an $\mathcal{O}$-algebra; this follows from Proposition 3.4.8, which shows that cartesian morphisms preserve cocartesian morphisms over inert maps in Δ^{op}. Since $\mathrm{Alg}_{\mathcal{O}}(-)$ preserves pullbacks, the fibre at $X \in \mathcal{M}_0$ can be identified with $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}_X^{\otimes})$, and from the description of the cartesian morphisms in [43, Proposition 3.1.2.1] it follows that for $f \colon Y \to X$ in $\mathcal{M}_0$ the functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}_X^{\otimes}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{M}_Y^{\otimes})$ is given by composition with the lax monoidal functor $\mathcal{M}_X^{\otimes} \to \mathcal{M}_Y^{\otimes}$ arising from the cartesian morphisms over f in $\mathcal{M}^{\otimes}$. □

Lemma 3.4.11 *If $\mathcal{M}$ is an $\mathcal{X}$-admissible double ∞-category where $\mathcal{X}$ is cocomplete, then $\widehat{\mathcal{M}}_{\mathcal{X}}$ is framed.*

Proof It suffices to show that the source-and-target projection $\widehat{\mathcal{M}}_{\mathcal{X},1} \to \widehat{\mathcal{M}}_{\mathcal{X},0}^{\times 2}$ is a cocartesian fibration. This is the functor

$$i^* \colon \mathrm{Fun}(\Sigma\mathcal{M}_1, \mathcal{X}) \to \mathrm{Fun}(\mathcal{M}_0, \mathcal{X})^{\times 2}$$

given by composition with the inclusion $i \colon \mathcal{M}_0 \amalg \mathcal{M}_0 \to \mathcal{M} \times_{\Delta^{\mathrm{op}}} \Sigma^1$. To see that this is a cocartesian fibration we use the criterion of [32, Corollary 4.52].

First observe that i^* has a left adjoint $i_!$, given by left Kan extension along i. Note that for $X \in \mathcal{M}_1 \subseteq \Sigma\mathcal{M}_1$ the ∞-category $(\mathcal{M}_0 \amalg \mathcal{M}_0)_{/X}$ is empty, and so for $F\colon \mathcal{M}_0 \amalg \mathcal{M}_0 \to \mathcal{X}$ we have $i_!F(X) \simeq \emptyset$. Given $G\colon \Sigma\mathcal{M}_1 \to \mathcal{X}$ and $\phi\colon i^*G \to F$ consider the pushout square

$$\begin{array}{ccc} i_!i^*G & \longrightarrow & i_!F \\ \downarrow & & \downarrow \\ G & \longrightarrow & G'. \end{array}$$

Since pushouts in $\mathrm{Fun}(\Sigma\mathcal{M}_1, \mathcal{X})$ are computed pointwise, we see that $F \xrightarrow{\sim} i^*i_!F \xrightarrow{\sim} i^*G'$ is an equivalence, which is what we need in order to apply [32, Corollary 4.52] to conclude that i^* is a cocartesian fibration (with $G \to G'$ being the cocartesian morphism over $i^*G \to F$). □

Remark 3.4.12 Suppose $\mathcal{M}$ is an $\mathcal{S}$-admissible double ∞-category. Given natural transformations $\phi\colon F \to F'$, $\gamma\colon G \to G'$ of functors $\mathcal{M}_0 \to \mathcal{S}$ and $\Phi\colon \mathcal{M}_{1,F,G} \to \mathcal{S}$, the description of the cocartesian pushforward of Φ along (ϕ, γ) above amounts to this being given by the left Kan extension along the induced functor $\mathcal{M}_{1,F,G} \to \mathcal{M}_{1,F',G'}$, i.e.

$$(\phi, \gamma)_!\Phi(x, p', q') \simeq \operatorname*{colim}_{p \in F(x_{00})_{p'}, q \in G(x_{11})_{q'}} \Phi(x, p, q).$$

It follows that the cartesian pullback of $\Psi\colon \mathcal{M}_{1,F',G'} \to \mathcal{S}$ is given by composition with this functor.

Applying Corollary 3.4.10 to $\widehat{\mathcal{M}}_{\mathcal{X}}$, we now get:

Corollary 3.4.13 *Let $\mathcal{O}$ be a non-symmetric ∞-operad such that the ∞-category $\mathcal{O}$ is weakly contractible and $\mathcal{M}$ an $\mathcal{X}$-admissible double ∞-category where $\mathcal{X}$ is cocomplete. Then*

$$\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathcal{M}}_{\mathcal{X}}) \to \mathrm{Fun}(\mathcal{M}_0, \mathcal{S})$$

is a cartesian fibration, corresponding to the functor

$$X \in \mathrm{Fun}(\mathcal{M}_0, \mathcal{S}) \mapsto \mathrm{Alg}_{\mathcal{O}}(\widehat{\mathcal{M}}_{\mathcal{X}}(X, X))$$

arising from the family of monoidal ∞-categories $\widehat{\mathcal{M}}^{\otimes}_{\mathcal{X},X}$. □

Remark 3.4.14 Suppose $\mathcal{M}$ is $\mathcal{S}$-admissible and $\mathcal{O}$ is as above. Then we can use the equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathcal{M}}_{\mathcal{X}}) \simeq \mathrm{Seg}_{\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{M}}(\mathcal{S})$$

and the description of the cartesian fibration from Proposition 2.1.19 to conclude that there is fibrewise a natural equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathcal{M}}_{\mathcal{S}}(X, X)) \simeq \mathrm{Mon}_{\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{M}_X}(\mathcal{S}) \simeq \mathrm{Alg}_{\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{M}_X}(\mathcal{S}).$$

3.5 Enriched day convolution

In this subsection we generalize our construction slightly by showing that if $\mathcal{M}$ is a double ∞-category and $\mathcal{V}^{\otimes}$ is an $\mathcal{M}$-monoidal ∞-category then in good cases there exists a double ∞-category $\widehat{\mathcal{M}}_{\mathcal{V}}$ such that we have a natural equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathcal{M}}_{\mathcal{V}}) \simeq \mathrm{Algd}_{\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{M}}(\mathcal{V}).$$

More generally, an object $\widehat{\mathcal{M}}_{\mathcal{V}}$ with this property will exist as a generalized non-symmetric ∞-operad that is not a double ∞-category. If $\mathcal{V}^{\otimes}$ is given by Day convolution, we can obtain $\widehat{\mathcal{M}}_{\mathcal{V}}$ from the following observation:

Proposition 3.5.1 *Suppose $\mathcal{M}$ is an $\mathcal{X}$-admissible double ∞-category where pullbacks in $\mathcal{X}$ preserve colimits, and $\mathcal{U}^{\otimes} \to \mathcal{M}$ is a small $\mathcal{M}$-monoidal ∞-category. Then $\mathcal{U}^{\otimes}$ is also an $\mathcal{X}$-admissible double ∞-category.*

Proof Given an active morphism $\phi\colon [m] \to [n]$ in Δ and an object $u_{0m} \in \mathcal{U}^{\otimes}_{m}$ over $x_{0m} \in \mathcal{M}_m$, the functor $\mathcal{U}^{\otimes}_{n/u_{0m}} \to \mathcal{M}_{n/x_{0m}}$ is a cocartesian fibration, whose fibre at y_{0n} and $\bar{\phi}\colon y_{0n} \to x_{0m}$ over ϕ we can identify with the slice $(\mathcal{U}^{\otimes}_{n,y_{0n}})_{/u_{0m}}$, defined using the cocartesian morphisms over $\bar{\phi}$. For $0 \leq k \leq m$ we can decompose this as a product

$$(\mathcal{U}^{\otimes}_{n,y_{0n}})_{/u_{0m}} \simeq (\mathcal{U}^{\otimes}_{\phi(k),y_{0\phi(k)}})_{/u_{0k}} \times (\mathcal{U}^{\otimes}_{n-\phi(k),y_{\phi(k)n}})_{/u_{km}}.$$

Given functors

$$\begin{aligned} F_{0k} &\colon \mathcal{U}^{\otimes}_{\phi(k)/u_{0k}} \to \mathcal{X}, \\ F_{kk} &\colon \mathcal{U}^{\otimes}_{0/u_{kk}} \simeq \mathcal{M}_{0/x_{kk}} \to \mathcal{X}, \\ F_{km} &\colon \mathcal{U}^{\otimes}_{n-\phi(k)/u_{km}} \to \mathcal{X}, \end{aligned}$$

together with natural transformations $F_{0k} \to F_{kk}|_{\mathcal{U}^{\otimes}_{\phi(k)/u_{0k}}}$, $F_{km} \to F_{kk}|_{\mathcal{U}^{\otimes}_{n-\phi(k)/u_{km}}}$, we have equivalences

$$\begin{aligned} \operatorname*{colim}_{\mathcal{U}^{\otimes}_{n/u_{0m}}} F_{0k} \times_{F_{kk}} F_{km} &\simeq \operatorname*{colim}_{y_{0n}\in\mathcal{M}_{n/x_{0m}}} \operatorname*{colim}_{(\mathcal{U}^{\otimes}_{n,y_{0n}})_{/u_{0m}}} F_{0k} \times_{F_{kk}} F_{km} \\ &\simeq \operatorname*{colim}_{y_{0n}\in\mathcal{M}_{n/x_{0m}}} \left(\operatorname*{colim}_{(\mathcal{U}^{\otimes}_{\phi(k),y_{0\phi(k)}})_{/u_{0k}}} F_{0k} \right) \times_{F_{kk}} \left(\operatorname*{colim}_{(\mathcal{U}^{\otimes}_{n-\phi(k),y_{\phi(k)n}})_{/u_{km}}} F_{km} \right) \\ &\simeq \left(\operatorname*{colim}_{y_{0\phi(k)}\mathcal{M}_{\phi(k)/x_{0k}}} \operatorname*{colim}_{(\mathcal{U}^{\otimes}_{\phi(k),y_{0\phi(k)}})_{/u_{0k}}} F_{0k} \right) \\ &\quad \times_{\left(\operatorname{colim}_{\mathcal{M}_{0/x_{kk}}} F_{kk}\right)} \left(\operatorname*{colim}_{y_{\phi(k)n}\in\mathcal{M}_{n-\phi(k)/x_{km}}} \operatorname*{colim}_{(\mathcal{U}^{\otimes}_{n-\phi(k),y_{\phi(k)n}})_{/u_{km}}} F_{km} \right) \\ &\simeq \left(\operatorname*{colim}_{\mathcal{U}^{\otimes}_{\phi(k)/u_{0k}}} F_{0k} \right) \times_{\left(\operatorname{colim}_{\mathcal{U}^{\otimes}_{0/u_{kk}}} F_{kk}\right)} \left(\operatorname*{colim}_{\mathcal{U}^{\otimes}_{n-\phi(k)/u_{km}}} F_{km} \right), \end{aligned}$$

where the second equivalence uses that fibre products in $\mathcal{X}$ preserve colimits in each variable and the third equivalence uses the $\mathcal{X}$-admissibility of $\mathcal{M}$. □

Corollary 3.5.2 *Let $\mathcal{M}$ be an $\mathcal{S}$-admissible double ∞-category and let $\mathcal{U}^{\otimes}$ be a small $\mathcal{M}$-monoidal ∞-category. Then there exists a double ∞-category $\widehat{\mathcal{U}^{\otimes}_{\mathcal{S}}}$ such that for any generalized non-symmetric ∞-operad $\mathcal{O}$ we have a natural equivalence*

$$\operatorname{Alg}_{\mathcal{O}}(\widehat{\mathcal{U}^{\otimes}_{\mathcal{S}}}) \simeq \operatorname{Seg}_{\mathcal{O}\times_{\Delta^{\mathrm{op}}}\mathcal{U}^{\otimes}}(\mathcal{S}) \simeq \operatorname{Algd}_{\mathcal{O}\times_{\Delta^{\mathrm{op}}}\mathcal{M}}(\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{U}^{\otimes}_{\mathcal{S}}).$$

Moreover, any $\mathcal{M}$-monoidal functor $\mathcal{U}^{\otimes} \to \mathcal{V}^{\otimes}$ induces a morphism of double ∞-categories $\widehat{\mathcal{U}^{\otimes}_{\mathcal{S}}} \to \widehat{\mathcal{V}^{\otimes}_{\mathcal{S}}}$.

