Arta Hamiti

Triangulated Derivators

Master's thesis in Mathematical Sciences
Supervisor: Steffen Oppermann
June 2022

Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
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Kunnskap for en bedre verden

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Abstract

In this thesis, we define Kan extensions and introduce the pointwise construction in order to define derivators in an appropriate context. By developing the theory of pointed and stable derivators, we prove that stable derivators induce additive categories. We establish the functors needed to construct the auto-equivalence and class of triangles in the main result, which states that stable and strong derivators give rise to triangulated categories. This provides a replacement for the well-known flaw of triangulated categories, namely the non-functorial cone construction. Lastly, the related functors between the induced triangulated categories are proven to be exact functors.

Sammendrag

I denne oppgaven definerer vi Kan-utvidelser og introduserer den punktvise konstruksjonen som lar oss definere derivatorer i en passende kontekst. Ved å utvikle teorien om punktede og stabile derivatorer, beviser vi at stabile derivatorer induserer additive kategorier. Vi etablerer funktorene som er nødvendige for å konstruere auto-ekvivalensen og klassen av trekanter i hovedresultatet, som sier at stabile og sterke derivatorer gir opphav til triangulerte kategorier. Dette gir en erstatning for den velkjente ulempen ved triangulerte kategorier, nemlig den ikke-funktoriale kjeglekonstruksjonen. Til slutt viser vi at de relaterte funktorene mellom de induserte triangulerte kategoriene er eksakte funktorer.

Contents

Introduction

The aim of this thesis is to introduce derivators, prove that they induce triangulated categories and understand how that solves, to some extent, the issue regarding the non-functorial cone construction.

The triangulated category was introduced by Jean-Louis Verdier [Ver96], based on ideas of Alexander Grothendieck, in order to axiomatise the extra structure of the derived category $\mathbf{D}(\mathcal{A})$, for an abelian category $\mathcal{A}$. Both for triangulated categories in general, and the derived category in particular, the respective cone constructions $\mathcal{T}^{[1]} \xrightarrow{Cone} \mathcal{T}$ and $\mathbf{D}(\mathcal{A})^{[1]} \xrightarrow{Cone} \mathbf{D}(\mathcal{A})$, are not functorial. However, for the derived category there is a functor $\mathbf{D}(\mathcal{A}^{[1]}) \to \mathbf{D}(\mathcal{A})$. Utilising this approach is the idea behind a derivator.

Although similar concepts were studied and introduced independently by several others, it was Grothendieck who first introduced the notion of a derivator [Gro]. A derivator can be thought of as a well-behaved 2-functor $\mathcal{C}at^{op} \to \mathcal{CAT}$ with some requirements, one being that the induced functors admit adjoints. These adjoints are inspired by Kan extensions, a notion initiated by Daniel M. Kan [KT76].

The main result of this thesis is theorem 6.9, stating that for some derivator $\mathbb{D}$ and small category $\mathcal{X}$, the category $\mathbb{D}(\mathcal{X})$ is triangulated. In proving this, we construct the functor $\mathbb{D}([1]) \to \mathbb{D}(\mathbb{1})$, providing the main ingredient of the solution to the non-functorial cone construction. This functor is exactly $\mathbf{D}(\mathcal{A}^{[1]}) \to \mathbf{D}(\mathcal{A})$ for the *derivator of* $\mathcal{A}$, denoted $\mathbb{D}_{\mathcal{A}}$.

In regards to sources, the author found the paper by Moritz Rahn (previously Moritz Groth) [Gro13] the most useful and it is therefore the primary source for the chapters 3-6. Most of chapter 1 is regarded as well-known homological algebra, see for example [Opp16], and [KS74] for the definition of a 2-category and 2-functor, specifically. The definitions in chapter 2 are based on Saunders Mac Lane [Lan71], see also [Rie14] for additional details. Simultaneously, a previous thesis on derivators [Bra21] proved a valuable source regarding details, and as a result of this, several proofs follow a similar structure. Following is an outline of how the thesis builds up the theory resulting in the main theorem.

The first chapter is meant as a gentle and particular introduction to the category theory necessary for the rest of the thesis. In addition to the more familiar definitions of categories, functors and natural transformations, there are also some concepts which might be new to some, such as slice categories, 2-categories and 2-functors.

In the second chapter we define Kan extensions, which at first sight are an abstract way of extending a functor F along a functor G. Section 2.1 provides another way of thinking of Kan extensions by showing that they generalise adjoints, and more importantly, that they generalise (co)limits. Kan extensions can also be calculated pointwise, which is done in section 2.2, and results in proposition 2.7, preparing us for the prime example of a derivator, namely the represented derivator $\mathbb{D}_{rep}$.

In the third chapter we introduce Beck-Chevalley transformations, leading to the definition of a derivator. Then we establish the opposite derivator and show some useful properties of derivators, before proving in theorem 3.19 that they give rise to categories which admit products and coproducts. The chapter ends with the definition of a shifted derivator and proposition 3.22 assuring us that this is a reasonable definition.

The fourth chapter is concerned with pointed derivators, that is, when the underlying category admits a zero object. In section 4.1 we prove lemma 4.9, a result often thought about as an extension by zero. This leads to the very important pairs of adjoints in section 4.2, namely the cone and fiber functor, and the suspension and loop functor. We see how these generate cartesian and cocartesian squares, and in section 4.3 we prove corollary 4.22, allowing us to calculate the (co)cartesian squares pointwise.

In the fifth chapter we define stable derivators and unravel the consequences of when cartesian and cocartesian squares coincide. We prove lemma 5.6, which emphasises the nuances in generating (co)cartesian squares and leads to the related two out of three property given by corollary 5.8. We also characterise stable derivators via certain squares in proposition 5.10, before proving that stable derivators induce pre-additive categories. With reference to the mentioned main source of this thesis, we state that these categories indeed are additive.

In the sixth chapter we introduce the triangulated derivator, through the notion of a strong derivator. Finally, we prove theorem 6.9, and discuss the consequences regarding the related cone constructions. The chapter ends with proposition 6.11, stating that the induced functors G^*, G^L and G^R are exact.

Notation

Categories	$\mathcal{X}, \mathcal{Y}$, etc.
Objects	a, b, c, A, B, etc.
Morphisms	f, g, h, id, etc.
Set of morphisms from a to b	$\mathcal{H}om(a, b)$
Functors	F, G, H, Id, etc.
Natural transformations	α, β, γ, etc.
Terminal category	$\mathbb{1}$
Limit of F	$limF$
Colimit of F	$colimF$
Slice category for functor G and object $y \in \mathcal{Y}$	$[G \to y]$
Vertical 2-cell composition	$\beta\|\alpha$
Horizontal 2-cell composition	$\beta\alpha$
Category of all small categories	$\mathcal{C}at$
Category of all categories	$\mathcal{C}AT$
Left Kan extension functor of F along G	LK_GF
Right Kan extension functor of F along G	RK_GF
Pre-derivator and derivator	$\mathbb{D}$
Induced restriction functor $\mathbb{D}(G)$	G^*
Right adjoint to G^*	G^R
Left adjoint to G^*	G^L
The derivator of $\mathcal{A}$	$\mathbb{D}_{\mathcal{A}}$
Category of chain complexes	$\mathbf{C}(\mathcal{A})$
The derived category	$\mathbf{D}(\mathcal{A})$
The represented derivator	$\mathbb{D}_{rep}$
The opposite derivator	$\mathbb{D}^{op}$
The diagonal functor	$\Delta_{\mathcal{K}}$
The shifted derivator	$\mathbb{D}^{\mathcal{K}}$
The arrow category	$[1]$
Subcategories of $[1] \times [1] = \square$	$\ulcorner, \lrcorner$
Suspension functor	Σ
Loop functor	Ω

1 Category theory

The first chapter begins with the definition of a category, and ends with the definition of a 2-functor, the preamble of a pre-derivator. Most of the definitions in between are from homological algebra, and might be familiar to the experienced reader. The examples and observations provided in this chapter are chosen such that they hopefully clear any confusion that might arise in the rest of the thesis.

1.1 Categories, functors and natural transformations

Definition 1.1. A *category* $\mathcal{C}$ consists of

i) a class of *objects*, denoted obj$(\mathcal{C})$

ii) for any two objects a, b, a set of *morphisms*, $\mathcal{H}om_{\mathcal{C}}(a, b)$

iii) for any three objects a, b, c, a *composition map*

$$\circ : \mathcal{H}om_{\mathcal{C}}(a, b) \times \mathcal{H}om_{\mathcal{C}}(b, c) \to \mathcal{H}om_{\mathcal{C}}(a, c), \quad (f, g) \mapsto g \circ f$$

which is associative. In other words, for any objects a, b, c, d, and any morphisms $f \in \mathcal{H}om_{\mathcal{C}}(a, b)$, $g \in \mathcal{H}om_{\mathcal{C}}(b, c), h \in \mathcal{H}om_{\mathcal{C}}(c, d)$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

iv) for any object a, a morphism called the *identity morphism* id_a in $\mathcal{H}om_{\mathcal{C}}(a, a)$ such that for $f \in \mathcal{H}om_{\mathcal{C}}(a, b)$ and $g \in \mathcal{H}om_{\mathcal{C}}(b, a)$ we have

$$f \circ id_a = f$$

$$id_a \circ g = g$$

Example 1.2. The *terminal category*, denoted $\mathbb{1}$, is defined as the category with a single object, and is of course a category.

Example 1.3. Any poset forms a category. For two elements p_1, p_2 in a poset $\mathcal{P}$ there is a morphism $p_1 \to p_2$ whenever there is a relation $p_1 \leq p_2$.

Example 1.4. Any set $\mathcal{S}$ forms a *discrete category* if one allows identity morphisms for each element in $\mathcal{S}$.

Definition 1.5. Let $\mathcal{X}$ be a category.

i) An *initial object* $i \in \mathcal{X}$ is an object such that for every other object $x \in \mathcal{X}$ there exists exactly one morphism $i \to x$.

i) An *terminal object* $t \in \mathcal{X}$ is an object such that for every other object $x \in \mathcal{X}$ there exists exactly one morphism $x \to t$.

Definition 1.6. A *small category* is a category where the class of objects forms a set.

The terminal category has an object which is both initial and terminal. Such an objects is a *zero object*. A poset category admits an initial object if it has a "smallest" element, and likewise for a terminal object. A discrete category has no initial or terminal object. All three of the mentioned categories are small.

Definition 1.7. Let $\mathcal{X}, \mathcal{Y}$ be two categories. Then a *functor* F from $\mathcal{X}$ to $\mathcal{Y}$ consists of two maps

$$F : \text{obj}(\mathcal{X}) \to \text{obj}(\mathcal{Y})$$

$$F : \mathcal{H}om_{\mathcal{X}}(x_1, x_2) \to \mathcal{H}om_{\mathcal{Y}}(F(x_1), F(x_2))$$

satisfying the following properties.

i) for all $x \in \mathcal{X}$ we have $F(id_x) = id_{F(x)}$

ii) for any two composable morphisms $f, g \in \mathcal{C}$ we have $F(gf) = F(g) \circ F(f)$

It is often useful to think of diagrams in categories as functors. For example, consider a diagram consisting of two objects, say a and b, in some category $\mathcal{C}$. If we let $\mathcal{P}$ be a category consisting of only two objects, then the diagram $\{a \quad b\}$ can be identified with a functor $\mathcal{P} \xrightarrow{F} \mathcal{C}$, given by sending one of the objects in $\mathcal{P}$ to a and the other to b.

Definition 1.8. Let $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ be a functor and x_1, x_2 objects in $\mathcal{X}$.

i) F is *full* if $\mathcal{H}om_{\mathcal{X}}(x_1, x_2) \xrightarrow{F} \mathcal{H}om_{\mathcal{Y}}(F(x_1), F(x_2))$ is surjective.

ii) F is *faithful* if $\mathcal{H}om_{\mathcal{X}}(x_1, x_2) \xrightarrow{F} \mathcal{H}om_{\mathcal{Y}}(F(x_1), F(x_2))$ is injective.

iii) F is *essentially surjective* if for every $y \in \mathcal{Y}$ there exists a $x \in \mathcal{X}$ such that $F(x) \cong y$.

iv) The *essential image* of F, denoted $essIm(F)$, is all objects $y \in \mathcal{Y}$ such that $F(x) \cong y$ for some $x \in \mathcal{X}$.

