Mathias Reiersen

Hölder Estimates of the $\bar{\partial}$-Equation

Master's thesis in Mathematical Sciences
Supervisor: Berit Stensønes
June 2022

Master's thesis

NTNU
Norwegian University of Science and Technology
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Acknowledgment

This thesis is the culmination of my master studies in mathematics at NTNU. It was written during the academic year 2021-2021 written under Professor Berit Stensønes.

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Summary

Solving and estimating the Cauchy-Riemann equations $\bar{\partial}u = f$ has been a staple in complex analysis since its inception and is central in many applications. In this thesis we will solve and find estimates for the $\bar{\partial}$-equation in two complex variables for bounded pseudoconvex domains with real analytic boundary of finite type. The techniques used are similar to a paper titled "Sup-Norm Estimates for $\bar{\partial}$" by Grundmeier, Stensønes and Simon and involves using a bumping to type of a domain.

Oppsummering

Det å løse og estimere Cauchy-Riemann-ligningene $\bar{\partial}u = f$ har vært grunnleggende i kompleks analyse siden feltets begynnelsen og er sentralt i mange applikasjoner. I denne oppgaven vil vi løse og estimere $\bar{\partial}$-ligningen i to komplekse variabler for begrensende pseudokonvekse områder med reell analytisk rand av endelig type. Teknikkene som brukes er som i en artikkel kalt "Sup-Norm Estimates for $\bar{\partial}$" av Grundmeier, Stensønes og Simon og innebærer bruk av bumping til type til et område.

Contents

Chapter 1

Introduction

In this thesis we will solve the $\bar{\partial}$-equation on bounded weakly pseudoconvex domains with real analytic boundary of finite type using methods from [5]. These methods are such that we will obtain sup norm and Hölder estimates for the solution operator.

The main result we will show is the following Theorem.

Main Theorem. *Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex domain with real analytic boundary of D'Angelo finite type $2k$ and let f be a $\bar{\partial}$-closed (0,1)-form on $\overline{\Omega}$. Then there exists a solution u of $\bar{\partial}u = f$ on Ω such that*

$$||u||_\infty \leq C_\Omega ||f||_\infty$$

where C_Ω is independent of f. Furthermore for every $\eta > 0$ there is a solution $u^{(\eta)}$ as above that satisfy $(\frac{1}{2k} - \eta)$-Hölder estimates with constant only depending on Ω and η.

1.1 Motivation

We recall that a function of one complex variable is called analytic if and only if it satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of a complex-valued function $f(z)$, respectively. Symbolically we can rewrite the Cauchy-Riemann equations from real and imaginary parts to holomorphic and anti-holomorphic parts by the relations

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

The requirement of being an analytic function then becomes $\partial f/\partial \overline{z} = 0$ meaning that the function does not depend on conjugate variables.

For a function $f(z_1,...,z_n)$ of several complex variables we say it is holomorphic if it is analytic in each variable $z_j \mapsto f(z_1,...,z_j,...,z_n)$. More generally, in a complex differential structure the exterior derivative d decomposes into operators ∂ and $\overline{\partial}$ by

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j,$$
$$\overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j.$$

Therefore, as in the one variable case, we can say that f is holomorphic if and only if $\overline{\partial} f = 0$. Naturally then, the $\overline{\partial}$-operator serves as a generalization to the Cauchy-Riemann equations.

The $\overline{\partial}$-operator appears in many areas of complex analysis. For example the Cauchy integral formula in $\mathbb{C}$ for C^1 functions on a domain D

$$f(\zeta) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)dz}{\zeta - z} - \frac{1}{2\pi i} \int_D \frac{\partial f(z)}{\partial \overline{z}} \frac{d\overline{z}}{\zeta - z},$$

and its generalization to several variables in the Bochner-Martinelli formula [6]

$$f(\zeta) = \int_{\partial D} f(\zeta)\omega(z) - \int_D \overline{\partial} f(z) \wedge \omega(z).$$

Here

$$\omega(\zeta, z) := \frac{(n-1)!}{(2\pi i)^n} \frac{1}{|\zeta - z|^{2n}} \sum_{j=1}^{n} (\overline{\zeta}_j - \overline{z}_j) d\overline{\zeta}_1 \wedge d\zeta_1 \wedge \ldots d\zeta_j \ldots \wedge d\overline{\zeta}_n \wedge d\zeta_n.$$

Other examples where the $\overline{\partial}$-operator arises are in the Hartogs phenomenon and in classification of peak points.

Proposition 1.1.1 (Hartogs Phenomenon)**.** *Let* $\Omega \subset\subset \mathbb{C}^n$, $n > 1$. *Let* K *be a compact* *in Ω so that $\Omega \setminus K$ is connected. If f is holomorphic on $\Omega \setminus K$, then there is a unique holomorphic function F on Ω such that $F|_{\Omega\setminus K} = f$.*

The proof of the Proposition relies on the following solution to the $\bar{\partial}$-problem.

Lemma 1.1.2. *Let ψ be a $\bar{\partial}$-closed (0,1)-from on $\mathbb{C}^n$ with compact support. Then there exits a function u with compact support such that $\bar{\partial} u = \psi$ where $u = 0$ on the unbounded component of supp(ψ).*

Proof of Hartogs Phenomenon. Let $\phi \in C_c^\infty(\Omega)$ be so that it is identically 1 on a neighborhood of K and is 0 in a neighborhood of the boundary $\partial\Omega$. Define the function

$$\tilde{f}(z) = \begin{cases} (1 - \phi(\mathbf{z})) \cdot f(z) & \text{if } \mathbf{z} \in \Omega \setminus K, \\ 0 & \text{if } \mathbf{z} \in K. \end{cases}$$

Then $\tilde{f} \in C^\infty(\Omega)$ and set $\omega = \bar{\partial}\tilde{f}$. We see that ω has C^∞ coefficients, $\bar{\partial}\omega = 0$ and that ω has compact support. We then find u so that $\bar{\partial} u = \omega$. We have also that u is identically 0 in a neighborhood U of $\partial\Omega$. We define $F = \tilde{f} - u$ and see that

$$\bar{\partial} F = \bar{\partial}\tilde{f} - \bar{\partial} u = \omega - \omega = 0$$

and so notice that

$$F|_U = (\tilde{f} - u)|_U = f|_U$$

yielding that $F = f$ on $\Omega \setminus K$ and thus F is an extension of f. □

The following Proposition and proof is as in [7] and shows how a solution to the $\bar{\partial}$-problem can be used show the existence of a function which peaks on the boundary of a pseudoconvex domain.

Proposition 1.1.3. *If Ω is a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ that is strongly pseudoconvex at $p \in \partial\Omega$ then there exists a function $f : \overline{\Omega} \longrightarrow \mathbb{C}$ holomorphic on Ω and smooth on $\overline{\Omega}$ with $f(p) = 1$ and $|f| < 1$ on $\overline{\Omega} \setminus \{p\}$.*

Proof. Since the boundary is smooth we can find a local strong support function $g(z)$ at p, namely a function $g \in C^\infty(U)$ where U is a neighborhood of p with $g(0) = p$ and $\operatorname{Re} g > 0$ on $\overline{\Omega \cap U} \setminus \{p\}$. Then by choosing a smooth cut off function χ in a small neighborhood of p we can define a $\overline{\partial}$-closed (0,1)-form

$$\psi(z) = \begin{cases} 0, & \text{when } z = p \\ \overline{\partial}\left(\frac{\chi}{g}\right), & \text{when } z \neq p. \end{cases}$$

Then ψ is smooth $\overline{\partial}$-closed with compact support. We solve $\overline{\partial}$ by finding u so that $\overline{\partial} u = \psi$. Define so f by

$$f(z) = \begin{cases} 0, & \text{when } z = p \\ \frac{1}{\chi/g - v}, & \text{when } z \neq p \end{cases}.$$

It is easy then to see that f satisfies the sought criteria. □

Remark 1.1.4. In this Proposition we are using that for strongly pseudoconvex domains we have solutions for $\overline{\partial}$ which satisfies sup-norm estimates. This will be discussed later.

The $\overline{\partial}$-problem has many aspects one can consider. In this thesis we will consider solutions when our input of data is a $\overline{\partial}$-closed (0,1)-form. So the question is: Given a (0,1)-form f on a domain $\overline{\Omega}$ with $\overline{\partial} f = 0$, does there exist a function u on $\overline{\Omega}$ so that $\overline{\partial} u = f$?

The question concerning existence of a solution to the $\overline{\partial}$-equation can be answered when the domain is *pseudoconvex.* Hörmander showed that given a $(0, q)$-form v on a pseudoconvex domain with $\overline{\partial} v = 0$, then there exists a solution.

Theorem 1.1.5 (Hörmander, Demailly, Theorem (8.4) [2]). *Let ρ be a plurisubharmonic function on a pseudoconvex domain $D \subset \mathbb{C}^n$, v be a $\overline{\partial}$-closed $(0,q)$-form. Then there exists a $(0,q-1)$- form u such that $\overline{\partial} u = v$ and*

$$\int_D |u|^2 e^{-\rho} \leq C \int_D \langle M^{-1} v, v\rangle e^{-\rho},$$

where M is a matrix that depends on ρ and q.

Remark 1.1.6. When $q = 1$ the matrix M is just the complex Hessian matrix of ρ.

The estimates for this solution are weighted L^2 estimates. For the solution to be the most applicable we would like to be able to produce solutions with stronger estimates and have a solution formula. A later result by Henkin and Ramirez [6] gives us this on strongly pseudoconvex domains in terms of an integral formula which satisfies Hölder estimates of order 1/2.

Theorem 1.1.7. *If $\Omega \subseteq \mathbb{C}^n$ is strongly pseudoconvex with C^4 boundary and f is a $\overline{\partial}$-closed (0,1)-form on a neighborhood of $\overline{\Omega}$ with C^1 coeffiecients, then the function*

$$H_\Omega f = c_n \int_{\partial\Omega\times[0,1]} f \wedge \eta(\mu) \wedge \omega(\zeta) - c_n \int_\Omega \frac{f(\zeta)}{||\zeta - z||^{2n}} \eta(\overline{\zeta} - \overline{z}) \wedge \omega(\zeta)$$

satisfies $\overline{\partial} H_\Omega f(z) = f(z)$. Here c_n is a constant dependent on dimension n, the functions

$$\omega(\zeta) := d\zeta_1 \wedge \ldots \wedge d\zeta_n$$
$$\eta(\xi) := \sum_{j=1}^{n} (-1)^{j+1} \xi_j d\xi_1 \wedge \ldots \wedge \wedge d\xi_{j-1} \wedge d\xi_{j+1} \wedge \ldots \wedge d\xi_n$$

for any $\xi = (\xi_1, ..., \xi_n) \in \mathbb{C}^n$, and for $\lambda \in [0,1]$

$$\mu_j = \frac{\overline{\zeta}_j - \overline{z}_j}{||\zeta - z||^2} \lambda + h_j (1 - \lambda)$$

where all $h_j(\zeta, z) : \partial\Omega \times \Omega \longrightarrow \mathbb{C}$ *are holomorphic in* z *and solves the Cauchy-Fantappiè equation*

$$\sum_{j=1}^{n} h_j(\zeta, z)(\zeta_j - z_j) \equiv 1$$

when $\zeta \in \partial\Omega, z \in \Omega$.

Remark 1.1.8. In the construction of the kernel one uses that strongly pseudoconvex domains are locally convex to construct the functions h_j. This is not the case for weakly pseudoconvex domains, as such domains are not locally convex.

Naturally the question of sup-norm estimates arises in the general pseudoconvex case aswell. However some additional assumptions need to be made. Sibony [10] gave an example of a C^∞ smooth pseudoconvex domain in $\mathbb{C}^3$ and a $\bar{\partial}$-closed (0,1)-form f which is bounded in the given domain, but no solution u of the $\bar{\partial}$-equation $\bar{\partial}u = f$ is bounded. This tells us that pseudoconvexity is not the sole requirement for solving $\bar{\partial}$ with sup norm estimates.

Fornæss have shown that for some domains of finite type one is able to find solutions with sup norm estimates [3]. An example of such is the Kohn-Nirenberg domain. Therefore one suspects that finite type for points in the boundary is a requirement to find a solution with sup norm estimates. For strongly pseudoconvex boundary points we have that the type is 2, thus this requirement holds for strongly pseudoconvex domains.

1.2 Definitions and Preliminaries

Most theory on the $\bar{\partial}$-equation involves the concept of *pseudoconvex* domains as already mentioned. Such domains are central in several complex variables. We can characterize them in different ways, and for this thesis we will use the notion of Levi-pseudoconvexity. This type of pseudoconvexity is dependent on the smoothness of the boundary.

Definition 1.2.1. We say that a domain $\Omega \subset\subset \mathbb{C}^n$ with boundary $\partial\Omega$ has C^k-differentiable boundary at the point $p \in \partial\Omega$ if there exists a real valued function $\rho \in C^k(U)$ defined in a neighborhood U of the boundary of Ω such that

$$\Omega \cap U = \{z \in U : \rho(z) < 0\}, \tag{1.1}$$

$$d\rho(z) \neq 0 \text{ for } z \in U. \tag{1.2}$$

We say the boundary is of class C^k if all points $p \in \partial\Omega$ is of class C^k.

Remark 1.2.2. We call the function ρ a *defining function* for the the domain Ω.

Definition 1.2.3. Consider the function $J : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ defined by

$$J(x_1, y_1, \ldots, x_n, y_n) = (-y_1, x_1, \ldots, -y_n, x_n).$$

The *complex tangent space* is defined as a subspace of the real tangent space

$$T_p^{\mathbb{C}} M := \{x \in T_p M | Jx \in T_p M\}.$$

Knowing these concepts we can define the concept of Levi-pseudoconvexity.

Definition 1.2.4 (Levi pseudoconvex)**.** Let $\Omega \subset \mathbb{C}^n$ be a domain with C^2 boundary. Then there is a C^2 defining function $r : \mathbb{C}^n \longrightarrow \mathbb{R}$. We say that Ω is Levi-pseudoconvex if for every $p \in \partial\Omega$ and $t \in T_p^{\mathbb{C}}\Omega$

$$\sum_{i,j} \frac{\partial^2 r(p)}{\partial z_i \partial \bar{z}_j} t_i \bar{t}_j \geq 0.$$

If the inequality is strict for all $p \in \partial\Omega$, we call Ω strongly Levi-pseudoconvex.

Remark 1.2.5. Levi-pseudoconvexity is independent of choice of defining function.

Remark 1.2.6. For domains not satisfying the required smoothness we classify pseudoconvexity differently. However, all domains in this thesis will have sufficiently smooth boundary. Therefore we will throughout this thesis refer to all Levi-pseudoconvex domains as just pseudoconvex.

We will also require that our domains have points of finite type in the boundary. We will consider the notion D'Angelo finite type.

Definition 1.2.7 (D'Angelo type [1])**.** Let $\Omega \subset\subset \mathbb{C}^n$ be pseudoconvex domain with C^∞ boundary and let r be the defining function for Ω. For a holomorphic function $f : \Delta \longrightarrow \mathbb{C}^n$ where $f(0) = p$ define the type at the point p as

$$\sup_f \frac{\nu(r \circ f)}{\nu(f)}$$

where $\nu(f)$ is the order of vanishing at the origin of $\mathbb{C}$.

Remark 1.2.8. We say a domain is of finite type m if all points $p \in \partial\Omega$ is of finite type m or less. Further if the domain in question is pseudoconvex the type of the point must be an even number as a result of computing the Levi-form.

To give estimates we will use a the concept of *bumping to type* which will be a larger pseudoconvex domain containing the original domain. Here we will utilize the extra room to find pointwise estimates which we can then translate to smooth ones.

Definition 1.2.9. Given a pseudoconvex domain Ω and $p \in \partial\Omega$, we say that Ω can be locally bumped at p if there exists a neighborhood U of p and another pseudoconvex domain Ω_p^* satisfying the inclusions $\overline{\Omega} \setminus \{p\} \cap U \subset \Omega_p^*$ with $p \in \partial\Omega_p^*$. We then say Ω_p^* is a local bumping at p.

Throughout the thesis we will be using that for functions $f(z)$ and $g(z)$ that $|f(z)| \lesssim |g(z)|$

if there is a constant $C > 0$ so that $|f(z)| \le C|g(z)|$. Also we will write $|f(z)| \sim |g(z)|$ when $|f(z)| \lesssim |g(z)|$ and $|g(z)| \lesssim |f(z)|$. When $|f| \sim |g|$ we call the functions proportional in size.

A crucial point is that whenever we have a defining function for a domain $\Omega = \{r(z) < 0\}$, we can use it to measure the distance to the boundary of the domain. This is because defining functions are not unique, rather they depend on one another. That is, if there are two C^k defining functions r_1 and r_2 for Ω, then there is an $h \in C^{k-1}(\Omega)$ with $h \neq 0$ such that $r_1 = hr_2$. This means that all defining functions are comparable in size and using the following Lemma gives us that a defining function $|r(z)| \sim \text{dist}(z, \partial\Omega)$.

Lemma 1.2.10. *Let Ω be a nonempty open subset of $\mathbb{R}^k$, where $k \ge 2$. Assume Ω is bounded and the boundary of Ω is of class C^2. Then the signed distance function*

$$d_\Omega(x) = \begin{cases} -dist(x, \partial\Omega), & if\, x \in \Omega \\ dist(x, \partial\Omega), & if\, x \not\in \Omega \end{cases}$$

is a defining function for Ω.

