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Operator algebras on Wiener amalgam spaces

Master's thesis in Mathematical sciences
Supervisor: Eduard Ortega
June 2022

Master's thesis

NTNU
Norwegian University of Science and Technology
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Kunnskap for en bedre verden

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Abstract

We give a thorough introduction to Wiener amalgam spaces on locally compact groups, considering both the continuous-type and the discrete-type norm. We prove several properties of Wiener amalgam spaces, including their set relations, duality and closure under products. Then we move on to operators on Wiener amalgam spaces, treating isometries through examples and non-examples, before moving to quasi-isometries. These lay the ground for the discussion of various representations of algebras onto Wiener amalgam spaces. Lastly, we define several types of operator algebras on Wiener amalgam spaces, namely group algebras and crossed products. We prove several properties of these, including duality theorems, identification theorems and structure theorems in certain special cases.

I denne oppgåva vert operatoralgebraar på Wiener-amalgamrom introdusert og behandla. Først kjem ei grundig innføring til Wiener-amalgamrom på lokalt kompakte grupper, som dreg nytte av både den kontinuerlege og diskrete normen til romma. Fleire av eigenskapane til Wiener-amalgamrom vert viste, mellom anna mengderelasjonar, dualitet og tillukking av både punktvis produkt og konvolusjonsprodukt. Vidare ser vi på operatorar på Wiener-amalgamrom, og først dei inverterbare isometriane. Det kjem fleire døme på at Lamperti-teoremet ikkje gjeld for Wiener-amalgamrom. Så vert kvasiisometriar introdusert og behandla. Kvasiisometriar gjev rammeverket for å diskutera representasjonar på Wiener-amalgamrom. Til slutt vert operatoralgebraar på Wiener-amalgamrom introdusert, og vi ser grundig på gruppealgebraar og kryssprodukt. Vi viser somme resultat for desse algebraane, mellom anna dualitetsteorem og strukturresultat.

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Preface

This thesis was written during the 2021-2022 study year, and concludes my master studies in mathematical sciences. The thesis was wrtitten under supervison by Eduard Ortega. I wish to thank Eduard for his excellent supervision. Without his enthusiastic involvement, this thesis would surely have imploded halfway through.

I also wish to thank several of my fellow students of mathematics for valuable academic discussions, as well as several not so academic discussions at lunch.

At last, a thanks to my family and other friends for huge moral support and encouragement.

Erling A.T. Svela, May 2022.

1 Introduction

The theory of C^*-algebras concerns Banach algebras represented on a Hilbert space. This is a well-studied field of operator algebras, and there is a lot of structure in the theroy. Starting with Herz [11], Banach algebras represented on L^p-spaces has also been studied, and the main idea has been to use techniques from C^*-algebras to study operator algebras on L^p-spaces. For example, Phillips has in [14] and [13] studied L^p-analogues of group algebras and cross products. In this thesis we wish to try a similar approach. Following the work of Gardella [5] and Phillips [14] we wish to study analogues of group L^p-operator algebras and cross product L^p-operator algebras for a different class of Banach spaces, the Wiener amalgam spaces.

The Wiener amalgam spaces were first introduced by Wiener in his work on generalized harmonic analysis [11]. They are spaces of functions with a norm that mixes a local criterion and a global criterion for inclusion into the space. In full generality, the Wiener amalgam $W(B, C)$ consists of functions on B that satisfies a local criterion in the Banach space B, and a global criterion in the Banach space C, but in this thesis we assume that both B and C are L^p-spaces on a locally compact group G. The appeal of this assumption is that the Wiener amalgams $W(L^p(G), L^q(G))$ (which we denote by $W^{p,q}(G)$) "generalizes" the L^p-spaces: setting $p = q$, $W^{p,q}(G) = L^p(G)$. Thus, just as the theory of algebras represented on Hilbert spaces (which are L^2-spaces) has been generalized to L^p-spaces, we should hopefully be able to generalize the theory of algebras represented on L^p-spaces (which are $W^{p,p}$-spaces) to Wiener amalgam spaces.

The thesis is outlined as follows:

Chapter 2 gives an introduction to Wiener amalgam spaces. We define the Wiener amalgam spaces $W^{p,q}(G)$ on a locally compact group G using a continuous-type norm, and show that the spaces are independent of the choice of localization window Q in the norm. We then show several nice properties of Wiener amalgam spaces, including set relations between the space and its local and global components, translation invariance, closure under pointwise products and that $W^{p,p}(G) = L^p(G)$. We then construct an equivalent discrete-type norm on $W^{p,q}(G)$ using partitions of unity. Using the discrete norm we prove a duality theorem for amalgams, as well as closure under convolutions.

Chapter 3 examines certain operators on Wiener amalgam spaces. First we examine the isometries of a Wiener amalgam space by attempting to adapt Lamperti's theorem for L^p-spaces. After showing that Lamperti's theorem we make a guess as to what the isometries of a Wiener amalgam space might be, before we weaken the concept of an isometry to that of a quasi-isometry (or bi-Lipschitz continuous quasi-isometric embedding). We state some nice quasi-isometric properties of Wiener amalgam spaces and then prove that a large class of operators are in fact quasi-isometries.

Chapter 4 defines operator algebras on Wiener amalgam spaces. After defining and introducing these algebras we move to consider two sppecial cases: Group algebras and crossed products. Following Gardella [5], we prove that there is a quasi-isometric anti-isomorphism between group algebras with dual exponents. We then give an example to show that the full group algebra is not well-defined in general. We then consider dynamical systems of groups acting continuously on a topological space. We consider representations of a dynamical system $G \curvearrowright X$ and construct the crossed product algebra as the image of a particular representation of the convolution algebra associated to the system. We prove that two conjugate dynamical systems have quasi-isometrically isomorphic crossed products. Following Phillips [14] we then examine crossed products for the special case where the group in the dynamical system is discrete. We adapt the techniques of Phillips to prove two structure theorems for crossed products by discrete groups.

1.1 Notation

Given a number $p \in (1, \infty)$, p' denotes its dual exponent. The identity element of a group is denoted by e. All groups are assumed to be metric groups. The Lebesgue measure on $\mathbb{R}$ is denoted by λ while μ denotes the Haar measure of a general locally compact group. We denote the p-norm of a function by $\|f\|_p$. If two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, we write $\|\cdot\|_1 \approx \|\|\cdot\|_2$. For a topological space X, $\mathbf{B}(X)$ denotes the Borel sets of X, while for a normed space X, $\mathcal{B}(X)$ denotes its bounded linear operators.

2 Wiener amalgam spaces

In this section we define our main item of interest: the Wiener amalgam spaces. This section is largely based on [9]. Our aim is to approach the topic from a more general viewpoint. In light of this, some preliminary terms must be introduced.

Definition 2.1. Let G be a group. If G is in addition a topological space such that the maps $\cdot : G \times G \to G$ and $(\cdot)^{-1} : G \to G$ given by multiplication and inversion, respectively, are continuous, then we call G a topological group.

A topological Hausdorff group G is said to be locally compact if for all $g \in G$ there is a compact neighbourhood K of g. That is, there is an open set $U \subseteq K$ such that $x \in U \subseteq K$.

For a fixed $a \in G$ we define the left translation by a, denoted T_a, by $T_a(x) = a^{-1}x$. T_a is a continuous mapping of G, and the induced map on functions on G is also denoted by T_a. For any $f : G \to \mathbb{C}$ we have $T_a f(t) = f(a^{-1}t)$. Right translation can be defined analogously.
It is well known that any locally compact group can be equipped with a left translation-invariant Borel measure, called a Haar measure. A group's Haar measure is uniquely determined up to some positive multiple. There is usually a canonical choice of Haar measure, and we will further on denote the space of p-integrable functions on a locally compact group by $L^p(G)$ whenever the choice of Haar measure is obvious.

Definition 2.2. Let G be a locally compact group with Haar measure μ, let $1 \leq p, q \leq \infty$ and fix a compact subset $Q \subset G$ with nonempty interior. The Wiener amalgam space $W_Q^{p,q}(G)$ consists of the functions $f : G \to \mathbb{C}$ such that $f \cdot \chi_K \in L^p(G)$ for all compact $K \subset G$, and such that the control function

$$F_f^Q(x) = \|f \cdot \chi_{xQ}\|_p, \qquad x \in G,$$

is in $L^q(G)$. The norm of f in $W_Q^{p,q}(G)$ is

$$\|f\|_{p,q} = \|F_f\|_q = \Big\| \|f \cdot \chi_{xQ}\|_p \Big\|_q$$

$$= \left(\int_G \left(\int_G |f(t)|^p \chi_Q(x^{-1}t)\, dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}.$$

Note. If p or q is infinite we use the supremum norm. If, for example, $p = \infty$ we get

$$\|f\|_{\infty,q} = \left(\int_G (\operatorname{esssup}_{t \in G} |f(t)\chi_{xQ}(t)\|)^q \right)^{\frac{1}{q}}.$$

If, instead $q=\infty$ we get

$$\|f\|_{p,\infty} = \operatorname{esssup}_{x\in G}\left(\int_G |f(t)|\chi_x Q(t)|\right)^{\frac{1}{p}}$$

We call the set Q in the definition the localization window of $W_Q^{p,q}(G)$, and a general compact set with non-empty interior is called an admissible window. The space $L^p(G)$ is called the local component of $W_Q^{p,q}(G)$, while $L^q(G)$ is its local component. Heuristically speaking, a function f is an element of $W_Q^{p,q}(G)$ if it is locally p-integrable and globally q-integrable.

The archetypical example of a Wiener amalgam space is when $G=\mathbb{R}$ with Lebesgue measure, and $Q=[0,1]$. In this case $W_{[0,1]}^{p,q}(\mathbb{R})$ consists of all functions $f:\mathbb{R}\to\mathbb{C}$ which are finite in the norm

$$\|f\|_{p,q} = \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(t)|^p\chi_{[0,1]}(t-x)\,dt\right)^{\frac{q}{p}}\,dx\right)^{\frac{1}{q}}$$

$$\left(\int_{\mathbb{R}}\left(\int_x^{x+1} |f(t)|^p\chi_{[0,1]}(t)\,dt\right)^{\frac{q}{p}}\,dx\right)^{\frac{1}{q}}.$$

These spaces, at least in the cases $p=1,\ q=2$ and $p=2,\ q=1$ were introduced by Wiener in 1926 [16].

$W_Q^{p,q}(G)$ is clearly a normed space, but we can in fact say more.

Proposition 2.3. *Let $1\leq p,q\leq\infty$, G a locally compact group, and $Q\subseteq G$ an admissible window. Then $W_Q^{p,q}(G)$ is a Banach space.*

Proof. To show completeness, we use the Cauchy summability criterion. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $W_Q^{p,q}(G)$ such that $\sum\|f_n\|_{p,q}<\infty$. Then $\sum\|f_n\|_{p,q}=\sum\|F_{f_n}\|_q<\infty$, and since $L^q(G)$ is complete, the series $\sum F_{f_n}$ converges in $L^q(G)$. This in turn means that $\sum F_{f_n}$ is finite almost everywhere:

$$\sum_{n=1}^{\infty} F_{f_n}(x) = \sum_{n=1}^{\infty}\|f_n\cdot\chi_{xQ}\|_p<\infty, \qquad \text{a.e.} x\in G.$$

By completeness of $L^p(G)$ the series $\sum f\cdot\chi_{xQ}$ converges for almost every x. Call the limit g_x. We define a function g almost everywhere by $g(t)=g_x(t)$. Note that this is well defined, as $g_x=g_y$ on $(xQ)\bigcap(yQ)$, and moreover that $g\cdot\chi_{xQ}=g_x$. We get

$$\|g\|_{p,q} = \Big\|\|g\cdot\chi_{xQ}\|_p\Big\|_q = \Big\|\|g_x\|_p\Big\|_q$$

$$= \left\| \left\| \sum_{n=1}^{\infty} f_n \cdot \chi_{xQ} \right\|_p \right\|_q \leq \sum_{n=1}^{\infty} \left\| \|f_n \cdot \chi_{xQ}\|_p \right\|_q$$

$$= \sum_{n=1}^{\infty} \|f_n\|_{p,q} < \infty.$$

So $g \in W_Q^{p,q}(G)$. To show that $\sum f_n$ in fact converges to g we pick an N and calculate

$$\left\| \sum_{n=1}^{N} f_n - g \right\|_{p,q} = \left\| \left\| \sum_{n=1}^{N} f_n \cdot \chi_{xQ} - g \cdot \chi_{xQ} \right\|_p \right\|_q$$

$$\left\| \left\| \sum_{n=1}^{N} f_n \cdot \chi_{xQ} - \sum_{n=1}^{\infty} f_n \cdot \chi_{xQ} \right\|_p \right\|_q = \left\| \left\| \sum_{n=N}^{\infty} f_n \cdot \chi_{xQ} \right\|_p \right\|_q$$

$$\leq \sum_{n=N}^{\infty} \|f_n\|_{p,q}.$$

which goes to 0 as N approaches ∞, by assumption. Therefore $\sum f_n$ converges in $W_Q^{p,q}(G)$, and $W_Q^{p,q}(G)$ is complete. $\square$

From the definition it seems like $W_Q^{p,q}(G)$ should depend on the window Q. This is not the case. The definition of the Wiener amalgam spaces is actually independent of Q.

Lemma 2.4. *Let $1 \leq p, q \leq \infty$ and G a locally compact group. The definition of $W_Q^{p,q}(G)$ is independent of Q. That is, if Q_1 and Q_2 are compact admissible windows, they define the same Wiener amalgam space $W^{p,q}(G)$ with equivalent norms.*

Proof. Let Q_1 and Q_2 be compact subsets with non-empty interior. The collection $\{\text{int}(xQ_1)\}_{x \in G}$ is an open cover of Q_2, so by compactness there is a finite set $\{x_i\}_{i=1}^{n}$ such that $Q_2 \subset \bigcup_{i=1}^{n} x_i Q_1$. Note that then $\chi_{Q_2}(t) \leq \chi_{\bigcup x_i Q_1}(t) \leq \sum \chi_{xQ_1}(t)$. We get

$$F_f^{Q_2}(x) = \|f \cdot \chi_{xQ_2}\|_p \leq \left\| f \cdot \sum_{i=1}^{n} \chi_{x_i x Q_1} \right\|_p$$

$$\leq \sum_{i=1}^{n} \|f \cdot \chi_{x_i x Q_1}\|_p = \sum_{i=1}^{n} F_f^{Q_1}(x_i x) = \sum_{i=1}^{n} T_{x_i^{-1}}(F_f^{Q_1})(x).$$

Since $L^q(G)$ is a translation-invariant space we find that

$$\|f\|_{W^{p,q}_{Q_2}} = \|F_f^{Q_2}\|_q \leq \sum_{i=1}^{n} \|T_{x_i^{-1}}(F_f^{Q_1})\|_q = \sum_{i=1}^{n} \|F_f^{Q_1}\|_q = n \cdot \|F_f^{Q_1}\|_q = n \cdot \|f\|_{W^{p,q}_{Q_1}}.$$

A symmetric argument gives $\|f\|_{W^{p,q}_{Q_1}} \leq m \cdot \|f\|_{W^{p,q}_{Q_2}}$, and so the norms are equivalent. $\square$

Note. Since the space $W^{p,q}_Q(G)$ is independent of the window Q we may omit Q in the notation and simply write $W^{p,q}(G)$. It will sometimes still be useful to distinguish between the same Wiener space with different windows. Whenever we want to emphasize the choice of window we write $W^{p,q}_Q(G)$ and denote the norm by either $\|\cdot\|_Q$ or $\|\cdot\|_{p,q,Q}$.

The above lemma has some interesting implications. If we let G be a finite group equipped with counting measure, then every finite subset of G will be both open and closed. That is, any subset will work as a localization window, since it is both compact and has non-empty interior. Let, for simplicity's sake, the order of G be n. If we pick any element $\gamma_0 \in G$ we get that $W^{p,q}_{\{\gamma_0\}}(G)$ consists of the functions $f : G \to \mathbb{C}$ such that

$$\begin{aligned}\|f\|^q_{p,q} &= \sum_{x \in G} \left(\sum_{t \in G} |f(t)\chi_{\{\gamma_0\}}(x^{-1}t)|^p \right)^{\frac{q}{p}} \\ &= \sum_{x \in G} (|f(x\gamma_0)|^p)^{\frac{q}{p}} = \|f\|^q_q \text{ is finite.}\end{aligned}$$

So $W^{p,q}_{\{\gamma_0\}}(G)$ is in fact just ℓ^q_n.
Let us now chose G to be a compact group with normalized Haar measure. Then G is an admissible window and we get that $W^{p,q}_G(G)$ consists of the functions $f : G \to \mathbb{C}$ such that

$$\|f\|^q_{p,q} = \int_G \left(\int_G |f(t)\chi_G(x^{-1}t)|^p \, d\mu(t) \right)^{\frac{q}{p}} \, d\mu(x)$$

$$= \int_G \left(\int_G |f(t)|^p \, d\mu(t) \right)^{\frac{q}{p}} \, d\mu(x) = \int_G \|f\|^q_p \, d\mu(x) = \|f\|^q_p \text{ is finite.}$$

So $W^{p,q}_G(G)$ is actually $L^p(G)$.

In the above proof we made use of the translation-invariance of L^q. The Wiener amalgam spaces also have this property, and translating a function leaves the norm unchanged. This will be of use later on.

Proposition 2.5. *Let $1 \leq p, q \leq \infty$, G a locally compact group, and $Q \subseteq G$ an admissible window. For any $a \in G$, the operator $T_a(f) = f(a^{-1}x)$ is an isometry of $W^{p,q}(G)$.*

Proof. Note first that since translation is an isometry of $L^p(G)$, we immediately get that $T_a(f) \cdot \chi_k \in L^p(G)$. Next we calculate the control function:

$$F_{T_a(f)}(x) = \left(\int_G |f(a^{-1}t)|^p \cdot \chi_Q(x^{-1}t)\ dt \right)^{\frac{1}{p}}$$

$$= \left(\int_G |f(u)|^p \cdot \chi_Q(x^{-1}au)\ du \right)^{\frac{1}{p}} = F_f(a^{-1}x) = T_a(F_f)(x).$$

Finally using that translation is an isometry of $L^q(G)$, we find that

$$\|T_a(f)\|_{p,q} = \|F_{T_a(f)}\|_q = \|T_a(F_f)\|_q = \|F_f\|_q = \|f\|_{p,q}.$$

So translation is an isometry of $W^{p,q}(G)$. $\square$

2.1 Inclusion properties

In this text, our reason for studying Wiener amalgam spaces comes from the fact that they "generalize" L^p-spaces. In particular, setting $p = q$ will actually result in an L^p-space.

Proposition 2.6. *Let $1 \leq p \leq \infty$, G a locally compact group, and $Q \subseteq G$ an admissible window. Then there is an isomorphism between $W^{p,p}(G)$ and $L^p(G)$.*

Proof. For simplicity, assume $1 \leq p < \infty$, and let $f \in W^{p,p}(G)$. Then, using Fubini's theorem in the second line,

$$\|f\|_{p,q} = \left(\int_G \left(\int_G |f(t)|^p \chi_Q(t-x)\ dt \right)^{\frac{p}{p}} dx \right)^{\frac{1}{p}} = \left(\int_G \int_G |f(t)|^p \chi_Q(t-x)\ dt\ dx \right)^{\frac{1}{p}}$$

$$= \left(\int_G \int_G |f(t)|^p \chi_Q(t-x)\ dx\ dt \right)^{\frac{1}{p}} = \left(\int_G \int_G |f(t)|^p \chi_Q(t-x)\ dx\ dt \right)^{\frac{1}{p}}$$

$$= \left(\int_G |f(t)|^p \int_G \chi_Q(t-x)\ dx\ dt \right)^{\frac{1}{p}} = \left(\int_G |f(t)|^p \cdot \mu(Q)\ dt \right)^{\frac{1}{p}}$$

$$= \mu(Q)^{\frac{1}{p}} \left(\int_G |f(t)|^p \, dt \right)^{\frac{1}{p}} = \mu(Q)^{\frac{1}{p}} \|f\|_p.$$

So the two norms are equivalent. That is, if a function is p-integrable, it is Wiener p-p-integrable and vice versa. $\square$

Note. While there is an isomorphism between $L^p(G)$ and $W_Q^{p,q}(G)$, for this map is not isometric for $p < \infty$. If $f \in L^p(G)$ has p-norm N, it will have Wiener p, p-norm $\mu(Q)^{\frac{1}{p}} \cdot N$.

A natural question is if inclusion properties of the local or global components affect the inclusion properties of the Wiener spaces. The question is easy to answer for local components.

Proposition 2.7. *Let $1 \leq q \leq \infty$, G be a locally compact group. If $1 \leq p_1 \leq p_2 \leq \infty$ then*

$$W^{p_2,q}(G) \subseteq W^{p_1,q}(G).$$

Proof. For any compact $Q \subset G$ we know that $L^{p_2}(Q) \subseteq L^{p_1}(Q)$, as

$$\|f\|_{L^{p_1}(Q)} \leq \mu(Q)^{\frac{1}{p_1} - \frac{1}{p_2}} \|f\|_{L^{p_2}(Q)},$$

which follows from Hölder's inequality. Repeating this argument with the control function F_f shows that

$$\|f\|_{p_1,q} = \left\| \|f \cdot \chi_{xQ}\|_{p_1} \right\|_q \leq \left\| \mu(xQ)^{\frac{1}{p_1} - \frac{1}{p_2}} \|f \cdot \chi_{xQ}\|_{p_2} \right\|_q$$
$$= \mu(Q)^{\frac{1}{p_1} - \frac{1}{p_2}} \|f\|_{p_2,q}.$$

So any function with finite p_2, q-norm will also have finite p_1, q-norm, and the inclusion is therefore true. $\square$

There is also set relations between the amalgam space and its global and local components.

Proposition 2.8. *Let G be a locally compact group. If $1 \leq p \leq q \leq \infty$, then $L^p(G) \cup L^q(G) \subseteq W^{p,q}(G)$.*

If $1 \leq q \leq p \leq \infty$, then $W^{p,q}(G) \subseteq L^p(G) \cup L^q(G)$.

Proof. For simplicity we only prove the case $1 \leq p \leq q < \infty$, as the other cases are similar. By local inclusion we have

$$\|f\|_{p,q} \leq C\|f\|_{p,q} \leq D\|f\|_q,$$

So $L^q \subseteq W^{p,q}(G)$. Moreover, by the integral version of Minkowski's inequality (see appendix C) we have

$$\|f\|_{p,q}^p = \left(\int_G \left(\int_G |f(t)|^p \chi_Q(x^{-1}t)\, dt\right)^{\frac{q}{p}} dx\right)^{\frac{p}{q}}$$

$$\leq \int_G \left(\int_G |f(t)|^q \chi_Q(x^{-1}t)\, dx\right)^{\frac{p}{q}} dt$$

$$= \int_G |f(t)|p \left(\int_G \chi_Q(x^{-1}t)\, dx\right)^{\frac{p}{q}} dt = \mu(Q)^{\frac{p}{q}} \int_G |f(t)|^p\, dt$$

$$= \mu(Q)^{\frac{p}{q}} \|f\|_p^p.$$

So $L^p \subseteq W^{p,q}(G)$ as well. $\square$

As a consequence of the above result, we have for instance that $L^1(\mathbb{R}) \cup L^2(\mathbb{R}) \subset W^{1,2}(\mathbb{R})$. Note that the inclusion is strict, as there are several functions that are in $W^{1,2}(\mathbb{R})$, but not in $L^1(\mathbb{R}) \cup L^2(\mathbb{R})$. One example is the function $f(t) = \frac{1}{t}^{5/9} \cdot \chi_{[0,1]}(t) + \frac{1}{t} \cdot \chi_{[1,\infty)}(t)$, that is

$$\begin{cases} 0, & x \leq 0 \\ \frac{1}{t}^{\frac{5}{9}}, & x \in [0,1] \\ \frac{1}{t}, & x \geq 1 \end{cases}.$$

Note that f is not in L^1, since the part where $x \geq 1$ diverges in 1-norm. Similarly the part where $x \in [0,1]$ diverges in 2-norm, and so $f \notin L^2$. However, $f \in W^{1,2}(\mathbb{R})$. The calculation can be done using $[0,1]$ as a window.

The Wiener amalgam spaces also behave nicely with regards to certain operations. For example, the spaces are closed under pointwise products.

Proposition 2.9. *Let G be a locally compact group. Let p_1, p_2, p_3, q_1, q_2 and q_3 be indices such that there are constants $C_1, C_2 > 0$ so that for all $h_1 \in L^{p_1}(G), h_2 \in L^{p_2}(G), k_1 \in L^{q_1}(G), k_2 \in L^{q_2}(G)$,*

$$\|h_1 h_2\|_{p_3} \leq C_1 \|h_1\|_{p_1} \|h_2\|_{p_2}$$

and

$$\|k_1 k_2\|_{q_3} \leq C_2 \|k_1\|_{q_1} \|k_2\|_{q_2}.$$

Then there is a constant $C > 0$ such that for all $f \in W^{p_1,q_1}(G)$ and $g \in W^{p_2,q_2}(G)$ we have

$$\|fg\|_{p_3,q_3} \leq C\|f\|_{p_1,q_1} \|g\|_{p_2,q_2}.$$

In other words, if $L^{p_1} \cdot L^{p_2} \subseteq L^{p_3}$ and $L^{q_1} \cdot L^{q_2} \subseteq L^{q_3}$ then

$$W^{p_1,q_1}(G) \cdot W^{p_2,q_2}(G) \subseteq W^{p_3,q_3}(G).$$

Proof. We let $f \in W^{p_1,q_1}(G)$ and $g \in W^{p_2,q_2}(G)$ and calculate.

$$\|fg\|_{p_3,q_3} = \left\| \|fg \cdot \chi_{xQ}\|_{p_3} \right\|_{q_3}$$

$$= \left\| \|(f \cdot \chi_{xQ}) \cdot (g \cdot \chi_{xQ})\|_{p_3} \right\|_{q_3}.$$

By assumption, this is less than

$$C_1 \left\| \|f \cdot \chi_{xQ}\|_{p_1} \cdot \|g \cdot \chi_{xQ}\|_{p_2} \right\|_{q_3}$$

$$= C_1 \left\|F_f \cdot F_g\right\|_{q_3} \leq C_1 C_2 \left\|F_f\right\|_{q_1} \left\|F_g\right\|_{q_2}$$

$$= C_1 C_2 \|f\|_{p_1,q_1} \|g\|_{p_2,q_2}.$$

□

Let $\phi \in L^\infty(G)$ We know that for any $p \in [1, \infty]$, if we pick a function $f \in L^p(G)$, then $\phi \cdot f \in L^p(G)$, and so $L^p \cdot L^\infty \subseteq L^p$. Thus, given any $\phi \in L^\infty(G) = W^{\infty,\infty}(G)$ we may define a linear map
$m_\phi : W^{p,q}(G) \to W^{p,q}(G)$ by

$$m_\phi f(t) = \phi(t) \cdot f(t),$$

and by the above proposition, this map is bounded. m_ϕ is called the multiplication operator of ϕ, or simply multiplication by ϕ.

