Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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Solitary waves in equations with fully mixed nonlocal and nonlinear terms

Master's thesis in Applied Physics and Mathematics
Supervisor: Mats Ehrnström
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Abstract. We study the existence of solitary waves in two classes of pseudo-differential evolution equations with fully mixed nonlinear and nonlocal terms. The nonlinearity is either cubic: $\partial_t u + \partial_x(Lu - uNu^2) = 0$ or quadratic: $\partial_t u + \partial_x(Lu - T(u,u)) = 0$. Here, L, N are linear Fourier multipliers, while T is a bilinear Fourier multiplier: $\widehat{T(u,u)}(\xi) = \int_{\mathbb{R}} p(\xi - \eta, \eta)\hat{u}(\xi - \eta)\hat{u}(\eta)\, d\eta$. The dispersive operator L is of positive order, while the orders of N, T (positive or negative) are restricted above by the order of L. We prove that there exist solitary-wave solutions u of small and large amplitude to these equations using constrained minimization and the concentration–compactness principle. We find that the solutions are in $H^\infty(\mathbb{R})$ and have subcritical wave speed c. For small solutions, we estimate the size of $\|u\|_{L^\infty}$ and the wave speed c.

Sammendrag. Vi studerer eksistens av solitære bølger i to familier av pseudodifferensialligninger med ikkelokale ikkelineariteter. Ikkelineariteten er enten kubisk: $\partial_t u + \partial_x(Lu - uNu^2) = 0$ eller kvadratisk: $\partial_t u + \partial_x(Lu - T(u,u)) = 0$. Her er L, N lineære Fourier-multiplikatorer, mens T er en bilineær Fourier-multiplikator: $\widehat{T(u,u)}(\xi) = \int_{\mathbb{R}} p(\xi - \eta, \eta)\hat{u}(\xi - \eta)\hat{u}(\eta)\, d\eta$. Den dispersive operatoren L har positiv orden, mens ordenen til N, T (positiv eller negativ) er oppad begrenset av ordenen til L. Vi beviser at det finnes solitær-bølgeløsninger u med små og store amplituder til disse ligningene ved hjelp av betinget minimering og konsentrasjons–kompakthetsprinsippet. Vi finner at løsningene er i $H^\infty(\mathbb{R})$ og har subkritisk bølgefart c. For små løsninger estimerer vi størrelsen til $\|u\|_{L^\infty}$ og bølgefarten c.

Preface

This work is the final outcome of the course "TMA4900–Industrial Mathematics, Master's Thesis" and concludes the five-year integrated master's program in "Applied Physics and Mathematics" at the Norwegian University of Science and Technology. The work was carried out in the spring of 2022 under the supervision of Professor Mats Ehrnström.

The thesis builds on my report in the course "TMA4500–Specialization Project" written in the fall of 2021 as a preparation for the master's thesis. In the project thesis, the main objective was to study relevant background theory and understand some of the methods used in existence proofs for solitary waves. We also presented a new existence result.

In the first part of this thesis, we improve and generalize that result. Since most parts of the proof contain substantial new work, it is not easy to separate the last part of the project thesis from the master's thesis. A detailed description of similarities and differences is presented at the beginning of the relevant chapter. In the second part of this thesis, similar methods are used to treat an equation with a different and more general operator. I appreciate being given a large degree of freedom in how to approach this.

I would like to express my gratitude to Mats for his help and guidance. During regular meetings, he has provided insight into questions I often didn't even know I had. In the fall, I will go on to do a PhD on the same subject and I would like to thank Mats and his research group for already having begun to include me.

Finally, I want to thank my family for their support and my friends in Trondheim for making the past five years so enjoyable. A special thank you to Eirik and my flatmates for keeping me company during various stages of lockdown and when I was again stuck inside the apartment for several months the past winter.

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Johanna Ulvedal Marstrander

Contents

CHAPTER 1

Introduction

The equations studied in this thesis are related to and inspired by the mathematical theory of water waves. To provide background, we very briefly review some history of this subject and introduce the basic governing equations. We discuss recent developments and related research motivating the problems considered in this thesis, and give an outline of the work at hand.

1.1. Early history

The mathematical study of water waves began in earnest after Leonhard Euler published his work on the governing equations of hydrodynamics in 1757 [**9**]. In the century that followed, several mathematicians worked on water waves. An outline of the early history can be found in [**3**]. Notable contributors include Laplace and Lagrange in the latter half of the 18th century and Poisson and Cauchy in the beginning of the 19th century. They mainly studied linearized water waves. Although there were some investigations into nonlinear theory at the time, linear theory dominated the field. Another influential figure within linear theory was Airy; linear wave theory is often called Airy wave theory due to his contributions [**3**].

In 1844, naval engineer John Scott Russel published his *Report on Waves* [**25**]. There, he describes wave phenomena he calls "waves of translation", which are now known as solitary waves.

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped — not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. [...] Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.
>
> (Russel, 1844)

A solitary wave is a single waveform propagating without changing shape. Russel reproduced the phenomenon in wave tanks and theorized about the properties of this type of wave. His observations were initially met with skepticism. This type of solution cannot arise from linearized equations and was not predicted by the existing theory at the time. Several mathematicians tried, unsuccessfully, to explain the phenomenon; among them were Airy and also Stokes (who later contributed to the derivation of the famous Navier-Stokes equations) [**3**].

The solitary wave could finally be mathematically explained when Boussinesq introduced the equations now known as the Boussinesq equations in 1872 and the Korteweg–de Vries equation in 1877 (Later rediscovered by Korteweg and de Vries in 1895) [**17**]. These types of solutions have since become a topic of mathematical research. They appear not only in water wave equations but may arise in a wide class of partial differential equations (PDEs) describing wave motion in physical systems.

1.2. The water waves problem

This section is based on the monograph [**17**]. There are many variations of the governing equations for water waves, the most general of them being the Navier-Stokes equations. These can be reduced to the incompressible Euler equations, where the flow is additionally assumed to be inviscid, incompressible, and irrotational. While these assumptions may seem restrictive, they are reasonable for a wide range of applications. The equations are supplemented by dynamic and kinematic boundary conditions; the domain is bounded by a fixed bottom and an unknown free surface, where the flow must satisfy certain conditions. The Euler equations combined with the set of boundary conditions are commonly referred to as the free-surface water waves problem.

The Euler equations with boundary conditions represent one formulation of this problem, but in fact, there exist many. We mention the Zakharov–Craig–Sulem formulation, a formulation where the unknowns are evaluated only at the surface. These equations are well suited for deriving models for different asymptotic regimes.

The water waves equations allow for a wide array of solutions with qualitatively different properties. A pertinent question is then whether our simplified models admit solutions that exhibit observed behaviors of physical waves.

One such behavior is that of traveling waves: Waves that propagate with constant speed while retaining their original shape. In one dimension, suppose that $\eta(x,t)$ denotes the surface elevation at position x and time t. We can formulate traveling waves mathematically by introducing a new *steady* variable $x - ct$, where $c \in \mathbb{R}$ is the wave speed; traveling waves are exactly those that can be written in the form

$$\eta(x,t) = \tilde{\eta}(x - ct)$$

for some function $\tilde{\eta}$.

(A) (B)

FIGURE 1. Illustration of periodic traveling waves (A) and a solitary wave (B).

If $\tilde{\eta}$ is periodic, then η is a *periodic traveling wave*. If $\tilde{\eta}$ is instead localized, that is $\tilde{\eta}(x-ct) \to 0$ as $|x-ct| \to \infty$, then η describes a *solitary wave*. This is the mathematical formulation of "the waves of translation" described by Russel and the solution type we concern ourselves with in this thesis.

Due to the complexity of the water waves equations, it is useful to classify the problem based on physical characteristics. In a fluid domain Ω, let h_0, a, λ denote the characteristic water depth, amplitude and horizontal scale respectively, see Figure 2.

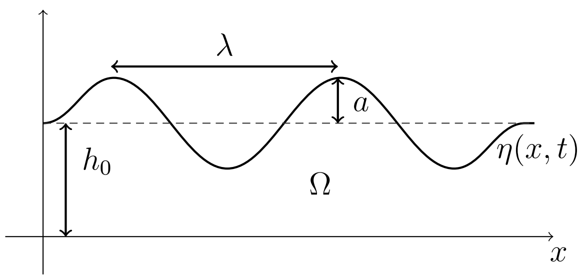

FIGURE 2. Illustration of the fluid domain Ω with characteristic sizes.

A nondimensionalized equation will typically include terms that represent the ratio between such characteristic sizes, examples being

$$\varepsilon = \frac{a}{h_0}, \quad \mu = \frac{h_0^2}{\lambda}.$$

The parameter ε is often referred to as the nonlinearity parameter, while μ is called the shallowness parameter.

Often, a simplified model equation can only be rigorously justified if one or both of these parameters are small. Furthermore, the nature of the equations will depend on the sizes of these parameters and their combinations. For example, $\varepsilon \ll 1$ and $\varepsilon \gg 1$ correspond to weakly and strongly nonlinear equations respectively, while $\mu \ll 1$ and $\mu \gg 1$ correspond to shallow and deep water respectively. Nonlinear effects are more important when μ is large, and dispersive effects play a larger role in deep than in shallow water. An equation is dispersive if waves of different spacial frequency propagate with different velocities.

Solitary waves arise when nonlinear and dispersive effects are balanced. The canonical example is the Korteweg–de Vries equation, which describes weakly nonlinear waves in shallow water (long waves):

$$\partial_t \eta + \partial_x^3 \eta + \eta \partial_x \eta = 0,$$

where $\eta(x, t)$ is the surface elevation at position x and time t. The last term is nonlinear, and the equation is dispersive due to the second term.

1.3. Recent developments

The dispersive properties depend on the dispersion relation. The Korteweg–de Vries equation does not preserve the dispersion relation of the full water waves equations, but is rather a second-order approximation. Incorporating the original dispersion relation leads to the Whitham equation [**31**], which is a nonlocal equation.

Nonlocal equations have increasingly been used to model water waves and other wave phenomena. In these equations, the solution in one point cannot be determined by the action in a neighborhood of that point. Instead, it depends on global information from the whole domain, making these types of equations suitable for capturing long-range effects. When reduced to the surface, model equations for water waves featuring the exact dispersion relation will include nonlocal terms; The Zacharov-Craig-Sulem formulation involves the Dirichlet-Neumann operator which is nonlocal. As already mentioned, the simpler Whitham equation is also nonlocal.

Nonlocal operators can be expressed as integral operators or *Fourier multiplier operators*. In fact, a large class of scalar evolution equations for water waves are of the form

$$\partial_t \eta + \partial_x (L\eta + n(\eta)) = 0, \tag{1.1}$$

where n is a nonlinear, local term and L is a Fourier multiplier operator. This means that the action of L on η may be expressed as multiplication with a function, called the *symbol*, on the Fourier side:

$$\widehat{L\eta}(\xi) = m(\xi)\hat{\eta}(\xi).$$

If m is a polynomial, then the action of L reduces to differentiation and the equation is local. Other m give truly nonlocal and far more general operators.

Typically, the search for solutions is carried out in a Sobolev space $H^t(\mathbb{R})$. If

$$L \colon H^t(\mathbb{R}) \to H^{t-s}(\mathbb{R})$$

is continuous, then L is said to be of order s. The Whitham equation is a special case of (1.1) where L is of negative order $s = -1/2$. For positive s, (1.1) covers for example the Korteweg–de Vries equation and the capillary–gravity Whitham equation.

The class of equations (1.1) has been widely studied under different assumptions on m and n. We briefly review relevant research. In 1987, Weinstein proved existence of solitary waves to an equation on the form 1.1 with $s \geq 1$ [**30**] using the method of concentration–compactness, introduced by Lions in [**20**]. The approach was put in a more general framework in [**1**]. Following this, the case $0 < s < 1$ has been studied by several authors. The fractional Korteweg–de Vries equation with $1/2 < s < 1$ was treated by Linares, Pilod & Saut in [**19**]. In [**21**], Maehlen proved existence of solitary waves to a class of model equations of low positive order (The order allowed depends on the nonlinearity: $s > 1/3$ corresponds to the case of a quadratic nonlinearity), where he allowed also for inhomogeneous nonlinearities and weaker regularity assumptions on the symbol of L. Other works treating positive-order operators include [**2, 16, 15**].

Similar techniques, albeit more technical, were used to show existence of small-amplitude solitary waves in the case where $s < 0$ in [**7**]. The class of equations covered there includes the Whitham equation, and it is shown that solitary waves in the Whitham equation are approximated by scalings of Korteweg–de Vries solitary waves. A similar existence result was also shown in [**27**] using a different method based on the implicit function theorem. Solutions in spaces of lower regularity were studied in [**14**].

Whitham-type equations and other full-dispersion models have been shown to exhibit a number of desirable features [**7, 18, 6**]. The balance between nonlinearity and dispersion can be investigated by fixing one and varying the other. In [**18**], Linares, Pilod & Saut fix a quadratic nonlinearity and vary the dispersion. They investigate several issues related to exitence and stability for fractional KdV-equations ($m = |\xi|^s, n(u) = u^2$), and show that solitary-wave solutions to this equation do not exist for $s \leq 1/3$.

Recently, it has been attempted to study the case when not only the linear term but also the nonlinearity n, is nonlocal. These kinds of equations can arise from the water waves equations, see for example [**26**], [**5**] or [**17**]. Traveling waves for equations of the form (1.1) where the nonlinearity n is of this type have been studied in e.g. [**8**] (capillary–gravity Whitham after inverting the linear operator) or [**23**] (a fractional Degasperis–Procesi equation), albeit from a different angle than in this thesis. We also mention results on Whitham-Boussinesq equations [**4, 22**] and a full-dispersion Green–Naghdi system [**6**] (introduced by Duchêne et al. in [**5**]). There, existence of small amplitude solitary-wave solutions is shown using concentration–compactness, and the equations include terms where a Fourier multiplier is entangled with the nonlinearity.

1.4. Problem description

We study two classes of equations with nonlocal nonlinearities and wish to show existence of solitary waves.

The first class has a cubic nonlinearity:

$$\partial_t u + \partial_x (Lu - uNu^2) = 0. \tag{1.2}$$

Here, the L is a Fourier multiplier of positive order $s > 1/2$, while N is a Fourier multiplier or order $r < s - 1$ which can be either positive or negative. Existence of solitary waves in an equation of this type was shown in the project thesis:

$$\partial_t u + \partial_x (\Lambda^s u - u\Lambda^r u^2) = 0.$$

There, L, N corresponded to Fourier multipliers Λ^s, Λ^r with symbols $(1+\xi^2)^{s/2}$ and $(1+\xi^2)^{r/2}$ respectively. Furthermore s, r had to satisfy $s > 1, 0 < r < s - 1$, and existence of small solutions was shown only for $0 < r < \min(s-1, 1)$. The novelty in the present work lies in showing existence of both big and small solutions for all r considered, allowing also for negative r, in relaxing the restriction on s, and in allowing for more general operators. We also include new regularity results and estimates of the wave speed c.

Existence is shown by means of minimization: We transform the problem into a constrained variational problem where we seek to minimize a functional $\mathcal{E}$ subject to a given constraint. The functional is chosen so that the minimizer solves the equation (1.2). Working on an unbounded domain $\mathbb{R}$, it is not clear that a minimizing sequence converges to a minimizer. To overcome this, we rely on the method of concentration–compactness (as in many of the aforementioned papers on solitary waves). While this is a common approach, see e.g. [**21**], [**7**], [**1**], the details along the way differ from case to case and the proofs in this thesis are modified or constructed independently by the author. Specifications are given throughout.

The second class of equations has a quadratic nonlinearity. This is perhaps more physically relevant, as the nonlinearity is quadratic in many problems arising from physics and fluid mechanics. We consider equations of the form:

$$\partial_t u + \partial_x (Lu - T(u, u)) = 0. \tag{1.3}$$

Again, L is a Fourier multiplier of positive order s, this time with $s > 0$. Analogously to the cubic case, the order of T, r, must satisfy $r < \min(s-1, (2s-1)/3)$ and can be either positive or negative. However, the nonlinear term $T(\cdot, \cdot)$ is now a bilinear Fourier multiplier with symbol p:

$$\widehat{T(u, v)} = \int_{\mathbb{R}} p(\xi - \eta, \eta)\hat{u}(\xi - \eta)\hat{v}(\eta)\, d\eta.$$

One reason for this is that we wanted to investigate quadratic nonlocal nonlinearities, but equations with a quadratic term on the simpler form

$$uNu, \qquad \text{where } N \text{ is a linear Fourier multiplier}$$

do not permit a variational formulation, see Section 4.1. We derive sufficient and necessary conditions on the symbol p for a variational formulation of (1.3) to exist.

Although uNu as above is a special case of a bilinear Fourier multiplier, we will see that the necessary conditions exclude this case.

We show existence of solitary waves to (1.3) using a similar variational formulation and method as in the cubic case. However, this has not before been done for an operator of the type 1.4 and many new or modified arguments were required.

We summarize our main results.

THEOREM 1.1. *For every $q > 0$ there exist solitary-wave solutions $u, v \in H^\infty(\mathbb{R})$ satisfying $\frac{1}{2}\|u\|_{L^2}, \frac{1}{2}\|v\|_{L^2} = q$ to* (1.2) *and* (1.3) *respectively. The corresponding wave speeds c_u, c_v are subcritical.*

Furthermore, if $q \in (0, q_0)$ for some $q_0 > 0$, then

$$\|u\|_{L^\infty} \eqsim \|u\|_{H^{s/2}} \eqsim q^{1/2}, \quad m(0) - c_u \eqsim q^\alpha,$$

and

$$\|v\|_{L^\infty} \eqsim \|v\|_{H^{s/2}} \eqsim q^{1/2}, \quad m(0) - c_v \eqsim q^\beta$$

where m is the symbol om L and α, β depend on L.

The study of equations (1.2) and (1.3) make up Chapter 3 and Chapter 4 respectively. For easy reference and to fix notation, we also include a chapter with preliminaries, Chapter 2, where basic notions and some important results are very briefly summarized.

CHAPTER 2

Preliminaries

The purpose of this section is to fix notation, review basic definitions, and collect certain results that will be explicitly referenced to later in the thesis. For background and details on Fourier analysis and Sobolev spaces, we refer to books by Grafakos[**10, 11**] and Triebel [**29**].

2.1. Definitions and spaces

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decaying smooth functions on $\mathbb{R}$

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R})\colon |f|_{k,l} < \infty \text{ for all } k, l \in \mathbb{N}_0\},$$

where

$$|f|_{k,l} = \sup_{x\in\mathbb{R}} (1 + |x|^2)^k \sum_{\alpha\leq l} |\partial_x^\alpha f(x)|.$$

Let $\mathcal{S}'(\mathbb{R})$ denote its dual space, the space of tempered distributions. Furthermore, let $\mathcal{F}$ be the Fourier transform, defined by

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \exp(-i\xi x)\, dx \quad \text{for } f \in \mathcal{S}(\mathbb{R}),$$

and extended by duality to $\mathcal{S}'(\mathbb{R})$. We shall often write $\mathcal{F}(f) = \hat{f}$.

We define $L^p(\mathbb{R})$ to be the set of all equivalence classes of Lebesgue-measurable functions on $\mathbb{R}$ with finite norm $\|f\|_{L^p} = (\int_{\mathbb{R}} |f|^p\, dx)^{1/p}$ or $\|u\|_{L^\infty} = \operatorname{ess\,supp}_{x\in\mathbb{R}} |f(x)|$ in the L^∞-case. We assume familiarity with these spaces as well as standard results on them and the Fourier transform.

Define the operator $\Lambda^s\colon \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ by

$$\widehat{\Lambda^s f}(\xi) = \langle\xi\rangle^s \hat{f}(\xi),$$

where we use the Japanese bracket $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. For $s \in \mathbb{R}, 1 < p < \infty$, we define the fractional Sobolev spaces $H_p^s(\mathbb{R})$ (also called Bessel-Potential Spaces) as

$$H_p^s = \{f \in \mathcal{S}'(\mathbb{R})\colon \|f\|_{H_p^s} = \|\Lambda^s f\|_{L^p} < \infty\}.$$

To simplify notation, we often omit the subscript when $p = 2$ and write $H^s(\mathbb{R})$ for $H_2^s(\mathbb{R})$. We define

$$H^\infty(\mathbb{R}) = \cap_{s\in\mathbb{R}} H^s(\mathbb{R}).$$

Recall that $\mathcal{F}$ is a unitary operator on $L^2(\mathbb{R})$, so that

$$\|f\|_{H_2^s} = \|\langle\cdot\rangle^s f\|_{L^2}.$$

This is not the case for other p. For all $1 < p < \infty$ we write $H^0_p(\mathbb{R}) = L^p(\mathbb{R})$. We will also need the homogeneous version of the fractional Sobolev norm $\|\cdot\|_{\dot{H}^s_p}$, which we define for $s \geq 0$ as

$$\|f\|_{\dot{H}^s_p}\|(-\partial^2)^{s/2}f\|_{L^p} < \infty,$$

where the fractional derivative is defined by $\widehat{(-\partial^2)^{s/2}}f(\xi) = |\xi|^s \hat{f}(\xi)$. The elements of homogeneous Sobolev spaces are equivalence classes of tempered distributions. For more information, we refer to [**11**, Chapter 6].

We will use the notation $a \lesssim b$ if there is a constant $C > 0$ such that $a \leq Cb$ and write $a \gtrsim b$ if $b \lesssim a$. If $b \lesssim a \lesssim b$ we write $a \approx b$.

2.2. Inequalities and embeddings

We will make use of the following inequalities and embedding results, most of which can be found in the book by Runst and Sickel[**24**].

PROPOSITION 2.1 ([**24**, Theorem 2.2.4/1 (i)]). *Suppose $t > 1/2$. Then*

$$H^t(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}).$$

PROPOSITION 2.2 ([**24**, Corollary 2.2.4/2]). *Let $2 < p < \infty$. Then*

$$H^t(\mathbb{R}) \hookrightarrow L^p(\mathbb{R}) \iff t \geq \frac{1}{2} - \frac{1}{p}.$$

PROPOSITION 2.3 ([**24**, Theorem 4.6.1/1]). *Assume that*

$$t_1 \leq t_2 \quad \text{and} \quad t_1 + t_2 > 0.$$

Furthermore, let $f \in H^{t_1}(\mathbb{R}), g \in H^{t_2}(\mathbb{R})$.

(i) *Let $t_2 > 1/2$. Then*

$$\|fg\|_{H^{t_1}} \lesssim \|f\|_{H^{t_1}}\|g\|_{H^{t_2}}.$$

(ii) *Let $t_2 < 1/2$. Then*

$$\|fg\|_{H^{t_1+t_2-1/2}} \lesssim \|f\|_{H^{t_1}}\|g\|_{H^{t_2}}.$$

PROPOSITION 2.4 ([**24**, Remark 2.5.3/2]). *Suppose that $s_0, s_1 \in \mathbb{R}, p_0, p_1 \geq 1, \theta \in (0,1)$ satisfy*

$$s = (1-\theta)s_0 + \theta s_1,$$
$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Suppose that $f \in H^{s_0}_{p_0}(\mathbb{R}) \cap H^{s_1}_{p_1}(\mathbb{R})$. Then

$$\|f\|_{H^s_p} \leq \|f\|^{1-\theta}_{H^{s_0}_{p_0}}\|f\|^{\theta}_{H^{s_1}_{p_1}}.$$

Proposition 2.5 (Young inequalities, [**10**, Theorem 1.2.12]). *Let* $p, q, r \in [1, \infty]$.

(i) If p, q, r *satisfy*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$

Then

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

(ii) If p, q, r *satisfy*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2,$$

then

$$\|f(g * h)\|_{L^1} \leq \|f\|_p \|g\|_q \|h\|_r.$$

Proposition 2.6 (Gagliardo-Nirenberg interpolation inequality, [**12**, Corollary 1.5]). *Suppose that* $1 < p, p_0, p_1 < \infty$, $s, s_1 \in \mathbb{R}$, $0 < \theta < 1$ *satisfy*

$$\frac{1}{p} - s = \frac{1-\theta}{p_0} + \theta(\frac{1}{p_1} - s_1), \quad s \leq \theta s_1.$$

Then

$$\|f\|_{\dot{H}^s_p} \lesssim \|f\|^{1-\theta}_{L^{p_0}} \|f\|^{\theta}_{\dot{H}^{s_1}_{p_1}}.$$

2.3. Other results

We will need the complex interpolation method, which allows us to extend properties to spaces with non-integer exponents by interpolation between two spaces with integer exponents.