Functors F such that $\mathcal{H}om_{\mathcal{X}}(x_1, x_2) \xrightarrow{F} \mathcal{H}om_{\mathcal{Y}}(F(x_1), F(x_2))$ is a bijection are called *fully faithful*.

Definition 1.9. Let $F, G : \mathcal{X} \to \mathcal{Y}$ be functors between two categories. A *natural transformation* $\alpha : F \to G$ consists of a morphism $\alpha_x : F(x) \to G(x)$ for all $x \in \mathcal{X}$ such that the following diagram commutes for $f : x_1 \to x_2$.

$$\begin{array}{ccc} F(x_1) & \xrightarrow{F(f)} & F(x_2) \\ {\scriptstyle \alpha_{x_1}}\downarrow & & \downarrow{\scriptstyle \alpha_{x_2}} \\ G(x_1) & \xrightarrow[G(f)]{} & G(x_2) \end{array}$$

It will be useful in many proofs to consider natural transformations that are the result of a natural transformation composed with a functor. For example, given $G \xrightarrow{\alpha} H$, we have a natural transformation αF from the functor GF to the functor HF.

$$\mathcal{X} \xrightarrow{F} \mathcal{Y} \underset{H}{\overset{G}{\underset{\alpha \Downarrow}{\rightrightarrows}}} \mathcal{Z}$$

This is also the same as using α only on objects in $\mathcal{Y}$ that are in the image of F.

Definition 1.10. Let $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ be a functor and $x \in \mathcal{X}$. The *identity transformation* on F, denoted Id_F, is defined by $(Id_F)_x = id_{F(x)}$.

1.2 Adjoints, limits and 2-categories

Definition 1.11. There are two equivalent ways of defining an *adjoint pair of functors*, $(F, G) : \mathcal{X} \rightleftarrows \mathcal{Y}$. Both definitions are useful, and in both cases F is the left adjoint while is G the right adjoint.

i) there are two natural transformations called the *counit* and *unit*, respectively,

$$\epsilon : FG \to Id$$

$$\eta : Id \to GF$$

such that the natural transformations

$$F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F$$

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G$$

compose to the identity transformations. These are often referred to as the *triangle identities*.

ii) there is an isomorphism

$$\mathcal{H}om_{\mathcal{Y}}(F(x), y) \cong \mathcal{H}om_{\mathcal{X}}(x, G(y))$$

which is natural for all objects $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

Following are some useful results which will be used multiple times throughout the thesis.

Observation 1.12. *Let $(F, G) : \mathcal{X} \rightleftarrows \mathcal{Y}$ be an adjoint pair of functors. Then F preserves initial objects and G preserves terminal objects.*

Proof. We prove the first claim, the second is dual. Let $i \in \mathcal{X}$ be an initial object. By definition there exists only one morphism in the set $\mathcal{H}om_{\mathcal{X}}(i, x)$ for all $x \in \mathcal{X}$, and in particular for $G(y)$ given some $y \in \mathcal{Y}$. And by definition of adjoint pair we have that

$$\mathcal{H}om_{\mathcal{X}}(i, G(y)) \cong \mathcal{H}om_{\mathcal{Y}}(F(i), y)$$

implying that $F(i)$ is an initial object in $\mathcal{Y}$. □

Observation 1.13. *Let $\mathbb{1} \xrightarrow{t} \mathcal{X}$ be the functor identifying the terminal object $t \in \mathcal{X}$, and $\mathcal{X} \xrightarrow{\pi} \mathbb{1}$ be the canonical projection functor. Then (π, t) is an adjoint pair.*

Proof. Let $\bullet$ be the only object in $\mathbb{1}$, and x some object in $\mathcal{X}$. Then both $\mathcal{H}om_{\mathbb{1}}(\pi(x), \bullet)$ and $\mathcal{H}om_{\mathcal{X}}(x, t(\bullet))$ consist of a single element due to the fact that t is terminal. Hence

$$\mathcal{H}om_{\mathbb{1}}(\pi(x), \bullet) \cong \mathcal{H}om_{\mathcal{X}}(x, t(\bullet))$$

□

Lemma 1.14. *Let $(F, G) : \mathcal{X} \rightleftarrows \mathcal{Y}$ be an adjoint pair of functors. Then*

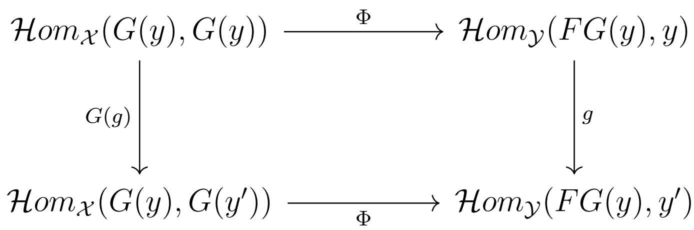
i) F is fully faithful if and only if the unit $Id_{\mathcal{X}} \xrightarrow{\eta} GF$ is an isomorphism.

ii) G is fully faithful if and only if the counit $FG \xrightarrow{\epsilon} Id_{\mathcal{Y}}$ is an isomorphism.

Proof. We prove ii). We have the following composition where Φ is the natural isomorphism related to the adjoint pair.

$$\mathcal{H}om_{\mathcal{Y}}(y, y') \xrightarrow{G} \mathcal{H}om_{\mathcal{X}}(G(y), G(y')) \xrightarrow{\Phi} \mathcal{H}om_{\mathcal{Y}}(FG(y), y')$$

In order to find out what this composition does on a map $y \xrightarrow{g} y'$, we use the naturality of Φ.



The square commutes, hence $\Phi(G(g)(id_{G(y)})) = g(\Phi(id_{G(y)})$ and since $\Phi(id_{G(y)}) = \epsilon_y$ we get that $\Phi(G(g)) = g\epsilon_y$. Now it is possible to see that G is fully faithful if and only if $g\epsilon_y$ is an isomorphism if and only if the counit ϵ is a natural isomorphism.

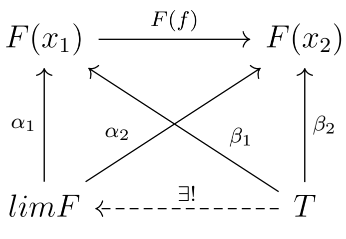
$\square$

In the next chapter we will see how limits and colimits form a bridge between the more concrete concepts such as a product and the relatively abstract definition of a Kan extension. The definition of a slice category will be important in creating that bridge.

Definition 1.15. For a functor $F : \mathcal{X} \to \mathcal{C}$, the *limit of* F, denoted $limF$, is an object in $\mathcal{C}$ together with morphisms $limF \xrightarrow{\alpha} F(x)$ for $x \in \mathcal{X}$ such that

i) given $x_1 \xrightarrow{f} x_2 \in \mathcal{X}$ we have that $F(f) \circ \alpha_1 = \alpha_2$

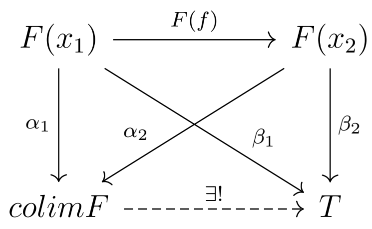
ii) for any other object $T \in \mathcal{C}$ with morphisms $T \xrightarrow{\beta} F(x)$ which also satisfies the previous criterion, there exists a unique map $\gamma : T \to limF$ s.t. $\alpha \circ \gamma = \beta$



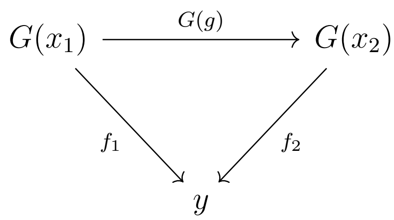
Definition 1.16. For a functor $F : \mathcal{X} \to \mathcal{C}$, the *colimit of* F, denoted $colimF$, is an object in $\mathcal{C}$ together with morphisms $F(x) \xrightarrow{\alpha} colimF$ for $x \in \mathcal{X}$ such that

i) given $x_1 \xrightarrow{f} x_2 \in \mathcal{X}$ we have that $\alpha_2 \circ F(f) = \alpha_1$

ii) for any other object $T \in \mathcal{C}$ with morphisms $F(x) \xrightarrow{\beta} T$ which also satisfies the previous criterion, there exists a unique map $\gamma : colimF \to T$ s.t. $\gamma \circ \alpha = \beta$ for all $x \in \mathcal{X}$



Definition 1.17. For a functor $\mathcal{X} \xrightarrow{G} \mathcal{Y}$ and object $y \in \mathcal{Y}$, the *slice category*, denoted $[G \to y]$, has objects of the form (x, f) for $x \in \mathcal{X}$ and $G(x) \xrightarrow{f} y \in \mathcal{Y}$. A morphism between two objects (x_1, f_1) and (x_2, f_2) is a morphism $x_1 \xrightarrow{g} x_2$ in $\mathcal{X}$ such that the following diagram commutes in $\mathcal{Y}$.



Together with a slice category $[G \to y]$ comes a canonical projection functor

$$\rho_y : [G \to y] \to \mathcal{X}$$

sending (x, f) to x, and $(x_1, f_1) \xrightarrow{g} (x_2, f_2)$ to $x_1 \xrightarrow{g} x_2$. Using this functor is the same as forgetting the extra structure a slice category carries. Given $y_1 \xrightarrow{h} y_2 \in \mathcal{Y}$ there also exists a functor

$$[G \to y_1] \xrightarrow{[G \to h]} [G \to y_2]$$

which sends (x, f) to (x, hf), and $(x_1, f_1) \xrightarrow{g} (x_2, f_2)$ to $(x_1, hf_1) \xrightarrow{g} (x_2, hf_2)$. Finally comes the definition of a 2-category and 2-functor, central concepts in chapter 3.

Definition 1.18. A *2-category* $\mathbb{C}$ consists of

i) *0-cells* $\mathcal{A}, \mathcal{B}, \mathcal{C}$, etc.

ii) *1-cells* F, G, H, etc.

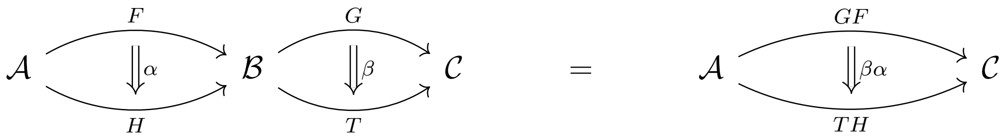
iii) *2-cells* α, β, γ, etc.

such that

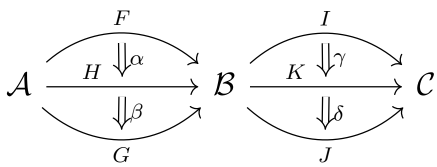
i) the 0-cells and 1-cells form a category $\mathbb{C}_0$ together with standard composition

ii) the 1-cells and 2-cells form a category $\mathbb{C}(\mathcal{A}, \mathcal{B})$ together with *vertical composition*

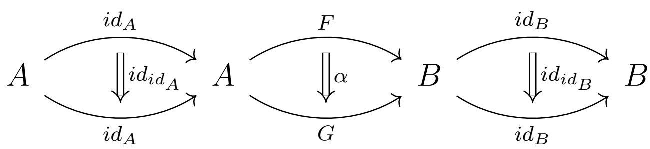
In addition we have *horizontal composition* of 2-cells



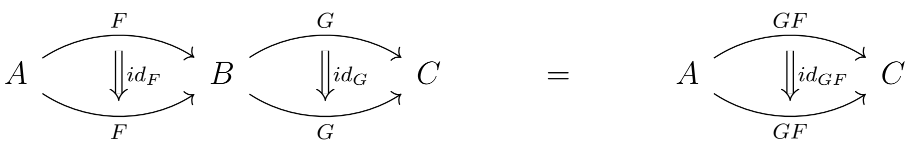
which is associative and s.t. $(\beta|\alpha)(\delta|\gamma) = (\gamma\alpha)|(\delta\beta)$ holds for the diagram



Lastly we need *horizontal identity*, i.e. for $id_{id_A} : id_A \to id_A$, $id_{id_B} : id_B \to id_B$ and $\alpha : F \to G$ we have $id_{id_A}\alpha = \alpha = \alpha id_{id_B}$ for the diagram



and the *2-identity* $id_G id_F = id_{GF}$



Definition 1.19. The category of all small categories is denoted $\mathcal{C}at$ and the category of (not necessarily small) categories is denoted $\mathcal{CAT}$.