Proof The only part that is not an immediate consequence of Proposition 3.5.1 and our results in the previous subsections is the claim about $\mathcal{M}$-monoidal functors, which follows using the criterion of Remark 3.2.19, where we can compute the colimits by decomposing them as in the proof of Proposition 3.5.1. $\square$

Remark 3.5.3 Let us describe the double ∞-category $\widehat{\mathcal{U}^{\otimes}_{\mathcal{S}}}$ a bit more explicitly. The objects of $\widehat{\mathcal{U}^{\otimes}_{\mathcal{S}}}$ are functors $\mathcal{U}^{\otimes}_0 \simeq \mathcal{M}_0 \to \mathcal{S}$, and the vertical morphisms are transformations of such functors. If $F, G \colon \mathcal{M}_0 \to \mathcal{S}$ are two objects, then by Proposition 3.2.15 the ∞-category $\widehat{\mathcal{U}^{\otimes}_{\mathcal{S}}}(F, G)$ of horizontal morphisms can be identified with $\mathrm{Fun}(\mathcal{U}^{\otimes}_{1,F,G}, \mathcal{S})$; here $\mathcal{U}^{\otimes}_{1,F,G} \simeq \mathcal{U}^{\otimes}_1 \times_{\mathcal{M}_1} \mathcal{M}_{1,F,G}$ where $\mathcal{U}^{\otimes}_1 \to \mathcal{M}_1$ is a cocartesian fibration. Using Remark 2.1.17 and [25, Proposition 7.3] we can identify this ∞-category with $\mathrm{Fun}_{/\mathcal{M}_1}(\mathcal{M}_{1,F,G}, \mathcal{U}^{\otimes}_{\mathcal{S},1})$. Translating the formula for composition of horizontal morphisms from Remark 3.2.16 in these terms, the composite of $\Phi \colon \mathcal{M}_{1,F,G} \to \mathcal{U}^{\otimes}_{\mathcal{S},1}$ and $\Psi \colon \mathcal{M}_{1,G,H} \to \mathcal{U}^{\otimes}_{\mathcal{S},1}$ is the functor from $\mathcal{M}_{1,F,H}$ given by

$$\begin{aligned}&(x, p \in F(x_{00}), q \in H(x_{11}))\\ &\quad\mapsto \operatorname*{colim}_{(\alpha\colon y \to x) \in \mathcal{M}_{2/x}} \operatorname*{colim}_{p' \in F(y_{00})_p} \operatorname*{colim}_{q' \in H(y_{22})_q} \operatorname*{colim}_{r \in H(y_{11})} \alpha_!(\Phi(y_{01}, p', r), \Psi(y_{12}, r, q')),\end{aligned}$$

where $\alpha_! \colon \mathcal{U}^{\otimes}_{\mathcal{S},1,y_{01}} \times \mathcal{U}^{\otimes}_{\mathcal{S},1,y_{12}} \simeq \mathcal{U}^{\otimes}_{\mathcal{S},2,y} \to \mathcal{U}^{\otimes}_{\mathcal{S},1,x}$ is the cocartesian pushforward along α (given by a left Kan extension along the corresponding operation for $\mathcal{U}^{\otimes}$).

Remark 3.5.4 For an $\mathcal{M}$-monoidal ∞-category of the form $\mathcal{M} \times_{\Delta^{\mathrm{op}}} \mathcal{C}^{\otimes}$, where $\mathcal{C}^{\otimes} \to \Delta^{\mathrm{op}}$ is a monoidal ∞-category, the description above simplifies considerably: We can identify $(\mathcal{M} \times_{\Delta^{\mathrm{op}}} \mathcal{C}^{\otimes})_{\mathcal{S}}$ with the pullback $\mathcal{M} \times_{\Delta^{\mathrm{op}}} \mathcal{C}^{\otimes}_{\mathcal{S}}$, so that horizontal morphisms reduce to functors $\mathcal{M}_{1,F,G} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})$. The composition of $\Phi \colon \mathcal{M}_{1,F,G} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ and $\Psi \colon \mathcal{M}_{1,G,H} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ is then given by the formula

$$\begin{aligned}(x, p \in F(x_{00}), q \in H(x_{11})) \mapsto &\operatorname*{colim}_{y \in \mathcal{M}_{2/x}} \operatorname*{colim}_{p' \in F(y_{00})_p} \operatorname*{colim}_{q' \in H(y_{22})_q} \operatorname*{colim}_{r \in H(y_{11})} \Phi(y_{01}, p', r)\\ &\otimes \Psi(y_{12}, r, q'),\end{aligned}$$

where $\otimes$ denotes the Day convolution.

More generally, we can obtain $\widehat{\mathcal{M}}_{\mathcal{V}}$ by passing to a larger universe:

Definition 3.5.5 Let $\widehat{\mathcal{S}}$ denote the (very large) ∞-category of large spaces. If $\mathcal{V}^{\otimes}$ is a locally small (but potentially large) $\mathcal{M}$-monoidal ∞-category, let $\widehat{\mathcal{M}}_{\mathcal{V}}$ denote the full subcategory of $\widehat{\mathcal{V}^{\mathrm{op},\otimes}_{\widehat{\mathcal{S}}}}$ spanned by the objects whose

- inert restrictions to the fibre at 0 are functors $\mathcal{M}_0 \to \widehat{\mathcal{S}}$ that factor through the full subcategory $\mathcal{S}$,
- inert restrictions to the fibre at 1 correspond to functors of the form

$$\mathcal{M}_{1,F,G} \to \mathcal{V}^{\mathrm{op},\otimes}_{\widehat{\mathcal{S}},1}$$

  that factor through the full subcategory $\mathcal{V}^{\otimes}_1$ (via the Yoneda embedding $\mathcal{V}^{\otimes} \hookrightarrow \mathcal{V}^{\mathrm{op},\otimes}_{\widehat{\mathcal{S}}}$).

Proposition 3.5.6 *Let $\mathcal{M}$ be an $\mathcal{S}$-admissible double ∞-category and $\mathcal{V}^{\otimes}$ a locally small $\mathcal{M}$-monoidal ∞-category.*

(i) *$\widehat{\mathcal{M}}_{\mathcal{V}}$ is a generalized non-symmetric ∞-operad.*

(ii) *For any generalized non-symmetric ∞-operad $\mathcal{O}$ we have a natural equivalence*

$$\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathcal{M}}_{\mathcal{V}}) \simeq \mathrm{Algd}_{\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{M}}(\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{V}^{\otimes}).$$

(iii) *An $\mathcal{M}$-monoidal functor $\mathcal{U}^{\otimes} \to \mathcal{V}^{\otimes}$ induces a natural morphism of generalized non-symmetric ∞-operads $\widehat{\mathcal{M}}_{\mathcal{U}} \to \widehat{\mathcal{M}}_{\mathcal{V}}$.*

(iv) *$\widehat{\mathcal{M}}_{\mathcal{V},1} \to \widehat{\mathcal{M}}_{\mathcal{V},0}^{\times 2} \simeq \mathrm{Fun}(\mathcal{M}_0, \mathcal{S})^{\times 2}$ is a cartesian fibration, and the inclusion $\widehat{\mathcal{M}}_{\mathcal{V},1} \to \widehat{\mathcal{V}_{\widehat{\mathcal{S}}}^{\mathrm{op},\otimes}}$ preserves cartesian morphisms.*

Proof Part (i) follows from Lemma 2.1.11, which also identifies $\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathcal{M}}_{\mathcal{V}})$ with the full subcategory of $\mathcal{O}$-algebras in $\widehat{\mathcal{V}_{\widehat{\mathcal{S}}}^{\mathrm{op},\otimes}}$ whose restrictions to $\mathcal{O}_0$ and $\mathcal{O}_1$ factor through $\mathcal{S} \subseteq \widehat{\mathcal{S}}$ and $\mathcal{V}_1^{\otimes} \subseteq (\widehat{\mathcal{V}_{\widehat{\mathcal{S}}}^{\mathrm{op},\otimes}})_1$. Under the equivalence between $\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathcal{V}_{\widehat{\mathcal{S}}}^{\mathrm{op},\otimes}})$ and $\widehat{\mathrm{Algd}}_{\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{M}}(\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{V}_{\widehat{\mathcal{S}}}^{\mathrm{op},\otimes})$ from Corollary 3.5.2 (where the latter denotes the ∞-category of $\mathcal{O}$-algebroids defined using $\widehat{\mathcal{S}}$), this full subcategory corresponds to that of $\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{M}$-algebroids $(\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{M})_X \to \mathcal{V}_{\widehat{\mathcal{S}}}^{\mathrm{op},\otimes}$ such that X is a functor $\mathcal{O}_0 \times \mathcal{M}_0 \to \mathcal{S}$ and the functor $(\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{M})_{X,1} \to \mathcal{V}_{\widehat{\mathcal{S}},1}^{\mathrm{op},\otimes}$ factors through $\mathcal{V}_1^{\otimes}$; since the Yoneda embedding is a fully faithful $\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{M}$-monoidal functor, this full subcategory is precisely $\mathrm{Algd}_{\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{M}}(\mathcal{O} \times_{\Delta^{\mathrm{op}}} \mathcal{V}^{\otimes})$, which gives (ii). From Corollary 3.5.2 we also know that an $\mathcal{M}$-monoidal functor induces a morphism of double ∞-categories $\widehat{\mathcal{U}_{\widehat{\mathcal{S}}}^{\mathrm{op},\otimes}} \to \widehat{\mathcal{V}_{\widehat{\mathcal{S}}}^{\mathrm{op},\otimes}}$; this evidently takes the full subcategory $\widehat{\mathcal{M}}_{\mathcal{U}}$ into $\widehat{\mathcal{M}}_{\mathcal{V}}$, which proves (iii). We know that $\widehat{\mathcal{V}_{\widehat{\mathcal{S}}}^{\mathrm{op},\otimes}}$ is a framed double ∞-category, so that $\widehat{\mathcal{V}_{\widehat{\mathcal{S}},1}^{\mathrm{op},\otimes}} \to (\widehat{\mathcal{V}_{\widehat{\mathcal{S}},0}^{\mathrm{op},\otimes}})^{\times 2}$ is a cartesian and cocartesian fibration. From the description of the cartesian morphisms in Remark 3.4.12 it follows that these restrict to $\widehat{\mathcal{M}}_{\mathcal{V}}$, which proves (iv). □

Proposition 3.5.7 *Let $\mathcal{M}$ be an $\mathcal{S}$-admissible double ∞-category.*

(i) *Suppose that $\mathcal{V}^{\otimes}$ is a locally small $\mathcal{M}$-monoidal ∞-category such that:*

  (1) *For every $x \in \mathcal{M}_1$ the ∞-category $\mathcal{V}_x$ has colimits indexed by ∞-groupoids and by the ∞-categories $\mathcal{M}_{n/y}$ for $y \in \mathcal{M}_1$,*

  (2) *These colimits are preserved by the functors $f_!\colon \mathcal{V}_x \to \mathcal{V}_{x'}$ induced by the cocartesian morphisms over $f\colon x \to x'$ in $\mathcal{M}_1$.*

  Then $\widehat{\mathcal{M}}_{\mathcal{V}} \to \Delta^{\mathrm{op}}$ is a locally cocartesian fibration.

(ii) *Suppose $\mathcal{V}^{\otimes}$ and $\mathcal{U}^{\otimes}$ are locally small $\mathcal{M}$-monoidal ∞-categories satisfying conditions (1) and (2) in (i), and $\mathcal{U}^{\otimes} \to \mathcal{V}^{\otimes}$ is an $\mathcal{M}$-monoidal functor such that:*

  (3) *For all x the functor $\mathcal{U}_x \to \mathcal{V}_x$ preserves colimits indexed by ∞-groupoids and by the ∞-categories $\mathcal{M}_{n/y}$.*

  Then the induced morphism $\widehat{\mathcal{M}}_{\mathcal{U}} \to \widehat{\mathcal{M}}_{\mathcal{V}}$ of generalized non-symmetric ∞-operads from Proposition 3.5.6(iii) preserves locally cocartesian morphisms.