Theorem 2.7 (Complex interpolation, [**28**, Chapter 4.2]). *Let* $s_1, s_2 \in \mathbb{R}$ *and let* $s = \theta s_1 + (1-\theta)s_2$ *for some* $\theta \in [0, 1]$. *Suppose* T *is a linear map, continuous from* $H^{s_1}(\mathbb{R})$ *to* $H^{s_1}(\mathbb{R})$ *and from* $H^{s_2}(\mathbb{R})$ *to* $H^{s_2}(\mathbb{R})$. *Then*

$$T \colon H^s(\mathbb{R}) \to H^s(\mathbb{R})$$

is also continuous.

It is well known that the derivative of a function is zero at its extrema. Generalizing this, it is reasonable that a minimizer of a functional $\mathcal{A}$ should solve the equation $\mathcal{A}'(u) = 0$. If we additionally impose a constraint on the solution, we arrive at the Lagrange multiplier principle:

THEOREM 2.8 (Lagrange multiplier principle, [**32**, Theorem 1.43]). *Let $c \in \mathbb{R}, f, g \in C^1(U, \mathbb{R})$ and let u_0 be a local extreme point of f on the constrained set $\{u \in U : g(u_0) = c\}$. Then either $g'(u_0) = 0$ or there exists a λ such that*

$$f'(u_0) = \lambda g'(u_0).$$

If a search for minimizers is carried out in a function space on a compact domain, any uniformly regular minimizing sequence will admit a subsequence converging to a minimizer. When working on $\mathbb{R}$, which is unbounded, one can instead use the concentration–compactness principle, first introduced by Lions in [**20**], to show that a sequence "concentrates". Informally, any bounded sequence admits a subsequence that either *vanishes*, meaning that the mass spreads out on an infinite domain, *dichotomizes*, meaning that the mass splits and separates in space, or it *concentrates*, i.e. most of the mass remains concentrated in a bounded domain. The theorem is stated below.

THEOREM 2.9 (The concentration–compactness principle, [**20**]). *Any sequence $\{\rho_n\}_{n \in \mathbb{N}} \subset L^1(\mathbb{R})$ satisfying*

$$\rho_n \geq 0 \text{ a.e. } on \ \mathbb{R},$$
$$\int_{\mathbb{R}} dx = \mu$$

for some $\mu > 0$ and for all $n \in \mathbb{N}$, admits a subsequence $\{\rho_{n_k}\}_{k \in \mathbb{N}}$ that satisfies either:

(i) *(Compactness) There exists a subsequence $\{y_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $r < \infty$ satisfying*

$$\int_{y_k - r}^{y_k + r} \rho_{n_k}(x)\, dx \leq \mu - \varepsilon \quad \text{for all } k \in \mathbb{N}.$$

(ii) *(Vanishing) For all $r < \infty$,*

$$\lim_{k \to \infty} \sup_{y \in \mathbb{R}} \int_{y-r}^{y+r} \rho_{n_k}(x)\, dx = 0.$$

(iii) *(Dichotomy) There exists $\bar{\mu} \in (0, \mu)$ such that for every $\varepsilon > 0$ there is a $k_0 \in \mathbb{N}, k_0 \geq 1$ and sequences of positive $L^1(\mathbb{R})$-functions*

$\{\rho_k^{(1)}\}_{k\in\mathbb{N}}, \{\rho_k^{(2)}\}_{k\in\mathbb{N}}$ *satisfying for all* $k \geq k_0$

$$\begin{aligned} \|\rho_{n_k} - (\rho_k^{(1)} + \rho_k^{(2)})\|_{L^1(\mathbb{R})} &\leq \varepsilon, \\ |\int_{\mathbb{R}} \rho_k^{(1)}\, dx - \bar{\mu}| &\leq \varepsilon, \\ |\int_{\mathbb{R}} \rho_k^{(2)}\, dx - (\mu - \bar{\mu})| &\leq \varepsilon, \\ \operatorname{dist}(\operatorname{supp}(\rho_k^{(1)}), \operatorname{supp}(\rho_k^{(2)})) &\to \infty. \end{aligned}$$

Elementary convergence results are assumed known, but we recall the Kolmogorov–Riez compactness theorem.

THEOREM 2.10 (Kolmogorov–Riesz compactness theorem [**13**]). *A subset* $\mathcal{F} \subset L^p(\mathbb{R})$, $1 \leq p < \infty$, *is totally bounded if, and only if,*

(i) $\mathcal{F}$ *is bounded,*

(ii) for all $\varepsilon > 0$, *there exists an* $R > 0$ *such that for every* $f \in \mathcal{F}$

$$\int_{|x|>R} |f(x)|^p\, dx < \varepsilon^p,$$

(iii) for all $\varepsilon > 0$, *there exists a* $\rho > 0$ *such that for every* $f \in \mathcal{F}$ *and* $y \in \mathbb{R}, |y| < \rho$,

$$\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p\, dx < \varepsilon^p.$$

CHAPTER 3

Solitary waves in equations with a nonlocal cubic term

In this chapter we study solitary-wave solutions to the equation

$$\partial_t u + \partial_x(Lu - uNu^2) = 0, \tag{3.1}$$

describing the evolution of a function u of time $t \in \mathbb{R}$ and space $x \in \mathbb{R}$. Here, L, N are Fourier multiplier operators of order s and r respectively, meaning

$$\widehat{Lu} = m(\xi)\hat{u}(\xi), \quad \widehat{Nu^2} = n(\xi)\widehat{u^2}(\xi)$$

for functions m, n and

$$L\colon H^t(\mathbb{R}) \to H^{t-s}(\mathbb{R}), \quad N\colon H^t(\mathbb{R}) \to H^{t-r}(\mathbb{R})$$

are continuous. We assume that L is of positive order, $s > 1/2$. The order of the operator N, r, can be either positive or negative but is restricted above by the order of L: $r < s - 1$. See Figure 1 for an overview. A detailed description and discussion of the assumptions are given in Section 3.1.

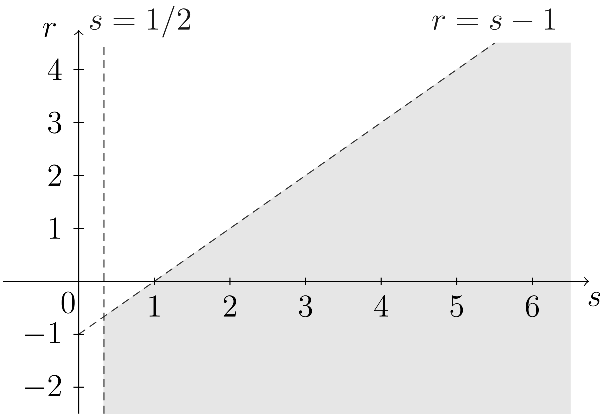

FIGURE 1. Illustration of the values of s, r for which we show existence of solitary waves. We show existence of both small- and large-amplitude solutions for values of s, r in the gray region. The boundaries are not included.

As we wish to study solitary-wave solutions, we insert

$$u(t, x) = \tilde{u}(x - ct), \quad \tilde{u}(x - ct) \to 0 \text{ as } |x - ct| \to \infty,$$

into the equation (3.1). Immediately denoting $\tilde{u}$ again by u and $x - ct$ by x and integrating, we arrive at the solitary-wave equation

$$-cu + Lu - uNu^2 = 0. \tag{3.2}$$

We reformulate the problem as a constrained minimization problem. To set up the problem, we define the functionals $\mathcal{E}, \mathcal{Q}, \mathcal{L}, \mathcal{N} \colon H^{s/2}(\mathbb{R}) \to \mathbb{R}$:

$$\begin{aligned}
\mathcal{E}(u) &= \mathcal{L}(u) - \mathcal{N}(u),\\
\mathcal{Q}(u) &= \frac{1}{2}\int_{\mathbb{R}} u^2\, dx,\\
\mathcal{L}(u) &= \frac{1}{2}\int_{\mathbb{R}} uLu\, dx = \frac{1}{2}\int_{\mathbb{R}} m(\xi)|\hat{u}(\xi)|^2\, d\xi,\\
\mathcal{N}(u) &= \frac{1}{4}\int_{\mathbb{R}} u^2Nu^2\, dx = \frac{1}{4}\int_{\mathbb{R}} n(\xi)|\widehat{u^2}(\xi)|^2\, d\xi.
\end{aligned}$$

We often omit the variable x for notational convenience, but always keep the variables in frequency space to avoid ambiguity. We will work in the Sobolev space $H^{s/2}(\mathbb{R})$, thus ensuring that the functionals are well-defined. We shall see that $\mathcal{E}, \mathcal{Q}$ are Fréchet differentiable with derivatives $\mathcal{E}'(u) = Lu - uNu^2$ and $\mathcal{Q}'(u) = u$. The Lagrange multiplier principle then implies that a minimizer of $\mathcal{E}$ subject to a constraint on $\mathcal{Q}$ solves (3.2) for some c, see Lemma 3.17 in Section 3.4.3 for details.

We therefore seek minimizers of the constrained variational problem

$$I_q := \inf\{\mathcal{E}(u) \colon u \in H^{s/2}(\mathbb{R}) \text{ and } \mathcal{Q}(u) = q\}. \tag{3.3}$$

We will construct a minimizing sequence for (3.3) and show that is *concentrates* according to Lions' concentration–compactness principle 2.9. This is then used to show convergence to a minimizer in $H^{s/2}(\mathbb{R})$.

In Section 3.1, we state and discuss the assumptions and main result. Sections 3.2 and 3.3 establish properties of the functionals and minimization problem I_q that we rely on in the subsequent analysis. In Section 3.4, we exclude vanishing and dichotomy and show how the concentration-alternative implies convergence and exitence of solutions. We conclude the chapter with estimates on the wave speed c and the regularity of the solutions in Section 3.5.

3.1. Assumptions and main theorem

Throughout the chapter, we assume the following::

(A) The symbol m of the Fourier multiplier L is real-valued, positive, even and satisfies the growth bounds

$$\text{(A1)} \qquad m(\xi) - m(0) \eqsim |\xi|^s \quad \text{for } |\xi| \geq 1,$$

$$\text{(A2)} \qquad m(\xi) - m(0) \eqsim |\xi|^{s'} \quad \text{for } |\xi| < 1,$$

with $s > 1/2, s' > 1$. Furthermore, we require that

$$\text{(A3)} \qquad \left|\frac{\partial m}{\partial \xi}(\xi)\right| \lesssim \langle \xi \rangle^{s-1} \quad \text{for all } \xi \in \mathbb{R}.$$

(B) The symbol n of the Fourier Multiplier N is real-valued, even, and satisfies the growth bound

$$\text{(B1)} \qquad n(\xi) \eqsim \langle \xi \rangle^{r} \quad \text{for all } \xi \in \mathbb{R},$$

where $r < s-1$. We also require that

$$\text{(B2)} \qquad \left|\frac{\partial n}{\partial \xi}(\xi)\right| \lesssim \langle \xi \rangle^{r-1} \quad \text{for all } \xi \in \mathbb{R}.$$

Given these assumptions, we obtain the following result:

THEOREM 3.1 (Existence of solitary-wave solutions). *For every $q > 0$, there is a solution $u \in H^\infty(\mathbb{R})$ of the solitary-wave equation* (3.2) *satisfying $\frac{1}{2}\|u\|_{L^2}^2 = q$. The corresponding wave speed c is subcritical, that is, $c < m(0)$.*

Furthermore, there is a $q_0 > 0$ such that for $q \in (0, q_0)$, the solution u and wave speed c additionally satisfy

(i) $\|u\|_{L^\infty} \eqsim \|u\|_{H^{s/2}} \eqsim q^{1/2}$,

(ii) $m(0) - c \eqsim q^{\alpha}, \ \alpha = \frac{s'}{s'-1}$.

We take a look at the assumptions and the parts they play in the proof.

3.1.1. The symbol m. The assumptions on L are inspired [**21**], where existence of solitary-wave solutions is shown for a class of equations with a positive-order Fourier multiplier on the linear term, such as here, but where the nonlinearity differs and is local. Concerning assumption (A2), the upper bound for the growth at zero is used to find a sufficiently low upper bound for I_q in 3.7. The lower bound is used to show properties of the solution u in Section 3.5 but is not necessary to show existence.

As for assumption (A1), the upper growth bound ensures that the minimization problem is well-defined in $H^{s/2}(\mathbb{R})$. Working in a higher-order Sobolev space would pose problems as $\|\cdot\|_{H^{s/2}}$ is still the highest-order norm that we can bound for minimizing sequences, see Lemma 3.8.

The lower bound for the growth of m is necessary to control the $H^{s/2}-$norm by $\mathcal{L}$. This is used to bound the L^4-norm by Sobolev embedding, using that $s > 1/2$, which is again used to exclude vanishing. The condition that $s > 1/2$ arises from the use of Sobolev embedding and interpolation theorems. However, it is also motivated by a comparison with the fractional Korteweg–de Vries equation. The condition $s > 1/2$ in the cubic case implies that $H^{s/2}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$. It corresponds to the condition that $s > 1/3$ in a setting where the nonlinearity is quadratic($s > 1/3$ implies $H^{s/2}(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$). If $s \leq 1/3$, the homogenous fKdV equation has no non-trivial solitary-wave solutions [**18**].

One could possibly exploit that the operator on the nonlinear term can be of negative order to allow for even smaller values of s. The minimization problem would still be well-defined with $I_q > -\infty$, see Lemma 3.6, and the main problem would be to exclude vanishing by bounding an $L^{\tilde{p}}$-norm for $2 < \tilde{p} < 2/(1-s)$ from below (see Lemma 3.9). We do something along these lines for an equation with a quadratic nonlinearity in Chapter 4, but the argument used to achieve that is not directly applicable to the cubic case here.

The assumption (A3) is needed to exclude dichotomy and used in Lemma 3.13. The point is to ensure that L is not "too" nonlocal, and this is related to the regularity of m [**21**]. The bound by a lower order, $s-1$ instead of s, is natural, but not necessary. Instead of (A3), one could assume that

$$|m(\xi+t) - m(\xi)| \lesssim \omega(t)\langle\xi\rangle^{s-1}$$

for some modulus of continuity ω.

Alternatively, one could assume that $\xi \mapsto m(\xi)/\langle\xi\rangle^s$ be uniformly continuous. This is done in [**21**]. It is a slightly weaker assumption, but would suffice for our purpose noting that (A3) holds for the symbol $\langle\xi\rangle^s$ and

$$\begin{aligned}|m(\xi) - m(\xi-t)| &= \left|\frac{m(\xi)}{\langle\xi\rangle^s} - \frac{m(\xi-t)}{\langle\xi-t\rangle^s}\right|\langle\xi\rangle^s + \frac{m(\xi-t)}{\langle\xi-t\rangle^s}|\langle\xi\rangle^s - \langle\xi-t\rangle^s| \\ &\leq (\tilde{\omega}_m(|t|) + |t|)\langle\xi\rangle^{s/2}\langle\xi-t\rangle^{s/2}\langle t\rangle^{s/2},\end{aligned}$$

for some bounded modulus of continuity ω_m.

In a completely different direction, one could consider an equation of type (3.2) with a negative-order operator L. However, this changes the nature of the problem and will not be pursued in this thesis.

3.1.2. The symbol n. The upper bound for the growth of n and the corresponding bound on r are used to bound $\mathcal{N}$ in terms of $\mathcal{L}$, which is crucial for obtaining both the lower bound for I_q in Lemma 3.6. Without this bound, we could not rely on the product estimates in Proposition 2.3 to do this. Several other lemmas also rely on the relation between s and r. The lower bound on the growth of n is used to bound I_q from above in Lemma 3.7. Furthermore, it implies that $\mathcal{N}(u) \eqsim \|u^2\|^2_{H^{r/2}}$, see Lemma 3.4. Although relaxing this assumption would require some (technical) changes throughout, the main issue is that n must be bounded below by a positive constant near zero in Lemma 3.7. Hence the lower bound could likely be replaced with a less strict requirement ensuring this.

As for assumption (B2), it plays the same role as assumption (A3) and the same modifications are possible.

3.1.3. Improvements from project thesis. The work in this chapter builds on the project thesis, where existence of solitary-wave solutions was shown for the equations

$$\partial_t u + \partial_x(\Lambda^s u - u\Lambda^r u^2) = 0,$$

where $s > 1, r < s - 1$. Equation 3.2 is clearly a generalization of this, with L, N generalizing Λ^s, Λ^r. A comment on the differences and similarities in the result and proof is therefore due.

The largest improvement is that we allow for a greater range of values of s and r. In particular, we allow s to be below the critical value $s = 1$ where the Sobolev embedding $H^{s/2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ ceases to hold. Instead, we rely on estimates from [**24**] that are summarized in Chapter 2 or use different arguments. We also allow for negative r and we show existence of both big and small solutions for all admissible values of s, r. The symbols m, n are allowed to be slightly more general, inspired by [**21**], the biggest change being that m can now be homogeneous.

To obtain the improved result, most Lemmas in Section 3.2 and Section 3.3 are new or significantly changed. Lemmas related to vanishing and dichotomy are similar, but generalized or otherwise improved. A few lemmas are only slightly rewritten and remain very similar but are included for completeness. This includes the proof for sub-additivity in Lemmas 3.11 and 3.12 and for existence from minimizers in Lemma 3.4.3.

The last section, on properties of the solutions and wave speed, is new.

3.2. Properties of symbols and functionals

We will repeatedly use that $\langle x\rangle^s \eqsim 1 + |x|^s$ for $s > 0$. We will also need the estimates in the next two lemmas.

Lemma 3.2. *For all $x, y \in \mathbb{R}$, we have the estimates*

$$\langle x + y\rangle \lesssim \langle x\rangle + \langle y\rangle \tag{3.4}$$

$$\langle x + y\rangle \lesssim \langle x\rangle\langle y\rangle \tag{3.5}$$

Proof.

$$\begin{aligned}\langle x + y\rangle^2 &= 1 + x^2 + 2xy + y^2\\ &\lesssim (1 + x^2) + (1 + y^2)\\ &= \langle x\rangle^2 + \langle y\rangle^2\\ &\le (\langle x\rangle + \langle y\rangle)^2.\end{aligned}$$

and

$$\begin{aligned}\langle x+y\rangle^2 &= 1+x^2+2xy+y^2\\ &\lesssim 1+x^2+y^2+x^2y^2\\ &\lesssim (1+x^2)(1+y^2)\\ &= \langle x\rangle^2\langle y\rangle^2.\end{aligned}$$

□

LEMMA 3.3. *The following estimates hold for all $\xi, \eta \in \mathbb{R}$,*

(i) The symbol m satisfies

$$|m(\xi)-m(\eta)| \lesssim |\xi-\eta|\langle\xi-\eta\rangle^{s/2}\langle\xi\rangle^{s/2}\langle\eta\rangle^{s/2}.$$

(ii) The symbol n satisfies

$$\begin{aligned}&|n(\xi)-n(\eta)| \lesssim |\xi-\eta| \quad \text{if } r\le 0,\\ &|n(\xi)-n(\eta)| \lesssim |\xi-\eta|\langle\xi-\eta\rangle^{r/2}\langle\xi\rangle^{r/2}\langle\eta\rangle^{r/2} \quad \text{if } r>0.\end{aligned}$$

PROOF. We begin by proving the estimate for m. The estimate for n when $r>0$ is proved in exactly the same manner. With $a=\xi-\eta, b=\eta$, (3.4) implies that

$$\langle\xi\rangle^{s/2} \lesssim \langle\xi-\eta\rangle^{s/2}\langle\eta\rangle^{s/2}, \tag{3.6}$$

and

$$\langle\eta\rangle^{s/2} \lesssim \langle\xi-\eta\rangle^{s/2}\langle\xi\rangle^{s/2}. \tag{3.7}$$

Using assumption (A3), the mean value theorem, and the estimates (3.6), (3.7), we obtain

$$\begin{aligned}|m(\xi)-m(\eta)| &\le |\xi-\eta| \sup_{|\theta|\le|\xi|,|\eta|}\langle\theta\rangle^{s-1}\\ &\le |\xi-\eta| \sup_{|\theta|\le|\xi|,|\eta|}\langle\theta\rangle^{s}\\ &\le |\xi-\eta|(\langle\xi\rangle^s+\langle\eta\rangle^s)\\ &\lesssim |\xi-\eta|\langle\xi\rangle^{s/2}\langle\eta\rangle^{s/2}\langle\xi-\eta\rangle^{s/2},\end{aligned}$$

which concludes the proof of (i).

We show (ii) for $r\le 0$ similarly.

$$\begin{aligned}|n(\xi)-n(\eta)| &\lesssim \sup_{|\theta|\le|\xi|,|\eta|}|\xi-\eta|\langle\theta\rangle^{r-1}\\ &\le |\xi-\eta|.\end{aligned}$$

□

Instead of working directly with the functionals $\mathcal{L}, \mathcal{N}$, it is often simpler to consider the related norms. We have the following result.

LEMMA 3.4. *We have the estimates*

$$\mathcal{L}(u) + \mathcal{Q}(u) \eqsim \|u\|_{H^{s/2}},$$

and

$$\mathcal{N}(u) \eqsim \|u^2\|_{H^{r/2}}.$$

PROOF. For $|\xi| \geq 1$, we have that

$$m(\xi) + 1 = m(\xi) - m(0) + m(0) + 1 \eqsim 1 + |\xi|^s \eqsim \langle\xi\rangle^s,$$

while for $|\xi| < 1$,

$$m(\xi) + 1 \eqsim 1 \eqsim \langle\xi\rangle^s.$$

It then follows automatically that

$$\mathcal{L}(u) + \mathcal{Q}(u) \eqsim \|u\|^2_{H^{s/2}}$$

since

$$\mathcal{L}(u) + \mathcal{Q}(u) = \frac{1}{2}\int_{\mathbb{R}} (1 + m(\xi))|\hat{u}(\xi)|^2 \, d\xi.$$

The estimate on $\mathcal{N}(u)$ follows trivially from the definition of $\mathcal{N}$ and assumption $(B1)$. □

The next lemma allows us to bound $\mathcal{N}$ in terms of $\mathcal{L}$. The result is closely related to the restriction $r < s - 1$ on r and crucial for providing a lower bound for I^q as well as other important bounds in Lemma 3.8.

LEMMA 3.5. *Let* $u \in H^{s/2}(\mathbb{R})$*. Then*

$$\|u^2\|_{H^{r/2}} \lesssim \|u\|^{2-\gamma}_{L^2} \|u\|^{\gamma}_{H^{s/2}}$$

for some $\gamma < 1$*.*

PROOF. *Step 1.* Assume $s > 1$ and $r > 0$. Then we can pick a $\tau \in \mathbb{R}$ satisfying

$$1 < \tau < s - r$$

and apply Propsition 2.3 (i) with $t_1 = r/2$ and $t_2 = \frac{\tau}{2}$:

$$\begin{aligned} \|u^2\|_{H^{r/2}} &\lesssim \|u\|_{H^{r/2}} \|u\|_{H^{\tau/2}} \\ &\lesssim \|u\|^{1-r/s}_{L^2} \|u\|^{r/s}_{H^{s/2}} \|u\|^{1-\tau/s}_{L^2} \|u\|^{\tau/s}_{H^{s/2}} \\ &= \|u\|^{2-(r+\tau)/s}_{L^2} \|u\|^{(r+\tau)/s}_{H^{s/2}}, \end{aligned}$$

which show the desired result with $\gamma = \frac{r+\tau}{s} < \frac{r+s-r}{s} = 1$.

If $r \leq 0$, pick $\tilde{r}$ such that $0 < \tilde{r} < s - 1$, apply Step 1 with $\tilde{r}$ instead of r and use that

$$\|u^2\|_{H^{r/2}} \leq \|u^2\|_{H^{\tilde{r}/2}}.$$

Step 2. Assume $s < 1$. If $r > -1$, then we can apply Proposition 2.3 (ii) with $t_1 = 0$ and $t_2 = 1/2 + r/2$:

$$\begin{aligned}\|u^2\|_{H^{r/2}} &\lesssim \|u\|_{L^2}\|u\|_{1/2+r/2}\\ &\lesssim \|u\|_{L^2}^{2-\frac{r+1}{s}}\|u\|_{H^{s/2}}^{\frac{r+1}{s}}.\end{aligned}$$

By assumption,

$$\frac{r+1}{s} < 1,$$

so we have shown the result of the lemma with $\gamma = \frac{r+1}{s}$ when $s < 1$ and $r > -1$.