There is also a Hölder-type inequality for Wiener amalgam spaces. Just like with Hölder's inequality in L^p it can be extended to a duality theorem. The duality of Wiener amalgam spaces will be proved after having defined discrete norms in the next section. For now we state and prove Hölder's inequality for Wiener amalgam spaces.

Proposition 2.10. *Let $1 \leq p, q \leq \infty$, G a locally compact group, and $Q \subseteq G$ an admissible window. For two measurable functions $f, g : G \to \mathbb{C}$, we have*

$$\|fg\|_{L^1} = \mu(Q)\|fg\|_{1,1} \leq \mu(Q)\|f\|_{p,q}\|g\|_{p',q'}.$$

Proof. The equality is clear, and the inequality follows from the closure of Wiener spaces under pointwise products. □

2.2 Discrete norms

While the natural definition of a Wiener space involves a "continuous" norm of the type given in definition 2.2, Wiener's original construction used a discrete norm, namely

$$\|f\|_{\mathbf{w}} = \left(\sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} |f(t)|^p \, dt \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}.$$

Remarkably, given $p, q \in [1, \infty]$, the continuous and discrete norms for $W^{p,q}(G)$ are in fact equivalent. The proceeding section will prove this.

We begin by introducing our main tool for creating discrete norms.

Definition 2.11. Let G be a locally compact group. Given some index set J, a set of functions $\Psi = \{\psi_i\}_{i\in J}$ on G is a bounded, uniform partition of unity (BUPU) if the following four properties are satisfied:

- $\sum_{i\in J} \psi_i \equiv 1$,
- $\sup_{i\in J} \|\psi_i\|_\infty < \infty$,
- there is a compact set $U \subset G$ with nonempty interior, and points $y_i \in G, i \in J$ such that $\mathrm{supp}(\psi_i) \subseteq y_i U$ for all i,
- for each compact $K \subset G$,

$$\sup_{x\in G} \#\{i \in J : x \in y_i K\} = \sup_{i\in J} \#\{j \in J : y_i K \cap y_j K \neq \emptyset\} < \infty.$$

The BUPU Ψ is implicitly associated with the set U and the points y_i.

Note. Heuristically speaking, a BUPU consists of compactly supported functions that sum to 1 at all points, and that do so in a "uniform" way. Notably, the sum is locally finite, and can therefore be interchanged with integrals and other sums.

$\mathbb{R}$ has several different BUPUs, for example, given any compact $U \subset \mathbb{R}$ and $\{y_i\}_{i\in J}$ such that $\{U + y_i\}_{i\in J}$ is a partition of $\mathbb{R}$, $\{\chi_{U+y_i}\}_{i\in J}$ is a BUPU. More generally, in a locally compact group any translate of a compact set $U \subset G$ defines a BUPU. In addition, a partition $\mathcal{P} = \{P_i\}$ of G defines a BUPU, regardless of the relation between the P_i.

If one wishes to measure more properties than just integrability, a set of bump functions with supports that cover G may also be used as a BUPU, as long as $\sum_{i\in\Lambda} \psi_i(x) = 1$ on the intersection of any two supports $\mathrm{supp}(\psi_j) \cap \mathrm{supp}(\psi_k)$.

The importance of BUPU's in our setting is that they induce a Wiener-type Banach space.

Definition 2.12. Let G be a locally compact group, and $\Psi = \{\psi\}_{i\in J}$ a BUPU on G associated with U and $\{y_i\}_{i\in J}$. For $1 \leq p, q \leq \infty$ the discrete Wiener amalgam space $\mathbf{w}^{p,q}(G, \Psi)$ consists of the functions $f : G \to \mathbb{C}$ such that for any compact set $K \subset G$, $f \cdot \chi_K \in L^p(G)$ and for which the control sequence

$$F_f(i) = F_f(i) = \|f \cdot \psi_i\|_p$$

is in ℓ^q_J. The norm of f in $\mathbf{w}^{p,q}(G, \Psi)$ is

$$\|f\|_{\mathbf{w}} = \|F_f\|_{\ell^q} = \left(\sum_{i\in J} \left(\int_G |f(t) \cdot \psi_i(t)|^p \, dt\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}$$

What we now want to show is that the norms of $W^{p,q}_Q(G)$ and $\mathbf{w}^{p,q}(G, \Psi)$ are equivalent, and that the two spaces therefore are equal. In order to deduce that the norms are equivalent the following result is the key observation.

Theorem 2.13. *Let $1 \leq p, q \leq \infty$, and G a locally compact group. If $\{\psi_i\}_{i\in J}$ is a BUPU on G associated to U and $\{y_i\}_{i\in J}$, then the norms*

$$\|f\|_{p,q} \qquad \text{and} \qquad \left\|\sum_{i\in J} \|f\psi_i\|_p \cdot \chi_{y_i U}\right\|_q$$

are equivalent.

Proof. We denote the norm on the right by $\|\cdot\|_U$. Define $G_U(x) = \sum \|f\psi_i\|_p \cdot \chi_{y_i U}(x)$, and note that then $\|f\|_U = \|G_U\|_q$. Fix some compact Q such that $UU^{-1} \subset Q$. That is, if $x \in y_i U$, then $y_i \in x^{-1}U$, so $y_i U \subset xUU^{-1} \subset xQ$. We therefore have $f\psi_i = f\psi_i \chi_{xQ}$ If we let $R = \sup \|\psi_i\|_\infty$, we have

$$\|f\psi_i\|_p = \|f\psi_i\chi_{xQ}\|_p \leq R\|f\chi_{xQ}\|_p.$$

We get the following relationship between the control functions.

$$G_U(x) = \sum_{i\in J} \|f\psi_i\|_p \cdot \chi_{y_i U}(x) = \sum_{i:x\in y_i U} \|f\psi_i\|_p \leq C_U R\|f \cdot \chi_{xQ}\|_p = F^Q_f(x),$$

where $C_U = \sup_{x\in G} \#\{i \in J : x \in y_i U\} < \infty$. Computing the q-norm of the control functions, we get

$$\|f\|_U = \|G_U\|_q \leq C_U R\|F^Q_f\|_p = \|F\|_{p,q}. \tag{1}$$

We now want to prove the opposite inequality.

$$\|f \cdot \chi_{xQ}\|_p = \left\| \sum_{i \in J} f \cdot \chi_{xQ} \psi_i \right\|_p \leq \sum_{i \in M_x} \|f\psi_i \cdot \chi_{xQ}\|_p$$
$$\leq \sum_{i \in M_x} \|f\psi_i\|_p = \sum_{i \in M_x} \|f\psi_i\|_p \chi_{y_i U}(x)$$
$$\leq \sum_{i \in J} \|f\psi_i\|_p \chi_{y_i U}(x) = G_U(x).$$

Just like with equation 1, this inequality carries over to the q-norms. We conclude that $\|\cdot\|_U$ and $\|\cdot\|_{p,q}$ are equivalent norms. $\square$

In the previously discussed case of a partition on $\mathbb{R}$, one can actually interchange the summation of the BUPU with the outer integral, which gives the equivalent norm

$$\|f\|_{p,q} \approx \left(\sum_{i \in J} \|f \chi_{U+y_i}\|_p^q \right)^{\frac{1}{q}} = \left\| \{ \|f \cdot \chi_{U+y_i}\|_p \}_{i \in J} \right\|_{\ell^q}.$$

We want to show that this is in fact true for any BUPU. In order to do this we need some preliminary terms and results.

Definition 2.14. A family of subsets $\{E_i\}_{j \in J} \subset X$ has a maximum of K overlaps if

$$K = \sup_{i \in J} \#\{j \in J : E_i \cap E_j \neq \emptyset\} = \sup_{x \in X} \sum_{i \in J} \chi_{E_i}(x) < \infty.$$

Note that putting $K = \operatorname{supp}(\psi_i)$ in the definition of a BUPU shows that the supports of the ψ_i have a maximum number of C_U overlaps. A collection of subsets with K overlaps has a particularly nice property.

Lemma 2.15 (Disjointization principle)**.** *If $\{E_i\}_{i \in J}$ is a family of subsets with at most K overlaps, there is a partition of J into finitely many subsets $\{J_r\}_{r=1}^K$ such that*

$$i \neq j \in J_r \quad \Rightarrow \quad E_i \cap E_j = \emptyset. \tag{2}$$

Proof. Let J_1 be a maximal subset of J with respect to property (2). Then we can define J_r for $r \geq 2$ inductively as a maximal subset of $J \setminus \bigcup_{i=1}^{r-1} J_i$ with respect to property (2). We now claim that this process stops after $r = K$. To prove this, suppose $s \in J \setminus \bigcup_{i=1}^{K} J_i$. Then, for any $1 \leq r \leq K$ we have $s \in J \setminus \bigcup_{i=1}^{r-1} J_i$, as well as $s \notin J_r$. As J_r is maximal in $J \setminus \bigcup_{i=1}^{r-1} J_i$ with respect to property (2), $J_r \bigcup \{s\}$

cannot satisfy property (2), that is, there is a $j_r \in J_r$ such that $E_s \bigcap E_{j_r} \neq \emptyset$. Doing this for each $r \in [1, K]$, we find a set $L = \{s, j_1, j_2, ..., j_K\}$ such that for any $l \in L$, $E_s \bigcap E_l \neq \emptyset$. But this means that $\{E_j\}_{j \in J}$ has at least $K+1$ overlaps, which is a contradiction! Therefore we get that $J = \bigcup_{i=1}^K J_i$. $\square$

With the disjointization principle in hand, we are able to further discretise the Wiener norm.

Proposition 2.16. *Let (X, μ) be a measure space, and $p \in [1, \infty]$. Assume $\{f_i\}_{i \in J} \subset L^p(X, \mu)$ are nonnegative functions such that $\{\text{supp}(f_i)\}_{i \in J}$ has a maximum of K overlaps. If $p \in [1, \infty)$ then for each finite $F \subset J$ we have*

$$\left(\sum_{n \in F} \|f_n\|_p^p \right)^{\frac{1}{p}} \leq \left\| \sum_{n \in F} f_n \right\|_p \leq K^{\frac{1}{p'}} \left(\sum_{n \in F} \|f_n\|_p^p \right)^{\frac{1}{p}}.$$

If any of the three terms are finite, the convergence is unconditional, and F can be replaced by J in the result. A similar result holds for $p = \infty$.

Proof. Using the disjointization principle, we partition J into $\bigcup_{i=1}^K J_i$, and therefore $\text{supp}(f_n) \bigcap \text{supp}(f_m) = \emptyset$ for $m \neq n \in J_i$. Moreover, as all f_n are nonnegative, we get, by using Hölder's inequality in ℓ_K^1,

$$\begin{aligned}
\left| \sum_{i=1}^k \sum_{n \in F \cap J_i} f_n \right|^p &= \left| \sum_{i=1}^K (1,1,1,..,1) \left(\sum_{n \in F \cap J_1} f_n, \sum_{n \in F \cap J_2} f_n, ..., \sum_{n \in F \cap J_K} f_n \right) \right|^p \\
&\leq \left(\|(1,1,1,...,1)\|_{p'} \left\| \left(\sum_{n \in F \cap J_1} f_n, \sum_{n \in F \cap J_2} f_n, ..., \sum_{n \in F \cap J_K} f_n \right) \right\|_p \right)^p \\
&= K^{\frac{p}{p'}} \sum_{i=1}^K \left(\sum_{n \in F \cap J_i} f_n \right)^p.
\end{aligned}$$

And so,

$$\left\| \sum_{n \in F} f_n \right\|_p^p = \int_X \left| \sum_{i=1}^K \sum_{n \in F \cap J_i} f_n \right|^p d\mu \leq K^{p/p'} \sum_{i=1}^K \int_X \left| \sum_{n \in F \cap J_i} f_n \right|^p d\mu.$$

Furthermore, since the supports of the f_n with $n \in F \cap J_i$ are pairwise disjoint, we can interchange sum and integral, yielding

$$= K^{\frac{p}{p'}} \sum_{i=1}^K \sum_{n \in F \cap J_i} \int_X |f_n|^p d\mu = K^{\frac{p}{p'}} \sum_{n \in F} \|f_n\|_p^p.$$

The opposite inequality follows in a similar way. ☐

From the preceeding results we now get our main result of the section, namely that we can interchange the norm and the sum in theorem 2.13.

Theorem 2.17. *Let $1 \leq p, q \leq \infty$, G a locally compact group, and $Q \subseteq G$ an admissible window. Let also $\Psi = \{\psi_i\}_{i \in J}$ be a BUPU on G. Then the norms of $W_Q^{p,q}(G)$ and $\mathbf{w}^{p,q}(G, \Psi)$ equivalent.*

Proof. If the BUPU is associated with the compact set U, then, since $\{\text{supp}(\psi_i)\}_{i \in J}$ has a maximum of C_U overlaps, we may invoke proposition 2.16 to get

$$\|(\cdot)\|_{p,q} \approx \left\| \sum_{i \in J} \|(\cdot)\psi_i\|_p \chi_{y_i U} \right\|_q$$

$$\approx \left(\sum_{i \in J} \left\| \|(\cdot)\psi_i\|_p \cdot \chi_{y_i U} \right\|_q^q \right)^{\frac{1}{q}}$$

$$= \left(\sum_{i \in J} \|(\cdot)\psi_i\|_p^q \cdot \|\chi_{y_i U}\|_q^q \right)^{\frac{1}{q}} = \mu(U) \left\| \left\{ \|(\cdot)\psi_i\|_p \right\}_{i \in J} \right\|_{\ell^q},$$

where the first equivalence is due to theorem 2.13. ☐

Being able to view the Wiener spaces in terms of a discrete-type norm will in some cases give us an easier way of understanding the relations between them. For instance, we get set relations for Wiener amalgam spaces based on the global component.

Corollary 2.18. *Let be G a locally compact group. Let $p, q, r \in [1, \infty]$, with $p \leq q$. Then $W^{r,p}(G) \subset W^{r,q}(G)$.*

Proof. Fixing a BUPU $\{\psi_i\}_{i \in J}$ and using the preceeding theorem there are positive constants C and D such that we get

$$\|f\|_{r,p} \geq C \left\| \{\|f\psi_i\|_r\}_{i \in J} \right\|_{\ell^p}$$

$$\geq C \left\| \{\|f\psi_i\|_r\}_{i \in J} \right\|_{\ell^q} \geq CD\|f\|_{r,q}.$$

☐

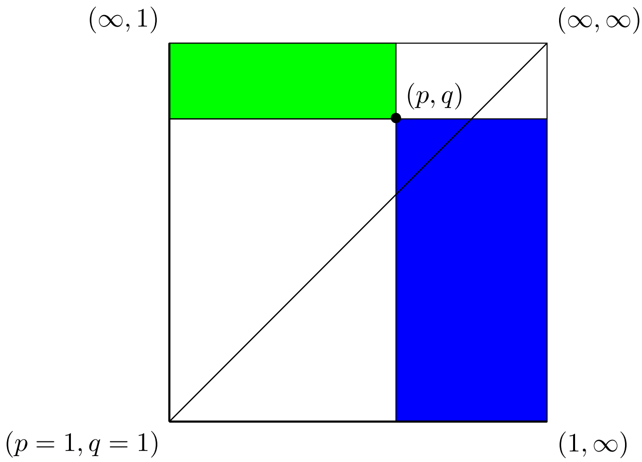

Figure 1: Schematic depiction of corrollary 2.19. The green area represents the Wiener amalgam spaces contained in $W^{p,q}$, while the blue area are the Wiener amalgam spaces containing $W^{p,q}$. The diagonal line is the L^p-spaces

Combining the inclusion theorems for global and local components of a Wiener amalgam space we get the following characterization of the Wiener amalgam space set relations.

Corollary 2.19. *Let $1 \leq p, q \leq \infty$, and G a locally compact group. $W^{p,q}(G)$ contains all Wiener amalgam spaces $W^{p_1,q_1}(G)$ such that $p \leq p_1$ and $q \geq q_1$. Similarly $W^{p,q}(G)$ is contained in all Wiener amalgam spaces $W^{p_2,q_2}(G)$ such that $p \geq p_2$ and $q \leq q_2$.*

The result is visualized in figure 1. Note that $W^{\infty,1}$ is contained in all other amalgam spaces, while $W^{1,\infty}$ contains all other amalgam spaces. Furthermore, the inclusion properties of the Wiener amalgam spaces show us that there are no inclusion relations between L^p-spaces in general, as expected.

Note. As a final remark on the set relations, note that if G is a discrete group, then the set relations are completely determined by the global structure. If $p_1 \leq p_2$ and $q_1 \leq q_2$ we get that

$$\|f\|_{p_1,q_1}^{q_1} = \sum_{x \in G} \left(\sum_{t \in G} |f(t)\|^{p_1} \chi_{xQ}(t) \right)^{\frac{q_1}{p_1}} \geq \sum_{x \in G} \left(\sum_{t \in G} |f(t)\|^{p_1} \chi_{xQ}(t) \right)^{\frac{q_2}{p_1}}$$

$$\geq \sum_{x\in G}\left(\sum_{t\in G}|f(t)\|^{p_2}\chi_{xQ}(t)\right)^{\frac{q_2}{p_2}} = \|f\|_{p_2,q_2}^{q_2}.$$

The first inequality is the global component inclusion of Wiener amalgams, and the second inequality follows from the inclusion of $\ell^{p_1}(G)$ into $\ell^{p_2}(G)$. Combining this with corollary 2.19 we get that $W^{p_1,q_1}(G) \subseteq W^{p_2,q_2}(G) \iff q_1 \leq q_2$.

Similarly, for a compact group G we use the inclusion of $L^{q_2}(G)$ into $L^{q_1}(G)$ and corollary 2.19, to get that $W^{p_1,q_1}(G) \subseteq W^{p_2,q_2}(G) \iff p_1 \geq p_2$. Thus the set relations between Wiener amalgams on compact groups is completely determined by the local structure, as expected.

It is well known that $C_c(G)$, the space of continuous functions on G with compact support, is a dense subspace of $L^p(G)$. It is also clear that any element $f \in C_c(G)$ will have finite Wiener norm for any p and q, so that $C_c(G) \subseteq W^{p,q}(G)$. By using the discrete norm, we are able to prove the following result.

Theorem 2.20. *Let G a locally compact group. Then $C_c(G)$ is a dense subspace of $W^{p,q}(G)$ for any $p, q \in [1, \infty)$.*

Proof. Once again, we will for simplicity assume $p, q < \infty$, and we will use the discrete-type norm for $W^{p,q}(G)$. Let $\{\psi_n\}_{n\in J}$ be a BUPU associated to Q and $\{y_n\}_{n\in J}$, where we've for further simplicity assumed that J is a countable index set. Let $f \in W^{p,q}(G)$ and let $f_n = f \cdot \psi_n$. By definition, $f_n \in L^p(y_nQ)$, so given any $\varepsilon > 0$ there is, by density of L^p, a sequence $\{f_{n,m}\} \in C_c(y_nQ)$ such that for $m > M_n$, $\|f_n - f_{n,m}\| < \left(\frac{(\varepsilon/4)}{2^{|n|}}\right)^{\frac{1}{q}}$. Moreover, as $\sum_{n\in\mathbb{Z}} \|f_n\|_p^q$ converges, there is a number $N \in \mathbb{N}$ such that $\sum_{|n|>N} \|f_n\|_p^q < \frac{\varepsilon}{4}$. With this in mind, we define

$$\overline{f_k} = \sum_{|n|\leq k} f_{n,k}\psi_n.$$

As a finite sum of compactly supported functions, $\overline{f_k} \in C_c(G)$. If we let $k \geq \max\{\{N\} \cup \{M_n\}_{n\in J}\}$, we get

$$\|\overline{f_k} - f\|_{\mathbf{w}}^q = \sum_{|n|=0}^{\infty} \|(f - f_{n,k})\psi_n\|_p^q = \sum_{|n|=0}^{\infty} \|f_n - f_{n,k}\|_p^q$$

$$= \sum_{|n|\leq k} \|f_n - f_{n,k}\|_p^q + \sum_{|n|>k} \|f_n\|_p^q \leq \sum_{|n|\leq k} \left(\frac{(\varepsilon/4)}{2^{|n|}}\right)^{\frac{1}{q}} + \frac{\varepsilon}{4}$$

$$= \sum_{|n|\leq k} \frac{\varepsilon/4}{2^{|n|}} + \frac{\varepsilon}{4} = \frac{\varepsilon}{4}\left(1 + 2\sum_{n=1}^{k} \frac{1}{2^n} + 1\right) < \frac{\varepsilon}{4}\left(1 + 2\sum_{n=1}^{\infty} \frac{1}{2^n} + 1\right) = \frac{\varepsilon}{4} \cdot 4 = \varepsilon.$$

Thus $\overline{f_k}$ converges towards f in $W^{p,q}(G)$. $\square$

2.3 Duality

To further demonstrate the usefulness of the discrete norm, we prove the following duality theorem for Wiener amalgam spaces.

Theorem 2.21. *Let G be a locally compact group, let $p, q \in [1, \infty)$ and let p', q' be the respective dual exponents. Then*

$$W^{p,q}(G)^* = W^{p',q'}(G).$$

Proof. We first construct a linear mapping from $W^{p',q'}(G)$ to $W^{p,q}(G)^*$, then we show that it is isometric. Finally we will show that the mapping is surjective.

For simplicity, assume that $\{K_i\}_{i\in J}$ is a partition of G and consider the associated BUPU $\{\chi_{K_i}\}_{i\in J}$. This can be assumed without loss of generality, as we will discuss later. Let $f \in W^{p,q}(G)$ and $g \in W^{p',q'}(G)$. We have that

$$\|f\|_{p,q} = \left(\sum_{i\in J} \|f \cdot \chi_{K_i}\|_p^q\right)^{\frac{1}{q}}$$

and

$$\|g\|_{p',q'} = \left(\sum_{i\in J} \|g \cdot \chi_{K_i}\|_{p'}^{q'}\right)^{\frac{1}{q'}}.$$

Firstly, we have by Hölder's inequality that

$$\int_G |f(t)g(t)|\, dt = \sum_{i\in J} \int_{K_i} |f(t)g(t)|\, dt \leq \sum_{i\in J} \int_G |f(t)\chi_{K_i} g(t)\chi_{K_i}|\, dt$$

$$\leq \sum_{i\in J} \|f \cdot \chi_{K_i}\|_p \|g \cdot \chi_{K_i}\|_{p'} \leq \left(\sum_{i\in J} \|f \cdot \chi_{K_i}\|_p^q\right)^{\frac{1}{q}} \left(\sum_{i\in J} \|g \cdot \chi_{K_i}\|_{p'}^{q'}\right)^{\frac{1}{q'}}$$

$$= \|f\|_{p,q}\|g\|_{p',q'}.$$

So $\int_G f\bar{g}\ dt$ is well-defined, and therefore g defines a bounded, linear functional ϕ_g on $W^{p,q}(G)$ with

$$\|\phi_g\| = \sup_{\|f\|=1} |\phi_g(f)| \leq \|g\|_{p',q'}.$$

We will now show that equality is achieved in this expression. For simplicity's sake, assume $p, q \in (1, \infty)$. Given a $g \in W^{p',q'}(G)$ we set $g_i = g \cdot \chi_{K_i}$. We define

$$f_i(t) = |g_i(t)|^{p'}/\overline{g_i(t)}.$$

Notice now that $\operatorname{supp}(f_i) \subset K_i$ and since $p = \frac{p'}{p'-1}$, we get that $|f_i(t)|^p = |g_i(t)|^{p'}$ and $\|f\|_p^p = \|g\|_{p'}^{p'} < \infty$. Furthermore, we have

$$\int_G f_i(t)\overline{g_i(t)}\ dt = \int_G |g_i(t)|^{p'}\ dt = \left(\int_G |g_i(t)|^{p'} dt\right)^{\frac{1}{p}+\frac{1}{p'}}$$

$$= \left(\int_G |g_i(t)|^{p'} dt\right)^{\frac{1}{p}} \left(\int_G |g_i(t)|^{p'} dt\right)^{\frac{1}{p'}}$$

$$= \left(\int_G |f_i(t)|^p\ dt\right)^{\frac{1}{p}} \left(\int_G |g_i(t)|^{p'}\ dt\right)^{\frac{1}{p'}} = \|f_i\|_p \|g_i\|_{p'}.$$

We set $a_i b_i = \|f_i\|_p \|g_i\|_{p'}$, and define

$$c_i = (b_i)^{q'}/(a_i b_i).$$

Just as with f_i we have that $(c_i a_i)^q = (b_i)^{q'}$ so we also get

$$\sum_{i\in J} c_i a_i b_i = \left(\sum_{i\in J}(c_i a_i)^q\right)^{\frac{1}{q}} \left(\sum_{i\in J}(b_i)^{q'}\right)^{\frac{1}{q'}}$$

$$= \left\|\{c_i a_i\}_{i\in J}\right\|_{\ell^q} \left\|\{b_i\}_{i\in J}\right\|_{\ell^{q'}} = \left\|\{c_i a_i\}_{i\in J}\right\|_{\ell^q} \|g\|_{p',q'}.$$

We can now define $f = \sum_J c_i f_i$ and note that

$$\|f\|_{p,q}^q = \left\|\{c_i a_i\}_{i\in J}\right\|_{\ell^q}^q = \left\|\{b_i\}_{i\in J}\right\|_{\ell^{q'}}^{q'} < \infty,$$

so $f \in W^{p,q}(G)$ and we have

$$\int_G f(t)\overline{g(t)}\ dt = \int_G \sum_{i\in J} c_i f_i(t)\overline{g(t)}\ dt = \sum_{i\in J} c_i \int_G f_i(t)\overline{g(t)}\ dt$$

$$= \sum_{i \in J} c_i \int_G f_i(t)\overline{g_i(t)}\, dt = \sum_{i \in J} c_i a_i b_i = \|f\|_{p,q} \|g\|_{p',q'}.$$

So equality is in fact obtained and therefore $\|\phi_g\| = \|g\|_{p',q'}$, so the map is isometric.