To prove the main theorem we will first show that a bounded domain with real analytic boundary of finite type can be bumped to type at each boundary point. We will use this fact to construct a solution operator and provide bounds.

Then to construct the solution operator we will use the Henkin Integral kernel. What we need to do is adjust the functions h_j involved in the Cauchy-Fantappiè equation

$$h_1(\zeta, z)(\zeta_1 - z_1) + h_2(\zeta, z)(\zeta_2 - z_2) \equiv 1.$$

By strong pseudoconvexity these functions satisfies bounds from below allowing the integral to be estimated. We then need to construct functions satisfying bounds when the domain is weakly pseudoconvex. This will be done by choosing smooth pointwise solutions to the Cauchy-Fantappiè equations and making them holomorphic.

Using the local bumping and Hörmanders theorem 1.1.5 we can show there are functions satisfy pointwise bounds on the boundary. By considering a smaller domain $\Omega_{-\varepsilon}$ these pointwise estimates can be replaced with estimates which are smooth on the boundary and satisfy the Cauchy-Fantappiè equation. These new functions can then be used in the Henkin Integral formula to get a solution operator on $\partial\Omega \times \Omega_{-\varepsilon}$ which will satisfy Hölder estimates.

The final step is then to extend this solution to a solution $\partial\Omega \times \Omega_{-\varepsilon}$.

Chapter 2

Bumping to Type

In this chapter we will show how to construct a local bumping of a bounded pseudo-convex domain with real analytic boundary of finite type.

2.1 Local Bumping

First we will show that weakly pseudoconvex, bounded domains of finite type with real analytic boundary can be bumped to type. Let $\Omega \subset\subset \mathbb{C}^{n+1}$ for $n \geq 1$ be such a domain. Then we can find a local description near a boundary points via a smooth defining function in a neighborhood of the domain. Let W be a neighborhood of 0 and let $\rho : W \longrightarrow \mathbb{R}$ be the smooth defining function for $W \cap \Omega$, then

$$\Omega \cap W = \{\mathbf{z} \in \mathbb{C}^{n+1} : \rho(\mathbf{z}) < 0\}.$$

Translating the domain we can assume without loss of generality that 0 is in the boundary and by a rotation assume that $t = (1, 0, ..., 0)$ is in the complex tangent space $T_0^{\mathbb{C}}(\Omega)$. Since the domain has real analytic boundary, by writing the series expansion around 0, we express

$$\rho(\mathbf{z}) = \rho(0) + 2\mathrm{Re}\Big(\sum_{j=1}^{n+1} \frac{\partial \rho}{\partial z_j} z_j + \frac{1}{2}\sum_{i=1,j=1}^{n+1} \frac{\partial^2 \rho}{\partial z_i \partial z_j} z_i z_j\Big) + \sum_{i=1,j=1}^{n+1} \frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j} z_i \overline{z}_j + O(|\mathbf{z}|^2).$$

Write $(w, \mathbf{z}) = (u + iv, x_1 + iy_1, ..., x_n + iy_n) \in \mathbb{C} \times \mathbb{C}^n$ and introduce a linear coordinate change so that

$$\frac{\partial \rho}{\partial u} = 1, \quad \frac{\partial \rho}{\partial v} = 0 \quad \text{and} \quad \frac{\partial \rho}{\partial z} = 0. \tag{2.1}$$

Set further

$$r(\mathbf{z}, \overline{\mathbf{z}}) = \sum_{m=2k}^{\infty} P_m(\mathbf{z}, \overline{\mathbf{z}})$$

where each $P_m(\mathbf{z})$ is homogeneous polynomial in z and $\overline{z}$ of order m and $P_{2k} \not\equiv 0$. The boundary is of finite type and the domain is pseudoconvex, therefore the number $2k$ must be less than or equal to the type at 0. The series expansion can then be expressed as

$$u + r(\mathbf{z}, \overline{\mathbf{z}}) + O(v^2, ||\mathbf{z}||\, v)$$

and we can write the domain locally as

$$\{(w, \mathbf{z}) \in \mathbb{C} \times \mathbb{C}^n : u + r(\mathbf{z}) + O(v^2, ||\mathbf{z}||v) < 0\}. \tag{2.2}$$

We want to control the higher order terms $O(v^2, ||\mathbf{z}||v)$. Calculating the Levi form $\mathrm{Lev}(\rho, \mathbf{z}, \mathbf{z})$ in a neighborhood of 0, we know from the pseudoconvexity of Ω must be greater than or equal to 0. The Levi form consists of terms of the form

$$\frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j} \sim \frac{\partial^2 r}{\partial z_i \partial \overline{z}_j} + s z_i^{s-1} \overline{z}_j^t + t z_i^s \overline{z}_j^{t-1} + stv z_i^{s-1} \overline{z}_j^{t-1} + s z_i^{s-1} + t \overline{z}_j^{t-1} \geq 0.$$

Therefore the terms which are of degree lower than the smallest degree of $r(\mathbf{z}, \overline{\mathbf{z}})$, in this case $2k$, cannot be negative as they would dominate $r(z, \overline{z})$ in the neighborhood. In other words when $s + t > 2k + 1$ the terms $vz^s \overline{z}^t$ are small and cannot affect the positivity of the levi form. This means that the bumped out domain can be assumed to have defining function

$$u + r(\mathbf{z}, \overline{\mathbf{z}}) + O(v^2, ||\mathbf{z}||^M v)$$

where $M \geq 2k$.

We now show how to deal with higher order terms. By a holomorphic change of coordinates $w = \tilde{w} - B\tilde{w}^2$ with large constant $B >> 0$

$$u = \tilde{u} - B(\tilde{u}^2 - \tilde{v}^2),$$
$$v = \tilde{v} - 2B\tilde{u}\tilde{v}$$

we can express

$$\rho(w, \mathbf{z}) = \tilde{u} - B\tilde{u}^2 + B\tilde{v}^2 + r(\mathbf{z}, \overline{\mathbf{z}}) + O(\tilde{v}^2, ||\mathbf{z}||^M \tilde{v}, \tilde{v}^2 \tilde{u}, ||\mathbf{z}||^M \tilde{u}\tilde{v}).$$

Multiplying the defining function with some nonzero function we can obtain a new defining function. Let s be a nonzero function, then

$$\nabla(\rho s) = \nabla(\rho)s + \rho\nabla(s)$$

and since $r = 0$ on the boundary, we have that $\nabla(\rho s) = \nabla(\rho)s$ and thus ρs also locally defines the domain.

For terms of the form $\tilde{u}\tilde{v}^2$, $\tilde{u}||\mathbf{z}||^M\tilde{v}$ and $-B\tilde{u}^2$ we can, for some $C > B$, multiply by $(1 + C\tilde{v}^2)$, $(1 + C||z||^M\tilde{v})$ and $(1 + C\tilde{u})$, respectively. Notice that then

$$C\tilde{u}(\tilde{v}^2 + ||\mathbf{z}||^M\tilde{v} + \tilde{u}) - B\tilde{u}^2 + O(\tilde{v}^2\tilde{u}, ||\mathbf{z}||^M\tilde{u}\tilde{v}) > 0.$$

Further consider the function

$$\tilde{u} + B\tilde{v}^2 + r(\mathbf{z},\overline{\mathbf{z}}) + O(\tilde{v}^2, ||\mathbf{z}||^M\tilde{v}). \tag{2.3}$$

Higher order terms are just those multiplied with $\tilde{v}$. For an $0 < \eta << 1$ add 0 to the expression by adding $\eta||\mathbf{z}||^{2k} - \eta||\mathbf{z}||^{2k} + 2\sqrt{B\eta}||\mathbf{z}||^k v - 2\sqrt{B\eta}||\mathbf{z}||^k v$. We want to complete the square in such a way that

$$\begin{aligned}(\sqrt{B}v - \sqrt{\eta}||\mathbf{z}||^k)^2 &= Bv^2 + \eta||\mathbf{z}||^{2k} - 2\sqrt{B\eta}||\mathbf{z}||^k v\\ &\leq Bv^2 + \eta||\mathbf{z}||^{2k} - 2\sqrt{B\eta}v\mathbf{z}^s\overline{\mathbf{z}}^t.\end{aligned}$$

This holds when $k > 2k + 1$ as $v||\mathbf{z}||^k \geq |v\mathbf{z}^s\overline{\mathbf{z}}^t|$ in a small neighborhood of 0. We then get that

$$(\sqrt{B}v - \sqrt{\eta}||\mathbf{z}||^{2k})^2 + O(v^2, ||\mathbf{z}||^M v) > 0.$$

Therefore the expression can

$$\tilde{u} + B\tilde{v}^2 + r(\mathbf{z},\overline{\mathbf{z}}) + (\sqrt{B}v - \sqrt{\eta}||\mathbf{z}||^k)^2 - \eta||\mathbf{z}||^{2k} + 2\sqrt{B\eta}v||\mathbf{z}||^{2k} + O(\tilde{v}^2, ||\mathbf{z}||^M\tilde{v}),$$

can be rewritten using

$$B\tilde{v}^2 + O(\tilde{v}^2) > 0;$$
$$2\sqrt{B\eta}\,\tilde{v}||\mathbf{z}||^{2k} + O(||\mathbf{z}||^M\tilde{v}) > 0;$$
$$(\sqrt{B}v - \sqrt{\eta}||\mathbf{z}||^k)^2 > 0.$$

We are left with a larger domain with defining function

$$\tilde{u} + r(\mathbf{z},\overline{\mathbf{z}}) - \eta||\mathbf{z}||^{2k}$$

where $\eta > 0$ is a small.

The local bumping will rely on the leading term in $r(\mathbf{z},\overline{\mathbf{z}})$. Namely, if P_{2k} is plurisubharmonic and not pluriharmonic we can bump the domain. To this end we will use a Proposition.

Proposition 2.1.1 (Bedford, Fornæss, Noell). *Suppose that $P(\mathbf{z})$ is a homogeneous plurisubharmonic polynomial on $\mathbb{C}^{n-1}$, and assume that P is not harmonic on any complex line through 0. Then there exists a function $F(\mathbf{z})$ which is C^∞, homogeneous of degree equal to that of P, positive away from 0 and which satisfies the condition that $P - \varepsilon F$ is strictly plurisubharmonic away from 0 for any* $0 < \varepsilon << 1$.

If the leading term in $r(\mathbf{z},\overline{\mathbf{z}})$ satisfy the requirements of the Proposition, we find F as in the Proposition. Setting $\tilde{P}_{2k} = P_{2k} - \varepsilon F$ we have from the Proposition that $\tilde{P}_{2k}$ is subharmonic. Since F is strictly positive away from 0 and of degree $2k$ we see that there is ab $\varepsilon' > 0$, so that we have $F \geq \varepsilon'||\mathbf{z}||^{2k}$. This leads us to the inequality

$$\tilde{P}_{2k} < P_{2k} - \varepsilon\varepsilon'||\mathbf{z}||^{2k} = P_{2k} - \varepsilon''||\mathbf{z}||^{2k}.$$

where $\varepsilon'' = \varepsilon\varepsilon'$. Setting $R_{2k+1} = \sum_{m=2k+1}^{\infty} P_m(\mathbf{z},\overline{\mathbf{z}})$ we see that

$$\begin{aligned}\tilde{u} + r(\mathbf{z},\overline{\mathbf{z}}) - \eta||\mathbf{z}||^{2k} &= \tilde{u} + P_{2k} + R_{2k+1} - \eta||\mathbf{z}||^{2k} \\ &> \tilde{u} + \tilde{P}_{2k}(\mathbf{z}) + \varepsilon''||\mathbf{z}||^{2k} + R_{2k+1} - \eta||\mathbf{z}||^{2k}.\end{aligned}$$

Now choosing $\eta > 0$ so that $\eta < \varepsilon''$ the inequality $(\varepsilon'' - \eta)||\mathbf{z}||^{2k} + R_{2k+1} > 0$ holds by (possibly) shrinking the neighborhood of 0. Then the local bumping can be given as

$$\Omega_0^* = \{(w,\mathbf{z}) \in \mathbb{C} \times \mathbb{C}^n : \mathrm{Re}\,(w) + \tilde{P}_{2k}(\mathbf{z}) < 0\}. \tag{2.4}$$

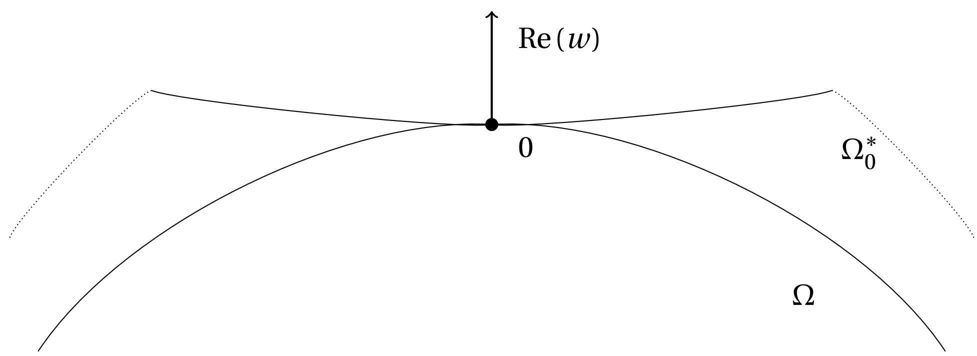

Figure 2.1: Visualization of a 2d slice of the local bumping Ω_0^*

2.2 A Pseudoconvex Extension

To extend the locally bumped domain to a pseudoconvex extension we will intersect our local bumping with a larger pseudoconvex domain coming from the Stein neighborhood basis for Ω.

Definition 2.2.1. Let $D \subset\subset \mathbb{C}^n$ be pseudoconvex, we say that $\overline{D}$ has a Stein neighborhood basis if for any open neighborhood U of $\overline{D}$ there is a pseudoconvex domain D_ε so that $\overline{D} \subset D_\varepsilon \subset U$.

We know Ω has a neighborhood basis from the following theorem.

Theorem 2.2.2. *[4] If D is a pseudoconvex domain with real analytic boundary, then $\overline{D}$ has a Stein neighborhood basis.*

The idea is that the pseudoconvexity from the local bumping needs to be continued when leaving the local area. Therefore we need D_ε coming from the Stein neighborhood basis. For a neighborhood U of 0, the domain $U \cap \Omega_0^*$ is pseudoconvex as we have constructed. Then taking the domain D_ε, we find a smaller neighborhood V of 0 so that $V \subset U$. We see thus that $V \cap D_\varepsilon \subset U \cap \Omega_0^*$. Then the intersection

$$D_\varepsilon \cap \Omega_0^*$$

defines the pseudoconvex domain which will be the extension satisfying pseudoconvexity everywhere on.

The intersection is pseudoconvex by the classic lemma.

Lemma 2.2.3. *If D_1 and D_2 are pseudoconvex domains, then the intersection $D_1 \cap D_2$ is pseudoconvex.*

Proof. A domain is pseudoconvex if and only if $-\log(\text{dist}(p, \partial D))$ is plurisubharmonic. Now $\text{dist}(p, \partial(D_1 \cap D_2)) = \min\{\text{dist}(p, \partial D_1), \text{dist}(p, \partial D_2)\}$ and further

$$-\log(\text{dist}(p, \partial(D_1 \cap D_2)) = \max\left\{-\log(\text{dist}(p, \partial D_1)), -\log(\text{dist}(p, \partial D_2))\right\}$$

which is plurisubharmonic. □

We want to define an intermediate bumped domain Ω_0^{**}. It will be a bumping in the same sense as Ω_0^*, however only “half”. We define this domain by in some sense halving the bumping function from Proposition 2.1.1. More explicitly by considering $2F$, we define $\tilde{\tilde{P}}_{2k} = P_{2k} - 2\varepsilon F$ and further notice that

$$\tilde{\tilde{P}}_{2k} \leq P_{2k} - 2\varepsilon'' ||\mathbf{z}||^{2k}$$

and

$$P_{2k} - 2\varepsilon'' ||\mathbf{z}||^{2k} < P_{2k} - \varepsilon'' ||\mathbf{z}||^{2k}$$

so $\tilde{\tilde{P}}_{2k} < \tilde{P}_{2k}$. Then define

$$\Omega_0^{**} = \{(w, \mathbf{z}) \in \mathbb{C} \times \mathbb{C}^n : \mathrm{Re}\,(w) + \tilde{\tilde{P}}_{2k}(\mathbf{z}) < 0\}.$$

It follows easily that $\Omega_0^{**} \subset \Omega_0^*$.