If $r \leq -1$, then we pick $\tilde{r}$ satisfying $s - 1 > \tilde{r} > -1$ and apply Step 2 with $\tilde{r}$ instead of r. Such an $\tilde{r}$ exists since by assumption $s > 0$.

Step 3. If $s = 1$, then pick $\tilde{s}$ satisfying $r + 1 < \tilde{s} < 1$, apply Step 2 with $\tilde{s}$ instead of s and use that

$$\|u\|_{H^{\tilde{s}/2}} \leq \|u\|_{H^{s/2}}.$$

□

3.3. Bounds for I_q and norm-estimates

In this section we lay the foundation for excluding vanishing and dichotomy later on. We provide upper and lower bounds for I_q and bound the norms of functions that are "close" to the minimizer in a way that will soon be made clear. It is the upper bound on I_q that allows us to show these bounds, while the lower bound $I_q > -\infty$ ensures that the minimization problem is meaningful.

LEMMA 3.6. *For all $q > 0$,*

$$I_q > -\infty.$$

PROOF. Let $u \in H^{s/2}(\mathbb{R})$ and satisfy $\mathcal{Q}(u) = q$. Using Lemma 3.4 and Lemma 3.5, then for some constants $C_1, C_2 > 0$ and $\gamma < 1$ we get that

$$\begin{aligned}\mathcal{E}(u) &= \mathcal{L}(u) + \mathcal{Q}(u) - \mathcal{N}(u) - \mathcal{Q}(u)\\ &> C_1\|u\|_{H^{s/2}}^2 - C_2 q^{2-\gamma}\|u\|_{H^{s/2}}^{2\gamma} - q\\ &> -\infty,\end{aligned}$$

where the last inequality holds since $\gamma < 1$ guarantees that the expression is positive as $\|u\|_{H^{s/2}} \to \infty$. □

To prove the next lemma, we introduce the ansatz function $\phi_t = \sqrt{t}\phi(tx)$. This is standard, see for example [**1**], [**21**], but the details differ.

LEMMA 3.7. *For all $q > 0$,*

$$I_q < m(0)q.$$

PROOF. It suffices to show that there exists a function $u \in H^{s/2}(\mathbb{R})$ with $\mathcal{Q}(u) = q$ such that $\mathcal{E}(u) < m(0)q$. To that end, pick a function $\phi \in \mathcal{S}(\mathbb{R})$ satisfying $\mathcal{Q}(\phi) = q$ and let $0 < t < 1$. Define

$$\phi_t(x) = \sqrt{t}\phi(tx)$$

and observe that $\mathcal{Q}(\phi_t) = q$.

Step 1. Finding lower bounds for $\mathcal{N}(\phi_t)$. Observe that if $r \geq 0$, then $\langle t\xi\rangle^r \geq 1$ for all ξ, while if $r < 0$, then $\langle t\xi\rangle^r > \langle \xi\rangle^r$ for all ξ. We have that

$$\begin{aligned}
\mathcal{N}(\phi_t) &= \frac{1}{4}\int_{\mathbb{R}} n(\xi)|\widehat{\phi_t^2}(\xi)|^2\, d\xi \\
&\gtrsim \frac{1}{4}\int_{\mathbb{R}} \langle \xi\rangle^r|\widehat{\phi^2}(\xi/t)|^2\, d\xi \\
&= \frac{1}{4}\int_{\mathbb{R}} t\langle t\xi\rangle^r|\widehat{\phi^2}(\xi)|^2\, d\xi \\
&\geq t\min(\|\phi^2\|_{L^2}^2, \|\phi^2\|_{H^{r/2}}^2)
\end{aligned}$$

Hence

$$\mathcal{N}(u) \geq C_1 t \tag{3.8}$$

for some constant $C_1 > 0$ depending on ϕ and r.

Step 2. Finding upper bounds for $\mathcal{L}(\phi_t)$. We have that

$$\begin{aligned}
\mathcal{L}(\phi_t) &= \frac{1}{2}\int_{\mathbb{R}} m(\xi)|\widehat{\phi_t}(\xi)|^2\, d\xi \\
&= m(0)q + \frac{t}{2}\int_{\mathbb{R}} (m(\xi) - m(0))|\widehat{\phi(t\cdot)}(\xi)|^2\, d\xi \\
&= m(0)q + \frac{1}{2t}\int_{\mathbb{R}} (m(\xi) - m(0))|\hat{\phi}(\xi/t)|^2\, d\xi \\
&= m(0)q + \frac{1}{2}\int_{\mathbb{R}} (m(t\xi) - m(0))|\hat{\phi}(\xi)|^2\, d\xi
\end{aligned}$$

If $s < s'$, then $m(t\xi) - m(0) \lesssim |t\xi|^{s'}$ for all $\xi \in \mathbb{R}$. On the other hand, if $s \geq s'$, then $m(t\xi) - m(0) \lesssim |t\xi|^{s'} + |t\xi|^s$. Hence, for constants $C_2, C_3 > 0$ depending on ϕ, s' and s, we get that

$$\begin{aligned}
\mathcal{L}(u)(\phi_t) &\leq \begin{cases} m(0)q + C_2 t^{s'} \int_{\mathbb{R}} |\xi|^{s'}|\hat{\phi}(\xi)|^2\, d\xi + C_2 t^s \int_{\mathbb{R}} |\xi|^s|\hat{\phi}(\xi)|^2\, d\xi & \text{if } s \geq s' \\ m(0)q + C_2 t^{s'} \int_{\mathbb{R}} |\xi|^{s'}|\hat{\phi}(\xi)|^2\, d\xi & \text{if } s < s' \end{cases} \\
&\leq \begin{cases} m(0)q + C_2 t^{s'}\|\phi\|_{H^{s'/2}}^2 + C_2 t^s\|\phi\|_{H^{s/2}}^2 & \text{if } s \geq s' \\ \leq m(0)q + C_2 t^{s'}\|\phi\|_{H^{s'/2}}^2 & \text{if } s < s' \end{cases} \\
&\leq m(0)q + C_3 t^{s'}.
\end{aligned} \tag{3.9}$$

Step 3. Finding an upper bound for $\mathcal{E}(\phi_t)$. Combining (3.8), and (3.9) gives

$$\begin{aligned}\mathcal{E}(\phi_t) &= \mathcal{L}(\phi_t) - \mathcal{N}(\phi_t)\\ &\le m(0)q + C_3 t^{s'} - C_1 t^1.\end{aligned}$$

The positive terms will go to zero faster than the negative term as $t \to 0$, since $s' > 1$ by assumption. Hence it is possible to pick a $t > 0$ such that

$$I_q \le \mathcal{E}(\phi_t) < m(0)q$$

for all $q > 0$. □

The upper bound on $\|\cdot\|_{H^{s/2}}$ is used repeatedly in the rest of the chapter, while the lower bound on $\|\cdot\|_{L^4}$ will be used to exclude vanishing.

LEMMA 3.8. *Any minimizing sequence for I_q has a subsequence satisfying*

$$\|u_n\|_{H^{s/2}}^{-1} + \|u_n^2\|_{H^{r/2}} + \|u_n\|_{L^4} \ge \delta \quad \text{for all } n \in \mathbb{N}.$$

for some $\delta > 0$.

PROOF. In light of Lemma 3.7, we can pick a subsequence $\{u_n\}_{n\in\mathbb{N}}$ satisfying

$$\mathcal{E}(u_n) < m(0)q \quad \text{for all } n \in \mathbb{N}. \tag{3.10}$$

Step 1. Finding an upper bound for $\|u_n\|_{H^{s/2}}$. We use (3.10), Lemma 3.4 and Lemma 3.5 and find that

$$\begin{aligned}\|u_n\|_{H^{s/2}}^2 &\eqsim \mathcal{L}(u_n) + \mathcal{Q}(u_n)\\ &= \mathcal{E}(u_n) + \mathcal{N}(u_n) + \mathcal{Q}(u_n)\\ &\lesssim (m(0)+1)q + \|u_n^2\|_{H^{r/2}}^2\\ &\lesssim q + q^{2-\gamma}\|u_n\|_{H^{s/2}}^{2\gamma}\end{aligned}$$

for some $\gamma < 1$. Since this means $\|u\|_{H^{s/2}}$ is bounded by itself to a smaller power, then it must be bounded. Picking $\delta_1 > 0$ small enough, then

$$\|u_n\|_{H^{s/2}} \le 1/\delta_1 \quad \text{for all } n \in \mathbb{N}.$$

Step 2. Finding a lower bound for $\|u_n^2\|_{H^{r/2}}$. Assuming $\{u_n\}_{n\in\mathbb{N}}$ has no subsequence such that $\|u_n^2\|_{H^{r/2}} \ge \delta_2$ for any $\delta_2 > 0$, then

$$\limsup_{n\to\infty} \|u_n^2\|_{H^{r/2}} \le 0.$$

But then there is a constant $C_1 > 0$ such that

$$\begin{aligned}I_q &= \liminf_{n\to\infty}(\mathcal{L}(u_n) - \mathcal{N}(u_n))\\ &\ge \liminf_{n\to\infty}(m(0)q + \int_{\mathbb{R}}(m(\xi) - m(0))|\widehat{u_n}|^2\,dx - C_1\|u_n^2\|_{H^{r/2}}^2)\\ &\ge m(0)q - C_1 \limsup_{n\to\infty}\|u_n^2\|_{H^{r/2}}^2\\ &= m(0)q,\end{aligned}$$

which contradicts (3.10). Hence, passing again to a subsequence if necessary,

$$\|u_n^2\|_{H^{r/2}} \geq \delta_2 \quad \text{for all } n \in \mathbb{N}$$

for some $\delta_2 > 0$.

Step 3. Finding a lower bound for $\|u_n\|_{L^4}$. If $r \leq 0$, then the result is immediate from

$$\|u_n\|_{L^4} = \|u_n^2\|_{L^2}^{1/2} \geq \|u_n^2\|_{H^{r/2}}^{1/2} \geq \delta_2^{1/2}.$$

If $r > 0$, then $s > 1$. By interpolation and using Lemma 2.3(i) with $t_1 = t_2 = s/2 > 1/2$, we obtain

$$\|u_n^2\|_{H^{r/2}} \leq \|u_n^2\|_{L^2}^{1-r/s} \|u_n^2\|_{H^{s/2}}^{r/s} \leq \|u_n\|_{L^4}^{2(1-r/s)} \|u_n\|_{H^{s/2}}^{2r/s},$$

which implies

$$\|u_n\|_{L^4} \geq \left(\frac{\|u_n^2\|_{H^{r/2}}}{\|u_n\|_{H^{s/2}}^{2r/s}} \right)^{\frac{1}{2(1-r/s)}} \geq \delta_1^{\frac{r}{s(1-r/s)}} \delta_2^{\frac{1}{2(1-r/s)}} = \delta_3.$$

Combining step 1,2,3 and picking $\delta = \min(\delta_1, \delta_2, \delta_2^{1/2}, \delta_3)$ gives the main result. □

We will restrict our attention to functions $u \in H^{s/2}(\mathbb{R})$, $\mathcal{Q}(u) = q$ satisfying

$$\mathcal{E}(u) < m(0)q \text{ and } \|u\|_{H^{s/2}}^{-1} + \|u^2\|_{r/2} + \|u\|_{L^4} \geq \delta,$$

henceforth referred to as *near minimizers*. Any minimizing sequence is from now on implicitly assumed to consist solely of near minimizers. Due to Lemma 3.7 and Lemma 3.8, such a sequence can be obtained from any minimizing sequence by passing to a subsequence.

3.4. Concentration–compactness and existence of solutions

We make use of the Concentration–compactness principle 2.9 to show existence of solutions to (3.2).

3.4.1. Excluding vanishing. The method for excluding vanishing is similar in several articles, including [**1, 14, 2**], and we use the same approach.

Lemma 3.9. *Let $v \in H^{s/2}(\mathbb{R})$ and assume that p^* satisfies $p^* > 2$ if $s \geq 1$ and $2 < p^* < 2/(1-s)$ if $s < 1$. Given $\delta > 0$, suppose that $\|v\|_{H^{s/2}} \leq 1/\delta$ and $\|v\|_{L^{p^*}} \geq \delta$. Then there exists $\varepsilon > 0$ such that*

$$\sup_{j \in \mathbb{Z}} \int_{j-2}^{j+2} |v(x)|^2 \, dx \geq \varepsilon.$$

PROOF. First, we wish to establish the inequality

$$\sum_{j\in\mathbb{Z}} \|\xi_j v\|_{H^{s/2}}^2 \lesssim \|v\|_{H^{s/2}}^2 \quad \text{for all } v \in H^{s/2}(\mathbb{R}). \tag{3.11}$$

To that end, let $\zeta\colon \mathbb{R} \to [0,1]$ denote a smooth function such that $\operatorname{supp}\zeta \subset [-2,2]$ and $\sum_{j\in\mathbb{Z}} \zeta(x-j) = 1$ for all $x \in \mathbb{R}$ and write $\zeta(x-j) = \zeta_j(x)$. Consider the operator $T\colon v \to \{\zeta_j v\}_j$.

Clearly, T is bounded from $H^n(\mathbb{R})$ to $l^2(H^n(\mathbb{R}))$ for any $n \in \mathbb{N}_0$ since at each x only finitely many $\zeta_j v$, $(\zeta_j v)^{(n)}$ are non-zero. This implies (3.11) for all $n \in \mathbb{N}_0$. Using complex interpolation 2.7, (3.11) follows for all $s > 0$.

Again using that only finitely many ζ_j are non-zero at any x, we establish that

$$\|v\|_{L^{p^*}}^{p^*} \eqsim \sum_{j\in\mathbb{Z}} \|\zeta_j v\|_{L^{p^*}}^{p^*} \quad \text{for all } v \in L^{p^*}(\mathbb{R}). \tag{3.12}$$

By assumption $s/2 > 1/2 - 1/p^*$. Hence there is an $1/2 < \tilde{s} < s$ such that by the Sobolev embedding theorem,

$$\|\zeta_j v\|_{L^{p^*}}^{p^*} \lesssim \|\zeta_j v\|_{H^{\tilde{s}/2}}^{p^*} \le \|\zeta_j v\|_{H^{s/2}}^{p^*-\tau} \|\zeta_j v\|_{L^2}^{\tau} \quad \text{for all } v \in H^{s/2}(\mathbb{R}), \tag{3.13}$$

where $\tau = p^*(1 - \tilde{s}/s) > 0$.

Combining (3.11), (3.12) and (3.13), then

$$\begin{aligned}
\|v\|_{L^{p^*}}^{p^*} &\eqsim \sum_{j\in\mathbb{Z}} \|\zeta_j v\|_{L^{p^*}}^{p^*} \\
&\lesssim \sum_{j\in\mathbb{Z}} \|\zeta_j v\|_{H^{s/2}}^{p^*-\tau} \|\zeta_j v\|_{L^2}^{\tau} \\
&\le \sup_{j\in\mathbb{Z}} \|\zeta_j v\|_{L^2}^{\tau} \|\zeta_j v\|_{H^{s/2}}^{p^*-2-\tau} \left(\sum_{j\in\mathbb{Z}} \|\zeta_j v\|_{H^{s/2}}^2 \right) \\
&\lesssim \sup_{j\in\mathbb{Z}} \|\zeta_j v\|_{L^2}^{\tau} \|v\|_{H^{s/2}}^{p^*-\tau}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sup_{j\in\mathbb{Z}} \int_{j-2}^{j+2} |v(x)|^2\, dx &\ge \sup_{j\in\mathbb{Z}} \|\zeta_j v\|_{L^2}^2 \\
&\gtrsim \|v\|_{L^{p^*}}^{2p^*/\tau} \|v\|_{H^{s/2}}^{2(\tau-p^*)/\tau} \\
&\ge \delta^{4p^*/\tau-2} = \varepsilon > 0,
\end{aligned}$$

which concludes the proof. □

Near minimizers u_n satisfy the assumptions of Lemma 3.9 with $p^* = 4$ and $\delta > 0$ from Lemma 3.8. We conclude that no minimizing sequence of I_q vanishes in accordance with the Concentration–Compactness principle 2.9.

COROLLARY 3.10. *Vanishing does not occur.*

3.4.2. Excluding dichotomy. The task of excluding dichotomy is quite technical and we split the proof into several parts. The main idea is to force a contradiction: We will first show that I_q is strictly sub-additive, that is $I_{q_1+q_2} < I_{q_1} + I_{q_2}$. Then we proceed to show that if dichotomy occurs, then $I_{q_1+q_2} \geq I_{q_1} + I_{q_2}$.

In the next two lemmas, we show that I_q is sub-homogeneous and that this implies sub-additivity.

LEMMA 3.11. *Suppose a function $f\colon \mathbb{R} \to \mathbb{R}$ is strictly sub-homogeneous, i.e.*

$$f(tx) < tf(x) \quad \text{for } t > 1.$$

Then f is strictly sub-additive:

$$f(x_1 + x_2) < f(x_1) + f(x_2).$$

PROOF. If $x_1 = x_2$, then the result is immediate with $t = 2$.

If $x_1 \neq x_2$, assume without loss of generality that $x_1 < x_2$. Then for some $t > 1$, we can write $x_2 = tx_1$ and hence

$$\begin{aligned} f(x_1 + x_2) &= f\left(x_2\left(1 + \frac{1}{t}\right)\right) < \left(1 + \frac{1}{t}\right) f(x_2) \\ &= f(x_2) + \frac{1}{t} f(t\frac{x_2}{t}) < f(x_2) + \frac{t}{t} f(\frac{x_2}{t}) \\ &= f(x_1) + f(x_2). \end{aligned}$$

□

LEMMA 3.12 (Sub-additivity). *I_q is strictly sub-additive, meaning*

$$I_{q_1+q_2} < I_{q_1} + I_{q_2}.$$

PROOF. By Lemma 3.11, it suffices to show sub-homogeneity of $q \mapsto I_q$, meaning

$$I_{tq} < tI_q \quad \text{for } t > 1, q > 0.$$

Let $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence of I_q, and let $\tilde{u}_n = \sqrt{t}u_n$ define a new sequence. Then $\mathcal{Q}(\tilde{u}_n) = tq$ and

$$\begin{aligned} I_{tq} &\leq \mathcal{L}(\tilde{u}_n) - \mathcal{N}(\tilde{u}_n) \\ &= t\mathcal{L}(u_n) - t^2\mathcal{N}(u_n) \\ &= t\mathcal{E}(u_n) + t(1-t)\mathcal{N}(u_n). \end{aligned}$$

Sub-homogeneity now follows since $(1 - t) < 0$ by assumption and $\mathcal{N}(u_n) \geq \delta > 0$ by Lemma 3.8. □

The next result is the key to excluding dichotomy.

LEMMA 3.13. *Let $u \in H^{s/2}(\mathbb{R})$ and let $\rho \in \mathcal{S}(\mathbb{R})$ be a non-negative Schwartz function. Define $\rho_R(x) = \rho(x/R)$.*

(i) For the operator L,

$$|\int_{\mathbb{R}} \rho_R u(\rho_R Lu - L(\rho_R u))\, dx| \to 0, \tag{3.14}$$

and

$$|\int_{\mathbb{R}} (1-\rho_R)u((1-\rho_R)Lu - L((1-\rho_R)u))\, dx| \to 0 \tag{3.15}$$

as $R \to \infty$.

(ii) For the operator N,

$$|\int_{\mathbb{R}} \rho_R u^2(\rho_R Nu^2 - N(\rho_R u^2))\, dx| \to 0$$

and

$$|\int_{\mathbb{R}} (1-\rho_R)u^2((1-\rho_R)Nu^2 - N((1-\rho_R)u^2))\, dx| \to 0$$

as $R \to \infty$.

PROOF. *(i)* Lemma 3.3 implies that

$$|m(\xi - t) - m(\xi)| \lesssim |t|\langle \xi\rangle^{s/2}\langle \xi - t\rangle^{s/2}\langle t\rangle^{s/2}$$

for all $\xi, t \in \mathbb{R}$.

Using this combined with Plancherel and Fubini's theorem, equation (3.14) follows from a direct calculation:

$$\begin{aligned}
&|\int_{\mathbb{R}} \rho_R u(\rho_R Lu - L(\rho_R u))\, dx| \\
&\leq \int_{\mathbb{R}} |\widehat{\overline{\rho_R u}}|(\xi) \int_{\mathbb{R}} |\widehat{\rho_R}(t)||\hat{u}(\xi - t)||m(\xi - t) - m(\xi)|\, dt d\xi \\
&\lesssim \int_{\mathbb{R}} |\widehat{\rho_R}(t)||t|\langle t\rangle^{s/2} \int_{\mathbb{R}} |\widehat{\overline{\rho_R u}}(\xi)||\hat{u}(\xi - t)|\langle \xi\rangle^{s/2}\langle \xi - t\rangle^{s/2} d\xi dt \\
&\leq \int_{\mathbb{R}} |\widehat{\rho_R}(t)||t|\langle t\rangle^{s/2}\|\rho_R u\|_{H^{s/2}}\|u\|_{H^{s/2}} dt \\
&= \|\rho_R u\|_{H^{s/2}}\|u\|_{H^{s/2}} \int_{\mathbb{R}} |R\hat{\rho}(Rt)||t|\langle t\rangle^{s/2} dt \\
&= \|\rho_R u\|_{H^{s/2}}\|u\|_{H^{s/2}} \int_{\mathbb{R}} \frac{1}{|R|}|\hat{\rho}(v)||v|\langle v/R\rangle^{s/2} dv \\
&\to 0
\end{aligned}$$

as $R \to \infty$ since $\rho \in \mathcal{S}(\mathbb{R})$.

To show (3.15), observe that

$$\begin{aligned}
&\int_{\mathbb{R}} v((1-\rho_R)Lu - L((1-\rho_R)u))\,dx \\
&= \int_{\mathbb{R}} v(-\rho_R Lu + L\rho_R u)\,dx \\
&\to 0
\end{aligned}$$

as $R \to \infty$, concluding the proof.

(ii) First, we show that $u^2 \in H^{s/2}(\mathbb{R})$ if $r > 0$ and that $u^2 \in L^2(\mathbb{R})$ if $r \leq 0$:

If $r > 0$, then $s > 1$ and $u^2 \in H^{s/2}(\mathbb{R})$ by Lemma 2.3(i) with $t_1 = t_2 = s/2 > 1/2$. If on the other hand $r \leq 0$, then

$$\|u^2\|_{L^2} = \|u\|_{L^4}^2 \lesssim \|u\|_{H^{s/2}},$$

by the Sobolev Embedding theorem 2.2.

Hence, if $r > 0$, the proof of *(ii)* is identical to the proof of *(ii)* since n satisfies

$$|n(\xi - t) - n(\xi)| \lesssim |t|\langle\xi\rangle^{r/2}\langle\xi - t\rangle^{r/2}\langle t\rangle^{r/2}$$

by Lemma 3.3 and since $u^2 \in H^{s/2}(\mathbb{R})$.

If on the other hand $r \leq 0$, then $u^2 \in L^2(\mathbb{R})$ and

$$|n(\xi - t) - n(\xi)| \lesssim |t|,$$

and hence

$$\begin{aligned}
&|\int_{\mathbb{R}} \rho_R u^2(\rho_R N u^2 - N(\rho_R u^2))\,dx| \\
&\leq \int_{\mathbb{R}} |\widehat{\rho_R u^2}|(\xi) \int_{\mathbb{R}} |\widehat{\rho_R}(t)||\widehat{u^2}(\xi - t)||n(\xi - t) - n(\xi)|\,dtd\xi \\
&\lesssim \int_{\mathbb{R}} |\widehat{\rho_R}(t)||t| \int_{\mathbb{R}} |\widehat{\rho_R u^2}(\xi)||\widehat{u^2}(\xi - t)|d\xi dt \\
&\leq \int_{\mathbb{R}} |\widehat{\rho_R}(t)||t|\|\rho_R u^2\|_{L^2}\|u^2\|_{L^2} dt \\
&= \|\rho_R u^2\|_{L^2}\|u^2\|_{L^2} \int_{\mathbb{R}} \frac{1}{|R|}|\hat{\rho}(v)||v|dv \\
&\to 0
\end{aligned}$$

as $R \to \infty$. The proof for $1 - \rho_R$ is as before. □

In the next lemma, we make concrete a consequence of dichotomy.