Proof We already know from Proposition 3.5.6 that $\widehat{\mathcal{M}}_{\mathcal{V}}$ is a generalized non-symmetric ∞-operad, so to prove (i) it suffices by Lemma 2.1.25 to show there are locally cocartesian morphisms over the active morphisms to $[1]$. Let $\alpha_n\colon [1] \to [n]$ denote the unique active morphism to $[n]$ in Δ. Given an object of $\widehat{\mathcal{M}}_{\mathcal{V}}$, which we can identify with $(X_1, \ldots, X_n)$ with

X_i in $\widehat{\mathcal{M}}_{\mathcal{V}}(F_{i-1}, F_i)$ for $F_i \colon \mathcal{M}_0 \to \mathcal{S}$, as well as Y in $\widehat{\mathcal{M}}_{\mathcal{V}}(G, H)$, we have

$$\begin{aligned} \mathrm{Map}^{\alpha_n}_{\widehat{\mathcal{M}}_{\mathcal{V}}}((X_1, \ldots, X_n), Y) &\simeq \mathrm{Map}^{\alpha_n}_{\widehat{\mathcal{V}^{\mathrm{op},\otimes}_{\mathcal{S}}}}((X_1, \ldots, X_n), Y) \\ &\simeq \mathrm{Map}_{\widehat{\mathcal{V}^{\mathrm{op},\otimes}_{\mathcal{S},1}}}(X_1 \odot_{F_1} \cdots \odot_{F_{n-1}} X_n, Y) \end{aligned}$$

Since $\widehat{\mathcal{V}^{\mathrm{op},\otimes}_{\mathcal{S}}}$ is a framed double ∞-category, if we take the fibre of the right-hand side over maps $g \colon F_0 \to G$, $h \colon F_n \to G$, we get

$$\begin{aligned} &\mathrm{Map}_{\widehat{\mathcal{V}^{\mathrm{op},\otimes}_{\mathcal{S},1}}}(X_1 \odot_{F_1} \cdots \odot_{F_{n-1}} X_n, Y)_{(g,h)} \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{M}_{1,F_0,F_n}, \mathcal{V}^{\mathrm{op},\otimes}_{\widehat{\mathcal{S}},1})}(X_1 \odot_{F_1} \cdots \odot_{F_{n-1}} X_n, (g,h)^*Y). \end{aligned}$$

Using [25, Proposition 5.1] and the notation of Remark 3.2.17, we can expand this out as the limit over $(x, p, q) \to (x', p', q') \in \mathrm{Tw}^r(\mathcal{M}_{1,F_0,F_n})$ of

$$\begin{aligned} &\mathrm{Map}_{\mathcal{V}^{\mathrm{op},\otimes}_{\widehat{\mathcal{S}},1}}\Big(\operatorname*{colim}_{\alpha \colon y \to x \in \mathcal{M}_{n/x}} \operatorname*{colim}_{(t_0,\ldots,t_n) \in F(y)_{p,q}} \alpha_!(X_1(y_{01}, t_0, t_1), \\ &\quad \ldots, X_n(y_{(n-1)n}, t_{n-1}, t_n)), \ (g,h)^*Y(x', p', q')\Big) \\ &\simeq \lim_{\alpha \colon y \to x \in \mathcal{M}^{\mathrm{op}}_{n/x}} \lim_{(t_0,\ldots,t_n) \in F(y)_{p,q}} \mathrm{Map}_{\mathcal{V}^{\otimes}_1}(\alpha_!(X_1(y_{01}, t_0, t_1), \\ &\quad \ldots, X_n(y_{(n-1)n}, t_{n-1}, t_n)), \ (g,h)^*Y(x', p', q')\big) \\ &\simeq \mathrm{Map}_{\mathcal{V}^{\otimes}_1}\Big(\operatorname*{colim}_{\alpha \colon y \to x \in \mathcal{M}_{n/x}} \operatorname*{colim}_{(t_0,\ldots,t_n) \in F(y)_{p,q}} \alpha_!(X_1(y_{01}, t_0, t_1), \\ &\quad \ldots, X_n(y_{(n-1)n}, t_{n-1}, t_n)), \ (g,h)^*Y(x', p', q')\Big), \end{aligned}$$

under our assumptions. Let $\alpha_{n,!}(X_1, \ldots, X_n)$ denote the functor $\mathcal{M}_{1,F_0,F_n} \to \mathcal{V}^{\otimes}_1$ given by

$$(x, p, q) \mapsto \operatorname*{colim}_{\alpha \colon y \to x \in \mathcal{M}_{n/x}} \operatorname*{colim}_{(t_0,\ldots,t_n) \in F(y)_{p,q}} \alpha_!(X_1(y_{01}, t_0, t_1), \ldots, X_n(y_{(n-1)n}, t_{n-1}, t_n)),$$

then we see using Proposition 3.5.6(iv) that the mapping space we started with is equivalent to $\mathrm{Map}_{\widehat{\mathcal{M}}_{\mathcal{V},1}}(\alpha_{n,!}(X_1, \ldots, X_n), Y)$, so that $(X_1, \ldots, X_n) \to \alpha_{n,!}(X_1, \ldots, X_n)$ is indeed a locally cocartesian morphism over α_n, as required. Moreover, part (ii) also follows immediately from this description of the locally cocartesian morphisms. □

For an arbitrary $\mathcal{S}$-admissible double ∞-category $\mathcal{M}$ it seems extremely awkward to formulate a condition on $\mathcal{V}$ such that $\widehat{\mathcal{M}}_{\mathcal{V}}$ is a double ∞-category. We therefore content ourselves with the following observation:

Proposition 3.5.8 *Let $\mathcal{M}$ be a double ∞-category such that $\mathcal{M}_0$ is an ∞-groupoid.*

(i) *If $\mathcal{V}^{\otimes}$ is an $\mathcal{M}$-monoidal ∞-category that is compatible with colimits indexed by ∞-groupoids and by the ∞-categories $\mathcal{M}_{n/x}$ for $x \in \mathcal{M}_1$, then $\widehat{\mathcal{M}}_{\mathcal{V}}$ is a framed double ∞-category.*

(ii) *If $\mathcal{U}^{\otimes} \to \mathcal{V}^{\otimes}$ is an $\mathcal{M}$-monoidal functor between $\mathcal{M}$-monoidal ∞-categories with colimits as in (i) such that each functor $\mathcal{U}_x \to \mathcal{V}_x$ preserves these colimits, then the natural morphism of generalized non-symmetric ∞-operads $\widehat{\mathcal{M}}_{\mathcal{U}} \to \widehat{\mathcal{M}}_{\mathcal{V}}$ preserves cocartesian morphisms.*

Proof We know from Proposition 3.5.7(i) that $\widehat{\mathcal{M}}_{\mathcal{V}} \to \mathbb{\Delta}^{\mathrm{op}}$ in (i) is a locally cocartesian fibration. If we can show this is actually a cocartesian fibration, then (ii) also follows since Proposition 3.5.7(ii) implies the functor preserves locally cocartesian morphisms.

By Lemma 2.1.25 to show we have a cocartesian fibration it suffices to check that for every active map $\phi\colon [2] \to [n]$ and for $\Phi \in \widehat{\mathcal{M}}_{\mathcal{V},n}$ the canonical map

$$\alpha_{n,!}\Phi = (\phi d_1)_!\Phi \to d_{1,!}\phi_!\Phi$$

is an equivalence.

For Φ over $F_0, \dots, F_n \colon \mathcal{M}_0 \to \mathcal{S}$, the object $\alpha_{n,!}\Phi$ is given at (x, p, q) by

$$\operatorname*{colim}_{y \xrightarrow{\gamma} x \in \mathcal{M}_{n/x}} \operatorname*{colim}_{(t_0,\dots,t_n)) \in F(y)_{p,q}} \gamma_!(\Phi_1(y_{01}, t_0, t_1), \dots, \Phi_n(y_{(n-1)n}, t_{n-1}, t_n)),$$

while if ϕ_{ij} denotes the active part of $\phi \circ \rho_{ij}$, we have

$$d_{1,!}\phi_!\Phi(x, p, q) \simeq \operatorname*{colim}_{z \xrightarrow{\zeta} x \in \mathcal{M}_{2/x}} \operatorname*{colim}_{t \in F_{\phi(1)}(z_{11})} \zeta_!(\phi_{01,!}\Phi_{0\phi(1)}(z_{01}, p, t), \phi_{12,!}\Phi_{\phi(1)n}(z_{12}, t, q)),$$

where (setting $\ell := \phi(1)$)

$$\begin{aligned}
&\phi_{01,!}\Phi_{0\ell}(z_{01}, p, t) \\
&\quad\simeq \operatorname*{colim}_{u \xrightarrow{\delta} z_{01} \in \mathcal{M}_{\ell/z_{01}}} \operatorname*{colim}_{(t_0,\dots,t_\ell) \in F_{0\ell}(u)} \delta_!(\Phi_1(u_{01}, t_0, t_1), \dots, \Phi_\ell(u_{(\ell-1)\ell}, t_{\ell-1}, t_\ell)), \\
&\phi_{12,!}\Phi_{\ell n}(z_{12}, t, q) \\
&\quad\simeq \operatorname*{colim}_{v \xrightarrow{\epsilon} z_{12} \in \mathcal{M}_{n-\ell/z_{12}}} \operatorname*{colim}_{(t_\ell,\dots,t_n) \in F_{\ell n}(u)} \epsilon_!(\Phi_{\ell+1}(v_{\ell(\ell+1)}, t_\ell, t_{\ell+1}), \dots, \Phi_n(v_{(n-1)n}, t_{n-1}, t_n)).
\end{aligned}$$

Since $\mathcal{V}^{\otimes}$ is compatible with these colimits, we can pass these colimits past $\zeta_!$ in the expression for $d_{1,!}\phi_!\Phi(x, p, q)$, obtaining an expression for this object as an iterated colimit of terms of the form

$$\begin{aligned}
&\zeta_!(\delta_!(\Phi_1(u_{01}, t_0, t_1), \dots, \Phi_\ell(u_{(\ell-1)\ell}, t_{\ell-1}, t_\ell)), \epsilon_!(\Phi_{\ell+1}(v_{\ell(\ell+1)}, t_\ell, t_{\ell+1}), \\
&\quad \dots, \Phi_n(v_{(n-1)n}, t_{n-1}, t_n))).
\end{aligned}$$

Note that we can rewrite this as

$$(\zeta \circ (\delta, \epsilon))_!(\Phi_1(u_{01}, t_0, t_1), \dots, \Phi_n(v_{(n-1)n}, t_{n-1}, t_n))$$

since $\mathcal{V}^{\otimes}$ is cocartesian. Thus we can rewrite the expression for $d_{1,!}\phi_!\Phi(x, p, q)$ as

$$\begin{aligned}
&\operatorname*{colim}_{\zeta \in \mathcal{M}_{2/x}} \operatorname*{colim}_{(\delta,\epsilon) \in \mathcal{M}_{\ell/z_{01}} \times \mathcal{M}_{n-\ell/z_{12}}} \operatorname*{colim}_{(t_0,\dots,t_n) \in F_{0\ell}(u) \times_{F_\ell(z)} F_{\ell n}(v)} (\zeta \circ (\delta, \epsilon))_!(\Phi_1(u_{01}, t_0, t_1), \\
&\quad \dots, \Phi_n(v_{(n-1)n}, t_{n-1}, t_n))
\end{aligned}$$

On the other hand, we can evaluate the colimit over $\mathcal{M}_{n/x}$ in the expression for $\alpha_{n,!}\Phi$ by first taking a left Kan extension along the functor $\phi_!\colon \mathcal{M}_{n/x} \to \mathcal{M}_{2/x}$ given by the cocartesian morphisms over ϕ. For a functor f out of $\mathcal{M}_{n/x}$ this gives

$$\operatorname*{colim}_{y \to x \in \mathcal{M}_{n/x}} f \simeq \operatorname*{colim}_{z \to x \in \mathcal{M}_{2/x}} \operatorname*{colim}_{(\mathcal{M}_{n/x})_{/z}} f,$$

where $(\mathcal{M}_{n/x})_{/z} \simeq \mathcal{M}_{n/z} \simeq \mathcal{M}_{\phi(1)/z_{01}} \times_{\mathcal{M}_{0/z_{11}}} \mathcal{M}_{n-\phi(1)/z_{12}}$, which is equivalent to $\mathcal{M}_{\phi(1)/z_{01}} \times \mathcal{M}_{n-\phi(1)/z_{12}}$ since $\mathcal{M}_0$ is an ∞-groupoid. Rewriting our expression for $\alpha_{n,!}\Phi$ using this, we get exactly our last formula for $d_{1,!}\phi_!\Phi$, as required. $\square$

Remark 3.5.9 We can describe the double ∞-category $\widehat{\mathcal{M}}_{\mathcal{V}}$ as follows:

- its objects are functors $\mathcal{M}_0 \to \mathcal{S}$, and its vertical morphisms are natural transformations of these,
- its horizontal morphisms from F to G are functors $\mathcal{M}_{1,F,G} \to \mathcal{V}_1^{\otimes}$ over $\mathcal{M}_1$,
- the composite of horizontal morphisms $\Phi\colon \mathcal{M}_{1,F,G} \to \mathcal{V}_1^{\otimes}$ and $\Psi\colon \mathcal{M}_{1,G,H} \to \mathcal{V}_1^{\otimes}$ is the functor $\mathcal{M}_{1,F,H} \to \mathcal{V}_1^{\otimes}$ given by

$$(x \in \mathcal{M}_1,\, p \in F(x_{00}),\, q \in F(x_{11})) \mapsto \operatorname*{colim}_{\alpha\colon y \to x \in \mathcal{M}_{2/x}} \operatorname*{colim}_{t \in H(y_{11})} \alpha_!(\Phi(y_{01}, p, t), \Psi(y_{12}, t, q)).$$

3.6 Example: enriched ∞-categories as associative algebras

In this subsection we illustrate our results on Day convolution by considering a simple example of our construction: we will give a description of enriched ∞-categories as associative algebras in a family of monoidal ∞-categories. An alternative construction of these monoidal ∞-categories is given in [39], where this perspective on enriched ∞-categories is developed extensively.