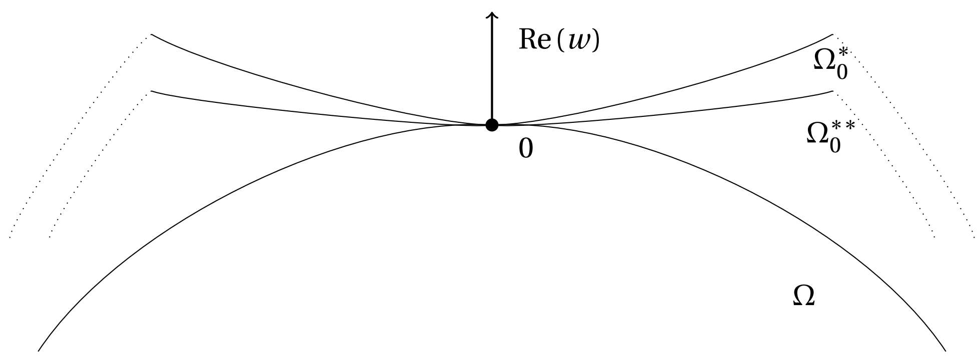

Figure 2.2: Visualization of a 2d slice of the intermediate Ω_0^{**}

So far we have defined a local bumping for finite type domains in $\mathbb{C}^{n+1}$ with $n \geq 1$ with the restriction that the leading polynomial term in $r(\mathbf{z}, \overline{\mathbf{z}})$ is not harmonic along any complex line through 0, but it is plurisubharmonic. For domains in $\mathbb{C}^2$ we can guarantee that this is the case for every bounded domain of finite type. This is because complex curves are easier to control and therefore finding a polynomial which satisfies the requirements in Proposition 2.1.1 is a then simpler ordeal.

In $\mathbb{C}^2$ the polynomial P_{2k} is only a polynomial in one complex variable. By a change of coordinates and using that a harmonic function in one variable is the real part of holomorphic function, we absorb harmonic terms into $\text{Re}(w)$ and get subharmonic, but not harmonic, and homogeneous polynomial of order $2k$.

In the rest of the thesis we will then only consider when $\Omega \subset \mathbb{C}^2$ is a bounded domain with real analytic boundary of finite type $2k$.

Chapter 3

A Solution Operator for $\overline{\partial}$

In this chapter we show how the bumping can be used to a solution operator which satisfies Hölder estimates on a slightly smaller domain.

Throughout this chapter $\Omega \subset \mathbb{C}^2$ will be a bounded pseudoconvex domain with real analytic boundary of finite type $2k$ and Ω_p^* will denote the bumped domain at p.

3.1 Koszul Complex

Given functions $g_1(p,z)$ and $g_2(p,z)$ on Ω_p^* which are smooth in z and solve the Cauchy-Fantappiè equation pointwise for each p, there is a way to modify them so that they become holomorphic *and* still satisfy the Cauchy-Fantappiè equation pointwise. This procedure is done via a Koszul complex and is what we will use this section to show.

Lemma 3.1.1. *For a fixed p and smooth solutions g_1, g_2 which satisfy*

$$g_1(p,z)(p_1 - z_1) + g_2(p,z)(p_2 - z_2) \equiv 1,$$

then there exists functions h_1, h_2 holomorphic in z which satisfy

$$h_1(p,z)(p_1 - z_1) + h_2(p_2 - z_2) \equiv 1.$$

Proof. Since

$$g_1(p,z)(p_1 - z_1) + g_2(p,z)(p_2 - z_2) \equiv 1,$$

we see that

$$\bar{\partial}\big(g_1(p,z)(p_1 - z_1) + g_2(p,z)(p_2 - z_2)\big) = 0.$$

By multiplying in 1

$$\overline{\partial}g_j \cdot 1 = \overline{\partial}g_j(g_1(p,z)(p_1 - z_1) + g_2(p,z)(p_2 - z_2))$$

which yields

$$\begin{aligned}\overline{\partial}g_1 \cdot 1 &= \overline{\partial}g_1\big(g_1(p,z)(p_1 - z_1) + g_2(p,z)(p_2 - z_2)\big),\\ \overline{\partial}g_2 \cdot 1 &= \overline{\partial}g_2\big(g_1(p,z)(p_1 - z_1) + g_2(p,z)(p_2 - z_2)\big).\end{aligned}$$

By using that $\overline{\partial}g_1(p_1 - z_1) = -\overline{\partial}g_2(p_2 - z_2)$ we rewrite

$$\begin{aligned}\overline{\partial}g_1 &= -g_1\overline{\partial}g_2(p_2 - z_2) + \overline{\partial}g_1 g_2(p_2 - z_2) = (g_2\overline{\partial}g_1 - g_1\overline{\partial}g_2)\cdot(p_2 - z_1),\\ \overline{\partial}g_2 &= \overline{\partial}g_2 g_1(p_1 - z_1) - g_2\overline{\partial}g_1(p_1 - z_1) = (g_1\overline{\partial}g_2 - g_2\overline{\partial}g_1)\cdot(p_1 - z_1).\end{aligned}$$

Now define a (0,1)-form

$$\omega := g_2\overline{\partial}g_1 - g_1\overline{\partial}g_2 \tag{3.1}$$

and notice that $\overline{\partial}\omega = 0$ so ω is a $\overline{\partial}$-closed form. Now, by using Theorem 1.1.5, we find a solution u satisfying $\overline{\partial}u = \omega$ lying in weighted L^2 space. Define

$$h_1 := g_1 - u(p_2 - z_2) \tag{3.2}$$

$$h_2 := g_2 + u(p_1 - z_1). \tag{3.3}$$

These functions satisfy the Cauchy-Fantappié equation

$$\begin{aligned}h_1(p_1 - z_1) + h_2(p_2 - z_2) &= g_1(p_1 - z_1) - u(p_2 - z_2)(p_1 - z_1) + g_2(p_2 - z_2) + u(p_1 - z_1)(p_2 - z_2)\\ &= g_1(p_1 - z_1) + g_2(p_2 - z_2) \equiv 1\end{aligned}$$

and by construction $\overline{\partial}h_j = 0$ for $j = 1,2$. □

3.2 Constructing a Support Function

Now we need to choose the smooth functions to modify via the Koszul complex. These function will chosen related to the bumped domain Ω_p^*. The function will reflect the type at 0 and the tangential direction. In a similar manner to the construction of the Henkin integral kernel, we seek local functions G_1, G_2 and Φ which will solve the division problem

$$G_1 + G_2 \equiv \Phi. \tag{3.4}$$

First we will construct $\Phi(w,z)$. We will require that it is holomorphic in the first variable and smooth in the second variable. We also require that it also satisfies

(1) $\Phi = (p_1 - z_1) - A \cdot F(p_2 - z_2)$ where $A > 0$;

(2) $F > 0$ away from p;

(3) $\{\Phi = 0\} \cap \Omega_p^* \setminus \{0\} = \emptyset$;

(4) $|\Phi| \sim \text{dist}(\cdot, \partial\Omega_p^*)$ on $\partial\Omega \setminus \{p\}$.

Indeed we can find such a function Φ on the bumping Ω_p^*. For ease of notation we set $p = 0$. For a constant $A > 0$ so large so that $A|z|^{2k} + \tilde{P}_{2k} \geq 0$ we define

$$\Phi(w,z) := w - A|z|^{2k}. \tag{3.5}$$

The first two criteria follow directly from how Φ is defined. When $\Phi = 0$ we then have that $w = A|z|^{2k}$ implying that w is real. It follows then that $\text{Re}\, w + \tilde{P}_{2k} = A|z|^{2k} + \tilde{P}_{2k} \geq 0$, so we get $\{\Phi = 0\} \cap \Omega_0^* \setminus \{0\} = \emptyset$.

On the boundary $\partial\Omega_0^*$ the defining function for Ω_0^* is 0. That is, $\text{Re}\, w + \tilde{P}_{2k} = 0$, meaning that near the boundary $|\text{Re}\, w| \sim |z|^{2k}$. Noting also that near 0 we have that the defining function ρ is dominated by $\operatorname{Re} w$ being the only first order term, in particular then $|\operatorname{Re} w| \geq |\operatorname{Im} w|$ near $\partial\Omega$. Therefore

$$|w| \leq |\operatorname{Re} w| + |\operatorname{Im} w| \leq 2|\operatorname{Re} w| \sim |z|^{2k}$$

and $|z|^{2k} \sim |\operatorname{Re} w| \leq |w|$ eventually showing that $|w| \sim |z|^{2k}$.

Similarly we can see that $\operatorname{dist}(\cdot, \partial\Omega_0^*) \sim |\operatorname{Re} w + \tilde{P}_{2k}| \sim |w|$. Using this we can then see that

$$\operatorname{dist}(\cdot, \partial\Omega_0^*) \sim \frac{1}{2}|w| + \frac{1}{2}|w| \sim |\Phi|.$$

Having constructed the supporting function Φ we now chose the accompanying functions G_1 and G_2 which will solve the division problems. Let

$$G_1 = 1,$$
$$G_2 = -Az^{k-1}\bar{z}^k.$$

These functions will, together with Φ, satisfy the division problem in (3.4). We see this by noting that

$$\frac{w}{\Phi} + \frac{-zAz^{k-1}\bar{z}^k}{\Phi} = \frac{w - A|z|^{2k}}{\Phi} \equiv 1.$$

We then want to apply the Koszul complex procedure in Lemma 3.1.1 to get holomorphic solutions. The (0,1)-form $\omega = g_2\bar{\partial}g_1 - g_1\bar{\partial}g_2$ from (3.1) is then

$$\omega = \frac{\bar{\partial}G_2}{\Phi^2} = \frac{-Ak|z|^{2k-2}}{(w - A|z|^{2k})^2}.$$

Finding the u which satisfy $\bar{\partial}u = \omega$ on Ω_0^* using Theorem 1.1.5 we get that the holomorphic functions relating to our smooth ones are given by

$$h_1 = \frac{1}{\Phi} + uz$$
$$h_2 = \frac{-Az^{k-1}\overline{z}^k}{\Phi} - uw.$$

as in equation (3.3).

3.3 Weighted L^2 Estimates

This section will be used to prove some lemmas which will allow us to pass from L^2 to pointwise estimates for the kernel. To do that what we need to find are estimates for the function u which is the solution of $\overline{\partial}u = \omega$ from (3.1) when using Theorem 1.1.5. We want to calculate an integral of the form

$$\int_D \langle M^{-1}v, v\rangle e^{-\psi}.$$

The matrix M appearing in the integral depends on ψ and the degree of the from v. In our case we have a (0,1)-form, meaning that M is only the Levi matrix of ψ. The choice of ψ needs to be carefully considered. An apt choice of weight should be such that the matrix M is simple to compute and gives an integrable integral.

In the case of our (0,1)-form ω we would ideally see that

$$\int_{\Omega_0^*} |\omega|^2 dw \wedge d\overline{w} \wedge dz \wedge d\overline{z} < \infty,$$

and thus have estimations for u. Inserting the expression for $|\omega|^2$ we see

$$\int_{\Omega_0^*} \frac{|z|^{4k-4}}{|w - A|z|^{2k}|^4} dw \wedge d\overline{w} \wedge dz \wedge d\overline{z}$$
$$= \int_{\Omega_0^*} \frac{|z|^{4k-4}}{(w - A|z|^{2k})^2(\overline{w} - A|z|^{2k})^2} dw \wedge d\overline{w} \wedge dz \wedge d\overline{z}.$$

Integrating over w and $\overline{w}$ gives essentially

$$\frac{|z|^{4k-4}}{(w-A|z|^{2k})(\overline{w}-A|z|^{2k})}dz\wedge d\overline{z} = \frac{|z|^{4k-4}}{|w-A|z|^{2k}|^2}dz\wedge d\overline{z}. \tag{3.6}$$

We now notice that if we gain $|z|^2$ this can be integrated and seen finite. This is then our aim when choosing the weight ψ.

The choice of ψ will rely on the classic plurisubharmonic function on pseudoconvex domains $-\log(\text{dist}(\cdot,\partial\Omega_0^*))$ and in the same manner as Φ be chosen related to how the bumped domain Ω_0^* looks like. We would like the weight to reflect the type at 0. Let $\varepsilon > 0$ and define the function $\rho = \log(|w|+|w|^{1+\varepsilon}+|z|^{2k})$. To avoid a vanishing Levi matrix we have introduced a term $|w|^{1+\varepsilon}$. The function ρ is plurisubharmonic and together with the $-\log(\text{dist}(\cdot,\partial\Omega_0^*))$ will form a suitable weight.

We integrate over the domain Ω_0^{**}, which is a bumping in the same sense as Ω_0^*, just "half" of the original bumping, but still having the $-\log(\text{dist}(\cdot,\partial\Omega_0^*)$ term in the weight. This is weight is only singular in 0 and not on the boundary $\partial\Omega_0^{**}$.

Lemma 3.3.1. *Let $(w,z)\in\Omega_0^{**}$ and for $\varepsilon>0$ and $\delta>0$ set*

$$\begin{aligned}\psi &= -(\varepsilon+\delta)\log(dist(\cdot,\partial\Omega_0^*))+\delta\rho\\ &= -(\varepsilon+\delta)\log(dist(\cdot,\partial\Omega_0^*))+\delta\log(|w|+|w|^{1+\varepsilon}+|z|^{2k}),\end{aligned}$$

then ψ is plurisubharmonic and the integral

$$\int_{\Omega_0^{**}}|u|^2e^{-\psi}dw\wedge d\overline{w}\wedge dz\wedge d\overline{z}$$

is finite.

To prove this Lemma we want to apply Theorem 1.1.5. This requires us to compute the Levi-matrix of ψ, whose expression is not easily computed. Seeing as we are going to get Hermitian matrices we can therefore simplify the computation by applying a result concerning such matrices.

Lemma 3.3.2. *Let T, L be complex valued Hermitian matrices with T positive semi definite and L positive definite. Then*

$$v^H (L+T)^{-1} v \leq v^H L^{-1} v. \tag{3.7}$$

Proof. Let L be a positive definite matrix, and T be a positive semi definite matrix. Writing the Cholsesky decomposition for $L^{-1} = UU^H$ we can write

$$v^H (L+T)^{-1} v = v^H (UU^H + T)^{-1} v = v^H U(I + U^H TU)^{-1} U^H v$$

since inverses of positive definite matrices are positive definite. Set $u = U^H v$. Now we notice that

$$\begin{aligned} u^H \big(I + U^H TU\big)^{-1} u &= \big(u^{-H}\big(I + U^H TU\big)u^{-1}\big)^{-1} \\ &= \frac{1}{(u^{-H} u^{-1} + u^{-H} U^H TU u^{-1})} \\ &\leq \big(u^{-1} u^{-H}\big)^{-1} = u^H u \end{aligned}$$

since

$$u^{-H} u^{-1} + u^{-1} U^H TU u^{-H} \geq u^{-H} u^{-1}$$

which leads to the result. □

Proof of Lemma 3.3.1. Let L be the Levi-matrix of ρ, and T be the Levi-matrix of ψ, then T is the matrix that is used in Theorem 1.1.5. The term $\delta\rho$ is plurisubharmonic. To see this we create a locally holomorphic function $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^3$ defined by

$$f(w, z) = (w^{\frac{1}{2}}, w^{\frac{1+\varepsilon}{2}}, z^k).$$

Then we can write $\rho = \log(||f||)$ which is a plurisubharmonic function.

When the functions are plurisubharmonic (strictly plurisubharmonic), the Levi matrices are positive semi definite (positive definite). Then we can apply Lemma 3.3.2 and we get the inequality

$$\int_{\Omega_0^{**}} \langle T^{-1}\omega, \omega\rangle e^{-\Psi} \leq \int_{\Omega_0^{**}} \langle L^{-1}\omega, \omega\rangle e^{-\Psi}.$$

We want to find an integrable expression which estimates $\langle L^{-1}\omega, \omega\rangle e^{-\Psi}$. As mentioned, L is the Levi-matrix corresponding to the function ρ, that is

$$L = \begin{bmatrix} \rho_{w\overline{w}} & \rho_{w\overline{z}} \\ \rho_{z\overline{w}} & \rho_{z\overline{z}} \end{bmatrix}.$$

To compute the inverse use the inverse formula for 2×2 matrices, giving

$$L^{-1} = \frac{1}{\det L} \begin{bmatrix} \rho_{z\overline{z}} & -\rho_{w\overline{z}} \\ -\rho_{z\overline{w}} & \rho_{w\overline{w}} \end{bmatrix}.$$

Being that ω is a (0,1)-form, the expression $L^{-1}\omega$ will be the form

$$\frac{1}{\det L}(-\rho_{w\overline{z}} + \rho_{w\overline{w}})\omega = \frac{(-\rho_{w\overline{z}} + \rho_{w\overline{w}})\omega}{\rho_{z\overline{z}}\rho_{w\overline{w}} - |\rho_{z\overline{w}}|^2}$$

and by Cauchy-Schwarz inequality we have an estimate

$$\langle L^{-1}\omega, \omega\rangle e^{-\Psi} \leq \Big(\frac{|\rho_{z\overline{w}}|}{|\rho_{z\overline{z}}\rho_{w\overline{w}} - |\rho_{z\overline{w}}|^2|} + \frac{|\rho_{w\overline{w}}|}{|\rho_{z\overline{z}}\rho_{w\overline{w}} - |\rho_{z\overline{w}}|^2|}\Big)|\omega|^2 e^{-\Psi}. \tag{3.8}$$

After some somewhat lengthy computations we can get a local estimate for this expression given by $|z|^{4k-2}/|\Phi|^{4-\varepsilon}$ ignoring constants. Roughly speaking integrating with respect to $dw \wedge d\overline{w}$ reduces the exponent in the denominator by 2 and we are left with $|z|^{4k-2}/|\Phi|^{2-\varepsilon}$ which will yield a finite integral. The details of the computations are now what follows.