LEMMA 3.14. *Suppose that dichotomy occurs for a sequence* $\{\frac{1}{2}u_n^2\}_{n\in\mathbb{N}}$ *with* $\mathcal{Q}(u_n) = q$.

Let $\phi \in C^\infty(\mathbb{R})$ be a non-negative function satisfying

$$\phi(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 2 \end{cases}$$

and let

$$\psi(x) = 1 - \phi(x) \quad \text{for all } x \in \mathbb{R}.$$

Let $\phi_n(x) = \phi(\frac{x-y_n}{R_n})$ and $\psi_n(x) = \psi(\frac{x-y_n}{R_n})$ for $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\{R_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ to be determined, and where $R_n \to \infty$ as $n \to \infty$. Define $u_n^{(1)} = \phi_n u_n, u_n^{(2)} = \psi_n u_n$.

Then for some $\bar{q} \in (0, q)$, $\{u_n\}_{n\in\mathbb{N}}$ and the two sequences $\{u_n^{(1)}\}_{n\in\mathbb{N}}$ and $\{u_n^{(2)}\}_{n\in\mathbb{N}}$ satisfy

$$\mathcal{Q}(u_n^{(1)}) \to \bar{q}, \tag{3.16}$$

$$\mathcal{Q}(u_n^{(2)}) \to (q - \bar{q}), \tag{3.17}$$

$$\frac{1}{2}\int_{R_n \le |x-y_n| \le 2R_n} u_n^2 \, dx \to 0. \tag{3.18}$$

as $n \to \infty$.

Proof. If Dichotomy occurs, then we can find $\bar{q} \in (0, q)$ and two sequences $\{\rho_n^{(1)}\}_{n\in\mathbb{N}}$ and $\{\rho_n^{(2)}\}_{n\in\mathbb{N}} \in L^1(\mathbb{R})$ satisfying

$$\|\frac{1}{2}u_n^2 - (\rho_n^{(1)} + \rho_n^{(2)})\|_{L^1} \to 0,$$

$$|\int_{\mathbb{R}} \rho_n^{(1)} \, dx - \bar{q}| \to 0,$$

$$|\int_{\mathbb{R}} \rho_n^{(2)} \, dx - (q - \bar{q})| \to 0,$$

as $n \to \infty$ and

$$\operatorname{supp} \rho_n^{(1)} \subset (y_n - R_n, y_n + R_n),$$

$$\operatorname{supp} \rho_n^{(2)} \subset (-\infty, y_n - 2R_n) \bigcup (y_n + 2R_n, \infty),$$

for $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\{R_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, where $R_n \to \infty$ as $n \to \infty$. Then clearly

$$\frac{1}{2}\int_{R_n \le |x-y_n| \le 2R_n} u_n^2 \, dx \to 0 \quad \text{as } n \to \infty,$$

which is (3.18).

Now only a simple calculation is required to establish (3.16)

$$\begin{aligned}
|\mathcal{Q}(u_n^{(1)}) - \bar{q}| &\leq |\int_{\mathbb{R}} \frac{1}{2}\phi_n^2 u_n^2 - \rho_n^{(1)}\,dx| \\
&\quad + |\int_{\mathbb{R}} \rho_n^{(1)}\,dx - \bar{q}| \\
&= |\int_{R_n \leq |x-y_n| < 2R_n} \frac{1}{2}u_n^2 - \rho_n^{(1)} - \rho_n^{(2)}\,dx| + |\int_{R_n \leq |x-y_n| < 2R_n} \frac{1}{2}\phi_n^2 u_n^2\,dx| \\
&\quad + \int_{\mathbb{R}} \rho_n^{(1)}\,dx - \bar{q}| \\
&\leq |\int_{\mathbb{R}} \frac{1}{2}u_n^2 - \rho_n^{(1)} - \rho_n^{(2)}\,dx| + |\int_{R_n \leq |x-y_n| < 2R_n} \frac{1}{2}u_n^2\,dx| \\
&\quad + |\int_{\mathbb{R}} \rho_n^{(1)}\,dx - \bar{q}| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. The result for $u_n^{(2)}$, (3.17), is found similarly. □

Finally, we combine the results of this subsection to show that dichotomy does not occur.

LEMMA 3.15. *Dichotomy does not occur.*

PROOF. Let $u_n^{(1)}, u_n^{(2)}, \phi_n, \psi_n$ be as in Lemma 3.14 and assume that dichotomy occurs. Seeking to contradict Lemma 3.12, we wish to show that

$$\mathcal{E}(u_n) \geq \mathcal{E}(u_n^{(1)}) + \mathcal{E}(u_n^{(2)}) \tag{3.19}$$

as $n \to \infty$. Using that $\phi_n + \psi_n = 1$ and the symmetry of L and N, we have that

$$\begin{aligned}
\mathcal{E}(u_n) &= \mathcal{L}(\phi_n u_n + \psi_n u_n) - \mathcal{N}(\phi_n u_n + \psi_n u_n) \\
&= \mathcal{E}(\phi_n u_n) + \mathcal{E}(\psi_n u_n) + \int_{\mathbb{R}} \phi_n u_n L(\psi_n u_n)\,dx \\
&\quad - (4\mathcal{N}(\sqrt{\psi_n \phi_n} u_n) + 2\int_{\mathbb{R}} \phi_n^2 u_n^2 N(\psi_n^2 u_n^2)\,dx \\
&\quad + 4\int_{\mathbb{R}} (\phi_n^2 + \psi_n^2) u_n^2 N(\psi_n \phi_n u_n^2)\,dx)
\end{aligned}$$

Thus, if we can show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi_n u_n L(\psi_n u_n)\, dx \geq 0, \tag{3.20}$$

$$\lim_{n\to\infty} |\mathcal{N}(\sqrt{\psi_n \phi_n} u_n)| = 0, \tag{3.21}$$

$$\lim_{n\to\infty} |\int_{\mathbb{R}} \phi_n^2 u_n^2 N(\psi_n^2 u_n^2)\, dx| = 0, \tag{3.22}$$

$$\lim_{n\to\infty} |\int_{\mathbb{R}} (\phi_n^2 + \psi_n^2) u_n^2 N(\psi_n \phi_n u_n^2)\, dx| = 0,, \tag{3.23}$$

then we are done.

We show (3.20) using (3.18) and Lemma 3.13 repeatedly with ϕ_n, ψ_n as $\rho_R, 1-\rho_R$:

$$\begin{aligned}
\lim_{n\to\infty} (\int_{\mathbb{R}} \phi_n u_n L(\psi_n u_n)\, dx) &= \lim_{n\to\infty} (\int_{\mathbb{R}} \sqrt{\phi_n \psi_n} u_n L(\sqrt{\phi_n \psi_n} u_n)\, dx) \\
&= \lim_{n\to\infty} (\mathcal{L}(\sqrt{\phi_n \psi_n} u_n) + \mathcal{Q}(\sqrt{\phi_n \psi_n} u_n) - \mathcal{Q}(\sqrt{\phi_n \psi_n} u_n) \\
&\gtrsim \lim_{n\to\infty} (\|\sqrt{\phi_n \psi_n} u_n\|_{H^{s/2}}^2 - \int_{\mathbb{R}} \phi_n \psi_n u_n^2\, dx) \\
&\geq \lim_{n\to\infty} (\|\sqrt{\phi_n \psi_n} u_n\|_{H^{s/2}}^2 - \int_{R_n \leq |x-y_n| \leq 2R_n} u_n^2\, dx) \\
&\geq 0.
\end{aligned}$$

By Lemma 3.13 with ϕ_n^2, ψ_n^2 as $\rho_R, 1-\rho_R$,

$$\begin{aligned}
\lim_{n\to\infty} |\int_{\mathbb{R}} \phi_n^2 u_n^2 N(\psi_n^2 u_n^2)\, dx| &= \lim_{n\to\infty} (|\int_{\mathbb{R}} \phi_n \psi_n u_n^2 N \phi_n \psi_n u_n^2)\, dx|) \\
&= \lim_{n\to\infty} (\mathcal{N}(\sqrt{\phi_n \psi_n} u_n)),
\end{aligned}$$

and, using Lemma 3.5(recall that $\gamma < 1$) and (3.18),

$$\begin{aligned}
\lim_{n\to\infty} (\mathcal{N}(\sqrt{\phi_n \psi_n} u_n)) &\eqsim \lim_{n\to\infty} (\|\phi_n \psi_n u_n^2\|_{H^{r/2}}^2) \\
&\leq \lim_{n\to\infty} (\|\sqrt{\phi_n \psi_n} u_n\|_{H^{s/2}}^{2\gamma} (\int_{R_n \leq |x-y_n| \leq 2R_n} u_n^2\, dx)^{2-\gamma}) \\
&= 0,
\end{aligned}$$

for some $\gamma < 1$. This shows (3.21) and (3.22).

For (3.23) observe that

$$\int_{\mathbb{R}} (\phi_n^2 + \psi_n^2) u_n^2 N(\psi_n \phi_n u_n^2)\, dx = \|\phi_n^{3/2} \psi_n^{1/2} u_n^2\|_{H^{r/2}}^2 + \|\phi_n^{1/2} \psi_n^{3/2} u_n^2\|_{H^{r/2}}^2 \to 0.$$

again using Lemma 3.5 and (3.18). Having established (3.19), clearly

$$I_q = \liminf_{n\to\infty}(\mathcal{E}(u_n)) \geq \liminf_{n\to\infty}(\mathcal{E}(u_n^{(1)})) \liminf_{n\to\infty}(\mathcal{E}(u_n^{(2)})) \geq I_{\bar{q}} + I_{q-\bar{q}}.$$

As I_q is sub-additive by Lemma 3.12, this cannot be the case and so we conclude that dichotomy does not occur. □

3.4.3. Existence of solutions.

LEMMA 3.16 (Existence of minimizer). *Let $\{u_n\}_n$ be a minimizing sequence of I_q, $q > 0$. Then there is a subsequence of $\{u_n(\cdot + y_n)\}_{n\in\mathbb{N}}$ that converges in $H^{s/2}(\mathbb{R})$ to a minimizer of I_q.*

PROOF. We have excluded vanishing and dichotomy, and so only the concentration–compactness alternative remains. Define $\tilde{u}_n(x) = u_n(x + y_n)$. Note that $\|\tilde{u}_n\|_{H^{s/2}} = \|u_n\|_{H^{s/2}} < 1/\delta$ for some $\delta > 0$ by Lemma 3.8. Since $\{\tilde{u}\}_{n\in\mathbb{N}}$ is bounded in $H^{s/2}(\mathbb{R})$, it admits a subsequence that converges weakly in $H^{s/2}(\mathbb{R})$ to $w \in H^{s/2}(\mathbb{R})$.

Since the sequence concentrates, then for each $\varepsilon > 0$, there is an $R > 0$ such that

$$\int_{\mathbb{R}} \tilde{u}_n^2 \, dx < \varepsilon \text{ as } n \to \infty.$$

If additionally $\tilde{u}_n$ were uniformly continuous with respect to translation in $L^2(\mathbb{R})$, then Kolmogorov-Riez-Sudakov's compactness theorem 2.10 would imply the existence of a subsequence that converges to a limit in $L^2(\mathbb{R})$ which must be $w \in H^{s/2}(\mathbb{R})$ by uniqueness of limits.

This is indeed the case, as

$$\begin{aligned}\int_{\mathbb{R}} |\tilde{u}_n(x+y) - \tilde{u}_n(x)|^2 \, dx &= \int_{\mathbb{R}} |(e^{-iy\xi} - 1)\hat{\tilde{u}}_n(\xi)|^2 \, d\xi \\ &= \int_{\mathbb{R}} |(e^{-iy\xi} - 1)\langle\xi\rangle^{-s/2})|^2 |\langle\xi\rangle^{s/2}\hat{\tilde{u}}_n(\xi)|^2 \, d\xi \\ &\leq \sup_{\xi\in\mathbb{R}} |(e^{-iy\xi} - 1)\langle\xi\rangle^{-s/2})|^2 \|u_n\|_{H^{s/2}} \\ &\xrightarrow[y\to 0]{} 0.\end{aligned}$$

We now demonstrate that

$$\mathcal{E}(w) = I_q.$$

Fatou's lemma implies that

$$\mathcal{L}(w) \leq \liminf \mathcal{L}(\tilde{u}_n).$$

Furthermore $\mathcal{N}(\tilde{u}_n) \to \mathcal{N}(w)$. As $n(\xi) \eqsim \langle \xi \rangle^r$, it suffices to show that $\tilde{u}_n^2 \to w^2$ in $H^{r/2}(\mathbb{R})$. Recall from Lemma 3.8 that $\|\tilde{u}_n\|_{H^{s/2}} < 1/\delta$. If $r \geq 0$, then $s > 1$ and

$$\begin{aligned}
\|w^2 - \tilde{u}_n^2\|_{H^{r/2}} &= \|(w - \tilde{u}_n)(w + \tilde{u}_n)\|_{H^{r/2}} \\
&\leq \|(w - \tilde{u}_n)(w + \tilde{u}_n\|_{L^2}^{1-r/s} \|(w - \tilde{u}_n)(w + \tilde{u}_n)\|_{H^{s/2}}^{r/s} \\
&\leq \|w - \tilde{u}_n\|_{L^4}^{1-r/s} \|w + \tilde{u}_n\|_{L^4}^{1-r/s} \|w - \tilde{u}_n\|_{H^{s/2}}^{r/s} \|w + \tilde{u}_n\|_{H^{s/2}}^{r/s} \\
&\leq \|w - \tilde{u}_n\|_{L^4}^{(1-r/s)} (2/\delta)^{(1+r/s)}
\end{aligned}$$

while if $r < 0$, then

$$\begin{aligned}
\|w^2 - \tilde{u}_n^2\|_{H^{r/2}} &\leq \|(w - \tilde{u}_n)(w + \tilde{u}_n)\|_{L^2} \\
&\leq \|w - \tilde{u}_n\|_{L^4} \|w + \tilde{u}_n\|_{L^4} \\
&\lesssim \|w - \tilde{u}_n\|_{L^4} (2/\delta)
\end{aligned}$$

In both cases, the expression goes to 0 as $n \to \infty$. To see this, let $1/2 < \tilde{s} < s$ and use the Sobolev embedding theorem and interpolation:

$$\begin{aligned}
\|w - \tilde{u}_n\|_{L^4} &\lesssim \|w - \tilde{u}_n\|_{H^{\tau/2}} \\
&\leq \|w - \tilde{u}_n\|_{L^2}^{1-\tilde{s}/s} \|w - \tilde{u}_n\|_{H^{s/2}}^{\tilde{s}/2} \\
&\leq \|w - \tilde{u}_n\|_{L^2}^{1-\tilde{s}/s} (1/\delta)^{\tilde{s}/2} \to 0
\end{aligned}$$

since $1 - \tilde{s}/s > 0$.

We conclude that $\mathcal{E}(w) = I_q = \lim_{n\to\infty} \mathcal{E}(\tilde{u}_n)$. Since $\mathcal{N}(\tilde{u}_n) \to \mathcal{N}(w)$, then we must have that $\mathcal{L}(\tilde{u}_n) \to \mathcal{L}(w)$, which together with the already established weak convergence $\tilde{u}_n \rightharpoonup w$ implies norm convergence in $H^{s/2}(\mathbb{R})$. □

LEMMA 3.17. *Any minimizer of the constrained variational problem I_q solves* (3.2)*, where the wave-speed c is the Lagrange multiplier.*

PROOF. According to the Lagrange multiplier principle 2.8, any minimizer of I_q satisfies

$$\mathcal{E}'(u) - c\mathcal{Q}'(u) = 0. \tag{3.24}$$

We find that the Fréchet derivatives of $\mathcal{L}, \mathcal{N}, \mathcal{Q}$ are Lu, uNu^2 and u respectively:

$$\begin{aligned}
\lim_{\|v\|_{H^{s/2}}\to 0} & \frac{|\mathcal{L}(u+v) - \mathcal{L}(u) - \int_{\mathbb{R}} vLu\,dx|}{\|v\|_{H^{s/2}}} \\
&= \lim_{\|v\|_{H^{s/2}}\to 0} \frac{|\int_{\mathbb{R}} \frac{1}{2}(uLu + uLv + vLu + vLv - \frac{1}{2}uLu - vLu\,dx|}{\|v\|_{H^{s/2}}} \\
&= \lim_{\|v\|_{H^{s/2}}\to 0} \frac{|\frac{1}{2}\int_{\mathbb{R}} vLv\,dx|}{\|v\|_{H^{s/2}}} \\
&\lesssim \lim_{\|v\|_{H^{s/2}}\to 0} \frac{\|v\|^2_{H^{s/2}}}{\|v\|_{H^{s/2}}} \\
&= 0,
\end{aligned}$$

while

$$\begin{aligned}
\lim_{\|v\|_{H^{s/2}}\to 0} & \frac{|\mathcal{Q}(u+v) - \mathcal{Q}(u) - \int_{\mathbb{R}} vu\,dx|}{\|v\|_{H^{s/2}}} \\
&= \lim_{\|v\|_{H^{s/2}}\to 0} \frac{|\int_{\mathbb{R}} \frac{1}{2}(u^2 + 2uv + v^2 - \frac{1}{2}u^2 - uv\,dx|}{\|v\|_{H^{s/2}}} \\
&= \lim_{\|v\|_{H^{s/2}}\to 0} \frac{|\frac{1}{2}\int_{\mathbb{R}} v^2\,dx|}{\|v\|_{H^{s/2}}} \\
&\lesssim \lim_{\|v\|_{H^{s/2}}\to 0} \frac{\|v\|^2_{H^{s/2}}}{\|v\|_{H^{s/2}}} \\
&= 0.
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\|v\|_{H^{s/2}}\to 0} \frac{|\mathcal{N}(u+v) - \mathcal{N}(u) - \int_{\mathbb{R}} vuNu^2\,dx|}{\|v\|_{H^{s/2}}} \\
&= \lim_{\|v\|_{H^{s/2}}\to 0} \frac{|\frac{1}{4}\int_{\mathbb{R}} u^2Nv^2 + 4uvN(uv) + 2uvNv^2 + v^2Nu^2 + 2v^2N(uv) + v^2Nu^2\,dx|}{\|v\|_{H^{s/2}}} \\
&\lesssim \lim_{\|v\|_{H^{s/2}}\to 0} \frac{\|v\|^2_{H^{s/2}}\|u\|^2_{H^{s/2}} + \|v\|^3_{H^{s/2}}\|u\|_{H^{s/2}}}{\|v\|_{H^{s/2}}} \\
&= 0.
\end{aligned}$$

Inserting the Fréchet derivatives into (3.24) gives the solitary-wave equation (3.2). □

3.5. Properties of solutions

We have established the first part of 3.1, namely that solutions $u \in H^{s/2}(\mathbb{R})$ to (3.2) satisfying $\frac{1}{2}\|u\|_{L^2}^2 = q$ exist for all $q > 0$. It remains to show the properties of the solutions u and wave speed c.

3.5.1. Improved estimates for $I^q, \|u\|_{H^{s/2}}, \mathcal{N}(u)$. We will refer to big and small solutions as solutions $u \in H^{s/2}(\mathbb{R})$ of (3.2) whose L^2-norms are respectively larger or smaller than a constant $q_0 > 0$ which will be introduced shortly.

If we consider small and big solutions separately, we can find improved upper bounds for I_q in both cases.

LEMMA 3.18. *There exists a $\tilde{q}_0 > 0$ such that for all $q \geq \tilde{q}_0$,*

$$I_q < 0.$$

PROOF. Pick a function $\phi \in H^{s/2}(\mathbb{R})$ satisfying $\mathcal{Q}(\phi) = 1$ and set

$$\phi_q(x) = \sqrt{q}\phi(x),$$

so that $\mathcal{Q}(\phi_q) = q$.

Since $\mathcal{L}(u) \lesssim \|u\|_{H^{s/2}}^2$ and $\mathcal{N}(u) \gtrsim \|u^2\|_{H^{r/2}}^2$ for $u \in H^{s/2}(\mathbb{R})$, then there are constants $C_1, C_2 > 0$ such that

$$\begin{aligned} \mathcal{E}(\phi_q) &\leq C_1\|\phi_q\|_{H^{s/2}}^2 - C_2\|\phi_q^2\|_{H^{r/2}}^2 \\ &= C_1 q\|\phi\|_{H^{s/2}}^2 - C_2 q^2\|\phi^2\|_{H^{r/2}}^2, \end{aligned}$$

which is negative for $q \geq \tilde{q}_0$ if $\tilde{q}_0$ is large enough. Thus there is a function $v = \phi_q \in H^{s/2}(\mathbb{R})$ such that

$$I_q \leq \mathcal{E}(v) < 0,$$

which is what we wanted to prove. □

The next two results are inspired by [**21**], where similar results are shown for equations where the nonlinearity is only local.

LEMMA 3.19. *Let $q_0 > 0$ be any positive constant. There exists $\kappa > 0$ such that for all $q \in (0, q_0)$,*

$$I_q < m(0)q - \kappa q^{1+\alpha},$$

where $\alpha = \frac{s'}{s'-1}$.

REMARK 3.20. If $m(\xi) = \langle\xi\rangle^s$, then the result holds with $\alpha = 2$.

PROOF. It suffices to show that there exists a function $u \in H^{s/2}(\mathbb{R}), \mathcal{Q}(u) = q$ such that $\mathcal{E}(u) < m(0)q - \kappa q^{1+\alpha}$. To that end, pick a function $\phi \in \mathcal{S}(\mathbb{R})$ satisfying $\mathcal{Q}(\phi) = 1$ and let $0 < t < 1$. Define

$$\phi_{q,t}(x) = \sqrt{qt}\phi(tx)$$

and observe that $\mathcal{Q}(\phi_{q,t}) = q$.

By a similar calculation as in Lemma 3.7,

$$\mathcal{N}(\phi_{q,t}) \eqsim \frac{q^2}{4}\int_{\mathbb{R}} t\langle t\xi\rangle^r |\widehat{\phi^2}(\xi)|^2\, d\xi$$

which implies

$$\mathcal{N}(u) \geq C_1 q^2 t$$

for some constant $C_1 > 0$.

Furthermore, again by a similar calculation as in Lemma 3.7,

$$\begin{aligned}\mathcal{L}(\phi_{q,t}) &= m(0)q + \frac{q}{2}\int_{\mathbb{R}}(m(t\xi) - m(0))|\hat{\phi}(\xi)|^2\, d\xi \\ &\leq m(0)q + C_2 t^{s'} q,\end{aligned}$$

for some $C_2 > 0$.

Combining this, we get that

$$\begin{aligned}\mathcal{E}(\phi_{q,t}) &= \mathcal{L}(\phi_{q,t}) - \mathcal{N}(\phi_{q,t}) \\ &\leq m(0)q + C_4 t^{s'} q - C_2 t q^2\end{aligned}$$

Set $t^s = C_5 q^\alpha$, where $C_5 > 0$ is small enough to have $t < 1$ for all $q < q_0$. Then

$$\begin{aligned}\mathcal{E}(\phi_{q,t}) &\leq m(0)q + q^{1+\alpha}(C_4 C_5 - C_2 C_5^{1/s'} q^{1+\alpha/s'-\alpha}) \\ &\leq m(0)q - q^{1+\alpha}\underbrace{(C_2 C_5^{1/s'} - C_4 C_5)}_{\kappa}.\end{aligned}$$

Since $1/s' < 1$, we can pick C_5 small enough to guarantee that $\kappa > 0$ which gives the desired result. □

LEMMA 3.21. *Let $\alpha = s'/(s'-1)$. For all $q < q_0$, near minimizers u of I_q satisfy*

$$\mathcal{L}(u) - m(0)q \eqsim \mathcal{N}(u) \eqsim q^{1+\alpha} \tag{3.25}$$

and

$$\|u\|^2_{H^{s/2}} \eqsim q. \tag{3.26}$$

PROOF. Let $u \in H^{s/2}(\mathbb{R})$. From Lemma 3.19, we get that

$$\mathcal{L}(u) - \mathcal{N}(u) \leq m(0)q - \kappa q^{1+\alpha}.$$

It follows directly that

$$\mathcal{N}(u) \gtrsim q^{1+\alpha}$$

for all values of r.