Example 1.20. $\mathcal{C}at$ and $\mathcal{CAT}$ are both 2-categories. The 0-cells are categories, the 1-cells are functors and the 2-cells are natural transformations.

Definition 1.21. A *2-functor* $\mathbb{D}$ is a map between two 2-categories $\mathbb{X}$ and $\mathbb{Y}$ such that

i) $\mathbb{D} : \mathbb{X}_0 \to \mathbb{Y}_0$ is a functor

ii) $\mathbb{D} : \mathbb{X}(\mathcal{X}_1, \mathcal{X}_2) \to \mathbb{Y}(\mathcal{Y}_1, \mathcal{Y}_2)$ is a functor

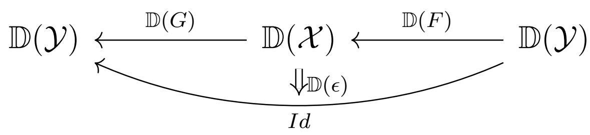
iii) given $\alpha : F \to H$ and $\beta : G \to T$ we require that $\mathbb{D}(\beta)\mathbb{D}(\alpha) = \mathbb{D}(\beta\alpha)$

iv) given $id_F : F \to F$ we require $\mathbb{D}(id_F) = id_{\mathbb{D}(F)}$

A 2-functor is a map between 2-categories that preserves all identities and compositions. Therefore, if a diagram commutes prior to applying a 2-functor, it will commute afterwards as well. Since a derivator will be defined as a 2-functor, this property will be utilised implicitly throughout. We end the chapter with another useful observation making use of this property.

Observation 1.22. *Let* $\mathbb{D} : \mathcal{C}at^{op} \to \mathcal{C}AT$ *be a 2-functor. If* $(F, G) : \mathcal{X} \rightleftarrows \mathcal{Y}$ *is an adjoint pair of functors, then* $(\mathbb{D}(G), \mathbb{D}(F)) : \mathbb{D}(\mathcal{X}) \rightleftarrows \mathbb{D}(\mathcal{Y})$ *is an adjoint pair of functors.*

Proof. By using the mentioned property of 2-functors and definition 1.11 ii), we see that the following now commutes



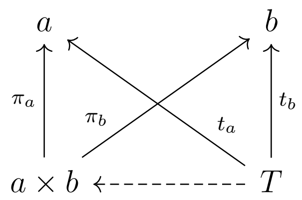
Likewise for $\mathbb{D}(\eta)$, and again by definition, we are done. □

2 Kan extensions

Saunders Mac Lane wrote that ”All concepts are Kan extensions” ([Lan71]). In section 2.1 we argue the same, first by generalising the concept of a product to that of a limit, then by introducing Kan extensions and showing how limits are right Kan extensions, tying all three concepts together. We end the section with an example regarding adjoints, again emphasising the quote. In section 2.2, we construct the pointwise Kan extension, giving us an example of a derivator for chapter 3.

2.1 All concepts are Kan extensions

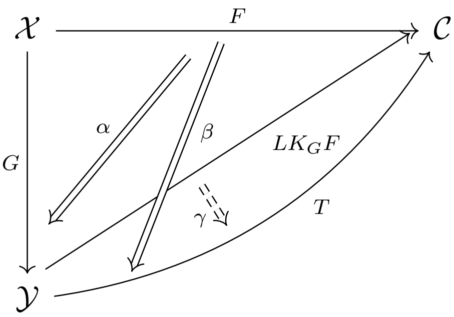
Assume we have two objects, a, b, in some category $\mathcal{C}$. The *product* of a and b is an object, $a \times b$, together with two morphisms $a \times b \xrightarrow{\pi_a} a$ and $a \times B \xrightarrow{\pi_b} b$, satisfying the universal property. That is, for any other object T and morphisms $(T \xrightarrow{t_a} a, T \xrightarrow{t_b} b)$ we get a unique map $T \to a \times b$ such that the following diagram commutes.

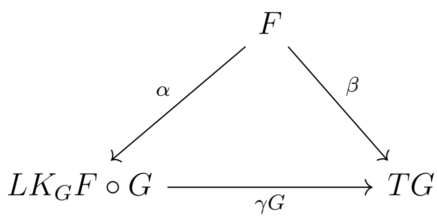


In order to generalise the concept of a product, we introduce a discrete category $\mathcal{P}$ consisting of two objects, 1 and 2. Let $\mathcal{P} \xrightarrow{F} \mathcal{C}$ be the functor sending 1 to a and 2 to b. Then, hopefully it is clear that the limit of F is $a \times b$. By changing the category $\mathcal{P}$, many similar concepts, such as pushouts and kernels, can be generalised and shown to be a limit.

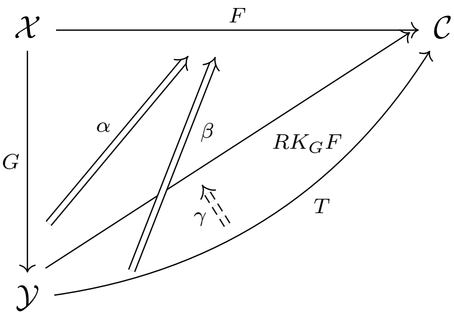
In a similar manner, the concept of a limit can be generalised and shown to be a right Kan extension. Dually, the colimit is a left Kan extension.

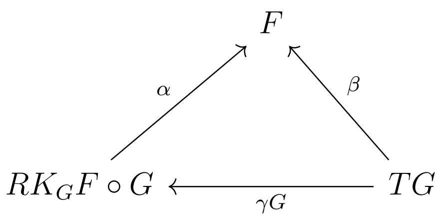
Definition 2.1. Given two functors $\mathcal{X} \xrightarrow{F} \mathcal{C}$ and $\mathcal{X} \xrightarrow{G} \mathcal{Y}$, a *left Kan extension* of F along G is a functor $\mathcal{Y} \xrightarrow{LK_GF} \mathcal{C}$ together with a natural transformation $F \xrightarrow{\alpha} LK_GF \circ G$ such that for any other functor and natural transformation ($\mathcal{Y} \xrightarrow{T} \mathcal{C}$, $F \xrightarrow{\beta} TG$) there exists a unique $LK_GF \xrightarrow{\gamma} T$ s.t. $\beta = G\gamma \circ \alpha$.



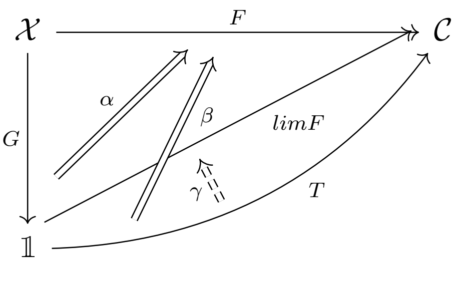


Definition 2.2. Given two functors $\mathcal{X} \xrightarrow{F} \mathcal{C}$ and $\mathcal{X} \xrightarrow{G} \mathcal{Y}$, a *right Kan extension* of F along G is a functor $\mathcal{Y} \xrightarrow{RK_GF} \mathcal{C}$ together with a natural transformation $RK_GF \circ G \xrightarrow{\alpha} F$ such that for any other functor and natural transformation ($\mathcal{Y} \xrightarrow{T} \mathcal{C}$, $TG \xrightarrow{\beta} F$) there exists a unique $T \xrightarrow{\gamma} RK_GF$ s.t. $\beta = \alpha \circ G\gamma$



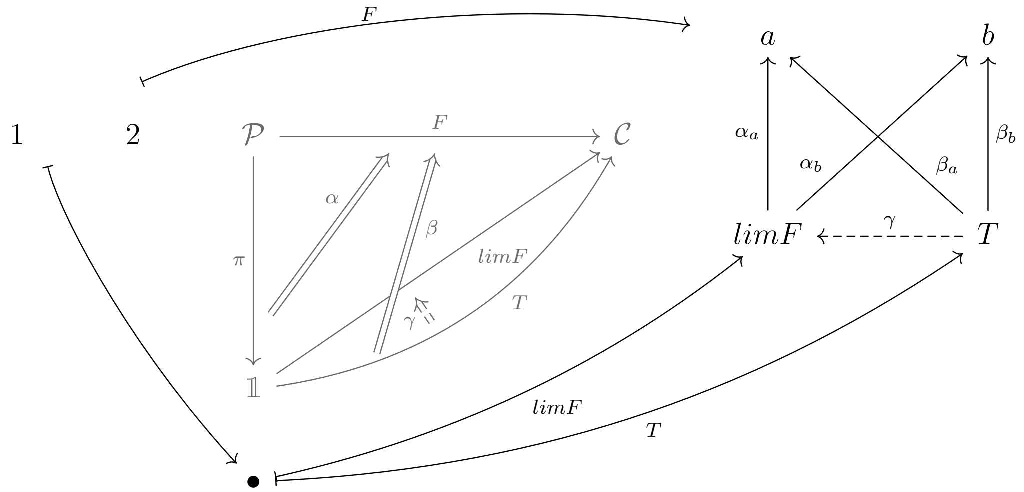


Let us see one perspective on how to think about Kan extensions. Given any functor $\mathcal{X} \xrightarrow{F} \mathcal{C}$ and the projection functor $\mathcal{X} \xrightarrow{\pi} \mathbb{1}$, the right Kan extension of F along π turns out to be the limit of F, i.e. $RK_\pi F = limF$.



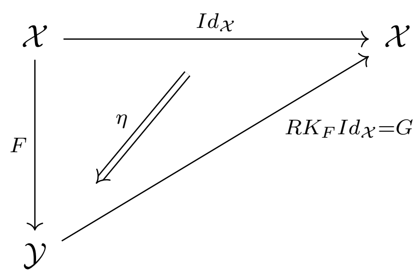
If we recall definition 1.15, we see that the notation matches up nicely. From the definition, i) is satisfied because α is a natural transformation, while ii) is satisfied by the universal property of the right Kan extension, namely the fact that $\beta = \alpha \circ G\gamma$.

If we try and tie all three concepts together, the product, limit and right Kan extension, the result might be slightly overwhelming or satisfying, depending on the reader. Since the limit is our middle ground, that is the notation used in the diagram. However, keep in mind that $RK_{\pi}F = limF = a \times b$ for our specific case.



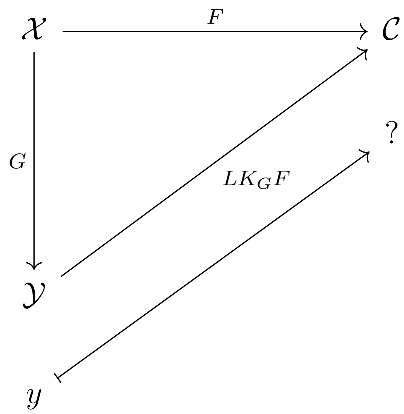
We end this section with another example of how an important construction actually is a Kan extension.

Example 2.3. Any functor $F : \mathcal{X} \to \mathcal{Y}$ has a right adjoint G, if and only if the right Kan extension of $Id_{\mathcal{X}}$ along F exists and it is preserved by any functor $H : \mathcal{X} \to \mathcal{Z}$ for some category $\mathcal{Z}$. What is meant by the latter criterion, is that $H \circ RK_F Id_{\mathcal{X}} = RK_F Id_{\mathcal{X}} \circ H$. In this case the right Kan extension is the right adjoint G and the related natural transformation is the unit. The natural transformation related to the left Kan extension is the counit.

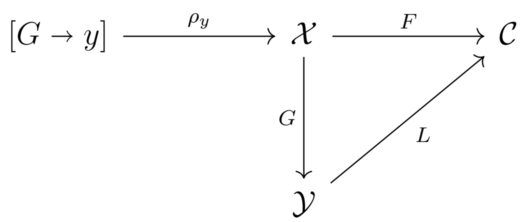


2.2 Pointwise Kan extensions

We want to know what the left Kan extension, LK_GF, does on objects. This is known as the pointwise construction of Kan extensions. In other words, given $y \in \mathcal{Y}$, what is $LK_GF(y) \in \mathcal{C}$?