Remark 3.6.1 Our construction will extend the following description of ordinary enriched categories: If S is a set and $\mathbf{V}$ is a monoidal category where the tensor product preserves coproducts in each variable, then there is a monoidal structure on $\mathrm{Fun}(S \times S, \mathbf{V})$ given by

$$(F \otimes G)(i,k) \cong \coprod_{j \in S} F(i,j) \otimes G(j,k),$$

with unit $\mathbb{1}$ the functor

$$\mathbb{1}(i,j) = \begin{cases} \mathbb{1}, & i = j \\ \emptyset, & i \neq j. \end{cases}$$

This is sometimes known as the "matrix multiplication" tensor product, since the formula is a "categorified" version of that for multiplication of matrices. An associative algebra A in the category $\mathrm{Fun}(S \times S, \mathbf{V})$ with this tensor product is the same thing as a $\mathbf{V}$-category with set of objects S:

- The multiplication map $A \otimes A \to A$ supplies composition maps $A(i,j) \otimes A(j,k) \to A(i,k)$.
- The unit map $\mathbb{1} \to A$ supplies identity maps $\mathbb{1} \to A(i,i)$.

Let us consider first the result of applying Day convolution to the simplest double ∞-category, namely $\mathbb{\Delta}^{\mathrm{op}}$. This is trivially $\mathcal{X}$-admissible for any ∞-category $\mathcal{X}$ with pullbacks, and so by Proposition 3.2.12 there is a double ∞-category $\widehat{\mathbb{\Delta}}^{\mathrm{op}}_{\mathcal{X}}$ which by Corollary 3.3.7 has the universal property that for any generalized non-symmetric ∞-operad $\mathcal{O}$ there is a natural equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathbb{\Delta}}^{\mathrm{op}}_{\mathcal{X}}) \simeq \mathrm{Seg}_{\mathcal{O}}(\mathcal{X}).$$

By construction, the fibre $\widehat{\mathbb{\Delta}}^{\mathrm{op}}_{\mathcal{X},n}$ is the ∞-category of functors $\mathbb{\Sigma}^n \to \mathcal{X}$ that are right Kan extended from $\mathbb{\Lambda}^n$. We thus see that:

- objects of $\widehat{\mathbb{\Delta}}^{\mathrm{op}}_{\mathcal{X}}$ are objects of $\mathcal{X}$,
- vertical morphisms (morphisms in $\widehat{\mathbb{\Delta}}^{\mathrm{op}}_{\mathcal{X},0}$) are morphisms in $\mathcal{X}$,

- horizontal morphisms (objects in $\widehat{\mathbb{A}}^{\mathrm{op}}_{\mathcal{X},1}$) are *spans* in $\mathcal{X}$, i.e. diagrams of shape

$$\bullet \leftarrow \bullet \rightarrow \bullet,$$

- squares (morphisms in $\widehat{\mathbb{A}}^{\mathrm{op}}_{\mathcal{X},1}$) are diagrams of shape

$$\begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet & \bullet \arrow[l] \arrow[r] & \bullet, \end{tikzcd}$$

- composition of horizontal morphisms is given by taking pullbacks.

Indeed, the double ∞-category $\widehat{\mathbb{A}}^{\mathrm{op}}_{\mathcal{X},n}$ is precisely the double ∞-category $\mathrm{SPAN}^+(\mathcal{X})$ of spans constructed in [33], and its universal property is that established in [34]. In particular, we have an equivalence

$$\mathrm{Alg}_{\mathbb{A}^{\mathrm{op}}}(\widehat{\mathbb{A}}^{\mathrm{op}}_{\mathcal{X}}) \simeq \mathrm{Seg}_{\mathbb{A}^{\mathrm{op}}}(\mathcal{X}),$$

identifying associative algebras in the double ∞-category $\widehat{\mathbb{A}}^{\mathrm{op}}_{\mathcal{X}}$ with category objects in $\mathcal{X}$. Specializing to the ∞-category $\mathcal{S}$ of spaces, this says that associative algebras in $\widehat{\mathbb{A}}^{\mathrm{op}}_{\mathcal{S}}$ are equivalent to *Segal spaces*, which describe the algebraic structure of ∞-categories.

By Corollary 3.4.13, the restriction $\mathrm{Alg}_{\mathbb{A}^{\mathrm{op}}}(\widehat{\mathbb{A}}^{\mathrm{op}}_{\mathcal{S}}) \to \mathcal{S}$ is a cartesian fibration, with fibre at a space X given by $\mathrm{Alg}_{\mathbb{A}^{\mathrm{op}}}(\widehat{\mathbb{A}}^{\mathrm{op}}_{\mathcal{S}}(X, X))$. Here $\widehat{\mathbb{A}}^{\mathrm{op}}_{\mathcal{S}}(X, X)$ is equivalent to $\mathcal{S}_{/X\times X} \simeq \mathrm{Fun}(X \times X, \mathcal{S})$. The monoidal structure is given by pullback of spans, which in terms of functors to $\mathcal{S}$ admits the following description:

Proposition 3.6.2 *For any space X there is a monoidal structure on the ∞-category $\mathrm{Fun}(X \times X, \mathcal{S})$ such that*

(i) *the tensor product of F and G is given by*

$$(F \otimes G)(x, x') \simeq \operatorname*{colim}_{y\in X} F(x, y) \times G(y, x').$$

(ii) *the unit $\mathbb{1}$ is given by*

$$\mathbb{1}(x, y) \simeq \mathrm{Map}_X(x, y) \simeq \begin{cases} \emptyset, & x \not\simeq y \\ \Omega_x X, & x \simeq y, \end{cases}$$

where $\mathrm{Map}_X(x, y)$ is the mapping space in the ∞-groupoid X, i.e. the space of paths from x to y in X.

(iii) *we have $\mathrm{Alg}_{\mathbb{A}^{\mathrm{op}}}(\mathrm{Fun}(X \times X, \mathcal{S})) \simeq \mathrm{Seg}(\mathcal{S})_X$.*

In other words, ∞-categories with space of objects X are associative algebras in $\mathrm{Fun}(X \times X, \mathcal{S})$ with this monoidal structure.

Now we want to consider the analogue of this result for *enriched* ∞-categories. Propositions 3.5.6 and 3.5.8 specialize to give the following:

Proposition 3.6.3 *Let $\mathcal{C}^{\otimes} \to \mathbb{A}^{\mathrm{op}}$ be a monoidal ∞-category compatible with colimits indexed by ∞-groupoids. Then there is a framed double ∞-category $\widehat{\mathbb{A}^{\mathrm{op}}_{\mathcal{C}}}$ such that for any generalized non-symmetric ∞-operad $\mathcal{O}$ there is an equivalence*

$$\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathbb{A}^{\mathrm{op}}_{\mathcal{C}}}) \simeq \mathrm{Algd}_{\mathcal{O}}(\mathcal{O} \times_{\mathbb{A}^{\mathrm{op}}} \mathcal{C}^{\otimes}).$$

A monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ induces a natural morphism of generalized non-symmetric ∞-operads $\widehat{\mathbb{A}^{\mathrm{op}}_{\mathcal{C}}} \to \widehat{\mathbb{A}^{\mathrm{op}}_{\mathcal{D}}}$, and this preserves cocartesian morphisms if the monoidal functor preserves colimits indexed by ∞-groupoids.

In particular, we get an equivalence

$$\mathrm{Alg}_{\mathbb{\Delta}^{\mathrm{op}}}(\widehat{\mathbb{\Delta}^{\mathrm{op}}_{\mathcal{C}}}) \simeq \mathrm{Algd}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathcal{C}^{\otimes}),$$

where as in Example 2.1.21 the right-hand side is the model of enriched ∞-categories considered in [24]. Specializing Remark 3.5.9 gives the following description of the double ∞-category $\widehat{\mathbb{\Delta}^{\mathrm{op}}_{\mathcal{C}}}$:

- its objects are spaces, and its vertical morphisms are morphisms of spaces,
- a horizontal morphism from X to Y is a functor $X \times Y \to \mathcal{C}$,
- the composite of the horizontal morphisms $\Phi\colon X \times Y \to \mathcal{C}$ and $\Psi\colon Y \times Z \to \mathcal{C}$ is the functor $X \times Z \to \mathcal{C}$ given by

$$(x, z) \mapsto \operatorname*{colim}_{y \in Y} \Phi(x, y) \otimes \Psi(y, z).$$

From Corollary 3.4.10 we know that the restriction $\mathrm{Alg}_{\mathbb{\Delta}^{\mathrm{op}}}(\widehat{\mathbb{\Delta}^{\mathrm{op}}_{\mathcal{C}}}) \to \mathcal{S}$ is a cartesian fibration, with fibre at a space X given by $\mathrm{Alg}_{\mathbb{\Delta}^{\mathrm{op}}}((\widehat{\mathbb{\Delta}^{\mathrm{op}}_{\mathcal{C}}})(X, X)) \simeq \mathrm{Alg}_{\mathbb{\Delta}^{\mathrm{op}}_X}(\mathcal{C})$. Here the ∞-category $(\widehat{\mathbb{\Delta}^{\mathrm{op}}_{\mathcal{C}}})(X, X)$ is equivalent to $\mathrm{Fun}(X \times X, \mathcal{C})$, giving:

Corollary 3.6.4 *Let $\mathcal{C}$ be a monoidal ∞-category compatible with colimits indexed by ∞-groupoids. Then there is a monoidal structure on the ∞-category* $\mathrm{Fun}(X \times X, \mathcal{C})$ *such that*

(i) *the tensor product of F and G is given by*

$$(F \otimes G)(p, q) \simeq \operatorname*{colim}_{x \in X} F(p, x) \otimes G(x, q),$$

(ii) *the unit $\mathbb{1}$ is given by*

$$\mathbb{1}(p, q) \simeq \mathrm{Map}_X(p, q) \otimes \mathbb{1} \simeq \begin{cases} \emptyset, & p \not\simeq q \\ \Omega_p X \otimes \mathbb{1}, & p \simeq q, \end{cases}$$

where $\mathbb{1}$ is the unit of $\mathcal{C}$,

(iii) *we have* $\mathrm{Alg}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathrm{Fun}(X \times X, \mathcal{C})) \simeq \mathrm{Alg}_{\mathbb{\Delta}^{\mathrm{op}}_X}(\mathcal{C})$.

4 The composition product and ∞-operads

In this section we apply our results on Day convolutions to describe ∞-operads as associative algebras in double ∞-categories. We first consider ordinary ∞-operads (in spaces) in Sect. 4.1, and then enriched ∞-operads in Sect. 4.2. We also briefly observe, in Sect. 4.3, that a version of the bar-cobar adjunction between ∞-operads and ∞-cooperads follows from this description of ∞-operads.

4.1 ∞-operads as associative algebras

In this subsection we will see that ∞-operads are given by associative algebras in a double ∞-category of symmetric collections (or coloured symmetric sequences) in $\mathcal{S}$. For this we use Barwick's model of ∞-operads from [3]; this is known to be equivalent to other models of ∞-operads thanks to the results of [3,13–15]. Before we recall Barwick's definition we first introduce some notation:

Definition 4.1.1 Write $\mathbb{F}$ for a skeleton of the category of finite sets, with objects $\mathbf{k} := \{1, \ldots, k\}$, $k = 0, 1, \ldots$. Let $\mathbb{\Delta}_{\mathbb{F}}$ be the category with objects pairs $([n], f : [n] \to \mathbb{F})$ with a morphism $([n], f) \to ([m], g)$ given by a morphism $\phi : [n] \to [m]$ in $\mathbb{\Delta}$ and a natural transformation $\eta : f \to g \circ \phi$ such that

(i) the map $\eta_i : f(i) \to g(\phi(i))$ is injective for all $i = 0, \ldots, m$,
(ii) the commutative square

$$\begin{tikzcd} f(i) \arrow[r, "\eta_i"] \arrow[d] & g(\phi(i)) \arrow[d] \\ f(j) \arrow[r, "\eta_j"'] & g(\phi(j)) \end{tikzcd}$$

is a pullback square for all $0 \leq i \leq j \leq m$.

Notation 4.1.2 For $I = ([n], f)$ in $\mathbb{\Delta}_{\mathbb{F}}$, we write $I|_{ij} := ([j-i], f|_{\{i,i+1,\ldots,j\}})$ for $0 \leq i < j \leq n$, and $I|_i := ([0], f(i))$. Moreover, for $x \in f(n)$ we write $I_x := ([n], f_x)$ where f_x is obtained by taking fibres at x.