We now find a crude upper bound $\mathcal{N}(u)$ that will be improved upon later. It follows from Lemma 3.5, where we recall that $\gamma \in (0,1)$, that

$$\begin{aligned}\mathcal{N}(u) &\eqsim \|u^2\|_{H^{r/2}}^2 \\ &\lesssim q^{2-\gamma}\|u\|_{H^{s/2}}^{2\gamma} \\ &\eqsim q^{2-\gamma}(\mathcal{L}(u) - m(0)q + \mathcal{Q}(u) + m(0)q)^{\gamma} \\ &\lesssim q^2 + q^{2-\gamma}\mathcal{N}(u)^{\gamma}.\end{aligned}$$

This implies that

$$\mathcal{N}(u) \lesssim q^2 + q^{\frac{2-\gamma}{1-\gamma}} \lesssim q^2 + q^{1+\frac{1}{1\gamma}} \lesssim q^2.$$

With this in hand, we establish the estimate (3.26):

$$q \lesssim \|u\|_{H^{s/2}}^2 \eqsim \mathcal{L}(u) + \mathcal{Q}(u) \lesssim \mathcal{Q}(u) + \mathcal{N}(u) + m(0)q - \kappa q^{1+\alpha} \lesssim q.$$

To show (3.25) we partition $u = u_1 + u_2$ where $\widehat{u}_1 = \chi_{[-1,1]}\hat{u}$ and $\widehat{u}_2 = (1 - \chi_{[-1,1]})\hat{u}$. Here χ_A denotes the function with value 1 at each point in the set A and 0 otherwise. Using Proposition 2.6, we find that

$$\begin{aligned}\|u_1\|_{L^4}^4 &\lesssim \|u_1\|_{L^2}^{4-2/s'}\|u_1\|_{\dot{H}^{s'/2}}^{2/s'} \\ &\lesssim q^{2-1/s'}(\mathcal{L}(u) - m(0)q)^{1/s'}.\end{aligned} \tag{3.27}$$

If $r > 0$, we have additionally that

$$\begin{aligned}\|u_1\|_{\dot{H}_4^{r/2}}^4 &\lesssim \|u_1\|_{L^2}^{4-\frac{4r+2}{s'}}\|u_1\|_{\dot{H}^{s'/2}}^{\frac{4r+2}{s'}} \\ &\lesssim q^{2-1/s'}(\mathcal{L}(u) - m(0)q)^{1/s'}.\end{aligned} \tag{3.28}$$

In the last line we used assumption (A1), that $\frac{2r+1}{s'} > \frac{1}{s'}$ and $\mathcal{N}(u) \lesssim q^2 \lesssim q$.

By Sobolev embedding on u_2,

$$\begin{aligned}\|u_2\|_{L^4}^4 &\lesssim \|u_2\|_{H^{s/2}}^4 \\ &\lesssim (\mathcal{L}(u) - m(0)q)^2 \\ &\lesssim (\mathcal{L}(u) - m(0)q)^{1/s'}q^{(2-1/s')},\end{aligned} \tag{3.29}$$

where in the last line we used that $\mathcal{L}(u) - m(0)q \lesssim \mathcal{N}(u) \lesssim q^2 \lesssim q$ for $q < q_0$.

If $r \leq 0$, we combine (3.27) and (3.29) and find that

$$\begin{aligned}\mathcal{N}(u) &\leq \|u\|_{L^4}^4 \\ &\lesssim \|u_1\|_{L^4}^4 + \|u_2\|_{L^4}^4 \\ &\lesssim q^{2-1/s'}(\mathcal{L}(u) - m(0)q)^{1/s'} \\ &= (q^{1+\alpha})^{1-1/s'}(\mathcal{L}(u) - m(0)q)^{1/s'}.\end{aligned} \tag{3.30}$$

If $r > 0$, we use the estimate (3.5) and obtain

$$
\begin{aligned}
\mathcal{N}(u) &\eqsim \int_{\mathbb{R}} \langle \xi \rangle^r |\widehat{u^2}|^2 \, d\xi \\
&\lesssim \int_{\mathbb{R}} |\langle \eta \rangle^{r/2} \langle \xi - \eta \rangle^{r/2} (\hat{u} * \hat{u})|^2 \, d\xi \\
&= \|(\Lambda^{r/2} u)^2\|_{L^2}^2 \\
&= \|u\|_{H_4^{r/2}}^4 \\
&\leq \|u_1\|_{L^4}^4 + \|u_1\|_{\dot{H}_4^{r/2}}^4 + \|u_2\|_{H_4^{r/2}}^4 .
\end{aligned}
\tag{3.31}
$$

Now, observe that

$$
\begin{aligned}
\|u_2\|_{H_4^{r/2}}^4 &= \|\langle \cdot \rangle^{r/2} \widehat{u_2} * \langle \cdot \rangle^{r/2} \widehat{u_2}\|_{L^2}^2 \\
&\leq \|\langle \cdot \rangle^{r/2} \widehat{u_2}\|_{L^2}^2 \|\langle \cdot \rangle^{r/2} \widehat{u_2}\|_{L^1}^2 \\
&\lesssim \|u_2\|_{H^{r/2}}^2 \|u_2\|_{H^{s/2}}^2 \\
&\leq \|u_2\|_{H^{s/2}}^4 \\
&\lesssim q^{2-1/s'} \|u_2\|_{\dot{H}^{s/2}}^{1/s'} \\
&\lesssim q^{2-1/s'} (\mathcal{L}(u) - m(0)q)^{1/s'} .
\end{aligned}
\tag{3.32}
$$

In the second to last line we used that $\|u_2\|_{H^{s/2}} \lesssim \|u\|_{H^{s/2}} \lesssim q$ and $(-1,1) \nsubseteq \operatorname{supp} \widehat{u_2}$.

Inserting (3.27), (3.28) and (3.32) into (3.31), we get that

$$
\mathcal{N}(u) \lesssim q^{2-1/s'} (\mathcal{L}(u) - m(0)q)^{1/s'} . \tag{3.33}
$$

Using (3.30) for $r \leq 0$ and (3.33) for $r > 0$, and combining this with the fact that $\mathcal{L}(u) - m(0)q \lesssim \mathcal{N}(u)$ and $\mathcal{N}(u) \gtrsim q^{1+\alpha}$, it now follows that

$$
\mathcal{L}(u) - m(0)q \eqsim \mathcal{N}(u) \eqsim q^{1+\alpha} .
$$

□

3.5.2. The wave speed c. We estimate the size of the wave speed c.

LEMMA 3.22. *Let q_0 be any constant larger than $\tilde{q}_0$ from Lemma 3.18. Any minimizer of I_q solves* (3.2) *with subcritical wave speed, that is $c < m(0)$. Furthermore,*

(i) if $q \geq q_0$, then

$$
c < -m(0).
$$

(ii) If $q < q_0$, then

$$
m(0) - c \eqsim q^{\alpha},
$$

where $\alpha = s'/(s'-1)$.

REMARK 3.23. The result implies that $c < m(0)$ for all $q > 0$ and that $c \to m(0)$ if and only if $q \to 0$.

PROOF. Minimizers u of I^q must satisfy

$$\mathcal{E}'(u) - c\mathcal{Q}'(u) = 0$$

in $H^{-s/2}(\mathbb{R})$. Pairing with u we obtain

$$\begin{aligned} c &= \frac{\langle \mathcal{E}'(u), u\rangle}{\langle \mathcal{Q}'(u), u\rangle} \\ &= \frac{\mathcal{E}(u) - \mathcal{N}(u)}{q} \\ &< m(0). \end{aligned} \tag{3.34}$$

(i) By Lemma 3.18, big solutions satisfy

$$\mathcal{L}(u) - \mathcal{N}(u) < 0,$$

so that

$$\mathcal{N}(u) > \mathcal{L}(u) \geq m(0)q.$$

Combined with (3.34), then

$$c < -\frac{\mathcal{N}(u)}{q} < -m(0).$$

(ii) By Lemma 3.19, small solutions satisfy

$$\mathcal{L}(u) - \mathcal{N}(u) < m(0)q - \kappa q^{1+\alpha}.$$

Combined with (3.34), then

$$c \leq m(0) - \kappa q^{\alpha}.$$

For the lower bound, observe that by Lemma 3.21 there is a constant $C_1 > 0$ such that,

$$\begin{aligned} c &= \frac{\mathcal{L}(u) - 2\mathcal{N}(u)}{q} \\ &\geq m(0) - C_1 q^{\alpha}. \end{aligned}$$

We conclude that

$$m(0) - c \eqsim q^{\alpha}$$

for small solutions. $\square$

3.5.3. Regularity of solutions. We conclude the chapter by showing the regularity estimates stated in Theorem 3.1. Combined with the existence proof in Section 3.4.3 and the estimates on the wave speed in Lemma 3.22, this completes the proof of Theorem 3.1.

LEMMA 3.24 (Regularity of solutions). *Any solution* $u \in H^{s/2}(\mathbb{R})$ *of the solitary-wave equation 3.2 is also in* $H^\infty(\mathbb{R})$. *Furthermore, if* $q \in (0, q_0)$, *then*

$$\|u\|_{L^\infty} \eqsim \|u\|_{H^{s/2}} \eqsim q.$$

PROOF. Rewriting (3.2), we have

$$(L - c)u = uNu^2.$$

Observe that $L - c$ is invertible since $c < m(0)$, so that $(L - c)^{-1} \colon H^t(\mathbb{R}) \to H^{t+s}(\mathbb{R})$ is well defined and continuous.

We first assume that $s > 1$. We want to show that

$$uNu^2 \in H^{1-s/2}(\mathbb{R}). \tag{3.35}$$

Then, we would have that $(L - c)u \in H^{1-s/2}(\mathbb{R})$ and consequently $u \in H^{1+s/2}(\mathbb{R})$. We could repeat the procedure with $\tilde{s} = 2 + s$ to find that $u \in H^{\tilde{s}/2+1}(\mathbb{R}) = H^{s/2+2}$. Continuing like this indefinitely, we would arrive at $u \in H^\infty(\mathbb{R})$.

Since $s > 1$, then $Nu^2 \in H^{s/2-r}$ by Proposition 2.3(i) with $t_1 = t_2 = s/2 > 1/2$. If $r \leq 0$, Proposition 2.3(i) also gives

$$\|uNu^2\|_{H^{s/2}} \lesssim \|u\|_{H^{s/2}} \|Nu^2\|_{H^{s/2-r}},$$

while if $r > 0$,

$$\|uNu^2\|_{H^{s/2-r}} \lesssim \|u\|_{H^{s/2}} \|Nu^2\|_{H^{s/2-r}}.$$

In both cases, this implies that (3.35) since

$$s/2 - r = s - r - s/2 > 1 - s/2 \quad \text{and} \quad s/2 = s - s/2 > 1 - s/2,$$

which completes the proof that $u \in H^\infty(\mathbb{R})$ if $s > 1$.

If $s < 1$, we can use 2.3(ii) with $t_1 = t_2 = s/2$ and find that $u^2 \in H^{s-1/2}(\mathbb{R})$, which implies that $Nu^2 \in H^{s-r-1/2}(\mathbb{R})$. Observing that $s - r - 1/2 > 1/2$, we then use 2.3(i)with $t_1 = s/2, t_2 = s - r - 1/2$ to see that

$$\|uNu^2\|_{s/2} \lesssim \|u\|_{s/2} \|Nu^2\|_{s-r-1/2}.$$

Thus $uNu^2 \in H^{s/2}(\mathbb{R})$. Consequently, $(L - c)u \in H^{s/2}(\mathbb{R})$ as well and hence $u \in H^{3s/2}(\mathbb{R})$. We can now proceed with $\tilde{s} = 3s > 1$ to arrive at $u \in H^\infty(\mathbb{R})$ also when $1/2 < s < 1$.

For the last part, suppose that $q \in (0, q_0)$. Then we know from Lemma 3.21 that

$$\|u\|_{H^{s/2}} \eqsim q^{1/2}.$$

Now $\|u\|_{L^\infty} \lesssim q^{1/2}$ by a similar argument as above:

$$\|u\|_{L^\infty} \lesssim \|u\|_{H^1} \lesssim \|u\|_{H^{s/2}} \eqsim q.$$

Moreover,

$$q = \|u^2\|_{L^1} \le \|u\|_{L^\infty} \|u\|_{L^1} \lesssim \|u\|_{L^\infty} \|u\|_{H^1} \lesssim \|u\|_{L^\infty} q^{1/2},$$

so that

$$\|u\|_{L^\infty} \gtrsim q^{1/2}.$$

Hence

$$\|u\|_{L^\infty} \eqsim q^{1/2}$$

and the proof is completed. □

CHAPTER 4

Solitary waves in equations with a nonlocal quadratic term

In the previous chapter, the nonlinearity was cubic. It is perhaps more physically relevant to study the case when the nonlinearity is quadratic. Our goal in this chapter is to prove existence of solitary waves in such an equation using similar techniques as in the cubic case.

As a first inquiry, one could replace the term uNu^2 in equation (3.2) with uNu, where N is still a linear Fourier multiplier. The next question is then what functional we would have to minimize to obtain this new equation

$$-cu + Lu - uNu = 0.$$

It is not obvious how one could construct such a functional and indeed, we will see that no functional with Fréchet derivative uNu, where N is a linear Fourier multiplier, can exist.

Instead, we will study the equation

$$\partial_t u + \partial_x (Lu - T(u, u)) = 0, \tag{4.1}$$

where L is still a linear Fourier multiplier, but T is now a *bilinear Fourier multiplier* with symbol p, meaning that

$$\widehat{T(u,u)} = \int_{\mathbb{R}} p(\xi - \eta, \eta)\hat{u}(\xi - \eta)\hat{u}(\eta)\, d\eta.$$

We wish to find out what conditions the symbol p has to satisfy in order to have a functional $\mathcal{T}$ with derivative

$$\mathcal{T}'(u) = T(u, u).$$

Operators of the form $u \mapsto uNu$ where N is a linear Fourier multiplier are special cases of bilinear Fourier multipliers where the symbol p only depends on one variable:

$$\widehat{T(u,v)}(\xi) = \int_{\mathbb{R}} p(\eta)\hat{u}(\xi - \eta)\hat{v}(\eta)\, d\eta = (p\hat{v} * \hat{u})(\xi) = \widehat{uNv}(\xi).$$

However, the condition on p we derive will exclude operators of this type.

We derive sufficient and necessary conditions on p in Section 4.1. The remainder of the chapter is devoted to proving existence and properties of solitary-wave solutions to (4.1).

4.1. Integrable bilinear Fourier multipliers

Let B_n be a bilinear Fourier multiplier with symbol n of order s, meaning

$$\widehat{B_n(u,v)}(\xi) = \int_{\mathbb{R}} n(\xi-\eta,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)\,d\eta$$

and

$$n(\xi-\eta,\eta) \lesssim \langle |\xi| + |\eta| \rangle^s.$$

Furthermore, let T_p be a trilinear Fourier multiplier with symbol p of order t, meaning

$$\widehat{T_p(u,v,w)} = \int p(\xi-\eta,\eta-\sigma,\sigma)\hat{u}(\xi-\eta)\hat{u}(\eta-\sigma)\hat{u}(\sigma)\,d\sigma\,d\eta$$

and

$$p(\xi-\eta,\eta-\sigma,\sigma) \lesssim \langle |\xi| + |\eta| + |\sigma| \rangle^t.$$

We begin by showing some simple properties of trilinear Fourier multipliers.

LEMMA 4.1. *Let $f,g,h \in H^t(\mathbb{R})$. Then*

$$T_p(f,g,h) = T_{p'}(f,g,h) = T_{p''}(f,g,h),$$

where

$$p'(\xi_1,\xi_2,\xi_3) = p(\xi_3,\xi_1,\xi_2),$$
$$p''(\xi_1,\xi_2,\xi_3) = p(\xi_2,\xi_3,\xi_1).$$

PROOF. To show the first equality, we use the substitution $\sigma \mapsto \eta - \sigma$ and $\eta \mapsto \xi - \sigma$ combined with Plancherel and Fubini's theorems:

$$\begin{aligned}
\widehat{T_p(f,g,h)}(\xi) &= \int_{\mathbb{R}^2} p(\xi-\eta,\eta-\sigma,\sigma)\hat{f}(\xi-\eta)\hat{g}(\eta-\sigma)\hat{h}(\sigma)\,d\sigma\,d\eta \\
&= \iint_{\mathbb{R}^2} p(\sigma,\xi-\eta,\eta-\sigma)\hat{f}(\sigma)\hat{g}(\xi-\eta)\hat{h}(\eta-\sigma)\,d\sigma\,d\eta \\
&= \int_{\mathbb{R}^2} p'(\xi-\eta,\eta-\sigma,\sigma)\hat{g}(\xi-\eta)\hat{h}(\eta-\sigma)\hat{f}(\sigma)\,d\sigma\,d\eta \\
&= \widehat{T_{p'}(g,h,f)}(\xi)
\end{aligned}$$

In the second to last line we used the definition of p'.

To show the second equality, let $p' = q$ and use the first equality:

$$\widehat{T_{p''}(h,f,g)} = \widehat{T'_q(h,f,g)}(\xi) = \widehat{T_q(g,h,f)}(\xi) = \widehat{T_{p'}(g,h,f)}(\xi),$$

where

$$p''(\xi_1,\xi_2,\xi_3) = q'(\xi_1,\xi_2,\xi_3) = q(\xi_3,\xi_1,\xi_2) = p'(\xi_3,\xi_1,\xi_2) = p(\xi_2,\xi_3,\xi_1).$$

□

Proposition 4.2. *Let B_n be a bilinear Fourier multiplier with symbol n of order s and let T_p be a trilinear Fourier multiplier with symbol p of order $t > 0$. Assume $\frac{t+1}{2} < s$. Define a functional $\mathcal{T}(u)$ by*

$$\mathcal{T}(u) = \int T_p(u, u, u)\, dx$$

Then then for all $u, v \in H_s(\mathbb{R})$, the Fréchet derivative

$$D\mathcal{T}[u](v) = \int vB_n(u, u)\, dx$$

if and only if

$$n(\xi - \eta, \eta) = p(\eta, \xi - \eta, -\xi) + p(-\xi, \eta, \xi - \eta) + p(\xi - \eta, -\xi, \eta).$$

Proof. Since the Fréchet derivative is unique,

$$D\mathcal{T}[u](v) = \int vB_n(u, u)\, dx$$

if and only if

$$\lim_{\|v\|_{H^s}\to 0} \frac{|\int 1 \cdot \mathcal{F}^{-1}(T_p(u + \widehat{v, u +} v, u + v) - T_p\widehat{(u, u,} u))\, dx - \int vB_n(u, u)\, dx|}{\|v\|_{H^s}} = 0. \tag{4.2}$$

We apply Lemma 4.1 and find that

$$\begin{aligned} T_p(u + v, u + v, u + v) - T_p(u, u, u) &\\ = T_p(u, u, v) + T_p(u, v, u) + T(u, v, v)&\\ + T_p(v, u, u) + T_p(v, u, v) + T_p(v, v, u) + T_p(v, v, v)&\\ = T_{p''}(v, u, u) + T_{p'}(v, u, u) + T_{p'}(v, v, u)&\\ + T_p(v, u, u) + T_{p''}(v, v, u) + T_p(v, v, u) + T_p(v, v, v).& \end{aligned}$$

In the next equations, p, p', p'' are functions of $(\xi-\eta, \eta-\sigma, \sigma)$ in every line. We omit the arguments for shorter notation. Inserting the above into (4.2), we get

$$\begin{aligned}
&\lim_{\|v\|_{H^s}\to 0} \frac{|\int_{\mathbb{R}} 1\cdot \mathcal{F}^{-1}(T_p(\widehat{u+v, u+v}, u+v) - T_p\widehat{(u,u,u)})\,dx - \int_{\mathbb{R}} vB_n(u,u)\,dx|}{\|v\|_{H^s}} \\
&\qquad \le \lim_{\|v\|_{H^s}\to 0} \frac{1}{\|v\|_{H^s}} \Big| \int_{\mathbb{R}} \delta(\xi) \int_{\mathbb{R}^2} (p''+p'+p)\hat{v}(\xi-\eta)\hat{u}(\eta-\sigma)\hat{u}(\sigma)\,d\sigma\,d\eta\,d\xi \\
&\qquad\qquad - \int_{\mathbb{R}^2} \hat{v}(-\xi) n(\xi-\eta,\eta)\hat{u}(\xi-\eta)\hat{u}(\eta)\,d\eta\,d\xi \Big| := A \\
&\qquad + \lim_{\|v\|_{H^s}\to 0} \frac{|\int_{\mathbb{R}} \delta(\xi)\int_{\mathbb{R}^2}(p''+p'+p)\hat{v}(\xi-\eta)\hat{v}(\eta-\sigma)\hat{u}(\sigma)\,d\sigma\,d\eta|}{\|v\|_{H^s}} := B \\
&\qquad + \lim_{\|v\|_{H^s}\to 0} \frac{|\int_{\mathbb{R}} \delta(\xi)\int_{\mathbb{R}^2} p\hat{v}(\xi-\eta)\hat{v}(\eta-\sigma)\hat{v}(\sigma)\,d\sigma\,d\eta|}{\|v\|_{H^s}} := C.
\end{aligned}$$

We wish to show that $A = B = C = 0$.

Since $T_p, T_{p'}, T_{p''}$ are all of order t and using that

$$\langle |\eta| + |\sigma| \rangle \lesssim \langle \eta \rangle + \langle \sigma \rangle \lesssim \langle -\eta \rangle^{1/2} \langle \eta - \sigma \rangle^{1/2} \langle \sigma \rangle^{1/2}$$

we get that

$$\begin{aligned}
B &= \lim_{\|v\|_{H^s}\to 0} \frac{|\int_{\mathbb{R}^2}(p''+p'+p)(-\eta,\eta-\sigma,\sigma)\hat{v}(-\eta)\hat{v}(\eta-\sigma)\hat{u}(\sigma)\,d\sigma\,d\eta|}{\|v\|_{H^s}} \\
&\lesssim \lim_{\|v\|_{H^s}\to 0} \frac{\int_{\mathbb{R}^2} \langle -\eta\rangle^{t/2}\langle \eta-\sigma\rangle^{t/2}\langle\sigma\rangle^{t/2} |\bar{\hat{v}}(\eta)||\hat{v}(\eta-\sigma)||\hat{u}(\sigma)|\,d\sigma\,d\eta}{\|v\|_{H^s}} \\
&\lesssim \lim_{\|v\|_{H^s}\to 0} \frac{\| |\langle\cdot\rangle^{t/2}\hat{v}| (|\langle\cdot\rangle^{t/2}\hat{u}| * |\langle\cdot\rangle^{t/2}\hat{u}|)\|_{L^1}}{\|v\|_{H^s}} \\
&\lesssim \lim_{\|v\|_{H^s}\to 0} \frac{\|v\|_{H^{t/2}}\|v\|_{H^{t/2}}\|\Lambda^{t/2}u\|_{L^1}}{\|v\|_{H^s}} \\
&= 0
\end{aligned}$$

since $\|\Lambda^{t/2}u\|_{L^1} \le \|u\|_{H^s}\|\langle\cdot\rangle^{t/2-s}\|_{L^2}$ is finite. That

$$C = 0$$

is shown in a similar manner.

Finally,

$$\begin{aligned}
A = \lim_{\|v\|_{H^s}\to 0} \frac{1}{\|v\|_{H^s}} \Big| &\int_{\mathbb{R}^2} (p''+p'+p)(-\eta,\eta-\sigma,\sigma)\hat{v}(-\eta)\hat{u}(\eta-\sigma)\hat{u}(\sigma)\,d\sigma\,d\eta \\
&- \int_{\mathbb{R}^2} \hat{v}(-\xi) n(\xi-\eta,\eta)\hat{u}(\xi-\eta)\hat{u}(\eta)\,d\eta\,d\xi \Big|
\end{aligned}$$

Renaming $\eta\mapsto\xi,\sigma\mapsto\eta$ in the first integral, we get

$$A=\lim_{\|v\|_{H^s}\to 0}\frac{|\int_{\mathbb{R}^2}((p''+p'+p)(-\xi,\xi-\eta,\eta)-n(\xi-\eta,\eta))\hat{v}(-\xi)\hat{u}(\xi-\eta)\hat{u}(\eta)d\eta d\xi|}{\|v\|_{H^s}}$$

This expression is only zero if

$$n(\xi-\eta,\eta)=(p''+p'+p)(-\xi,\xi-\eta,\eta)=p(\xi-\eta,\eta,-\xi)+p(\eta,-\xi,\xi-\eta)+p(-\xi,\xi-\eta,\eta),$$

which is what we wanted to show. □

If the operators satisfy the conditions of the last proposition, then the functional $\mathcal{T}$ is on a specific form.