It turns out that if we define any map L by sending y to a particular colimit, then it is indeed a functor and left Kan extension. This section is about proving this and we therefore start with the set up as follows.



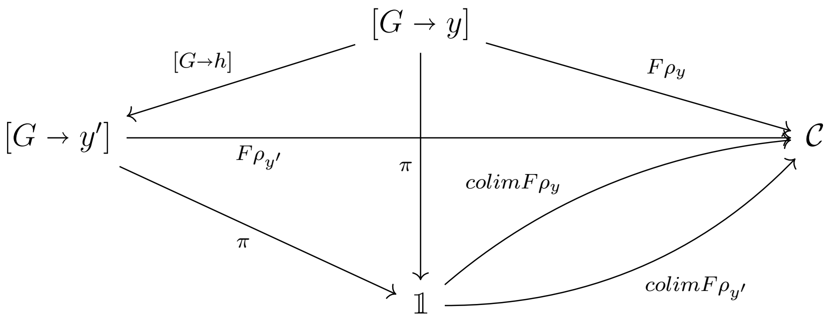
$[G \to y]$ is a slice category, ρ_y is the corresponding projection functor and L is given by sending any $y \in \mathcal{Y}$ to $colimF\rho_y$.

Lemma 2.4. *L, as described above, is a functor.*

Proof. We need to show that for $y \xrightarrow{h} y' \xrightarrow{h'} y''$ we have that $L(h'h) = L(h')L(h)$

$$colimF\rho_y \xrightarrow{L(h)} colimF\rho_{y'} \xrightarrow{L(h')} colimF\rho_{y''}$$

It suffices to show that $L(h)$ is unique, because then, by similar reasoning, $L(h')$ will also be unique implying $L(h'h)$ and $L(h')L(h)$ will both be the same unique map. In addition, if $L(h)$ is unique, it will be clear that $L(id_y) = id_{L(y)}$. In aid of this, consider the diagram

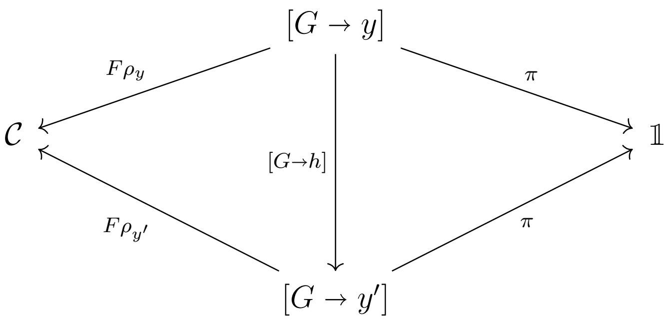


(2.1)

where $colimF\rho_y$ sends the only object in $\mathbb{1}$ to the object $colimF\rho_y$ in $\mathcal{C}$, and likewise for $colimF\rho_{y'}$. Hence, finding a unique map between the objects $colimF\rho_y$ and $colimF\rho_{y'}$ in $\mathcal{C}$ is the same as finding a unique natural transformation between the functors $colimF\rho_y$ and $colimF\rho_{y'}$.

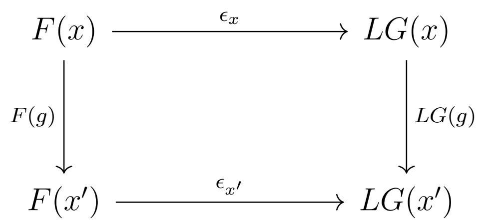
We know from earlier discussions that $colimF\rho_y$ is the left Kan extension of $F\rho_y$ along π. We have another functor from $\mathbb{1}$ to $\mathcal{C}$, namely $colimF\rho_{y'}$. If we can find a natural transformation from $F\rho_y$ to $colimF\rho_{y'} \circ \pi$ we can invoke the universal property of $colimF\rho_y$. We also know $colimF\rho_{y'}$ is a left Kan extension, and therefore have a natural transformation $colimF\rho_{y'} \circ \pi \xrightarrow{\alpha'} F\rho_{y'}$.

We see that $colimF\rho_{y'}\pi = colimF\rho_{y'}\pi[G \to h] \xrightarrow{\alpha'[G \to h]} F\rho_{y'}[G \to h] = F\rho_y$ using the two commuting triangles

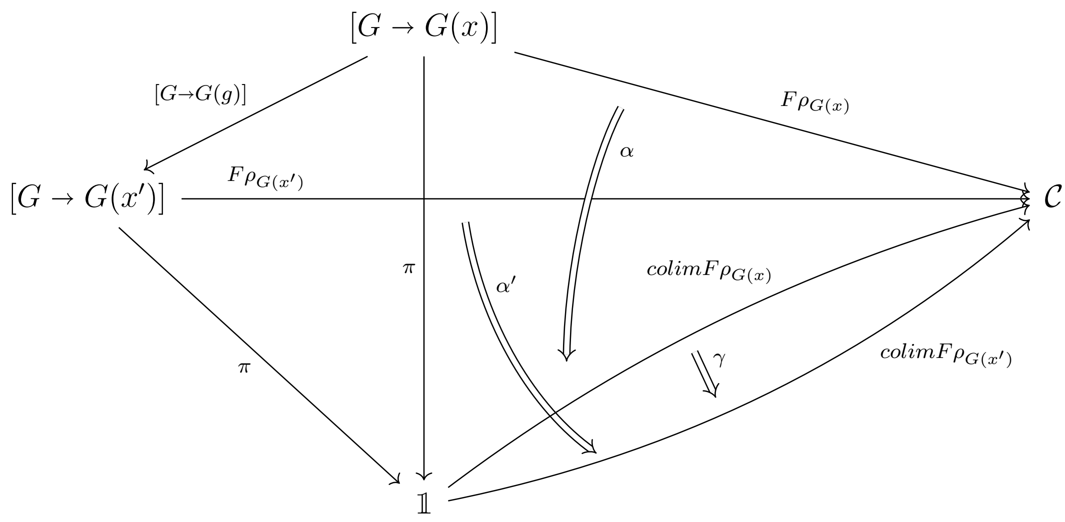


from diagram 2.1. Now, by the universal property of $colimF\rho_y$, there is a unique natural transformation $colimF\rho_y \xrightarrow{\gamma} colimF\rho_{y'}$ which is what was needed. □

Now that we know L is a functor, we want a natural transformation $\epsilon : F \to LG$ in order to show later that L indeed is a Kan extension. Our goal is, given $x \xrightarrow{g} x'$, to end up with a commuting diagram of the form

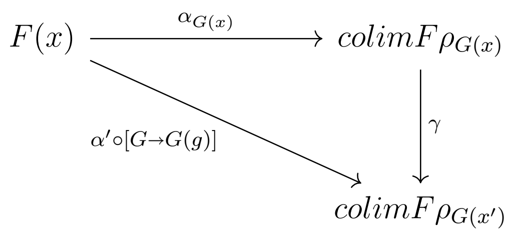


We adopt diagram 2.1 from the proof of lemma 2.4, and replace y and y' with $G(x)$ and $G(x')$. The transformations α and α' are the natural transformations associated with their respective Kan extensions, and γ is as in the previous proof.



Lemma 2.5. *There exists a natural transformation $\epsilon : F \to LG$.*

Proof. Using the three functors $F\rho_{G(x)}, colimF\rho_{G(x)} \circ \pi$ and $colimF\rho_{G(x')} \circ \pi$ on the object $(x, id_{(G(x))}) \in [G \to G(x)]$ we get the triangle



(2.2)

which commutes by definition of γ. Using the two functors $F\rho_{G(x')}$ and $colimF\rho_{G(x')} \circ \pi$

on the morphism $(x, G(g)) \xrightarrow{g} (x', id_{G(x')}) \in [G \to G(x')]$ we get the triangle

$$\begin{array}{ccc} F(x) & & \\ {\scriptstyle F(g)}\downarrow & \searrow^{\alpha'_{G(x)}} & \\ F(x') & \xrightarrow[\alpha'_{G(x')}]{} & colimF\rho_{G(x')} \end{array} \tag{2.3}$$

which commutes because α' is a natural transformation. Noticing that $\alpha' \circ [G \to G(g)]$ and $\alpha'_{G(x)}$ are the same morphism, we get a commuting square which is a combination of the triangles 2.2 and 2.3.

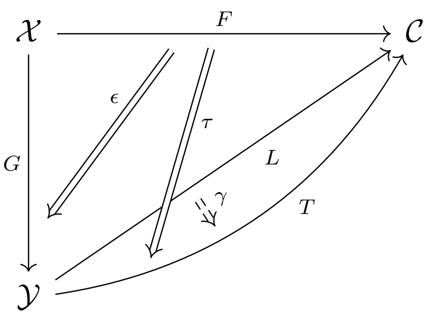
$$\begin{array}{ccc} F(x) & \xrightarrow{\alpha_{G(x)}} & colimF\rho_{G(x)} \\ {\scriptstyle F(g)}\downarrow & & \downarrow{\scriptstyle \gamma} \\ F(x') & \xrightarrow[\alpha'_{G(x')}]{} & colimF\rho_{G(x')} \end{array}$$

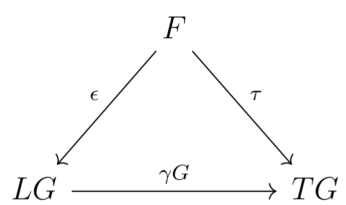
Remembering that $L(G(x)) = colimF\rho_{G(x)}$ by definition, completes the proof. $\square$

Now that we have a functor L together with a natural transformation ϵ, it remains to show that the pair is a Kan extension.

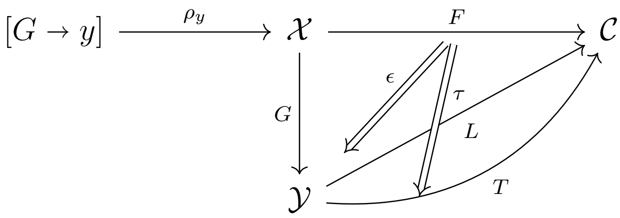
Theorem 2.6. (L, ϵ) *as constructed above is a left Kan extension.*

Proof. We assume $(\mathcal{Y} \xrightarrow{T} \mathcal{C}, F \xrightarrow{\tau} TG)$ is another pair of functor and natural transformation and need to show that there exists a unique $L \xrightarrow{\gamma} T$ such that the triangle on the right commutes.

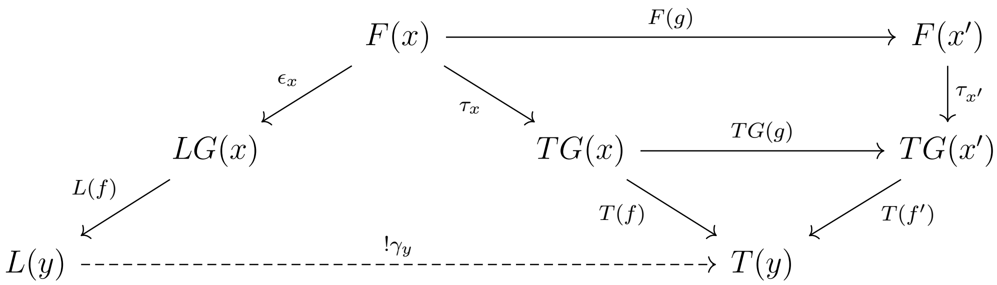




The fact that using L on objects is defined as a colimit, will be used in order to construct γ. Fixing some $y \in \mathcal{Y}$ results in the following diagram



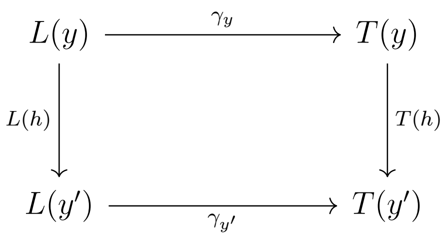
Using the different functors on the objects and morphism $(x, f) \xrightarrow{g} (x', f') \in [G \to y]$, gives us the following diagram, allowing us to use the universal property of $L(y)$.