It remains to show that $g \to \phi_g$ is a surjective map. Firstly, note that for any $i \in J$ the space $L^p(K_i)$ is contained in $W^{p,q}(G)$. Given a $\psi \in W^{p,q}(G)^*$ restricting ψ to K_i gives a bounded linear functional $\psi_i \in L^p(K_i)^*$. By duality of L^p, there is a $g_i \in L^{p'}(K_i)$ such that $\psi_i(h) = \int_G h\overline{g_i}\, dt$. We define $g = \sum_{i \in J} g_i$, and g is well-defined as the g_i's have disjoint support. We now want to show that $g \in W^{p',q'}(G)$, and our first step is to show that $\left\{ \|g_i\|_{p'} \right\}_{i \in J} \in \ell^{p'}$. Given a sequence $\{c_i\}i \in J$ and $\varepsilon > 0$ there are $f_i \in L^p(K_i)$ such that $\|f_i\|_p \leq 1$ and

$$\int_G f_i(t)\overline{g_i(t)}\, dt \geq \|g_i\|_{p'} - \frac{\varepsilon}{2^i |c_i|}.$$

Now, the function $f = \sum_J c_i f_i$ is in $W^{p,q}(G)$, as $\|f\|_{p,q} \leq \left\| \{c_i\}_{i \in J} \right\|_q < \infty$. Therefore,

$$\left| \sum_{i \in J} c_i \int_G f_i(t)\overline{g_i(t)}\, dt \right| = \left| \sum_{i \in J} c_i \psi_i(f_i) \right| = \left| \sum_{i \in J} c_i \psi(f_i) \right|$$
$$= \left| \psi \left(\sum_{i \in J} f_i \right) \right| = |\psi(f)| \leq \left\| \{c_i\}_{i \in J} \right\|_{\ell^q} \|\psi\|.$$

Without loss of generality, assume that $c_i \int_G f_i(t)\overline{g_i(t)}\, dt > 0$. Then,

$$\sum_{i \in J} |c_i| \|g_i\|_{p'} \leq \sum_{i \in J} |c_i| \left(\left| \int_G f_i(t)\overline{g_i(t)}\, dt \right| + \frac{\varepsilon}{2^i |c_i|} \right)$$
$$= \left| \sum_{i \in J} c_i \int_G f_i(t)\overline{g_i(t)}\, dt \right| + \varepsilon \leq \|\{c_i\}_J\|_{\ell^q} \|\psi\| + \varepsilon.$$

Therefore $\{\|g_i\|_{p'}\}_{i \in J} \in \ell^{q'}$. Now

$$\|g\|_{p',q'} = \left(\sum_{i \in J} \|g_i\|_{p'}^{q'} \right)^{\frac{1}{q'}} = \left\| \left\{ \|g_i\|_{p'} \right\}_{i \in J} \right\|_{\ell^{q'}} < \infty.$$

Therefore $g \in W^{p',q'}(G)$. Lastly, note that

$$\psi(f) = \psi(f \cdot 1) = \psi \left(\sum_{i \in J} f_i \right) = \sum_{i \in J} \psi(f_i) = \sum_{i \in J} \int_G f_i(t)\overline{g_i(t)}\, dt$$

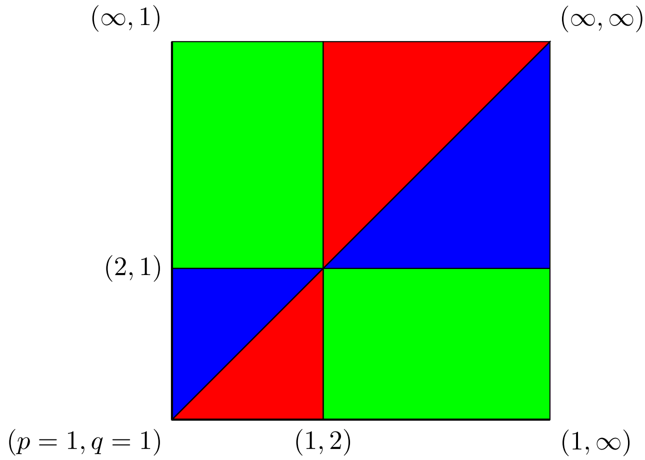

Figure 2: Illustration of the duality of Wiener amalgam spaces. Any Wiener space will have its dual in the other area of the same colour.

$$= \sum_{i \in J} \int_G f_i(t)\overline{g(t)}\, dt = \int_G f(t)\overline{g(t)}\, dt.$$

So in fact, $\psi = \phi_g$ for a $g \in W^{p',q'}(G)$. □

Note. When used later on, the duality pairing $W^{p,q}(G) \times W^{p',q'}(G) \to \mathbb{C}$ given by $(f,g) \to \phi_g(f)$ will usually be denoted by $\langle f, g \rangle$.

Note that while there is in general no inclusion relation between an L^p-space and its dual, there are several Wiener amalgam spaces that contain its dual. Using corollary 2.19 we see that whenever $p \leq 2$ and $q \geq 2$,

$$W^{p',q'}(G) = (W^{p,q}(G))^* \subseteq W^{p,q}(G).$$

If $p \geq 2$ and $q \leq 2$ the opposite inclusion holds.

The duality theorem is visualized in figure 2. We see that dual pairs of Wiener amalgam spaces "mirror" each other across L^2. Given a Wiener amalgam space $W^{p,q}$ with $p, g < \infty$ its dual will be in the other area of the figure with the same colour. The spaces that contain or is contained in its dual are marked in green. A space in the uppermost green area will be contained in its dual, while a space in the lowermost green area will contain its dual.

2.4 Convolution

A fundamental operation on $L^p(G)$ is convolution. For two functions f and g on G, their convolution is defined by

$$f * g(x) = \int_G f(t)g(x^{-1}t)\ dt.$$

Recall the following result for L^p-spaces.

Theorem 2.22 (Young's convolution inequality)**.** *Let G be a locally compact group, and let p, q and r be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f \in L^p(G)$ and $g \in L^q(G)$, then*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

In other words, $L^p(G) * L^q(G) \subseteq L^r(G)$.

Proof. See [10]. □

Note. In the special case $p = r$, $q = 1$, Young's inequality implies that
$L^1 * L^p \subseteq L^1$.

Young's inequality concretizes the well-behavedness of L^p-spaces with respect to convolution. Certain Wiener amalgam spaces are also well behaved with respect to convolution.

Definition 2.23. A locally compact group G is called an invariant neighbourhood group (IN-group) if there is a neighbourhood Q of e such that $xQx^{-1} = Q$ for any $x \in G$.

Note that being an IN-group is equivalent to there being a neighbourhood of the identity such that $Qx = xQ$ for all $x \in G$. Examples of IN-groups include the abelian groups and the reduced Heisenberg group $\mathbb{R}\times\mathbb{R}\times\mathbb{T}$. The statement of the theorem below, theorem 2.24, is true for any IN-group. However, for simplicity we only prove the special case $G = \mathbb{R}$. A proof of the general case can be found in the appendix.

Theorem 2.24. *For $i \in 1, 2, 3$, let p_i, q_i be such that there is constants $C_1, C_2 > 0$ such that*

$$\|h * k\|_{p_3} \leq C_1 \|h\|_{p_1} \|k\|_{p_2} \qquad \forall h \in L^{p_1}(\mathbb{R}), \forall k \in L^{p_2}(\mathbb{R})$$

and

$$\|h * k\|_{q_3} \leq C_2 \|h\|_{q_1} \|k\|_{q_2} \qquad \forall h \in L^{q_1(\mathbb{R})}, \forall k \in L^{p_2}(\mathbb{R}).$$

Then, there is another constant $C > 0$ such that

$$\|f * g\|_{p_3,q_3} \leq C\|f\|_{p_1,q_1}\|g\|_{p_2,q_2}$$

for all $f \in W^{p_1,q_1}(\mathbb{R})$ and $g \in W^{p_2,q_2}(\mathbb{R})$. So if $L^{p_1}(\mathbb{R}) * L^{p_2}(\mathbb{R}) \subseteq L^{p_3}(\mathbb{R})$ and $L^{q_1}(\mathbb{R}) * L^{q_2}(\mathbb{R}) \subseteq L^{q_3}(\mathbb{R})$ then

$$W^{p_1,q_1}(\mathbb{R}) * W^{p_2,q_2}(\mathbb{R}) \subseteq W^{p_3,q_3}(\mathbb{R}).$$

Proof. This proof uses the equivalent discrete norms for the respective Wiener spaces. Let $\chi_n = \chi_{[n,n+1)} = \chi_{[0,1)+n}$, and consider the BUPU $\{\chi_n\}_{n=1}^{\infty}$. We write the "discrete control function" of f in $L^p(\mathbb{R})$ as

$$F_{f,p}(n) = \|f \cdot \chi_n\|_p \qquad n \in \mathbb{Z}.$$

In this setting, the $W^{p,q}(\mathbb{R})$-norm of f is

$$\|f\|_{p,q} = \|F_{f,p}\|_{\ell^q} = \left(\sum_{n\in\mathbb{Z}} |F_{f,p}(n)|^q\right)^{\frac{1}{q}}.$$

Now, given $f \in W^{p_1,q_1}(\mathbb{R})$ and $g \in W^{p_2,q_2}(\mathbb{R})$ and note that

$$\operatorname{supp}(f \cdot \chi_n * g \cdot \chi_m) \subseteq [m, m+1] + [n, n+1] = [m+n, m+n+2] = [0,2] + m + n.$$

$$\implies ((f \cdot \chi_n) * (g \cdot \chi_m)) \cdot \chi_k \neq 0 \iff k = m+n \text{ or } k = m+n+1.$$

Using linearity of the convolution we estimate F_{f*g,p_3}.

$$\begin{aligned}
F_{f*g,p_3}(k) &= \|(f*g)\cdot\chi_k\|_{p_3} \\
&= \left\|\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}((f\cdot\chi_n)*(g\cdot\chi_m))\cdot\chi_k\right\|_{p_3} \\
&= \left\|\sum_{m\in\mathbb{Z}}((f\cdot\chi_{k-m})*(g\cdot\chi_m))\cdot\chi_k + ((f\cdot\chi_{k-m+1})*(g\cdot\chi_m))\cdot\chi_k\right\|_{p_3} \\
&\leq \sum_{m\in\mathbb{Z}}\|((f\cdot\chi_{k-m})*(g\cdot\chi_m))\cdot\chi_k\|_{p_3} + \sum_{m\in\mathbb{Z}}\|((f\cdot\chi_{k-m+1})*(g\cdot\chi_m))\cdot\chi_k\|_{p_3} \\
&\leq \sum_{m\in\mathbb{Z}}\|(f\cdot\chi_{k-m})*(g\cdot\chi_m)\|_{p_3} + \sum_{m\in\mathbb{Z}}\|(f\cdot\chi_{k-m+1})*(g\cdot\chi_m)\|_{p_3} \\
&\leq C_1\sum_{m\in\mathbb{Z}}\|f\cdot\chi_{k-m}\|_{p_1}\|g\cdot\chi_m\|_{p_2} + C_1\sum_{m\in\mathbb{Z}}\|f\cdot\chi_{k-m+1}\|_{p_1}\|g\cdot\chi_m\|_{p_2}
\end{aligned}$$

$$= C_1 \sum_{m\in\mathbb{Z}} F_{f,p_1}(k-n)F_{g,p_2}(n) + C_1 \sum_{m\in\mathbb{Z}} F_{f,p_1}(k-n+1)F_{g,p_2}(n)$$

$$= C_1(F_{f,p_1} * F_{g,p_2})(k) + C_1(F_{f,p_1} * F_{g,p_2})(k+1)$$

$$= C_1(F_{f,p_1} * F_{g,p_2})(k) + C_1 T_{-1}(F_{f,p_1} * F_{g,p_2})(k).$$

Using that translation is an isometry of ℓ^q we compute the $W^{p_3,q_3}(\mathbb{R})$-norm.

$$\|f * g\|_{p_3,q_3} = \|F_{f*g,p_3}\|_{\ell^{q_3}}$$

$$\leq C_1\|F_{f,p_1} * F_{g,p_2}\|_{\ell^{q_3}} + C_1\|T_{-1}(F_{f,p_1} * F_{g,p_2})\|_{\ell^{q_3}}$$

$$= 2C_1\|F_{f,p_1} * F_{g,p_2}\|_{\ell^{q_3}} \leq C_2 \cdot 2C_1\|F_{f,p_1}\|_{\ell^{q_1}}\|F_{g,p_2}\|_{\ell^{q_2}}$$

$$= C\|f\|_{p_1,q_1}\|g\|_{p_2,q_2}.$$

$\square$

3 Operators of $W^{p,q}(G)$

In this section we examine certain operators on Wiener amalgam spaces. More specifically, we show that Lamperti's theorem does not hold for Wiener amalgam spaces by showing that several automorphisms of the Borel sets of G will not be isometries of $W^{p,q}(G)$. We also state our guess on what the isometries of $W^{p,q}(G)$ are in general. Then we consider a more general class of operators that is hopefully more suitable when considering Wiener amalgam spaces.

3.1 Isometries

On an L^p-space, at least with $p \neq 2$, Lamperti's theorem characterizes its invertible isometries. In order to state the theorem we first give some definitions.

Definition 3.1. A Boolean algebra is a set A with two distinguished elements $\emptyset$ and I, together with two associative and commutative binary operations $\vee$ and $\wedge$ and a unary operation $(\cdot)^c$ such that the folowing properties hold for all $E, F \in A$.

- $E \vee E = E = E \wedge E$,
- $E \vee (E \wedge F) = E = E \wedge (E \vee F)$,
- $E \vee \emptyset = E = E \wedge I$,
- $E \wedge \emptyset = \emptyset, \qquad E \vee I = I$,
- $E \vee E^c = I, \qquad E \wedge E^c = \emptyset$.

A Boolean algebra homomorphism is a function $\phi : A \to B$ between two Boolean algebras such that for all $E, F \in A$,

- $\phi(E \vee_A F) = \phi(E) \vee_B \phi(F), \qquad \phi(E \wedge_A F) = \phi(E) \wedge_B \phi(F)$,
- $\phi(E^c) = \phi(E)^c$,
- $\phi(\emptyset_A) = \emptyset_B, \qquad \phi(I_A) = I_B$.

Given a measure space (X, B, μ) we let $N = \{E \in B \text{ such that } \mu(E) = 0\}$. The quotient B/N will be a Boolean algebra with binary operations given by union and intersection, unary operation given by complementation, $\emptyset$ given by the residue class $\emptyset + N$ and I given by $X + N$.

Definition 3.2. Let (X, B, μ) be a measure space.

We denote by $\mathcal{U}(L^\infty(X))$ the L^∞-functions on X such that $|f(t)| = 1$. $\mathcal{U}(L^\infty(X))$ is called the set of characters on X.

We denote by $\mathrm{Aut}(B)$ the automorphisms of the Boolean algebra B/N.

Lemma 3.3. *Let (X, B, μ) be a measure space, and $p \in [1, \infty)$. Given an $f \in \mathcal{U}(L^\infty(X))$, the map $m_f : L^p(X) \to L^p()$ given by*

$$m_f(\xi)(t) = f(t) \cdot \xi(t), \qquad \forall\, \xi \in L^p(X)$$

is an isometry of $L^p(X)$. Furthermore, the map $m : \mathcal{U}(L^p(X)) \to \mathrm{Isom}(L^p(X))$ given by $m(f) = m_f$ is a group homomorphism.
Moreover. Given a $\phi \in \mathrm{Aut}(B)$, the map $u_\phi : L^p(X) \to L^p(X)$ given by

$$u_\phi(\xi)(t) = \xi \circ \phi(t) \cdot \left(\frac{d(\mu \circ \phi^{-1})}{d\mu}(t) \right)^{\frac{1}{p}}, \qquad \forall\, \xi \in L^p(X)$$

is also an isometry of L^p, and the corresponding map $u : \mathrm{Aut}(B) \to \mathrm{Isom}(L^p(X))$ is a group homomorphism.

Proof. Showing that $\|m_f \xi\|_p = \|\xi\|_p$ is a straightforward calculation, and the homomorphism property is easily checked. We prove that $\|u_\phi \xi\|_p = \|\xi\|_p$ using the change of variables formula (see appendix B). Let $\phi \in \mathrm{Aut}(B)$ and $f \in L^p(X)$.

$$\|u_\phi(f)\|_p^p = \int_X \left| f(\phi(t)) \cdot \left(\frac{d(\mu \circ \phi^{-1})}{d\mu}(t) \right)^{\frac{1}{p}} \right|^p dt$$

$$= \int_X |f(\phi(t))|^p \cdot \frac{d(\mu \circ \phi^{-1})}{d\mu}(t)\, dt = \int_X |f(x)|^p\, dx = \|f\|_p^p.$$

To show that u is a group homomorphism, let in addition $\psi \in \mathrm{Aut}(B)$. Then we have

$$(u_\phi \circ u_\psi)(f)(t) = u_\psi(f)(\phi(t)) \cdot \left(\frac{d(\mu \circ \phi^{-1})}{d\mu}(t) \right)^{\frac{1}{p}}$$

$$= f(\psi(\phi(t))) \cdot \left(\frac{d(\mu \circ \psi^{-1})}{d\mu}(\phi(t)) \right)^{\frac{1}{p}} \cdot \left(\frac{d(\mu \circ \phi^{-1})}{d\mu}(t) \right)^{\frac{1}{p}},$$

which, using the properties of the Radon-Nikodym derivatives, equals

$$f((\phi\circ\psi)(t))\left(\frac{d(\mu\circ\psi^{-1}\circ\phi^{-1})}{d(\mu\circ\phi^{-1})}(t)\cdot\frac{d(\mu\circ\phi^{-1})}{d\mu}(t)\right)^{\frac{1}{p}}$$

$$=f((\phi\circ\psi)(t))\left(\frac{d(\mu\circ\psi^{-1}\circ\phi^{-1})}{d\mu}(t)\right)^{\frac{1}{p}}=u_{\phi\circ\psi}(f)(t).$$

□

The groups $\mathcal{U}(L^\infty(X))$ and $\mathrm{Aut}(B)$ are subgroups of $\mathrm{Isom}(L^p(X))$, and it can also be verified that the semi-direct product $\mathcal{U}\left(L^\infty(X)\right)\rtimes\mathrm{Aut}(B)$ also consists of isometries of $L^p(X)$. Lamperti's theorem states that in most cases, these are in fact all the isometries of $L^p(X)$.

Theorem 3.4 (Lamperti's theorem). *Let (X,B,μ) be a measure space, and $p\in[1,\infty)\setminus\{2\}$. If $T:L^p(X)\to L^p(X)$ is an invertible isometry, there exists $f\in\mathcal{U}\left(L^p(X)\right)$ and $\phi\in\mathrm{Aut}(B)$ such that $T=m_f u_\phi$. In other words, there is a group isomorphism*

$$\mathcal{U}\left(L^\infty(X)\right)\rtimes\mathrm{Aut}(B)\cong\mathrm{Isom}(L^p(X)).$$

Proof. See [8]. □

When $X=\mathbb{R}$ with Lebesgue measure, the invertible isometries of $L^p(\mathbb{R})$ include multiplication by any character, translations, reflection and dilations. It is already proven that translations are isometries of $W_Q^{p,q}(G)$, and it is also easy to see that multiplication by a character is also an isometry. However, not all $\phi\in\mathrm{Aut}(\mathbf{B}(G))$ give isometries of the Wiener amalgam space $W^{p,q}(G)$. Consider for example $G=\mathbb{R}$ with Lebesgue measure, and consider the admissible window $[0,1]$. Then the dilation operator D_2 defined by $D_2(f)(t)=\sqrt{2}f(2t)$ is not an isometry of $W_{[0,1]}^{p,q}(\mathbb{R})$. Note that for any interval I, $F_{\chi_I}(X)=\left(\int_G\chi_I(t)\chi_{Q+x}(t)dt\right)^{\frac{1}{p}}=\lambda(I\cap(Q+x))^{\frac{1}{p}}$, while $F_{D_2(\chi_I)}(x)$ can similarly be shown to equal $\lambda(I\cap(2Q+2x))^{\frac{1}{p}}$. Let $f=\chi_{[0,1]}$. Calculating the control functions, we get that $(F_f(x))^p=\max((1-|x|)^{\frac{1}{p}},0)$, and

$$F_{D_2(f)}(x)=\begin{cases}0, & x\le-1\text{ or }x\ge\frac{1}{2}\\ 2+2x, & x\in[-1,-\frac{1}{2}]\\ 1, & x\in[-\frac{1}{2},0]\\ 1-2x, & x\in[0,\frac{1}{2}]\end{cases}.$$

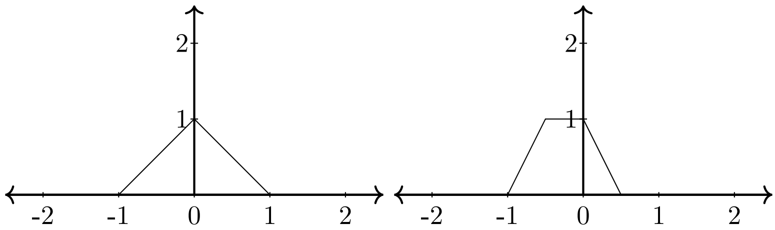

Figure 3: The p-th powers of the control functions of $\chi_{[0,1]}$ and $D_2(\chi_{[0,1]})$ with respect to the window $[0,1]$.

The control functions are described in figure 3. Calculating the q-norm of F_f and $F_{D_2(f)}$ shows that

$$\|f\|_{[0,1]}^q = \frac{2p}{q+p} \qquad \text{while} \qquad \|D_2(f)\|_{[0,1]}^q = \frac{q+3p}{2q+2p}.$$

Now let $g = \chi_{[-1,1]}$. Proceeding as above, we get that

$$(F_g(x))^p = \begin{cases} 0, & x \le -2 \text{ or } x \ge 1 \\ 2+x, & x \in [-2,-1] \\ 1, & x \in [-1,0] \\ 1-x, & x \in [0,1] \end{cases},$$

$$\left(F_{D_2(g)}(x)\right)^p = \begin{cases} 0, & x \le -\frac{3}{2} \text{ or } x \ge \frac{1}{2} \\ 3+2x, & x \in [-\frac{3}{2},-\frac{1}{2}] \\ 1-2x, & x \in [-\frac{1}{2},\frac{1}{2}] \end{cases},$$

and so

$$\|g\|_{[0,1]}^q = \frac{q+3p}{q+p} \qquad \text{and} \qquad \|D_2(f)\|_{[0,1]}^q = \frac{2^{\frac{p}{q+p}}p}{q+p}.$$

Thus there is no way to rescale D_2 such that the Wiener norms of f and $D_2(f)$ and g and $D_2(g)$ agree at the same time. It follows that D_2 cannot be an isometry of $W^{p,q}_{[0,1]}(\mathbb{R})$.

Restricting ourselves to the same Wiener amalgam space, note that the reflection operator $R(f)(t) = f(t^{-1})$ is actually an isometry of $W^{p,q}_{[0,1]}(\mathbb{R})$. This holds in more general cases as well. Consider the covering of a locally compact group G given by $\{\gamma Q\}_{\gamma \in G}$. If $Q^{-1} \in \{\gamma Q\}_{\gamma \in G}$ then R is an isometry of $W^{p,q}_Q(G)$. Let $Q^{-1} = zQ$. We calculate:

$$\|R(f)\|_Q = \left(\int_G \left(\int_G |f(t^{-1})|^p \chi_Q(tx^{-1})\, dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}$$

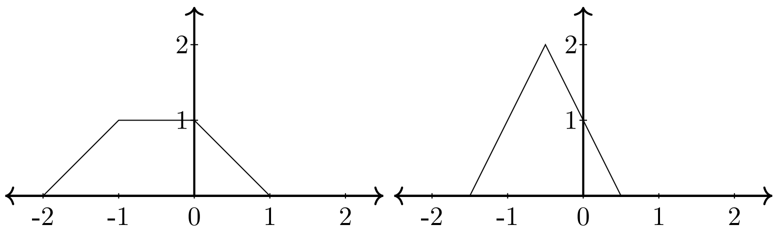

Figure 4: The p-th powers of the control functions of $\chi_{[-1,1]}$ and $D_2(\chi_{[-1,1]})$ with respect to the window $[0,1]$.

$$= \left(\int_G \left(\int_G |f(u)|^p \chi_Q(u^{-1}x^{-1})\, du \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}$$

$$= \left(\int_G \left(\int_G |f(u)|^p \chi_{x^{-1}Q^{-1}}(u)\, du \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}$$

$$= \left(\int_G \left(\int_G |f(u)|^p \chi_{(zv^{-1})^{-1}Q^{-1}}(u)\, du \right)^{\frac{q}{p}} dv \right)^{\frac{1}{q}}$$

$$= \left(\int_G \left(\int_G |f(u)|^p \chi_{vQ}(u)\, du \right)^{\frac{q}{p}} dv \right)^{\frac{1}{q}} = \|f\|_Q.$$

In the second line the substitution $u = t^{-1}$ is used, while the substitution $v = zx^{-1}$ is used in the fourth line.