Corollary 4.3. *Let $B_n, T_p, \mathcal{T}$ be as in Proposition 4.2. Then*

$$\mathcal{T}(u) = \int_{\mathbb{R}} T_p(u,u,u)\,dx = \frac{1}{3}\int_{\mathbb{R}} uB_n(u,u)\,dx.$$

Proof. Similarly to the proof of 4.2,

$$\begin{aligned}
\mathcal{T}(u) &= \int_{\mathbb{R}} T_p(u,u,u)\,dx \\
&= \int_{\mathbb{R}^2} p(-\eta,\eta-\xi,\xi)\hat{u}(-\eta)\hat{u}(\eta-\xi)\hat{u}(\xi)\,d\eta\,d\xi \\
&= \int_{\mathbb{R}^2} p(\eta,\xi-\eta,-\xi)\hat{u}(\eta),\hat{u}(\eta-\xi)\hat{u}(-\xi)\,d\eta\,d\xi \qquad (\xi\mapsto-\xi,\eta\mapsto-\eta) \\
&= \frac{1}{3}\int_{\mathbb{R}^2} p(\eta,\xi-\eta,-\xi)\hat{u}(\eta),\hat{u}(\xi-\eta)\hat{u}(-\xi)\,d\eta\,d\xi \\
&\quad+\frac{1}{3}\int_{\mathbb{R}^2} p(-\xi,\eta,\xi-\eta)\hat{u}(-\xi),\hat{u}(\xi-\eta)\hat{u}(\eta)\,d\eta\,d\xi \quad (\xi\mapsto\eta-\xi,\eta\mapsto-\xi) \\
&\quad+\frac{1}{3}\int_{\mathbb{R}^2} p(\xi-\eta,-\xi,\eta)\hat{u}(\xi-\eta),\hat{u}(-\xi)\hat{u}(\eta)\,d\eta\,d\xi \quad (\xi\mapsto-\eta,\eta\mapsto\xi-\eta) \\
&= \frac{1}{3}\int_{\mathbb{R}} \bar{\hat{u}}(\xi)n(\xi-\eta,\eta)\hat{u}(\xi-\eta)\hat{u}(\eta)\,d\eta\,d\xi \\
&= \frac{1}{3}\int_{\mathbb{R}} uB_n(u,u)\,dx.
\end{aligned}$$

□

4.2. Assumptions and main theorem

As in the previous chapter, we search for solitary-wave solutions. We follow the same strategy. From (4.1), we get the solitary-wave equation

$$-cu+Lu-T(u,u)=0. \tag{4.3}$$

Keeping in mind the results of last section, we define several functionals from $H^{s/2}(\mathbb{R}) \to \mathbb{R}$:

$$\begin{aligned}
\mathcal{Q}(u) &= \frac{1}{2}\int_{\mathbb{R}} u^2\,dx,\\
\mathcal{J}(u) &= \mathcal{L}(u) - \mathcal{T}(u),\\
\mathcal{L}(u) &= \frac{1}{2}\int_{\mathbb{R}} uLu\,dx = \frac{1}{2}\int_{\mathbb{R}} m(\xi)|\hat{u}(\xi)|^2\,d\xi,\\
\mathcal{T}(u) &= \frac{1}{3}\int_{\mathbb{R}} uT(u,u)\,dx = \frac{1}{3}\int_{\mathbb{R}^2} p(\xi-\eta,\eta)\hat{u}(-\xi)\hat{u}(\xi-\eta)\hat{u}(\eta)\,d\eta\,d\xi.
\end{aligned}$$

Observe that $\mathcal{Q}$ is identical here as in the cubic case in the previous chapter. The operator $\mathcal{L}$ has the same form, but the assumptions on m will be slightly different.

As in Chapter 3 we seek minimizers to a constrained variational problem, namely

$$\Gamma_q := \inf\{\mathcal{J}(u)\colon u \in H^{s/2} \text{ and } \mathcal{Q}(u) = q\}.$$

We shall again use the concentration–compactness principle to show existence of such minimizers, and use the Lagrange multiplier principle to show that they solve (4.3) with the wave speed being the Lagrange multiplier.

We make the following assumptions:

(A) The symbol m of the Fourier multiplier L is real-valued, positive, even and satisfies the growth bounds

$$\text{(A1)} \qquad m(\xi) \eqsim \langle\xi\rangle^s \quad \text{for } \xi \in \mathbb{R},$$

$$\text{(A2)} \qquad m(\xi) - m(0) \eqsim |\xi|^{s'} \quad \text{for } |\xi| < 1,$$

where

$$s > 0, \quad s' > 1/2.$$

Furthermore, we require that

$$\text{(A3)} \qquad \left|\frac{\partial m}{\partial \xi}(\xi)\right| \lesssim \langle\xi\rangle^{s-1} \quad \text{for all } \xi \in \mathbb{R}.$$

(B) The symbol p of the bilinear Fourier multiplier T satisfies

$$\text{(B1)} \qquad \langle|\xi|+|\eta|\rangle^{r'} \lesssim p(\xi-\eta,\eta) \lesssim \langle|\xi|+|\eta|\rangle^{r}$$

for all $\xi, \eta \in \mathbb{R}$ and where

$$r' \le r \le \min(s-1, \frac{2s-1}{3}).$$

We further require that

$$\text{(B2)} \qquad \left|\frac{\partial p(\xi-\eta,\eta)}{\partial \xi}\right| \lesssim \langle|\xi|+|\eta|\rangle^{r-1} \quad \text{for all } \xi \in \mathbb{R}.$$

Finally, we require that

(B3) $p(\xi - \eta, \eta)$ should be symmetric in $\xi - \eta, \eta$ and $-\xi$.

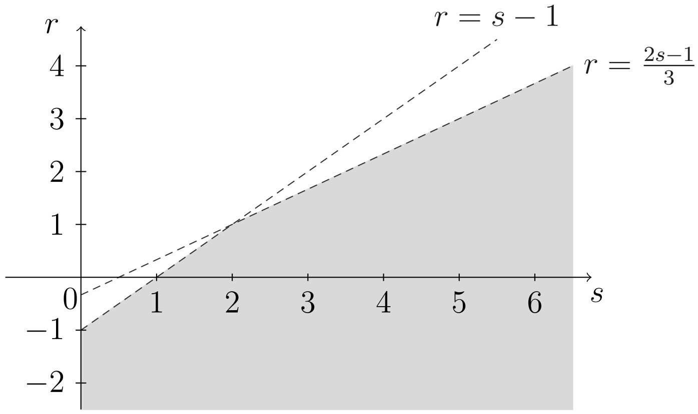

Figure 1. Illustration of the values of s, r for which we show existence of solitary waves. We show existence of both small- and large-amplitude solutions for values of s, r in the gray region. The boundaries are not included.

Given these assumptions, we show the result stated below.

Theorem 4.4 (Existence of solitary-wave solutions). *For every* $q > 0$, *there is a solution* $u \in H^\infty(\mathbb{R})$ *of the solitary-wave equation* (4.3) *satisfying* $\frac{1}{2}\|u\|_{L^2}^2 = q$. *The corresponding wave speed* c *is subcritical, that is,* $c < m(0)$.

Furthermore, there is a $q_0 > 0$ *such that for* $q \in (0, q_0)$, *the solution* u *and wave speed* c *additionally satisfy*

(i) $\|u\|_{L^\infty} \eqsim \|u\|_{H^{s/2}} \eqsim q^{1/2}$

(ii) $m(0) - c \eqsim q^\beta,\ \beta = \frac{s'}{2s'-1}$.

Remark 4.5. In the final stages of preparation, an error related to the signs of the Fourier transforms was discovered. We present the argument as is here, but discuss ways to remedy this in Section 4.2.2.

4.2.1. Discussion of assumptions. The assumptions on the symbol m are similar to those in Chapter 3 and we refer to the discussion of those in Section 3.1.1. Here, we comment only on the differences. In assumption (A2), s' is allowed to be lower ($s' > 1/2$ as opposed to $s' > 1$). This is related to the lower order of the nonlinearity (quadratic as opposed to cubic), see the proof of Lemma 4.11. Furthermore, s is now allowed to be an arbitrarily small positive number. We achieve this by exploiting the decay of p in ξ and η when $r < s - 1 < 0$ for small s.

As opposed to Chapter 3, the symbol m is now assumed to be inhomogeneous (this follows from (A1)). This simplifies many of the calculations. However, the same techniques as in Chapter 3 could easily be applied to L in this chapter to allow also for homogeneous symbols m.

As for the symbol p, we first remark that the symmetry of p in $-\xi, \xi-\eta, \eta$ is a sufficient condition for

$$D\mathcal{T}(u) = T(u,u),$$

which allows us to use the Lagrange multiplier principle and minimization techniques. This of course follows from the first section of this chapter, Section 4.1.

The growth bound on the derivative, (B2), is used to exclude dichotomy. Modifications akin to (A3) and (B2) in Chapter 3 are possible.

The lower bound for the growth of p ensures that p is bounded below by a positive constant near zero, which is necessary to find an upper bound for Γ_q in Lemma 4.11. It could be replaced by another condition ensuring this. The upper bound for the growth of p and the corresponding bounds on r are used in the proof of Lemma 4.8 which is again used to bound Γ_q from below and to show $H^{s/2}$-convergence in Lemma 4.5.3.

One could likely find more appropriate growth restrictions on the symbol p by considering different types of symmetries separately, see Section 4.2.2.

4.2.2. Discussion of method and assumption (B). In the final stages of preparations, we discovered certain faults in the argument, all of the same type: Estimates on the symbol p were in some calculations used to find estimates on $|\mathcal{T}(u)|$ without regards to the sign of $\hat{u}(\xi)$. In the cubic case (Chapter 3), this was not an issue, as we automatically end up with an absolute value on the Fourier side:

$$\mathcal{N}(u) = \frac{1}{4}\int_{\mathbb{R}} n(\xi)|\widehat{u^2}(\xi)|^2\, d\xi.$$

Of course, the same does not apply to $\mathcal{T}(u)$. We have reduced the problem to two lemmas: Lemma 4.12 where we find lower bounds for $\|u\|_{L^3}, \|u\|_{L^{\tilde{p}}}$ (for some $\tilde{p} > 2$) and Lemma 4.21 were we show that $\mathcal{T}(u) \lesssim q^{1+\beta}$. The former is used mainly to exclude vanishing, while the latter is used to estimate the size of solutions and the wave speed. In particular, the following inequalities do not necessarily hold:

(i) $\mathcal{T}(u) \lesssim \|u\|_{L^3}^3$ if $r \leq 0$ in equation (4.14),
(ii) $\mathcal{T}(u) \lesssim \|u\|_{H_3^{r/2}}^3$ if $r \geq 0$ in equations (4.15) and (4.27),
(iii) $\mathcal{T}(u) \lesssim \int_{\mathbb{R}} u(\Lambda^{r/2}u)^2\, dx$ if $r \leq 0$ in (4.16),
(iv) $\mathcal{T}(u) \lesssim \|u\|_{H_3^{r/3}}^3$ if $r \leq 0$ in equation (4.30).

Time did not permit to fully resolve this issue, but we briefly discuss what happens if the symmetry of p in $-\xi, \xi-\eta, \eta$ is multiplicative, (4.4), or additive, (4.6) (other symmetries are also possible).

If p satisfies the assumptions under (B) and is additionally of the form

$$p(\xi-\eta,\eta)=\tilde{p}(-\xi)\tilde{p}(\xi-\eta)\tilde{p}(\eta), \tag{4.4}$$

then

$$\begin{aligned}\mathcal{T}(u)&=\frac{1}{3}\int_{\mathbb{R}}\tilde{p}(-\xi)\tilde{p}(\xi-\eta)p(\eta)\hat{u}(-\xi)\hat{u}(\xi-\eta)\hat{u}(\eta)\,d\xi\\&=\frac{1}{3}\int_{\mathbb{R}}(\tilde{p}\hat{u})(-\xi)(\tilde{p}\hat{u}*\tilde{p}\hat{u})\,d\xi\\&=\frac{1}{3}\int_{\mathbb{R}}(T_{\tilde{p}}u)^3\,dx, \end{aligned}\tag{4.5}$$

where $T_{\tilde{p}}$ is now a linear Fourier multiplier with symbol $\tilde{p}$. To satisfy assumption (B1), the order of $\tilde{p}$ must be lower than $r/3$ if $r\leq 0$ due to equation 4.11 and lower than $r/2$ if $r>0$ due to (4.9). This implies (i), (ii) and (iv) if $\tilde{p}$ is as described, and the whole proof goes through with $s>1/3$ and $r<\min(s-1,\frac{2s-1}{3})$. The inequality (iii) does still not hold, but (4.5) will serve the same purpose provided the order of $\tilde{p}$ is lower than $r/2$. Then the proof holds for $s>0$.

Now consider the case when p is instead of the form

$$p(\xi-\eta,\eta)=\tilde{p}(-\xi)+\tilde{p}(\xi-\eta)+\tilde{p}(\eta). \tag{4.6}$$

Then

$$\mathcal{T}(u)=\int_{\mathbb{R}}u^2T_{\tilde{p}}u\,dx,$$

where $T_{\tilde{p}}$ is again a Fourier multiplier with symbol $\tilde{p}$. Suppose first that $r\geq 0$. Then assumption (B1) is satisfied if it is satisfied for $\tilde{p}$, but this is not necessarily enough to get the wanted estimates on $\mathcal{T}(u)$. However, if we instead use the stricter assumption that

$$\tilde{p}(\xi)\lesssim\langle\xi\rangle^{r/2},$$

then (ii) holds by a simple application of Hölders inequality:

$$|\mathcal{T}(u)|\lesssim\|u^2T_{\tilde{p}}u\|_{L^1}\leq\|u^2\|_{L^{3/2}}\|\tilde{p}u\|_{L^3}\lesssim\|u\|_{L^3}\|u\|_{H^{r/2}}\leq\|u\|_{H_3^{r/2}}.$$

For $r<0$ an assumption of the type (B1) is never satisfied if p is a sum as (4.6). However, an estimate like Lemma 4.8 goes through also in this case, provided

$$\tilde{p}(\xi)\lesssim\langle\xi\rangle^{r/2},$$

Furthermore, the bounds on Γ_q still hold, and clearly

$$|\mathcal{T}(u)|\lesssim\|u^2\|_{L^{3/2}}\|\tilde{p}u\|_{L^3}\lesssim\|u\|_{L^3}^3.$$

Hence we could likely relax the restriction on p to allow for this type of symbol also when $r<0$.

When continuing to resolve these issues in future work, it is perhaps more sensible to impose different growth restrictions on p depending on the type of symmetry and on the sign of r.

With these reservations, we can begin proving Theorem 4.4. The proof is structured in the same way as the proof of Theorem 3.1 in Chapter 3. The parts that are identical are referenced to and left out. Parts that are similar, but where the details differ are written out for clarity.

4.3. Properties of symbols and functionals

We begin by verifying that $\mathcal{J}$ is a conserved property in time.

PROPOSITION 4.6. *The functional $\mathcal{J}$ is a conserved property in time for any solution u of* (4.1) *that is in $\mathcal{S}(\mathbb{R})$ for each t and continuously differentiable in time.*

PROOF. Using the symmetry of L and T, we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{J}(u) &= \frac{d}{dt}\mathcal{L}(u) - \frac{d}{dt}\mathcal{T}(u) \\
&= \int_{\mathbb{R}} (\partial_t u) Lu\, dx - \frac{1}{3}\int_{\mathbb{R}} (\partial_t u) S(u,u)\, dx - \frac{1}{3}\int_{\mathbb{R}} u \partial_t S(u,u)\, dx \\
&= \int_{\mathbb{R}} (\partial_t u) Lu\, dx - \frac{1}{3}\int_{\mathbb{R}} (\partial_t u) S(u,u)\, dx \\
&\quad - \frac{1}{3}\int_{\mathbb{R}} u S(\partial_t u, u)\, dx - \frac{1}{3}\int_{\mathbb{R}} u S(u, \partial_t u)\, dx \\
&= \int_{\mathbb{R}} (\partial_t u)(Lu - S(u,u))\, dx \\
&= \frac{1}{2}\int_{\mathbb{R}} \partial_x (Lu - S(u,u))^2\, dx \\
&= (Lu - S(u,u))\Big|_{x=-\infty}^{x=\infty} \\
&= 0,
\end{aligned}$$

since $u(\cdot, t) \in \mathcal{S}(\mathbb{R})$ for all $t \in \mathbb{R}$. □

We establish some estimates for $\langle \cdot \rangle$ that will be used repeatedly in the rest of this chapter.

For easy reference, we recall the estimates from Lemma 3.2 in chapter 3

$$\langle a + b \rangle \lesssim \langle a \rangle + \langle b \rangle \tag{4.7}$$

$$\langle a + b \rangle \lesssim \langle a \rangle \langle b \rangle \tag{4.8}$$

LEMMA 4.7. *The following estimates hold for all $\xi, \eta \in \mathbb{R}$:*

$$\langle |\xi| + |\eta| \rangle \lesssim \langle -\xi \rangle^{1/2} \langle \eta \rangle^{1/2} \langle \xi - \eta \rangle^{1/2} \tag{4.9}$$
$$\langle |\xi| + |\eta| \rangle \gtrsim \langle -\xi \rangle^{1/2} \langle \eta \rangle^{1/2} \tag{4.10}$$

and

$$\langle \xi \rangle^{1/3} \langle \xi - \eta \rangle^{1/3} \langle \eta \rangle^{1/3} \lesssim \langle |\xi| + |\eta| \rangle \tag{4.11}$$

PROOF. It follows from (4.8) with $a = \xi - \eta, b = \eta$ that

$$\langle \xi \rangle \lesssim \langle \xi \rangle^{1/2} \langle \eta \rangle^{1/2} \langle \xi - \eta \rangle^{1/2},$$

and similarly for $\langle \eta \rangle$.

Using this and (4.7) with $a = |\xi|, b = |\eta|$ and (4.8), we find

$$\langle |\xi| + |\eta| \rangle \lesssim \langle |\xi| \rangle + \langle |\eta| \rangle = \langle \xi \rangle + \langle \eta \rangle \lesssim \langle \xi \rangle^{1/2} \langle \xi - \eta \rangle^{1/2} \langle \eta \rangle^{1/2},$$

which establishes (4.9).

For (4.10), simply observe that

$$\langle -\xi \rangle \langle \eta \rangle \leq \langle |\xi| + |\eta| \rangle \langle |\xi| + |\eta| \rangle \leq \langle |\xi| + |\eta| \rangle^2.$$

The first inequality of (4.11) is found by applying (4.8), while the last is found in the same way as (4.10). □

The next lemma plays the same role as Lemma 3.5 in the previous Chapter. It allows us to bound $\mathcal{T}$ in terms of $\mathcal{L}$ and will be used repeatedly in the rest of the chapter.

LEMMA 4.8. *Let $u \in H^{s/2}(\mathbb{R})$. Then*

$$|\mathcal{T}(u)| \lesssim \|u\|_{L^2}^{3-\gamma} \|u\|_{H^{s/2}}^{\gamma},$$

for some $\gamma < 2$.

PROOF. Assume first that $r > 0$. Pick τ such that $r + 1 < \tau < 2s - 2r$. This is possible since

$$r < \frac{2s-1}{3} \quad \Longrightarrow \quad r + 1 < 2s - 2r.$$

Using the estimate (4.9), Hölders inequality and Youngs convolution inequality we obtain

$$\begin{aligned}
|\mathcal{T}(u)| &\eqsim |\int_{\mathbb{R}} uT(u,u)\,dx| \\
&\lesssim \int_{\mathbb{R}^2} |\bar{\hat{u}}(\xi)||\hat{u}(\xi-\eta)||\hat{u}(\eta)|\langle|\xi|+|\eta|\rangle^r \,d\eta\,d\xi \\
&\lesssim \int_{\mathbb{R}^2} |\hat{u}(-\xi)\langle-\xi\rangle^{r/2}||\hat{u}(\xi-\eta)\langle\xi-\eta\rangle^{r/2}||\hat{u}(\eta)\langle\eta\rangle^{r/2}|\,d\eta\,d\xi \\
&\leq \||\langle\cdot\rangle^{r/2}\hat{u}|(|\langle\cdot\rangle^{r/2}\hat{u}| * |\langle\cdot\rangle^{r/2}\hat{u}|)\|_{L^1} \\
&\leq \|\langle\cdot\rangle^{r/2}\hat{u}\|_{L^2}^2 \|\langle\cdot\rangle^{r/2}\hat{u}\|_{L^1} \\
&\leq \|u\|_{H^{r/2}}^2 \|\langle\cdot\rangle^{(r-\tau)/2}\|_{L^2} \|u\|_{H^{\tau/2}} \\
&\lesssim \|u\|_{L^2}^{3-(2r+\tau)/s} \|u\|_{H^{s/2}}^{(2r+\tau)/s},
\end{aligned}$$

which shows the result with $\gamma = \frac{2r+\tau}{s} < 2$.

If $r \leq 0$, then (4.10) gives that

$$\langle|\xi|+|\eta|\rangle^r \lesssim \langle-\xi\rangle^{r/2}\langle\eta\rangle^{r/2}.$$

Recall that $(r-s)/2 < -1/2$. Proceeding in a similar manner as before, we get

$$\begin{aligned}
|\mathcal{T}(u)| &\lesssim \||\langle\cdot\rangle^{r/2}\hat{u}|(|\langle\cdot\rangle^{r/2}\hat{u}| * |\hat{u}|)\|_{L^1} \\
&\lesssim \|u\|_{H^{r/2}} \|u\|_{L^2} \|\langle\cdot\rangle^{(r-s)/2}\|_{L^2} \|u\|_{H^{s/2}} \\
&\lesssim \|u\|_{L^2}^2 \|u\|_{H^{s/2}},
\end{aligned}$$

which shows the statement when $r \leq 0$ with $\gamma = 1 < 2$. □

The next lemma will be used to bound $\|\cdot\|_{L^3}$ from below in Lemma 4.12.

LEMMA 4.9. *Let $u \in H^{s/2}(\mathbb{R})$ and suppose $r > 0$. Then the estimate*

$$\|u\|_{H_3^{r/2}} \lesssim \|u\|_{L^3}^{1-r/\tau} \|u\|_{H^{s/2}}^{r/\tau},$$

holds for $\tau \in (r, s-1)$.

PROOF. Using Proposition 2.4 with $s = r/2, s_0 = 0, s_1 = \tau/2, p = p_0 = p_1 = 3$, we obtain

$$\|u\|_{H_3^{r/2}} \leq \|u\|_{L^3}^{1-r/\tau} \|u\|_{H_3^{\tau/2}}^{r/\tau}. \tag{4.12}$$

Furthermore, using Plancherel and Youngs convolutions inequality

$$\begin{aligned}
\|u\|^3_{H_3^{\tau/2}} &= \int_{\mathbb{R}} |\Lambda^{\tau/2} u|^3 \, dx \\
&= \int_{\mathbb{R}} \overline{(\Lambda^{\tau/2} u)}(\Lambda^{\tau/2} u)|\Lambda^{\tau/2} u| \, dx \\
&= \int_{\mathbb{R}} \langle \xi \rangle^{\tau/2} \hat{u}(\xi)(\langle \cdot \rangle^{\tau/2} \hat{u} * \mathcal{F}(|\Lambda^{\tau/2} u|)(\xi) \, d\xi \\
&\lesssim \|\langle \cdot \rangle^{\tau/2} \hat{u}\|_{L^2} \|\mathcal{F}(|\Lambda^{\tau/2} u|)\|_{L^2} \|\langle \cdot \rangle \hat{u}\|_{L^1} \\
&\leq \|u\|^2_{H^{\tau/2}} \|\langle \cdot \rangle^{(\tau - s)/2}\|_{L^2} \|\langle \cdot \rangle^{s/2} \hat{u}\|_{L^2} \\
&\leq \|u\|^3_{H^{s/2}}. \qquad (4.13)
\end{aligned}$$

Combining (4.12) and (4.13) gives the desired estimate. □

4.4. Bounds for Γ_q and norm-estimates

As in the previous chapter, we find upper and lower bounds for Γ_q.