Definition 4.1.3 A presheaf $F : \mathbb{\Delta}_{\mathbb{F}}^{\mathrm{op}} \to \mathcal{S}$ is a *Segal operad* if it satisfies the following three "Segal conditions":

(1) for every object $I = ([n], f)$ of $\mathbb{\Delta}_{\mathbb{F}}$, the natural map

$$F(I) \to F(I|_{01}) \times_{F(I|_1)} \cdots \times_{F(I|_{n-1})} F(I|_{(n-1)n})$$

is an equivalence,
(2) for every object $I = ([1], \mathbf{k} \to \mathbf{l})$, the natural map

$$F(I) \to \prod_{x \in \mathbf{l}} F(I_x)$$

is an equivalence,
(3) for every object $I = ([0], \mathbf{k})$, the natural map

$$F(I) \to \prod_{x \in \mathbf{k}} F(I_x)$$

is an equivalence.

We write $\mathrm{Seg}^{\mathrm{opd}}_{\mathbb{\Delta}_{\mathbb{F}}^{\mathrm{op}}}(\mathcal{S})$ for the full subcategory of $\mathcal{P}(\mathbb{\Delta}_{\mathbb{F}})$ spanned by the Segal operads.

Remark 4.1.4 In the presence of condition (1), conditions (2) and (3) can be replaced by the following more general version:

For every object $I = ([n], f)$ of $\mathbb{\Delta}_{\mathbb{F}}$, the natural map

$$F(I) \to \prod_{x \in f(n)} F(I_x)$$

is an equivalence.

Segal presheaves on $\mathbb{\Delta}_{\mathbb{F}}$ describe the algebraic structure of ∞-operads: If we write $\mathfrak{e} := ([0], \mathbf{1})$ and $\mathfrak{c}_n := ([1], \mathbf{n} \to \mathbf{1})$, then the Segal conditions describe how $F(I)$ decomposes as a limit of $F(\mathfrak{e})$ and $F(\mathfrak{c}_n)$. We can think of an object of $\mathbb{\Delta}_{\mathbb{F}}$ as a forest with levels; then

$\mathfrak{e}$ correponds to a plain edge and $\mathfrak{c}_n$ to a corolla with n leaves, while the Segal condition corresponds to the decomposition of a forest into its edges and corollas. If F is viewed as an ∞-operad, the value $F(\mathfrak{e})$ is the space of objects of F, while $F(\mathfrak{c}_n)$ is the space of n-ary morphisms.

The following is the starting point for our construction of the composition product on symmetric sequences:

Proposition 4.1.5 *The projection $\mathbb{\Delta}_{\mathbb{F}}^{\mathrm{op}} \to \mathbb{\Delta}^{\mathrm{op}}$ is an $\mathfrak{X}$-admissible double ∞-category for any cocomplete ∞-category $\mathfrak{X}$ with pullbacks where colimits are universal.*

For the proof we need some notation and a lemma:

Definition 4.1.6 Suppose $\phi\colon [n] \to [m]$ is an active map in $\mathbb{\Delta}$ and $A = (\mathbf{a}_0 \to \cdots \to \mathbf{a}_n)$ is an object of $(\mathbb{\Delta}_{\mathbb{F}})_{[n]}$. An object of the slice $(\mathbb{\Delta}_{\mathbb{F}})_{[m],A/}$, defined using ϕ, is an object $(\mathbf{b}_0 \to \cdots \to \mathbf{b}_m)$ of $(\mathbb{\Delta}_{\mathbb{F}})_{[m]}$ together with injective maps $\mathbf{a}_i \to \mathbf{b}_{\phi(i)}$ such that the squares

$$\begin{tikzcd} \mathbf{a}_i \arrow[r] \arrow[d] & \mathbf{a}_{i+1} \arrow[d] \\ \mathbf{b}_{\phi(i)} \arrow[r] & \mathbf{b}_{\phi(i+1)} \end{tikzcd}$$

are cartesian. Let $(\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[m],A/}$ denote the full subcategory of $(\mathbb{\Delta}_{\mathbb{F}})_{[m],A/}$ containing those objects where the map $\mathbf{a}_n \to \mathbf{b}_{\phi(m)}$ is an isomorphism.

Lemma 4.1.7 *Let $\phi\colon [n] \to [m]$ be an active morphism in $\mathbb{\Delta}$, and let $A = (\mathbf{a}_0 \to \cdots \to \mathbf{a}_n)$ be an object of $(\mathbb{\Delta}_{\mathbb{F}})_{[n]}$. For $0 \le i \le j \le n$ let $A_{ij} := (\mathbf{a}_i \to \mathbf{a}_{i+1} \to \cdots \to \mathbf{a}_j)$.*

(i) *The inclusion $(\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[m],A/} \hookrightarrow (\mathbb{\Delta}_{\mathbb{F}})_{[m],A/}$ is coinitial.*

(ii) *For every k, $0 \le k \le n$, the functor $(\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[m],A/} \to (\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[\phi(k)],A_{0k}/} \times (\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[m-\phi(k)],A_{kn}/}$ is an equivalence.*

Proof By [43, Theorem 4.1.3.1] for part (i) it suffices to check that for all $B \in (\mathbb{\Delta}_{\mathbb{F}})_{[m],A/}$ the category $((\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[m],A/})_{/B}$ is weakly contractible. Observe that the projection $(\mathbb{\Delta}_{\mathbb{F}})_{[m]} \to \mathbb{F}_{\mathrm{inj}}$ given by evaluation at $[m]$ is a cartesian fibration, where $\mathbb{F}_{\mathrm{inj}}$ denotes the subcategory of $\mathbb{F}$ containing only the injective maps. The category $((\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[m],A/})_{/B}$ therefore has a terminal object, given by the cartesian morphism $B' \to B$ over the map $A_n \to B_{\phi(m)}$, which implies that it is weakly contractible. (Indeed this is the unique object of $((\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[m],A/})_{/B}$, which is actually a contractible ∞-groupoid.) Part (ii) is immediate from the definition. □

Proof of Proposition 4.1.5 The functor $\mathbb{\Delta}_{\mathbb{F}} \to \mathbb{\Delta}$ is a cartesian fibration, and the corresponding functor $\mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ takes $[n]$ to the category $(\mathbb{\Delta}_{\mathbb{F}})_{[n]}$ where

- an object is a sequence $\mathbf{a}_0 \to \cdots \to \mathbf{a}_n$ of morphisms in $\mathbb{F}$,
- a morphism is a commutative diagram

$$\begin{tikzcd} \mathbf{a}_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{a}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \cdots \arrow[r] & \mathbf{a}_{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{a}_n \arrow[d] \\ \mathbf{b}_0 \arrow[r] & \mathbf{b}_1 \arrow[r] & \cdots \arrow[r] & \mathbf{b}_{n-1} \arrow[r] & \mathbf{b}_n \end{tikzcd}$$

where the squares are cartesian and the maps $\mathbf{a}_i \to \mathbf{b}_i$ are injective.

This clearly satisfies the Segal condition, i.e. $(\mathbb{\Delta}_{\mathbb{F}})_{[n]} \simeq (\mathbb{\Delta}_{\mathbb{F}})_{[1]} \times_{(\mathbb{\Delta}_{\mathbb{F}})_{[0]}} \cdots \times_{(\mathbb{\Delta}_{\mathbb{F}})_{[0]}} (\mathbb{\Delta}_{\mathbb{F}})_{[1]}$. It follows that $\mathbb{\Delta}_{\mathbb{F}}^{\mathrm{op}} \to \mathbb{\Delta}^{\mathrm{op}}$ is a cocartesian fibration corresponding to the functor $\mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ taking $[n]$ to $(\mathbb{\Delta}_{\mathbb{F}})_{[n]}^{\mathrm{op}}$ and so is also a double ∞-category.

Suppose $\phi\colon [n] \to [m]$ is an active map in $\mathbb{\Delta}$. If $A = (\mathbf{a}_0 \to \cdots \to \mathbf{a}_n)$ and $A' = (\mathbf{a}_0 \to \cdots \to \mathbf{a}_k)$ and $A'' = (\mathbf{a}_k \to \cdots \to \mathbf{a}_n)$ then we must show that the natural map

$$\operatorname*{colim}_{((\mathbb{\Delta}_{\mathbb{F}})_{[m],A/})^{\mathrm{op}}} F \to \operatorname*{colim}_{((\mathbb{\Delta}_{\mathbb{F}})_{[\phi(k)],A'/})^{\mathrm{op}}} F \times_{\operatorname{colim}_{((\mathbb{\Delta}_{\mathbb{F}})_{[0],\mathbf{a}_k/})^{\mathrm{op}}} F} \operatorname*{colim}_{((\mathbb{\Delta}_{\mathbb{F}})_{[n-\phi(k)],A''/})^{\mathrm{op}}} F$$

is an equivalence for any appropriate functor F.

We have a commutative square

$$\begin{tikzcd}
\operatorname{colim}_{((\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[m],A/})^{\mathrm{op}}} F \arrow[r] \arrow[d, "\sim"] & \operatorname{colim}_{((\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[\phi(k)],A'/})^{\mathrm{op}}} F \times_{\operatorname{colim}_{((\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[0],\mathbf{a}_k/})^{\mathrm{op}}} F} \operatorname{colim}_{((\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[n-\phi(k)],A''/})^{\mathrm{op}}} F \arrow[d, "\sim"] \\
\operatorname{colim}_{((\mathbb{\Delta}_{\mathbb{F}})_{[m],A/})^{\mathrm{op}}} F \arrow[r] & \operatorname{colim}_{((\mathbb{\Delta}_{\mathbb{F}})_{[\phi(k)],A'/})^{\mathrm{op}}} F \times_{\operatorname{colim}_{((\mathbb{\Delta}_{\mathbb{F}})_{[0],\mathbf{a}_k/})^{\mathrm{op}}} F} \operatorname{colim}_{((\mathbb{\Delta}_{\mathbb{F}})_{[n-\phi(k)],A''/})^{\mathrm{op}}} F,
\end{tikzcd}$$

where the vertical maps are equivalences by Lemma 4.1.7(i). To see that the bottom horizontal map is an equivalence it hence suffices to show the top horizontal map is an equivalence.

Here $(\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[0],\mathbf{a}_k/}$ is contractible, since it only contains the identity map of $\mathbf{a}_k$, while the functor $(\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[m],A/} \to (\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[\phi(k)],A'/} \times (\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[n-\phi(k)],A''/}$ is an equivalence by Lemma 4.1.7(ii). Thus the top horizontal functor is

$$\begin{aligned}
&\operatorname*{colim}_{(X,Y)\in((\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[\phi(k)],A'/})^{\mathrm{op}}\times((\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[n-\phi(k)],A''/})^{\mathrm{op}}} F(X) \times_{F([0],\mathbf{a}_k)} F(Y) \\
&\to \operatorname*{colim}_{X\in((\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[\phi(k)],A'/})^{\mathrm{op}}} F(X) \times_{F([0],\mathbf{a}_k)} \operatorname*{colim}_{Y\in((\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[n-\phi(k)],A''/})^{\mathrm{op}}} F(Y),
\end{aligned}$$

which is an equivalence since colimits in $\mathcal{X}$ are universal. □

Notation 4.1.8 Given a morphism $f: \mathbf{a} \to \mathbf{b}$ of finite sets, we write $\mathrm{Fact}(f)$ for the groupoid $((\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{iso}}_{[2],\mathbf{a}\to\mathbf{b}/})^{\mathrm{op}}$ of factorizations of f.

Applying Proposition 3.2.12 and Corollary 3.3.7 we get:

Corollary 4.1.9 *There is a double ∞-category $\widehat{\mathbb{\Delta}}^{\mathrm{op}}_{\mathbb{F},\mathcal{S}}$ with the universal property that for any generalized non-symmetric ∞-operad $\mathcal{O}$ there is a natural equivalence*

$$\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathbb{\Delta}}^{\mathrm{op}}_{\mathbb{F},\mathcal{S}}) \simeq \mathrm{Seg}_{\mathcal{O}\times_{\mathbb{\Delta}^{\mathrm{op}}}\mathbb{\Delta}^{\mathrm{op}}_{\mathbb{F}}}(\mathcal{S}).$$

In particular, $\mathrm{Alg}_{\mathbb{\Delta}^{\mathrm{op}}}(\widehat{\mathbb{\Delta}}^{\mathrm{op}}_{\mathbb{F},\mathcal{S}}) \simeq \mathrm{Seg}_{\mathbb{\Delta}^{\mathrm{op}}_{\mathbb{F}}}(\mathcal{S})$.

