

SEGAL SPACES, SPANS, AND SEMICATEGORIES

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ABSTRACT. We show that Segal spaces, and more generally category objects in an ∞ -category \mathcal{C} , can be identified with associative algebras in the double ∞ -category of spans in \mathcal{C} . We use this observation to prove that “having identities” is a property of a non-unital (∞, n) -category.

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1. INTRODUCTION

A “semicategory” or non-unital category is a category without identity morphisms. It is an easy exercise to show that “having identities” is a property of a semicategory, and “preserving identities” is a property of functors of semicategories. More precisely, the forgetful functor $\text{Cat} \rightarrow \text{Semicat}$ gives an equivalence between Cat and a subcategory of Semicat .

The analogues of this statement for higher categories turn out to be very useful: To define particular examples of higher categories or functors between them, it can be extremely convenient to first ignore the identities and then at the end check that the resulting non-unital structure has the required property. For $(\infty, 1)$ -categories (which we will refer to as ∞ -categories), such a result is already known: it is due to Harpaz [Har15] in the context of Segal spaces¹, and for quasicategories it is a combination of work of Tanaka [Tan18] and Steimle [Ste18]. Similar results have also been proved for other higher-categorical structures, including A_∞ -categories (see [LM06] for a comparison of different notions of weak units in this setting) and monoidal 2-categories [JK13].

The goal of the present paper is to show that “having identities” is also a property of (∞, n) -categories for all n :

Theorem 1.1. *Let $\text{Seg}^n(\mathcal{S})$ denote the ∞ -category of n -fold Segal spaces and $\text{Seg}_{\text{nu}}^n(\mathcal{S})$ its non-unital analogue. Then the forgetful functor $\text{Seg}^n(\mathcal{S}) \rightarrow \text{Seg}_{\text{nu}}^n(\mathcal{S})$ induces an equivalence*

$$\text{Seg}^n(\mathcal{S}) \xrightarrow{\sim} \text{Seg}_{\text{qu}}^n(\mathcal{S})$$

where $\text{Seg}_{\text{qu}}^n(\mathcal{S}) \subseteq \text{Seg}_{\text{nu}}^n(\mathcal{S})$ is a subcategory of quasi-unital n -fold Segal spaces and quasi-unital functors between them.

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¹See Warning 4.22 for the precise relation of our result to that of Harpaz.

We will prove this in Section 4 by first proving the case $n = 1$ for category objects (or internal ∞ -categories) in any ∞ -category with finite limits; the general statement then follows easily by iterating this case.

In the case $n = 1$ we will deduce the theorem from the analogous statement for non-unital associative algebras in monoidal ∞ -categories, which has been proved by Lurie [Lur17]. To do so, we must first identify category objects as certain associative algebras. For ordinary categories, it seems to have been first observed by Bénabou [Bén67] that a category can be viewed as an associative algebra (or monad) in a 2-category $\text{Span}^+(\text{Set})$ of spans of sets; this has

- sets as objects,
- spans $I \leftarrow S \rightarrow J$ as 1-morphisms from I to J , with composition given by taking pullbacks, i.e.

$$\begin{array}{c} S \\ \swarrow \quad \searrow \\ J \quad \quad K \end{array} \circ \begin{array}{c} T \\ \swarrow \quad \searrow \\ I \quad \quad J \end{array} := \begin{array}{c} T \times_J S \\ \swarrow \quad \searrow \\ I \quad \quad K \end{array},$$

- and morphisms of spans

$$\begin{array}{ccc} & S & \rightarrow & S' \\ & \swarrow & & \searrow \\ I & & & & J \end{array}$$

as 2-morphisms, composing in the obvious way.

In particular, a category with S as its set of objects is the same thing as an associative algebra in the “double slice” $\text{Set}_{/S,S}$ with the tensor product defined by pullbacks over S . However, functors are not the same thing as morphisms of algebras in $\text{Span}^+(\text{Set})$. To remedy this, we can upgrade to a double category $\text{SPAN}^+(\text{Set})$ whose objects are sets, vertical morphisms are functions, horizontal morphisms are spans, and whose squares are diagrams of the form

$$\begin{array}{ccccc} \bullet & \leftarrow & \bullet & \rightarrow & \bullet \\ \downarrow & & \downarrow & & \downarrow \\ \bullet & \leftarrow & \bullet & \rightarrow & \bullet \end{array}$$

(This example of a double category is discussed in some detail in [GP99, §3.2]; the earliest reference is perhaps [Bur71, Remarque on p. 294].) We can then consider associative algebras (also known as monoids or monads) in the double category $\text{SPAN}^+(\text{Set})$, which are the same thing as algebras in its horizontal 2-category and so again give categories. However, the vertical morphisms give a new notion of morphisms of algebras, which in this case recovers functors between categories; thus Cat is equivalent to the category of associative algebras in $\text{SPAN}^+(\text{Set})$. (This observation can be found in [Lei04, Example 5.3.5], [Shu08, Example 11.2], and [FGK11, Example 2.6].) In Section 3 we prove an ∞ -categorical version of this statement, using the double ∞ -category of spans constructed in [Hau18]:

Theorem 1.2. *Let \mathcal{C} be an ∞ -category with finite limits. There is an equivalence of ∞ -categories*

$$\text{Cat}(\mathcal{C}) \simeq \text{Alg}(\text{SPAN}^+(\mathcal{C}))$$

between category objects in \mathcal{C} and associative algebras in the double ∞ -category of spans in \mathcal{C} .

2. PRELIMINARIES

In this section we briefly review the higher-algebraic structures we will make use of below.

Notation 2.1. We write $\mathbf{\Delta}$ for the simplex category of ordered sets $[n] := \{0, \dots, n\}$. A morphism $\phi: [m] \rightarrow [n]$ is called *inert* if it is the inclusion of a subinterval, i.e. $\phi(i) = \phi(0) + i$ for $i = 0, \dots, n$. For $0 \leq i \leq j \leq n$ we write $\rho_{i,j}: [j-i] \rightarrow [n]$ for the inert morphism with $\rho_{i,j}(0) = i$, $\rho_{i,j}(j-i) = j$.

Definition 2.2. Let \mathcal{C} be an ∞ -category with pullbacks. A *category object* in \mathcal{C} is a simplicial object $X: \mathbf{\Delta}^{\text{op}} \rightarrow \mathcal{C}$ such that for all $[n] \in \mathbf{\Delta}$ the morphism

$$X_n \rightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1,$$

induced by the morphisms $\rho_{i,(i+1)}$ and $\rho_{i,i}$, is an equivalence. We write $\text{Cat}(\mathcal{C})$ for the full subcategory of $\text{Fun}(\mathbf{\Delta}^{\text{op}}, \mathcal{C})$ spanned by the category objects. Category objects in the ∞ -category \mathcal{S} of spaces are called *Segal spaces* [Rez01].

Remark 2.3. Category objects in \mathcal{C} model the algebraic structure of a (homotopy-coherent) category internal to \mathcal{C} : If we think of X_0 as the objects of X and X_1 as the morphisms then we have:

- $d_1, d_0: X_1 \rightarrow X_0$, assigning source and target objects to morphisms,
- $s_0: X_0 \rightarrow X_1$, assigning identity morphisms to objects,
- $X_1 \times_{X_0} X_1 \xleftarrow{\sim} X_2 \xrightarrow{d_1} X_1$, assigning composites to composable pairs of morphisms.

The remaining structure in the simplicial object X ensures that the composition is homotopy-coherently associative and unital.

Remark 2.4. If \mathcal{C} has a terminal object $*$, we can identify the category objects $X: \mathbf{\Delta}^{\text{op}} \rightarrow \mathcal{C}$ such that $X_0 \simeq *$ with *associative monoids* in \mathcal{C} .