As shown above, dilation is not an isometry in $W^{p,q}_{[0,1]}(\mathbb{R})$. From this, one might expect that the suitable subset of automorphisms of $\mathbf{B}(G)$ that define isometries are the measure-preserving ones. However, this is still not a sufficient condition. If we consider $G = \mathbb{Z}_4$ with $p = 1$, $q = 2$ $Q_1 = \{0,1\}$ and $Q_2 = \{0,2\}$. Given an element $f \in W^{1,2}(\mathbb{Z}_4)$ its norm with respect to Q_1 is

$$\|f\|_{1,2,Q_1} = \left(\sum_{x=0}^{3} \left(\sum_{t=0}^{3} |f(t)| \chi_{Q_1}(t-x) \right)^2 \right)^{\frac{1}{2}} = \left(\sum_{x=0}^{3} (|f(x)| + |f(x+1)|)^2 \right)^{\frac{1}{2}}$$

$$= \left(\sum_{x=0}^{3} |f(x)|^2 + 2|f(x)||f(x+1)| + |f(x+1)|^2 \right)^{\frac{1}{2}},$$

while the norm with respect to Q_2 is

$$\|f\|_{1,2,Q_2} = \left(\sum_{x=0}^{3} \left(\sum_{t=0}^{3} |f(t)| \chi_{Q_2}(t-x) \right)^2 \right)^{\frac{1}{2}} = \left(\sum_{x=0}^{3} (|f(x)| + |f(x+2)|)^2 \right)^{\frac{1}{2}}$$

$$= \left(\sum_{x=0}^{3} |f(x)|^2 + 2|f(x)||f(x+2)| + |f(x+2)|^2 \right)^{\frac{1}{2}}.$$

We see that the cross terms don't agree, and consequently that the automorphism of $\mathbf{B}(\mathbb{Z}_4)$ defined by $\{0,1,2,3\} \to \{0,2,1,3\}$ does not define an isometry of $W^{1,2}(\mathbb{Z}_4)$.

At this point a problem seems to appear. While the Wiener amalgam space $W^{p,q}(G)$ does not depend on the window Q or the BUPU Ψ, its isometries will. Isometric operators of $W^{p,q}_Q(G)$ may not be isometric operators of $W^{p,q}_{\tilde{Q}}(G)$, as we will show using the reflection operator R on $W^{p,q}(\mathbb{R})$. There are several admissible windows Q that do not make R an isometry. Let for instance $Q = [0,1] \cup [\frac{3}{2}, 2]$. If $f = \chi_Q$, then

$$(F_f(x))^p = \begin{cases} 0, & x \leq -2 \text{ or } x \geq 2 \\ 2+x, & x \in [-2, -\frac{3}{2}] \\ \frac{1}{2}, & x \in [-\frac{3}{2}, -\frac{1}{2}] \\ \frac{3}{2} - |2x|, & x \in [-\frac{1}{2}, \frac{1}{2}] \\ \frac{1}{2}, & x \in [\frac{1}{2}, \frac{3}{2}] \\ 2-x, & x \in [\frac{3}{2}, 2] \end{cases}$$

and

$$(F_{Rf}(x))^p = \begin{cases} 0, & x \leq -4 \text{ or } x \geq 0 \\ 4+x, & x \in [-4, -\frac{7}{2}] \\ -3-x, & x \in [-\frac{7}{2}, -3] \\ 6+2x, & x \in [-3, -\frac{5}{2}] \\ 1, & x \in [-\frac{5}{2}, -2] \\ -1-x, & x \in [-2, -\frac{3}{2}] \\ 2+x, & x \in [-\frac{3}{2}, -1] \\ -x, & x \in [-1, 0] \end{cases}.$$

It follows that $\|f\|_Q^q = \frac{p}{q+p} \left(\left(\frac{1}{2}\right)^{\frac{q+p}{p}} + \left(\frac{3}{2}\right)^{\frac{q+p}{p}} \right) + \left(\frac{1}{2}\right)^{\frac{q}{p}}$ while $\|Rf\|_Q^q = \frac{7}{2} \frac{p}{q+p}$.

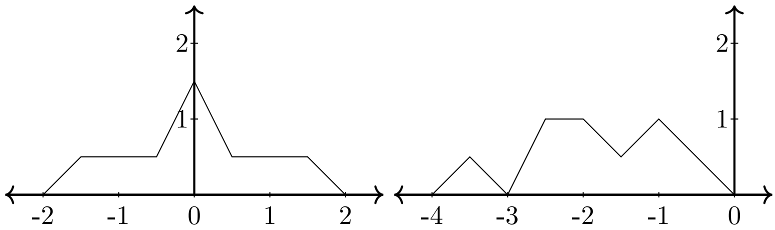

Figure 5: The p-th powers of the control functions of $\chi_{[0,1]\cup[\frac{3}{2},2]}$ and $R(\chi_{[0,1]\cup[\frac{3}{2},2]})$ with respect to the window $[0,1]\cup[\frac{3}{2},2]$.

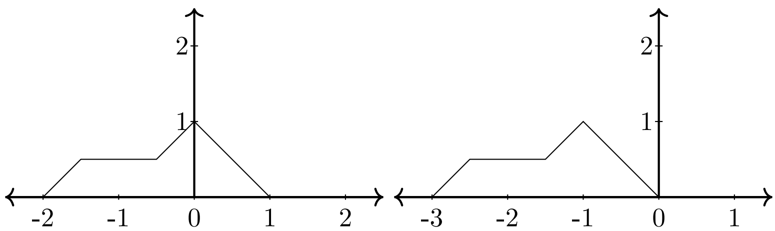

Figure 6: The p-th powers of the control functions of $\chi_{[0,1]}$ and $R(\chi_{[0,1]})$ with respect to the window $[0,1]\cup[\frac{3}{2},2]$.

On the other hand, if we let $g=\chi_{[0,1]}$, then

$$F_g(x)^p=\begin{cases}0, & x\leq -2 \text{ or } x\geq 1,\\ 2+x, & x\in[-2,-\frac{3}{2}]\\ \frac{1}{2}, & x\in[-\frac{3}{2},-\frac{1}{2}]\\ 1-|x|, & x\in[-\frac{1}{2},1]\end{cases}$$

and

$$F_{Rg}(x)^p=\begin{cases}0, & x\leq -3 \text{ or } x\geq 0\\ 3+x, & x\in[-3,-\frac{5}{2}]\\ \frac{1}{2}, & x\in[-\frac{5}{2},-\frac{3}{2}]\\ 2+x, & x\in[-\frac{3}{2},-1]\\ -x, & x\in[-1,0]\end{cases}.$$

From this we see that $\|g\|_Q^q=\frac{2p}{q+p}+\left(\frac{1}{2}\right)^{\frac{q}{p}}=\|Rg\|_Q^q$. Therefore R cannot be an isometry of $W_Q^{p,q}(\mathbb{R})$.
As shown, classifying the invertible isometries on a Wiener amalgam space is notably more difficult than classifying the isometries of an L^p-space. This stems

from the mappings' dependence of the covering Q, and it seems that in order to get a better theory, isometries must be abandoned in favour of a weaker notion. This is the theme of the next subsection. Before that, we will state our guesses for the set of invertible isometries on a Wiener amalgam space.

Definition 3.5. Let G be a locally compact group, and Q an admissible window. We let $\mathrm{Inv}_Q(G)$ be the subset of $\mathrm{Aut}(\mathbf{B}(G))$ such that $\phi \in \mathrm{Inv}_Q(G) \implies \forall\, g \in G\,, \exists\, h \in G$ such that $\phi(gQ) = hQ$.

The following result is a generalization of the example with $Q^{-1} \in \{\gamma Q\}_{\gamma \in G}$.

Lemma 3.6. *Let $1 \leq p, q < \infty$, G a locally compact group and Q an admissible window. Then any element of the semi-direct product $\mathcal{U}(L^\infty) \rtimes \mathrm{Inv}_Q(G)$ is an invertible isometry of $W_Q^{p,q}(G)$.*

Proof. For a pair $(f, \phi) \in \mathcal{U}(L^\infty) \rtimes \mathrm{Inv}_Q(G)$ proceed exactly as in the case above. Use first the substitution $u = \phi(t)$. Then, since $\phi \in \mathrm{Inv}_Q(G)$ there is a $z \in G$ such that $\phi(Q) = zQ$. Use then the substitution $v = z\phi(x)$. $\square$

We know that elements of $\mathcal{U}(L^\infty(G))$ and elements of Inv_Q are isometries. We have yet to find out if there are any other isometries of $W_Q^{p,q}(G)$ than the ones generated by these two groups.

Question 1. Are there invertible isometries on $W_Q^{p,q}(G)$ that do not satisfy the property $|T(\chi_Q)| = \chi_{\gamma Q}$? In other words, is the inclusion

$$\mathcal{U}(L^\infty(G)) \rtimes \mathrm{Inv}_Q(G) \subseteq \mathrm{Isom}(W_Q^{p,q}(G))$$

strict?

Question 2 (Isometries on the real line)**.** Let Γ be the subgroup of $\mathrm{Aut}(\mathbf{B}(\mathbb{R}))$ generated by the translations $\{T_x\}_{x\in\mathbb{R}}$ and the reflection R. For $1 \leq p, q < \infty$, is the inclusion of groups

$$\mathcal{U}(L^\infty(\mathbb{R})) \rtimes \Gamma \subseteq \mathrm{Isom}\left(W_{[0,1]}^{p,q}(\mathbb{R})\right)$$

strict?

3.2 Quasi-Isometries

Definition 3.7. A map $T : X \to Y$ between normed spaces is called a quasi-isometric embedding if there are constants $A > 0,\ B \geq 0$ such that

$$\frac{1}{A}\|x_1 - x_2\|_X - B \leq \|T(x_1) - T(x_2)\|_Y \leq A\|x_1 - x_2\|_X + B$$

for all $x_1, x_2 \in X$. A and B are called the parameters of the embedding. T is called a quasi-isometry if in addition, there is another constant $C > 0$ such that

$$\inf_{x \in X} \|T(x) - y\|_Y \leq C$$

for all $y \in Y$. A quasi-isometry has parameters A, B, C.
If T is a quasi-isometry with $B = 0$, it is called bi-Lipschitz continuous.

Proposition 3.8. *Given a normed space X with two equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$, the identity map* $\mathrm{id} : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ *is a bi-Lipschitz quasi-isometry.*

Proof. As the norms are equivalent, there are constants $C, D > 0$ such that

$$C\|x - y\|_1 \leq \|x - y\|_2 \leq D\|x - y\|_1.$$

Letting $A = \max(\frac{1}{C}, D)$ we see that id is a quasi-isometry with parameters $A, 0$ and 1, and is therefore also bi-Lipschitz. □

Note. Even when a quasi-isometry T is bi-Lipschitz continuous, we will for simplicity's sake just call it a quasi-isometry. In this thesis all quasi-isometries will have $B = 0$ anyways.

Two spaces being quasi-isometric constitutes an equivalence relation, and the quasi-isometry class of a normed space X is referred to as its large-scale geometry. As shown in section 2, any admissible continuous or discrete-type norm on $W^{p,q}(G)$ is equivalent. The identity map of $W^{p,q}(G)$ is therefore a quasi-isometry when considered as a map between $W^{p,q}_Q(G)$ and $W^{p,q}_{\tilde{Q}}(G)$ or $\mathbf{w}^{p,q}(G, \Psi)$. In the large-scale geometry of $W^{p,q}(G)$ it is therefore possible to work with any desired norm. We have seen that there are advantages to having several different norms on $W^{p,q}(G)$ available, and being able to pick a desired norm will be of use to us when discussing operator algebras on $W^{p,q}(G)$.
We state some of the facts shown in section 2 in the large-scale language.

Proposition 3.9. *Let G be a locally compact group, and $p, q \in [1, \infty]$.*

- *For any admissible windows Q_1 and Q_2, the corresponding Wiener amalgam spaces $W^{p,q}_{Q_1}(G)$ and $W^{p,q}_{Q_2}(G)$ are quasi-isometric.*
- *There is a quasi-isometry between $L^p(G)$ and $W^{p,p}(G)$*
- *Given a window Q and a BUPU Ψ on G, not necessarily subordinate to Q, the space $W^{p,q}_Q(G)$ defined by the continuous norm with respect to Q is quasi-isometric to the space $\mathbf{w}^{p,q}(G, \Psi)$ defined by the discrete norm with respect to Ψ.*

- *There is a quasi-isometry between $W^{p,q}(G)^*$ and $W^{p',q'}(G)$.*

It is apparent that any isometry of $W^{p,q}(G)$ will be a quasi-isometry, but are there more? Following Lamperti's theorem one might think that any element of $\mathcal{U}(L^\infty(G)) \rtimes \mathrm{Aut}(\mathbf{B}(G))$ will be a quasi-isometry. We know that multiplying with a character gives an isometry of $W^{p,q}(G)$, and thus a quasi-isometry, but what about $\mathrm{Aut}(\mathbf{B}(G))$? We have not been able to prove that any $\phi \in \mathrm{Aut}(\mathbf{B}(G))$ gives a quasi-isometry, but we know that this is true in a special case, namely for the subgroup $\mathrm{Aff}(G)$ of affine bijections.

Definition 3.10. Let G be a group. A map $T : G \to G$ is affine if there is a group homomorphism $\phi : G \to G$ so that for all $x, y \in G$ we have

$$T(x)^{-1}T(y) = \phi(x^{-1}y).$$

The set of affine bijections includes the group homomorphisms like reflection, rotation and scaling, but it also includes the translations. Thus most of the important elements of $\mathrm{Aut}(\mathbf{B}(G))$ are affine bijections.

When T is an affine bijection, then one can see that the associated homomorphism ϕ is an automorphism. Furthermore, if we let $b = T(e)$, then note that for any $x \in G$

$$b^{-1}T(x) = T(e)^{-1}T(x) = \phi(e^{-1}x) = \phi(x).$$

So actually, any affine transformation may be written as $T(X) = b\phi(x)$ for a $b \in G$ and a group endomorphism ϕ. We associate an affine bijection T with the pair $(b, \phi) \in G \times \mathrm{Aut}(G)$. Note also that if T and S are affine bijections associated to (a, ϕ), and (b, ψ) then TS will also be an affine bijection, as it is associated with the pair $(a\phi(b), \phi\psi)$. Moreover, it can also be seen that T^{-1} is affine and can be associated with $(\phi^{-1}(a^{-1}), \phi^{-1})$, and the identity map is also affine and associated to the pair (e, id). Thus the set $\mathrm{Aff}(G)$ is a subgroup of $\mathrm{Aut}(\mathbf{B}(G))$.

As usual, we denote by u_T the operator on $W^{p,q}(G)$ defined by

$$u_T f(t) = u(T(t)) \cdot \left(\frac{d(\mu \circ T^{-1})}{d\mu}(t)\right)^{\frac{1}{p}},$$

but note that for any Borel set E,

$$\begin{aligned}\mu(T^{-1}(xE)) &= \mu(\phi^{-1}(b^{-1})\phi^{-1}(xE)) = \mu(\phi^{-1}(b^{-1})\phi^{-1}(x)\phi^{-1}(E)) \\ &= \mu(\phi^{-1}(E)) = \mu((\phi^{-1}(b^{-1})\phi^{-1}(E) = \mu(T^{-1}(E)).\end{aligned}$$

So $\mu \circ T^{-1}$ is in fact a Haar measure, and by uniqueness, there is a constant c such that the Radon-Nikodym derivative $\frac{d(\mu \circ T^{-1})}{d\mu}(t)$ is constant and equal to c. We may thus write u_T as

$$u_T f(t) = c^{\frac{1}{p}} \cdot f(T(t)).$$

We now show that any affine bijection gives a quasi-isometry.

Proposition 3.11. *Let G be a locally compact group, let $T \in \mathrm{Aff}(G)$, and $1 \leq p, q \leq \infty$. The mapping $u_T : W^{p,q}(G) \to W^{p,q}(G)$ is a quasi-isometry.*

Proof. Fix an admissible window Q, and within this proof, denote the Wiener norm of $f \in W^{p,q}(G)$ with respect to Q by $\|f\|_Q$. Let $T \in \mathrm{Aff}(G)$ be an affine bijection of G, and write T as $b\phi$. For an $f \in W_Q^{p,q}(G)$ we calculate the norm of $u_T f$. We first compute the control function.

$$\left(F_{u_T f}^Q(x)\right)^p = \int_G |u_T f(t)|^p \, \chi_Q(x^{-1}t)\, dt = \int_G |f(T(t))|^p \cdot c\chi_Q(x^{-1}t)\, dt.$$

We do a change of variables $u = T(t)$ and get

$$\begin{aligned}\left(F_{u_T}^Q(f)(x)\right)^p &= \int_G |f(T(t))|^p \cdot c\chi_Q(x^{-1}t)\, dt \\ &= \int_G |f(u)|^p \, \chi_Q(x^{-1}T^{-1}(u))\, du \\ &= \int_G |f(u)|^p \, \chi_{\phi(Q)}((T(x))^{-1} u)\, du = \left(F_f^{\phi(Q)}(T(x))\right)^p.\end{aligned}$$

The second-to-last equality follows by noting that since T is affine,

$$T^{-1}(u) \in xQ \iff u \in T(xQ) = b\phi(xQ) = b\phi(x)\phi(Q) = T(x)\phi(Q).$$

We now calculate the Wiener norm of $u_T f$. Doing a change of variables $y = T(x)$ in the second line we get

$$\begin{aligned}\|u_T f\|_Q^q &= \int_G \left(F_{u_T f}^Q(x)\right)^q \, dx = \int_G \left(F_f^{\phi(Q)}(T(x))\right)^q \, dx \\ &= \int_G \left(F_f^{\phi(Q)}(y)\right)^q \cdot \frac{1}{c}\, dy = \frac{1}{c}\|F_f^{\phi(Q)}\|_q^q = \frac{1}{c}\|f\|_{\phi(Q)}^q.\end{aligned}$$

Since $\|\cdot\|_Q$ and $\|\cdot\|_{\phi(Q)}$ are equivalent norms there are constants A, B so that

$$A\|f\|_Q \leq \|f\|_{\phi(Q)} \leq B\|f\|_Q,$$

and so

$$\frac{A}{c}\|f\|_Q \leq \|u_T f\|_Q \leq \frac{B}{c}\|f\|_Q$$

Letting $L = \max(\frac{A}{c}, \frac{c}{B})$ gives us

$$\frac{1}{L}\|f\|_Q \leq \|u_T f\|_Q \leq L\|f\|_Q$$

which proves that u_T is a quasi-isometry. $\square$

4 Operator algebras on Wiener amalgam spaces

4.1 Banach algebras and operator algebras

Definition 4.1. A vector space A over $\mathbb{K}$ is called an algebra if there is an operation $\cdot : A \times A \to A$ called multiplication such that for $a, b, c \in A$ and $\lambda \in \mathbb{K}$:

- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
- $(a + b) \cdot c = a \cdot c + b \cdot c$,
- $a \cdot (b + c) = a \cdot b + a \cdot c$,
- $\lambda(a \cdot b) = (\lambda a) \cdot b = a \cdot (\lambda b)$.

Moreover, if A is a complete normed space and

$$\|a \cdot b\| \leq \|a\| \|b\|,$$

for all $a, b \in A$, A is called a Banach algebra.

Note. A Banach algebra A is called unital if there is a unique element $\mathbf{1} \in A$ such that $a \cdot \mathbf{1} = a = \mathbf{1} \cdot a$ for any $a \in A$, and furthermore, $\|\mathbf{1}\| = 1$.

There are several examples of Banach algebras. Both the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are unital Banach algebras with respect to their standard Banach space structure and standard multiplication, and so are $\mathbb{R}^n$ and $\mathbb{C}^n$. For a locally compact Hausdorff space X the set $C_0(X)$ of continuous functions vanishing at infinity is a Banach space with the supremum norm, and with multiplication defined by $(f \cdot g)(x) = f(x) \cdot g(x)$ it is a Banach algebra. Likewise, for a measure space (X, B, μ), $L^\infty(X)$ is a Banach algebra under multiplication. It is even unital, with unit $\mathbf{1} = \chi_X$. There are Banach algebras that does not use pointwise multiplication as a product. Let for example G be a locally compact group with Haar measure, and consider$L^1(G)$. We already know that it is a Banach space, and by Young's inequality we have that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$, so convolution gives a submultiplicative product on $L^1(G)$.

If we let G be a locally compact group, one interesting Banach algebra from our perspective is the Wiener amalgam $W^{\infty,1}(G)$. This space is called the Wiener algebra, and was introduced by Wiener in 1932. It is a Banach space, and it is an algebra under pointwise multiplication, as

$$\|fg\|_{\infty,1} = \int_G \|fg\chi_{xQ}\|_\infty \, dx \leq \int_G \|f\chi_{xQ}\|_\infty \|g\chi_{xQ}\|_\infty \, dx$$

$$\leq \sqrt{\int_G \|f\chi_{xQ}\|_\infty^2 \, dx \int_G \|g\chi_{xQ}\|_\infty^2 \, dx} \leq \|f\|_{\infty,1}\|g\|_{\infty,1}$$

for all $f, g \in W^{\infty,1}(G)$. The second inequality is Cauchy-Schwarz (as $W^{\infty,1} \subseteq L^2$) and the third inequality follows from the global component inclusion of amalgams. Thus, the Wiener algebra is in fact a Banach algebra.

Given a wiener amalgam space $W^{p,q}(G)$ with any norm, continuous or discrete, the set $\mathcal{B}(W^{p,q}(G))$ of bounded linear operators on $W^{p,q}(G)$ is a Banach space under the induced operator norm. With multiplication given by composition it is also a Banach algebra. Our main object of study will be the so-called operator algebras of $W^{p,q}(G)$. Informally speaking, these algebras are closed subspaces of $\mathcal{B}(W^{p,q}(G))$. This definition will be formalized in the following way.

Definition 4.2. Let $1 \leq p, q \leq \infty$. A Banach space X is a $W^{p,q}$-space if there is a locally compact group G and either an admissible window Q or a BUPU Ψ such that there is either

- An isometric isomorphism $X \to W_Q^{p,q}(G)$, or
- An isometric isomorphism $X \to \mathbf{w}^{p,q}(G, \Psi)$.

Definition 4.3. Let A be a Banach algebra, and E a $W^{p,q}$-space.

- A representation of A on E is a continuous algebra homomorphism $\phi : A \to \mathcal{B}(E)$.
- If $\phi : A \to \mathcal{B}(E)$ is quasi-isometric, A is said to be quasi-isometrically represented on E, and ϕ is a quasi-isometric representation.
- If $\phi : A \to \mathcal{B}(E)$ is isometric, A is said to be isometrically represented on E, and ϕ is an isometric representation.

Definition 4.4. Let A be a Banach algebra, and let $1 \leq p, q \leq \infty$. A is a $W^{p,q}$-operator algebra if there is a $W^{p,q}$-space E such that A can be quasi-isometrically represented on E.

The theoretical advantage of defining Wiener space operator algebras in this way, rather than as any closed subspace of $\mathcal{B}(E)$ for some $W^{p,q}$-space E is that it allows us to consider more abstractly defined Banach algebras, not just algebras of functions.

Note. While the operator algebra is required to be quasi-isometrically represented on a Wiener amalgam space, it may be difficult to prove that a given representation is quasi-isometric. The standard way to do this is to pick a window Q

or BUPU Ψ for $W^{p,q}(G)$ so that $A \to \mathcal{B}(W_Q^{p,q}(G))$ or $A \to \mathcal{B}(\mathbf{w}^{p,q}(G, \Psi)$ is an isometric representation. It will then be quasi-isometric in any other norm, and thus a quasi-isometric representation of $W^{p,q}(G)$.

Many of our already mentioned Banach algebras can be represented on a Wiener amalgam space. Trivially, $\mathcal{B}(W^{p,q}(G))$ can be represented on $W^{p,q}(G)$ using the identity, but there are others as well. We give some examples.

Let $1 \leq p, q \leq \infty$. $\mathbb{C}$ can be represented on any $W^{p,q}$-space E by the representation $\phi : \mathbb{C} \to \mathcal{B}(E)$ defined by

$$\phi_a \xi = a \cdot \xi$$

for $a \in \mathbb{C}$, $\xi \in E$, and where the dot is scalar multiplication. By homogeneity of the norm, $\|\phi_a(\xi)\|_{p,q} = \|a\xi\|_{p,q} = |a|\|\xi\|_{p,q}$, so ϕ is in fact an isometric representation. It follows that $\mathbb{C}$ is a $W^{p,q}$-operator algebra for any p and q.

Let $1 \leq p, q \leq \infty$ and consider a locally compact group. We want to represent $L^\infty(G)$ quasi-isometrically on a $W^{p,q}$-space. Consider an admissible window Q by defining the multiplication operator $m : L^\infty(G) \to \mathcal{B}(W^{p,q}(G))$ as in section 2.1 by

$$m_f \xi(t) = f(t) \cdot \xi(t)$$

for $f \in L^\infty(G), \xi \in W^{p,q}(G)$ and $t \in G$. Since, by proposition 2.9, the Wiener amalgam spaces are closed with respect to pointwise products, we have that $\|m_f \xi\|_{p,q} \leq \mu(Q)\|f\|_\infty \|\xi\|_{p,q}$, and so if one picks a Q with $\mu(Q) = 1$, m is an isometric representation of $L^\infty(G)$ on $W_Q^{p,q}(G)$. Thus m is a quasi-isometric representation on $W^{p,q}(G)$.