LEMMA 4.10. *For all $q > 0$,*

$$\Gamma_q > -\infty.$$

PROOF. Let $u \in H^{s/2}(\mathbb{R})$, $\mathcal{Q}(u) = q$. Using Lemma 4.8, then for $\gamma < 2$,

$$\begin{aligned}
\mathcal{J}(u) &= \mathcal{L}(u) - \mathcal{T}(u) \\
&> C_1 \|u\|^2_{H^{s/2}} - C_2 q^{3-\gamma} \|u\|^{\gamma}_{H^{s/2}} \\
&> -\infty
\end{aligned}$$

since the expression is positive as $\|u\|_{H^{s/2}} \to \infty$. □

LEMMA 4.11. *For all $q > 0$,*

$$\Gamma_q < m(0)q.$$

PROOF. Pick a function $\phi \in \mathcal{S}(\mathbb{R})$ satisfying $\mathcal{Q}(\phi) = q$ and $\hat{\phi}(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. Let $0 < t < 1$ and define

$$\phi_t(x) = \sqrt{t}\phi(tx).$$

Then $\mathcal{Q}(\phi_t) = q$.

The upper bound for $\mathcal{L}(\phi_t)$ in Chapter 3, Lemma 3.7 used no properties of s except $s > 0$. Hence for some constant $C_1 > 0$,

$$\mathcal{L}(\phi_t) \leq m(0)q + C_1 t^{s'}.$$

We now find a lower bound for $\mathcal{T}(\phi_t)$.

$$\begin{aligned}
\mathcal{T}(\phi_t) &= \frac{1}{3}\int_{\mathbb{R}} \widehat{\phi}_t(-\xi)p(\xi-\eta,\eta)\widehat{\phi}_t(\xi-\eta)\widehat{\phi}_t(\xi-\eta)\,d\eta\,d\xi \\
&\eqsim t^{3/2-3}\int_{\mathbb{R}} \hat{\phi}(-\xi/t)p(\xi-\eta,\eta)\hat{\phi}((\xi-\eta)/t)\hat{\phi}(\eta/t)\,d\eta\,d\xi \\
&= t^{1/2}\int_{\mathbb{R}} \hat{\phi}(-\xi)p(t(\xi-\eta),t\eta)\hat{\phi}(\xi-\eta)\hat{\phi}(\eta)\,d\eta\,d\xi \\
&\gtrsim t^{1/2}\int_{\mathbb{R}} \hat{\phi}(-\xi)\langle|t\xi|+|t\eta|\rangle^{r'}\hat{\phi}(\xi-\eta)\hat{\phi}(\eta)\,d\eta\,d\xi \\
&\gtrsim t^{1/2}\min(\|u\|_{L^3}^3,\|u\|_{H_3^{r'/2}}^3) \\
&\gtrsim t^{1/2}.
\end{aligned}$$

Hence for constants C_1, C_2,

$$\begin{aligned}
\mathcal{J}(\phi_t) &= \mathcal{L}(\phi_t) - \mathcal{T}(\phi_t) \\
&\le m(0)q + C_1 t^{s'} - C_2 t^{1/2} \\
&< m(0)q
\end{aligned}$$

for $t > 0$ small enough since $s' > 1/2$ by assumption. □

The next Lemma gives bounds for $\|\cdot\|_{H^{s/2}}$ and $\mathcal{T}$ that are used throughout the rest of the chapter, among other things to exclude vanishing and dichotomy and to show convergence from concentration. The lower bound for $\|\cdot\|_{L^3}$ will be used to exclude vanishing when $s > 1/3$, while the lower bound on $\|\cdot\|_{L^{\tilde{p}}}$ is used to exclude vanishing when $s \le 1/3$.

LEMMA 4.12. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for Γ_q.*

(i) There is a subsequence, again denoted by $\{u_n\}_{n\in\mathbb{N}}$ and a $\delta > 0$ such that

$$\|u_n\|_{H^{s/2}}^{-1} + \mathcal{T}(u_n) + \|u_n\|_{L^3} \ge \delta.$$

(ii) If $s \le 1/3$, then additionally

$$\|u_n\|_{L^{\tilde{p}}} \ge \delta,$$

for some $\tilde{p} \in (2, 2/(1-s))$.

PROOF. *Step 1. Bounding $\|u_n\|_{H^{s/2}}$ from above.* Lemma 4.8 and Lemma 4.11 together imply that

$$\begin{aligned}
\|u_n\|_{H^{s/2}}^2 &\eqsim \mathcal{L}(u_n) \\
&= \mathcal{J}(u_n) + \mathcal{T}(u) \\
&\lesssim m(0)q + q^{(3-\gamma)/2}\|u_n\|_{H^{s/2}}^{\gamma},
\end{aligned}$$

for some $\gamma < 2$. Dividing both sides by $\|u_n\|_{H^{s/2}}^{\gamma}$ it is clear that $\|u_n\|_{H^{s/2}}$ must be bounded:

$$\|u_n\|_{H^{s/2}} \leq 1/\delta_1,$$

provided $\delta_1 > 0$ is small enough.

Step 2. Bounding $\mathcal{T}(u_n)$ from below. If no subsequence satisfies $\mathcal{T}(u_n) \geq \delta_2$ for any $\delta_2 > 0$, then

$$\limsup_{n\to\infty} \mathcal{T}(u_n) \leq 0.$$

this is a contradiction, since it would imply that

$$\begin{aligned} I_q &= \liminf_{n\to\infty}(\mathcal{L}(u_n) - \mathcal{T}(u_n) \\ &\geq m(0)q, \end{aligned}$$

contrary to Lemma 4.11.

Step 3. Bounding $\|u_n\|_{L^3}$ from below. If $r \leq 0$, then

$$\|u_n\|_{L^3}^3 \gtrsim \mathcal{T}(u_n) \geq \delta_2, \tag{4.14}$$

while if $r > 0$, then it follows from Lemma 4.9 and equation (4.9) that

$$\|u_n\|_{L^3}^{1-r/\tau} \geq \|u_n\|_{H_3^{r/2}} \|u_n\|_{H^{s/2}}^{-r/\tau} \gtrsim \mathcal{T}(u_n)^{1/3} \|u_n\|_{H^{s/2}}^{-r/\tau} \geq \delta_2^{1/3} \delta_1^{r/\tau} = \delta_3^{1-r/\tau} \tag{4.15}$$

for some $\tau \in (r, s-1)$.

Picking $\delta = \min(\delta_1, \delta_2, \delta_2^{1/3}, \delta_3)$ concludes the proof of *(i)*.

To show *(ii)*, suppose that $s \leq 1/3$. Then $r < 0$. It follows from properties of $\mathcal{T}$, Hölders inequality and Proposition 2.4 that

$$\begin{aligned} \mathcal{T}(u_n) &\lesssim \int_{\mathbb{R}} u_n (\Lambda^{r/2} u_n)^2 \, dx \\ &\leq \|u_n\|_{L^{\tilde{p}}} \|(\Lambda^{r/2} u_n)^2\|_{L^{\tilde{q}}} \\ &= \|u_n\|_{L^{\tilde{p}}} \|u_n\|_{H_{2\tilde{q}}^{r/2}}^2 \\ &\leq \|u_n\|_{L^{\tilde{p}}} \|u_n\|_{H_3^{r/2}}^{12/\tilde{q}-6} \|u_n\|_{H_4^{r/2}}^{8-12/\tilde{q}} \end{aligned} \tag{4.16}$$

where $2 < \tilde{p} < 2/(1-s) < 3, 3 < 2\tilde{q} < 4$ and $1/\tilde{p} + 1/\tilde{q} = 1$. Now, $\|u_n\|_{H_3^{r/2}} \lesssim \|u_n\|_{H^{s/2}}$ by equation (4.13). The same holds for $\|u_n\|_{H_4^{r/2}}$ by a similar calculation:

$$\begin{aligned}
\|u_n\|_{H_4^{r/2}}^4 &= \int_{\mathbb{R}} |\Lambda^{r/2} u_n)|^4 \, dx \\
&= \int_{\mathbb{R}} (\widehat{\Lambda^{r/2} u_n})^2 (\widehat{\Lambda^{r/2} u_n})^2 \, dx \\
&= \|\langle \cdot \rangle^{r/2} \widehat{u_n} * \langle \cdot \rangle^{r/2} \widehat{u_n}\|_{L^2}^2 \\
&\leq \|u_n\|_{H^{r/2}}^2 \|\langle \cdot \rangle^{r/2} \widehat{u_n}\|_{L^1}^2 \\
&\leq \|u_n\|_{H^{s/2}}^2 \|\langle \cdot \rangle^{(r-s)/2}\|_{L^2}^2 \|u_n\|_{H^{s/2}}^2 \\
&\lesssim \|u_n\|_{H^{s/2}}^4.
\end{aligned}$$

Hence

$$\|u_n\|_{L^{\tilde{p}}} \gtrsim \mathcal{T}(u_n) \|u_n\|_{H^{s/2}}^{-2} \geq \delta_1^2 \delta_2.$$

□

4.5. Concentration–compactness and existence of solutions

We use the Concentration–Compactness principle and the results from the last section to exclude vanishing and dichotomy, before showing existence of solutions.

4.5.1. Excluding vanishing.

Lemma 4.13. *Vanishing does not occur.*

Proof. The result is a direct consequence of Lemma 3.9 from Chapter 3, applied to near minimizers, with $\delta > 0$ from Lemma 4.12, $p^* = 3$ if $s > 1/3$ and $p^* = \tilde{p}$ if $s \leq 1/3$. □

4.5.2. Excluding dichotomy. To exclude dichotomy, we reuse some of the results from Chapter 3.

Lemma 4.14 (Sub-additivity). Γ_q *is strictly sub-additive:*

$$\Gamma_{q_1+q_2} < \Gamma_{q_1} + \Gamma_{q_2}.$$

Proof. Let $t > 1$. Let $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence and define a new sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ by $\tilde{u}_n = \sqrt{t} u_n$, such that $\mathcal{Q}(\tilde{u}_n) = tq$. We find that

$$\begin{aligned}
I_{tq} &= \liminf_{n\to\infty} t\mathcal{L}(u_n) - t^{3/2}\mathcal{T}(u_n) \\
&= \liminf_{n\to\infty} (t\mathcal{J}(u_n) - (t^{3/2} - t)\mathcal{T}(u_n)) \\
&\leq t\Gamma_q - (t^{3/2} - t)\delta \\
&< t\Gamma_q.
\end{aligned}$$

In the second to last line we used that $\mathcal{T}(u_n) \geq \delta > 0$, see Lemma 4.12.

We conclude that $q \mapsto \Gamma^q$ is sub-homogeneous, which implies sub-additivity by Lemma 3.11. □

The next Lemma is the equivalent of Lemma 3.13, but for the operator T.

Lemma 4.15. *Let $u, v \in H^{s/2}(\mathbb{R})$ and let $\rho \in \mathcal{S}(\mathbb{R})$ be a non-negative Schwartz function. Define $\rho_R(x) = \rho(x/R)$. Then*

$$\text{(i)} \qquad \left| \int_{\mathbb{R}} v(\rho_R T(u,u) - T(\rho_R u, u)\, dx \right| \to 0$$

and

$$\text{(ii)} \qquad \left| \int_{\mathbb{R}} v((1-\rho_R)T(u,u) - T((1-\rho_R)u, u)\, dx \right| \to 0$$

as $R \to \infty$.

Proof. Combining Plancherel and Fubinis theorem, we have that

$$\begin{aligned}
&|\int_{\mathbb{R}} v(\rho_R T(u,u) - T(\rho_R u,u) dx| \\
&\quad = |\int_{\mathbb{R}} \overline{\hat{v}(\xi)}(\hat{\rho}_R * \int_{\mathbb{R}} p(\cdot-\eta,\eta)\hat{u}(\cdot-\eta)\hat{u}(\eta)d\eta - \int_{\mathbb{R}} p(\xi-\eta,\eta)(\hat{\rho}_R*\hat{u})(\xi-\eta)\hat{u}(\eta)d\eta d\xi| \\
&\quad \leq \int_{\mathbb{R}^3} |\overline{\hat{v}(\xi)}\hat{\rho}_R(t)\hat{u}(\eta)\hat{u}(\xi-t-\eta)||(p(\xi-t-\eta,\eta)-p(\xi-\eta,\eta)|d\eta dt d\xi. \qquad (4.17)
\end{aligned}$$

By assumption on p and the mean value theorem,

$$\begin{aligned}
|p(\xi-t-\eta,\eta) - p(\xi-\eta,\eta)| &\leq \sup_{|\theta|\leq|\xi-t|,|\xi|} |t|\frac{\partial p}{\partial \xi}(\theta) \\
&\lesssim \begin{cases} |t| & \text{if } r \leq 0 \\ |t|(\langle|\xi-t|+|\eta|\rangle^r + \langle|\xi|+|\eta|\rangle^r) & \text{if } r > 0. \end{cases}
\end{aligned}$$

Furthermore, applying estimates on $\langle\cdot\rangle$ (4.8) and (4.9), then

$$|p(\xi-t-\eta,\eta) - p(\xi-\eta,\eta)| \lesssim \begin{cases} |t| & \text{if } r \leq 0 \\ |t|\langle\xi\rangle^{r/2}\langle t\rangle^{r/2}\langle\eta\rangle^{r/2}\langle\xi-t-\eta\rangle^{r/2} & \text{if } r > 0. \end{cases}$$

Assume first that $r > 0$. Inserting this into equation (4.17), we get that

$$
\begin{aligned}
&|\int_{\mathbb{R}} v(\rho_R T(u,u) - T(\rho_R u,u)dx| \\
&\lesssim \int_{\mathbb{R}} |\hat{\rho}_R(t)||t|\langle t\rangle^{r/2}| \int_{\mathbb{R}} |\overline{\hat{v}(\xi)}\langle\xi\rangle^{r/2}| \int_{\mathbb{R}} |\hat{u}(\eta)\langle\eta\rangle^{r/2}||\hat{u}(\xi - t - \eta)\langle\xi - t - \eta\rangle^{r/2}| d\eta d\xi dt \\
&\leq \|\overline{\hat{v}}\langle\cdot\rangle^{r/2}\|_{L^2} \|\hat{u}\langle\cdot\rangle^{s/2}\|_{L^2} \|\langle\cdot\rangle^{(r-s)/2}\|_{L^2} \|\hat{u}\langle\cdot\rangle^{r/2}\|_{L^2} \int_{\mathbb{R}} |\hat{\rho}(Rt)|t|\langle t\rangle^{r/2} dt \\
&\leq \|v\|_{H^{r/2}} \|u\|_{H^{s/2}} \|u\|_{H^{r/2}} \frac{1}{|R|} \int_{\mathbb{R}} |\hat{\rho}(t)|t|\langle t/R\rangle^{r/2} dt.
\end{aligned}
$$

This expression goes to zero as R goes to infinity since $\rho \in \mathcal{S}(\mathbb{R})$.

If $r \leq 0$, then

$$
\begin{aligned}
&|\int_{\mathbb{R}} v(\rho_R T(u,u) - T(\rho_R u,u)\, dx| \\
&\quad \lesssim \int_{\mathbb{R}} |\hat{\rho}_R(t)||t| \int_{\mathbb{R}^2} |\overline{\hat{v}(\xi)}||\hat{u}(\eta)||\hat{u}(\xi - t - \eta)|\, d\eta\, d\xi\, dt \\
&\quad \leq \|\overline{\hat{v}}\|_{L^2} \|\hat{u}\|_{L^2}^2 \||\hat{\rho}(Rt)|t|\|_{L^2} \\
&\quad \leq \frac{1}{|R|} \|v\|_{L^2} \|u\|_{L^2}^2 \|t\hat{\rho}(t)\|_{L^2}.
\end{aligned}
$$

This expression also goes to zero since $\rho \in \mathcal{S}(\mathbb{R})$, concluding the proof of (i).

Now (ii) follows from a simple calculation using that T is a bilinear operator:

$$
\begin{aligned}
|\int_{\mathbb{R}} v((1-\rho_R)T(u,u) - T((1-\rho_R)u,u)\, dx| &= |\int_{\mathbb{R}} -v\rho_R T(u,u) + vT(\rho_R u,u)\, dx| \\
&= |\int_{\mathbb{R}} v(T(u,u) - T(\rho_R u,u))\, dx| \\
&\to 0.
\end{aligned}
$$

□

With this result in hand, and using the results from Chapter 3 for L, we can exclude dichotomy.

Lemma 4.16. *Dichotomy does not occur.*

Proof. The proof of Lemma 3.14 did not rely on properties of the near minimizers of I_q except for $\mathcal{Q}(u_n) = q$. Hence the result also holds for near minimizers of Γ_q. Hence, let $u_n^{(1)} = \phi_n u_n, u_n^{(2)} = \psi_n u_n$ as in Lemma 3.14, but where $\{u_n\}_{n\in\mathbb{N}}$ minimizes Γ_q. Suppose that dichotomy occurs. We conclude that

$$
\begin{aligned}
&\mathcal{Q}(u_n^{(1)}) \to \bar{q},\\
&\mathcal{Q}(u_n^{(2)}) \to (q-\bar{q}),\\
&\frac{1}{2}\int_{R_n \le |x-y_n| \le 2R_n} u_n^2 \, dx \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

We now wish to show that

$$\mathcal{J}(u_n) \ge \mathcal{J}(u_n^{(1)}) + \mathcal{J}(u_n^{(2)}) \tag{4.18}$$

as $n \to \infty$. We have that

$$
\begin{aligned}
\mathcal{J}(u_n) &= \mathcal{L}(\phi_n u_n + \psi_n u_n) - \mathcal{T}(\phi_n u_n + \psi_n u_n)\\
&= \mathcal{L}(\phi_n u_n) + \mathcal{L}(\psi_n u_n) + \int_{\mathbb{R}} \phi u_n L(\psi_n u_n)\, dx - \mathcal{T}(\phi_n u_n) - \mathcal{T}(\psi_n u_n)\\
&\quad - \int_{\mathbb{R}} \phi_n u_n T(\psi_n u_n, \psi_n u_n)\, dx - \int_{\mathbb{R}} \psi_n u_n T(\phi_n u_n, \phi_n u_n).
\end{aligned}
$$

In the last line, we used the symmetry of L and T.

It then suffices to show that

$$\lim_{n\to\infty} (\int_{\mathbb{R}} \phi_n u_n L(\psi_n u_n)\, dx) \ge 0, \tag{4.19}$$

$$\lim_{n\to\infty} (\int_{\mathbb{R}} \phi_n u_n T(\psi_n u_n, \psi_n u_n)\, dx) = 0,$$

$$\lim_{n\to\infty} (\int_{\mathbb{R}} \psi_n u_n T(\phi_n u_n, \phi_n u_n)) = 0. \tag{4.20}$$

To show (4.19) we use 3.13, where ϕ_n and ψ_n corresponds to ρ_R and $1-\rho_R$ respectively:

$$
\begin{aligned}
\lim_{n\to\infty} (\int_{\mathbb{R}} \phi_n u_n L(\psi_n u_n)\, dx) &= \lim_{n\to\infty} (\int_{\mathbb{R}} \sqrt{\phi_n \psi_n} u_n L(\sqrt{\phi\psi}_n u_n)\, dx)\\
&\eqsim \lim_{n\to\infty} \| \sqrt{\phi_n \psi_n} u_n \|_{H^{s/2}}^2\\
&\ge 0.
\end{aligned}
$$

To show equation 4.5.2, we use Lemma 4.15 repeatedly with $(\phi_n)^{1/3}$ as ρ_R and $(\psi_n)^{1/3}$ as $(1-\rho_R)$. Then we apply Lemma 4.8, where $\gamma < 2$. We get that

$$\begin{aligned}
\lim_{n\to\infty} \left| \int_{\mathbb{R}} \phi_n u_n T(\psi_n u_n, \psi_n u_n)\, dx \right| &= \lim_{n\to\infty} |\mathcal{T}(\sqrt[3]{\phi_n}\ \sqrt[3/2]{\psi_n} u_n)| \\
&\lesssim \lim_{n\to\infty} \|\sqrt[3]{\phi_n}\ \sqrt[3/2]{\psi_n} u_n\|_{L^2}^{3-\gamma} \|\sqrt[3]{\phi_n}\ \sqrt[3/2]{\psi_n} u_n\|_{H^{s/2}}^{\gamma} \\
&\lesssim \|u_n\|_{H^{s/2}}^{\gamma} \lim_{n\to\infty} \left(\int_{R_n \le |x-y_n| \le 2R_n} u_n^2\, dx \right)^{(3-\gamma)/2} \\
&= 0.
\end{aligned}$$

Equation (4.20) is shown in the same manner, and we conclude that (4.18) holds if dichotomy occurs.

However, this contradicts Lemma 4.14, since

$$\Gamma_q = \liminf_{n\to\infty} \mathcal{J}(u_n) \ge \liminf_{n\to\infty} \mathcal{J}(u_n^1) + \liminf_{n\to\infty} \mathcal{J}(u_n^2) \ge \Gamma_{\bar{q}} + \Gamma_{(q-\bar{q})}.$$

Hence dichotomy cannot occur. □

4.5.3. Existence of solutions. We first show existence of a minimizer of Γ_q for each $q > 0$.

LEMMA 4.17 (Existence of minimizer). *For any minimizing sequence* $\{u_n\}_{n\in\mathbb{N}}$ *of* Γ_q *where* $q > 0$*, there is a subsequence of* $\{u_n(\cdot + y_n)\}_{n\in\mathbb{N}}$ *that converges in* $H^{s/2}(\mathbb{R})$ *to a minimizer of* Γ_q*.*

PROOF. Let $\tilde{u}_n(x) = u_n(x + y_n)$. Existence of a subsequence of $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ that converges strongly in $L^2(\mathbb{R})$ and weakly in $H^{s/2}(\mathbb{R})$ to $w \in H^{s/2}(\mathbb{R})$ is shown in exactly the same manner as in Lemma 3.4.3.

As before, we then wish to show that

$$\mathcal{J}(\omega) = \Gamma^q.$$

Since $\tilde{u}_n \rightharpoonup \omega$ in $H^{s/2}$ and $\mathcal{L}(\tilde{u}_n) \eqsim \|u_n\|_{H^{s/2}}^2$, then

$$\mathcal{L}(\omega) \le \liminf_{n\to\infty} \mathcal{L}(\tilde{u}_n).$$

Furthermore,

$$\mathcal{T}(\tilde{u}_n) \xrightarrow[n\to\infty]{} \mathcal{T}(\omega),$$

since

$$\begin{aligned}
|\mathcal{T}(\tilde{u}_n) - \mathcal{T}(\omega)| &= |\mathcal{T}(\tilde{u}_n - \omega) - \int_{\mathbb{R}} \tilde{u}_n T(\omega, \omega)\, dx + \int_{\mathbb{R}} \tilde{u}_n T(\omega, \tilde{u}_n)\, dx| \\
&\le |\mathcal{T}(\tilde{u}_n - \omega)| + \left| \int_{\mathbb{R}} p(\xi - \eta, \eta)(\widehat{\tilde{u}_n}(-\xi)\widehat{\omega}(\xi - \eta)(\widehat{\omega}(\eta) - \widehat{\tilde{u}_n}(\eta))\, dx \right|
\end{aligned}$$

Using Lemma 4.8, where $\gamma < 2$, and the known L^2-convergence,

$$|\mathcal{T}(\tilde{u}_n - \omega)| \lesssim \|\tilde{u}_n - \omega\|_{L^2}^{3-\gamma} \|\tilde{u}_n - \omega\|_{H^{s/2}}^{\gamma} \xrightarrow[n\to\infty]{} 0.$$

By a similar argument as in the proof of 4.8,

$$\begin{aligned} &| \int_{\mathbb{R}} p(\xi - \eta, \eta)(\widehat{\tilde{u}_n}(-\xi)\widehat{\omega}(\xi - \eta)(\widehat{\omega}(\eta) - \widehat{\tilde{u}_n}(\eta))\,dx| \\ &\qquad \lesssim \begin{cases} \|\omega\|_{H^{r/2}} \|\tilde{u}_n - \omega\|_{H^{s/2}}^{r/s} \|\tilde{u}_n - \omega\|_{L^2}^{1-r/s} \|\tilde{u}_n\|_{H^{s/2}} & \text{if } r > 0, \\ \|\omega\|_{H^{r/2}} \|\tilde{u}_n - \omega\|_{L^2} \|\tilde{u}_n\|_{H^{s/2}} & \text{if } r \le 0 \end{cases} \\ &\qquad \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

since $\tilde{u}_n \to \omega$ in $L^2(\mathbb{R})$.