To compute L^{-1}, we for ease of notation set $Q = |w| + |w|^{1+\varepsilon} + |z|^{2k}$. Then

$$\frac{\partial Q}{\partial z} = kz^{k-1}\overline{z}^{k},$$
$$\frac{\partial Q}{\partial w} = \frac{1}{2}w^{-\frac{1}{2}}\overline{w}^{\frac{1}{2}} + (\frac{1+\varepsilon}{2})w^{\frac{1+\varepsilon}{2}-1}\overline{w}^{\frac{1+\varepsilon}{2}}.$$

and the partials for ρ

$$\rho_z = \frac{1}{Q}kz^{k-1}\overline{z}^{k},$$
$$\rho_w = \frac{1}{Q}\frac{1}{2}w^{-\frac{1}{2}}\overline{w}^{\frac{1}{2}} + (\frac{1+\varepsilon}{2})w^{\frac{1+\varepsilon}{2}-1}\overline{w}^{\frac{1+\varepsilon}{2}}$$

We then see that

$$\begin{aligned}\rho_{w\overline{w}} &= \frac{1}{Q^2}\Big[\big(\frac{1}{4|w|} + (\frac{1+\varepsilon}{2})^2 w^{-\frac{1-\varepsilon}{2}}\overline{w}^{-\frac{1-\varepsilon}{2}}\big)Q \\ &\quad - \big(\frac{w^{-\frac{1}{2}}\overline{w}^{\frac{1}{2}}}{2} + (\frac{1+\varepsilon}{2})w^{\frac{1+\varepsilon}{2}-1}\overline{w}^{\frac{1+\varepsilon}{2}}\big)\big(\frac{\overline{w}^{-\frac{1}{2}}w^{\frac{1}{2}}}{2} + (\frac{1+\varepsilon}{2})\overline{w}^{\frac{1+\varepsilon}{2}-1}w^{\frac{1+\varepsilon}{2}}\big)\Big] \\ &= \frac{1}{Q^2}\Big[\big(\frac{1}{4|w|} + (\frac{1+\varepsilon}{2})^2|w|^{\varepsilon-1}\big)Q \\ &\quad - \big(\frac{w^{-\frac{1}{2}}\overline{w}^{\frac{1}{2}}}{2} + (\frac{1+\varepsilon}{2})w^{\frac{1+\varepsilon}{2}-1}\overline{w}^{\frac{1+\varepsilon}{2}}\big)\overline{\big(\frac{w^{-\frac{1}{2}}\overline{w}^{\frac{1}{2}}}{2} + (\frac{1+\varepsilon}{2})w^{\frac{1+\varepsilon}{2}-1}\overline{w}^{\frac{1+\varepsilon}{2}}\big)}\Big].\end{aligned}$$

Using that $(z+w)\overline{(z+w)} = |z|^2 + |w|^2 + 2\mathrm{Re}\,(zw)$ can rewrite

$$\begin{aligned}&= \frac{1}{Q^2}\Big[\big(\frac{1}{4|w|} + \big(\frac{1+\varepsilon}{2}\big)^2|w|^{\varepsilon-1}\big)\big(|w| + |w|^{1+\varepsilon} + |z|^{2k}\big) \\ &\quad - \frac{1}{4} - (\frac{1+\varepsilon}{2})^2|w|^{2\varepsilon} - 2\mathrm{Re}\big(\frac{1}{2}w^{\frac{-1}{2}}\overline{w}^{\frac{1}{2}}(\frac{1+\varepsilon}{2})w^{\frac{\varepsilon+1}{2}}\overline{w}^{\frac{\varepsilon-1}{2}}\big)\Big] \\ &= \frac{1}{Q^2}\Big[\frac{1}{4} + \frac{|w|^{\varepsilon}}{4} + \frac{|z|^{2k}}{4|w|} + \big(\frac{1+\varepsilon}{2}\big)^2|w|^{\varepsilon} + \big(\frac{1+\varepsilon}{2}\big)^2|w|^{2\varepsilon} + \big(\frac{1+\varepsilon}{2}\big)^2|w|^{\varepsilon-1}|z|^{2k} \\ &\quad - \frac{1}{4} - (\frac{1+\varepsilon}{2})^2|w|^{2\varepsilon} - 2\mathrm{Re}\big(\frac{1}{2}w^{\frac{-1}{2}}\overline{w}^{\frac{1}{2}}(\frac{1+\varepsilon}{2})w^{\frac{\varepsilon+1}{2}}\overline{w}^{\frac{\varepsilon-1}{2}}\big)\Big]\end{aligned}$$

Notice that

$$2\mathrm{Re}\big(\frac{1}{2}w^{\frac{-1}{2}}\overline{w}^{\frac{1}{2}}(\frac{1+\varepsilon}{2})w^{\frac{\varepsilon+1}{2}}\overline{w}^{\frac{\varepsilon-1}{2}}\big) = \big(\frac{1+\varepsilon}{2}\big)|w|^{\varepsilon}$$

so we can further rewrite

$$\begin{aligned}
&\frac{1}{Q^2}\Big[\frac{1}{4}+\frac{|w|^\varepsilon}{4}+\frac{|z|^{2k}}{4|w|}+\big(\frac{1+\varepsilon}{2}\big)^2|w|^\varepsilon+\big(\frac{1+\varepsilon}{2}\big)^2|w|^{2\varepsilon}+\big(\frac{1+\varepsilon}{2}\big)^2|w|^{\varepsilon-1}|z|^{2k}\\
&\qquad -\frac{1}{4}-(\frac{1+\varepsilon}{2})^2|w|^{2\varepsilon}-\big(\frac{1+\varepsilon}{2}\big)|w|^\varepsilon\Big]\\
&=\frac{1}{Q^2}\Big[\frac{|w|^\varepsilon}{4}+\frac{|z|^{2k}}{4|w|}+\big(\frac{1+\varepsilon}{2}\big)^2|w|^\varepsilon+\big(\frac{1+\varepsilon}{2}\big)^2|w|^{\varepsilon-1}|z|^{2k}-\big(\frac{1+\varepsilon}{2}\big)|w|^\varepsilon\Big]
\end{aligned}$$

by grouping together terms so that we can use

$$\Big((\frac{1+\varepsilon}{2})^2-\big(\frac{1+\varepsilon}{2}\big)+\frac{1}{4}\Big)=\frac{\varepsilon^2}{4} \tag{3.9}$$

we can see that

$$\begin{aligned}
&=\frac{1}{Q^2}\Big[\frac{|z|^{2k}}{4|w|}+\big(\frac{1+\varepsilon}{2}\big)^2|w|^{\varepsilon-1}|z|^{2k}+\big(\frac{1+\varepsilon}{2}\big)^2|w|^\varepsilon-\big(\frac{1+\varepsilon}{2}\big)|w|^\varepsilon+\frac{|w|^\varepsilon}{4}\Big]\\
&=\frac{1}{Q^2}\Big[\frac{|z|^{2k}}{4|w|}+\big(\frac{1+\varepsilon}{2}\big)^2|w|^{\varepsilon-1}|z|^{2k}+|w|^\varepsilon\Big(\big(\frac{1+\varepsilon}{2}\big)^2-\big(\frac{1+\varepsilon}{2}\big)+\frac{1}{4}\Big)\Big]\\
&=\frac{1}{Q^2}\Big[\frac{|z|^{2k}}{4|w|}+\big(\frac{1+\varepsilon}{2}\big)^2|w|^{\varepsilon-1}|z|^{2k}+\frac{\varepsilon^2}{4}|w|^\varepsilon\Big]
\end{aligned} \tag{3.10}$$

For the partials with respect to the z-variable we get

$$\begin{aligned}
\rho_{z\overline{z}}&=\frac{k^2|z|^{2k-2}(|w|+|w|^{1+\varepsilon}+|z|^{2k})-k^2|z|^{2k-2}|z|^{2k}}{(|w|+|w|^{1+\varepsilon}+|z|^{2k})^2}\\
&=k^2|z|^{2k-2}\frac{(|w|+|w|^{1+\varepsilon})}{(|w|+|w|^{1+\varepsilon}+|z|^{2k})^2}
\end{aligned} \tag{3.11}$$

Finally we compute the mixed terms

$$\rho_{z\overline{w}}=\frac{kz^{k-1}\overline{z}^k}{Q^2}\big(\frac{w^{\frac{1}{2}}\overline{w}^{-\frac{1}{2}}}{2}+(\frac{1+\varepsilon}{2})w^{\frac{1+\varepsilon}{2}}\overline{w}^{\frac{\varepsilon-1}{2}}\big) \tag{3.12}$$

$$\rho_{w\overline{z}}=\frac{kz^k\overline{z}^{k-1}}{Q^2}\big(\frac{w^{-\frac{1}{2}}\overline{w}^{\frac{1}{2}}}{2}+(\frac{1+\varepsilon}{2})w^{\frac{\varepsilon-1}{2}}\overline{w}^{\frac{1+\varepsilon}{2}}\big). \tag{3.13}$$

Then we have all the terms which appear in the Levi matrix. Now for computing the determinant we see that

$$\begin{aligned}
\rho_{z\overline{z}}\rho_{w\overline{w}} &= \frac{k^2|z|^{2k-2}}{Q^4}\Big(|w|+|w|^{1+\varepsilon}\Big)\Big[\frac{|z|^{2k}}{4|w|}+\big(\frac{1+\varepsilon}{2}\big)^2|w|^{\varepsilon-1}|z|^{2k}+\frac{\varepsilon^2}{4}|w|^{\varepsilon}\Big] \\
&= \frac{k^2|z|^{2k-2}}{Q^4}\Big[\frac{|z|^{2k}}{4}+\big(\frac{1+\varepsilon}{2}\big)^2|w|^{\varepsilon}|z|^{2k}+\frac{\varepsilon^2}{4}|w|^{1+\varepsilon} \\
&\qquad +\frac{|z|^{2k}|w|^{\varepsilon}}{4}+\big(\frac{1+\varepsilon}{2}\big)^2|w|^{2\varepsilon}|z|^{2k}+\frac{\varepsilon^2}{4}|w|^{1+2\varepsilon}\Big]
\end{aligned}$$

and that

$$\begin{aligned}
|\rho_{z\overline{w}}|^2 &= \Big|\frac{kz^{k-1}\overline{z}^k\big(\frac{\overline{w}^{-\frac{1}{2}}w^{\frac{1}{2}}}{2}+(\frac{1+\varepsilon}{2})\overline{w}^{\frac{1+\varepsilon}{2}-1}w^{\frac{1+\varepsilon}{2}}\big)}{(|w|+|w|^{1+\varepsilon}+|z|^{2k})^2}\Big|^2 \\
&= \frac{k^2|z|^{2k-2}}{Q^4}\Big[\frac{|z|^{2k}}{4}+\big(\frac{1+\varepsilon}{2}\big)^2|w|^{2\varepsilon}|z|^{2k}+\big(\frac{1+\varepsilon}{2}\big)|w|^{\varepsilon}|z|^{2k}\Big]
\end{aligned}$$

which leads to an expression for $\rho_{z\overline{z}}\rho_{w\overline{w}}-|\rho_{z\overline{w}}|^2$ given by

$$\begin{aligned}
&\frac{k^2|z|^{2k-2}}{Q^4}\Big[\frac{|z|^{2k}}{4}+\big(\frac{1+\varepsilon}{2}\big)^2|w|^{\varepsilon}|z|^{2k}+\frac{\varepsilon^2}{4}|w|^{1+\varepsilon}+\frac{|z|^{2k}|w|^{\varepsilon}}{4}+\big(\frac{1+\varepsilon}{2}\big)^2|w|^{2\varepsilon}|z|^{2k}+\frac{\varepsilon^2}{4}|w|^{1+2\varepsilon} \\
&\qquad -\frac{|z|^{2k}}{4}-(\frac{1+\varepsilon}{2})^2|w|^{2\varepsilon}|z|^{2k}-\big(\frac{1+\varepsilon}{2}\big)|w|^{\varepsilon}|z|^{2k}\Big] \\
&= \frac{k^2|z|^{2k-2}}{Q^4}\Big[\big(\frac{1+\varepsilon}{2}\big)^2|w|^{\varepsilon}|z|^{2k}+\frac{\varepsilon^2}{4}|w|^{1+\varepsilon}+\frac{|z|^{2k}|w|^{\varepsilon}}{4}+\frac{\varepsilon^2}{4}|w|^{1+2\varepsilon}-\big(\frac{1+\varepsilon}{2}\big)|w|^{\varepsilon}|z|^{2k}\Big]
\end{aligned}$$

Now by grouping the terms containing $|w|^{\varepsilon}|z|^{2k}$ and again using (3.9)

$$\frac{k^2|z|^{2k-2}}{Q^4}\Big[\big(\frac{1+\varepsilon}{2}\big)^2|w|^{\varepsilon}|z|^{2k}+\frac{\varepsilon^2}{4}|w|^{1+\varepsilon}+\frac{|z|^{2k}|w|^{\varepsilon}}{4}+\frac{\varepsilon^2}{4}|w|^{1+2\varepsilon}-\big(\frac{1+\varepsilon}{2}\big)|w|^{\varepsilon}|z|^{2k}\Big]$$
$$=\frac{k^2|z|^{2k-2}}{Q^4}\Big[\big(\frac{1+\varepsilon}{2}\big)^2|w|^{\varepsilon}|z|^{2k}-\big(\frac{1+\varepsilon}{2}\big)|w|^{\varepsilon}|z|^{2k}+|z|^{2k}|w|^{\varepsilon}\frac{1}{4}+\frac{\varepsilon^2}{4}|w|^{1+\varepsilon}+\frac{\varepsilon^2}{4}|w|^{1+2\varepsilon}\Big]$$
$$=\frac{k^2|z|^{2k-2}}{Q^4}\Big[|w|^{\varepsilon}|z|^{2k}\Big(\big(\frac{1+\varepsilon}{2}\big)^2-\big(\frac{1+\varepsilon}{2}\big)+\frac{1}{4}\Big)+\frac{\varepsilon^2}{4}|w|^{1+\varepsilon}+\frac{\varepsilon^2}{4}|w|^{1+2\varepsilon}\Big]$$
$$=\frac{k^2|z|^{2k-2}}{Q^4}\Big[\frac{\varepsilon^2}{4}|w|^{\varepsilon}|z|^{2k}+\frac{\varepsilon^2}{4}|w|^{1+\varepsilon}+\frac{\varepsilon^2}{4}|w|^{1+2\varepsilon}\Big]$$
$$=\frac{k^2|z|^{2k-2}}{Q^4}\cdot\frac{\varepsilon^2}{4}\Big[|w|^{\varepsilon}|z|^{2k}+|w|^{1+\varepsilon}+|w|^{1+2\varepsilon}\Big]$$

Now that we have computed all the entries in the matrix L we can express the reciprocal of the determinant as

$$\frac{1}{\det L}=\frac{4}{\varepsilon^2}\cdot\frac{Q^4}{k^2|z|^{2k-2}}\Big(|w|^{1+\varepsilon}+|w|^{1+2\varepsilon}+|z|^{2k}|w|^{\varepsilon}\Big)^{-1}.$$

Note also that

$$|w|^{1+\varepsilon}+|w|^{1+2\varepsilon}+|z|^{2k}|w|^{\varepsilon}=|w|^{\varepsilon}(|w|^{1+\varepsilon}+|z|^{2k}+|w|)$$

so we can further rewrite

$$\frac{1}{\det L}=\frac{4}{\varepsilon^2}\frac{Q^3}{k^2|z|^{2k-2}|w|^{\varepsilon}}.$$

We need to consider the expression

$$\frac{|\rho_{w\overline{w}}|}{|\rho_{z\overline{z}}\rho_{w\overline{w}}-|\rho_{z\overline{w}}|^2|}$$

which after inserting (3.10) turns out to be

$$= \frac{Q}{k^2|z|^{2k-2}} \frac{1}{\varepsilon^2/4} \frac{1}{|w|^{\varepsilon}} \Big[\frac{|z|^{2k}}{4|w|} + \big(\frac{1+\varepsilon}{2}\big)^2 |w|^{\varepsilon-1}|z|^{2k} + \frac{\varepsilon^2}{4}|w|^{\varepsilon} \Big].$$

Rewriting the expression some

$$\begin{aligned}
&\frac{Q}{k^2|z|^{2k-2}} \frac{1}{\varepsilon^2/4} \frac{1}{|w|^{\varepsilon}} \Big[\frac{|z|^{2k}}{4|w|} + \big(\frac{1+\varepsilon}{2}\big)^2 |w|^{\varepsilon-1}|z|^{2k} + \frac{\varepsilon^2}{4}|w|^{\varepsilon} \Big] \\
&= Q\frac{1}{|w|^{\varepsilon}}\big[\frac{|w|^{\varepsilon}}{k^2|z|^{2k-2}} + \frac{1}{\varepsilon^2/4}\frac{1}{k^2}\frac{|z|^2}{4|w|} + (1+\varepsilon)^2\frac{1}{k^2\varepsilon^2}|w|^{\varepsilon-1}|z|^2\big] \\
&= Q\big[\frac{1}{k^2|z|^{2k-2}} + \frac{1}{\varepsilon^2}\frac{1}{k^2}\frac{|z|^2}{|w|^{1+\varepsilon}} + (1+\varepsilon)^2\frac{1}{k^2\varepsilon^2}\frac{|z|^2}{|w|}\big]
\end{aligned}$$

and by using that $\frac{1}{\varepsilon^2} > 1$ and $(1+\varepsilon)^2 > 1$ we get that

$$\begin{aligned}
&Qe^{-\psi}\big[\frac{1}{k^2|z|^{2k-2}} + \frac{1}{\varepsilon^2}\frac{1}{k^2}\frac{|z|^2}{|w|^{1+\varepsilon}} + (1+\varepsilon)^2\frac{1}{k^2\varepsilon^2}\frac{|z|^2}{|w|}\big] \\
&< Qe^{-\psi}\big[\frac{(1+\varepsilon)^2}{\varepsilon^2}\frac{1}{k^2|z|^{2k-2}} + \frac{(1+\varepsilon)^2}{\varepsilon^2}\frac{1}{k^2}\frac{|z|^2}{|w|^{1+\varepsilon}} + \frac{(1+\varepsilon)^2}{\varepsilon^2}\frac{1}{k^2\varepsilon^2}\frac{|z|^2}{|w|}\big] \\
&= Qe^{-\psi}\frac{(1+\varepsilon)^2}{\varepsilon^2}\frac{1}{k^2}\Big[\frac{|z|^2}{|z|^{2k}} + \frac{|z|^2}{|w|^{1+\varepsilon}} + \frac{|z|^2}{|w|}\Big]
\end{aligned}$$