For any $n \in \mathbb{N}$, the complex-valued $n \times n$-matrices is a Banach algebra, and it is also a Wiener space operator algebra. We let $E = W^{p,q}(\mathbb{Z}_n)$, and construct a representation of M_n on E by the mapping $\phi : M_n \to \mathcal{B}(E)$ defined by

$$\phi(M)(\xi) = M \cdot \xi,$$

where the elements of E are regarded as vectors in $\mathbb{C}^n$. This mapping is a homomorphism of algebras, and therefore a representation. Thus M_n is an algebra of operators on E with respect to the operator norm on E. If we want to emphasize what Wiener space we are taking the norm with respect to, we write $M_n^{p,q}$ or M_n^p.

It is well known that the set of compact operators on any Banach space X is an ideal of $\mathcal{B}(X)$. Thus, given any Wiener amalgam space E the inclusion map $\iota : K(E) \to \mathcal{B}(E)$ composed with the evaluation representation of $\mathcal{B}(E)$ on E is an isometric representation of $K(E)$ and $K(E)$ is a $W^{p,q}$-operator algebra.

Similarly, since $W^{\infty,1}(G)$ is a closed subalgebra of $L^\infty(G)$ we can use the inclusion of $W^{\infty,1}(G)$ into $L^\infty(G)$ together with the representation m of $L^\infty(G)$ to get a representation of $W^{\infty,1}(G)$ on $W_Q^{p,q}(G)$.

As already mentioned, $\mathbb{C}$ is a Banach algebra with its usual product. However, one can define several products on $\mathbb{C}$. One of them is the trivial product, $a \cdot b = 0$ for all $a, b \in \mathbb{C}$. The complex numbers under this product is denoted by $\mathbb{C}_0$. $\mathbb{C}_0$ is also an operator algebra. This can easily be seen by noting that the mapping $\phi : \mathbb{C}_0 \to M_2^p$ given by

$$\phi(z) = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}$$

is an injective homomorphism of algebras. Thus the closure of its image is a subalgebra of M_2^p, and thus an operator algebra. However, any representation of $\mathbb{C}_0$ on a Wiener amalgam space will be the zero representation. Algebras like $\mathbb{C}_0$ are called degenerate.

4.2 Group algebras

Let us now consider representations of groups on a Wiener amalgam space. We define representations of groups similarly to representations of algebras.

Definition 4.5. Let $1 \leq p, q \leq \infty$, G a locally compact group, and E a $W^{p,q}$-space. A representation of a group G on a $W^{p,q}$-space E is a quasi-isometric group homomorphism $\phi : G \to \mathcal{B}(E)$.

In this thesis, the most important case of group representations is the following.

Definition 4.6. Let G be locally compact, and E a $W^{p,q}$-space. A group representation $\phi : G \to \mathcal{B}(E)$ is called isometric if ϕ_g is an isometry on E. We denote isometric representations by $\phi : G \to \mathrm{Isom}(E)$.

As an example of isometric group representations, consider any locally compact group G, an admissible window Q and the corresponding Wiener amalgam space $W^{p,q}(G)$. As shown in section 2, the Wiener amalgam spaces are translation invariant for any $1 \leq p, q \leq \infty$. Thus, the left regular representation $\mathrm{Lt} : G \to \mathcal{B}(W^{p,q}(G))$ defined by

$$\mathrm{Lt}_g f(t) = T_g f(t) = f(g^{-1}t)$$

is an isometric representation on G.

The importance of isometric group representations comes from the fact that they induce representations of a particular algebra on the same Wiener amalgam space. As noted above, the space $L^1(G)$ is a Banach algebra under convolution and we will now show that there is a correspondance between isometric representations of G on a $W^{p,q}$-space E and quasi-isometric representations of the algebra $L^1(G)$ on E. Recall that we denote the duality pairing $\phi_g(f)$ of dual Wiener amalgam spaces by $\langle f, g\rangle$.

Proposition 4.7. *Let $1 \leq p, q \leq \infty$, G be a locally compact group, and Q an admissible window. Consider the $W^{p,q}$-space $E = W_Q^{p,q}(G)$ and let $\pi : G \to \mathrm{Isom}(W^{p,q}(G))$ be an isometric representation of G on E. For every $f \in L^1(G)$ there is a unique $\phi_f^\pi \in \mathcal{B}(E)$ such that*

$$\langle \phi_f^\pi v, w\rangle = \int_G f(x)\langle \pi_x v, w\rangle \, dx$$

for any $v \in E$ and $w \in E^*$. The induced map $\phi^\pi : L^1(G) \to \mathcal{B}(E)$ is a representation of $L^1(G)$ on E.

Proof. For a fixed $v \in E$ the map $\psi(w) = \int_G f(x)\langle \pi_x v, w\rangle \, dx$ is linear, and as

$$\left| \int_G f(x)\langle \pi_x v, w\rangle \, dx \right| \leq \int_G |f(x)\langle \pi_x v, w\rangle| \, dx$$

$$\leq \int_G |f(x)| \|\pi_x v\| \|w\| \, dx = \|f\|_1 \|v\|_E \|w\|_{E^*},$$

it is also bounded. By duality, there is a unique function $\phi_f^\pi v \in E$ so that

$$\langle \phi_f^\pi v, w\rangle = \psi(w) = \int_G f(x)\langle \pi_x v, w\rangle \, dx.$$

The induced map $v \to \phi_f^\pi v$ is linear and, as shown above, also bounded. Lastly

$$\langle \phi_{f*g}^\pi v, w\rangle = \int_G f * g(x)\langle \pi_x v, w\rangle \, dx = \int_G \int_G f(y) g(y^{-1}x) \, dy \langle \pi_x v, w\rangle \, dx$$

$$= \int_G f(y) \int_G g(y^{-1}x)\langle \pi_x v, w\rangle \, dx \, dy = \int_G f(y) \int_G g(y^{-1}x)\langle \pi_y \pi_{y^{-1}} \pi_x v, w\rangle \, dx \, dy$$

$$= \int_G f(y)\langle \pi_y \phi_g^\pi v, w\rangle \, dy = \langle \phi_f^\pi \phi_g^\pi v, w\rangle$$

for all $w \in E^*$, so ϕ^π is also an algebra homomorphism. $\square$

Note. We usually denote the action of ϕ_f^π on v by $\int_G f(x)\pi_x v\ dx$ and interpret the integral weakly (see appendix D).

In the special case where $\pi = \mathrm{Lt}$ we can write the representation explicitly. We let $f \in L^1(G), \xi \in E$ and $\eta \in E^*$ and calculate.

$$\langle \phi_f^{\mathrm{Lt}}(\xi), \eta\rangle = \int_G f(g)\langle \mathrm{Lt}_g \xi, \eta\rangle\ dg$$

$$= \int_G f(g) \int_G \xi(g^{-1}h)\overline{\eta(h)}\ dh\ dg = \int_G \left(\int_G f(g)\xi(g^{-1}h)\ dg\right) \overline{\eta(h)}\ dh.$$

Thus $\phi_f^{\mathrm{Lt}}\xi(t) = \int_G f(s)\xi(s^{-1}t)\ ds = f * \xi(t)$. We also call this representation the left regular representation, but we denote it by $\lambda^{p,q} : L^1(G) \to \mathcal{B}(E)$. We emphasize the exponents p and q of E because unlike Lt, which is isometric, the norm of $\lambda^{p,q}$ depends on E. Note that the proof of theorem 2.24 only guarantees that $\|\lambda^{p,q}\| \leq 2$ while using the integrated form, we actually get that $\|\lambda^{p,q}\|$ is a quasi-isometric representation.

The left regular representation $\lambda^{p,q}$ is really all we need to define interesting operator algebras, but for completion's sake, we prove the 1-1 correspondance of isometric representations of G and representations of $L^1(G)$.

We say that a representation $\phi : L^1(G) \to \mathcal{B}(E)$ is non-degenerate if

$$\overline{\mathrm{Span}\{\phi(f)e, f \in L^1(G), e \in E\}} = E.$$

Moreover, if ϕ is a representation of $L^1(G)$ on E, we denote by ϕ' the representation of $L^1(G)$ on E^* defined by $(\phi'_f)(b) = (\phi_f)'(b)$.

Proposition 4.8. *Let ϕ be a non-degenerate representation of $L^1(G)$ on E. Then there is a unique isometric representation π^ϕ of G on E such that*

$$\langle \phi_f v, w\rangle = \int_G f(x)\langle \pi_x^\phi v, w\rangle\ dx.$$

Furthermore, the maps $\phi \to \pi^\phi$ and $\pi \to \phi^\pi$ are bijections between the set of isometric representations of G on E and the non-degenerate representations of $L^1(G)$ on E, and mutual inverses of each other.

Proof. We define a representation on the dense subspace $\phi(L^1(G))E$. Define π^ϕ by

$$\pi_x(\phi_f v) = \phi_{T_x f} v.$$

Note that $T_{x^{-1}}g * T_x f = g * f$. Given $f, g \in L^1(G)$, $v \in E$ and $u \in E^*$ we get that

$$\begin{aligned}\langle \pi_x^\phi \phi_f v, \pi_x^\phi (\phi_g)' w\rangle &= \langle \phi_{T_x f} v, (\phi_{T_x g})' w\rangle = \langle \phi_{T_x g} \phi_{T_x f} v, w\rangle \\ &= \langle \phi_{T_x g * T_x f} v, w\rangle = \langle \phi_{g*f} v, w\rangle = \langle \phi_g \phi_f v, w\rangle \\ &= \langle \phi_f v, (\phi_g)' w\rangle.\end{aligned}$$

It follows that π_x^ϕ extends to a well defined isometric operator on E with inverse $\pi_{x^{-1}}^\phi$ and that $\pi_x^\phi \pi_y^\phi = \pi_{xy}^\phi$. Thus the map $x \to \pi_x^\phi v$ is continuous for all $v \in E$ and π^ϕ is a unitary representation of G.

Given $f \in L^1(G)$ we want that $\phi_f^{\pi^\phi} = \phi_f$. By continuity, it is sufficient to show that $\langle \phi_f^{\pi^\phi} \phi_g v, w\rangle = \langle \phi_f \phi_g v, w\rangle$ for all $f, g \in C_c(G)$, $v \in E$ and $w \in E^*$. We get that

$$\begin{aligned}\langle \phi_f^{\pi^\phi} \phi_g v, w\rangle &= \int_G f(x) \langle \pi_x^\phi \pi_g v, w\rangle \, dx = \int_G \langle \phi(f(x) T_x g) v, w\rangle \, dx \\ &= \left\langle \phi_{\int_G f(x) T_x g dx} v, w \right\rangle = \langle \phi_{f*g} v, w\rangle \\ &= \langle \phi_f \phi_g v, w\rangle.\end{aligned}$$

Conversely, we also have that

$$\int_G f(x) \langle \pi_x^{\phi^\pi} v, w\rangle \, dx = \langle \phi_f^\pi v, w\rangle = \int_G f(x) \langle \pi_x v, w\rangle \, dx,$$

So there is actually a 1-1 correspondance between isometric representations of G and representations of $L^1(G)$ on E. ☐

Let us now get back to defining some $W^{p,q}$-operator algebras.

Definition 4.9. Let G be a locally compact abelian group, Q an admissible window, and $p, q \in [1, \infty)$.

We define the algebra of p, q-pseudofunctions to be

$$F_{\lambda,Q}^{p,q}(G) = \overline{\lambda^{p,q}(L^1(G))}^{\|\cdot\|} \subseteq \mathcal{B}(W_Q^{p,q}(G)).$$

Just as with the definition of a Wiener amalgam space, it is initially unclear whether the definition of the pseudofunctions depend on the window Q or not. While the norm of the algebra of p, q-pseusofunctions does depend on the choice of Q, it is in a large-scale sense independent of Q.

Lemma 4.10. *Let G be a locally compact group. If Q and P are admissible windows, there is a quasi-isometry between $F_{\lambda,Q}^{p,q}(G)$ and $F_{\lambda,P}^{p,q}(G)$.*

To prove independence of window we need another lemma.

Lemma 4.11. *Let X be a normed space, and $\|\cdot\|_1$ and $\|\cdot\|_2$ two equivalent norms on X. Then the induced operator norms $\|\cdot\|_{\mathrm{op},1}$ and $\|\cdot\|_{\mathrm{op},2}$ on $\mathcal{B}(X)$ are equivalent.*

Proof. Since $\|\cdot\|_1 \approx \|\cdot\|_2$ there are constants $C, D > 0$ such that

$$C\|x\|_1 \leq \|x\|_2 \leq D\|x\|_1,$$

for all $x \in X$. For a $T \in \mathcal{B}(X)$ and an $x \in X$ with $\|x\|_2 \leq 1$ we then get

$$\|Tx\|_2 \leq D\|Tx\|_1 \leq D\|T\|_{\mathrm{op},1}\|x\|_1 \leq \frac{D}{C}\|T\|_{\mathrm{op},1}\|x\|_2 \leq \frac{D}{C}\|T\|_{\mathrm{op},1}.$$

Similarly, if $\|x\|_1 \leq 1$ we get

$$\|Tx\|_1 \leq \frac{1}{C}\|Tx\|_2 \leq \frac{1}{C}\|T\|_{\mathrm{op},2}\|x\|_2 \leq \frac{D}{C}\|T\|_{\mathrm{op},2}\|x\|_1 \leq \frac{D}{C}\|T\|_{\mathrm{op},2}.$$

By taking the supremum in both expressions above we therefore get

$$\frac{C}{D}\|T\|_{\mathrm{op},1} \leq \|T\|_{\mathrm{op},2} \leq \frac{D}{C}\|T\|_{\mathrm{op},1}$$

and the operator norms are equivalent. □

Let us now prove independence of localization window.

Proof. As Q and P induce equivalent norms on $W^{p,q}(G)$, the operator norms are also equivalent. As $\lambda_P^{p,q}(L^1(G))$ and $\lambda_Q^{p,q}(L^1(G))$ are identical when regarded as sets, their closures in the respective operator norm will therefore also be identical. It follows that the identity map of $L^1(G)$ extends to a quasi-isometry of algebras. □

Quasi-isometries allow us to right away classify the p, q-pseudofunctions of certain groups. If G is discrete, we know that $W^{p,q}(G)$ is quasi-isometric to $L^q(G)$. It follows that the p, q-pseudofunctions on G are quasi-isometric to the q-pseudofunctions on G in the sense of Gardella's paper [5]. Similarly, if G is compact then $F_\lambda^{p,q}(G) \cong_q F_\lambda^p(G)$. Some of these algebras can in fact be explicitly described.

Following Gardella we know that the p-pseudofunctions on an abelian group can all be characterized by a set of generators with relations. As an application we characterize the p, q-pseudofunctions on $\mathbb{Z}_n$.

$F_\lambda^{p,q}(\mathbb{Z}_n)$ is the universal L^p-operator algebra generated by invertible isometries $\{u_i\}_{i=0}^{n-1}$ on $W^{p,q}(\mathbb{Z}_n)$ subject to the relations $u_i u_j = u_{i+j \mod n}$. This algebra is quasi-isometric to $\mathbb{C}^n$.

4.2.1 Varying the exponents of $F_\lambda^{p,q}(G)$

In [5] Gardella proves that there is an isometric anti-isomorphism between the algebra of p-pseudofunctions and the algebra of p'-pseudofunctions. We wish to extend this result to p, q-pseudofunctions. We first introduce an important tool.

Definition 4.12. Let A be a Banach algebra. The opposite algebra A^{opp} is the Banach space A equipped with the opposite multiplication $a \cdot_{\text{opp}} b = b \cdot a$. Any representation of A^{opp} is naturally represented with an anti-representation of A.

The importance of the opposite algebra comes from its properties whenever A is a $W^{p,q}$-operator algebra.

Lemma 4.13. *Given $p, q \in (1, \infty)$, an algebra A is a $W^{p,q}$-operator algebra if and only if A^{opp} is a $W^{p',q'}$-operator algebra.*

Proof. We only show one direction, as the other is analogous. Recall that for a bounded operator $f \in \mathcal{B}(X)$, where X is an arbitrary Banach space, there exists an adjoint operator $f' \in \mathcal{B}(X')$. The adjoint operator satisfies $\|f'\| = \|f\|$ and $(fg)' = g'f'$. Now assume that A is a $W^{p,q}$-operator algebra. Fix a quasi-isometric representation $\phi : A \to \mathcal{B}(E)$ for a $W^{p,q}$-space E. We then define a linear map $\phi' : A \to \mathcal{B}(E')$ by $\phi'(a) = \phi(a)'$. Since ϕ is quasi-isometric we have that for any two elements $a, b \in A$

$$\|\phi'(a)\|_{\text{op},p',q'} = \|\phi(a)'\|_{\text{op},p',q'} = \|\phi(a)\|_{\text{op},p,q} \leq D\|a\| \qquad \text{and}$$

$$\|\phi'(a)\|_{\text{op},p',q'} = \|\phi(a)\|_{\text{op},p,q} \geq \frac{1}{D}\|a\|.$$

So ϕ' is quasi-isometric. We also have

$$\phi'(a \cdot b) = \phi(a \cdot b)' = \phi(b)' \cdot \phi(a)' = \phi'(b) \cdot \phi'(a).$$

So ϕ' is a quasi-isometric anti-representation of A on E', or in other words, a quasi-isometric representation of A^{opp} on E'. Therefore A^{opp} is a $W^{p',q'}$-operator algebra. $\square$

We want to use anti-representations to find isometric isomorphisms between algebras of p, q-pseudofuntions. We let G be a locally compact group, Δ its modular function, and fix a Q such that $Q^{-1} \in \{\gamma Q\}_{\gamma \in G}$. Given $f \in W_Q^{1,1}(G)$ we define $f^{\#} : G \to \mathbb{C}$ by $f^{\#}(s) = \Delta(s^{-1})\overline{Rf(s)} = \Delta(s^{-1})\overline{f(s^{-1})}$. As reflection and conjugation are isometries, $\#$ is a well defined map from $W_Q^{1,1}(G)$ into itself. We collect some properties of the function $\#$ in the following proposition.

Proposition 4.14. *Let G and Q be as above.*

- *$\#$ is an anti-multiplicative isometric map of order* 2.
- *Let $p, q \in (1, \infty)$. Then $\lambda_Q^{p,q}(f)' = \lambda_Q^{p',q'}(f^{\#})$.*

Proof. $\#$ is clearly a linear map of order 2. Moreover, for any $f \in W^{1,1}(G)$, we have that

$$\|f^{\#}\|_{1,1} = |Q| \int_G |\Delta(s^{-1})\overline{f(s^{-1})}| \, ds = |Q| \int_G |f(s)| = \|f\|_{1,1}.$$

So $\#$ is also an isometry. Lastly, if $g \in W^{1,1}(G)$ as well, then

$$(f * g)^{\#}(s) = \Delta(s^{-1})(\overline{f * g})(s^{-1}) = \Delta(s^{-1}) \int_G \overline{f(t)g(t^{-1}s^{-1})} \, dt$$

$$= \Delta(s^{-1}) \int_G \overline{f(s^{-1}u)g(u^{-1})} \, du = \int_G \Delta(s^{-1})\Delta(u)\Delta(u^{-1})\overline{f(s^{-1}u)g(u^{-1})} \, du$$

$$= \int_G \Delta(u^{-1})\overline{g(u^{-1})}\Delta(s^{-1}u)\overline{f(s^{-1}u)} \, du = \int_G g^{\#}(u)f^{\#}(u^{-1}s) \, du = \left(g^{\#} * f^{\#}\right)(s)$$

Where we have used equation 5 in the second line. Thus $\#$ is an anti-multiplicative isometric isomorphism. In order to prove the second claim, note that since $\#$ is a map of order two, the statement is equivalent to showing that $\lambda_Q^{p,q}(f^{\#})' = \lambda_Q^{p',q'}(f)$ Given $u \in W^{p,q}(G)$ and $v \in W^{p',q'}(G)$ and $f \in W^{1,1}(G)$, then if we use equation 6 from appendix A twice, we get,

$$\langle \lambda_Q^{p,q}(f^{\#})(u), v \rangle = \langle f^{\#} * u, v \rangle = \int_G (f^{\#} * u)(s)\overline{v(s)} \, ds$$

$$= \int_G \left(\int_G \Delta(t^{-1})f^{\#}(st^{-1})u(t) \, dt \right) \overline{v(s)} \, ds$$

$$= \int_G \int_G \Delta(t^{-1})\Delta(ts^{-1})\overline{f(ts^{-1})}u(t)\overline{v(s)} \, ds \, dt$$

$$= \int_G \left(\int_G \Delta(s^{-1}) \overline{f(ts^{-1})v(s)}\, ds \right) u(t)\, dt = \int_G \overline{(f * v)(t)} u(t)\, dt$$

$$= \langle u, f * v \rangle = \langle u, \lambda_Q^{p',q'}(f)(v) \rangle.$$

It follows that $\left(\lambda_Q^{p,q}(f^{\#})\right)' = \lambda_Q^{p',q'}(f)$. ☐

Note. The second statement of the proposition says that the following diagram commutes.

$$\begin{tikzcd} L^1(G) \arrow[r, "\lambda^{p,q}"] \arrow[d, "\#"] & \mathcal{B}(W_Q^{p,q}(G)) \arrow[d, "\mathrm{adj}"] \\ L^1(G) \arrow[r, "\lambda^{p',q'}"] & \mathcal{B}(W_Q^{p',q'}(G)) \end{tikzcd}$$

In light of the first part of proposition 4.14, we immediately get the following theorem.

Theorem 4.15. *Let G be locally compact and $p, q \in (1, \infty)$. Then $\#$ extends to a quasi-isometric anti-isomorphism of $F_\lambda^{p,q}(G)$ and $F_\lambda^{p',q'}(G)$. If G is abelian, then $\#$ extends to a proper quasi-isomorphism. For any fixed Q, $\#$ extends to an isometric anti-isomorphism between the corresponding algebras $F_{\lambda,Q}^{p,q}(G)$ and $F_{\lambda,Q}^{p',q'}(G)$.*

Proof. $\#$ induces a quasi-isometric anti-isomorphism between the images of the two representations, which extends to a quasi-isometric anti-isomorphism between the closures. ☐

A consequence of the preceeding theorem is that we can limit ourselves only to the p, q-pseudofunctions where the exponents p, q are in $[1, 2]$.

4.3 The full group algebra

When considering L^p-operator algebras, one can also define a "universal" group algebra, called the full group algebra. It is defined to be the completion of $L^1(G)$ in the following norm.

$$\|f\|_{F^p(G)} = \sup\{\|\pi(f)\|_{\mathrm{op}},\ \pi : L^1(G) \to \mathcal{B}(E) \text{ contractive}\},$$

where E varies over all L^p-spaces.

In our setting we are tempted to define the full group $W^{p,q}$-operator algebra of G as the completion of $L^1(G)$ in the norm

$$\|f\|_{F^{p,q}(G)} = \sup\{\|\pi(f)\|_{\mathrm{op}},\ \pi : L^1(G) \to \mathcal{B}(E) \text{ continuous}\},$$

where E varies over all possible $W^{p,q}$-spaces. However, while this construction does work in some cases, like for discrete or compact groups, it does not work in general. As an example let us fix p and q, let $G = \mathbb{R}$ and consider the norm $\|\cdot\|_{F^{p,q}(\mathbb{R})}$.

Firstly note that the norm of any non-zero $f \in W^{p,q}(\mathbb{R})$ may be made arbitrarily large by considering a larger window. As an illustrative example, consider the characteristic function $f = \chi_{[a,b]} \in W^{p,q}(\mathbb{R})$. We consider the windows $Q_n = [0,n]$. We pick an n such that $\lambda([a,b]) \leq n$. We denote the p,q-norm with respect to the window Q_n by $\|\cdot\|_{Q_n}$. We calculate the control function $F_f^{Q_n}$.

$$F_f^{Q_n}(x) = \left(\int_{\mathbb{R}} \left|\chi_{[a,b]}(t)\chi_{Q_n}(t-x)\right|^p\ dt\right)^{\frac{1}{p}}$$

$$= \left(\int_{\mathbb{R}} \chi_{[a,b]\cap(Q_n+x)}(t)\ dt\right)^{\frac{1}{p}} = \left(\lambda\left([a,b]\cap[x,x+n]\right)\right)^{\frac{1}{p}}.$$

The final expression can be explicitly calculated, and so we get that

$$F_f^{Q_n}(x) = \begin{cases} 0, & x+n \leq a \text{ or } x \leq b \\ (x+n-a)^{\frac{1}{p}}, & x \in [a-n, b-n] \\ (b-a)^{\frac{1}{p}}, & x \in [b-n, a] \\ (b-x)^{\frac{1}{p}}, & x \in [a,b] \end{cases}.$$

Thus we have

$$\|f\|_{Q_n}^q = \int_{\mathbb{R}} \left(F_f^{Q_n}(x)\right)^q\ dx$$

$$= \int_{a-n}^{b-n} (x+n-a)^{\frac{q}{p}}\ dx + \int_{b-n}^{a} (b-a)^{\frac{q}{p}}\ dx + \int_a^b (b-x)^{\frac{q}{p}}\ dx.$$

Focusing our attention on the middle term we have that

$$\int_{b-n}^{a} (b-a)^{\frac{q}{p}}\ dx = (b-a)^{\frac{q}{p}}(n-b+a).$$

This expression diverges as $n \to \infty$, and so $\|f\|_{Q_n} \to \infty$. Since the norm of any non-zero characteristic function can be made arbitrarily large, so can the norm of any simple function, by linearity. Extending this by density proves the claim for any $f \in L^1(\mathbb{R})$.