Definition 2.5. A *double ∞ -category* is a cocartesian fibration $\mathcal{M} \rightarrow \mathbf{\Delta}^{\text{op}}$ such that the corresponding functor $\mathbf{\Delta}^{\text{op}} \rightarrow \text{Cat}_\infty$ is a category object. A *monoidal ∞ -category* is a double ∞ -category \mathcal{M} such that \mathcal{M}_0 is contractible, corresponding to an associative monoid in Cat_∞ .

A double ∞ -category is thus an ∞ -categorical analogue of a category internal to categories, or a double category. This notion has a useful generalization:

Definition 2.6. A *generalized non-symmetric ∞ -operad* is a functor $p: \mathcal{O} \rightarrow \mathbf{\Delta}^{\text{op}}$ such that:

- (i) For every object X in $\mathcal{O}_n := \mathcal{O} \times_{\mathbf{\Delta}^{\text{op}}} \{[n]\}$, and every inert morphism $\phi: [m] \rightarrow [n]$ in $\mathbf{\Delta}$, there exists a p -cocartesian morphism $X \rightarrow \phi_! X$ lying over ϕ .
- (ii) For every object $[n] \in \mathbf{\Delta}$, the functor

$$\mathcal{O}_n \rightarrow \mathcal{O}_1 \times_{\mathcal{O}_0} \cdots \times_{\mathcal{O}_0} \mathcal{O}_1$$

induced by the cocartesian morphisms over the maps $\rho_{i,i+1}$ and $\rho_{i,i}$, is an equivalence.

- (iii) Given X in \mathcal{O}_n , choose compatible cocartesian lifts $X \rightarrow X_{i,j}$ over $\rho_{i,j}$. Then for any $Y \in \mathcal{O}_m$, the commutative square

$$\begin{array}{ccc} \text{Map}_{\mathcal{O}}(Y, X) & \longrightarrow & \text{Map}_{\mathcal{O}}(Y, X_{0,1}) \times_{\text{Map}_{\mathcal{O}}(Y, X_{1,1})} \cdots \times_{\text{Map}_{\mathcal{O}}(Y, X_{n-1, n-1})} \text{Map}_{\mathcal{O}}(Y, X_{n-1, n}) \\ \downarrow & & \downarrow \\ \text{Map}_{\mathbf{\Delta}^{\text{op}}}([m], [n]) & \longrightarrow & \text{Map}_{\mathbf{\Delta}^{\text{op}}}([m], [1]) \times_{\text{Map}_{\mathbf{\Delta}^{\text{op}}}([m], [0])} \cdots \times_{\text{Map}_{\mathbf{\Delta}^{\text{op}}}([m], [0])} \text{Map}_{\mathbf{\Delta}^{\text{op}}}([m], [1]) \end{array}$$

is cartesian.

Remark 2.7. Generalized non-symmetric ∞ -operads are an ∞ -categorical analogue of the *virtual double categories* of [CS10] or *fc-multicategories* of [Lei02]; see [GH15, §2] for further discussion and motivation. We can identify the double ∞ -categories as the generalized non-symmetric ∞ -operads that are cocartesian fibrations.

Definition 2.8. A *non-symmetric ∞ -operad* is a generalized non-symmetric ∞ -operad \mathcal{O} such that \mathcal{O}_0 is contractible.

Definition 2.9. Let \mathcal{O} be a generalized non-symmetric ∞ -operad, and suppose \mathcal{C} is an ∞ -category with pullbacks. A *Segal \mathcal{O} -object* in \mathcal{C} is a functor $\Phi: \mathcal{O} \rightarrow \mathcal{C}$ such that for all $X \in \mathcal{O}_n$ the natural map

$$\Phi(X) \rightarrow \Phi(X_{0,1}) \times_{\Phi(X_{1,1})} \cdots \times_{\Phi(X_{n-1,n-1})} \Phi(X_{n-1,n})$$

is an equivalence, where $X \rightarrow X_{i,j}$ is a cocartesian morphism over $\rho_{i,j}$. We write $\text{Seg}_{\mathcal{O}}(\mathcal{C})$ for the full subcategory of $\text{Fun}(\mathcal{O}, \mathcal{C})$ spanned by the Segal \mathcal{O} -objects.

Lemma 2.10. *Let \mathcal{C} be an ∞ -category with finite limits and \mathcal{O} a generalized non-symmetric ∞ -operad, and let $i_{\mathcal{O}}$ denote the inclusion $\mathcal{O}_0 \hookrightarrow \mathcal{O}$. Then right Kan extension along $i_{\mathcal{O}}$ gives a functor*

$$i_{\mathcal{O},*}: \text{Fun}(\mathcal{O}_0, \mathcal{C}) \rightarrow \text{Seg}_{\mathcal{O}}(\mathcal{C}),$$

right adjoint to the restriction $i_{\mathcal{O}}^*: \text{Seg}_{\mathcal{O}}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{O}_0, \mathcal{C})$.

Proof. For $F: \mathcal{O}_0 \rightarrow \mathcal{C}$ and $X \in \mathcal{O}_n$, we have

$$i_{\mathcal{O},*}F(X) \simeq \lim_{(Y, X \rightarrow Y) \in \mathcal{O}_0 \times_{\mathcal{O}} \mathcal{O}_{X/}} F(Y),$$

provided this limit exists in \mathcal{C} . If $X \rightarrow X_{i,i}$ is a cocartesian morphism over $\rho_{i,i}: [0] \rightarrow [n]$ then the discrete set $\{X \rightarrow X_{i,i}\}$ is a coinital subcategory of $\mathcal{O}_0 \times_{\mathcal{O}} \mathcal{O}_{X/}$, and so we have

$$i_{\mathcal{O},*}F(X) \simeq \prod_{i=0}^n F(X_{i,i}),$$

which exists provided \mathcal{C} has finite limits and clearly gives a Segal object. Since the right Kan extension $i_{\mathcal{O},*}$ is the right adjoint to $i_{\mathcal{O}}^*: \text{Fun}(\mathcal{O}, \mathcal{C}) \rightarrow \text{Fun}(\mathcal{O}_0, \mathcal{C})$, it follows that the adjunction restricts to the full subcategory $\text{Seg}_{\mathcal{O}}(\mathcal{C})$. \square

Remark 2.11. The object $i_{\mathcal{O},*}F$ is the terminal Segal \mathcal{O} -object whose restriction to \mathcal{O}_0 is F . For $\mathcal{C} = \mathcal{S}$, we can think of the object $i_{\mathcal{O},*}F(X) \simeq \prod_{i=0}^n F(X_{i,i})$ as the space of “labels” over \mathcal{O}_0 that a Segal \mathcal{O} -object Φ in \mathcal{S} with $\Phi|_{\mathcal{O}_0} \simeq F$ would assign to points of $\Phi(X)$ (with the assignment of such given by the unit map $\Phi \rightarrow i_{\mathcal{O},*}F$). For example, when \mathcal{O} is Δ^{op} (where $(\Delta^{\text{op}})_0$ is just a point) we have for $X \in \mathcal{S} \simeq \text{Fun}((\Delta^{\text{op}})_0, \mathcal{S})$ that $(i_{\Delta^{\text{op},*}}X)_n \simeq X^{\times n+1}$; if $\Phi: \Delta^{\text{op}} \rightarrow \mathcal{S}$ is a Segal space with $\Phi_0 \simeq X$ then we can think of Φ_n as a space of strings of n composable morphisms, and the unit map $\Phi_n \rightarrow (i_{\Delta^{\text{op},*}}X)_n \simeq X^{\times n+1}$ assigns to such a string the list of $n+1$ objects appearing in it.