Recall that in the local bumping Ω_0^* we have that $|w| \sim |z|^{2k}$. So we see that

$$\begin{aligned}
\frac{|w| + |w|^{1+\varepsilon} + |z|^{2k}}{|z|^{2k}} &\sim 1 + |w|^{\varepsilon}, \\
\frac{|w| + |w|^{1+\varepsilon} + |z|^{2k}}{|w|} &\sim 1 + |w|^{\varepsilon}, \\
\frac{|w| + |w|^{1+\varepsilon} + |z|^{2k}}{|w|^{1+\varepsilon}} &\sim 1 + \frac{1}{|w|^{\varepsilon}}.
\end{aligned}$$

Recall that in the local bumping we had that $|w| \sim |z|^{2k}$. Therefore we bound locally

$$\frac{|\rho_{w\overline{w}}|}{|\rho_{z\overline{z}}\rho_{w\overline{w}} - |\rho_{z\overline{w}}|^2|} < \frac{|z|^2}{k^2}\frac{(1+\varepsilon)^2}{\varepsilon^2}\Big[1 + |w|^{\varepsilon} + \frac{1}{|w|^{\varepsilon}}\Big].$$

The term

$$\frac{|\rho_{z\overline{w}}|}{|\rho_{z\overline{z}}\rho_{w\overline{w}} - |\rho_{z\overline{w}}|^2|}$$

we can also bound in the local bumping

$$\begin{aligned}
\frac{2Q}{k\varepsilon^2}\frac{|z|}{|w|^\varepsilon}(1+(1+\varepsilon)|w|^\varepsilon) &= \frac{2}{k\varepsilon^2}\frac{|w||z|+|w|^{1+\varepsilon}|z|+|z|^{2k+1}}{|w|^\varepsilon}(1+(1+\varepsilon)|w|^\varepsilon) \\
&\sim |z|^{2k+1}\frac{1+|w|^\varepsilon}{|w|^\varepsilon}(1+|w|^\varepsilon) \\
&= |z|^{2k+1}(2+\frac{1}{|w|^\varepsilon}+|w|^\varepsilon) \\
&\leq |z|^2(2+\frac{1}{|w|^\varepsilon}+|w|^\varepsilon).
\end{aligned}$$

All these computations then lead to the final local estimate of (3.8)

$$\langle L^{-1}\omega,\omega\rangle e^{-\psi} \lesssim \Big((1+\frac{1}{|w|^\varepsilon}+|w|^\varepsilon)+\frac{1}{k^2}\frac{(1+\varepsilon)^2}{\varepsilon^2}\Big[1+|w|^\varepsilon+\frac{1}{|w|^\varepsilon}\Big]\Big)|z|^2 e^{-\psi}|\omega|^2.$$

We see here that we have "gained" $|z|^2$ to the integrand which makes (3.6) integrable. The weight in the integral can be handle by noting that $Q = |w|+|w|^{1+\varepsilon}+|z|^{2k} \geq |w|+|z|^{2k} \gtrsim |\Phi|$ and that $\operatorname{dist}(\cdot,\partial\Omega_0^*) \sim |\Phi|$ in Ω_0^*, so we see that

$$\begin{aligned}
e^{-\psi} &= exp[(\varepsilon+\delta)\log(\operatorname{dist}(\cdot,\partial\Omega_0^*)) - \delta\rho] \\
&= (\operatorname{dist}(\cdot,\partial\Omega_0^*))^{\varepsilon+\delta}Q^{-\delta} \\
&\lesssim |\Phi|^\varepsilon
\end{aligned}$$

The estimates we have achieved are all local, however this is of no problem to us because away from a neighborhood of 0 the integral is bounded, so we need only consider the integral in a small ball around 0. Let therefore $B_R(0)$ be a ball around 0 of radius $R << 1$. The integrals we consider are

$$\int_{\Omega_0^{**}\cap B_R(0)} \frac{1}{k^2}\frac{(1+\varepsilon)^2}{\varepsilon^2}\Big[1+|w|^\varepsilon+\frac{1}{|w|^\varepsilon}\Big]\frac{|z|^{4k-2}}{|w-A|z|^{2k}|^{4-\varepsilon}}dw\wedge d\overline{w}\wedge dz\wedge d\overline{z}$$

and

$$\int_{\Omega_0^{**}\cap B_R(0)} |z|^2(1+\frac{1}{|w|^\varepsilon}+|w|^\varepsilon)\frac{|z|^{4k-4}}{|w-A|z|^{2k}|^{4-\varepsilon}}dw\wedge d\overline{w}\wedge dz\wedge d\overline{z}$$

after inserting for ω. We know that $|w|^\varepsilon$ is a bounded term in Ω_0^*, what we essentially need to compute are the integrals:

$$\int_{\Omega_0^{**}\cap B_R(0)} \frac{|z|^{4k-2}}{|w-A|z|^{2k}|^{4-\varepsilon}}; \tag{3.14}$$

$$\int_{\Omega_0^{**}\cap B_R(0)} \frac{1}{|w|^\varepsilon}\frac{|z|^{4k-2}}{|w-A|z|^{2k}|^{4-\varepsilon}}. \tag{3.15}$$

Using this we want to compute

$$\int_{\Omega_0^{**}\cap B_R(0)} \frac{|z|^{4k-2}}{|w-A|z|^{2k}|^{4-\varepsilon}}dw\wedge d\overline{w}\wedge dz\wedge d\overline{z}.$$

Containing the domain of integration in a sufficiently large polidisk we can in an easier fashion compute the integral. Choose an R' dependent on R and Ω_0^{**} so that we can contain $\Omega_0^{**}\cap B_R(0)\subset\{0\le|w|\le R', 0\le|z|\le R'\}$. Use that $|\Phi|\gtrsim|\mathrm{Re}\, w-A|z|^{2k}|+|\mathrm{Im}\, w|$ the fact that $\mathrm{Re}\, w<0$ to see $|\Phi|\gtrsim|w|+A|z|^{2k}$. Then using polar coordinates for $|w|$ we see that

$$\begin{aligned}\int_0^{R'}\frac{|z|^{4k-2}}{(r+A|z|^{2k})^{4-\varepsilon}}dr &= \frac{1}{(\varepsilon-3)(\varepsilon-2)}\Big[\frac{-|z|^{4k-2}((3-\varepsilon)r+A|z|^{2k})}{(r+A|z|^{2k})^{3-\varepsilon}}\Big]_0^{R'}\\ &\le \frac{1}{(\varepsilon-3)(\varepsilon-2)}\frac{|z|^{4k-2}}{(A|z|^{2k})^{2-\varepsilon}}\end{aligned}$$

Using polar coordinates for $|z|$ we get

$$\int_0^{R'}\frac{r^{4k-1}}{(Ar^{2k})^{2-\varepsilon}}dr\le\frac{(AR'^{2k})^\varepsilon}{2A^2\varepsilon k}<\infty.$$

The second integral can be computed in the same manner, but we have to deal with the potential singularity in $|w|^{-\varepsilon}$. Let $|w| \geq v$, then

$$\int_{\Omega_0^{**} \cap B_R(0)} \frac{1}{|w|^{\varepsilon}} \frac{|z|^{4k-2} e^{-\psi}}{|w - A|z|^{2k}|^4} \leq \int_{\Omega_0^{**} \cap B_R(0)} \frac{1}{v^{\varepsilon}} \frac{|z|^{4k-2} e^{-\psi}}{|w - A|z|^{2k}|^4}$$

Then we want to compute

$$\lim_{v \to 0} \frac{1}{v^{\varepsilon}} \int_0^{R'} \int_v^{R'} \frac{r s^{4k-1}}{(r + A s^{2k})^{4-\varepsilon}} dr ds$$

We get then that for some constant C the expression $Cv^{\varepsilon-\varepsilon} = C$ dominates the integral proof is done.

□

Remark 3.3.3. The terms when calculating the weight involves a term ε^{-2}. So when $\varepsilon \longrightarrow 0$ the weighted integral blows up. This means that to use the weighted estimates to gain $|z|^2$ we also lose the sharp Hölder estimate $1/(2k)$ by a small amount.

3.4 Passing to Pointwise Estimates

We want to pass from the L^2 weighted estimates to pointwise estimates. To do this we will show that it is possible to fit a polidisk into the intermediate bumping domain Ω_0^{**} centered at a point $q \in \partial\Omega$. Because of the way we chose the Φ we will can fit a polidisk such that the volume is similar to $|\Phi(q)||q_2|$. We can use this to find a pointwise bound for u involving Φ. We show these estimates locally where the domain is given in the local coordinates $\Omega = \{\text{Re}\, w + r(z, \bar{z}) + O(\text{Im}\, w^2, |z| \text{Im}\, w) < 0\}$ as in 2.2.

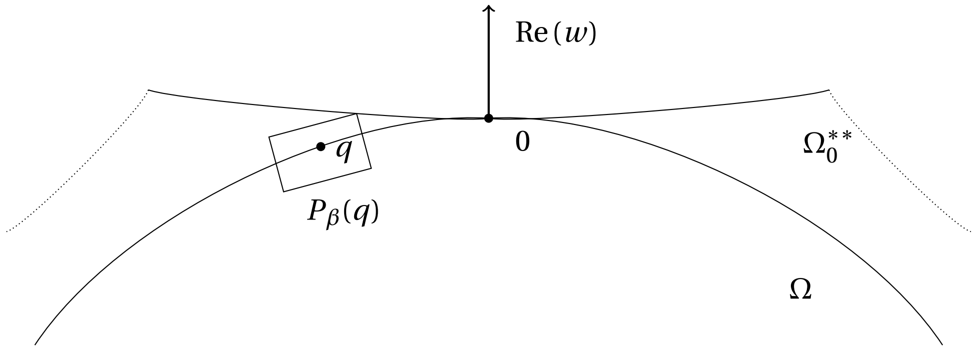

Figure 3.1: Visualization of 2d slice of Ω_0^{**} with the polidisk $P_\beta(q)$ centered at a boundary point.

For $r > 0$, define a polidisk by

$$P_r(q) := \{(w,z) \in \mathbb{C}^2 : |w - q_1| < r|\Phi(q)|, |z - q_2| < \frac{1}{2}|q_2|\}.$$

In the local bumping we know that for a point $q \in \partial\Omega$ we have that $\text{dist}(q, \partial\Omega_0^*) \sim |\Phi(q)|$. Then we can find $\beta' > 0$ so that $\beta'|\Phi(q)| \le \text{dist}(q, \partial\Omega_0^*)$. In the intermediate bumping we can then find a $0 < \beta < \beta'$ so that $\beta|\Phi(q)| \le \text{dist}(q, \partial\Omega_0^{**})$. Then for any $z \in P_\beta(q)$ we get

$$|z_1 - q_1| < \text{dist}(q, \partial\Omega_0^{**}),$$
$$|z_2 - q_2| < \frac{1}{2}|q_2|,$$

showing that $||z - q|| \lesssim \text{dist}(q, \partial\Omega_0^{**})$ and therefore $P_\beta(q) \subset \Omega_0^{**}$.

When $z \in \Omega$ is away from $\partial\Omega$ we also want to fit a polidisk in Ω_0^{**}. show that we still have that $|\Phi(z)| \le \text{dist}(z, \partial\Omega_0^{**})$. This will follow from Φ being a continuously differentiable function and therefore also locally Lipschitz. This means that there is an $\alpha > 0$ so that $|\Phi(\xi_1) - \Phi(\xi_2)| \lesssim ||\xi_1 - \xi_2||$ when $||\xi_1 - \xi_2|| < \alpha$.

Notice that $\text{dist}(z, \partial\Omega_0^{**}) \ge ||z - q||$ since $q \in \partial\Omega$ and $\text{dist}(z, \partial\Omega_0^{**}) \ge \text{dist}(q, \partial\Omega_0^{**})$.

Hence we get that

$$\begin{aligned}|\Phi(z)| &\le |\Phi(q)| + |\Phi(z) - \Phi(q)| \\ &\le |\Phi(q)| + ||z-q|| \\ &\lesssim \operatorname{dist}(q, \partial\Omega_0^{**}) + \operatorname{dist}(z, \partial\Omega_0^{**}) \\ &\lesssim \operatorname{dist}(z, \partial\Omega_0^{**})\end{aligned}$$

when $||z-q|| < \alpha$. This then implies that there is a $\gamma > 0$ so that $P_\gamma(z) \subset \Omega_0^{**}$.

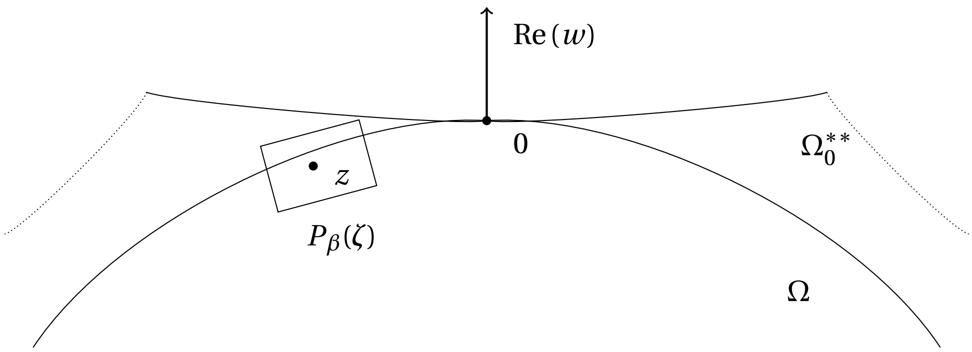

Figure 3.2: Visualization of 2d slice of Ω_0^{**} with the polidisk $P_\beta(\zeta)$ centered at point $\zeta \in \Omega$.

Having fit a polidisk in Ω_0^{**} we can use sub-averaging to get pointwise estimates for u by utilizing that u is part of a holomorphic function. We want to show the following Proposition.

Proposition 3.4.1. *Let $p \in \partial\Omega$ and $z = (z_1, z_2) \in \overline{\Omega} \setminus \{p\}$ with $||z-p|| < \alpha$, then for each $\eta > 0$ with $\eta << \frac{1}{2k}$ we have*

$$|u(p-z)| \le C_\eta \cdot \frac{1}{|p_2 - z_2|} \frac{1}{|\Phi(p,z)|^{1+\eta}},$$

where $C_\eta > 0$ is a constant depending on η.

Proof. Without loss of generality set $p = 0$. We choose $\eta > 0$ and $\delta > 0$ and set

$$\psi = -2(\eta+\delta)\log(\text{dist}(z, \partial\Omega_0^*)) + 2\delta\log(|z_1| + |z_2|^{1+\eta} + |z_2|^{2k}).$$

From Lemma 3.3.1, we have seen that the integral

$$\int_{\Omega_0^{**}} |u|^2 e^{-\psi} \leq B. \tag{3.16}$$

is finite.

Let $q \in \partial\Omega$. We have seen earlier that $\text{dist}(q, \partial\Omega_0^*) \sim |\Phi(q)|$. By the discussion above we can fit a polidisk into the domain Ω_0^{**} and there is a $c > 0$ so that

$$\sqrt{vol(P_\beta)} \geq c|\Phi(q)||q_2|. \tag{3.17}$$

Using that $\psi(q) - \psi(z)$ is bounded in Ω_0^* we can find a suitable constant $B > 0$ so that we increase the bound by multiplying in $Be^{\psi(q)-\psi(z)}$. Hence

$$\begin{aligned} |u(q)|^2 &\leq \frac{B}{vol(P_\beta)} \int_{P_\beta(q)} |u|^2 e^{-\psi(z)+\psi(q)} \\ &\leq \frac{Be^{\psi(q)}}{vol(P_\beta)} \int_{\Omega_0^{**}} |u|^2 e^{-\psi}. \end{aligned}$$

The integral is finite and thus

$$|u(q)| \lesssim \frac{e^{\frac{1}{2}\psi(q)}}{\sqrt{vol(P_\beta)}}.$$

Inserting in for the weight ψ and using the volume estimate (3.17) we get

$$|u(q)| \lesssim \frac{\text{dist}(q, \partial\Omega_0^*)^{-\eta-\delta}(|q_1| + |q_1|^{1+\varepsilon} + |q_2|^{2k})^\delta}{|\Phi(q)||q_2|}.$$

Now $\text{dist}(q, \partial\Omega_0^*) \sim |\Phi(q)|$ when $q \in \partial\Omega$, therefore $\text{dist}(q, \partial\Omega_0^*) \gtrsim |\Phi(q)|$, and because

$$Q = |q_1| + |q_1|^{1+\eta} + |q_2|^{2k} \geq |q_1| + |q_2|^{2k} \geq |\Phi(q)|$$

we finally get that

$$\begin{aligned} |u(q)| &\lesssim \frac{|\Phi(q)|^{-\eta-\delta}|\Phi(q)|^{\delta}}{|q_2||\Phi(q)|} \\ &\leq \frac{1}{|q_2||\Phi(q)|^{1+\eta}}. \end{aligned}$$

Now when we have a point z is away from $\partial\Omega$, we can then still fit a polidisk of size $|\Phi(z)||z_2|$ in Ω_0^{**} as by the discussion preceding the Proposition. The estimates for u then also hold for $z \in \Omega$ when $||z|| < \alpha$, so

$$|uz_2| \leq \frac{C}{|\Phi(z)|^{1+\eta}}.$$

□

Remark 3.4.2. We cannot achieve better estimate than $|\Phi|^{1+\eta}$. There will always be a loss of η coming from the weighted estimates in Lemma 3.3.1. This means that if $\eta \longrightarrow 0$ the estimates blow up.