With this in mind, we consider the sequence of left regular representations

$$\left\{\lambda_n^{p,q} : L^1(\mathbb{R}) \to \mathcal{B}\left(W^{p,q}_{Q_n}(\mathbb{R})\right)\right\}_{n=1}^{\infty}.$$

Once again, we first consider a characteristic function $\chi_{[a,b]}$. Note that since $e^{-x^2} \in W^{p,q}(\mathbb{R})$ we may construct an approximate identity $\{K_m\}_{m=1}^{\infty}$ that is normalized in $W^{p,q}_{Q_n}(\mathbb{R})$. We then get that

$$\|\chi_{[a,b]} * K_m\|_{Q_n} \to \|\chi_{[a,b]}\|_{Q_n}$$

as m approaches infinity. This implies that

$$\|\lambda_n^{p,q}(\chi_{[a,b]})\|_{\text{op}} \geq \|\chi_{[a,b]}\|_{Q_n}$$

and thus we get

$$\begin{aligned}\|\chi_{[a,b]}\|_{F^{p,q}(\mathbb{R})} &= \sup\{\|\pi(f)\|_{\text{op}},\ \pi : L^1(G) \to \mathcal{B}(E) \text{ continuous}\} \\ &\geq \|\lambda_n^{p,q}(f)\|_{\text{op}} \geq \|\chi_{[a,b]}\|_{Q_n}\ \forall\, n \in \mathrm{N}.\end{aligned}$$

Since this holds for all n, it means that $\|\chi_{[a,b]}\|_{F^{p,q}(\mathbb{R})} = \infty$.

Since $\|\cdot\|_{F^{p,q}(\mathbb{R})} = \infty$ for all non-zero characteristic functions, we can extend the result to hold for simple functions by linearity, and to all non-zero L^1-functions by density. It follows that $\|\cdot\|_{F^{p,q}(\mathbb{R})}$ is not a norm. If we want a general Wiener amalgam analogue of the full group L^p-operator algebra we need to approach it differently.

4.4 Crossed products

The goal of this section is to define cross product algebras. These are generalizations of group algebras to G-spaces, that is, a pair (G, X) consisting of a group G acting on a locally compact topological space X. We want to see how the action of G on X is incorporated into the cross product algebra. This section is inspired by [14].

4.4.1 Group actions and dynamical systems

We first recall some terminology related to group actions. A continuous group action of G on a topological space X is a group homomorphism $\sigma : G \to \text{Homeo}(X)$.

Given $g \in G$ and $x \in X$, the action of g on x is denoted by $\sigma(g, x), g \cdot x$ or simply by gx. If G acts on X we will write $G \curvearrowright X$, and call X a G-space. The action is said to be free if for a $g \in G$, the existence of an $x \in X$ such that $gx = x$ implies that $g = e$. The action is said to be essentially free if for any $g \in G \setminus \{e\}$ the set $\{x \in X \text{ s.t } gx = x\}$ has empty interior. If X is in addition a normed space, then a group action α of G on X is called isometric if α_g is an isometry for each $g \in G$. A G-space is called minimal if for every $x \in X$, $\overline{\{g \cdot x, g \in G\}} = X$. We give some examples of group actions.

One example is the trivial action, $g \cdot x = x \; \forall g \in G$, for any group G and any topological space X. Thus a group can act on any topological space X, at least in a trivial way.

If $X = G$ then left multiplication is an action of G on G: $g \cdot h = gh$. Subgroups can also act on a larger group by left multiplication. For example, $\mathbb{Z}$ may act on $\mathbb{R}$ by integer translation.

The multiplicative group of positive reals $\mathbb{R}_{\geq 0}$ acts on $\mathbb{R}$ by dilation. $a \cdot x = ax$ for all $a \geq 0$ and $x \in \mathbb{R}$.

The symmetric group on n symbols acts on the set $\{1, 2, 3, ..., n\}$ by permuting the elements.

If G acts on X, one can also define a group action of G on the Banach algebra $C_0(X)$ in the following way: For any $g \in G$, the map $\alpha_g : C_0(X) \to C_0(X)$ is defined, for $f \in C_0(X)$ and $x \in X$, by

$$\alpha_g(f)(x) = f(g^{-1} \cdot x).$$

This is well defined as since $x \to g \cdot x$ is a homeomorphism $\alpha_g f$ will be continuous and vanish at infinity. Note that for any $f \in C_0(X)$ the map $g \to \alpha_g f$ is continuous. Moreover, we also have that α_g is an isometry on $C_0(X)$, and thus an isometric action. We call the triple $(G, C_0(X), \alpha)$ a Banach algebra dynamical system.

Given a dynamical system $(G, C_0(X), \alpha)$ we want to construct a "product-$W^{p,q}$-operator algebra that has information about both G and X, as well as the action α. From now on, we assume that G is a locally compact group, and X is a locally compact Hausdorff space.

Just as with group algebras, crossed products are constructed as complections of a convolution algebra represented on a Wiener amalgam space. The convolution algebra in question is the space $L^1(G, C_0(X), \alpha)$ of absolutely integrable $C_0(X)$-valued functions on G. We take this opportunity to also introduce $C_0(X)$-valued Wiener amalgam spaces.

Definition 4.16. Let G be a locally compact group, and X a Hausdorff space. The space $L^1(G, C_0(X))$ consists of all functions $f : G \to C_0(X)$ which are finite in the norm

$$\|f\|_1 = \int_G \|f(g)\|_\infty \, d\mu(g).$$

The space $W^{p,q}(G, C_0(X))$ consists of the functions $f : G \to C_0(X)$ with finite Wiener norm, that is,

$$\|f\|_{p,q} = \left(\int_G \left(\int_G \|f(t)\|_\infty^p \cdot \chi_{xQ}(t) \, dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}$$

The function-valued Wiener amalgam spaces are complete, just as before. In addition, when G acts isometrically on $C_0(X)$, $L^1(G, C_0(X))$ is also a Banach algebra.

Proposition 4.17. *Let $(G, C_0(X), \alpha)$ be a dynamical system. Then the α-twisted convolution on $L^1(G, C_0(X))$ defined by*

$$f *_\alpha g(t)(x) = \int_G f(s)(x)\alpha_s(g(s^{-1}t)(x)) \, ds = \int_G f(s)(x)(g(s^{-1}t)(s^{-1} \cdot x) \, ds$$

for $f, g \in L^1(G, C_0(X)), t \in G$ and $x \in X$, makes $L^1(G, C_0(X))$ a Banach algebra. We denote it by $L^1(G, C_0(X), \alpha)$.

Proof. The only thing we must show is that the product is associative and submultiplicative.

Firstly, given $f, g \in L^1(G, C_0(X))$ we get

$$\|f *_\alpha g\|_1 = \int_G \| \left(f *_\alpha g\right)(t)\|_\infty \, dt = \int_G \left\| \int_G f(s)\alpha_s(g(t^{-1}s)) \, ds \right\|_\infty dt$$

$$\leq \int_G \int_G \left\|f(s)\alpha_s(g(t^{-1}s))\right\|_\infty \, ds \, dt \leq \int_G \int_G \|f(s)\|_\infty \|\alpha_s(g(t^{-1}s))\|_\infty \, ds \, dt$$

$$= \int_G \int_G \|f(s)\|_\infty \|g(t^{-1}s)\|_\infty \, ds \, dt$$

$$= \int_G \|f(s)\|_\infty \, ds \int_G \|g(t^{-1}s)\|_\infty \, dt = \|f\|_1 \|g\|_1.$$

This shows that the norm is submultiplicative.

Furthermore, if $h \in L^1(G, C_0(X), \alpha)$ we get that

$$
\begin{aligned}
((f *_\alpha g) *_\alpha h)(t)(x) &= \int_G (f *_\alpha g)(s)(x) \cdot h(s^{-1}t)(s^{-1}x)\, ds \\
&= \int_G \int_G f(r)(x) g(r^{-1}s)(r^{-1}x)\, dr\, h(s^{-1}t)(s^{-1}x)\, ds \\
&= \int_G f(r)(x) \int_G g(r^{-1}s)(r^{-1}x)\, h(s^{-1}t)(s^{-1}x)\, ds\, dr \\
&= \int_G f(r)(x) \int_G g(u)(r^{-1}x)\, h(u^{-1}r^{-1}t)(u^{-1}r^{-1}x)\, du\, dr \\
&= \int_G f(r)(x) \cdot (g *_\alpha h)(r^{-1}t)(r^{-1}x)\, dr = (f *_\alpha (g *_\alpha h))(t)(x),
\end{aligned}
$$

where the fourth equality comes from the substitution $u = r^{-1}s$. Thus the product is associative and $L^1(G, C_0(X), \alpha)$ is a Banach algebra. □

4.4.2 Crossed products

$L^1(G, C_0(X), \alpha)$ is a convolution algebra that encodes the dynamical structure of $G \curvearrowright X$. By representing this algebra on some ($C_0(X)$-valued) Wiener amalgam space E we will get operator algebras on Wiener space. However, in order for the algebras to still encode structure of $G \curvearrowright X$ we must represent $L^1(G, C_0(X), \alpha)$ in a way that preserves the dynamical structure of the action. We take the opportunity to define representations for a more general dynamical system (G, A, α) where A is any Banach algebra.

Note. From now on we broaden our definition of a $W^{p,q}$-space. A Banach space X is a $W^{p,q}$-space if there is a locally compact group G, either an admissible window Q or a BUPU Ψ on G and a Banach algebra B such that there is either

- An isometric isomorphism $X \to W_Q^{p,q}(G, B)$, or
- An isometric isomorphism $X \to \mathbf{w}^{p,q}(G, \Psi, B)$.

When $B = \mathbb{C}$ this is exactly our old definition of a $W^{p,q}$-space. We will mostly concern ourselves with the case where $B = C_0(X)$ for some Hausdorff space X.

Definition 4.18. Let (G, A, α) be a Banach algebra dynamical system, and E a $W^{p,q}$-space. A covariant representation of (G, A, α) is a pair of representations (u, π) on E, where $u : G \to \mathrm{Isom}(E)$ and $\pi : A \to \mathcal{B}(E)$ and such that for all $g \in G$ and $a \in A$,

$$u_g \pi(a) = \pi(\alpha_g(a)) u_g. \tag{3}$$

Note. The property 3 says that the diagram

$$\begin{tikzcd} E \arrow[r, "\pi(a)"] \arrow[d, "u_g"] & E \arrow[d, "u_g"] \\ E \arrow[r, "\pi(\alpha_g(a))"] & E \end{tikzcd}$$

commutes.

If the dynamical system is of the form $(G, C_0(X), \alpha)$, and $Q \subseteq G$ is an admissible window, one can represent $(G, C_0(X), \alpha)$ on $\mathcal{B}(W_Q^{p,q}(G, C_0(X)))$ in a particularly nice way. We define a covariant representation (u, π) of $(G, C_0(X), \alpha)$ on $W_Q^{p,q}(G, C_0(X))$ by

$$u_g(\xi)(t, x) = \xi(g^{-1}t, x)$$

and

$$m_f(\xi)(t, x) = f(t \cdot x) \cdot \xi(t, x)$$

for all $f \in C_0(X), g, t \in G$, and $\xi \in W_Q^{p,q}(G, C_0(X))$. The representation (u, m) is called the canonical covariant representation of $(G, C_0(X), \alpha$.

$u : G \to \mathcal{B}(W_Q^{p,q}(G, C_0(X))$ is clearly an isometric representation, since it is pretty much just the same representation as $\mathrm{Lt} : G \to \mathcal{B}(W_Q^{p,q}(G)$ from section 4.2. It might be less clear that $m : C_0(X) \to \mathcal{B}(W_Q^{p,q}(G, C_0(X))$ is isometric, but it actually is.

Lemma 4.19. *Let $1 \leq p, q \leq \infty$, G a locally compact group, $Q \subset G$ an admissible window and X a Hausdorff space. The representation $m : C_0(X) \to \mathcal{B}(W_Q^{p,q}(G, C_0(X))$ defined above is isometric.*

Proof. Let $f \in C_0(X)$. Since f vanishes at infinity its maximum is attained at an interior point x_{max}. One can readily check that the function $\xi(t, x) = \chi_Q(t) f(t)$ is in $W_Q^{p,q}(G, C_0(X))$ since it is compactly supported in G. Let $\tilde{\xi}$ be the normalization of ξ. For any $t \in Q$ we now have that

$$\|f \cdot \tilde{\xi}(t, -)\|_\infty = \max_{x \in X} \left| f(x) \frac{f(x)}{\|\xi\|_{p,q}} \chi_Q(t) \right| = \frac{1}{\|\xi\|_{p,q}} \max_{x \in X} \left|(f(x))^2\right|$$

$$= \frac{1}{\|\xi\|_{p,q}} |f(x_{max})|^2 = \|f\|_\infty |\tilde{\xi}(t, -)\|_\infty.$$

Thus

$$\|m_f \tilde{\xi}\|_{p,q} = \left(\int_G \left(\int_G \|f \tilde{\xi}(t, -)\|_\infty^p \chi_Q(x^{-1}t) dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}$$

$$= \left(\int_G \left(\int_G \|f\|_\infty^p \|\tilde{\xi}(t,-)\|_\infty^p \chi_Q(x^{-1}t) dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}$$

$$= \|f\|_\infty \left(\int_G \left(\int_G \|\tilde{\xi}(t,-)\|_\infty^p \chi_Q(x^{-1}t) dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} = \|f\|_\infty \|\tilde{\xi}\|_{p,q}.$$

Since we also have that $\|m_f\|_{\text{op}} \leq \|f\|_\infty$ we thus get that $\|m_f\|_{\text{op}} = \|f\|_\infty$, and so m is an isometric representation of $(G, C_0(X), \alpha)$ on $W_Q^{p,q}(G, C_0(X))$, and thus a quasi-isometric representation on $W^{p,q}(G, C_0(X))$. $\square$

Lastly, a quick calculation shows that for any $g, t \in G$, $f \in C_0(X)$ and $\xi \in W_Q^{p,q}(G, C_0(X))$ we have that

$$u_g\left(m(f)\right)\xi(t,x) = m(f)\xi(g^{-1}t,x) = f((g^{-1}t)\cdot x)\xi(g^{-1}t,x)$$

and similarly,

$$\begin{aligned} m(\alpha_g f)\left(u_g \xi\right)(t,x) &= \alpha_g f(t\cdot x) u_g \xi(t,x) \\ &= f((g^{-1}t)\cdot x)\xi(g^{-1}t,x). \end{aligned}$$

So the pair (u, m) is in fact a covariant representation.

Given any covariant representation (u, π) of (G, A, α) we may also define its integrated representation $u \ltimes \pi : L^1(G, A, \alpha) \to \mathcal{B}(E)$ by

$$(u \ltimes \pi)_f(\xi) = \int_G \pi(f(g))(u_g(\xi))\, dg.$$

For $f \in L^1(G, A, \alpha), g \in G$ and $\xi \in E$ for some $W^{p,q}$-space E.

Again, if the dynamical system is of the form $(G, C_0(X), \alpha)$, $Q \subseteq G$ is an admissible window, and (u, π) is the canonical representation (u, m), then the integrated form is

$$(u \ltimes m)_f(\xi)(t,x) = \int_G f(s, t\cdot x)\cdot \xi(s^{-1}t, x)\, ds$$

for $f \in L^1(G, C_0(X), \alpha), \xi \in W_Q^{p,q}(G, C_0(X)), t \in G$ and $x \in X$.

We now have a homomorphic image of the algebra $L^1(G, C_0(X), \alpha)$ in the space of bounded operators $\mathcal{B}(E)$, thus an algebra. Taking its completion results in the desired Banach algebra.

Definition 4.20. Let $(G, C_0(X), \alpha)$ be a Banach algebra dynamical system, Q an admissible window, and (u, m) the canonical covariant representation of

$(G, C_0(X), \alpha)$ on $W_Q^{p,q}(G, C_0(X))$. We define the reduced crossed product $F_\alpha^{p,q}(G, X)$ as

$$F_\alpha^{p,q}(G, X) = \overline{(u \ltimes m)(L^1(G, C_0(X), \alpha)}^{\|\cdot\|} \subseteq \mathcal{B}(W_Q^{p,q}(G, C_0(X))).$$

Note. Like before, because the operator norms are equivalent, the definition of the cross product does not depend on the choice of Q.

Let us now calculate the cross product in a simple special case.

If we let $X = \{*\}$, then any action of G on X will be trivial. Furthermore, $C_0(*) = \mathbb{C}$, so $L^1(G, \mathbb{C}) = L^1(G)$ and the convolution in $L^1(G, \mathbb{C}, \alpha)$ is

$$f * g(s) = \int_G f(t)\alpha_t(g(s^{-1}t))\, dt = \int_G f(t)g(s^{-1}t)\, dt,$$

So $L^1(G, C_0(*), \mathrm{id}) = L^1(G)$ as Banach algebras. Furthermore,$u \ltimes m$, the canonical representation of $L^1(G, \mathbb{C}, \alpha)$ on $W^{p,q}(G, \mathbb{C})$ agrees with the left regular representation of $L^1(G)$ on $W^{p,q}(G)$, and the norms agree up to a scalar multiple. It follows that $F_\alpha^{p,q}(G, *) \cong F_\lambda^{p,q}(G)$.

Sadly, crossed products do not in general encode all information of the action $G \curvearrowright^\sigma X$. One example of lost information is the group structure of G, as shown in the following example.

Proposition 4.21. *Let G be a finite abelian group, and let the action $G \curvearrowright^\sigma G$ be given by translation. Then*

$$F_\alpha^{p,q}(G) \cong \mathcal{B}(\mathbb{C}^{|G|}).$$

Proof. Denote the group order of G by n. Because G is finite, every $f \in L^1(G, C_0(G), \alpha)$ may be represented as an $n \times n$-matrix $F_{t,x} = f(t, x)$ enumerated by G. We also know that $\mathcal{B}(\mathbb{C}^n) \cong M_n$, the set of $n \times n$-matrices under regular matrix multiplication. Note that in the current setting twisted convolution is given by

$$f *_\alpha g(t, x) = \sum_{s \in G} f(s, x)g(t - s, x + s).$$

We construct an isomomorphism ϕ between the two algebras $L^1(G, C_0(G), \alpha)$ and M_n. This is sufficient because since G is finite, the completion $F_\alpha^{p,q}(G)$ is equal to $L^1(G, C_0(G), \alpha)$. Given any $f \in L^1(G, C_0(G), \alpha)$ we let $\phi : L^1(G, C_0(G), \alpha) \to M_n$ be given by

$$\phi(f)_{t,x} = f(x - t, t).$$

Since any $f \in L^1(G, C_0(G), \alpha)$ may be identified with a matrix, ϕ is trivially bijective. It is also easily seen to be linear. It only remains to show that ϕ is multiplicative. Given any two $f, g \in L^1(G, C_0(G), \alpha)$, we want that

$$\phi(f *_\alpha g) = \phi(f) \cdot \phi(g),$$

where the operation on the left is twisted convolution in $L^1(G, C_0(G), \alpha)$ and the operation on the right is matrix multiplication. We show this component-wise. For any $t, x \in G$ we have

$$\phi(f *_\alpha g)_{t,x} = (f *_\alpha g)(x - t, t) = \sum_{s \in G} f(s,t) g(x - t - s, t + s).$$

We re-index using $u = s + t$ and get

$$\sum_{s \in G} f(s,t) g(x - t - s, t + s) = \sum_{u \in G} f(u - t, t) g(x - u, u)$$

$$= \sum_{u \in G} \phi(f)_{t,u} \phi(g)_{u,x} = (\phi(f) \cdot \phi(g))_{t,x} .$$

Thus ϕ is indeed an isomorphism, and the result is proved. $\square$

The result above shows that $F^p_\alpha(G, X)$ only remembers the cardinality of G. We will now show that while the cross product does not remember G, it does remember X and, in special cases, even the action. The special case in mention is that of conjugate systems.

Definition 4.22. Two continuous group actions $G \curvearrowright^\sigma X$ and $H \curvearrowright^\rho Y$ are said to be conjugate actions if there is a group homomorphism $\phi : G \to H$ and a homeomorphism $\psi : X \to Y$ such that $\psi(\sigma(g, x)) = \rho(\phi(g), \psi(x))$ for all $g \in G$ and $x \in X$. In other words, the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "\sigma"] \arrow[d, "\phi"] & X \arrow[d, "\psi"] \\ H \arrow[r, "\rho"] & Y \end{tikzcd}$$

If, in addition, ϕ is a homeomorphism between G and H, the two actions are called topologically conjugate. We will now show that if two actions are conjugate, their cross products will be quasi-isometric. Firstly note that since ϕ is a topological isomorphism of groups, the pushforward measure $\mu \circ \phi^{-1}$ on H will be a translation-invariant Borel measure. By uniqueness of the Haar measure of

H there is a positive constant c such that $\mu \circ \phi^{-1}(E) = c \cdot \nu(E)$ for all measurable subsets of H. It follows that the Radon-Nikodym derivative $\frac{d(\mu\circ\phi^{-1})}{d\nu}$ is constant and equals c. We now let α denote the induced action of G on $C_0(X)$, and β denote the induced action of H on $C_0(Y)$, and seek to define define a map $F : L^1(G, C_0(X), \alpha) \to L^1(H, C_0(Y), \beta)$ that is an isomorphism of algebras. To that matter, define F by

$$F(f)(h, y) = f(\phi^{-1}(h), \psi^{-1}(y)) \cdot c$$

for all $f \in L^1(G, C_0(X), \alpha), h \in H$ and $y \in Y$. Here ϕ and ψ are the conjugacy maps, Furthermore, for clarity we will in the calculation below distingush between integration with respect to different Haar measures on G and H. For $f \in L^1(G, C_0(X), \alpha)$ we have that for any $h \in H$

$$\max_{y\in Y} |f(\phi^{-1}(h), \psi^{-1}(y))| = \max_{x\in X} |f(\phi^{-1}(h), x)|.$$

Therefore,

$$\|F(f)\|_{L^1(H,C_0(Y),\beta)} = \int_H \max_{y\in Y} |F(f)(h, y)| \, d\nu(h)$$

$$= \int_H \max_{y\in Y} \left|f(\phi^{-1}(h), \psi^{-1}(y)) \cdot c\right| \, d\nu(h)$$

$$= \int_H \max_{x\in X} |f(\phi^{-1}(h), x) \; c \; d\nu(h) = \int_H \max_{x\in X} |f(\phi^{-1}(h), x)| \, d\left(\mu \circ \phi^{-1}\right)(h)$$

$$= \int_G \max_{x\in X} |f(g, x)| \, d\mu(x) = \|f\|_{L^1(G,C_0(X),\alpha)} < \infty,$$

Where we have used the change of variables formula from appendix B in the fifth line. So F actually defines a map from $L^1(G, C_0(X), \alpha)$ to $L^1(H, C_0(Y), \beta)$, and it is also easily seen to be linear. We also present the rest of the important properties of F.

Proposition 4.23. *Let $G \curvearrowright^\sigma X$ and $H \curvearrowright^\rho Y$ be conjugate actions, let $Q \subseteq G$ be an admissible window, and let F be as above. Then*

- *F is an isomorphism of algebras.*
- *Given a $T \in \mathcal{B}(W^{p,q}_Q(G, C_0(X)))$ we let $CT = F \circ T \circ F^{-1} \in \mathcal{B}(W^{p,q}_{\phi(Q)}(H, C_0(Y)))$. Then $(u \ltimes m)^H(F(f)) = C(u \ltimes m)^G(f)$.*

Proof. F is a bounded by the above calculation, and it is linear as discussed before. All that remains to show is that it is bijective and multiplicative. To show bijectiveness, consider the linear map $G : L^1(H, C_0(Y), \beta) \to L^1(G, C_0(X), \alpha)$ given by

$$G(\xi)(g,x) = \xi(\phi(g), \psi(x)) \cdot \frac{1}{c}.$$

For $f \in L^1(G, C_0(X), \alpha)$ we have that

$$\begin{aligned} G(F(f))(g,x) &= F(f)(\phi(g), \psi(x)) \cdot \frac{1}{c} \\ &= f(\phi^{-1}(\phi(g)), \psi^{-1}(\psi(x))) \cdot c \cdot \frac{1}{c} \\ &= f(g,x). \end{aligned}$$

F is therefore bijective. To show multiplicativeness, let $f_1, f_2 \in L^1(G, C_0(X), \alpha)$. We have that

$$F(f_1) *_\beta F(f_2)(h,y) = \int_H F(f_1)(s,y) F(f_2)(s^{-1}h, \rho(s^{-1}y))\, d\nu(s)$$

$$= \int_H f_1(\phi^{-1}(s), \psi^{-1}(y)) \cdot c \cdot f_2(\phi^{-1}(s^{-1}h), \psi^{-1}(\rho(s^{-1}, y)) \cdot c\, d\nu(s)$$

$$= \int_H f_1(\phi^{-1}(s), \psi^{-1}(y)) \cdot f_2(\phi^{-1}(s^{-1})\phi^{-1}(h)), \psi^{-1}(\rho(s^{-1}, y)) \cdot c\, d(\mu \circ \phi^{-1})(s).$$

By conjugacy, $\sigma(\phi^{-1}(t^{-1}), \psi^{-1}(y)) = \psi^{-1}(\rho(t^{-1}, y)$, so the above integral equals

$$c \cdot \int_H f_1(\phi^{-1}(s), \psi^{-1}(y)) \cdot f_2(\phi^{-1}(s^{-1})\phi^{-1}(h)), \sigma(\phi^{-1}(s^{-1}), \psi^{-1}(y))\, d(\mu \circ \phi^{-1})(s).$$

Since ϕ is bijective, there is an element $t \in G$ such that $\phi(t) = s$. We do a change of variables in the above integral and get

$$c \cdot \int_H f_1(\phi^{-1}(s), \psi^{-1}(y)) \cdot f_2(\phi^{-1}(s^{-1})\phi^{-1}(h)), \sigma(\phi^{-1}(s^{-1}), \psi^{-1}(y))\, d(\mu \circ \phi^{-1})(s).$$

$$\begin{aligned} &= c \cdot \int_G f_1(t, \psi^{-1}(y)) \cdot f_2(t^{-1}\phi^{-1}(h), \sigma(t^{-1}, \psi^{-1}(y)))\, d\mu(t) \\ &= (f_1 *_\alpha f_2)(\phi^{-1}(h), \psi^{-1}(y)) \cdot c = F(f_1 *_\alpha f_2)(h,y). \end{aligned}$$

So F is multiplicative, and thus a bijective bounded homomorphism of algebras. By the bounded inverse theorem, F is therefore an isomorphism of Banach algebras.