Proposition 2.12. *Let \mathcal{C} be an ∞ -category with finite limits and \mathcal{O} a generalized non-symmetric ∞ -operad, and let $i_{\mathcal{O}}$ denote the inclusion $\mathcal{O}_0 \hookrightarrow \mathcal{O}$. Then the functor*

$$i_{\mathcal{O}}^*: \text{Seg}_{\mathcal{O}}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{O}_0, \mathcal{C})$$

is a cartesian fibration. For $\Phi \in \text{Seg}_{\mathcal{O}}(\mathcal{C})$ and $\eta: F \rightarrow i_{\mathcal{O}}^*\Phi$, the cartesian morphism over η is given by the pullback

$$\begin{array}{ccc} \eta^*\Phi & \longrightarrow & \Phi \\ \downarrow & & \downarrow \\ i_{\mathcal{O},*}F & \xrightarrow{i_{\mathcal{O},*\eta}} & i_{\mathcal{O},*}i_{\mathcal{O}}^*\Phi \end{array}$$

in $\text{Fun}(\mathcal{O}, \mathcal{C})$.

Proof. The functor $i_{\mathcal{O}}^*: \text{Seg}_{\mathcal{O}}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{O}_0, \mathcal{C})$ has a right adjoint by Lemma 2.10. To see that $i_{\mathcal{O}}^*$ is a cartesian fibration, we apply the criterion of [Hau17, Corollary 4.52]. We must check that for $\Phi \in \text{Seg}_{\mathcal{O}}(\mathcal{C})$ and $\eta: F \rightarrow i_{\mathcal{O}}^*\Phi$, if we define $\eta^*\Phi$ by the pullback square above, then the composite $i_{\mathcal{O}}^*\eta^*\Phi \rightarrow i_{\mathcal{O}}^*i_{\mathcal{O},*}F \rightarrow F$ is an equivalence.

Since $i_{\mathcal{O}}$ is fully faithful we have $i_{\mathcal{O}}^*i_{\mathcal{O},*} \simeq \text{id}$ and as $i_{\mathcal{O}}^*$ preserves limits we see that $i_{\mathcal{O}}^*\eta^*\Phi$ is the pullback

$$\begin{array}{ccc} i_{\mathcal{O}}^*\eta^*\Phi & \longrightarrow & i_{\mathcal{O}}^*\Phi \\ \downarrow & & \parallel \\ F & \longrightarrow & i_{\mathcal{O}}^*\Phi, \end{array}$$

whence $i_{\mathcal{O}}^*\eta^*\Phi \rightarrow F$ is indeed an equivalence. The characterization of cartesian morphisms follows from [Hau17, Proposition 4.51]. \square

Definition 2.13. A morphism of generalized non-symmetric ∞ -operads is a commutative triangle

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{f} & \mathcal{P} \\ & \searrow & \swarrow \\ & \Delta^{\text{op}}, & \end{array}$$

where f preserves cocartesian morphisms over inert maps in Δ^{op} . We write $\text{Opd}_{\infty}^{\text{gns}}$ for the ∞ -category of generalized non-symmetric ∞ -operads, defined as a subcategory of $\text{Cat}_{\infty/\Delta^{\text{op}}}$. A morphism of generalized non-symmetric ∞ -operads from \mathcal{O} to \mathcal{P} is also called an \mathcal{O} -algebra in \mathcal{P} , and we write $\text{Alg}_{\mathcal{O}}(\mathcal{P})$ for the ∞ -category of \mathcal{O} -algebras in \mathcal{P} , defined as a full subcategory of $\text{Fun}_{/\Delta^{\text{op}}}(\mathcal{O}, \mathcal{P})$.

Definition 2.14. For the terminal (generalized) non-symmetric ∞ -operad Δ^{op} , we refer to Δ^{op} -algebras in a generalized non-symmetric ∞ -operad \mathcal{O} as *associative algebras*, and write

$$\text{Alg}(\mathcal{O}) := \text{Alg}_{\Delta^{\text{op}}}(\mathcal{O}).$$

Definition 2.15. For \mathcal{C} an ∞ -category, we write $\Delta_{\mathcal{C}}^{\text{op}} \rightarrow \Delta^{\text{op}}$ for the cocartesian fibration corresponding to the right Kan extension $i_{\Delta^{\text{op}},*}\mathcal{C}: \Delta^{\text{op}} \rightarrow \text{Cat}_{\infty}$. Lemma 2.10 shows that $\Delta_{\mathcal{C}}^{\text{op}}$ is a double ∞ -category.

If $\mathcal{M} \rightarrow \Delta^{\text{op}}$ is any double ∞ -category then there is a canonical morphism $\mathcal{M} \rightarrow \Delta_{\mathcal{M}_0}^{\text{op}}$ over Δ^{op} , corresponding to the unit morphism $M \rightarrow i_{\Delta^{\text{op}},*}i_{\Delta^{\text{op}}}^*M$ where M is the functor $\Delta^{\text{op}} \rightarrow \text{Cat}_{\infty}$ associated to \mathcal{M} . This preserves all cocartesian morphisms, and so is in particular a morphism of generalized non-symmetric ∞ -operads.

Lemma 2.16. *For any generalized non-symmetric ∞ -operad \mathcal{O} , we have a natural equivalence*

$$\text{Alg}_{\mathcal{O}}(\Delta_{\mathcal{C}}^{\text{op}}) \simeq \text{Fun}(\mathcal{O}_0, \mathcal{C}),$$

given by restriction to the fibre over $[0]$.

Proof. Since we can replace \mathcal{O} by $\mathcal{O} \times \Delta^n$, it suffices to show that we have a natural equivalence of mapping spaces

$$\text{Map}_{\text{Opd}_{\infty}^{\text{gns}}}(\mathcal{O}, \Delta_{\mathcal{C}}^{\text{op}}) \simeq \text{Map}_{\text{Cat}_{\infty}}(\mathcal{O}_0, \mathcal{C}).$$

The generalized non-symmetric ∞ -operad \mathcal{O} has an enveloping double ∞ -category (see [Hau17, §A.8]) $\text{Env}(\mathcal{O})$ with a morphism of generalized non-symmetric ∞ -operads $\mathcal{O} \rightarrow \text{Env}(\mathcal{O})$ such that composition with this gives an equivalence between \mathcal{O} -algebras in a double ∞ -category \mathcal{M} and functors $\text{Env}(\mathcal{O}) \rightarrow \mathcal{M}$ that preserve all

cocartesian morphisms. Moreover, $\text{Env}(\mathcal{O})_0 \simeq \mathcal{O}_0$. It therefore suffices to prove that the natural map

$$\text{Map}_{\text{Cat}_{\infty/\Delta^{\text{op}}}}^{\text{cocart}}(\mathcal{M}, \Delta_{\mathcal{C}}^{\text{op}}) \rightarrow \text{Map}_{\text{Cat}_{\infty}}(\mathcal{M}_0, \mathcal{C})$$

is an equivalence, where \mathcal{M} is now a double ∞ -category. If M is the corresponding functor $\Delta^{\text{op}} \rightarrow \text{Cat}_{\infty}$, then we can rewrite this as

$$\begin{aligned} \text{Map}_{\text{Cat}_{\infty/\Delta^{\text{op}}}}^{\text{cocart}}(\mathcal{M}, \Delta_{\mathcal{C}}^{\text{op}}) &\simeq \text{Map}_{\text{Fun}(\Delta^{\text{op}}, \text{Cat}_{\infty})}(M, i_*\mathcal{C}) \\ &\simeq \text{Map}_{\text{Cat}_{\infty}}(i^*M, \mathcal{C}) \\ &\simeq \text{Map}_{\text{Cat}_{\infty}}(\mathcal{M}_0, \mathcal{C}), \end{aligned}$$