Here $\mathrm{Seg}_{\mathbb{\Delta}^{\mathrm{op}}_{\mathbb{F}}}(\mathcal{S})$ is the ∞-category of presheaves on $\mathbb{\Delta}_{\mathbb{F}}$ that satisfy condition (1) in Definition 4.1.3. The double ∞-category $\widehat{\mathbb{\Delta}}^{\mathrm{op}}_{\mathbb{F},\mathcal{S}}$ can be described as follows:

- Objects are functors $\mathbb{F}^{\mathrm{op}}_{\mathrm{inj}} \simeq \mathbb{\Delta}^{\mathrm{op}}_{\mathbb{F},[0]} \to \mathcal{S}$,
- Vertical morphisms are natural transformations of such functors.
- A horizontal morphism Φ from F to $G\colon \mathbb{F}^{\mathrm{op}}_{\mathrm{inj}} \to \mathcal{S}$ assigns to $([1], \mathbf{a} \to \mathbf{b})$ a span

$$F(\mathbf{a}) \leftarrow \Phi(\mathbf{a} \to \mathbf{b}) \to G(\mathbf{b}).$$

- Squares are natural transformations of such diagrams.

- If Φ is a horizontal morphism from F to G and Ψ is a horizontal morphism from G to H then their composite assigns to $([1], \mathbf{a} \xrightarrow{f} \mathbf{b})$ the space over $F(\mathbf{a})$ and $H(\mathbf{b})$ given by

$$\operatorname*{colim}_{\mathbf{a}\to\mathbf{x}\to\mathbf{b}\in\mathrm{Fact}(f)} \Phi(\mathbf{a} \to \mathbf{x}) \times_{G(\mathbf{x})} \Psi(\mathbf{x} \to \mathbf{b}).$$

Definition 4.1.10 Let $\mathrm{COLL}(\mathcal{S})$ denote the full subcategory of $\widehat{\mathbb{\Delta}}^{\mathrm{op}}_{\mathbb{F},\mathcal{S}}$ spanned by the functors $(\Sigma\mathbb{\Delta}^{\mathrm{op}}_{\mathbb{F}})_n \to \mathcal{S}$ such that their inert restrictions to $(\Sigma\mathbb{\Delta}^{\mathrm{op}}_{\mathbb{F}})_0 \simeq \mathbb{F}^{\mathrm{op}}_{\mathrm{inj}}$ take coproducts of finite sets to products, and their inert restrictions to $(\Sigma\mathbb{\Delta}^{\mathrm{op}}_{\mathbb{F}})_1$ moreover satisfy condition (2) in Definition 4.1.3 when restricted to $(\mathbb{\Delta}^{\mathrm{op}}_{\mathbb{F}})_{[1]}$.

Lemma 4.1.11 $\mathrm{COLL}(\mathcal{S})$ *is a sub-double ∞-category of* $\widehat{\mathbb{\Delta}}^{\mathrm{op}}_{\mathbb{F},\mathcal{S}}$.

Proof It follows from Lemma 2.1.11 that $\mathrm{COLL}(\mathcal{S})$ is a generalized non-symmetric ∞-operad, so it only remains to check that the cocartesian morphisms restrict to $\mathrm{COLL}(\mathcal{S})$. To see this it suffices to check that the horizontal morphisms in $\mathrm{COLL}(\mathcal{S})$ are closed under composition. If Φ is a horizontal morphism from F to G and Ψ is one from G to H, and both lie in $\mathrm{COLL}(\mathcal{S})$, then we have

$$\begin{aligned}
(\Phi \odot_G \Psi)(\mathbf{a} \to \mathbf{b}) &\simeq \operatorname*{colim}_{\mathbf{a}\to\mathbf{x}\to\mathbf{b}} \Phi(\mathbf{a} \to \mathbf{x}) \times_{G(\mathbf{x})} \Psi(\mathbf{x} \to \mathbf{b}) \\
&\simeq \operatorname*{colim}_{(\mathbf{a}_i\to\mathbf{x}_i\to\mathbf{1})\in\prod_{i\in\mathbf{b}}\mathrm{Fact}(\mathbf{a}_i\to\mathbf{1})} \left(\prod_{i\in\mathbf{b}} \Phi(\mathbf{a}_i \to \mathbf{x}_i)\right) \\
&\qquad \times_{(\prod_{i\in\mathbf{b}} G(\mathbf{x}_i))} \left(\prod_{i\in\mathbf{b}} \Psi(\mathbf{x}_i \to \mathbf{1})\right) \\
&\simeq \operatorname*{colim}_{(\mathbf{a}_i\to\mathbf{x}_i\to\mathbf{1})\in\prod_{i\in\mathbf{b}}\mathrm{Fact}(\mathbf{a}_i\to\mathbf{1})} \prod_{i\in\mathbf{b}} \Phi(\mathbf{a}_i \to \mathbf{x}_i) \times_{G(\mathbf{x}_i)} \Psi(\mathbf{x}_i \to \mathbf{1}) \\
&\simeq \prod_{i\in\mathbf{b}} \operatorname*{colim}_{\mathbf{a}_i\to\mathbf{x}_i\to\mathbf{1}\in\mathrm{Fact}(\mathbf{a}_i\to\mathbf{1})} \Phi(\mathbf{a}_i \to \mathbf{x}_i) \times_{G(\mathbf{x}_i)} \Psi(\mathbf{x}_i \to \mathbf{1}) \\
&\simeq \prod_{i\in\mathbf{b}} (\Phi \odot_G \Psi)(\mathbf{a}_i \to \mathbf{1}),
\end{aligned}$$

i.e. $\Phi \odot_G \Psi$ also lies in $\mathrm{COLL}(\mathcal{S})$, as required. □

The double ∞-category $\mathrm{COLL}(\mathcal{S})$ can be described as follows:

- Its objects can be identified with spaces (since functors $\mathbb{F}^{\mathrm{op}}_{\mathrm{inj}} \to \mathcal{S}$ in $\mathrm{COLL}(\mathcal{S})_0$ are determined by their value at $\mathbf{1}$).
- Its vertical morphisms are maps of spaces.
- A horizontal morphism Φ from X to Y is determined by assigning to $([1], \mathbf{n} \to \mathbf{1})$ a span

$$X^{\times n} \leftarrow \Phi(n) \to Y,$$

  where $\Phi(n)$ has a Σ_n-action compatible with permuting the factors of $X^{\times n}$.
- A square is a natural transformation of such diagrams.
- If Φ is a horizontal morphism from X to Y and Ψ is one from Y to Z, then their composite assigns to $([1], \mathbf{n} \to \mathbf{1})$ the space over $X^{\times n} \times Z$ given by

$$\operatorname*{colim}_{\mathbf{n}\to\mathbf{m}\to\mathbf{1}} \Phi(\mathbf{n} \to \mathbf{m}) \times_{Y^{\times m}} \Psi(m),$$

  where $\Phi(\mathbf{n} \to \mathbf{m}) \simeq \prod_{i=1}^{m} \Phi(n_i)$.

Restricting Corollary 4.1.9 to $\mathrm{COLL}(\mathcal{S})$, we get:

Corollary 4.1.12 *There is an equivalence* $\mathrm{Alg}_{\Delta^{\mathrm{op}}}(\mathrm{COLL}(\mathcal{S})) \simeq \mathrm{Seg}^{\mathrm{opd}}_{\Delta^{\mathrm{op}}_{\mathbb{F}}}(\mathcal{S})$. □

In other words, ∞-operads can be described as associative algebras in the double ∞-category $\mathrm{COLL}(\mathcal{S})$. Moreover, applying Corollary 3.4.13 we see that the restriction $\mathrm{Alg}_{\Delta^{\mathrm{op}}}(\mathrm{COLL}(\mathcal{S})) \to \mathcal{S}$ is a cartesian fibration. If we write

$$\mathrm{Coll}_X(\mathcal{S}) := \mathrm{COLL}(\mathcal{S})(X, X)$$

for the ∞-category of X-collections in $\mathcal{S}$, the fibre at $X \in \mathcal{S}$ is given by $\mathrm{Alg}_{\Delta^{\mathrm{op}}}(\mathrm{Coll}_X(\mathcal{S}))$. To describe the monoidal structure on $\mathrm{Coll}_X(\mathcal{S})$ we first need to introduce some notation:

Notation 4.1.13 Let $\mathbb{F}^{\simeq}$ denote the maximal subgroupoid of $\mathbb{F}$, and write $j\colon \mathbb{F}^{\simeq} \to (\Delta^{\mathrm{op}}_{\mathbb{F}})_{[1]}$ for the fully faithful functor taking $\mathbf{n}$ to $\mathbf{n} \to \mathbf{1}$. Given $X \in \mathcal{S}$, we write $(\Delta^{\mathrm{op}}_{\mathbb{F}})_{[0],X} \to (\Delta^{\mathrm{op}}_{\mathbb{F}})_{[0]}$ for the left fibration corresponding to the unique product-preserving functor $(\Delta^{\mathrm{op}}_{\mathbb{F}})_{[0]} \simeq \mathbb{F}^{\mathrm{op}}_{\mathrm{inj}} \to \mathcal{S}$ that takes $\mathbf{1}$ to X. Moreover, for $X, Y \in \mathcal{S}$ let $(\Delta^{\mathrm{op}}_{\mathbb{F}})_{[1],X,Y}$ denote the pullback $(\Delta^{\mathrm{op}}_{\mathbb{F}})_{[1]} \times_{(\Delta^{\mathrm{op}}_{\mathbb{F}})^{\times 2}_{[0]}} \big((\Delta^{\mathrm{op}}_{\mathbb{F}})_{[0],X} \times (\Delta^{\mathrm{op}}_{\mathbb{F}})_{[0],Y}\big)$. If we define $\mathbb{F}^{\simeq}_X := \coprod_{n=0}^{\times} X^{\times n}_{h\Sigma_n}$ to be the free commutative monoid on the ∞-groupoid X, then we have a pullback

$$\begin{tikzcd} \mathbb{F}^{\simeq}_X \times Y \arrow[r, "j_{X,Y}"] \arrow[d] & (\Delta^{\mathrm{op}}_{\mathbb{F}})_{[1],X,Y} \arrow[d] \\ \mathbb{F}^{\simeq} \arrow[r, "j"] & (\Delta^{\mathrm{op}}_{\mathbb{F}})_{[1]}. \end{tikzcd}$$

Lemma 4.1.14 *For any $X \in \mathcal{S}$, the ∞-category* $\mathrm{COLL}(\mathcal{S})(X, Y)$ *of horizontal morphisms from X to Y is equivalent to the full subcategory of* $\mathrm{Fun}((\Delta^{\mathrm{op}}_{\mathbb{F}})_{[1],X,Y}, \mathcal{S})$ *spanned by functors that are right Kan extensions along $j_{X,Y}$, so that*

$$\mathrm{COLL}(\mathcal{S})(X, Y) \simeq \mathrm{Fun}(\mathbb{F}^{\simeq}_X \times Y, \mathcal{S}).$$

Proof By Proposition 3.2.15 we may identify $\widehat{\Delta}^{\mathrm{op}}_{\mathbb{F},\mathcal{S}}(F, G)$ with $\mathrm{Fun}((\Delta^{\mathrm{op}}_{\mathbb{F}})_{[1],F,G}, \mathcal{S})$ for any functors $F, G\colon (\Delta^{\mathrm{op}}_{\mathbb{F}})_{[0]} \to \mathcal{S}$. For the objects that lie in $\mathrm{COLL}(\mathcal{S})$ these are functors $(\Delta^{\mathrm{op}}_{\mathbb{F}})_{[1],X,Y} \to \mathcal{S}$, and under this identification it is easy to see that the functors that lie in $\mathrm{COLL}(\mathcal{S})(X, Y)$ are precisely those that are right Kan extended from $\mathbb{F}^{\simeq}_X \times Y$. □

Remark 4.1.15 In particular, we may identify the ∞-category $\mathrm{Coll}_*(\mathcal{S})$ of horizontal endomorphisms of the point with the ∞-category $\mathrm{Fun}(\mathbb{F}^{\simeq}, \mathcal{S})$ of symmetric sequences in $\mathcal{S}$. More generally, the ∞-category $\mathrm{Coll}_X(\mathcal{S})$ is equivalent to $\mathrm{Fun}(\mathbb{F}^{\simeq}_X \times X, \mathcal{S})$, the ∞-category of X-collections, or X-coloured symmetric sequences, in $\mathcal{S}$.

Notation 4.1.16 For a functor $F\colon \mathbb{F}^{\simeq}_X \times Y \to \mathcal{C}$, we will denote its value at $((x_1, \ldots, x_n), y) \in X^{\times n}_{h\Sigma_n} \times Y$ by $F\binom{x_1,\ldots,x_n}{y}$.