Note that the second property is the same as showing that the following diagram commutes.

$$\begin{array}{ccc} L^1(G, C_0(X), \alpha) & \xrightarrow{(u \ltimes m)_G} & \mathcal{B}(W_Q^{p,q}(G, C_0(X))) \\ \Big\downarrow F & & \Big\downarrow \mathrm{C} \\ L^1(H, C_0(Y), \beta) & \xrightarrow{(u \ltimes m)_H} & \mathcal{B}(W_{\phi(Q)}^{p,q}(H, C_0(Y))) \end{array}$$

To show this, we let $f \in L^1(G, C_0(X), \alpha), \xi \in W_{\phi(Q)}^{p,q}(H, C_0(Y)), h \in H$ and $y \in Y$ and calculate directly:

$$(C(u \ltimes m)^G f)\xi(h, y) = (u \ltimes m)^G f(F^{-1}\xi)\left(\phi^{-1}(h), \psi^{-1}(y)\right) \cdot c$$

$$= \int_G f(t, \sigma(\phi^{-1}(h), \psi^{-1}(y))(F^{-1}\xi)(t^{-1}\phi^{-1}(h), \psi^{-1}(y))\, d\mu(t) \cdot c$$

$$= \int_G f(t, \sigma(\phi^{-1}(h), \psi^{-1}(y))\xi(\phi(t^{-1}\phi^{-1}(h)), y) \cdot \frac{1}{c}\, d\mu(t) \cdot c$$

$$= \int_G f(t, \sigma(\phi^{-1}(h), \psi^{-1}(y)))\xi(\phi(t^{-1})h, y)\, d\mu(t).$$

Using the change of variables formula, the above integral equals

$$= \int_H f(\phi^{-1}(s), \sigma(\phi^{-1}(h), \psi^{-1}(y))\xi(s^{-1}h, y)\, d(\mu \circ \phi^{-1})(s)$$

$$= \int_H f(\phi^{-1}(s), \sigma(\phi^{-1}(h), \psi^{-1}(y))\xi(s^{-1}h, y) \cdot c\, d\nu(s).$$

Again, by conjugacy this integral equals

$$\int_H f(\phi^{-1}(s), \psi^{-1}(\rho(h, y)))\xi(s^{-1}h, y) \cdot c\, d\nu(s)$$

$$= \int_H F(f)(s, \rho(h, y))\xi(s^{-1}h, y)\, d\nu(s)$$

$$= (u \ltimes m)^H (Ff)(\xi)(h, y).$$

$\square$

Using the above proposition and a similar argument to the proof of theorem 4.14, we immediately get the following result.

Theorem 4.24. *Let $G \curvearrowright^{\sigma} X$ and $H \curvearrowright^{\rho} Y$ be conjugate actions. Then the map*

$$F : L^1(G, C_0(X), \alpha) \to L^1(H, C_0(Y), \beta)$$

extends to a quasi-isometric isomorphism

$$F^{p,q}_{\alpha}(G, X) \cong F^{p,q}_{\beta}(H, Y).$$

Proof. From the above proposition we know that the isometry F induces a quasi-isometry of $(u \ltimes m)^G(L^1(G, C_0(X), \alpha))$ and $(u \ltimes m)^H(L^1(H, C_0(Y), \beta)$. Taking the closure proves the theorem. □

Conjugacy is a very strong condition and states that the actions act on the "same" space by the same "group". We want a weaker condition so that we can relate actions on different systems. There is a weaker equivalence notion for continuous group actions that we will make use of.

Definition 4.25. Two continuous group actions $G \curvearrowright^{\sigma} X$ and $H \curvearrowright^{\rho} Y$ are continuously orbit equivalent if there is a homeomorphism $\psi : X \to Y$ and two continuous maps $c_H : G \times X \to H$ and $c_G : H \times Y \to G$ so that for all $x \in X, y \in Y, g \in G$ and $h \in H$,

$$\psi(\sigma(g, x)) = \rho(c_H(g, x), \psi(x)) \qquad \text{and} \qquad \psi^{-1}(\rho(h, y)) = \sigma(c_G(h, y), \psi^{-1}(y)).$$

In other words, the following diagrams commute.

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ \big\downarrow{\scriptstyle (c_H, \psi)} & & \big\downarrow{\scriptstyle \psi} \\ H \times Y & \xrightarrow{\rho} & Y \end{array} \qquad\qquad \begin{array}{ccc} H \times Y & \xrightarrow{\rho} & Y \\ \big\downarrow{\scriptstyle (c_G, \psi^{-1})} & & \big\downarrow{\scriptstyle \psi^{-1}} \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

Unlike conjugate group actions, we do not immediately get mappings between G and H in the continuously orbit equivalent case. Our best attempt is to fix an $x \in X$ and consider the mapping $G \to H$ given by $g \to c_H(g, x)$, and similarly for the mapping $H \to G$. Still, these maps will not be isomorphisms or even homomorphisms between G and H. However, if we assume that both group actions are essentially free, we get bijections between G and H.

Lemma 4.26. *Let $G \curvearrowright^{\sigma} X$ and $H \curvearrowright^{\rho} Y$ be continuously orbit equivalent group actions, and assume that both actions are essentially free. Then we have*

$$c_G(c_H(g, x), \psi(x)) = g \qquad \text{and} \qquad c_H(c_G(h, y), \psi^{-1}(y)) = h$$

for all $x \in X, y \in Y, g \in G$ and $h \in H$.

Proof. The following proof was adapted from [12]. We prove one identity, as the other is analogous. Fix a $g \in G$ and an $x \in X$. Let $h = c_H(g, x)$. By continuous orbit equivalence, $\psi(\sigma(g, x)) = \rho(h, \psi(x))$, and thus

$$\sigma(g, x) = \psi^{-1}(\rho(h, \psi(x))) = \sigma(c_G(h, \psi(x)), x) = \sigma(c_G(c_h(g, x), \psi(x)), x),$$

where the second equality is by continuous orbit equivalence. We thus have that

$$g\left(c_G(c_H(g, x), \psi(x)\right)^{-1} \cdot x = x.$$

However, this equation holds for all $x \in X$, so by essential freeness, $g\left(c_G(c_H(g, x), \psi(x)\right)^{-1} = e$, or rather,

$$c_G(c_H(g, x), \psi(x) = g.$$

□

We do not know if two general continuously orbit equivalent systems have isomorphic cross products, but this is indeed the case when the groups are discrete. We show this, proceeding like we did for conjugate actions.

Given an $f \in L^1(G, C_0(X), \alpha)$ we define the function Ff by

$$Ff(h, y) = f(c_G(h, y), \psi^{-1}(y)) \cdot \frac{d(\mu \circ c_{G_y})}{d\nu}(h)$$

for any $h \in H$ and $y \in Y$. Like before, $\mu \circ c_{G_y}$ is defined to be the pushforward measure of μ on H by the mapping $c_G(-, y)$, but since the discrete groups are equipped with counting measure, the Radon-Nikodym derivative will be constant and equal to 1, so we may just as well define F by

$$Ff(h, y) = f(c_G(h, y), \psi^{-1}(y)).$$

Similarly for a $\phi \in L^1(H, C_0(Y), \beta)$ we define $G\phi$ by

$$G\phi(g, x) = \phi(c_H(g, x), \psi(x))$$

for $g \in G$ and $x \in X$. Like before it is easily seen that we get bounded linear maps F and G between $L^1(G, C_0(X), \alpha)$ and $L^1(H, C_0(Y), \beta)$. We can also verify that F and G are inverses. Given $f \in L^1(G, C_0(X), \alpha)$,

$$G(F(f))(g, x)$$

$$= f(c_G\left(c_H(g, x), \psi(x)\right), \psi^{-1}(\psi(x)))$$

$$= f(g, x),$$

where the last equality follows from lemma 4.26. Using continuous orbit equivalence and lemma 4.26 one can also show that F and G give Banach algebra isomorphisms between $L^1(G, C_0(X), \alpha)$ and $L^1(H, C_0(Y), \beta)$, and that these homomorphisms respect the canonical covariant representation. We get the following result.

Theorem 4.27. *Let $G \curvearrowright^{\sigma} X$ and $H \curvearrowright^{\rho} Y$ be continuously orbit equivalent group actions on topological spaces, where G and H are discrete groups. If both σ and ρ are essentially free group actions, there is an isomorphism between their crossed products. In other words,*

$$F_{\alpha}^{p,q}(G, X) \cong F_{\beta}^{p,q}(H, Y).$$

Proof. Analogous to the proof for conjugate systems. $\square$

4.5 Crossed products by discrete groups

Given any homeomorphism $\psi : X \to X$, the integers act on X by $z \cdot x = h^z(x)$, similarly, for any homeomorphism of order n, $\mathbb{Z}_n$ acts on X by $i \cdot x = h^i(x)$. In fact, any discrete group may act on a topological space in a similar way. The next section is dedicated to a further study of crossed products by discrete groups.

We denote by $\tilde{C}_0(X)$ the unitization of $C_0(X)$. If X is compact we do not add another unit. Note now that $C_c(G)$ may be included into $C_c(G, \tilde{C}_0(X), \alpha)$ by, for any $f \in C_c(G)$, letting $\tilde{f}(t, x) = f(t)\mathbf{1}(x) = f(t) \ \forall \ x \in X$. It is easily seen that $\tilde{f} *_{\alpha} \tilde{g} = \tilde{f * g}$, so the inclusion is as algebras, not just as vector spaces. In particular, this means that $\chi_t \in C_c(G, \tilde{C}_0(X), \alpha)$ for any $t \in G$. Thus, given $f \in C_c(G, C_0(X), \alpha)$ we may write f as a sum

$$f(t, x) = \sum_{g \in G} f_g(x) \chi_g(t)$$

Where $f_g = f(g, -) \in C_0(X)$. In this case we can also include $C_0(X)$ into $C_c(G, C_0(X), \alpha)$ by identifying any $f \in C_0(X)$ with $f(x) \cdot \chi_e(t)$ in $L^1(G, C_0(X), \alpha)$. Under this identification we have that, for $f, g \in C_0(X)$,

$$f *_{\alpha} g(t, x) = \sum_{s \in G} f(x) \chi_e(s) g(s^{-1} \cdot x) \chi_e(s^{-1} t).$$

The last expression is non-zero if and only if $s = e$, so we get

$$f *_{\alpha} g(t, x) = f(x) g(e \cdot x) \chi_e(t) = f(x) g(x) \chi_e(t).$$

Thus twisted convolution of two elements of $C_0(X)$ corresponds to pointwise multiplication of the functions. Thus we have a proper inclusion $C_0(X) \subset F^{p,q}_\alpha(G,X)$ of algebras.

We now want to see how the decomposition of functions in $C_c(G, C_0(X), \alpha)$ behave with respect to the canonical covariant representation. In order to do this we define two auxilliary maps. For any $g \in G$ define $s_g : C_0(X) \to W^{p,q}(G, C_0(X))$ by

$$s_g(f)(h,x) = \begin{cases} f(x), & h = g \\ 0, & h \neq g \end{cases}$$

for any $f \in C_0(X), h \in G$ and $x \in X$. Furthermore, we define $t_g : W^{p,q}(G, C_0(X)) \to C_0(X)$ by

$$t_g(\xi)(x) = \xi(g,x)$$

for any $\xi \in W^{p,q}(G, c_0(X))$ and $x \in X$. Note that $t_g s_g f = f$ and $s_g t_g \xi = \xi \cdot \chi_{\{g\}\times X}$ for f and ξ in the respective domains. We now collect some results on the interaction between s_g, t_g and $u \ltimes m$.

Lemma 4.28. *Let $1 \leq p,q < \infty$, G a discrete group acting on X, and $Q \subseteq G$ an admissible window. Let s_g and t_g be as above, and let $u \ltimes m : L^1(G,X,\alpha) \to \mathcal{B}(W^{p,q}_Q(G, C_0(X)))$ be the canonical covariant representation. Then:*

- *For $f \in C_c(G, C_0(X), \alpha), \xi \in W^{p,q}_Q(G, C_0(X)), t \in G$ and $x \in X$ we have*

$$\left((u \ltimes m)(f)\xi\right)(t,x) = \sum_{g \in G} f_g(t \cdot x)\xi(g^{-1}t, x).$$

- *If $c \in \mathcal{B}(W^{p,q}_Q(G, C_0(X)))$ satisfies $t_h c s_k = 0$ for all $h,k \in G$, then $c = 0$.*
- *If $f \in C_c(G, C_0(X), \alpha)$ and $h,k \in G$, then*

$$t_h(u \ltimes m)(f)s_k = m(\alpha_{h^{-1}}(f_{hk^{-1}}))$$

Proof. The first statement is just the definition of $u \ltimes m$ in the special case where G is discrete and $f \in C_c(G, C_0(X), \alpha)$, making the sum convergent.
To prove the second statement, let $\eta \in C_0(X)$).If we multiply from the right by t_k, $t_h c s_k = 0$ is equivalent to $0 = t_h c(\chi_{\{k\}\times X})(\eta)(x) = c(\eta\chi_k)(h,x)$. This holds for all $k,h \in G, x \in X$ and $\eta \in C_c(X)$, and so, by linearity, $c(f) = 0$ for all $f \in C_c(G,X,\alpha)$. By continuity it follows that $c(f) = 0$ for all $f \in W^{p,q}_Q(G, C_0(X))$.
To prove the last statement, we let $\eta \in C_0(X)$ and use the identity from the first statement.

$$t_h\left(u \ltimes m\right)(f)s_k(\eta)(x) = \left(u \ltimes m\right)(f)s_k(\eta)(h,x)$$

$$= \sum_{g \in G} f_g(h \cdot x)(s_k \eta)(g^{-1}h, x).$$

The summand is non-zero only if $g^{-1}h = k$, so the sum equals

$$f_{hk^{-1}}(h \cdot x)\eta(x) = m(\alpha_{h^{-1}}(f_{hk^{-1}}))(\eta)(x).$$

□

For each $g \in G$, we may now define a linear map $E_g : C_c(G, C_0(X), \alpha) \to C_0(X)$ by

$$E_g(f) = E_g\left(\sum_{h \in G} f_h \chi_h\right) = f_g,$$

whenever $f \in C_c(G, C_0(X), \alpha)$. Note that for $f \in C_c(G, C_0(X), \alpha)$ we have that

$$\|E_g(f)\|_\infty = \|f_g\|_\infty = (\|f_g\|_\infty^q)^{\frac{1}{q}}$$

$$\leq \left(\sum_{g \in G} \|f_g\|_\infty\right)^{\frac{1}{q}} = \|f\|_p = \|f\|_{p,q},$$

so E_g is a continuous linear map, and we may thus extend E_g continuously to an operator $E_g : F^{p,q}_\alpha(G, X) \to C_0(X)$ with operator norm $\|E_g\| \leq 1$. In this way the functions $\{E_g(f)\}_{g \in G}$ determine some sort of coefficients of f. These are not coefficients in the general sense, as the series $\sum_{g \in G} f_g \chi_g$ does not converge in general, but at least f is uniquely determined by its coefficients.

Proposition 4.29. *If $f \in F^{p,q}_\alpha(G, X)$ and $E_g(f) = 0$ for all $g \in G$, then $f = 0$.*

Proof. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in $C_c(G, C_0(X), \alpha)$ such that $\|(u \ltimes m)f_n - f\| \to 0$. It follows by continuity that for any $g \in G$

$$0 = E_g(f) = \lim_{n \to \infty} E_g((u \ltimes m)f_n) = \lim_{n \to \infty} f_n(g, -).$$

Thus, by part 2 of the previous lemma, $(u \ltimes m)f_n \to 0$, and so $f = 0$. □

The case E_e is special. It is called the standard conditional expectation from $F^{p,q}_\alpha(G, X)$ to $C_0(X)$ and is usually denoted by E.

When restricting ourselves to crossed products by discrete groups, we are able to deduce some interesting results about their structure. To discover further structure we will in addition impose that G acts on X freely. We then have the following lemma.

Lemma 4.30. *Let G be a discrete group, X a free, compact G-space, and let $F \subseteq G \setminus \{e\}$ be finite. Then there is an $n \in \mathbb{N}$ and functions $\{s_k\}_{k=1}^n$ in $C(X)$ such that $|s_k(x)| = 1$ for all k and x, and such that for all $x \in X$ and $g \in F$ we have*

$$\frac{1}{n}\sum_{k=1}^{n} s_k(x)\overline{s_k(g^{-1}x)} = 0. \tag{4}$$

Proof. For use within this proof only, we define a pair (F, U), where F is a finite subset of $G \setminus \{e\}$ and $U \subseteq X$ is open to be inessential if there are functions like in the proof, and such that the property 4 is satisfied for all $g \in F$ and $x \in U$. Proving the lemma is thus the same as showing that (F, X) is inessential for all F. To begin, we prove the following three claims:

- For every $g \in G \setminus \{e\}$ and $x \in X$ there is an open set $U \subseteq X$ such that $(\{g\}, U)$ is inessential.
- If (F, U) and (F, V) are inessential, then $(F, U \cup V)$ is also inessential.
- If (E, U) and (F, U) are inessential, then $(E \cup F, U)$ is also inessential.

For the first claim, U be a neighbourhood of x so that $\overline{U} \cup g^{-1}\overline{U} = \emptyset$. We set $n = 2$, and let $s_1 \equiv 1$. Moreover, pick a continuous function $r : X \to \mathbb{R}$ such that $r|_{\overline{U}} = 0$ and $r|_{g^{-1}\overline{U}}$. If we now set $s_2(x) = \exp(ir(x))$, then

$$\frac{1}{2}\left(s_1(x)\overline{s_1(g^{-1}x)} + s_2(x)\overline{s_2(g^{-1}x)}\right) = \frac{1}{2}\left(1 \cdot 1 + 1 \cdot -1\right) = 0.$$

So then $(\{g\}, U)$ is inessential. To prove the other two claims, note that, almost trivially, if $\{r_k\}_{k=1}^n, \{s_i\}_{i=1}^m \in C(X)$ then

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n}(r_k s_i)(x)\overline{(r_k s_i)(g^{-1}x)}$$

$$= \left(\frac{1}{n}\sum_{k=1}^{n} r_k(x), \overline{r_k(g^{-1}x)}\right)\left(\frac{1}{m}\sum_{i=1}^{m} s_i(x)\overline{s_i(g^{-1}x)}\right)$$

for all g and x. To prove the second claim, let $\{r_k\}_{k=1}^n$ be the functions corresponding to the inessential pair (F, U), and $\{s_i\}_{i=1}^m$ be the functions corresponding to the inessential pair (F, V). If we now set $N = nm$ and $t_j = t_{k \cdot i} = r_k \cdot s_i$, then by the previous identity,

$$\frac{1}{N}\sum_{j=1}^{N} t_j(x)\overline{t_j(g^{-1}x)} = \frac{1}{mn}\sum_{k=1}^{n}\sum_{i=1}^{m}(r_k s_i)(x)\overline{(r_k s_i)(g^{-1}x)}$$

$$= \left(\frac{1}{n} \sum_{k=1}^{n} r_k(x) \overline{r_k(g^{-1}x)} \right) \left(\frac{1}{m} \sum_{i=1}^{m} s_i(x) \overline{s_i(g^{-1}x)} \right).$$

By inessentiality, the last expression is 0 for all $g \in F$ and $x \in U \cup V$. Thus $(F, U \cup V)$ is inessential as well. The proof of the third claim is similar to the second claim.

The next step is to show that $(\{g\}, X)$ is inessential for all $g \in G \setminus \{e\}$. By the first claim, for every $x \in X$ there is a neighbourhood U_x of x so that $(\{g\}, U_x)$ is inessential. Clearly, $\{U_x\}_{x \in X}$ is an open cover of X, so by compactness, there is a finite subcover $\{U_i\}_{i=1}^n$ of X such that $(\{g\}, U_i)$ is inessential. We now use the second claim $n-1$ times, and get that $(\{g\}, \cup_{i=1}^n U_i) = (\{g\}, X)$ is inessential.

Lastly, let $F \subseteq G \setminus \{e\}$ be an arbitrary finite set. Then $F = \{f_1, f_2, ..., f_m\}$, and since $(\{f_j\}, X)$ is inessential for all $1 \leq j \leq m$, we may use the third claim $m-1$ times to get that $(F, X) = (\cup_{j=1}^m \{f_j\}, X)$ is inessential, as desired. □

Proposition 4.31. *Let G be discrete, and X a free compact G-space. For every $a \in F_\alpha^{p,q}(G, X)$ and $\varepsilon > 0$ there is an $n \in \mathbb{N}_0$ and functions $s_1, s_2, ..., s_n \in C(X)$ so that $|s_k(x)| = 1$ for all integers $k \leq n$ and all $x \in X$, and also such that*

$$\left\| E(a) - \sum_{k=1}^{n} s_k *_\alpha a *_\alpha \overline{s_k} \right\| < \varepsilon.$$

Proof. By density, pick a finite set $F \subseteq G$ and functions $b_g \in C(X)$ such that if $b = \sum_{g \in F} b_g \chi_g$ then $\|a - b\| < \frac{1}{2}\varepsilon$. Assume without loss of generality that $e \in F$. By lemma 4.30 there is an $n \in \mathbb{N}$ and functions $\{s_k\}_{k=1}^n \subset C(X)$ so that $|s_k(x)| = 1$ for all k and x and that for all $x \in X$ and $g \in F \setminus \{e\}$ we have

$$\frac{1}{n} \sum_{k=1}^{n} s_k(x) \overline{s_k(g^{-1}x)} = 0.$$

Now define a map $P : F_\alpha^{p,q}(G, X) \to F_\alpha^{p,q}(G, X)$ by

$$P(f) = \sum_{k=1}^{n} s_k *_\alpha f *_\alpha \overline{s_k}$$

for $f \in F_\alpha^{p,q}(G, X)$. To prove the theorem we must show that $\|E(a) - P(a)\| < \varepsilon$. Note that since $\|s_k\| = 1$ for all k we have that $\|P\| \leq 1$ and so

$$\|E(a) - P(a)\| \leq \|E(a) - E(b)\| + \|E(b) - P(b)\| + \|P(b) - P(a)\|$$

$$\leq \|a - b\| + \|E(b) - P(b)\| + \|a - b\| < \varepsilon + \|E(b) - P(b)\|.$$

Thus, it is sufficient to show that $P|_{C_c(G,C(X),\alpha)} = E|_{C_c(G,C(X),\alpha)}$.
Now note first that

$$P(b_e\chi_e) = \frac{1}{n}\sum_{k=1}^{n} s_k *_\alpha b_e\chi_e *_\alpha \overline{s_k} = \frac{1}{n}\sum_{k=1}^{n} s_k\chi_e *_\alpha b_e\chi_e *_\alpha \overline{s_k\chi_e}$$

$$= b_e\frac{1}{n}\sum_{k=1}^{n} s_k\overline{s_k} = b_e = E(b).$$

Now if $g \in F \setminus \{e\}$, then using (4) and the definition of multiplication in $F^{p,q}_\alpha(G,X)$ we have

$$\left(s_k *_\alpha b_g\chi_g *_\alpha \overline{s_k}\right)(t,x) = \sum_{s\in G} s_k(x)\chi_e(s)\left(b_g\chi_g *_\alpha \overline{s_k}\right)(s^{-1}t, s^{-1}\cdot x)$$

$$= s_k(x)\left(b_g\chi_g *_\alpha \overline{s_k}\right)(t,x) = s_k(x)\sum_{r\in G} b_g(x)\chi_g(r)\overline{s_k}(r^{-1}\cdot x)\chi_e(r^{-1}t)$$

$$= s_k(x)b_g(x)\overline{s_k}(g^{-1}\cdot x)\chi_e(g^{-1}t) = b_g(x)s_k(x)\alpha_g(\overline{s_k})(x)\chi_g(t).$$

We thus get that

$$P(b_g\chi_g) = \frac{1}{n}\sum_{k=1}^{n} s_k *_\alpha b_g\chi_g *_\alpha \overline{s_k} = b_g\left(\frac{1}{n}\sum_{k=1}^{n} s_k\alpha_g(\overline{s_k})\right)\chi_g = 0.$$

So by linearity $P(b) = E(b)$ and the result is proved. $\square$

We can use the above proposition to classify the structure of crossed products of group actions on minimal G-spaces.

Theorem 4.32. *Let G be a discrete group, and X a compact, metrizable G-space. Assume furthermore that G acts freely on X and that X is a minimal G-space. Then $F^{p,q}_\alpha(G,X)$ is a simple algebra.*

Proof. Let $I \subset F^{p,q}_\alpha(G,X)$ be a proper, closed ideal. We want to show that then $I = \{0\}$. We first claim that $I \cap C(X) = \{0\}$.
To prove this. Assume the opposite, and let $f \in I \cap C(X)$ be non-zero. Then there is a non-empty open set $U \subset X$ so that $f|_U \neq 0$. By minimality, we have that $\cup_{g\in G}\, gU = X$, and by compactness, there is a finite subset $F \subset G$ so that $\cup_{g\in F}\, gU = X$. We can now define a $b \in C(X)$ by

$$b(x) = \sum_{g\in F} f(g^{-1}\cdot x)\overline{f(g^{-1}\cdot x)}$$

for all $x \in X$. Because $\cup_{g\in F} gU = X$ there will always be one non-zero term in the sum, and since every summand is non-negative we have that $b(x) > 0$ for all $x \in X$. Thus b is invertible in $C(X)$ and thus also in $F^{p,q}_\alpha(G, X)$.