as required. \square

Remark 2.17. Let \mathcal{M} be a double ∞ -category and X an object of \mathcal{M}_0 . Then X induces a functor $\Delta^{\text{op}} \rightarrow \Delta_{\mathcal{M}_0}^{\text{op}}$ over Δ^{op} (corresponding to the morphism of right Kan extensions $i_{\Delta^{\text{op}},*}\{X\} \rightarrow i_{\Delta^{\text{op}},*}\mathcal{M}_0$), which preserves cocartesian morphisms. We can then define a monoidal ∞ -category \mathcal{M}_X^{\otimes} as the pullback

$$\begin{array}{ccc} \mathcal{M}_X^{\otimes} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \Delta^{\text{op}} & \longrightarrow & \Delta_{\mathcal{M}_0}^{\text{op}} \end{array}$$

of ∞ -categories, which is also a pullback of cocartesian fibrations over Δ^{op} . If we think of objects of \mathcal{M}_1 as ‘‘horizontal morphisms’’ in the double ∞ -category, then this gives a monoidal structure on the ∞ -category $\mathcal{M}_1(X, X) := \mathcal{M}_1 \times_{\mathcal{M}_0 \times \mathcal{M}_0} \{(X, X)\}$ of horizontal endomorphisms of X , given by composition of horizontal morphisms.

3. CATEGORY OBJECTS AS ALGEBRAS IN SPANS

In this section we will prove that category objects in an ∞ -category \mathcal{C} can be identified with associative algebras in the double ∞ -category of spans in \mathcal{C} . We first recall the construction of this double ∞ -category, following [Hau18] (see also [Bar17, DK19, GR17] for alternative approaches).

Definition 3.1. Let Σ^n denote the partially ordered set of pairs (i, j) with $0 \leq i \leq j \leq n$, where $(i, j) \leq (i', j')$ if $i \leq i' \leq j' \leq j$. These give a functor $\Sigma^\bullet: \Delta \rightarrow \text{Cat}$, where for $\phi: [n] \rightarrow [m]$ the functor $\Sigma^n \rightarrow \Sigma^m$ takes (i, j) to $(\phi(i), \phi(j))$. If \mathcal{C} is an ∞ -category, we write $\overline{\text{SPAN}}^+(\mathcal{C}) \rightarrow \Delta^{\text{op}}$ for the cocartesian fibration corresponding to the functor

$$\text{Fun}(\Sigma^\bullet, \mathcal{C}): \Delta^{\text{op}} \rightarrow \text{Cat}_{\infty}.$$

If \mathcal{C} is an ∞ -category with pullbacks, we write $\text{SPAN}^+(\mathcal{C})$ for the full subcategory of $\overline{\text{SPAN}}^+(\mathcal{C})$ spanned by the objects $F: \Sigma^n \rightarrow \mathcal{C}$ such that the canonical morphism

$$F(i, j) \rightarrow F(i, i+1) \times_{F(i+1, i+1)} \cdots \times_{F(j-1, j-1)} F(j-1, j)$$

is an equivalence for all i, j .

Proposition 3.2 ([Hau18, Proposition 5.14]). *For any ∞ -category \mathcal{C} with pullbacks, the restricted functor $\text{SPAN}^+(\mathcal{C}) \rightarrow \Delta^{\text{op}}$ is a double ∞ -category.*

Definition 3.3. For $X \in \mathcal{C}$, we can define a monoidal ∞ -category $\mathcal{C}_{/X, X}^{\otimes} := \text{SPAN}^+(\mathcal{C})_X$ as in Remark 2.17. This gives a monoidal structure on the ∞ -category $\mathcal{C}_{/X, X} \simeq \text{SPAN}^+(\mathcal{C})_1(X, X)$ of objects of \mathcal{C} equipped with two maps to X . The tensor product of $X \leftarrow Y \rightarrow X$ and $X \leftarrow Z \rightarrow X$ is defined by the pullback $X \leftarrow Y \times_X Z \rightarrow X$, and the unit is $X \xleftarrow{\text{id}} X \xrightarrow{\text{id}} X$.

Definition 3.4. Let $p: \widehat{\Sigma} \rightarrow \Delta^{\text{op}}$ denote the cartesian fibration for the functor $\Sigma^\bullet: \Delta \rightarrow \text{Cat}$. We can identify objects of $\widehat{\Sigma}$ with pairs $([n], (i, j))$ where $[n] \in \Delta$ and $0 \leq i \leq j \leq n$; a morphism $([n], (i, j)) \rightarrow ([m], (i', j'))$ is given by a morphism $\phi: [m] \rightarrow [n]$ in Δ such that $(i, j) \leq (\phi(i'), \phi(j'))$ in the partially ordered set Σ^n . Note that this morphism is cartesian precisely when $(i, j) = (\phi(i'), \phi(j'))$.

Proposition 3.5. For any ∞ -category \mathcal{J} over Δ^{op} , there is a natural equivalence

$$\text{Fun}_{/\Delta^{\text{op}}}(\mathcal{J}, \overline{\text{SPAN}}^+(\mathcal{C})) \simeq \text{Fun}(\mathcal{J} \times_{\Delta^{\text{op}}} \widehat{\Sigma}, \mathcal{C}).$$

A functor $F: \mathcal{J} \rightarrow \overline{\text{SPAN}}^+(\mathcal{C})$ takes a morphism $\phi: x \rightarrow y$ in \mathcal{J} to a cocartesian morphism in $\overline{\text{SPAN}}^+(\mathcal{C})$ if and only if the corresponding functor $F': \mathcal{J} \times_{\Delta^{\text{op}}} \widehat{\Sigma} \rightarrow \mathcal{C}$ (which satisfies $F'(x, (i, j)) \simeq F(x)(i, j)$) takes all morphisms (ϕ, γ) where γ in $\widehat{\Sigma}$ is cartesian to equivalences in \mathcal{C} .

Proof. The equivalence follows from [GHN17, Proposition 7.3], which identifies $\overline{\text{SPAN}}^+(\mathcal{C})$ with the cocartesian fibration defined in [Lur09, Corollary 3.2.2.13(1)] by this universal property. The second statement then follows from the description of the cocartesian morphisms in [Lur09, Corollary 3.2.2.13(2)]. \square

Definition 3.6. We define a functor $\Pi: \widehat{\Sigma} \rightarrow \Delta^{\text{op}}$ by setting $\Pi([n], (i, j)) := [j - i]$ and sending a map $([n], (i, j)) \rightarrow ([m], (i', j'))$ lying over $\phi: [m] \rightarrow [n]$ to the map $[j' - i'] \rightarrow [j - i]$ given by $t \mapsto \phi(t + i') - i$. (In other words, we restrict ϕ to a map $\{i', i' + 1, \dots, j'\} \rightarrow \{i, i + 1, \dots, j\}$.)

Definition 3.7. We can define another functor $\Psi: \Delta^{\text{op}} \rightarrow \widehat{\Sigma}$ by sending $[n]$ to $([n], (0, n))$ and a morphism $\phi: [m] \rightarrow [n]$ in Δ to the morphism $([n], (0, n)) \rightarrow ([m], (0, m))$ lying over ϕ . Observe that we have $\Pi\Psi \simeq \text{id}_{\Delta^{\text{op}}}$. We can also define a natural transformation $\eta: \text{id}_{\widehat{\Sigma}} \rightarrow \Psi\Pi$ by taking $\eta_{([n], (i, j))}$ to be the map $([n], (i, j)) \rightarrow ([j - i], (0, j - i))$ lying over $\rho_{i, j}$. We also have $p\Psi = \text{id}_{\Delta^{\text{op}}}$, and a natural transformation $\epsilon: \Psi p \rightarrow \text{id}_{\widehat{\Sigma}}$ given by the natural maps $([n], (0, n)) \rightarrow ([n], (i, j))$.