Using the pointwise estimates from Proposition 3.4.1 we then readily achieve pointwise estimates for $h_1(p,z)$ and $h_2(p,z)$. We see that for $z \in \overline{\Omega}$ with $||z-p|| < \alpha$ and $p \in \partial\Omega$

$$\begin{aligned} |h_1(p,z)| &= |\frac{1}{\Phi} - u(p_2-z_2)| \\ &\leq \frac{1}{|\Phi|} + |u(p_2-z_2)| \\ &\lesssim \frac{1}{|\Phi|^{1+\eta}} \end{aligned} \tag{3.18}$$

$$\begin{aligned} |h_2(p,z)| &= |\frac{-A(p_2-z_2)^{k-1}\overline{(p_2-z_2)}^k}{\Phi} - u(p_1-z_1)| \\ &\lesssim \frac{|p_2-z_2|^{2k-1}}{|\Phi|^{1+\eta}} \\ &\lesssim \frac{1}{|\Phi|^{1+\eta}}. \end{aligned} \tag{3.19}$$

by recalling that $|p_1-z_1| \sim |p_2-z_2|^{2k}$ in Ω_0^*. When $||z|| > \alpha$ the functions are bounded, which follow from the fact that h_1 and h_2 are holomorphic in Ω so we can apply standard Cauchy estimates to u on Ω_0^*.

We also seek estimates for the derivatives $d(h_1)$ and $d(h_2)$. Because the functions h_1 and h_2 are holomorphic, we apply Cauchy estimates on the fitted polidisk to see that

$$\begin{aligned}\left|\frac{\partial h_1}{\partial z_1}\right| &\leq \frac{1}{2\pi}\int_{|\xi_1-z_1|=\beta|\Phi|}\left|\frac{h_1(\xi)}{(\xi_1-z_1)^2}d\xi_1\right| \\ &\leq \frac{\sup|h_1|}{\beta|\Phi|}, \\ \left|\frac{\partial h_1}{\partial z_2}\right| &\leq \frac{1}{2\pi}\int_{|\xi_2-z_2|=\frac{1}{2}|p_2-z_2|}\left|\frac{h_1(\xi)}{(\xi_2-z_2)^2}d\xi_2\right| \\ &\leq \frac{2\sup|h_1|}{|p_2-z_2|}.\end{aligned}$$

This readily implies that

$$|d(h_1)| \lesssim \frac{1}{|\Phi|^{2+\eta}} + \frac{1}{|z_2||\Phi|^{1+\eta}}. \tag{3.20}$$

The same argument holds for h_2 since $|p_2 - z_2|^{2k-1}$ is bounded in Ω_p^*, thus

$$|d(h_2)| \lesssim \frac{1}{|\Phi|^{2+\eta}} + \frac{1}{|z_2||\Phi|^{1+\eta}}. \tag{3.21}$$

3.5 Constructing an Integral Kernel

To prove the main Theorem we want estimates which are smooth in the boundary variable. The functions h_j are dependent on solving $\overline{\partial}u = \omega$, so the smoothness involves checking the smoothness of Hörmander's solution. We bypass this problem by showing that the functions can be replaced with similar functions which depend smoothly on the boundary variable.

Similarly to [9] we will create the integral kernel on a smaller domain $\Omega_{-\varepsilon} \subset \Omega$ to ensure the functions h_j depend smoothly on the boundary variable. As we have estimated and constructed our h_j's locally we will thereafter use a partition of unity to glue together a solution operator on $\Omega_{-\varepsilon}$.

Proposition 3.5.1. *Let $p \in \partial\Omega$. Then there is an open neighborhood U of p and functions $\tilde{h}_1(\zeta, z)$ and $\tilde{h}_2(\zeta, z)$ on $(U \cap \partial\Omega) \times \Omega_{-\varepsilon}$ which are smooth in ζ, holomorphic in z which satisfies*

$$\tilde{h}_1(\zeta, z)(\zeta_1 - z_1) + \tilde{h}_2(\zeta, z)(\zeta_2 - z_2) \equiv 1,$$

and when $||\zeta - z|| < \alpha$ there is a constant C_η so that

$$|\tilde{h}_j(\zeta, z)| \leq \frac{C_\eta}{|\Phi(\zeta, z)|^{1+\eta}} \tag{3.22}$$

$$|d\tilde{h}_j(\zeta, z)| \leq \frac{C_\eta}{|\Phi(\zeta, z)|^{2+\eta}} + \frac{C_\eta}{|\zeta_2 - z_2||\Phi(\zeta, z)|^{1+\eta}} \tag{3.23}$$

for $j = 1, 2$.

When $||\zeta - z|| \geq \alpha$, the functions $\tilde{h}_j(\zeta, z)$ and $d\tilde{h}_j$ are bounded for $j = 1, 2$.

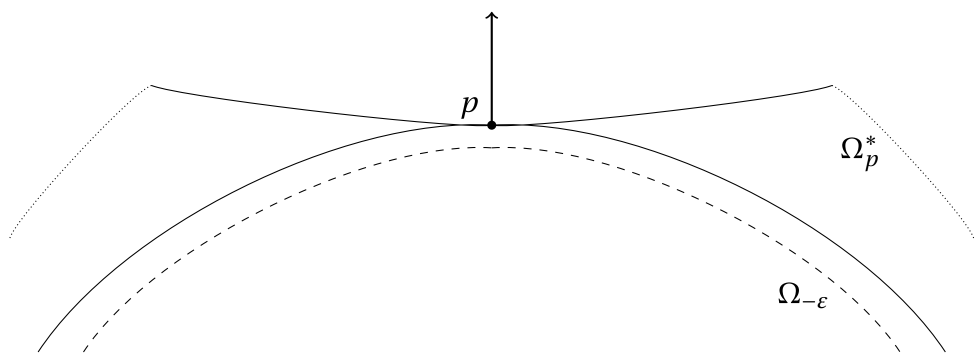

Figure 3.3: Visualization of $\Omega_{-\varepsilon}$.

Proof. Fix a point $p \in \partial\Omega$ and choose a small $\varepsilon > 0$. In the same manner as in Chapter

2 we set with out loss of generality $p = 0$ and define for a neighborhood W of 0

$$W \cap \Omega_{-\varepsilon} = \{z \in \mathbb{C}^2 : \rho(z) < -\varepsilon\}.$$

This is still pseudoconvex if ε is sufficiently small and we have $\Omega_{-\varepsilon} \subset\subset \Omega$.

We have the pointwise holomorphic solutions $h_1(p,z)$ and $h_2(p,z)$ to the Cauchy-Fantappiè equation for $p \in \partial\Omega$ and $z \in \Omega$ with estimates in $p - z$ with $(p,z) \in \partial\Omega \times \Omega$. We now show these functions can be translated to smooth estimates on $\partial\Omega \times \Omega_{-\varepsilon}$. We define $\Psi : \partial\Omega \times \overline{\Omega}_{-\varepsilon} \longrightarrow \mathbb{C}$ by

$$\Psi(\zeta, z) = h_1(p,z)(\zeta_1 - z_1) + h_2(p,z)(\zeta_2 - z_2).$$

This is continuous in the ζ-variable, holomorphic in the z-variable and with $\Psi(p,z) \equiv 1$. By continuity in ζ we can find a neighborhood U of p so that $|\Psi(\zeta,z)| \geq \frac{1}{2}$ on $(U(p) \cap \partial\Omega) \times \overline{\Omega}_{-\varepsilon}$. We can so define a new function

$$\tilde{h}_j(\zeta, z) = \frac{h_j(p,z)}{\Psi(\zeta,z)}$$

for $j = 1,2$. Then $\tilde{h}$ is smooth in ζ, holomorphic in z and $\tilde{h}_1(\zeta,z)(\zeta_1 - z_1) + \tilde{h}_2(\zeta,z)(\zeta_2 - z_2) \equiv 1$ on $(U(p) \cap \partial\Omega) \times \overline{\Omega}_{-\varepsilon}$. Each of the functions $\tilde{h}_j(\zeta,z)$ will satisfy the bound

$$|\tilde{h}_j(\zeta,z)| \leq 2|h_j(p,z)|.$$

From (3.18) and (3.19) we know the functions are then pointwise bounded by

$$|\tilde{h}_1(\zeta,z)| \lesssim \frac{1}{|\Phi(p,z)|^{1+\eta}}$$
$$|\tilde{h}_2(\zeta,z)| \lesssim \frac{|p_2 - z_2|^{2k-1}}{|\Phi(p,z)|^{1+\eta}}$$

when $||p - z|| < \alpha$. By possibly shrinking U further to ensure that $||p - \zeta|| \leq ||p - z||$ we may assume that $||p - z|| \geq \frac{1}{2}||\zeta - z||$ because

$$||\zeta - z|| \leq ||p - \zeta|| + ||p - z|| \leq 2||p - z||.$$

In this neighborhood we also have that $|\Phi(p,z)| \gtrsim |\Phi(\zeta,z)|$ as seen by

$$\begin{aligned}
2|\Phi(p,z)| &\geq |\mathrm{Re}\,\Phi| + |\mathrm{Im}\,\Phi| \\
&= |-|\mathrm{Re}\,(p_1 - z_1)| - A|p_2 - z_2|^{2k}| + |\mathrm{Im}\,(p_2 - z_2)| \\
&= |\mathrm{Re}\,(p_1 - z_1)| + |\mathrm{Im}\,(p_2 - z_2)| + A|p_2 - z_2|^{2k} \\
&\geq |p_1 - z_1| + A|p_2 - z_2|^{2k} \\
&\geq \frac{1}{2}|\zeta_1 - z_1| + \frac{A}{2^{2k}}|\zeta_2 - z_2|^{2k} \\
&\gtrsim |\Phi(\zeta,z)|
\end{aligned}$$

so the estimates then become

$$\begin{aligned}
|\tilde{h}_1(\zeta,z)| &\lesssim \frac{1}{|\Phi(\zeta,z)|} + \frac{1}{|\Phi(\zeta,z)|^{1+\eta}} \\
&\lesssim \frac{1}{|\Phi(\zeta,z)|^{1+\eta}} \qquad (3.24)\\
|\tilde{h}_2(\zeta,z)| &\lesssim \frac{|\zeta_2 - z_2|^{2k-1}}{|\Phi(\zeta,z)|} + \frac{|\zeta_2 - z_2|^{2k-1}}{|\Phi(\zeta,z)|^{1+\eta}} \\
&\lesssim \frac{1}{|\Phi(\zeta,z)|^{1+\eta}}. \qquad (3.25)
\end{aligned}$$

when $\zeta \in U(p)$, $z \in \overline{\Omega}_{-\varepsilon}$ and $||\zeta - z|| < \alpha$. The smooth estimates for the differentials dh_j are obtained in the same manner. □

To get a solution operator on the entirety of the domain we need to glue these local constructions together. Our domain is bounded, meaning that the boundary $\partial\Omega$ is compact. For each $p \in \partial\Omega$ we know that there is an open neighborhood V of p so that Proposition 3.5.1 holds. Then letting $\{V_p\}_{p\in\partial\Omega}$ be an open cover of the boundary $\partial\Omega$ we use the compactness to reduce to a finite subcover $V_{p_1}, ..., V_{p_m}$.

To glue these functions together we will use a partition of unity. Therefore choose $\{\chi_j(\zeta)\}_{j=1}^m$ to be a family of functions with each $\chi_j \in C^\infty(V_{p_j})$, where $0 \leq \chi_j(\zeta) \leq 1$ so that $\sum_{j=1}^m \chi_j(\zeta) \equiv 1$ on $\partial\Omega$. This is a partition of unity subordinate to the finite cover

$\{V_{p_j}\}_{j=1}^{m}$. Note that Proposition 3.5.1 hold for each of the sets V_{p_j}. We set

$$\widehat{h}_1 = \sum_{j=1}^{m} \chi_j(\zeta)\tilde{h}_{1,p_j}(\zeta,z),$$
$$\widehat{h}_2 = \sum_{j=1}^{m} \chi_j(\zeta)\tilde{h}_{2,p_j}(\zeta,z).$$

These new functions satisfy the Cauchy-Fantappiè equation on $\partial\Omega$ as seen by

$$\begin{aligned}\widehat{h}_1(\zeta,z)(\zeta_1-z_1)+\widehat{h}_2(\zeta,z) &= \sum_{j=1}^{m} \chi_j(\zeta)\tilde{h}_{1,q_j}(\zeta_1-z_1)+\sum_{j=1}^{m} \chi_j(\zeta)\tilde{h}_{2,q_j}(\zeta_2-z_2)\\ &= \sum_{j=1}^{m} \chi_j(\zeta)\big(\tilde{h}_{1,q_j}(\zeta,z)(\zeta_1-z_1)+\tilde{h}_{2,q_j}(\zeta,z)(\zeta_2-z_2)\big)\equiv 1.\end{aligned}$$

Recall from Theorem 1.1.7 the Henkin integral formula

$$H_\Omega f = c_n\int_{\partial\Omega\times[0,1]} f(\zeta)\wedge\eta(\mu)\wedge\omega(\zeta) - c_n\int_\Omega \frac{f(\zeta)}{||\zeta-z||^{2n}}\eta(\overline{\zeta}-\overline{z})\wedge\omega(\zeta).$$

Using the now constructed functions $\widehat{h}_j$ we set

$$\widehat{\mu}_j = \frac{\overline{\zeta}_j-\overline{z}_j}{||\zeta-z||^2}\lambda+\widehat{h}_j(1-\lambda)$$

and because $\widehat{h}_1$ and $\widehat{h}_2$ satisfy the Cauchy-Fantappiè equation, we then get an integral operator $H_\Omega^{(\varepsilon)}$ on $\partial\Omega\times\Omega_{-\varepsilon}$ which maps $\overline{\partial}$-closed $(0,1)$-forms f on $\overline{\Omega}$ into functions on $\Omega_{-\varepsilon}$. This operator is given by the modified Henkin formula

$$H_\Omega^{(\varepsilon)} f(z) = c_2\int_{\partial\Omega\times[0,1]} f(\zeta)\wedge\eta(\widehat{\mu})\wedge\omega(\zeta) - c_2\int_\Omega \frac{f(\zeta)}{||\zeta-z||^{4}}\eta(\overline{\zeta}-\overline{z})\wedge\omega(\zeta). \tag{3.26}$$

Note that the estimates from Proposition 3.5.1 still hold locally which we will use when finding estimates for the operator

3.6 Hölder Estimates

Now that we have constructed the solution operator $H_\Omega^{(\varepsilon)}$ which maps $\overline{\partial}$-closed (0,1)-forms on $\overline{\Omega}$ to functions on $\Omega_{-\varepsilon}$ with $\overline{\partial}(H_\Omega^{(\varepsilon)} f) = f$ on $\Omega_{-\varepsilon}$. We have yet to prove any estimates for this operator. To do this we need to use the local estimates which we found in Proposition 3.5.1.What we will show now are Hölder estimates of order $1/(2k) - \eta$.

Theorem 3.6.1. *Let $\Omega \subset \mathbb{C}^2$ be a pseudoconvex bounded domain with real analytic boundary of D'Angelo finite type $2k$. Then for every $\overline{\partial}$-closed (0,1)-form f on $\overline{\Omega}$ and for each $\eta > 0$ there is a constant $C_{\Omega,\eta} > 0$ such that one has*

$$|(H_\Omega^{(\varepsilon)} f)(\xi_1) - (H_\Omega^{(\varepsilon)} f)(\xi_2)| \leq C_{\Omega,\eta} ||f||_\infty |\xi_1 - \xi_2|^{\frac{1}{2k} - \eta}$$

for all sufficiently small $\varepsilon > 0$ and all $\xi_1, \xi_2 \in \Omega_{-\varepsilon}$.

The proof requires us to estimate the integrals in the operator $H_\Omega^{(\varepsilon)}$. However note that the integral over Ω is unchanged from the that in the Henkin integral formula. The Hölder estimates then must arise from the integral over the boundary. This is as expected in that the local estimates reflect the type of the boundary.