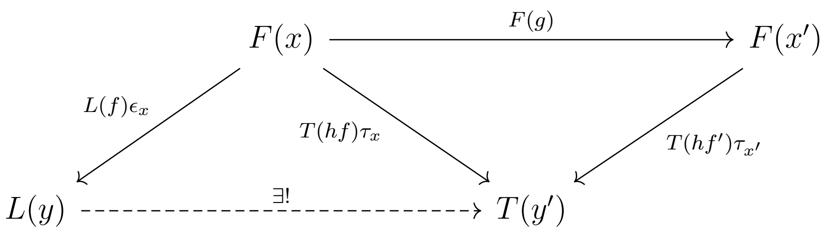
(2.4)

The right top square commutes because τ is a natural transformation, and the right lower triangles commutes because T is a functor. This implies that there is a unique map $L(y) \xrightarrow{\gamma_y} T(y)$ by definition 1.16. It is unique in the sense that for all $(x, f) \in [G \to y]$ it is the unique morphism such that $\gamma_y \circ L(f)\epsilon_x = T(f)\tau_x$.

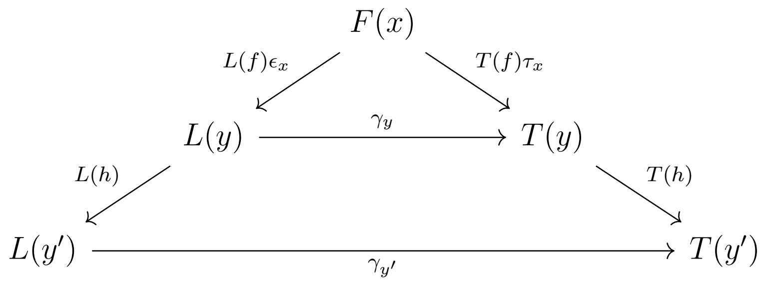
The next goal is to show that γ is a natural transformation, i.e. that for $y \xrightarrow{h} y' \in \mathcal{Y}$ the following square commutes in $\mathcal{C}$, where $\gamma_{y'}$ is constructed in the same manner as γ_y.



In order to achieve this, we turn our attention to a new diagram, where the right triangle is the same commuting triangle as in diagram 2.4, only extended with the morphism $y \xrightarrow{h} y'$.



The universal property of $L(y)$ implies a unique morphism from $L(y)$ to $T(y')$, say ϕ, making the left triangle commute. Both $\gamma_{y'}L(h)$ and $T(h)\gamma_y$ also make that particular triangle commute, as shown in the following diagram.

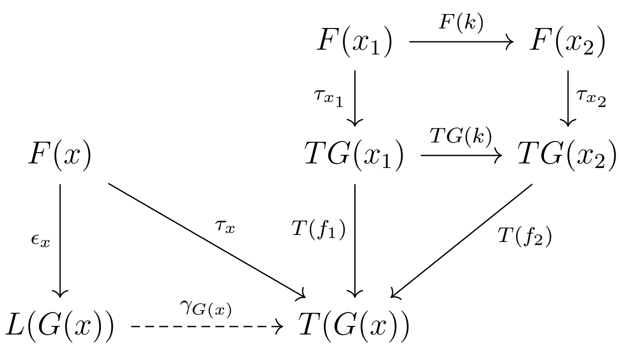


Therefore, $\gamma_{y'}L(h)$ and $T(h)\gamma_y$ must be the same unique morphism. The reason that the latter diagram commutes is the observation

$$\gamma_{y'}L(h)L(f)\epsilon_x = T(h)T(f)\tau_x = T(h)\gamma_y L(f)\epsilon_x$$

where the first equality is by the colimit property of $\gamma_{y'}$, and the second is from the top commuting triangle.

Lastly, it is required that γ is such that $\gamma G\epsilon = \tau$. We fix the morphism $(x_1, f_1) \xrightarrow{k} (x_2, f_2)$ and object $(x, id_{G(x)})$ in the slice category $[G \to G(x)]$. Using the functors F, TG and LG, we get the diagram



where the right hand square and triangle commute because τ is a natural transformation and T is a functor, respectively. This then allows us to use the colimit property of $L(G(x))$, giving us a unique morphism $\gamma_{G(x)}$. The last observation that $\gamma_{G(x)} = (\gamma G)_x$ finishes the proof. □

Proposition 2.7. Let $\mathcal{X}, \mathcal{Y}, \mathcal{C}$ be small categories, let $\mathcal{C}$ admit colimits, and let $\mathcal{X} \xrightarrow{G} \mathcal{Y}$ be a functor. Then there is a left Kan extension functor $\mathcal{C}^{\mathcal{X}} \xrightarrow{LK} \mathcal{C}^{\mathcal{Y}}$, given by $LK(F) = LK_G F$, which is left adjoint to $\mathcal{C}^{\mathcal{Y}} \xrightarrow{G^*} \mathcal{C}^{\mathcal{X}}$, given by $G^*(T) = TG$.

$$\mathcal{C}^{\mathcal{X}} \underset{G^*}{\overset{LK}{\rightleftarrows}} \mathcal{C}^{\mathcal{Y}}$$

Proof. We need to show that

$$\mathcal{H}om_{\mathcal{C}^{\mathcal{Y}}}(LK_G F, T) \xrightarrow{\phi} \mathcal{H}om_{\mathcal{C}^{\mathcal{X}}}(F, TG), \quad \gamma \longmapsto (G\gamma)\alpha$$

is a bijection, where α is the natural transformation associated to $LK_G F$. Let the inverse of ϕ, say ψ, be such that it sends a natural transformation $F \xrightarrow{\tau} TG$ to some $LK_G F \xrightarrow{\hat{\gamma}} T$, where $\hat{\gamma}$ is constructed as the γ from the previous theorem. That is, such that $\tau = (G\hat{\gamma})\alpha$ and $\hat{\gamma}$ is unique such that $T(f)\tau_x = \hat{\gamma}_y LK_G F(f)\alpha_x$ for all $(x, f) \in [G \to y]$. It remains to show ϕ and ψ are inverses. The first direction is the easiest as $\phi(\psi(\tau)) = \phi(\hat{\gamma}) = (\hat{\gamma}G)\alpha = \tau$ where the last equality is by definition of $\hat{\gamma}$.

Conversely, $\psi(\phi(\gamma)) = \psi((\gamma G)\alpha) = \hat{\gamma}$ s.t. $(\gamma G)\alpha = (\hat{\gamma}G)\alpha$. If γ also is such that $T(f)((\gamma G)\alpha)_x = \gamma_y LK_G F(f)\alpha_x$ then we are done. Since $\gamma_{G(x)} = \gamma G_x$, it is the same as asking the following to commute, which it does because γ is a natural transformation.

$$\begin{array}{lccc}
F(x) & & & \\
& \searrow^{\alpha_x} & & \\
& LK_G F(G(x)) & \xrightarrow{\gamma_{G(x)}} & TG(x) \\
& {\scriptstyle LK_G F(f)}\downarrow & & \downarrow{\scriptstyle T(f)} \\
& LK_G F(y) & \xrightarrow{\gamma_y} & T(y)
\end{array}$$

□

Example 2.8. For $\mathcal{Y} = \mathbb{1}$ and the projection functor $\mathcal{X} \to \pi$, the previous proposition implies an adjunction

$$\mathcal{C}^{\mathcal{X}} \underset{\pi^*}{\overset{LK}{\rightleftarrows}} \mathcal{C}$$

where LK sends a diagram in $\mathcal{C}$ with shape $\mathcal{X}$ to $LK_\pi F$ which, as we saw in the previous section, is the colimit of F. Hence, the left adjoint of π^* is the colimit functor.

In later chapters, whenever the reader sees π^L, the notation for the left adjoint of π^*, it it safe to think of it as a variation the colimit functor. Likewise, π^R can be thought of as a variation of a limit functor.

3 Derivators

In order to define a derivator, we first introduce pre-derivators, Beck-Chevalley transformations, and in particular those transformations induced by slice squares. After defining the opposite derivator, we investigate some properties of general derivators. We also construct the (partial) underlying diagram functor, an essential piece to proving that derivators induce categories that admit products and coproducts. Lastly, we prove that it makes sense to define the shifted derivator.

3.1 Pre-derivators and Beck-Chevalley transformations

Definition 3.1. A *pre-derivator* $\mathbb{D}$ is a 2-functor

$$\mathbb{D} : \mathcal{C}at^{op} \to \mathcal{CAT}$$

Before we give examples of some pre-derivators, we introduce some notation which will be used throughout this thesis. Given a morphism $x \xrightarrow{h} y$ in some small category $\mathcal{X}$, there are two induced functors $x, y : \mathbb{1} \to \mathcal{X}$ identifying their respective object, and a natural transformation h between them. Applying the pre-derivator results in the functors $\mathbb{D}(x), \mathbb{D}(y)$ and the natural transformation $\mathbb{D}(h)$, which we denote by x^*, y^* and h^*, respectively.

$$\mathbb{1} \underset{y}{\overset{x}{\rightrightarrows}} \mathcal{X} \;\; (\Downarrow h) \qquad\qquad \mathbb{D}(\mathcal{X}) \underset{y^*}{\overset{x^*}{\rightrightarrows}} \mathbb{D}(\mathbb{1}) \;\; (\Downarrow h^*)$$

For $a \xrightarrow{f} b \in \mathbb{D}(\mathcal{X})$ we denote $x^*(a) \xrightarrow{x^*(f)} x^*(b)$ as $a_x \xrightarrow{f_x} b_x$, and likewise for y^*. Using h pointwise between x^* and y^* results in the following commuting square.

$$\begin{array}{ccc} a_x & \xrightarrow{f_x} & b_x \\ {\scriptstyle h_a^*}\downarrow & & \downarrow{\scriptstyle h_b^*} \\ a_y & \xrightarrow[f_y]{} & b_y \end{array}$$

These are all objects in $\mathbb{D}(\mathbb{1})$, which often is called the *underlying category*. Given an object a in $\mathcal{X}$, we evaluate a pointwise in the underlying category by using the functors $\mathbb{D}(\mathcal{X}) \xrightarrow{x^*} \mathbb{D}(\mathbb{1})$ for $x \in \mathcal{X}$. If we do this for all $x \in \mathcal{X}$, we get the *underlying shape* of a.

We are already in a position to construct *the underlying diagram functor* $dia_{\mathcal{X}}$, which sends an object in $\mathbb{D}(\mathcal{X})$ to its underlying diagram or shape in $\mathbb{D}(\mathbb{1})$.

Lemma 3.2. *Let $\mathbb{D}$ be a pre-derivator and $\mathcal{X}$ a small category. For $a \xrightarrow{f} b \in \mathbb{D}(\mathcal{X})$ and $x \xrightarrow{h} y \in \mathcal{X}$ as above*

i) $\mathcal{X} \xrightarrow{dia_{\mathcal{X}}(a)} \mathbb{D}(\mathbb{1})$ given by $x \mapsto a_x$ and $h \mapsto h_a^*$ is a functor.

ii) $dia_{\mathcal{X}}(a) \xrightarrow{dia_{\mathcal{X}}(f)} dia_{\mathcal{X}}(b)$, given by f component-wise, is a natural transformation.

iii) $\mathbb{D}(\mathcal{X}) \xrightarrow{dia_{\mathcal{X}}} \mathbb{D}(\mathbb{1})^{\mathcal{X}}$ with $a \mapsto dia_{\mathcal{X}}(a), f \mapsto dia_{\mathcal{X}}(f)$ is a functor.