Proposition 3.8. The functor $\Pi: \widehat{\Sigma} \rightarrow \Delta^{\text{op}}$ exhibits Δ^{op} as the localization of $\widehat{\Sigma}$ at the set I of cartesian morphisms that lie over inert maps in Δ^{op} .

Proof. Let W be the set of morphisms in $\widehat{\Sigma}$ that are mapped to isomorphisms (i.e. identity morphisms) by Π . A morphism $([n], (i, j)) \rightarrow ([m], (i', j'))$ over $\phi: [m] \rightarrow [n]$ is in W if and only if we have $j' - i' = j - i$ and $\phi(i' + t) = i + t$ for $t = 0, \dots, j' - i'$. In this case we have a commutative triangle

$$\begin{array}{ccc} ([n], (i, j)) & \xrightarrow{\quad\quad\quad} & ([m], (i', j')) \\ & \searrow & \swarrow \\ & ([j - i], (0, j - i)) & \end{array}$$

where the two diagonal morphisms are in I . Thus by the 2-of-3 property for equivalences, any functor $\widehat{\Sigma} \rightarrow \mathcal{C}$ that takes the maps in I to equivalences takes all the maps in W to equivalences. The localizations of $\widehat{\Sigma}$ at I and W are therefore the same. On the other hand, the components of the natural transformation η are all in W , so using η we see that composition with Ψ is an inverse to

$$\Pi^*: \text{Fun}(\Delta^{\text{op}}, \mathcal{C}) \rightarrow \text{Fun}_W(\widehat{\Sigma}, \mathcal{C}),$$

for any ∞ -category \mathcal{C} , where $\text{Fun}_W(\widehat{\Sigma}, \mathcal{C})$ denotes the full subcategory of functors $\widehat{\Sigma} \rightarrow \mathcal{C}$ that take the morphisms in W to equivalences. In other words, Π exhibits Δ^{op} as the localization of $\widehat{\Sigma}$ at W . \square

Proposition 3.9. *Suppose $f: \mathcal{J} \rightarrow \Delta^{\text{op}}$ is a functor such that \mathcal{J} has f -cocartesian morphisms over inert maps in Δ^{op} . Then there is a functor $\bar{\Pi}: \mathcal{J} \times_{\Delta^{\text{op}}} \widehat{\Sigma} \rightarrow \mathcal{J}$ (where the fibre product is over p) lying over Π , which exhibits \mathcal{J} as the localization of $\mathcal{J} \times_{\Delta^{\text{op}}} \widehat{\Sigma}$ at the set $I_{\mathcal{J}}$ of morphisms $(x, (i, j)) \rightarrow (x', (i', j'))$ such that $f(x) \rightarrow f(x')$ is inert, $I \rightarrow I'$ is cocartesian, and $(f(I), (i, j)) \rightarrow (f(I'), (i', j'))$ is cartesian.*

Proof. The functor $\Psi: \Delta^{\text{op}} \rightarrow \widehat{\Sigma}$ satisfies $p \circ \Psi \cong \text{id}_{\Delta^{\text{op}}}$, and so induces a functor $\bar{\Psi}: \mathcal{J} \rightarrow \mathcal{J} \times_{\Delta^{\text{op}}} \widehat{\Sigma}$. If $\bar{p}: \mathcal{J} \times_{\Delta^{\text{op}}} \widehat{\Sigma} \rightarrow \mathcal{J}$ is the projection (which lies over $p: \widehat{\Sigma} \rightarrow \Delta^{\text{op}}$), then we have an equivalence $\bar{p}\bar{\Psi} \simeq \text{id}$, and we also get a natural transformation $\bar{\epsilon}: \bar{\Psi}\bar{p} \rightarrow \text{id}$ over ϵ .

Since the components of $\eta: \widehat{\Sigma} \times \Delta^1 \rightarrow \widehat{\Sigma}$ lie over inert morphisms in Δ^{op} , there is a unique cocartesian lift of η to a natural transformation $\bar{\eta}: \mathcal{J} \times_{\Delta^{\text{op}}} \widehat{\Sigma} \rightarrow \mathcal{J} \times_{\Delta^{\text{op}}} \widehat{\Sigma}$, where $\bar{\eta}_0$ is the identity and $\bar{\eta}(x, (i, j))$ is $(x, (i, j)) \rightarrow (x_{i,j}, (0, j-i))$ where $x \rightarrow x_{i,j}$ is a cocartesian morphism over $\rho_{i,j}$. (This follows from the lifting property obtained by combining [Lur09, Propositions 3.1.1.6 and 3.1.2.3].) We define $\bar{\Pi}: \mathcal{J} \times_{\Delta^{\text{op}}} \widehat{\Sigma} \rightarrow \mathcal{J}$ to be the composite $\bar{p}\bar{\eta}_1$. Then $\bar{\Pi}$ lies over $p\Psi\Pi$, which is Π as $p\Psi = \text{id}_{\Delta^{\text{op}}}$. We can identify $\bar{\eta}_1$ with $\bar{\Psi}\bar{\Pi}$ since

$$\bar{\Psi}\bar{\Pi} \simeq \bar{\Psi}\bar{p}\bar{\eta}_1 \xrightarrow{\bar{\epsilon}\bar{\eta}_1} \bar{\eta}_1$$

is an equivalence. Moreover, $\bar{\eta}\bar{\Psi}: \bar{\Psi} \rightarrow \bar{\eta}_1\bar{\Psi}$ is an equivalence, being given by cocartesian morphisms over identities, hence $\bar{\Pi}\bar{\Psi} \simeq \bar{p}\bar{\Psi} \simeq \text{id}$.

Let $W_{\mathcal{J}}$ be the set of morphisms in $\mathcal{J} \times_{\Delta^{\text{op}}} \widehat{\Sigma}$ that are sent to equivalences by $\bar{\Pi}$. If $(x, (i, j)) \rightarrow (x', (i', j'))$ is such a morphism, then we have a commutative square

$$\begin{array}{ccc} (x, (i, j)) & \longrightarrow & (x', (i', j')) \\ \downarrow & & \downarrow \\ (x_{i,j}, (0, j-i)) & \xrightarrow{\sim} & (x'_{i',j'}, (0, j'-i')), \end{array}$$

where the vertical maps are in $I_{\mathcal{J}}$. By the 2-of-3 property for equivalences, this means that any functor that takes the morphisms in $I_{\mathcal{J}}$ to equivalences must take all morphisms in $W_{\mathcal{J}}$ to equivalences. Thus the localizations of $\mathcal{J} \times_{\Delta^{\text{op}}} \widehat{\Sigma}$ at $I_{\mathcal{J}}$ and $W_{\mathcal{J}}$ are the same. The same argument as in the proof of Proposition 3.8 now shows that $\bar{\Pi}$ exhibits \mathcal{J} as the localization of $\mathcal{J} \times_{\Delta^{\text{op}}} \widehat{\Sigma}$ at $W_{\mathcal{J}}$. \square

Combining Proposition 3.5 with Proposition 3.9, we get:

Corollary 3.10. *Suppose $f: \mathcal{J} \rightarrow \Delta^{\text{op}}$ is a functor such that \mathcal{J} has f -cocartesian morphisms over inert maps in Δ^{op} . Then there is a fully faithful functor of ∞ -categories*

$$\text{Fun}(\mathcal{J}, \mathcal{C}) \hookrightarrow \text{Fun}_{\Delta^{\text{op}}}(\mathcal{J}, \overline{\text{SPAN}}^+(\mathcal{C}))$$

that identifies $\text{Fun}(\mathcal{J}, \mathcal{C})$ with the functors that preserve cocartesian morphisms over inert morphisms in Δ^{op} . \square