Further we define

$$B_\Omega^{(\varepsilon)}(f) := c_2 \int_{\partial\Omega \times [0,1]} f(\zeta) \wedge \eta(\mu^{(\varepsilon)}) \wedge \omega(\zeta),$$
$$I_\Omega^{(\varepsilon)}(f) := c_2 \int_\Omega \frac{f(\zeta)}{||\zeta - z||^4} \eta(\overline{\zeta} - \overline{z}) \wedge \omega(\zeta).$$

Thus we can write $H_\Omega^{(\varepsilon)} f = B_\Omega^{(\varepsilon)} f - I_\Omega^{(\varepsilon)} f$. If we then show that both $B_\Omega^{(\varepsilon)} f$ and $I_\Omega^{(\varepsilon)} f$ satisfy Hölder estimates of order $1/(2k) - \eta$ the theorem will follow easily.

First we show that $I_{\Omega}^{(\varepsilon)}$ satisfies

$$|(I_{\Omega}^{(\varepsilon)} f)(\xi_1) - (I_{\Omega}^{(\varepsilon)} f)(\xi_2)| \le C_{\Omega,\eta} ||f||_\infty |\xi_1 - \xi_2|^{\frac{1}{2k}-\eta} \tag{3.27}$$

for for all sufficiently small $\varepsilon > 0$ and all $\xi_1, \xi_2 \in \Omega_{-\varepsilon}$.

We know that $I_{\Omega}^{(\varepsilon)} f$ satisfies Hölder estimates of order $1/2$ by the properties of the Henkin integral kernel [6]. This means $I_{\Omega}^{(\varepsilon)}$ will also satisfy estimates of lower order, in particular of order $1/2k - \eta$. This follows from embedding of Hölder spaces on bounded sets.

Next we want to show that

$$|(B_{\Omega}^{(\varepsilon)} f)(\xi_1) - (B_{\Omega}^{(\varepsilon)} f)(\xi_2)| \le C_{\Omega,\eta} ||f||_\infty |\xi_1 - \xi_2|^{\frac{1}{2k}-\eta} \tag{3.28}$$

for for all sufficiently small $\varepsilon > 0$ and all $\xi_1, \xi_2 \in \Omega_{-\varepsilon}$.

To see this we apply a known result from real function theory as stated and proved in chapter V.3.1 of [8].

Lemma 3.6.2. *Let $D \subset\subset \mathbb{R}^N$ be a bounded domain with C^1 boundary. Suppose $g \in C^1(D)$ and that for some $0 < \alpha < 1$ there is a constant c so that*

$$|dg(x)| \le c \cdot dist(x, \partial D)^{\alpha-1},$$

for $x \in D$. Then there is a constant C so that

$$|g(x) - g(y)| \le C||x - y||^\alpha$$

for $x, y \in D$.

Proof. Let $\delta > 0$ and define the set $U_\delta = \{z \in \mathbb{R}^N : 0 < z_1 < \delta, \text{ and } ||z|| < \delta\}$. Suppose that $|dg(z)| \leq c z_1^{\alpha-1}$. For any $x, y \in U_{\delta/2}$ we see that

$$\begin{aligned}
|g(x_1, x) - g(x_1 + ||x-y||, x)| &\leq \int_{x_1}^{x_1+||x-y||} \Big|\frac{\partial g}{\partial x_1}(t,x)\Big| dt \\
&\leq c \int_{x_1}^{x_1+||x-y||} |dg(t,x)| dt \\
&\leq Cc \int_{x_1}^{x_1+||x-y||} t^{\alpha-1} dt \\
&\leq Cc ||x-y||^{\alpha}
\end{aligned}$$

By the Mean Value Theorem we see that

$$\begin{aligned}
|g(x_1 + ||x-y||, x) - g(y_1 + ||x-y||, y)| &\leq |dg(z)|\,||x-y|| \\
&\leq c z_1^{\alpha-1} ||x-y|| \\
&\leq c||x-y||^{\alpha-1}||x-y||.
\end{aligned}$$

Then

$$\begin{aligned}
|g(x) - g(y)| \leq\, &|g(x_1, x) - g(x_1 + ||x-y||, x)| \\
&+ |g(x_1 + ||x-y||, x) - g(y_1 + ||x-y||, x)| \\
&+ |g(y_1 + ||x-y||, y) - g(y_1, y)| \\
&\leq 3Cc||x-y||^{\alpha}
\end{aligned}$$

for all $x, y \in U_{\delta/2}$ with $|x-y| < \delta/2$.

Now we translate the domain so that $0 \in \partial D$ and change coordinates so that $T_0(\partial D) = \{x \in \mathbb{R}^N : x_1 = 0\}$. Then in a neighborhood U_δ around 0 we have $\text{dist}(x, \partial D)^{\alpha-1} \leq |x_1|^{\alpha-1}$ for all $x \in U \cap D$. This gives

$$|dg(x)| \leq c x_1^{\alpha-1}$$

and we apply the local result giving that $|g(x) - g(y)| \lesssim ||x-y||^{\alpha}$ when $||x-y|| < \delta/2$.

By compactness of ∂D we can reduce an open cover of sets $\{V_{\delta/2,\nu}\}_{\nu\in I}$ to a finite cover of open sets $V_{\delta/2}$ which cover ∂D. Then we get that $|g(x)-g(y)| \lesssim ||x-y||^{\alpha}$ when $x, y \in \bigcup_i V_{\delta/2,i}$ with $||x-y|| < r$ where $r > 0$ and depends on α and D.

Define the compact set $E = \{x \in D : \text{dist}(x, \partial D) \geq r\}$ and so

$$|g(x)-g(y)| \leq C||x-y|| < \infty$$

when $x, y \in E$. Therefore when $||x-y|| \geq r$ we can write

$$\frac{|g(x)-g(y)|}{||x-y||^{\gamma}} \leq \frac{|g(x)-g(y)|}{r^{\gamma}} < \infty$$

and we are done. □

To apply Lemma 3.6.2 we want to fix a point $z \in \Omega_{-\varepsilon}$ and show that

$$\left| d_z \Big(\int_{\partial\Omega\times[0,1]} f(\zeta) \wedge \eta\big(\widehat{\mu}\big) \wedge \omega(\zeta) \Big) \right| \lesssim \text{dist}(z, \partial\Omega)^{\frac{1}{2k}-\eta-1}. \tag{3.29}$$

Recall the functions

$$\eta(\xi) = \sum_{j=1}^{n} (-1)^{j+1} \xi_j d\xi_1 \wedge \ldots \wedge \wedge d\xi_{j-1} \wedge d\xi_{j+1} \wedge \ldots \wedge d\xi_n,$$

$$\widehat{\mu}_j = \frac{\overline{\zeta}_j - \overline{z}_j}{||\zeta - z||^2} \lambda + \widehat{h}_j (1-\lambda).$$

We see that

$$d\widehat{\mu}_j = \frac{\partial}{\partial\lambda} \widehat{\mu}_j d\lambda + \sum_i \frac{\partial}{\partial\overline{\zeta}_i} \widehat{\mu}_j d\overline{\zeta}_i.$$

We can then express

$$\begin{aligned}\eta(\widehat{\mu}) &= \Big[\Big(\frac{\overline{\zeta}_1-\overline{z}_1}{||\zeta-z||^2}\lambda+\widehat{h}_1(1-\lambda)\Big)\Big(\frac{\overline{\zeta}_2-\overline{z}_2}{||\zeta-z||^2}-\widehat{h}_2\Big)\\ &\quad -\Big(\frac{\overline{\zeta}_2-\overline{z}_2}{||\zeta-z||^2}\lambda+\widehat{h}_2(1-\lambda)\Big)\Big(\frac{\overline{\zeta}_1-\overline{z}_1}{||\zeta-z||^2}-\widehat{h}_1\Big)\Big]\\ &= \Big(\frac{\overline{\zeta}_2-\overline{z}_2}{||\zeta-z||^2}\widehat{h}_1-\frac{\overline{\zeta}_1-\overline{z}_1}{||\zeta-z||^2}\widehat{h}_2\Big)d\lambda. \end{aligned} \tag{3.30}$$

We see that only terms containing $d\lambda$ appear in the final expression. The reason for this is that the integral cannot support the added differentials $d\overline{\zeta}_j$.

Differentiating under the integral sign we see that $B^{(\varepsilon)}_{\Omega}(f)$ becomes

$$\begin{aligned}&\Big|\int_{\partial\Omega} f(\zeta)\wedge d_z\Big(\frac{\overline{\zeta}_2-\overline{z}_2}{||\zeta-z||^2}\widehat{h}_1-\frac{\overline{\zeta}_1-\overline{z}_1}{||\zeta-z||^2}\widehat{h}_2\Big)\wedge\omega(\zeta)\Big|\\ &\le ||f||_\infty\int_{\partial\Omega}\Big|d_z\Big(\frac{\overline{\zeta}_2-\overline{z}_2}{||\zeta-z||^2}\widehat{h}_1\Big)\Big|+\Big|d_z\Big(g\frac{\overline{\zeta}_1-\overline{z}_1}{||\zeta-z||^2}\widehat{h}_2\Big)\Big|dS(\zeta).\end{aligned}$$

Apply the quotient rule to see that (setting $(w,z)=(\zeta_1-z_1,\zeta_2-z_2)$ to shorten the expression)

$$\begin{aligned}d\Big(\frac{\overline{z}}{||\zeta-z||^2}\widehat{h}_1\Big) &= \frac{d(\overline{z}\widehat{h}_1)||\zeta-z||^2-\overline{z}\widehat{h}_1 d(||\zeta-z||^2)}{||\zeta-z||^4}\\ &= \frac{1}{||\zeta-z||^4}((\widehat{h}_1 d\overline{z}+\overline{z}d\widehat{h}_1)||\zeta-z||^2-\overline{z}\widehat{h}_1(\overline{z}dz+zd\overline{z}+\overline{w}dw+wd\overline{w})),\\ d\Big(\frac{\overline{w}}{||\zeta-z||^2}\widehat{h}_2\Big) &= \frac{1}{||\zeta-z||^4}((\widehat{h}_2 d\overline{w}+\overline{w}d\widehat{h}_2)||\zeta-z||^2+\overline{w}\widehat{h}_2(\overline{z}dz+zd\overline{z}+\overline{w}dw+wd\overline{w})).\end{aligned}$$

Taking the norm we get an estimate

$$\Big|d(\frac{\overline{z}}{||\zeta-z||^2}\widehat{h}_1)\Big|\le\frac{1}{||\zeta-z||^4}\Big((|\widehat{h}_1|+|zd\widehat{h}_1|)||\zeta-z||^2+|\widehat{h}_1|\big(2|z|^2+2|zw|\big)\Big).$$

Using that $|z|\le||\zeta-z||$ and $|w|\le||\zeta-z||$ we can simplify the expression to

$$\Big|d\Big(\frac{\overline{z}\widehat{h}_1}{||\zeta-z||^2}\Big)\Big|\le\frac{|\widehat{h}_1|}{||\zeta-z||^2}+\frac{|z||d\widehat{h}_1|}{||\zeta-z||^2}+\frac{4|\widehat{h}_1|}{||\zeta-z||^2} \tag{3.31}$$

and similarly we have

$$\left|d\Big(\frac{\overline{w}\widehat{h}_2}{||\zeta - z||^2}\Big)\right| \le \frac{|\widehat{h}_2|}{||\zeta - z||^2} + \frac{|w||d\widehat{h}_2|}{||\zeta - z||^2} + \frac{4|\widehat{h}_2|}{||\zeta - z||^2}. \tag{3.32}$$

We will use these estimates to show (3.29). We will consider the integral in two cases: when $||\zeta - z|| < \alpha$ and $||\zeta - z|| \ge \alpha$ as in Proposition 3.5.1.

Lemma 3.6.3. *Fix $z \in \Omega_{-\varepsilon}$, then the estimate*

$$\int_{\partial\Omega \cap \{||\zeta - z|| \ge \alpha\}} \left| d_z\Big(\frac{\overline{\zeta}_2 - \overline{z}_2}{||\zeta - z||^2}\widehat{h}_1 - \frac{\overline{\zeta}_1 - \overline{z}_1}{||\zeta - z||^2}\widehat{h}_2\Big) dS(\zeta)\right| \le C_{\Omega,\eta}\, dist(z, \partial\Omega)^{\frac{1}{2k} - \eta - 1} \tag{3.33}$$

holds.

Proof. When $||\zeta - z|| \ge \alpha$ we have from Proposition 3.5.1 that $\widehat{h}_j$ and $d(\widehat{h}_j)$ are bounded functions. Hence

$$\left|d\Big(\frac{(\overline{\zeta}_2 - \overline{z}_2)\widehat{h}_1}{||\zeta - z||^2}\Big)\right| + \left|d\Big(\frac{(\overline{\zeta}_1 - \overline{z}_1)\widehat{h}_2}{||\zeta - z||^2}\Big)\right| \lesssim \frac{1}{||\zeta - z||^2}. \tag{3.34}$$

Since $||\zeta - z|| \ge \mathrm{dist}(z, \partial\Omega)$ and $||\zeta - z|| \ge \alpha$ we can rewrite

$$||\zeta - z||^{-2} = \frac{||\zeta - z||^{\frac{1}{2k} - \eta - 1}}{||\zeta - z||^{-\frac{1}{2k} + \eta - 1}} \le \frac{\mathrm{dist}(z, \partial\Omega)^{\frac{1}{2k} - \eta - 1}}{\alpha^{-\frac{1}{2k} + \eta - 1}}$$

and therefore

$$\int_{\partial\Omega \cap \{||\zeta - z|| \ge \alpha\}} \frac{1}{||\zeta - z||^2} dS(\zeta) \le C_{\Omega,\eta}\mathrm{dist}(z, \partial\Omega)^{\frac{1}{2k} - \eta - 1} \tag{3.35}$$

□

Lemma 3.6.4. *Fix $z \in \Omega_{-\varepsilon}$, then the estimate*

$$\int_{\partial\Omega\cap\{||\zeta-z||<\alpha\}} \left| d_z\Big(\frac{\overline{\zeta}_2-\overline{z}_2}{||\zeta-z||^2}\widehat{h}_1 - \frac{\overline{\zeta}_1-\overline{z}_1}{||\zeta-z||^2}\widehat{h}_2\Big) dS(\zeta)\right| \leq C_{\Omega,\eta}\, dist(z,\partial\Omega)^{\frac{1}{2k}-\eta-1}$$

holds.

To show this we want to use a local coordinate system to integrate over the boundary. The coordinate system will be such that we can use ImΦ and the defining function ρ as coordinates. This will be useful in coming estimates.

Lemma 3.6.5. *There is a constant $\gamma > 0$ so that for each $z \in \Omega_{-\varepsilon}$ there is a C^1 coordinate system in $B(z,\gamma)$ with*

$$\begin{aligned} x_1 &= \rho(\zeta) + |\rho(z)| \\ x_2 &= Im\Phi(\zeta,z) \\ x_3 &= Re(\zeta_2 - z_2) \\ x_4 &= Im(\zeta_2 - z_2). \end{aligned}$$

Proof. Fix $z \in \partial\Omega$. As in (2.1) from section 2.2 we have a coordinate change so that

$$\frac{\partial\rho}{\partial u} = 1, \quad \frac{\partial\rho}{\partial v} = 0 \quad \text{and} \quad \frac{\partial\rho}{\partial z} = 0.$$

Notice then that

$$\frac{\partial\rho}{\partial\zeta_1} = \frac{\partial\rho}{\partial u}\frac{\partial u}{\partial\zeta_1} = \frac{1}{2}$$

as $2u = \zeta_1 - z_1 + \overline{\zeta_1 - z_1}$ and therefore

$$\frac{1}{2} = \frac{\partial \rho}{\partial \zeta_1} = \frac{\partial \rho}{\partial z_1}\frac{\partial z_1}{\partial \zeta_1} = \frac{1}{2}\frac{\partial z_1}{\partial \zeta_1}.$$

Similarly we see that $\frac{\partial \rho}{\partial \zeta_2} = 0$. Then

$$\begin{aligned} dx_1 \wedge dx_2 &= d\rho(\zeta) \wedge d\mathrm{Im}\,(\zeta_1 - z_1) \\ &= \frac{\partial \rho}{\partial \zeta_1} d\zeta_1 \wedge (\partial + \overline{\partial})(\frac{w - \overline{w}}{2}) \\ &= \frac{1}{4} d\zeta_1 \wedge d\overline{\zeta}_1 \neq 0 \end{aligned}$$

The result holds now for the fixed z as follows from the inverse function theorem, so we can use $\{x_1, x_2, x_3, x_4\}$ as local coordinates in a neighborhood around z. □

Proof of Lemma 3.6.4. When $||\zeta - z|| < \alpha$ we have the local estimates

$$\begin{aligned} |\widehat{h}_j(\zeta, z)| &\leq \frac{C_\eta}{|\Phi(\zeta, z)|^{1+\eta}} \\ |d\widehat{h}_j(\zeta, z)| &\leq \frac{C_\eta}{|\Phi(\zeta, z)|^{2+\eta}} + \frac{C_\eta}{|\zeta_2 - z_2||\Phi(\zeta, z)|^{1+\eta}} \end{aligned}$$

from Proposition 3.5.1. This then gives that

$$\begin{aligned} \left| d\Big(\frac{(\overline{\zeta}_2 - \overline{z}_2)\widehat{h}_1}{||\zeta - z||^2}\Big)\right| &\lesssim \frac{1}{||\zeta - z||^2 |\Phi|^{1+\eta}} + \frac{|\zeta_2 - z_2|}{||\zeta - z||^2}\Big(\frac{1}{|\Phi|^{2+\eta}} + \frac{1}{|\zeta_2 - z_2||\Phi|^{1+\eta}}\Big) + \frac{1}{||\zeta - z||^2 |\Phi|^{1+\eta}} \\ &\lesssim \frac{1}{||\zeta - z||^2 |\Phi|^{1+\eta}} + \frac{1}{||\zeta - z|||\Phi|^{2+\eta}}. \end{aligned}$$