Moreover, we claim that $b \in I$. This is because a direct calculation shows that $b = \sum_{g\in F} f(g^{-1} \cdot x)\overline{f(g^{-1} \cdot x)} = \sum_{g\in F} \chi_g *_\alpha f *_\alpha \overline{f} *_\alpha \chi_{g^{-1}}$, which is in I by the absorption property. Thus I contains an invertible element, and it follows that the unit of $F^{p,q}_\alpha(G, X)$ is in I. By the absorption property we therefore have $I = F^{p,q}_\alpha(G, X)$, contradicting the assumption that I was proper. This shows that $I \cap C(X) = \{0\}$.

Moving on, we view $E : F^{p,q}_\alpha(G, X) \to C(X)$ as a map $F^{p,q}_\alpha(G, X) \to F^{p,q}_\alpha(G, X)$. We claim that $E(a) = 0$ for all $a \in I$. By disjointness, it is sufficient to show that $E(a) \in I$. Given any $\varepsilon > 0$ we use the proposition 4.31 to pick $\{s_k\}_{k=1}^n \subset C(X)$ so that $|s_k(x)| = 1$ and $b = \frac{1}{n}\sum_{k=1}^n s_k *_\alpha a *_\alpha \overline{s_k}$ satisfies $\|E(a) - b\| < \varepsilon$. Since $b \in I$ by absorption and ε is arbitrary, it follows that $E(a) \in \overline{I} = I$, and so $E|_I = 0$.

Now let $a \in I$. For any $g \in G$ we have $a *_\alpha \chi_{g^{-1}} \in I$ by absorption. Thus $E(a *_\alpha \chi_{g^{-1}}) = 0$. Note that if $a \in C_c(G, C(X), \alpha)$ then

$$\begin{aligned}\left(a *_\alpha \chi_{g^{-1}}\right)(t, x) &= \sum_{s\in G} a(s, x)\chi_{g^{-1}}(s^{-1}t)\mathbf{1}(s^{-1} \cdot x)\\ &= \sum_{s\in G} a_s(x)\chi_{sg^{-1}}(t),\end{aligned}$$

and so

$$E(a *_\alpha \chi_{g^{-1}}) = a_g(x) = E_g(a).$$

This extends to any element $a \in F^{p,q}_\alpha(G, X)$. Since $E(a *_\alpha \chi_{g^{-1}}) = 0$ for all $a \in I$ we have that $E_g(a) = 0$ for all $a \in I$. By uniqueness of coefficients this implies that $a = 0$ and so $I = \{0\}$. □

Definition 4.33. Let A be a unital Banach algebra. A normalized trace on A is a linear functional τ satisfying the following conditions:

- $\tau(\mathbf{1}) = 1$.
- $\|\tau\| = 1$.
- $\tau(ab) = \tau(ba) \quad \forall\, a, b \in A$.

Using the same strategies as in the proof of theorem 4.32, we are able to classify all normalized traces on $F^{p,q}_\alpha(G, X)$. We will use the following proposition.

Proposition 4.34. *Let G be a countable discrete group, and X a free compact metrizable G-space. If $A \subset F^{p,q}_\alpha(G,X)$ is a subalgebra such that $C(X) \subseteq A$, and $\tau : A \to \mathbb{C}$ is a normalized trace. Then there exists a unique Borel probability measure μ on X such that for all $a \in A$ we have*

$$\tau(a) = \int_X E(a)\, d\mu,$$

where E is the standard conditional expectation from $F^{p,q}_\alpha(G,X)$ to $C(X)$.

Proof. We first show that $\tau = \tau|_{C(X)} \circ E$. We let $a \in A$ and $\varepsilon > 0$ we want to prove that $|\tau(a) - \tau(E(a))| < \varepsilon$. We use proposition 4.31 to pick $\{s_k\}_{k=1}^n \subset C(X)$ so that $|s_k(x)| = 1$ and $\|E(a) - \frac{1}{n}\sum_{k=1}^n s_k *_\alpha a *_\alpha \overline{s_k}\| < \varepsilon$. Since $C(X) \subseteq A$ and $\overline{s_k} \in C(X)$ for all k we use the trace property to get that $\tau(s_k *_\alpha a *_\alpha \overline{s_k}) = \tau(a)$ for all k. We therefore get

$$\begin{aligned} |\tau(a) - \tau(E(a))| &= \left|\tau\left(\frac{1}{n}\sum_{k=1}^n s_k *_\alpha a *_\alpha \overline{s_k}\right) - \tau(E(a))\right| \\ &\leq \left\|E(a) - \frac{1}{n}\sum_{k=1}^n s_k *_\alpha a *_\alpha \overline{s_k}\right\| < \varepsilon. \end{aligned}$$

Thus $\tau = \left(\tau|_{C(X)}\right) \circ E$. We now apply the Riesz representation theorem on $\tau|_{C(X)}$ to find the unique Borel Probability measure μ on X such that $\tau|_{C(X)}(f) = \int_X f\, d\mu$. Finally we get

$$\begin{aligned} \tau(a) &= \left(\left(\tau|_{C(X)}\right) \circ E\right)(a) \\ &= \tau|_{C(X)}(E(a)) = \int_X E(a)\, d\mu. \end{aligned}$$

□

We can now prove our second structure theorem.

Theorem 4.35. *Let G be a discrete group, and X a free, compact, metrizable G-space. Let $M(G,X)$ be the set of G-invariant Borel probability measures on X, and $T^{p,q}(G,X)$ be the normalized traces on $F^{p,q}_\alpha(G,X)$. The mapping $\phi : M(G,X) \to T^{p,q}(G,X)$, defined by*

$$\phi_\mu(a) = \int_X E(a)\, d\mu$$

whenever $\mu \in M(G,X)$, is an affine bijection. Its inverse sends a normalized trace τ to the Riesz representation of $\tau|_{C(X)}$.

Proof. We first show that ϕ_μ in fact is a normalized trace on $F^{p,q}_\alpha(G,X)$. Firstly,

$$\phi_\mu(\mathbf{1}) = \int_X E(\mathbf{1})\ d\mu = \int_X \mathbf{1}\ d\mu = \mu(X) = 1.$$

If we now let $f \in F^{p,q}_\alpha(G,X)$ be such that $\|f\| \leq 1$ then

$$|\phi_\mu(f)| = \left| \int_X E(f)\ d\mu \right| \leq \int_X \|E(f)\|\ d\mu$$

$$\leq \int_X \|f\|\ d\mu \leq \|f\| \int_x d\mu = 1.$$

So we have $\|\phi_\mu\| \leq 1$, and since $\phi_\mu(1) = 1$ we know that equality is attained, and so $\|\phi_\mu\| = 1$. Lastly, note that if $a, b \in C_c(G, C(X), \alpha)$, then

$$a *_\alpha b(t,x) = \sum_{g\in G}\sum_{h\in G} (a_g\chi_g *_\alpha b_h\chi_h)(t,x)$$

$$= \sum_{g\in G}\sum_{h\in G}\sum_{s\in G} a_g(x)\chi_g(s) \cdot b_h(s^{-1}\cdot x)\chi_h(s^{-1}t)$$

$$= \sum_{g\in G}\sum_{h\in G} a_g(x)\chi_g(g) \cdot b_h(g^{-1}\cdot x)\chi_h(g^{-1}t)$$

$$\sum_{g\in G}\sum_{h\in G} a_g(x) \cdot b_h(g^{-1}\cdot x)\chi_{gh}(t).$$

We now evaluate $\phi_\mu(a *_\alpha b)$:

$$\phi_\mu(a *_\alpha b) = \int_X E(a *_\alpha b)\ d\mu$$

$$= \int_X E\left(\sum_{g\in G}\sum_{h\in G} a_g(x) b_h(g^{-1}\cdot x)\chi_{gh}(t)\right) dx$$

$$= \int_X \sum_{g\in G} a_g(x) b_{g^{-1}}(g^{-1}\cdot x)\ dx.$$

Since μ is G-invariant, we can do a change of variables $u = g^{-1}\cdot x$ and get that

$$\int_X \sum_{g\in G} a_g(x) b_{g^{-1}}(g^{-1}\cdot x)\ dx = \int_X \sum_{g\in G} a_g(g\cdot u) b_{g^{-1}}(u) du$$

$$= \int_X \sum_{h\in G} a_{h^{-1}}(h^{-1}\cdot u)b_h(u)\,du = \int_X E(b *_\alpha a)\,d\mu = \phi_\mu(b *_\alpha a),$$

where we have re-indexed the sum using $h = g^{-1}$. Thus the trace property is satisfied for all $a, b \in C_c(G, C(X), \alpha)$, and by continuity it thus holds for all $f, g \in F^{p,q}_\alpha(G, X)$. So ϕ_μ is a normalized trace for all $\mu \in M(G, X)$.

Furthermore, we have that for any $f \in C(X)$, $\phi_\mu(f) = \int_X f\ d\mu$. This, using linearity and density to extend to $F^{p,q}_\alpha(G, X)$, implies that ϕ is injective, and that the descripiton of its inverse is correct. Thus it only remains to show that ϕ is surjective. Let $\tau \in T^{p,q}(G, X)$. By the previous proposition there is a Borel probability measure μ on X so that $\tau(f) = \int_X E(f)\ d\mu$ for all $f \in F^{p,q}_\alpha(G, X)$. Note that for any $g \in G$ and $f \in C(X)$ we have that

$$\begin{aligned}
&\left(\chi_g *_\alpha f *_\alpha \chi_{g^{-1}}\right)(t, x)\\
&= \sum_{s\in G} \chi_g(s)\mathbf{1}(x)\left(f *_\alpha \chi_{g^{-1}}\right)(s^{-1}t, s^{-1}\cdot x)\\
&= \sum_{s\in G}\sum_{r\in G} \chi_g(s)\left(f(s^{-1}\cdot x)\chi_e(r)\right)\left(\chi_{g^{-1}}(r^{-1}s^{-1}t)\mathbf{1}(\left(r^{-1}s^{-1}\right)\cdot x)\right)\\
&= \sum_{s\in G}\sum_{r\in G} f(s^{-1}\cdot x)\chi_g(s)\chi_e(r)\chi_{g^{-1}}(r^{-1}s^{-1}t).
\end{aligned}$$

The summand is non-zero if and only if $r = e$ and $s = g$, so we get that

$$\begin{aligned}
\left(\chi_g *_\alpha f *_\alpha \chi_{g^{-1}}\right)(t, x) &= f(g^{-1}\cdot x)\chi_g(g)\chi_e(e)\chi_{g^{-1}}(e^{-1}g^{-1}t)\\
&= f(g^{-1}\cdot x)\chi_e(t).
\end{aligned}$$

Thus, using the trace property, we find that

$$\begin{aligned}
&\int_X f\ d\mu = \tau(f) = \tau(f *_\alpha \chi_{g^{-1}} *_\alpha \chi_g)\\
&= \tau(\chi_g *_\alpha f *_\alpha \chi_{g^{-1}}) = \int_X f(g^{-1}\cdot x)\ d\mu(x).
\end{aligned}$$

Since the measure μ is unique by the Riesz representation theorem, the above identity shows that μ is G-invariant, so ϕ is in fact surjective, and this completes the proof. □

5 Concluding remarks

Just as moving from Hilbert spaces to Banach spaces changes the whole C^*-theory, moving from L^p-spaces to Wiener amalgam spaces changes a lot of the theory of L^p-operator algebras. The main issue is that while the properties of $W^{p,q}(G)$ does not depend on the choice of norm, the properties of its operators does. We have tried one way of correcting this, namely by moving from isometric representations to quasi-isometric representations. This has allowed us to define operator algebras on Wiener amalgam spaces, and made us able to generalize some of the L^p-theory to amalgams. It may be interesting to see how far this approach can be stretched. Can the L^p-theory developed for groupoid algebras ([7]), AF-algebras ([6] and [15]) and graph algebras([3]) be adapted for $W^{p,q}$-operator algebras?

The approach we have used has a shortcoming, namely that universal objects like full group algebras fail to be well-defined in general. Could there be a different approach to operator algebras on Wiener amalgams that allows us to construct universal objects?

We have also seen that Lamperti's theorem cannot be carried over to Wiener amalgams. An interesting question that arises from this is how one can actually classify the invertible isometries of $W_Q^{p,q}(G)$. We have made a guess as to what the isometries might be, but it might be interesting to further investigate this. Likewise, classifying the quasi-isometries of $W_Q^{p,q}(G)$ could also be interesting. We have proved that a large subgroup of the group $\mathcal{U}(L^\infty(G)) \rtimes \mathrm{Aut}(\mathbf{B}(G))$ define quasi-isometries, but could there be more? Are there any quasi-isometries that are not elements of the group $\mathcal{U}(L^\infty(G)) \rtimes \mathrm{Aut}(\mathbf{B}(G))$?

In this thesis we have restricted ourselves to Wiener amalgams defined on locally compact groups, but there is a more general theory for amalgam spaces. It may be interesting to consider approaches that allows us to represent Banach algebras on more general amalgams.

Appendix

A The modular function

This section defines and collects the basic results about the modular function and is largely based on chapter 1.4 of [4].
Let G be a locally compact group, and μ a Haar measure on G. Given any $g \in G$ the map μ_g defined by

$$\mu_g(A) = \mu(Ag)$$

is again a Haar measure on G, as

$$\mu_g(hA) = \mu(hAg) = \mu(Ag) = \mu_g(A).$$

As a group's Haar measure is uniquely determined up to some positive multiple, there exists a number $\Delta(g)$ such that

$$\mu = \Delta(g)\mu_g.$$

Δ thus defines a function $G :\to \mathbb{R}_+$, called the modular function of G. We state its most important properties without proof.

Proposition A.1.

- *Δ is a continuous group homomorphism $G \to \mathbb{R}_+^\times$.*
- *If G is abelian or compact, then $\Delta \equiv 1$. Such groups are called unimodular.*
- *For $x, y \in G$ and $f \in L^1(G)$ we have*

$$\int_G R_y f(x)\, dx = \int_G f(xy)\, dx = \Delta(y^{-1}) \int_G f(x)\, dx.$$

- *For all $f \in L^1(G)$ we have*

$$\int_G f(x^{-1})\Delta(x^{-1})\, dx = \int_G f(x)\, dx.$$

A Convolution in non-abelian groups

Given a locally compact group with left translation-invariant Haar measure μ, and two functions $f, g \in L^1(G)$, their convolution product is defined by

$$f * g(x) = \int_G f(y)g(y^{-1}x)\, dy.$$

As the measure is left translation-invariant, we can do a change of variables $u = x^{-1}y$, leaving the infinitesimal unchanged and yielding

$$f * g(x) = \int_G f(y)g(y^{-1}x)\, dy = \int_G f(xy)g(y^{-1})\, dy. \tag{5}$$

Combining this with the fourth property of the modular function, we get that

$$f * g(x) = \int_G \Delta(y^{-1})f(y^{-1})g(yx)\, dy = \int_G \Delta(y^{-1})f(xy^{-1})g(y)\, dy. \tag{6}$$

B The Radon-Nikodym Theorem

This appendix states the Radon-Nikodym Theorem, and collects some important properties of Radon-Nikodym derivatives. The theorem is from [2], and the properties are gathered from [8].

Let μ and ν be measures on a measurable space (X, B). ν is said to be absolutely continuous with respect to μ if for any measurable set $E \in B$, $\mu(E) = 0$ implies $\nu(E) = 0$. If, in addition, μ is σ-finite, there exists a unique measurable function $\frac{d\nu}{d\mu}$, the Radon-Nikodym derivative, such that

$$\int_X f d\nu = \int_X f \frac{d\nu}{d\mu}\, d\mu$$

for all $f \in L^1(X, B, \nu)$. This also means that

$$f \cdot \frac{d\nu}{d\mu} \in L^1(X, B, \nu) \qquad \forall f \in L^1(X, B, \nu).$$

Proposition B.1. *The Radon-Nikodym derivative satisfies the following properties*

- *If μ, ν, ρ are measures on (X, B), then*

$$\frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\rho} = \frac{d\nu}{d\rho}.$$

- *If $\phi \in \mathrm{Aut}(B)$*

$$\frac{d\nu}{d\mu} \circ \phi = \frac{d(\nu \circ \phi^{-1})}{d(\mu \circ \phi^{-1})}.$$

In particular,

$$\frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\nu} \equiv 1.$$

- *If ϕ instead is an isomorphism between measure spaces, then also*

$$\frac{d\nu}{d\mu} \circ \phi = \frac{d(\nu \circ \phi^{-1})}{d(\mu \circ \phi^{-1})}.$$

A The change of variables formula

Let $(X, B, \mu))$ be a measure space, (Y, Σ) a measurable space, and $\phi : X \to Y$ a measurable mapping. Given any $E \in \Sigma$, we can measure the pre-image of E under ϕ, and it can be verified that the function $\mu \circ \phi^{-1} : \Sigma \to [0, \infty]$ given by

$$\mu \circ \phi^{-1}(E) = \mu(\phi^{-1}(E))$$

is a measure on (Y, Σ). This measure is called the pushforward measure of μ by ϕ.

The pushforward measure allow us to change between integrating on X and integrating on Y, as described by the following result, whose proof is found in [1]

Theorem B.2 (Change of variables)**.** *Let $(X, B, \mu))$ be a measure space, (Y, Σ) a measurable space, and $\phi : X \to Y$ a measurable mapping. A measurable function f on Y is integrable with respect to the measure $\mu \circ \phi^{-1}$ if and only if the composition $f \circ \phi$ is integrable with respect to μ. In that case, the integrals coincide, and we have*

$$\int_Y f \, d(\mu \circ \phi^{-1}) = \int_X f \circ \phi \, d\mu.$$

Note. If (Y, Σ) has another measure ν, then the Radon-Nikodym theorem gives us that

$$\int_Y f \, d(\mu \circ \phi^{-1}) = \int_Y f \cdot \frac{d(\mu \circ \phi^{-1})}{d\nu} \, d\nu.$$

Combining this with the change of variables formula we get that

$$\int_{\phi^{-1}(X)} f \circ \phi \, d\mu = \int_X f \circ \phi \, d\mu = \int_Y f \cdot \frac{d(\mu \circ \phi^{-1})}{d\nu} \, d\nu,$$

which gives a more "familiar" change of variables formula.

C Minkowski's integral inequality

The classical Minkowski inequality relates the norms of f, g and $f+g$ for $f, g \in L^p(G)$. We have that

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p,$$

with equality if and only if f and g are positively linearly dependent. This is the special case of a more general result from measure theory.

Lemma C.1. *Minkowski's integral inequality Let (S_1, μ_1) and (S_2, μ_2) be σ-finite measure spaces, $p \in [1, \infty)$, and suppose that $F : S_1 \times S_2 \to \mathbb{R}$ is measurable. Then we have*

$$\left(\int_{S_2} \left| \int_{S_1} F(x,y)\, d\mu_1(x) \right|^p d\mu_2(y)\right)^{\frac{1}{p}} \leq \int_{S_1} \left(\int_{S_2} |F(x,y)|^p \, d\mu_2(y)\right)^{\frac{1}{p}} d\mu_1(x).$$

If $p > 1$ and both sides are finite, then equality holds only if $|F(x,y)| = \phi(x)\psi(y)$ for some non-negative measurable functions ϕ and ψ. This result can be extended to $p = \infty$ in the obvious way.

Note. Assume $f, g \in L^p(S_2)$. Let $S_1 = \{1, 2\}$, let μ_1 be the counting measure on S_1, and define $F(x,y)$ by

$$F(1,y) = f(y), \qquad F(2,y) = g(y).$$

Then Minkowski's integral inequality yields the classical Minkowski inequality.

D Vector-valued integrals

When dealing with Banach spaces it is sometimes useful to discuss integral-like constructions for functions taking values in an arbitrary Banach space. There are several different ways to construct an integral for Banach space-valued functions. The following is called the weak integral.

Definition D.1. A function $f : E \to F$ between Banach spaces is called weakly integrable if for all $\phi \in F^*$, the composition $\phi \circ f : E \to \mathbb{C}$ is Lebesgue measurable, that is, an element of $L^1(X)$

Definition D.2. Given a weakly integrable function $f : E \to F$ its weak integral is the unique element $e \in F$ such that for all $\phi \in F^*$,

$$\langle e, \phi \rangle = \int_X \langle f(x), \phi \rangle \, dx.$$

We denote the weak integral of f by $\int_X f(x)\, dx$, and it equals e.

The weak integral has most of the properties of the regular integral, but comes with a special Banach space property as well.

Proposition D.3. *Let $f : E \to F$ be weakly integrable. Then*

$$\left\| \int_X f(x)\, dx \right\| \le \int_X \|f(x)\|\, dx.$$

If $\psi : F \to G$ is a bounded, linear map, then $\psi \circ f$ is weakly integrable and

$$\int_X \psi \circ f(x)\, dx = \psi\left(\int_X f(x)\, dx\right).$$

As a special case, let G be a locally compact group, and X a Banach space. If a function $\psi : G \to \mathcal{B}(X)$ is compactly supported and strongly continuous, then the weak integral can be computed explicitly:

$$\int_G \psi(t)\, dt = \left[x \to \int_G \psi(t)(x)\, dt\right] \in \mathcal{B}(X).$$

E A general version of theorem 2.24

Theorem E.1. *Let G be an IN-group. For $i \in 1,2,3$, let p_i, q_i be such that there is constants $C_1, C_2 > 0$ such that*

$$\|h * k\|_{p_3} \le C_1 \|h\|_{p_1} \|k\|_{p_2} \qquad \forall h \in L^{p_1}, \forall k \in L^{p_2}$$

and

$$\|h * k\|_{q_3} \le C_2 \|h\|_{q_1} \|k\|_{q_2} \qquad \forall h \in L^{q_1}, \forall k \in L^{p_2}.$$

Then, there is another constant $C > 0$ such that

$$\|f * g\|_{p_3,q_3} \le C \|f\|_{p_1,q_1} \|g\|_{p_2,q_2}$$

for all $f \in W^{p_1,q_1}(G)$ and $g \in W^{p_2,q_2}(G)$. So if $L^{p_1}(G) * L^{p_2}(G) \subseteq L^{p_3}(G)$ and $L^{q_1}(G) * L^{q_2}(G) \subseteq L^{q_3}(G)$ then

$$W^{p_1,q_1}(G) * W^{p_2,q_2}(G) \subseteq W^{p_3,q_3}(G).$$

Proof. We will use the discrete norm. Let Q be the invariant neighbourhood, and assume that the translates x_nQ constitute a BUPU. Let $\chi_n = \chi_{x_nQ}$. We write the "discrete control function" of f in L^p as

$$F_{f,p}(n) = \|f \cdot \chi_n\|_p \qquad n \in \mathbb{Z}.$$

In this setting, the $W^{p,q}(G)$-norm of f is

$$\|f\|_{p,q} = \|F_{f,p}\|_{\ell^q} = \left(\sum_{n\in\mathbb{Z}} |F_{f,p}(n)|^q\right)^{\frac{1}{q}}.$$

Now, given $f \in W^{p_1,q_1}(G)$ and $g \in W^{p_2,q_2}(G)$ and note that since Q is an invariant neighbourhood

$$\operatorname{supp}(f \cdot \chi_n * g \cdot \chi_m) \subseteq x_mQx_nQ = x_{m+n}\tilde{Q}.$$

Since $\tilde{Q}$ is compact, there is a finite subset of $\{x_n\}_{\mathbb{Z}}$, say $\{x_j\}_{j=1}^T$ such that $\tilde{Q} \subseteq \cup_{j=1}^T x_jQ$. Thus

$$((f \cdot \chi_n) * (g \cdot \chi_m)) \cdot \chi_k \neq 0 \iff k = m+n+i, i \in 1, 2, ..., T.$$

Using linearity of the convolution we estimate F_{f*g,p_3}.

$$F_{f*g,p_3}(k) = \|(f*g) \cdot \chi_k\|_{p_3}$$

$$= \left\|\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}((f \cdot \chi_n) * (g \cdot \chi_m)) \cdot \chi_k\right\|_{p_3}$$

$$= \left\|\sum_{m\in\mathbb{Z}}\sum_{j=1}^{T}((f \cdot \chi_{k-m+j}) * (g \cdot \chi_m)) \cdot \chi_k\right\|_{p_3}$$

$$\leq \sum_{m\in\mathbb{Z}}\sum_{j=1}^{T}\|((f \cdot \chi_{k-m+j}) * (g \cdot \chi_m)) \cdot \chi_k\|_{p_3}$$

$$\leq \sum_{m\in\mathbb{Z}}\sum_{j=1}^{T}\|(f \cdot \chi_{k-m+j}) * (g \cdot \chi_m)\|_{p_3}$$

$$\leq C_1 \sum_{m\in\mathbb{Z}}\sum_{j=1}^{T}\|f \cdot \chi_{k-m+j}\|_{p_1}\|g \cdot \chi_m\|_{p_2}$$

$$
\begin{aligned}
&= C_1 \sum_{m\in\mathbb{Z}} \sum_{j=1}^{T} F_{f,p_1}(k-m+j)F_{g,p_2}(n) \\
&= C_1 \sum_{j=1}^{T} (F_{f,p_1} * F_{g,p_2})(k+j) \\
&= C_1 \sum_{j=1}^{T} T_{x_j{}^{-1}}(F_{f,p_1} * F_{g,p_2})(k).
\end{aligned}
$$

Using that translation is an isometry of ℓ^q we compute the $W^{p_3,q_3}(G)$-norm.

$$
\begin{aligned}
\|f * g\|_{p_3,q_3} &= \|F_{f*g,p_3}\|_{\ell^{q_3}} \\
&\leq C_1 \sum_{j=1}^{T} \|F_{f,p_1} * F_{g,p_2}\|_{\ell^{q_3}} \\
&= TC_1 \|F_{f,p_1} * F_{g,p_2}\|_{\ell^{q_3}} \leq C_2 \cdot TC_1 \|F_{f,p_1}\|_{\ell^{q_1}} \|F_{g,p_2}\|_{\ell^{q_2}} \\
&= C\|f\|_{p_1,q_1}\|g\|_{p_2,q_2}.
\end{aligned}
$$

□

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