Under this equivalence, functors $\mathcal{J} \rightarrow \text{SPAN}^+(\mathcal{C})$ over Δ^{op} that preserve cocartesian morphisms over inert maps correspond to functors $F: \mathcal{J} \rightarrow \mathcal{C}$ such that for all $x \in \mathcal{J}$ over $[n] \in \Delta^{\text{op}}$, the morphism

$$F(x) \rightarrow F(x_{0,1}) \times_{F(x_{1,1})} \cdots \times_{F(x_{n-1,n-1})} F(x_{n-1,n})$$

is an equivalence, where $x \rightarrow x_{i,j}$ denotes the cocartesian morphism over $\rho_{i,j}$. In particular, we have:

Corollary 3.11. *Let $\mathcal{O} \rightarrow \Delta^{\text{op}}$ be a (generalized) non-symmetric ∞ -operad and \mathcal{C} an ∞ -category with pullbacks. Then there is a natural equivalence of ∞ -categories*

$$\text{Seg}_{\mathcal{O}}(\mathcal{C}) \simeq \text{Alg}_{\mathcal{O}}(\text{SPAN}^+(\mathcal{C})).$$

Proposition 3.12. *For $X \in \mathcal{C}$, the fibre of $\text{ev}_{[0]}: \text{Seg}_{\mathcal{O}}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{O}_0, \mathcal{C})$ at the constant functor with value X is naturally equivalent to $\text{Alg}_{\mathcal{O}}(\mathcal{C}_{/X,X}^{\otimes})$.*

Proof. Combining the equivalences of Corollary 3.11 and Lemma 2.16, we can identify $\text{Seg}_{\mathcal{O}}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{O}_0, \mathcal{C})$ with the functor $\text{Alg}_{\mathcal{O}}(\text{SPAN}^+(\mathcal{C})) \rightarrow \text{Alg}_{\mathcal{O}}(\Delta_{\mathcal{C}}^{\text{op}})$; the constant functor $\mathcal{O}_0 \rightarrow \mathcal{C}$ corresponds to the composite $\mathcal{O} \rightarrow \Delta^{\text{op}} \rightarrow \Delta_{\mathcal{C}}^{\text{op}}$ where the second morphism is the associative algebra in $\Delta_{\mathcal{C}}^{\text{op}}$ associated to $X \in \mathcal{C}$. Since $\text{Alg}_{\mathcal{O}}(-)$ preserves limits, the fibre we want is given by the pullback square

$$\begin{array}{ccc} \text{Alg}_{\mathcal{O}}(\mathcal{C}_{/X,X}^{\otimes}) & \longrightarrow & \text{Alg}_{\mathcal{O}}(\text{SPAN}^+(\mathcal{C})) \\ \downarrow & & \downarrow \\ * \simeq \text{Alg}_{\mathcal{O}}(\Delta^{\text{op}}) & \longrightarrow & \text{Alg}_{\mathcal{O}}(\Delta_{\mathcal{C}}^{\text{op}}), \end{array}$$

as required. \square

4. QUASI-UNITAL CATEGORY OBJECTS

Our goal in this section is to prove that “having identities” is a property of a category object. We will prove this by using the results of the previous section to reduce to the case of associative algebras, which has already been proved by Lurie. We begin by recalling Lurie’s result, which requires introducing some notation:

Definition 4.1. Let $\Delta_{\text{inj}}^{\text{op}}$ denote the subcategory of Δ^{op} containing only the injective maps; this is a non-symmetric ∞ -operad, and its algebras are non-unital associative algebras. If \mathcal{O} is a generalized non-symmetric ∞ -operad, we write $\text{Alg}_{\text{nu}}(\mathcal{O})$ for the ∞ -category $\text{Alg}_{\Delta_{\text{inj}}^{\text{op}}}(\mathcal{O})$ of non-unital associative algebras in \mathcal{O} . The inclusion $j: \Delta_{\text{inj}}^{\text{op}} \rightarrow \Delta^{\text{op}}$ is a morphism of non-symmetric ∞ -operads, and induces the expected forgetful functor $j^*: \text{Alg}(\mathcal{O}) \rightarrow \text{Alg}_{\text{nu}}(\mathcal{O})$.

Definition 4.2. Let $\mathcal{V}^{\otimes} \rightarrow \Delta^{\text{op}}$ be a monoidal ∞ -category. If A is a non-unital associative algebra in \mathcal{V} , a *quasi-unit* for A is a morphism $u: I \rightarrow A$ such that the composite

$$A \simeq I \otimes A \xrightarrow{u \otimes A} A \otimes A \xrightarrow{m} A,$$

where m is the algebra multiplication, is equivalent to id_A , and similarly for the map with u on the other side. We say that a non-unital algebra A is *quasi-unital* if there exists a quasi-unit for A . If A and B are quasi-unital algebras, we say that a morphism $f: A \rightarrow B$ in $\text{Alg}_{\text{nu}}(\mathcal{V})$ is *quasi-unital* if $f \circ u$ is a quasi-unit for B , where $u: I \rightarrow A$ is a quasi-unit for A .

Warning 4.3. We emphasize that being quasi-unital is a *property* of a non-unital algebra. In particular, the data of a quasi-unit is *not* part of the structure of a quasi-unital algebra, we are merely asserting that it is possible to choose one.

Definition 4.4. Let $\text{Alg}_{\text{qu}}(\mathcal{V})$ denote the subcategory of $\text{Alg}_{\text{nu}}(\mathcal{V})$ whose objects are the quasi-unital algebras, and whose morphisms are the quasi-unital ones.

Theorem 4.5 (Lurie, [Lur17, Theorem 5.4.3.5]). *If \mathcal{V} is a monoidal ∞ -category, then the functor $j^*: \text{Alg}(\mathcal{V}) \rightarrow \text{Alg}_{\text{nu}}(\mathcal{V})$ induces an equivalence*

$$\text{Alg}(\mathcal{V}) \xrightarrow{\sim} \text{Alg}_{\text{qu}}(\mathcal{V})$$

onto the quasi-unital subcategory.

For the rest of this section we fix an ∞ -category \mathcal{C} with finite limits. We can then extend the definitions above to category objects in \mathcal{C} :

Definition 4.6. A *non-unital category object* in \mathcal{C} is a Segal $\Delta_{\text{inj}}^{\text{op}}$ -object, i.e. a functor $\Delta_{\text{inj}}^{\text{op}} \rightarrow \mathcal{C}$ satisfying the same limit condition as a category object. We write $\text{Cat}_{\text{nu}}(\mathcal{C})$ for the ∞ -category $\text{Seg}_{\Delta_{\text{inj}}^{\text{op}}}(\mathcal{C})$ of non-unital category objects.

Definition 4.7. Let $X: \Delta_{\text{inj}}^{\text{op}} \rightarrow \mathcal{C}$ be a non-unital category object. A *quasi-unit* for X is a commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{u} & X_1 \\ & \searrow & \swarrow \\ & X_0 & \swarrow \\ & & X_0 \end{array} \quad \begin{array}{c} d_1 \\ d_0 \end{array}$$

such that the composite

$$X_1 \simeq X_0 \times_{X_0} X_1 \xrightarrow{u \times_{X_0} X_1} X_1 \times_{X_0} X_1 \simeq X_2 \xrightarrow{d_1} X_1$$

is equivalent to the identity, and similarly for the morphism with u on the other side. We say a non-unital category object X is *quasi-unital* if there exists a quasi-unit for X .