As in the previous section, we conclude that $\mathcal{J}(\omega) = \Gamma_q = \lim_{n\to\infty} \mathcal{J}(\tilde{u}_n)$, that $\mathcal{T}(\tilde{u}_n) \to \mathcal{T}(\omega)$, and hence that $\mathcal{L}(\tilde{u}_n) \to \mathcal{L}(\omega)$. Combined, this and the weak convergence of $\tilde{u}_n \rightharpoonup \omega$ implies norm convergence in $H^{s/2}(\mathbb{R})$. □

We verify that minimizers of Γ_q are in fact solutions of the solitary-wave equation.

LEMMA 4.18. *Any minimizer $u \in H^{s/2}(\mathbb{R})$ of the constrained variational problem Γ_q solves the solitary-wave equation* (4.3), *where c is the Langrange multiplier.*

PROOF. By the Lagrange multiplier principle 2.8, any minimizer of Γ_q satisfies

$$\mathcal{J}'(u) - c\mathcal{Q}'(u) = 0, \tag{4.21}$$

where the wave speed c is the Lagrange multiplier in the constrained variational problem.

The estimates (4.9) and (4.10) and symmetry assumptions on T guarantees that the Fréchet derivative $\mathcal{T}'(u) = T(u,u)$:

$$\begin{aligned}
\mathcal{T}'(u) &= \lim_{\|v\|_{H^{s/2}}\to 0} \frac{|\mathcal{T}(u+v) - \mathcal{T}(u) - vT(u,u)|}{\|u\|_{H^{s/2}}} \\
&= \lim_{\|v\|_{H^{s/2}}\to 0} \frac{|\int_{\mathbb{R}} uT(v,v)\,dx|}{\|v\|_{H^{s/2}}} \\
&\lesssim \lim_{\|v\|_{H^{s/2}}\to 0} \frac{|\int_{\mathbb{R}} \overline{|\hat{u}(\xi)|}\langle|\xi|+|\eta|\rangle^r|\hat{v}(\xi-\eta)||\hat{v}(\eta)|\,dx|}{\|v\|_{H^{s/2}}} \\
&\lesssim \begin{cases} \lim_{\|v\|_{H^{s/2}}\to 0} \frac{\||\overline{\hat{u}}\langle\cdot\rangle^{r/2}||\hat{v}\langle\cdot\rangle^{r/2}|*|\hat{v}\langle\cdot\rangle^{r/2}|\|_{L^1}}{\|v\|_{H^{s/2}}} & \text{if } r > 0 \\ \lim_{\|v\|_{H^{s/2}}\to 0} \frac{\||\overline{\hat{u}}||\hat{v}\langle\cdot\rangle^{r/2}|*|\hat{v}\langle\cdot\rangle^{r/2}|\|_{L^1}}{\|v\|_{H^{s/2}}} & \text{if } r \le 0 \end{cases} \\
&\lesssim \begin{cases} \lim_{\|v\|_{H^{s/2}}\to 0} \frac{\|u\|_{H^{r/2}}\|v\|_{H^{r/2}}\|v\|_{H^{s/2}}}{\|v\|_{H^{s/2}}} & \text{if } r > 0 \\ \lim_{\|v\|_{H^{s/2}}\to 0} \frac{\|u\|_{L^2}\|v\|_{H^{r/2}}\|v\|_{H^{s/2}}}{\|v\|_{H^{s/2}}} & \text{if } r \le 0 \end{cases} \\
&= 0.
\end{aligned}$$

The Fréchet derivatives of $\mathcal{L}(u)$ and $\mathcal{Q}(u)$ are known from Chapter 3, see Lemma 3.17:

$$\mathcal{L}'(u) = Lu \qquad \mathcal{Q}'(u) = u.$$

Inserting the derivatives into (4.21) gives (4.3). □

4.6. Properties of solutions

As in the previous chapter, we will distinguish between big and small solutions, that is solutions $u \in H^{s/2}(\mathbb{R})$ of the solitary-wave equation (4.3) with L^2-norm bigger or smaller than a constant q_0 respectively. We then find improved estimates of Γ_q, $\|u\|_{H^{s/2}}$ and the wave speed in the two cases separately. Finally, we find that all solutions are H^∞-regular and estimate $\|u\|_{L^\infty}$ for small solutions.

4.6.1. Improved estimates for $\Gamma_q, \|u\|_{H^{s/2}}, \mathcal{T}(u)$. The next three Lemmas are similar to the corresponding results in Chapter 3, and some details are left out.

LEMMA 4.19. *There exists a $\tilde{q}_0 > 0$ such that for all $q \geq \tilde{q}_0$,*

$$\Gamma_q < 0.$$

PROOF. Pick $\phi \in H^{s/2}(\mathbb{R})$ that satisfies $\mathcal{Q}(\phi) = 1$ and $\mathcal{T}(\phi) > 0$. Set

$$\phi_q(x) = \sqrt{q}\phi(x),$$

so that $\mathcal{Q}(\phi_q) = q$. We find that

$$\mathcal{J}(\phi_q) = q(\mathcal{L}(\phi) - \sqrt{q}\mathcal{T}(\phi)).$$

This expression is negative for big q, proving the existence of a $\tilde{q}_0$ as described. $\square$

LEMMA 4.20. *Let q_0 be any positive constant and let $\beta = s'/(2s'-1)$. There is a $\kappa > 0$ such that for all $q \in (0, q_0)$,*

$$\Gamma_q < m(0)q - \kappa q^{1+\beta}.$$

PROOF. Pick a function $\phi \in \mathcal{S}(\mathbb{R})$ satisfying $\mathcal{Q}(\phi) = 1$ and define

$$\phi_{q,t} = \sqrt{qt}\phi(tx),$$

for $t \in (0, 1)$. In a similar manner as in Lemma 4.11 and Lemma 3.19, we find that the following holds for some constants $C_1, C_2 > 0$

$$\begin{aligned} \mathcal{Q}(\phi_{q,t}) &= q, \\ \mathcal{L}(\phi_{q,t}) &\leq m(0)q + C_1 q t^{s'}, \\ \mathcal{T}(\phi_{q,t}) &\geq C_2 q^{3/2} t^{1/2}. \end{aligned}$$

Since $q < q_0$, we can set $t^{s'} = C_3 q^\beta$ and pick C_3 small enough to guarantee $t < 1$.

Combined with the above, we then get

$$\begin{aligned} \mathcal{J}(\phi_{q,t}) &\leq m(0)q + C_1 q t^{s'} - C_2 q^{3/2} t^{1/2} \\ &= m(0)q - q^{1+\beta}(C_2 C_3^{1/2s'} q^{1/2+\beta/2s'-\beta} - C_1 C_3) \\ &= m(0)q - \kappa q^{1+\beta}. \end{aligned}$$

In the last line, we used that $1/2 + \beta/2s' - \beta = 0$ by the definition of β and set $\kappa = C_2 C_3^{1/2s'} - C_1 C_3$. Since $1/2s' < 1$, we can always pick C_3 small enough to guarantee that $\kappa > 0$ which concludes the proof. $\square$

With this result in hand, we can estimate the size of $\mathcal{T}(u)$ and $\|u\|_{H^{s/2}}$ for small solutions.

LEMMA 4.21. *Let $\beta = s'/(2s'-1)$ and let q_0 be as in Lemma 4.19. For all $q \in (0, q_0)$, near minimizers satisfy*

$$\begin{aligned} \|u\|_{H^{s/2}(\mathbb{R})} &\eqsim q^{1/2}, \\ \mathcal{T}(u) &\eqsim q^{1+\beta} \end{aligned} \tag{4.22}$$

PROOF. We begin by finding a crude upper bound for $\mathcal{T}(u)$. For some $\gamma \in (0,2)$, Lemma 4.8 gives that

$$\begin{aligned}\mathcal{T}(u) &\lesssim q^{\frac{3-\gamma}{2}}\|u\|_{H^{s/2}}^{\gamma}\\ &\lesssim q^{\frac{3-\gamma}{2}}(m(0)q+\mathcal{L}(u)-m(0)q)^{\frac{\gamma}{2}}\\ &\lesssim q^{3/2}+q^{\frac{3-\gamma}{2}}\mathcal{T}(u)^{\frac{\gamma}{2}}.\end{aligned}$$

This implies that

$$\mathcal{T}(u)\lesssim q^{3/2}+q^{\frac{3-\gamma}{2-\gamma}}=q^{3/2}+q^{1+\frac{1}{2-\gamma}}\lesssim q^{3/2}.$$

A simple argument now gives (4.22):

$$q\lesssim \|u\|_{L^2}^2\lesssim \|u\|_{H^{s/2}}^2\eqsim \mathcal{L}(u)\lesssim \mathcal{T}(u)+q\lesssim q.$$

The lower bound for $\mathcal{T}(u)$ is a direct consequence of Lemma 4.20:

$$\mathcal{T}(u)>\mathcal{L}(u)-m(0)q+\kappa q^{1+\beta}\gtrsim q^{1+\beta}.$$

It remains to show the improved upper bound

$$\mathcal{T}(u)\lesssim q^{1+\beta}.$$

Partition $u=u_1+u_2$ where $\widehat{u_1}=\chi_{[-1,1]}\hat{u}, \widehat{u_2}=(1-\chi_{[-1,1]})\hat{u}$. Observe that by the Gagliardo-Nirenberg interpolation inequality 2.6,

$$\|u_1\|_{L^3}^3\lesssim \|u_1\|_{L^2}^{3-1/s'}\|u_1\|_{\dot{H}^{s'/2}}^{1/s'}\lesssim q^{3/2-1/2s'}(\mathcal{L}(u)-m(0)q)^{1/2s'}\lesssim q^{3/2-1/2s'}\mathcal{T}(u)^{1/2s'} \tag{4.23}$$

Suppose now that $r>0$. Then using again the Gagliardo-Nirenberg interpolation inequality, we have that

$$\begin{aligned}\|u_1\|_{\dot{H}_3^{r/2}}^3 &\lesssim \|u_1\|_{L^2}^{3-\frac{3r+1}{s'}}\|u_1\|_{\dot{H}^{s'/2}}^{\frac{3r+1}{s'}}\\ &\lesssim q^{3/2-\frac{3r+1}{2s'}}(\mathcal{L}(u)-m(0)q)^{\frac{3r+1}{2s'}}\\ &\lesssim q^{3/2-\frac{3r+1}{2s'}}(\mathcal{T}(u))^{\frac{3r+1}{2s'}}\\ &\lesssim q^{3/2-1/2s'}\mathcal{T}(u)^{1/2s'}.\end{aligned} \tag{4.24}$$

In the last line, we used that $\mathcal{T}(u)\lesssim q^{3/2}\lesssim q$ and $1/2s'<(3r+1)/2s'$ when $r>0$.

Furthermore, by Lemma 4.8, there is a $\gamma\in(0,2)$ such that

$$\|u_2\|_{H_3^{r/2}}^3\lesssim \|u_2\|_{L^2}^{3-\gamma}\|u_2\|_{H^{s/2}}^{\gamma}\lesssim \|u_2\|_{H^{s/2}}^3.$$

Since

$$\|u_2\|_{H^{s/2}}\leq \|u\|_{H^{s/2}}\lesssim q^{1/2} \tag{4.25}$$

as established in 4.22, then also

$$\|u_2\|_{H_3^{r/2}}^3\lesssim q^{3/2-1/2s'}\|u_2\|_{H^{s/2}}^{1/s'}\lesssim q^{3/2-1/2s'}(\mathcal{L}(u)-m(0)q)^{1/2s'}\lesssim q^{3/2-1/2s'}(\mathcal{T}(u))^{1/2s'}. \tag{4.26}$$

Combining (4.9), (4.23), (4.24) and (4.26), we conclude that for $r > 0$,

$$\begin{aligned}\mathcal{T}(u) &\lesssim \|u\|^3_{H_3^{r/2}} \\ &\lesssim \|u_1\|^3_{H_3^{r/2}} + \|u_2\|^3_{H_3^{r/2}} \\ &\lesssim \|u_1\|^3_{\dot{H}_3^{r/2}} + \|u_1\|^3_{L^3} + \|u_2\|_{H_3^{r/2}} \\ &\lesssim q^{3/2-1/2s'}(\mathcal{T}(u))^{1/2s'}. \end{aligned} \tag{4.27}$$

Suppose now that $r \le 0$. Using the estimate (4.8) repeatedly,

$$\langle \eta\rangle^{1/2} \lesssim \langle \eta\rangle^{1/3}\langle \xi-\eta\rangle^{1/6}\langle \xi\rangle^{1/6},$$

so that

$$\langle \xi-\eta\rangle^{r/3}\langle \eta\rangle^{r/3} = \langle \xi\rangle^{r/6}\langle \xi-\eta\rangle^{r/6}\langle \eta\rangle^{r/3}\langle \xi-\eta\rangle^{r/6}\langle \xi\rangle^{-r/6} \lesssim \langle \eta\rangle^{r/2}\langle \xi\rangle^{-r/6}\langle \xi-\eta\rangle^{r/6}. \tag{4.28}$$

Then

$$\begin{aligned}\|u_2\|^3_{H_3^{r/3}} &= \int_{\mathbb{R}} \overline{(\Lambda^{r/3}u_2)}(\Lambda^{r/3}u_2)|\Lambda^{r/3}u_2|\,dx \\ &= \int_{\mathbb{R}} (\langle\cdot\rangle^{r/3}\widehat{u_2} * \overline{\langle\cdot\rangle^{r/3}\widehat{u_2}})\mathcal{F}(|\Lambda^{r/3}u_2|)\,d\xi \\ &\lesssim \int_{\mathbb{R}^2} \langle \xi-\eta\rangle^{r/3}\langle\eta\rangle^{r/3}\overline{\widehat{u_2}(\eta)}\widehat{u_2}(\xi-\eta)\mathcal{F}(|\Lambda^{r/3}u_2|)\,d\eta\,d\xi \\ &\lesssim \int_{\mathbb{R}^2} \langle\eta\rangle^{r/2}\overline{\widehat{u_2}(\eta)}\langle \xi-\eta\rangle^{r/6}\widehat{u_2}(\xi-\eta)\langle\xi\rangle^{-r/6}\mathcal{F}(|\Lambda^{r/3}u_2|)\,d\eta\,d\xi \\ &\lesssim \int_{\mathbb{R}^2} (\langle\cdot\rangle^{r/2}\overline{\widehat{u_2}} * \langle\cdot\rangle^{r/6}\widehat{u_2})\langle\xi\rangle^{-r/6}\mathcal{F}(|\Lambda^{r/3}u_2|)\,d\xi \\ &\le \|\langle\cdot\rangle^{r/2}\widehat{u_2}\|_{L^1}\|\langle\cdot\rangle^{r/6}\widehat{u_2}\|_{L^2}\|\langle\xi\rangle^{-r/6}\mathcal{F}(|\Lambda^{r/3}u_2|\|_{L^2} \\ &\le \|u_2\|_{H^{s/2}}\|u_2\|_{L^2}\|\Lambda^{r/3}u_2\|_{H^{-r/6}} \\ &\lesssim \|u_2\|^2_{H^{s/2}}\|u_2\|_{H^{r/6}} \\ &\lesssim \|u_2\|^3_{H^{s/2}} \\ &\lesssim q^{3/2-1/2s'}\|u_2\|^{1/s'}_{H^{s/2}}, \end{aligned} \tag{4.29}$$

where in the last line we used (4.25).

We combine (4.23), (4.28) and (4.29) and conclude for that for $r \leq 0$

$$\begin{aligned}\mathcal{T}(u) &\lesssim \|u_1 + u_2\|^3_{H_3^{r/3}} \\ &\lesssim \|u_1\|^3_{H_3^{r/3}} + \|u_2\|^3_{H_3^{r/3}} \\ &\lesssim \|u_1\|^3_{L^3} + q^{3/2-1/2s'}\|u_2\|^{1/s'}_{H^{s/2}} \\ &\lesssim q^{3/2-1/2s'}(\mathcal{L}(u) - m(0)q)^{1/2s'} \\ &\lesssim q^{3/2-1/2s'}(\mathcal{T}(u))^{1/2s'}. \end{aligned} \tag{4.30}$$

Dividing (4.27) and (4.30) by $\mathcal{T}(u)^{1/2s'}$ for $r \leq 0$ and $r > 0$ respectively, we arrive at

$$\mathcal{T}(u) \lesssim q^{\frac{3/2-1/2s'}{1-1/2s'}} = q^{1+\beta},$$

which is what we wanted to show. □

4.6.2. The wave speed c.

LEMMA 4.22. *Let q_0 be any constant larger than $\tilde{q}_0$ from Lemma 4.19. Any minimizer of Γ_q solves* (4.3) *with subcritical wave speed. Furthermore,*

(i) if $q \geq q_0$, then

$$c < -\frac{m(0)}{2}. \tag{4.31}$$

(ii) If $q < q_0$, then

$$m(0) - c \lesssim q^{\beta}, \tag{4.32}$$

where $\beta = s'/(2s'-1)$.

PROOF. That a minimizer u of Γ_q is a solution to (4.3) was shown in Lemma 4.18. For any q, such a minimizer satisfies

$$T(u,u) - cu - Lu = 0,$$

Multiplying by u and integrating gives

$$3\mathcal{T}(u) - 2cq - 2\mathcal{L}(u) = 0,$$

which implies that

$$\begin{aligned} c &= \frac{\mathcal{J}(u) - 1/2\mathcal{T}(u)}{q} \\ &< m(0). \end{aligned}$$

Suppose now that $q \geq q_0$. By Lemma 4.19,

$$\mathcal{L}(u) - \mathcal{T}(u) < 0$$

so that

$$\mathcal{T}(u) > \mathcal{L}(u) \geq m(0)q.$$

Hence

$$\begin{aligned} c &= \frac{\mathcal{J}(u) - 1/2\mathcal{T}(u)}{q} \\ &\leq -\frac{m(0)}{2}, \end{aligned}$$

which is (4.31).

For $q \in (0, q_0)$, observe that for a constant $C_1 > 0$,

$$\begin{aligned} c &= \frac{\mathcal{J}(u) - 1/2\mathcal{T}(u)}{q} \\ &\leq m(0) - C_1 q^\beta, \end{aligned} \tag{4.33}$$

where the lower bound on $\mathcal{T}(u)$ is from Lemma 4.21. Furthermore, using the upper bound on $\mathcal{T}(u)$ from the same Lemma, we obtain for some constant C_2,

$$\begin{aligned} c &= \frac{\mathcal{L}(u) - 3/2\mathcal{T}(u)}{q} \\ &\geq m(0) - C_2 q^\beta. \end{aligned} \tag{4.34}$$

Combining (4.33) and (4.34) gives (4.32). □

4.6.3. Regularity of solutions. We show the regularity results stated in Theorem 4.4.

LEMMA 4.23 (Regularity of solutions). *Any solution $u \in H^{s/2}(\mathbb{R})$ of (4.3) with $\mathcal{Q}(u) = q$ is also in $H^\infty(\mathbb{R})$. Furthermore, if $q < q_0$ then*

$$\|u\|_{L^\infty}^2 \eqsim \|u\|_{H^{s/2}}^2 \eqsim q.$$

PROOF. Rewriting (4.3), we have

$$(L - c)u = T(u, u). \tag{4.35}$$

Since $c < m(0)$, then $(L - c)\colon H^t(\mathbb{R}) \to H^{t-s}(\mathbb{R})$ is an invertible linear operator with continuous inverse $(L - c)^{-1}\colon H^t(\mathbb{R}) \to H^{t+s}(\mathbb{R})$.

Assume first that $r > 0$. We wish to show that

$$T(u, u) \in H^{-r/2}(\mathbb{R}). \tag{4.36}$$

To see this, use estimate 4.9 and observe that

$$\begin{aligned}\|T(u,u)\|_{H^{-r/2}} &= \|\langle\cdot\rangle^{-r/2}\int_{\mathbb{R}} p(\cdot-\eta,\eta)\hat{u}(\cdot-\eta)\hat{u}(\eta)\,d\eta\|_{L^2}\\ &\le \|\int_{\mathbb{R}} |\langle\cdot-\eta\rangle^{r/2}\hat{u}(\cdot-\eta)||\langle\eta\rangle^{r/2}\hat{u}(\eta)|\,d\eta\|_{L^2}\\ &\le \|\langle\cdot\rangle^{r/2}\hat{u}\|_{L^1}\|\langle\cdot\rangle^{r/2}\hat{u}\|_{L^2}\\ &\le \|u\|_{H^{s/2}}^2.\end{aligned}$$

Equation (4.36) combined with (4.35) implies that $(L-c)u \in H^{-r/2}(\mathbb{R})$ which in turn implies $u \in H^{s/2+(s-r)/2}(\mathbb{R}) \subset H^{s/2+1/2}(\mathbb{R})$. Repeating the procedure with $\tilde{s} = s/2+(s-r)/2$, we obtain that $u \in H^{s/2+1}(\mathbb{R})$. Continuing this indefinitely, we conclude that $u \in H^{\infty}(\mathbb{R})$.

Now suppose $r \le 0$. Then $T(u,u) \in L^2(\mathbb{R})$ since

$$\begin{aligned}\|T(u,u)\|_{L^2}^2 &= \int_{\mathbb{R}} |\int_{\mathbb{R}} p(\xi-\eta,\eta)\hat{u}(\xi-\eta)\hat{u}(\eta)\,d\eta|^2\,d\xi\\ &\lesssim \||\langle\cdot\rangle^{r/2}\hat{u}| * |\langle\cdot\rangle^{r/2}\hat{u}|\|_{L^2}\\ &\le \|\langle\cdot\rangle^{r/2}\hat{u}\|_{L^1}\|\langle\cdot\rangle^{r/2}\hat{u}\|_{L^2}\\ &\le \|\langle\cdot\rangle^{s/2}\hat{u}\|_{L^2}\|\langle\cdot\rangle^{(r-s)/2}\|_{L^2}\|u\|_{H^{r/2}}\\ &\lesssim \|u\|_{H^{s/2}}^2.\end{aligned}$$

This implies that that $(L-c)u \in L^2(\mathbb{R})$, which in turn implies that that $u \in H^s(\mathbb{R})$. Analogously to earlier, we can then repeat the procedure with $\tilde{s} = 2s$ and obtain $u \in H^{2s}(\mathbb{R})$. Continuing indefinitely, we conclude that $u \in H^{\infty}(\mathbb{R})$ also when $r \le 0$.

In the remainder of the proof, we assume that $q \in (0, q_0)$. We know from Lemma 4.21 that then

$$\|u\|_{H^{s/2}} \eqsim q^{1/2}.$$

Using the same bootstrap argument as above, we get

$$\|u\|_{L^\infty} \lesssim \|u\|_{H^1} \lesssim \|u\|_{H^{s/2}} \lesssim q^{1/2}.$$

Furthermore, we have that

$$q = \|u^2\|_{L^1} \le \|u\|_{L^\infty}\|u\|_{L^1} \lesssim \|u\|_{L^\infty}\|\langle\cdot\rangle^{-1}\|_{L^2}\|u\|_{H^1} \lesssim \|u\|_{L^\infty}\|u\|_{H^{s/2}}$$

so that

$$\|u\|_{L^\infty} \gtrsim \frac{q}{\|u\|_{H^{s/2}}} \gtrsim q^{1/2}.$$

□

We have proved all the components needed to establish Theorem 4.4: Lemma 4.5.3 guarantees the existence of minimizers of Γ_q for all $q > 0$, while Lemma 4.18

shows that these minimizers satisfy (4.3). The regularity of the solutions follows from Lemma 4.23 and the estimates on the wave speed c are found in Lemma 4.22.

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