Proof. i) For $x \xrightarrow{h} y \xrightarrow{i} z$ and $x \xrightarrow{id_x} x \in \mathcal{X}$ we have that

$$dia_{\mathcal{X}}(a)(ih) = (ih)_a^* = (i)_a^*(h)_a^* = dia_{\mathcal{X}}(a)(i)dia_{\mathcal{X}}(a)(h)$$

and

$$dia_{\mathcal{X}}(a)(id_x) = (id_x)_a^* = (id_{x^*})_a = id_{a_x} = id_{dia_{\mathcal{X}}(a)(x)}$$

using that $\mathbb{D}$ is a 2-functor, and the definition of an identity transformation.

ii) In the following square, f_x and f_y are the components of $dia_{\mathcal{X}}(f)$.

$$\begin{array}{ccc} a_x & \xrightarrow{f_x} & b_x \\ {\scriptstyle h_a^*}\downarrow & & \downarrow{\scriptstyle h_b^*} \\ a_y & \xrightarrow[f_y]{} & b_y \end{array}$$

As h^* is a natural transformation, the square commutes, and $dia_{\mathcal{X}}(f)$ is therefore a natural transformation.

iii) For $a \xrightarrow{f} b \xrightarrow{g} c$ and $a \xrightarrow{id_a} a \in \mathbb{D}(\mathcal{X})$, we look at the natural transformations component-wise

$$(dia_{\mathcal{X}}(gf))_x = (gf)_x = (g)_x(f)_x = (dia_{\mathcal{X}}(g)dia_{\mathcal{X}}(f))_x$$

and

$$(dia_{\mathcal{X}}(id_a))_x = (id_a)_x = x^*(id_a) = id_{x^*(a)} = id_{a_x} = (Id_{dia_{\mathcal{X}}(a)})_x$$

In these steps we are using that x^* is a functor.

□

Now that we are more familiar with the notation that will be used for functors and natural transformations induced by a derivator, namely $-^*$, we look at some examples of pre-derivators.

Example 3.3. Let $\mathcal{C}$ be a category which admits (co)limits. We define the *represented pre-derivator* for a small category $\mathcal{X}$ by $\mathbb{D}_{rep}(\mathcal{X}) := \mathcal{C}^{\mathcal{X}}$. Given a functor $\mathcal{X} \xrightarrow{G} \mathcal{Y}$ between small categories, the represented pre-derivator induces the pre-composition functor $\mathcal{C}^{\mathcal{Y}} \xrightarrow{G^*} \mathcal{C}^{\mathcal{X}}$, as seen in proposition 2.7, and is indeed a pre-derivator.

$$\mathcal{X} \underset{H}{\overset{G}{\rightrightarrows}} \mathcal{Y} \quad (\Downarrow \alpha) \qquad \longmapsto \qquad \mathcal{C}^{\mathcal{Y}} \underset{H^*}{\overset{G^*}{\rightrightarrows}} \mathcal{C}^{\mathcal{X}} \quad (\Downarrow \alpha^*)$$

Example 3.4. What follows is a rough introduction to another example of a pre-derivator. For more details, see [Gro19] and [Kra21].

Given an abelian category $\mathcal{A}$, one can form the category of chain complexes, $\mathbf{C}(\mathcal{A})$. Objects are chain complexes $\ldots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} \ldots$ for $A^i \in \mathcal{A}$ and such that $d^n \circ d^{n-1} = 0$. A morphism between two complexes $A \xrightarrow{\phi} B$ consists of levelwise morphisms between them $A^i \xrightarrow{\phi_i} B^i$ such that the obvious squares commute. ϕ is said to be a quasi-isomorphism if the induced map in homology $H^n\phi : H^nA \to H^nB$ is an isomorphism for all $n \in \mathbb{Z}$.

Given a small category $\mathcal{X}$, we can also form the category $\mathbf{C}(\mathcal{A})^{\mathcal{X}}$. As the objects now are functors, a quasi-isomorphism in this category is a natural transformation that is levelwise a quasi-isomorphism. We denote the class of quasi-isomorphisms in $\mathbf{C}(\mathcal{A})^{\mathcal{X}}$ by $W_{\mathcal{A}}^{\mathcal{X}}$. Localisation of a category $\mathcal{C}$ with respect to a class of morphisms in $\mathcal{C}$, say S is a way of creating a category $\mathcal{C}[S^{-1}]$ where all the morphisms in S are invertible. This leads to the definition of the *derivator of (chain complexes in) $\mathcal{A}$.*

$$\mathbb{D}_{\mathcal{A}}(\mathcal{X}) := \mathbf{C}(\mathcal{A})^{\mathcal{X}}[(W_{\mathcal{A}}^{\mathcal{X}})^{-1}]$$

This is the derived category $\mathbf{D}(\mathcal{A})$. For a functor $\mathcal{X} \xrightarrow{G} \mathcal{Y}$, there is a pre-composition functor $\mathbf{C}(\mathcal{A})^{\mathcal{Y}} \xrightarrow{G^*} \mathbf{C}(\mathcal{A})^{\mathcal{X}}$ as seen in the previous example. This will preserve quasi-isomorphisms and in turn induce a functor

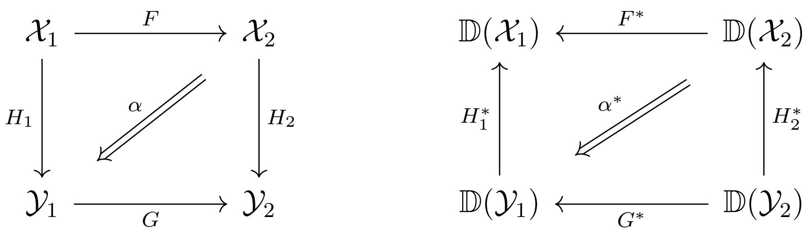
$$\mathbb{D}_{\mathcal{A}}(\mathcal{Y}) = \mathbf{C}(\mathcal{A})^{\mathcal{Y}}[(W_{\mathcal{A}}^{\mathcal{Y}})^{-1}] \xrightarrow{\mathbb{D}_{\mathcal{A}}(G)} \mathbf{C}(\mathcal{A})^{\mathcal{X}}[(W_{\mathcal{A}}^{\mathcal{X}})^{-1}] = \mathbb{D}_{\mathcal{A}}(\mathcal{X})$$

Definition 3.5. Let $\mathbb{D}$ be a pre-derivator and $\mathcal{X} \xrightarrow{G} \mathcal{Y}$ a functor. We say $\mathbb{D}$ *admits a left Kan extension* along G if there exists a left adjoint, G^L, to the induced restriction functor

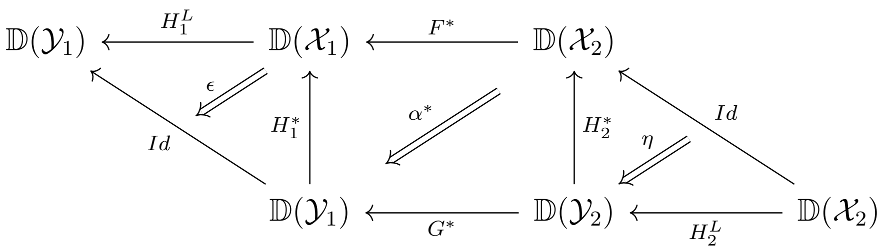
G^*. Likewise, $\mathbb{D}$ *admits a right Kan extension* along G if there exists a right adjoint, G^R, to G^*.

Remark 3.6. This definition is not directly connected with the definitions of Kan extensions. However, recalling proposition 2.7 it is clear that the represented pre-derivator admits both left and right Kan extensions, and that these are precisely the Kan extensions from section 2.1.

We are almost ready to define a derivator. Before we do so, consider a pre-derivator $\mathbb{D}$ and some small categories $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$. Applying the pre-derivator to the square on the left induces the square on the right.

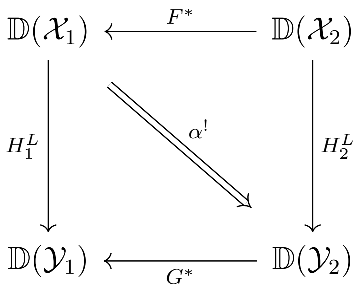


If we assume $\mathbb{D}$ admits left Kan extensions along its functors, it results in the diagram

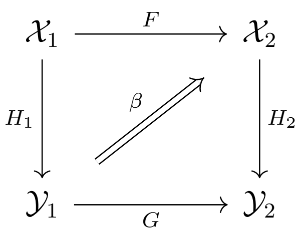


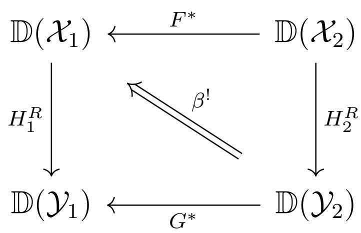
where η and ϵ are the counit and unit of they respective adjunctions. We can then compose η, α^* and ϵ to get the following composition and simplified square

$$H_1^L F^* \xRightarrow{(\epsilon \circ G^* H_2^L)(H_1^L \circ \alpha^* \circ H_2^L)(H_1^L F^* \circ \eta)} G^* H_2^L$$

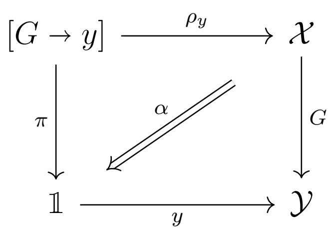


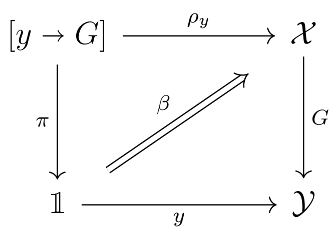
This is called the *Beck-Chevalley transformation* associated to α. In order to avoid writing the long composition in the future, it is denoted by $\alpha^!$. The dual construction begins with the following square on the left, and its associated Beck-Chevalley transformation is on the right.





We will now take a look at a special case of the Beck-Chevalley transformations, which will show up in the definition of a derivator. We look at the following diagrams, which will be referred to as a *slice squares*.





Consider the left slice square. For the functor $\mathcal{X} \xrightarrow{G} \mathcal{Y}$, we get the slice category $[G \to y]$ and its corresponding projection functor ρ_y. The functor y identifies the objects y in $\mathcal{Y}$, and π is the canonical projection functor. We should also notice that for an object (x, f) in $[G \to y]$ we get $\alpha_{(x,f)} = f$, which is the reason α might be denoted as the morphism f.

If $\mathbb{D}$ admits left and right Kan extensions, it results in the corresponding Beck-Chevalley transformations $\alpha^!$ and $\beta^!$.

$$\begin{array}{ccc} \mathbb{D}([G \to y]) & \xleftarrow{\rho_y^*} & \mathbb{D}(\mathcal{X}) \\ {\scriptstyle \pi^L}\downarrow & \overset{\alpha^!}{\searrow} & \downarrow{\scriptstyle G^L} \\ \mathbb{D}(\mathbb{1}) & \xleftarrow[y^*]{} & \mathbb{D}(\mathcal{Y}) \end{array} \qquad \begin{array}{ccc} \mathbb{D}([y \to G]) & \xleftarrow{\rho_y^*} & \mathbb{D}(\mathcal{X}) \\ {\scriptstyle \pi^R}\downarrow & \overset{\beta^!}{\nwarrow} & \downarrow{\scriptstyle G^R} \\ \mathbb{D}(\mathbb{1}) & \xleftarrow[y^*]{} & \mathbb{D}(\mathcal{Y}) \end{array}$$

Asking that the Beck-Chevalley transformations associated to slice squares are natural isomorphisms, is a way of integrating pointwise Kan extensions into derivators.

3.2 Derivators

Definition 3.7. Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}, \mathcal{Y}$ be small categories, and $y \in \mathcal{Y}$. A pre-derivator $\mathbb{D}$ is a *derivator* if it satisfies the following

Der i) The inclusions $\mathcal{X}_1 \to \mathcal{X}_1 \sqcup \mathcal{X}_2$ and $\mathcal{X}_2 \to \mathcal{X}_1 \sqcup \mathcal{X}_2$ induce an equivalence of categories $\mathbb{D}(\mathcal{X}_1 \sqcup \mathcal{X}_2) \to \mathbb{D}(\mathcal{X}_1) \times \mathbb{D}(\mathcal{X}_2)$, and $\mathbb{D}(\emptyset)$ is equivalent to $\mathbb{1}$.

Der ii) A morphism $a \xrightarrow{f} b \in \mathbb{D}(\mathcal{X})$ is an isomorphism if and only if $a_x \xrightarrow{f_x} b_x$ is an isomorphism in $\mathbb{D}(\mathbb{1})$ for all $x \in \mathcal{X}$.