Remark 4.8. A non-unital category object X in \mathcal{C} is quasi-unital if and only if X is quasi-unital when viewed as a non-unital associative algebra in $\mathcal{C}_{/X_0, X_0}^{\otimes}$. Any two quasi-units are therefore equivalent, by [Lur17, Remark 5.4.3.2].

Remark 4.9. The functor $\text{ev}_{[0]}: \text{Cat}_{\text{nu}}(\mathcal{C}) \rightarrow \mathcal{C}$ is a cartesian fibration by Proposition 2.12. Suppose X is a non-unital category object and $u: X_0 \rightarrow X_1$ is a quasi-unit for X . For $f: Y \rightarrow X_0$ in \mathcal{C} , let $f^*X \rightarrow X$ denote the cartesian morphism in $\text{Cat}_{\text{nu}}(\mathcal{C})$ over f . Then we have a diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X_0 \\ \downarrow f^*u & & \downarrow u \\ (f^*X)_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ Y_0 \times Y_0 & \longrightarrow & X_0 \times X_0, \end{array}$$

where the morphism f^*u exists since the bottom square is cartesian. The morphism f^*u is then a quasi-unit for f^*X . In other words, if X is quasi-unital then so is f^*X for any $f: Y \rightarrow X_0$.

Definition 4.10. Suppose X and $Y \in \text{Cat}_{\text{nu}}(\mathcal{C})$ are quasi-unital. A morphism $\phi: X \rightarrow Y$ is *quasi-unital* if there exists a commutative diagram

$$\begin{array}{ccccc} X_0 & \xrightarrow{\phi_0} & Y_0 & & \\ & \searrow u & \searrow v & & \\ & & X_1 & \xrightarrow{\phi_1} & Y_1 \\ & \searrow \Delta & \searrow \Delta & & \\ & & X_0 \times X_0 & \xrightarrow{\phi_0 \times \phi_0} & Y_0 \times Y_0, \end{array} \quad \begin{array}{c} (d_0, d_1) \\ (d_0, d_1) \end{array}$$

where u and v are quasi-units for X and Y , respectively. We write $\text{Cat}_{\text{qu}}(\mathcal{C})$ for the subcategory of $\text{Cat}_{\text{nu}}(\mathcal{C})$ containing the quasi-unital objects and the quasi-unital morphisms between them.

Remark 4.11. From Remark 4.9 and the uniqueness of quasi-units we see that a morphism $\phi: X \rightarrow Y$ is quasi-unital if and only if $X \rightarrow \phi_0^*Y$ is quasi-unital.

Moreover, the latter is quasi-unital if and only if it corresponds to a quasi-unital morphism between non-unital algebras in $\mathcal{C}^{\otimes}/X_0, X_0$.

Proposition 4.12. *Suppose $p: \mathcal{E} \rightarrow \mathcal{B}$ is a cartesian fibration, and \mathcal{E}_0 is a subcategory of \mathcal{E} such that*

- (i) *for $x \in \mathcal{E}_0$ and $f: b \rightarrow p(x)$ in \mathcal{B} , the cartesian morphism $f^*x \rightarrow x$ lies in \mathcal{E}_0 .*
- (ii) *if x and y are objects of \mathcal{E}_0 then a morphism $\phi: x \rightarrow y$ in \mathcal{E} lies in \mathcal{E}_0 if and only if $x \rightarrow p(\phi)^*y$ lies in \mathcal{E}_0 .*

Then $p|_{\mathcal{E}_0}: \mathcal{E}_0 \rightarrow \mathcal{B}$ is a cartesian fibration, and a morphism in \mathcal{E}_0 is cartesian if and only if its image in \mathcal{E} is cartesian.

Proof. Given $x \in \mathcal{E}_0$ and $f: b \rightarrow p(x)$ we must show that the cartesian morphism $f^*x \rightarrow x$ in \mathcal{E} is cartesian when viewed as a morphism in \mathcal{E}_0 . For $y \in \mathcal{E}_0$ we have a commutative diagram

$$\begin{array}{ccc} \text{Map}_{\mathcal{E}_0}(y, f^*x) & \longrightarrow & \text{Map}_{\mathcal{E}_0}(y, x) \\ \downarrow & & \downarrow \\ \text{Map}_{\mathcal{E}}(y, f^*x) & \longrightarrow & \text{Map}_{\mathcal{E}}(y, x) \\ \downarrow & & \downarrow \\ \text{Map}_{\mathcal{B}}(p(y), b) & \longrightarrow & \text{Map}_{\mathcal{B}}(p(y), p(x)). \end{array}$$

Here the top square is cartesian by assumption (ii) and the bottom square is cartesian since $f^*x \rightarrow x$ is cartesian. It follows that the composite square is cartesian, which completes the proof. \square

Combined with Remark 4.11 and Proposition 2.12, this gives:

Corollary 4.13. *The functor $\text{ev}_{[0]}: \text{Cat}_{\text{qu}}(\mathcal{C}) \rightarrow \mathcal{C}$ is a cartesian fibration; a morphism in $\text{Cat}_{\text{qu}}(\mathcal{C})$ is cartesian if and only if its image in $\text{Cat}_{\text{nu}}(\mathcal{C})$ is cartesian. \square*

Theorem 4.14. *The functor $j^*: \text{Cat}(\mathcal{C}) \rightarrow \text{Cat}_{\text{nu}}(\mathcal{C})$ induces an equivalence*

$$\text{Cat}(\mathcal{C}) \xrightarrow{\sim} \text{Cat}_{\text{qu}}(\mathcal{C})$$

onto the quasi-unital subcategory.

Proof. We have a commutative triangle

$$\begin{array}{ccc} \text{Cat}(\mathcal{C}) & \xrightarrow{j^*} & \text{Cat}_{\text{nu}}(\mathcal{C}) \\ & \searrow i_{\Delta^{\text{op}}}^* & \swarrow i_{\Delta_{\text{inj}}^{\text{op}}}^* \\ & \mathcal{C} & \end{array}$$

Here the diagonal functors are both cartesian fibrations by Proposition 2.12. We claim that j^* also preserves cartesian morphisms. Using the description of the cartesian morphisms in Proposition 2.12, this amounts to observing that the canonical natural transformation $j^*i_{\Delta^{\text{op},*}} \rightarrow i_{\Delta_{\text{inj},*}^{\text{op}}}$ is clearly an equivalence. The functor j^* obviously factors through the subcategory $\text{Cat}_{\text{qu}}(\mathcal{C})$, so by Corollary 4.13 we get a commutative triangle

$$\begin{array}{ccc} \text{Cat}(\mathcal{C}) & \xrightarrow{j_{\text{qu}}^*} & \text{Cat}_{\text{qu}}(\mathcal{C}) \\ & \searrow i_{\Delta^{\text{op}}}^* & \swarrow i_{\Delta_{\text{inj},\text{qu}}^{\text{op}}}^* \\ & \mathcal{C} & \end{array}$$

where the diagonal functors are cartesian fibrations, and the horizontal functor preserves cartesian morphisms. To prove that j_{qu}^* is an equivalence, it therefore suffices to prove that for every object $X \in \mathcal{C}$ the functor

$$\text{Cat}(\mathcal{C})_X \rightarrow \text{Cat}_{\text{qu}}(\mathcal{C})_X$$

on fibres over X is an equivalence. But by Remark 4.11 and Proposition 3.12 we can identify this with the restriction of $j^*: \text{Alg}(\mathcal{C}_{/X,X}^\otimes) \rightarrow \text{Alg}_{\text{nu}}(\mathcal{C}_{/X,X}^\otimes)$ to a functor

$$\text{Alg}(\mathcal{C}_{/X,X}^\otimes) \rightarrow \text{Alg}_{\text{qu}}(\mathcal{C}_{/X,X}^\otimes),$$

which is an equivalence by Theorem 4.5. \square

We can inductively define the ∞ -category of n -uple category objects in \mathcal{C} as

$$\text{Cat}^n(\mathcal{C}) := \text{Cat}(\text{Cat}^{n-1}(\mathcal{C}));$$

this corresponds to a full subcategory of $\text{Fun}(\Delta^{n,\text{op}}, \mathcal{C})$. Similarly, we can define $\text{Cat}_{\text{nu}}^n(\mathcal{C})$ and $\text{Cat}_{\text{qu}}^n(\mathcal{C})$. Applying Theorem 4.14 inductively, we get:

Corollary 4.15. *The restriction functor $\text{Cat}^n(\mathcal{C}) \rightarrow \text{Cat}_{\text{nu}}^n(\mathcal{C})$ factors through an equivalence*

$$\text{Cat}^n(\mathcal{C}) \xrightarrow{\sim} \text{Cat}_{\text{qu}}^n(\mathcal{C}).$$

The n -uple category objects in the ∞ -category \mathcal{S} of spaces are known as n -uple Segal spaces. By imposing constancy conditions, we can restrict to the class of n -fold Segal spaces [Bar05], which model (the algebraic structure of) (∞, n) -categories. This notion makes sense more generally:

Definition 4.16. A 1-fold Segal object in \mathcal{C} is a category object. For $n > 1$, an n -fold Segal object is an n -uple category object X such that $X_{0,\bullet,\dots,\bullet}: \Delta^{n-1,\text{op}} \rightarrow \mathcal{C}$ is constant and $X_{i,\bullet,\dots,\bullet}$ is an $(n-1)$ -fold Segal object for all $i > 0$.

If we write $\text{Seg}^n(\mathcal{C})$ for the full subcategory of $\text{Cat}^n(\mathcal{C})$ spanned by the n -fold Segal objects, and $\text{Seg}_{\text{nu}}^n(\mathcal{C})$ for the analogously defined non-unital n -fold Segal objects, restricting the equivalence of Corollary 4.15 gives:

Corollary 4.17. *The restriction functor $\text{Seg}^n(\mathcal{C}) \rightarrow \text{Seg}_{\text{nu}}^n(\mathcal{C})$ factors through an equivalence*

$$\text{Seg}^n(\mathcal{C}) \xrightarrow{\sim} \text{Seg}_{\text{qu}}^n(\mathcal{C}),$$

where $\text{Seg}_{\text{qu}}^n(\mathcal{C})$ is the subcategory of non-unital n -fold Segal spaces that are quasi-unital (when viewed as n -uple category objects) and quasi-unital morphisms between them.

The ∞ -category $\text{Cat}_{(\infty,n)}$ of (∞, n) -categories can be described as the full subcategory $\text{CSS}^n(\mathcal{S}) \subseteq \text{Seg}^n(\mathcal{S})$ consisting of the *complete* n -fold Segal spaces. The equivalence of Corollary 4.17 thus identifies $\text{Cat}_{(\infty,n)}$ this with a certain full subcategory of $\text{Seg}_{\text{qu}}^n(\mathcal{S})$. This full subcategory can also be described without reference to this equivalence, as follows:

Definition 4.18. If X is a non-unital Segal space, we write $X(x, y)$ for the fibre of $(d_1, d_0): X_1 \rightarrow X_0 \times X_0$ at (x, y) ; we refer to the points of $X(x, y)$ as *morphisms* from x to y , and denote these in the usual way. If X is quasi-unital with quasi-unit $u: X_0 \rightarrow X_1$ we say that a morphism $f: x \rightarrow y$ is an *equivalence* in X if there exists $g: y \rightarrow x$ such that $g \circ f \simeq u(x)$ and $f \circ g \simeq u(y)$. Let X_1^{eq} denote the subspace of X_1 containing those components that correspond to equivalences. It is clear from the definition of a quasi-unit that u restricts to a map $X_0 \rightarrow X_1^{\text{eq}}$, and we say that X is *complete* if this map is an equivalence in \mathcal{S} .

Remark 4.19. More generally, we can define an equivalence in a non-unital Segal space X to be a morphism $f: x \rightarrow y$ such that the morphisms

$$f^*: X(y, z) \rightarrow X(x, z), \quad f_*: X(z, x) \rightarrow X(z, y)$$

given by composition with f are equivalences in \mathcal{S} for all $z \in X_0$. It is easy to see that this is equivalent to the previous definition when X is quasi-unital. Moreover, if X is quasi-unital then X is complete if and only if either (or both) of the face maps $X_1^{\text{eq}} \rightarrow X_0$ is an equivalence, since the quasi-unit is always a section of both. Thus we can characterize the complete quasi-unital Segal spaces without explicitly referring to the quasi-unit.

Definition 4.20. Suppose $X: \mathbf{\Delta}_{\text{inj}}^{n, \text{op}} \rightarrow \mathcal{S}$ is an n -fold quasi-unital Segal space. Then X is *complete* if

- the quasi-unital Segal space $X_{\bullet, 0, \dots, 0}$ is complete,
- the quasi-unital $(n-1)$ -fold Segal space $X_{1, \bullet, \dots, \bullet}$ is complete.

Equivalently, X is complete if the n quasi-unital Segal spaces of the form $X_{1, \dots, 1, \bullet, 0, \dots, 0}$ are all complete. We write $\text{CSS}_{\text{qu}}^n(\mathcal{S})$ for the full subcategory of $\text{Seg}_{\text{qu}}^n(\mathcal{S})$ spanned by the complete quasi-unital n -fold Segal spaces.

Corollary 4.21. *The restriction functor $\text{Seg}^n(\mathcal{S}) \rightarrow \text{Seg}_{\text{nu}}^n(\mathcal{S})$ restricts to an equivalence*

$$\text{CSS}^n(\mathcal{S}) \rightarrow \text{CSS}_{\text{qu}}^n(\mathcal{S}).$$

Proof. This is immediate from Corollary 4.17, since the definition of completeness for a quasi-unital n -fold Segal space clearly restricts to the usual notion of completeness for an n -fold Segal space. \square

Warning 4.22. Our notion of a quasi-unital Segal space is not quite the same as that studied by Harpaz [Har15]. Let us say that a morphism $u: x \rightarrow x$ in a non-unital Segal space X is a *quasi-identity* if there are equivalences $f \circ u \simeq f$ for f in $X(x, y)$ and $u \circ g \simeq g$ for g in $X(y, x)$, natural in f and g (more precisely, for all y the maps $u_*: X(y, x) \rightarrow X(y, x)$ and $u^*: X(x, y) \rightarrow X(x, y)$ are equivalent to the corresponding identities). We can then call X *weakly quasi-unital* if for every object x there exists a quasi-identity $u: x \rightarrow x$; these objects correspond to those called quasi-unital in [Har15]. In general this seems to be a strictly weaker notion than ours, which essentially requires there to exist a choice of quasi-identities $u_x: x \rightarrow x$ that is continuous in $x \in X_0$; we expect that this stronger condition is needed to get an equivalence between Segal spaces and quasi-unital Segal spaces (whereas Harpaz only gets an equivalence between the subcategories of complete objects).

Remark 4.23. We can define a non-unital Segal space X to be *complete* if the two face maps $X_1^{\text{eq}} \rightarrow X_0$ are both equivalences, following Remark 4.19 and [Har15, §3]; it is easy to see that if X is complete then it is also weakly quasi-unital. It follows from Harpaz's work that every complete non-unital Segal space is in the image of the restriction functor from complete Segal spaces, and thus is in particular quasi-unital in our sense. Similarly, a morphism of complete non-unital Segal spaces is quasi-unital if and only if it preserves quasi-identities.

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