Corollary 4.1.17 *The ∞-category* $\mathrm{Fun}(\mathbb{F}^{\simeq}_X \times X, \mathcal{S})$ *has a monoidal structure such that*

(i) *the tensor product of F and G is given by*

$$(F \circ G)\begin{pmatrix} x_1, \ldots, x_n \\ z \end{pmatrix} \simeq \operatorname*{colim}_{\substack{\mathbf{n}\to\mathbf{m}\to\mathbf{1} \\ (y_i)\in X^m}} \prod_{i\in\mathbf{m}} F\begin{pmatrix} x_k : k \in \mathbf{n}_i \\ y_i \end{pmatrix} \times G\begin{pmatrix} y_1, \ldots, y_m \\ z \end{pmatrix},$$

where the colimit is over $((\Delta_{\mathbb{F},X})^{\mathrm{iso}}_{[2],(\mathbf{n}\to\mathbf{1},(x_i),z)/})^{\mathrm{op}}$,

(ii) *the unit* $\mathbb{1}_X$ *is given by*

$$\mathbb{1}_X\begin{pmatrix} x_1, \ldots, x_n \\ y \end{pmatrix} \simeq \begin{cases} \emptyset, & \mathbf{n} \ncong \mathbf{1}, \\ \mathrm{Map}_X(x_1, y), & \mathbf{n} \cong \mathbf{1}, \end{cases}$$

(iii) *we have* $\mathrm{Alg}_{\Delta^{\mathrm{op}}}(\mathrm{Fun}(\mathbb{F}^{\simeq}_X \times X, \mathcal{S})) \cong \mathrm{Seg}^{\mathrm{opd}}_{\Delta^{\mathrm{op}}_{\mathbb{F}}}(\mathcal{S})_X$.

Remark 4.1.18 In particular, the ∞-category $\mathrm{Fun}(\mathbb{F}^{\simeq}, \mathcal{S})$ of symmetric sequences has a monoidal structure with tensor product given by

$$(F \circ G)(\mathbf{n}) \simeq \operatorname*{colim}_{(\mathbf{n}\to\mathbf{m}\to\mathbf{1})\in\mathrm{Fact}(\mathbf{n}\to\mathbf{1})} \prod_{i\in\mathbf{m}} F(\mathbf{n}_i) \times G(\mathbf{m}).$$

This formula is easily seen to agree with the usual formula for the composition product of symmetric sequences by expanding out $\mathrm{Fact}(\mathbf{n} \to \mathbf{1})$ as a coproduct of its components, cf. [18, Lemma A.4].

Remark 4.1.19 In [3], Barwick defines Φ-∞-operads for *operator categories* Φ as Segal presheaves on categories Δ_Φ, of which $\Delta_{\mathbb{F}}$ is a special case. The proof of Corollary 4.1.17 works for any operator category, giving a monoidal structure on the ∞-category $\mathrm{Fun}(\Phi^{\simeq}_X, \mathcal{S})$ of "X-coloured Φ-symmetric sequences" where associative algebras are Φ-∞-operads with X as their space of objects. In particular, replacing $\mathbb{F}$ with the category $\mathbb{O}$ of ordered finite sets we obtain the analogous results for non-symmetric ∞-operads.

4.2 Enriched ∞-operads as associative algebras

In this subsection we extend the results of the previous subsection to ∞-operads enriched in a symmetric monoidal ∞-category. The starting point is the following analogue of Proposition 3.5.8 for $\Delta^{\mathrm{op}}_{\mathbb{F}}$-monoidal ∞-categories:

Proposition 4.2.1 *Let* $\mathcal{U}^{\otimes}$ *be a* $\Delta^{\mathrm{op}}_{\mathbb{F}}$*-monoidal* ∞*-category that is compatible with colimits indexed by* ∞*-groupoids. Then* $\widehat{\Delta}^{\mathrm{op}}_{\mathbb{F},\mathcal{U}}$ *is a framed double* ∞*-category. If* $\mathcal{U}^{\otimes} \to \mathcal{V}^{\otimes}$ *is a* $\Delta^{\mathrm{op}}_{\mathbb{F}}$*-monoidal functor between such* $\Delta^{\mathrm{op}}_{\mathbb{F}}$*-monoidal* ∞*-categories such that each functor* $\mathcal{U}_X \to \mathcal{V}_X$ *for* $X \in (\Delta_{\mathbb{F}})_{[1]}$ *preserves colimits indexed by* ∞*-groupoids, then the natural morphism of generalized non-symmetric* ∞*-operads* $\widehat{\Delta}^{\mathrm{op}}_{\mathbb{F},\mathcal{U}} \to \widehat{\Delta}^{\mathrm{op}}_{\mathbb{F},\mathcal{V}}$ *preserves cocartesian morphisms.*

Proof Follows as in the proof of Proposition 3.5.8, using Lemma 4.1.7 to restrict to colimits indexed by ∞-groupoids. □

We now recall some definitions from [10]; we refer the reader there for motivation for these definitions.

Definition 4.2.2 Let $V \colon \Delta^{\mathrm{op}}_{\mathbb{F}} \to \mathbb{F}_*$ be the functor of [10, Definition 2.2.11], taking $([n], \mathbf{a}_0 \to \cdots \to \mathbf{a}_n)$ to $(\coprod_{i=1}^n \mathbf{a}_i)_+$, and a morphism $([n], \mathbf{a}_0 \to \cdots \to \mathbf{a}_n) \to ([m], \mathbf{b}_0 \to \cdots \to \mathbf{b}_m)$ over $\phi \colon [n] \to [m]$ in Δ^{op} to the map $(\coprod_{i=1}^m \mathbf{a}_i)_+ \to (\coprod_{j=1}^n \mathbf{b}_j)_+$ given on the component $\mathbf{a}_i$ by the map $\mathbf{a}_i \to (\coprod_{j=1}^n \mathbf{b}_j)_+$ taking $x \in \mathbf{a}_i$ to an object $y \in \mathbf{b}_j$ if $\phi(j-1) < i \leq \phi(j)$ and the map $\mathbf{a}_i \to \mathbf{a}_{\phi(j)}$ takes x to the image of y under the map $\mathbf{b}_j \to \mathbf{a}_{\phi(j)}$, and to the base point $*$ otherwise. The functor V assigns to a forest its set of vertices with an added basepoint. Note that V assigns every morphism in $\Delta^{\mathrm{op}}_{\mathbb{F}}$ that lies over an identity morphism in Δ^{op} to an inert morphism in $\mathbb{F}_*$.

Definition 4.2.3 If $\mathcal{V}^{\otimes} \to \mathbb{F}_*$ is a symmetric monoidal ∞-category, and $\mathcal{V}_{\otimes} \to \mathbb{F}_*^{\mathrm{op}}$ is the corresponding cartesian fibration, then we define the ∞-category $\mathbb{\Delta}_{\mathbb{F}}^{\mathcal{V}}$ by the pullback square

$$\begin{tikzcd} \mathbb{\Delta}_{\mathbb{F}}^{\mathcal{V}} \arrow[r] \arrow[d] & \mathcal{V}_{\otimes} \arrow[d] \\ \mathbb{\Delta}_{\mathbb{F}} \arrow[r, "V^{\mathrm{op}}"'] & \mathbb{F}_*^{\mathrm{op}}. \end{tikzcd}$$

Note that $V\colon \mathbb{\Delta}_{\mathbb{F}}^{\mathrm{op}} \to \mathbb{F}_*$ satisfies

$$V([n], \mathbf{a}_0 \to \cdots \to \mathbf{a}_n) \cong V([1], \mathbf{a}_0 \to \mathbf{a}_1) + \cdots + V([1], \mathbf{a}_{n-1} \to \mathbf{a}_n),$$

which implies that $\mathbb{\Delta}_{\mathbb{F}}^{\mathcal{V},\mathrm{op}}$ is a $\mathbb{\Delta}_{\mathbb{F}}$-monoidal ∞-category.

Remark 4.2.4 Let $\mathcal{V}$ be a symmetric monoidal ∞-category compatible with colimits indexed by ∞-groupoids. The double ∞-category $\widehat{\mathbb{\Delta}}^{\mathrm{op}}_{\mathbb{F},\mathbb{\Delta}_{\mathbb{F}}^{\mathcal{V},\mathrm{op}}}$ can be described as follows:

- The objects are functors $(\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[0]} \to \mathcal{S}$, and the vertical morphisms are natural transformations of such functors.
- A horizontal morphism from F to G is a functor

$$\Phi\colon (\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[1],F,G} \to (\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[1]} \times_{\mathbb{F}_*} \mathcal{V}^{\otimes}$$

  over $(\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[1]}$. This thus assigns to an object $(\mathbf{n} \xrightarrow{f} \mathbf{m}, p \in F(\mathbf{n}), q \in G(\mathbf{m}))$ an object $(\Phi(f, p, q)_i)_{i\in\mathbf{m}}$ of $\mathcal{V}^{\times|\mathbf{m}|}$.
- If Φ is a horizontal morphism from F to G and Ψ is one from G to H, then their composite is the functor from $(\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[1],F,H}$ given by

$$\begin{aligned} &(\mathbf{n} \xrightarrow{f} \mathbf{m}, p \in F(\mathbf{n}), q \in H(\mathbf{m})) \\ &\mapsto \left(\operatorname*{colim}_{(\mathbf{n}\xrightarrow{g}\mathbf{x}\xrightarrow{h}\mathbf{m})\in\mathrm{Fact}(f)} \; \operatorname*{colim}_{t\in G(\mathbf{x})} \bigotimes_{j\in h^{-1}(i)} \Phi(g, p, t)_j \otimes \Psi(h, t, q)_i \right)_{i\in\mathbf{m}} \end{aligned}$$

Definition 4.2.5 Let $\mathcal{V}$ be a symmetric monoidal ∞-category compatible with colimits indexed by ∞-groupoids. We denote by $\mathrm{COLL}(\mathcal{V})$ the full subcategory of $\widehat{\mathbb{\Delta}}^{\mathrm{op}}_{\mathbb{F},\mathbb{\Delta}_{\mathbb{F}}^{\mathcal{V},\mathrm{op}}}$ spanned by the objects

(1) whose inert restrictions to $[0]$ are given by functors $(\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[0]} \simeq \mathbb{F}^{\mathrm{op}}_{\mathrm{inj}} \to \mathcal{S}$ that take coproducts of finite sets to products,
(2) whose inert restrictions to $[1]$ correspond to functors

$$(\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[1],F,G} \to (\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[1]} \times_{\mathbb{F}_*} \mathcal{V}^{\otimes}$$

that send all morphisms to cocartesian morphisms in the target.

This is a sub-double ∞-category of $\widehat{\mathbb{\Delta}}^{\mathrm{op}}_{\mathbb{F},\mathbb{\Delta}_{\mathbb{F}}^{\mathcal{V},\mathrm{op}}}$ by a variant of the proof of Lemma 4.1.11.

Remark 4.2.6 A functor $F\colon \mathbb{F}^{\mathrm{op}}_{\mathrm{inj}} \to \mathcal{S}$ that satisfies condition (1) is the right Kan extension of its restriction to the object $\mathbf{1}$. Thus the objects of $\mathrm{COLL}(\mathcal{V})$ can equivalently be described as spaces. Since the restriction of the functor V to $(\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[1]}$ sends all morphisms to inert morphisms in $\mathbb{F}_*$, the functor $(\mathbb{\Delta}_{\mathbb{F}})^{\mathrm{op}}_{[1]} \to \mathrm{Cat}_\infty$ corresponding to the cocartesian fibration

$(\Delta_{\mathbb{F}})^{\mathrm{op}}_{[1]} \times_{\mathbb{F}_*} \mathcal{V}^{\otimes}$ is also a right Kan extension of its restriction to the full subcategory $\mathbb{F}^{\simeq}$, and this restriction is the constant functor with value $\mathcal{V}$. Thus a horizontal morphism from X to Y in $\mathrm{COLL}(\mathcal{V})$ is uniquely determined by its restriction to a functor $\mathbb{F}^{\simeq}_{X} \times Y \to \mathcal{V}$.

Notation 4.2.7 We say a morphism in $\Delta^{\mathrm{op}}_{\mathbb{F}}$ is *operadic inert* if it lies over an inert morphism in Δ^{op}. Let $\mathrm{Alg}^{\mathrm{opd}}_{\Delta^{\mathrm{op}}_{\mathbb{F},X}}(\mathcal{V})$ denote the full subcategory of $\mathrm{Alg}_{\Delta^{\mathrm{op}}_{\mathbb{F},X}/\Delta^{\mathrm{op}}_{\mathbb{F}}}(V^*\mathcal{V})$ spanned by morphisms that take operadic inert morphisms to cocartesian morphisms; we call such objects *operadic* $\Delta^{\mathrm{op}}_{\mathbb{F},X}$*-algebras*. We then write $\mathrm{Algd}^{\mathrm{opd}}_{\Delta^{\mathrm{op}}_{\mathbb{F}}}(\mathcal{V})$ for the full subcategory of $\mathrm{Algd}_{\Delta^{\mathrm{op}}_{\mathbb{F}}}(\mathcal{V})$ corresponding to operadic algebras for all $\Delta^{\mathrm{op}}_{\mathbb{F},X}$, $X \in \mathcal{S}$. Similarly, we can define operadic algebras and algebroids for $\mathcal{O} \times_{\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}_{\mathbb{F}}$ where $\mathcal{O}$ is any generalized non-symmetric ∞-operad, by taking the operadic inert morphisms in $\mathcal{O} \times_{\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}_{\mathbb{F}}$ to be those that lie over inert morphisms in $\mathcal{O}$ and operadic inert morphisms in $\Delta^{\mathrm{op}}_{\mathbb{F}}$.