Der iii) Each functor $\mathcal{X} \xrightarrow{G} \mathcal{Y}$ induces a restriction functor $\mathbb{D}(\mathcal{Y}) \xrightarrow{G^*} \mathbb{D}(\mathcal{X})$ which admits left and right Kan extensions G^L and G^R.

$$\mathbb{D}(\mathcal{X}) \underset{G^*}{\overset{G^L}{\rightleftarrows}} \mathbb{D}(\mathcal{Y}) \ (adj) \qquad\qquad \mathbb{D}(\mathcal{Y}) \underset{G^R}{\overset{G^*}{\rightleftarrows}} \mathbb{D}(\mathcal{X}) \ (adj)$$

Der iv) The Beck-Chevalley transformations induced by slice squares are isomorphisms.

$$\pi^L \rho_y^* \xrightarrow{\quad iso \quad} y^* G^L \qquad\qquad y^* G^R \xrightarrow{\quad iso \quad} \pi^R \rho_y^*$$

Remark 3.8. We will use the notation *Der i)* when referring to the first derivator axiom, and similarly for the other axioms.

Example 3.9. It is now possible to see that the previous chapter was concerned with showing that the represented pre-derivator, $\mathbb{D}_{rep}$, actually is a derivator. Der i) is satisfied because $\mathcal{C}^{\mathcal{X}_1 \sqcup \mathcal{X}_2} \cong \mathcal{C}^{\mathcal{X}_1} \times \mathcal{C}^{\mathcal{X}_2}$ and $\mathcal{C}^{\emptyset} \cong \mathbb{1}$. Der ii) is true by definition of a natural isomorphism. Der iii) is satisfied as mentioned in remark 3.6. Der iv) also holds if we take a closer look at the Beck-Chevalley transformation given when applying the represented pre-derivator to a slice square.

$$\begin{array}{ccc} \mathcal{C}^{[G\to y]} & \xleftarrow{\ \rho_y^*\ } & \mathcal{C}^{\mathcal{X}} \\ {\scriptstyle \pi^L}\downarrow & \searrow\!\!\searrow & \downarrow{\scriptstyle G^L} \\ \mathcal{C} & \xleftarrow[\ y^*\ ]{} & \mathcal{C}^{\mathcal{Y}} \end{array}$$

Recall that π^L is the colimit, as seen in example 2.8. For a functor $F \in \mathcal{C}^{\mathcal{X}}$ we have that $\pi^L \rho_y^*(F) = colim F\rho_y$ and $y^* G^L(F) = LK_G F(y)$. In section 2.2 we defined a map L such that $L(y) = colim F\rho_y$ and then showed that L actually was the left Kan extension $LK_G F$. Hence the Beck-Chevalley transformation $\pi^L \rho_y^* \to y^* G^L$ is a natural transformation. The arguments are dual for the other transformation, meaning Der iv) is satisfied. The conclusion is that the represented pre-derivator $\mathbb{D}(\mathcal{X}) = \mathcal{C}^{\mathcal{X}}$ is a derivator.

Example 3.10. With some restriction on $\mathcal{A}$, the derivator of $\mathcal{A}$ is in fact a derivator. In addition to the mentioned sources, [Bra21] provides a detailed explanation.

Definition 3.11. The *opposite derivator*, $\mathbb{D}^{op}$, is defined as $\mathbb{D}^{op}(\mathcal{X}) = (\mathbb{D}(\mathcal{X}^{op}))^{op}$ for some small category $\mathcal{X}$.

Proposition 3.12. $\mathbb{D}$ is a derivator if and only if $\mathbb{D}^{op}$ is a derivator.

Proof. Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}$ and $\mathcal{Y}$ be small categories, $a \xrightarrow{f} b$ a morphism in $\mathbb{D}^{op}(\mathcal{X})$, and fix $y \in \mathcal{Y}$.
(⇒) Assume $\mathbb{D}$ is a derivator.
[Der i)]

$$\begin{aligned}
\mathbb{D}^{op}(\mathcal{X}_1 \sqcup \mathcal{X}_2) &= (\mathbb{D}(\mathcal{X}_1 \sqcup \mathcal{X}_2)^{op})^{op} && \text{(by def)}\\
&= (\mathbb{D}(\mathcal{X}_1^{op} \sqcup \mathcal{X}_2^{op}))^{op} \\
&= (\mathbb{D}(\mathcal{X}_1^{op}) \sqcap \mathbb{D}(\mathcal{X}_2^{op})^{op} && (\mathbb{D}\text{ derivator})\\
&= (\mathbb{D}(\mathcal{X}_1^{op}))^{op} \sqcap (\mathbb{D}(\mathcal{X}_2^{op}))^{op} \\
&= \mathbb{D}^{op}(\mathcal{X}_1) \sqcap \mathbb{D}^{op}(\mathcal{X}_2) && \text{(by def)}
\end{aligned}$$

and $\mathbb{D}^{op}(\emptyset) = (\mathbb{D}(\emptyset^{op}))^{op} = (\mathbb{D}(\emptyset))^{op} = \mathbb{1}^{op} = \mathbb{1}$

[Der ii)]

$$
\begin{aligned}
& a \xrightarrow{f} b \text{ isomorphism in } \mathbb{D}^{op}(\mathcal{X}) \\
\iff \; & a \xrightarrow{f} b \text{ isomorphism in } \mathbb{D}(\mathcal{X}^{op})^{op} \\
\iff \; & a \xleftarrow{f^{op}} b \text{ isomorphism in } \mathbb{D}(\mathcal{X}^{op}) \\
\iff \; & a_x \xleftarrow{f_x^{op}} b_x \text{ isomorphism in } \mathbb{D}(\mathbb{1}) \text{ for all } x \in \mathcal{X}^{op} \\
\iff \; & a_x \xrightarrow{f_x} b_x \text{ isomorphism in } \mathbb{D}(\mathbb{1})^{op} \text{ for all } x \in \mathcal{X}^{op} \\
\iff \; & a_x \xrightarrow{f_x} b_x \text{ isomorphism in } \mathbb{D}^{op}(\mathbb{1}) \text{ for all } x \in \mathcal{X}
\end{aligned}
$$

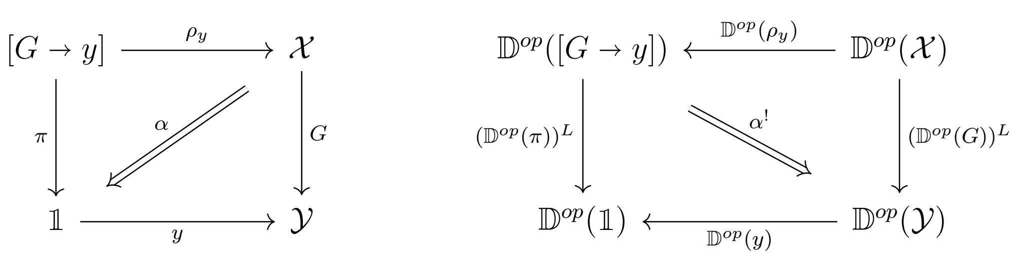
The last step is true because the objects in $\mathcal{X}$ and $\mathcal{X}^{op}$ are the same.

[Der iii)]
Given $\mathcal{X} \xrightarrow{G} \mathcal{Y}$ we want to define the induced functor $\mathbb{D}^{op}(\mathcal{Y}) \xrightarrow{\mathbb{D}^{op}(G)} \mathbb{D}^{op}(\mathcal{X})$ such that it admits left and right Kan extensions, i.e. left and right adjoints. We have that $\mathcal{X}^{op} \xrightarrow{G^{op}} \mathcal{Y}^{op}$ induces $\mathbb{D}(\mathcal{Y}^{op}) \xrightarrow{(G^{op})^*} \mathbb{D}(\mathcal{X}^{op})$ with left adjoint $(G^{op})^L$ and right adjoint $(G^{op})^R$. Taking the opposite functor results in

$$(\mathbb{D}(\mathcal{Y}^{op}))^{op} \xrightarrow{((G^{op})^*)^{op}} (\mathbb{D}(\mathcal{X}^{op}))^{op}$$

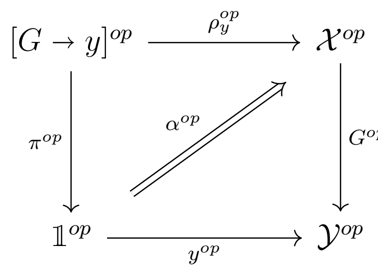
with left adjoint $((G^{op})^R)^{op}$ and right adjoint $((G^{op})^L)^{op}$. Hence, $\mathbb{D}^{op}(G) = ((G^{op})^*)^{op}$ with left adjoint $(\mathbb{D}^{op}(G))^L = ((G^{op})^R)^{op}$ and right adjoint $(\mathbb{D}^{op}(G))^R = ((G^{op})^L)^{op}$.

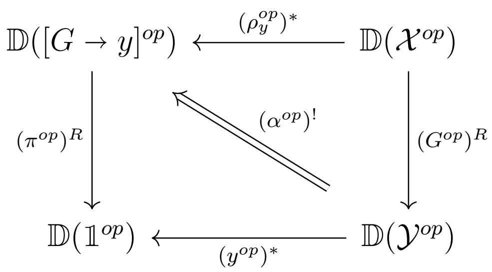
[Der iv)]
Given the slice square on the left, we want to show that the induced Beck-Chevalley transformation, $\alpha^!$, is a natural isomorphism.



(3.1)

Taking the opposite of the slice square induces the Beck-Chevalley transformation $(\alpha^{op})^!$. Because $[G \to y]^{op} \cong [y \to G^{op}]$, the induced $(\alpha^{op})^!$ is a natural isomorphism by Der iv).

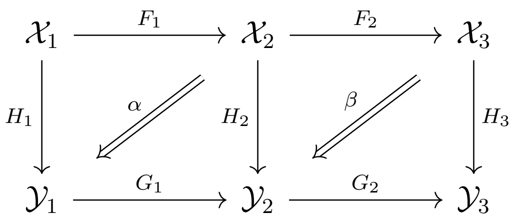


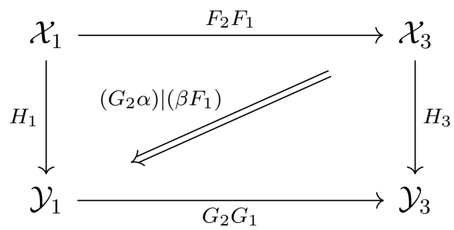


Taking the opposite of the right square gives exactly the right square in diagram 3.1 using the definitions from [Der iii)], and $((\alpha^{op})^!)^{op} = \alpha^!$ is a natural isomorphism. The other Beck-Chevalley transformation is dual.