Near each boundary point we have that $|\Phi| \leq |\zeta_1 - z_1| + A|\zeta_2 - z_2|^{2k} \leq ||\zeta - z|| + A||\zeta - z||^{2k} \lesssim ||\zeta - z||$ and so

$$\begin{aligned} \frac{1}{||\zeta - z||^2 |\Phi|^{1+\eta}} &= \frac{|\Phi|}{||\zeta - z||^2 |\Phi|^{2+\eta}} \\ &\lesssim \frac{1}{||\zeta - z|| \, |\Phi|^{2+\eta}}. \end{aligned}$$

Likewise we see that

$$\left|d\Big(\frac{(\overline{\zeta}_1-\overline{z}_1)\widehat{h}_2}{||\zeta-z||^2}\Big)\right| \lesssim \frac{1}{||\zeta-z||\,|\Phi|^{2+\eta}}.$$

Then we get the bound

$$\int_{\partial\Omega\cap\{||\zeta-z||<\alpha\}} \frac{1}{||\zeta-z||\,|\Phi|^{2+\eta}} dS(\zeta) \tag{3.36}$$

We know that $|\Phi| \geq \text{dist}(z,\partial\Omega)$, so $|\Phi|^\eta \geq \text{dist}(z,\partial\Omega)^\eta$ and

$$\int_{\partial\Omega\cap\{||\zeta-z||<\alpha\}} \frac{1}{||\zeta-z||\,|\Phi|^{2+\eta}} dS(\zeta)| \leq \frac{1}{\text{dist}(z,\partial\Omega)^\eta}\int_{\partial\Omega\cap\{||\zeta-z||<\alpha\}} \frac{1}{||\zeta-z||\,|\Phi|^{2}} dS(\zeta).$$

From the local bumping we know that the defining function for the bumped domain is smaller than the defining function ρ for Ω. So we have $\text{Re}\,(\zeta_1-z_1)+\tilde{P}_{2k}<\rho<0$. Rearranging we get that $\text{Re}\,(\zeta_1-z_1)\leq \rho-\tilde{P}_{2k}\lesssim \rho-|\zeta_2-z_2|^{2k}$. Now this means that we can use the local defining function in our expression because

$$|\text{Re}\,(\zeta_1-z_1)-A|\zeta_2-z_2|^{2k}|\geq|\rho-(c+A)|\zeta_2-z_2|^{2k}|.$$

and as $\rho<0$ we have

$$|\rho-(A+c)|\zeta_2-z_2|^{2k}|=\left|-1(|\rho|+(A+c)|\zeta_2-z_2|^{2k})\right|\sim|\rho|+|\zeta_2-z_2|^{2k}.$$

What we seek to compute is then

$$\int_{\partial\Omega\cap\{||\zeta-z||<\alpha\}} \frac{1}{||\zeta-z||\big(|\rho|+|\zeta_2-z_2|^{2k}\big)^2} dS(\zeta).$$

We use the local coordinates form Lemma 3.6.5. We choose an R, dependent on α and Ω, such that

$$\partial\Omega\cap\{||\zeta-z||<\alpha\}\subseteq\{x_1=|\rho(z)|,|x_2|,|x_3|,|x_4|\leq R\}$$
$$\left|\left|dS(\zeta)\big|_{\partial\Omega\cap\{||\zeta-z||<\alpha\}}\right|\right|\lesssim||dx_2\wedge dx_3\wedge dx_4||.$$

This means we can bound the integral in terms of the new local coordinate system and the defining function for Ω and the inequality

$$\int_{\partial\Omega\cap\{||\zeta-z||<\alpha\}} \frac{1}{||\zeta-z||\big(|\rho|+|\zeta_2-z_2|^{2k}\big)^2}\,dS(\zeta)$$
$$\lesssim \int_{0\le x_2,x_3,x_4\le R} \frac{1}{(|\rho|+A(x_3^2+x_4^2)^k+|x_2|)^2\sqrt{x_3^2+x_4^2}}\,dx_2dx_3dx_4.$$

holds. Integrating up x_2 gives us

$$\int_{0\le x_3,x_4\le R}\Big(\frac{1}{(|\rho|+A(x_3^2+x_4^2)^k)}-\frac{1}{(|\rho|+A(x_3^2+x_4^2)^k+R)}\Big)\frac{1}{\sqrt{x_3^2+x_4^2}}\,dx_3dx_4$$
$$\le \int\Big(\frac{1}{|\rho|+A(x_3^2+x_4^2)^k}\Big)\frac{1}{\sqrt{x_3^2+x_4^2}}\,dx_3dx_4.$$

Introducing polar coordinates for x_3 and x_4 we can further increase the bound and thus we need to consider

$$\int \frac{1}{|\rho|+Ar^{2k}}\,dr.$$

Using a substitution $r=|\rho|^{\frac{1}{2k}}s$ we get

$$\frac{1}{|\rho|^{1-\frac{1}{2k}}}\int_0^{R|\rho|^{-1/(2k)}}\frac{1}{1+s^{2k}}\,ds$$

and since the integral is finite we have the following bound

$$\int_{\partial\Omega\cap\{||\zeta-z||<\alpha\}}\frac{1}{||\zeta-z||\,|\Phi|^2}\,dS(\zeta)\lesssim|\rho(z)|^{\frac{1}{2k}-1}.$$

Because $|\rho|\sim \mathrm{dist}(z,\partial\Omega)$ we see that

$$\int_{\partial\Omega\cap\{||\zeta-z||<\alpha\}}\frac{1}{||\zeta-z||\,|\Phi|^{2+\eta}}\,dS(\zeta)\lesssim \mathrm{dist}(z,\partial\Omega)^{\frac{1}{2k}-\eta-1}$$

and Lemma 3.6.4 holds. □

Now proving that $B_\Omega^{(\varepsilon)}$ satisfies Hölder estimates only requires us to apply our Lemmas and proving Theorem 3.6.1 is just combing the estimates for $B_\Omega^{(\varepsilon)}$ and $I_\Omega^{(\varepsilon)}$.

Proof of Theorem 3.6.1. We have seen that we have the inequality

$$|d(B_\Omega^{(\varepsilon)} f)(z)| \leq ||f||_\infty \int_{\partial\Omega} \left| d_z \left(\frac{\overline{\zeta}_2 - \overline{z}_2}{||\zeta - z||^2} \widehat{h}_1 \right) \right| + \left| d_z \left(\frac{\overline{\zeta}_1 - \overline{z}_1}{||\zeta - z||^2} \widehat{h}_2 \right) \right| dS(\zeta).$$

Applying Lemmas 3.6.3 and 3.6.4 we further see that

$$|d(B_\Omega^{(\varepsilon)} f)(z)|| \leq ||f||_\infty C_{\Omega,\eta} \mathrm{dist}(z, \partial\Omega)^{\frac{1}{2k} - \eta - 1}$$

and applying Lemma 3.6.2 and we get that

$$|(B_\Omega^{(\varepsilon)} f)(\xi_1) - (B_\Omega^{(\varepsilon)} f)(\xi_2)| \leq ||f||_\infty C_{\Omega,\eta} |\xi_1 - \xi_2|^{\frac{1}{2k} - \eta}.$$

Now Theorem 3.6.1 follow from (3.27) and (3.28) since

$$|(B_\Omega - I_\Omega)(f)(\xi_1) - (B_\Omega - I_\Omega)(f)(\xi_2)| \leq ||f||_\infty C_{\Omega,\eta} |\xi_1 - \xi_2|^{\frac{1}{2k} - \eta}.$$

□

Chapter 4

The Main Result

This chapter will prove the Main Theorem by extending the integral kernel to entire domain Ω.

4.1 Proof of Main Theorem

We will now extend the solution operator from Theorem 3.6.1 to a solution operator which yields functions defined on Ω which also achieves Hölder estimates of order $1/(2k)-\eta$ and sup-norm estimates. Recall first the Main Theorem.

Main Theorem. *Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex domain with real analytic boundary of D'Angelo finite type $2k$ and let f be a $\bar\partial$-closed (0,1)-form on $\overline{\Omega}$. Then there exists a solution u of $\bar\partial u = f$ on Ω such that*

$$||u||_\infty \leq C_\Omega ||f||_\infty$$

where C_Ω is independent of f. Furthermore for every $\eta > 0$ there is a solution $u^{(\eta)}$ as above that satisfy $(\frac{1}{2k}-\eta)$-Hölder estimates with constant only depending on Ω and η.

We will construct a sequence of functions u_k which will be solutions to $\bar\partial u_k = f$ on each $\Omega_{-\varepsilon_k}$ and show that we can obtain u as a limit of a convergent sequence. We will create a family of functions and find the sequence from there. We will use the Arzela-Ascoli theorem to then show convergence.

Theorem 4.1.1 (Arzela-Ascoli Theorem)**.** *If K is a compact metric space and $\{f_i\}$ is a sequence of complex valued functions which is uniformly bounded and uniformly equicontinuous on K, then $\{f_i\}$ has a uniformly convergent subsequence.*

We also want to define what it means to exhaust Ω by compacts.

Definition 4.1.2. We say a topological space M can be exhausted by compacts if there is a nested sequence of compact sets $K_1 \subseteq K_2 \subseteq \ldots$ such that $K_i \subseteq \text{int}(K_{i+1})$ for each i and $M = \bigcup_{i=1}^{\infty} K_i$.

We need to define what it means to be *equicontinuous*.

Definition 4.1.3. Let $\Gamma = \{f_i : \Omega \longrightarrow \mathbb{C}\}_{\in I}$ be a family of functions. We say that the family is uniformly equicontinuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that when $z, w \in \Omega$ with $|z - w| < \delta$, $|f(z) - f(w)| < \varepsilon$ for all $f \in \Gamma$.

First we show that solutions arising from $H_{\Omega}^{(\varepsilon)}$ equicontinuous. Let $v > 0$, then there is a δ and $\xi_1, \xi_2 \in \Omega_{-\varepsilon}$ so that when $|\xi_1 - \xi_2| < \delta$, then

$$|u_m^{(\eta)}(\xi_1) - u_m^{(\eta)}(\xi_2)| \leq C_{\Omega,\eta}||f||_{\infty}|\xi_1 - \xi_2|^{\frac{1}{2k}-\eta} < v$$

for all $u_m^{(\eta)} \in \{u_k^{(\eta)} | k \geq 1\}$ by choosing $\delta = \frac{v^{\eta - \frac{1}{2k}}}{||f||_{\infty} C_{\Omega,\eta}}$.

Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a sequence of positive numbers so that $\varepsilon_{k+1} < \varepsilon_k$. Then as $k \longrightarrow \infty$ we have that $\varepsilon_k \longrightarrow 0$. Let further f be $\bar{\partial}$-closed (0,1)-form on Ω and set $u_k^{(\eta)} = H_{\Omega}^{(\varepsilon_k)} f$ which solves $\bar{\partial} u_k^{(\eta)} = f$ on Ω_{ε_k}. The family of functions $\{u_k^{(\eta)} : \Omega_{\varepsilon_k} \longrightarrow \mathbb{C}\}$ is uniformly bounded and uniformly equicontinuous functions on Ω_{ε_k}.

Further let $\{\overline{\Omega}_{\varepsilon_k}\}_{k=1}^{\infty}$ be sequence of compact domains dependent on the sequence $\{\varepsilon_k\}_{k=1}^{\infty}$. This sequence is nested since $\varepsilon_{k+1} < \varepsilon_k$ and also $\overline{\Omega}_{\varepsilon_i} \subseteq \Omega_{\varepsilon_{i+1}}$. The family of functions $\{u_k | k \geq m\}$ is normal on $\overline{\Omega}_{\varepsilon_m}$ from the Arzela-Ascoli Theorem. We therefore can find a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ which converges uniformly to a function u on $\overline{\Omega}_{\varepsilon_{k_j}}$.

For an arbitrary compact $K \subset \Omega$ since $\{\Omega_{\varepsilon_k}\}$ is a nested open cover of Ω, some Ω_{ε_m} must contain K. This is because the sequence exhaust Ω. Then since $u_m^{(\eta)}$ has a convergent subsequence in Ω_{ε_m}, the same subsequence converges in K. This then implies that there exists a a $u^{(\eta)}$ on Ω which then satisfies $\bar{\partial} u^{(\eta)} = f$.

On an arbitrary compact K, we can then find a subsequence converging to $u^{(\eta)}$. Then the limit u must also satisfy $\overline{\partial}u = f$ because of

$$|\overline{\partial}u - f| = |\overline{\partial}(u - u_{k_j}) + \overline{\partial}u_{k_j} - f| \leq |\overline{\partial}(u - u_{k_j})| + |\overline{\partial}u_{k_j} - f| \longrightarrow 0 + 0.$$

and

$$\begin{aligned} |u^{(\eta)}(z) - u^{(\eta)}(w)| &= |u^{(\eta)}(z) - u^{(\eta)}(w) + u_{k_j}^{(\eta)}(z) - u_{k_j}^{(\eta)}(z) + u_{k_j}^{(\eta)}(w) - u_{k_j}^{(\eta)}(w)| \\ &\leq |u^{(\eta)}(z) - u_{k_j}^{(\eta)}(z)| + |u_{k_j}^{(\eta)}(w) - u^{(\eta)}(w)| + |u_{k_j}^{(\eta)}(z) - u_{k_j}^{(\eta)}(w)| \\ &\longrightarrow 0 + 0 + C_{\Omega,\eta}||f||_\infty |z - w|^{\frac{1}{2k} - \eta} \end{aligned}$$

Sup-norm estimates now follow from the Hölder continuity. If the domain of a Hölder continuous function is a bounded subset of $\mathbb{R}^n$, then the function is also bounded.

Proposition 4.1.4. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. If a function $f : \Omega \longrightarrow \mathbb{C}$ satisfies*

$$|f(z) - f(w)| \leq c|z - w|^\alpha$$

for an $\alpha \in (0,1)$ and all $z, w \in \Omega$, then f is bounded.

Proof. Suppose f is unbounded. Then we can find a sequence $\{z_n\}_{n=1}^\infty \subset \Omega$ so that $|f(z_n)| > n$ for all n. As Ω is a bounded domain, the sequence is bounded. Then there exists a convergent subsequence $\{z_{n_k}\}_{k=1}^\infty \subset \{z_n\}_{n=1}^\infty$ satisfying $|f(z_{n_k})| > n_k$. Now since

$$|f(z_{n_j}) - f(z_{n_i})| \leq c|z_{n_j} - z_{n_i}|^\alpha$$

and $\{z_{n_k}\}_{k=1}^\infty$ is a Cauchy sequence we get that the sequence $\{f(z_{n_k})\}_{k=1}^\infty$ is a Cauchy sequence. Cauchy sequences are bounded, however since we supposed f unbounded we have that $|f(z_{n_k})| > n_k$ for all k. This is a contradiction. □

Applying this result to our solution $u^{(\eta)}$ we immediately see that

$$||u||_\infty \leq C||f||_\infty$$

and we have proved the main theorem.

Chapter 5

Final Remarks

5.1 Conclusion

We have proved that we can modify Henkins integral formula to become a solution operator for $\overline{\partial}$ on bounded weakly-pseudoconvex domains with real analytic boundary of finite type in $\mathbb{C}^2$. This operator satisfied Hölder estimates reflecting the type of the boundary. We did this by showing that for bounded domains of finite type in $\mathbb{C}^2$ we can always find a local bumping to type of the domain around any boundary point. Using then an appropriate choice of smooth solutions to the Cauchy-Fantappiè equation we used Hörmanders solution of $\overline{\partial}$ to modify them into holomorphic functions with weighted L^2 estimates also satisfying the Cauchy-Fantappiè equation. Selecting an appropriate weight and using Cauchy estimates we achieved pointwise estimates for h_1 and h_2.

On a slightly smaller domain $\Omega_{-\varepsilon}$ we replace the functions with functions where the local estimates are smooth on the boundary. Patching these together we obtained an integral formula which satisfied Hölder estimates which reflects the type at the boundary, with some small loss coming from the weighted estimates.

Further we extended the solution operator to the entire domain Ω using exhaustion of compacts, the Arzela-Ascoli and a normal families argument.

5.2 Future Work

In this thesis we have shown the main theorem when we have a pseudoconvex domain of finite type in $\mathbb{C}^2$. Naturally one seeks solutions when domains lie in $\mathbb{C}^n$ with $n \geq 3$. The challenge lies in the geometric behavior of the boundary. This is easier for domains lying $\mathbb{C}^2$ as the only complex curves which are tangent to the boundary is $\mathbb{C}$. Moving up in dimension there exists possibilities for curves touching to much higher order, meaning the type might change depending on direction. Finding a bumping to type of the domain and a corresponding support function is then a central question. For similar domains in $\mathbb{C}^3$ the paper this thesis is based on [5] proves the same result.

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