Restricting the equivalence from Proposition 3.5.6 to $\mathrm{COLL}(\mathcal{V})$, we get:

Corollary 4.2.8 *Let $\mathcal{V}$ be a symmetric monoidal ∞-category compatible with colimits indexed by ∞-groupoids. There is a framed double ∞-category* $\mathrm{COLL}(\mathcal{V})$ *where:*

- *objects are spaces and vertical morphisms are morphisms of spaces,*
- *horizontal morphisms from X to Y are functors $\mathbb{F}^{\simeq}_{X} \times Y \to \mathcal{V}$,*
- *if $\Phi\colon \mathbb{F}^{\simeq}_{X} \times Y \to \mathcal{V}$ and $\Psi\colon \mathbb{F}^{\simeq}_{Y} \times Z \to \mathcal{V}$ are two horizontal morphisms, their composite is the functor $\mathbb{F}^{\simeq}_{X} \times Z \to \mathcal{V}$ given by*

$$\begin{pmatrix} x_1, \ldots, x_n \\ z \end{pmatrix} \mapsto \operatorname*{colim}_{\mathbf{n} \xrightarrow{f} \mathbf{m} \to \mathbf{1}} \operatorname*{colim}_{(y_j) \in Y^m} \bigotimes_j \Phi\begin{pmatrix} x_i : i \in f^{-1}(j) \\ y_j \end{pmatrix} \otimes \Psi\begin{pmatrix} y_1, \ldots, y_m \\ z \end{pmatrix}.$$

We have $\mathrm{Alg}_{\mathcal{O}}(\mathrm{COLL}(\mathcal{V})) \simeq \mathrm{Algd}^{\mathrm{opd}}_{\mathcal{O} \times_{\Delta^{\mathrm{op}}} \Delta^{\mathrm{op}}_{\mathbb{F}}}(\mathcal{V})$. *Moreover, if $\phi\colon \mathcal{V} \to \mathcal{W}$ is a symmetric monoidal functor that preserves colimits indexed by ∞-groupoids, then ϕ induces a morphism of double ∞-categories* $\mathrm{COLL}(\mathcal{V}) \to \mathrm{COLL}(\mathcal{W})$ *given on horizontal morphisms by composition with ϕ.*

Let $\mathrm{Coll}_X(\mathcal{V}) := \mathrm{COLL}(\mathcal{V})(X, X)$; then $\mathrm{Coll}_X(\mathcal{V})$ is equivalent to the ∞-category $\mathrm{Fun}(\mathbb{F}^{\simeq}_{X} \times X, \mathcal{V})$ of symmetric X-collections in $\mathcal{V}$. The monoidal structure on $\mathrm{Coll}_X(\mathcal{V})$ has the following description:

Corollary 4.2.9 *Let $\mathcal{V}$ be a symmetric monoidal ∞-category compatible with colimits indexed by ∞-groupoids. The ∞-category* $\mathrm{Fun}(\mathbb{F}^{\simeq}_{X} \times X, \mathcal{V})$ *has a monoidal structure such that*

(i) *the tensor product of F and G is given by*

$$(F \circ G)\begin{pmatrix} x_1, \ldots, x_n \\ z \end{pmatrix} \simeq \operatorname*{colim}_{\substack{\mathbf{n} \to \mathbf{m} \to \mathbf{1} \\ y_i \in X, i \in \mathbf{m}}} \bigotimes_{i \in \mathbf{m}} F\begin{pmatrix} x_k : k \in \mathbf{n}_i \\ y_i \end{pmatrix} \otimes G\begin{pmatrix} y_1, \ldots, y_k \\ z \end{pmatrix},$$

where the colimit is over $((\Delta_{\mathbb{F},X})^{\mathrm{iso}}_{[2],(\mathbf{n} \to \mathbf{1},(x_i),z)/})^{\mathrm{op}}$

(ii) *the unit $\mathbb{1}_X$ is given by*

$$\mathbb{1}_X\begin{pmatrix} x_1, \ldots, x_n \\ y \end{pmatrix} \simeq \begin{cases} \emptyset, & \mathbf{n} \ncong \mathbf{1}, \\ \mathrm{Map}_X(x_1, y) \otimes \mathbb{1}, & \mathbf{n} \cong \mathbf{1}, \end{cases}$$

(iii) *we have* $\mathrm{Alg}_{\Delta^{\mathrm{op}}}(\mathrm{Fun}(\mathbb{F}^{\simeq}_{X} \times X, \mathcal{V})) \simeq \mathrm{Alg}_{\Delta^{\mathrm{op}}_{\mathbb{F},X}}(\mathcal{V})$.

Moreover, if $\phi\colon \mathcal{V} \to \mathcal{W}$ *is a symmetric monoidal functor that preserves colimits indexed by* ∞*-groupoids, then composition with* ϕ *gives a monoidal functor* $\mathrm{Fun}(\mathbb{F}_X^{\simeq} \times X, \mathcal{V}) \to \mathrm{Fun}(\mathbb{F}_X^{\simeq} \times X, \mathcal{W})$.

4.3 ∞-cooperads and a bar-cobar adjunction

In this subsection we will apply Lurie's bar-cobar adjunction for associative algebras [44, Sect. 5.2.2] to obtain a version of the bar-cobar adjunction between ∞-operads and ∞-cooperads with a fixed space of objects. We first spell out the variant of ∞-cooperads that this applies to:

Definition 4.3.1 For $X \in \mathcal{S}$ and $\mathcal{V}$ a symmetric monoidal ∞-category compatible with colimits indexed by ∞-groupoids, a $\mathcal{V}$*-enriched* ∞*-cooperad* with space of objects X is a coassociative coalgebra in $\mathrm{Fun}(\mathbb{F}_X^{\simeq} \times X, \mathcal{V})$, equipped with the monoidal structure of Corollary 4.2.9. We write

$$\mathrm{Coopd}_X(\mathcal{V}) := \mathrm{Alg}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathrm{Fun}(\mathbb{F}_X^{\simeq} \times X, \mathcal{V})^{\mathrm{op}})^{\mathrm{op}}$$

for the ∞-category of $\mathcal{V}$-∞-cooperads with space of objects X, and

$$\mathrm{Coopd}_X^{\mathrm{coaug}}(\mathcal{V}) := \mathrm{Coopd}_X(\mathcal{V})_{\mathbb{1}_X/}$$

for the ∞-category of coaugmented $\mathcal{V}$-∞-cooperads. Similarly, we write

$$\mathrm{Opd}_X(\mathcal{V}) := \mathrm{Alg}_{\mathbb{\Delta}^{\mathrm{op}}}(\mathrm{Fun}(\mathbb{F}_X^{\simeq} \times X, \mathcal{V}))$$

and

$$\mathrm{Opd}_X^{\mathrm{aug}}(\mathcal{V}) := \mathrm{Opd}_X(\mathcal{V})_{/\mathbb{1}_X}.$$

Corollary 4.3.2 *Let* $\mathcal{V}$ *be a symmetric monoidal* ∞*-category compatible with small colimits. There is an adjunction*

$$\mathrm{Bar} : \mathrm{Opd}_X^{\mathrm{aug}}(\mathcal{V}) \rightleftarrows \mathrm{Coopd}_X^{\mathrm{coaug}}(\mathcal{V}) : \mathrm{Cobar},$$

where on underlying symmetric sequences $\mathrm{Bar}(\mathcal{O})$ *is given by*

$$\mathbb{1}_X \circ_{\mathcal{O}} \mathbb{1}_X \simeq \operatorname*{colim}_{\mathbb{\Delta}^{\mathrm{op}}} \left(\mathbb{1} \leftleftarrows \mathcal{O} \,\overleftarrow{\overrightarrow{\leftleftarrows}}\, \mathcal{O} \circ \mathcal{O} \cdots \right) \tag{1}$$

and $\mathrm{Cobar}(\mathcal{Q})$ *is given by*

$$\lim_{\mathbb{\Delta}} \left(\mathbb{1} \rightrightarrows \mathcal{Q} \,\overrightarrow{\overleftarrow{\rightrightarrows}}\, \mathcal{Q} \circ \mathcal{Q} \cdots \right). \tag{2}$$

Proof Apply [44, Theorem 5.2.2.17] to the monoidal ∞-category $\mathrm{Fun}(\mathbb{F}_X^{\simeq} \times X, \mathcal{V})_{\mathbb{1}_X//\mathbb{1}_X}$. □

Remark 4.3.3 Here we have defined ∞-cooperads as coalgebras in symmetric sequences, following the definition proposed in, for instance, [20]. However, the notion of cooperad in **V** that is relevant in bar-cobar duality for operads often seems to be that of operads enriched in $\mathbf{V}^{\mathrm{op}}$ (as for example used by Ching to define the bar–cobar adjunction for operads in spectra

[7]). In general these two versions of cooperads are quite different: an ∞-cooperad $\mathcal{O}$ with one object in our sense has a comultiplication $\mathcal{O} \to \mathcal{O} \circ \mathcal{O}$, which is given by morphisms

$$\mathcal{O}(n) \to \coprod_{k=0}^{\infty} \left(\left(\coprod_{i_1+\cdots+i_k=n} \mathrm{Ind}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_k}}^{\Sigma_n} (\mathcal{O}(i_1) \otimes \cdots \otimes \mathcal{O}(i_k)) \right) \otimes \mathcal{O}(k) \right)_{h\Sigma_k},$$

while an ∞-operad enriched in $\mathcal{V}^{\mathrm{op}}$ would be given by morphisms

$$\mathcal{O}(n) \to \prod_{k=0}^{\infty} \left(\left(\prod_{i_1+\cdots+i_k=n} \mathrm{CoInd}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_k}}^{\Sigma_n} (\mathcal{O}(i_1) \otimes \cdots \otimes \mathcal{O}(i_k)) \right) \otimes \mathcal{O}(k) \right)^{h\Sigma_k};$$

here $\mathrm{Ind}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_k}}^{\Sigma_n}$ and $\mathrm{CoInd}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_k}}^{\Sigma_n}$ denote induction and coinduction, respectively, or in other words left and right Kan extension along the functor $B(\Sigma_{i_1} \times \cdots \times \Sigma_{i_k}) \to B\Sigma_n$.

However, if we make some assumptions on both the ∞-operads we consider and on the ∞-categories we enrich in, then the two notions do agree: First suppose $\mathcal{V}$ is a *semiadditive* ∞-category, meaning it has a zero object and finite biproducts (i.e. finite products and coproducts coincide). (For example, this holds in any stable ∞-category, such as those of spectra or chain complexes.) If we then restrict ourselves to consider only *reduced* ∞-operads $\mathcal{O} \in \mathrm{Opd}_*(\mathcal{V})$, meaning ∞-operads such that $\mathcal{O}(0) \simeq 0$, then the coproducts in $\mathcal{O} \circ \mathcal{O}$ are finite and hence are equivalent to products. Moreover, for such reduced symmetric sequences we can rewrite the formula for the composition product without taking any homotopy orbits:[6]

$$(\mathcal{O} \circ \mathcal{O})(n) \simeq \bigoplus_{\mathbf{n} \twoheadrightarrow \mathbf{k} \twoheadrightarrow *} \mathcal{O}(i_1) \otimes \cdots \otimes \mathcal{O}(i_k) \otimes \mathcal{O}(k),$$

where $i_j = |\mathbf{n}_j|$. This is easy to see in the coordinate-free description as discussed in Sect. 1.1: passing to reduced symmetric sequences means only surjective maps of sets appear, and these have no automorphisms in $\mathbb{F}^{[2],\simeq}$.

Therefore for reduced $\mathcal{O}$ a comultiplication $\mathcal{O} \to \mathcal{O} \circ \mathcal{O}$ is equivalently described by Σ_n-equivariant maps

$$\mathcal{O}(n) \to \mathcal{O}(i_1) \otimes \cdots \otimes \mathcal{O}(i_k) \otimes \mathcal{O}(k)$$

where $n = i_1 + \cdots + i_k$. This is precisely the structure of an ∞-operad enriched in $\mathcal{V}^{\mathrm{op}}$ with the same n-ary operations as $\mathcal{O}$.

For reduced ∞-operads enriched in semiadditive ∞-categories, we therefore expect that the bar–cobar adjunction arising from the composition product is the correct one for understanding bar–cobar duality for enriched ∞-operads. One might wonder if there exists some more general version of a bar-cobar adjunction without these restrictions, but this setting does in fact seem to cover all the cases of bar–cobar duality for operads in the literature that we are aware of.

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