($\Leftarrow$) We have that $(\mathbb{D}^{op})^{op}(\mathcal{X}) = (\mathbb{D}^{op}(\mathcal{X}^{op}))^{op} = ((\mathbb{D}((\mathcal{X}^{op})^{op}))^{op})^{op} = \mathbb{D}(\mathcal{X})$. We assume $\mathbb{D}^{op}$ is a derivator and see that its opposite, $\mathbb{D}$, must also be a derivator by ($\Rightarrow$). $\square$

Lemma 3.13. *The two diagrams*

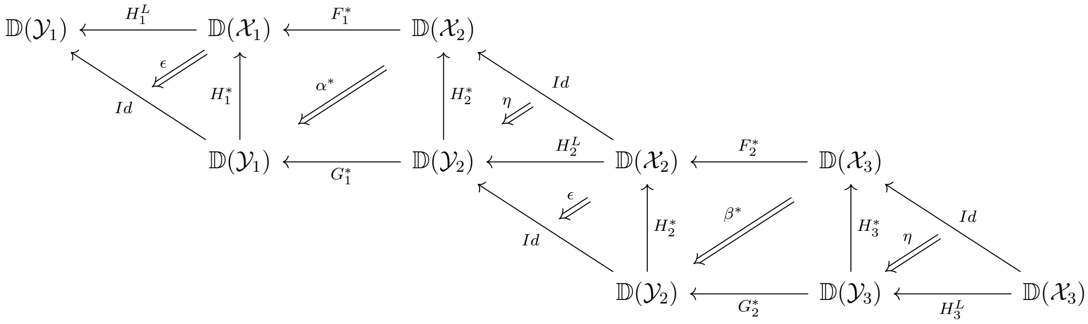




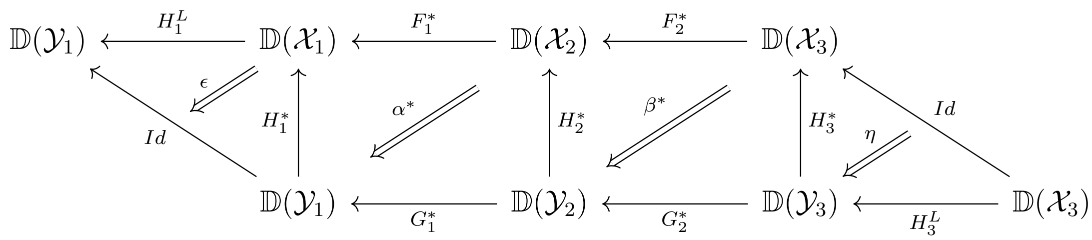
induce three different Beck-Chevalley transformations, $\alpha^!, \beta^!$ and $((G_2\alpha)|(\beta F_1))^!$, where the two first can be vertically composed to induce the third. In other words, that

$$(G_1^*\beta^!)|(\alpha^! F_2^*) = ((G_2\alpha)|(\beta F_1))^!$$

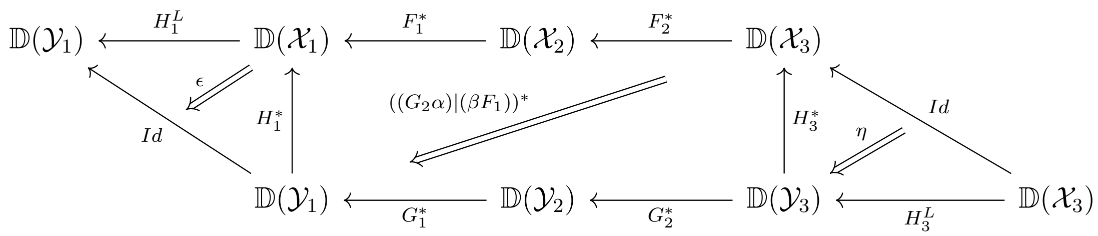
Proof. It is possible, but perhaps not insightful to brute force this proof. Instead, we use diagrams and notice $(G_1^*\beta^!)|(\alpha^! F_2^*)$ is the same as composing all of the natural transformations in the diagram



which, by using the triangle identities of adjoints (definition 1.11), is the same as



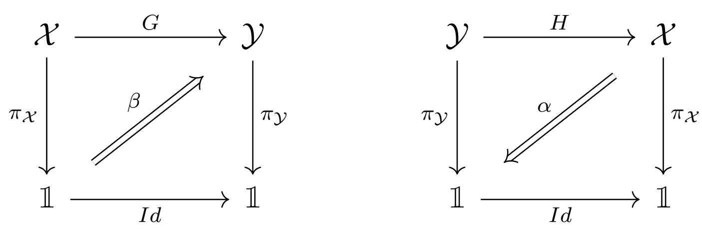
which is the same as the diagram



because $((\alpha^* G_2^*)|(F_1^* \beta^*)) = ((G_2\alpha)|(\beta F_1))^*$. Realising that the last diagram is the definition of $((G_2\alpha)|(\beta F_1))^!$ completes the proof. □

This result implies that if two of the three transformations $\alpha^!, \beta^!$ and $((G_2\alpha)|(\beta F_1))^!$ are natural isomorphisms, then the last one must also be a natural isomorphism.

Lemma 3.14. *Let $(G, H) : \mathcal{X} \rightleftarrows \mathcal{Y}$ be a pair of adjoint functors. Given the squares*



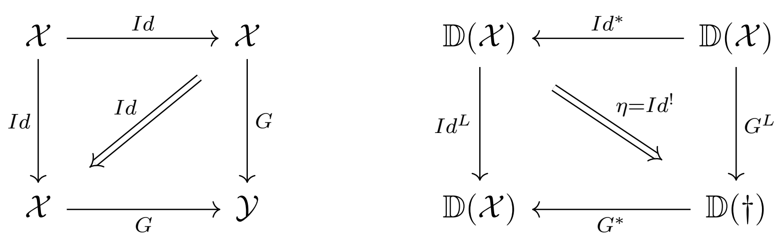
the Beck-Chevalley transformations $\beta^! : \pi_{\mathcal{X}}^R G^* \to Id^* \pi_{\mathcal{Y}}^R$ and $\alpha^! : \pi_{\mathcal{Y}}^L H^* \to Id^* \pi_{\mathcal{X}}^L$ are natural isomorphisms.

Proof. We prove that $\alpha^!$ is a natural isomorphism. Since (G, H) is an adjoint pair, by observation 1.22, (H^*, G^*) is also an adjoint pair. This implies $H^* \cong G^L$ and allows us to see that $\pi_{\mathcal{Y}}^L H^* = \pi_{\mathcal{Y}}^L G^L = (\pi_{\mathcal{Y}} G)^L = (Id \pi_{\mathcal{X}})^L = \pi_{\mathcal{X}}^L$ by the left commuting square in the diagram. Since $\mathbb{D}(-) = -^*$ is a functor, it means $Id^* = Id$, hence $Id^* \pi_{\mathcal{X}}^L = \pi_{\mathcal{X}}^L$.

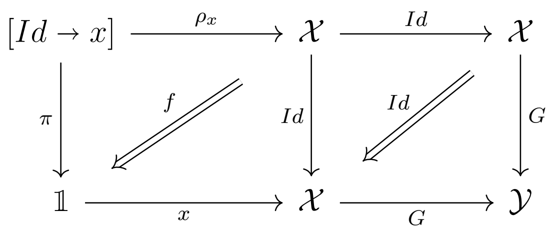
□

Lemma 3.15. *If $\mathcal{X} \xrightarrow{G} \mathcal{Y}$ is a functor, then G^L and G^R are fully faithful.*

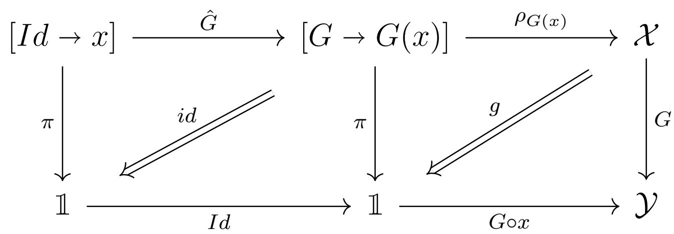
Proof. We prove the result for G^L. The Beck-Chevalley transformation induced by the following left square is the unit of the adjunction (G^L, G^*), namely $Id \xrightarrow{\eta} G^L G^*$.



Therefore, if η is an isomorphism, we are done by lemma 1.14. We paste a slice square to the left of our original square and obtain the diagram



By Der iv) we know that $f^!$ is an isomorphism, and by lemma 3.13 we know that if the outer square induces an isomorphism, we are done. In aid of this, consider the following diagram where the outer square agrees with the outer square of the previous diagram.



Notice that $\hat{G}$, given by sending (x, f) to $(x, G(f))$, is an equivalence. This is true because $\hat{G}$ is dense, and G is fully faithful, so $\mathcal{H}om(x, x') \cong \mathcal{H}om(G(x), G(x'))$ meaning that $\hat{G}$ is also fully faithful.

As $\hat{G}$ is an equivalence, the Beck-Chevalley transformation of the left diagram, $id^!$, is an isomorphism by lemma 3.14. In addition, $g^!$ is an isomorphism by Der iv), so the Beck-Chevalley transformation of the outer diagram is an isomorphism by lemma 3.13. $\square$

3.3 $\mathbb{D}(\mathcal{X})$ admits products and coproducts

The underlying diagram functor used functors induced by $\mathbb{1} \xrightarrow{x} \mathcal{X}$ for x in $\mathcal{X}$. In the next lemma, we want to construct the *partial underlying diagram functor*, and instead use functors induced by

$$\mathbb{1} \times \mathcal{X} \xrightarrow{k \times Id_{\mathcal{X}}} \mathcal{K} \times \mathcal{X}$$

Lemma 3.16. *$\mathbb{D}$ is a derivator and $\mathcal{K}, \mathcal{X}$ are small categories. Let $a \xrightarrow{f} b$ be a morphism in $\mathbb{D}(\mathcal{K} \times \mathcal{X})$, and let $k_1 \xrightarrow{h} k_2$ be a morphism in $\mathcal{K}$.*

i) $\mathcal{K} \xrightarrow{dia_{\mathcal{K},\mathcal{X}}(a)} \mathbb{D}(\mathcal{X})$ given by $k \mapsto (k \times Id_{\mathcal{X}})^*(a)$ and $h \mapsto h_a^*$ is a functor.

ii) $dia_{\mathcal{K},\mathcal{X}}(a) \xrightarrow{dia_{\mathcal{K},\mathcal{X}}(f)} dia_{\mathcal{K},\mathcal{X}}(b)$, given by f component-wise, is a natural transformation.

iii) $\mathbb{D}(\mathcal{K} \times \mathcal{X}) \xrightarrow{dia_{\mathcal{K},\mathcal{X}}} \mathbb{D}(\mathcal{X})^{\mathcal{K}}$ with $a \mapsto dia_{\mathcal{K},\mathcal{X}}(a)$, $f \mapsto dia_{\mathcal{K},\mathcal{X}}(f)$ is a functor.

Proof. Analogous to lemma 3.2. □

Definition 3.17. The *diagonal functor* $\Delta_{\mathcal{K}} : \mathcal{X} \to \mathcal{X}^{\mathcal{K}}$ sends an object A in $\mathcal{X}$ to the constant diagram of shape $\mathcal{K}$ in $\mathcal{X}$.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad\Delta_{\mathcal{K}}\quad} & \mathcal{X}^{\mathcal{K}} \\ A & \longmapsto & \mathcal{K} \longrightarrow \mathcal{X} \\ & & k \longmapsto A \end{array}$$

Consider $dia_{\mathcal{K}} : \mathbb{D}(\mathcal{X}) \to \mathbb{D}(\mathcal{X})^{\mathcal{K}}$. Finding left and right adjoints to $\Delta_{\mathcal{K}}$ is something we want to do in general, as it would mean that $\mathbb{D}(\mathcal{X})$ admitted colimits and limits of shape $\mathcal{K}$. There is a case in particular when these adjoints exist. We will use $dia_{\mathcal{K},\mathcal{X}}$ and the functor $\mathbb{D}(\mathcal{X}) \xrightarrow{\pi_{\mathcal{K}}^*} \mathbb{D}(\mathcal{K} \times \mathcal{X})$ induced by $\mathcal{K} \times \mathcal{X} \xrightarrow{\pi_{\mathcal{K}}} \mathcal{X}$.

Lemma 3.18. *Let $A \in \mathbb{D}(\mathcal{X})$.*

i) The composition $\mathbb{D}(\mathcal{X}) \xrightarrow{\pi_{\mathcal{K}}^*} \mathbb{D}(\mathcal{K} \times \mathcal{X}) \xrightarrow{dia_{\mathcal{K},\mathcal{X}}} \mathbb{D}(\mathcal{X})^{\mathcal{K}}$ is equal to $\Delta_{\mathcal{K}}$.

ii) If $dia_{\mathcal{K},\mathcal{X}}$ is an equivalence, then $\Delta_{\mathcal{K}}$ admits left and right adjoints.

Proof. i) Writing out what the composition does on objects, we get the following diagram.

$$\begin{array}{ccccccc} \mathbb{D}(\mathcal{X}) & \xrightarrow{\pi_{\mathcal{K}}^*} & \mathbb{D}(\mathcal{K}\times\mathcal{X}) & \xrightarrow{dia_{\mathcal{K},\mathcal{X}}} & \mathbb{D}(\mathcal{X})^{\mathcal{K}} & & \\ A & \longmapsto & \pi_{\mathcal{K}}^*(A) & \longmapsto & \mathcal{K} & \xrightarrow{dia_{\mathcal{K},\mathcal{X}}(\pi_{\mathcal{K}}^*(A))} & \mathbb{D}(\mathcal{X}) \\ & & & & k & \longmapsto & (k\times id_{\mathcal{X}})^*(\pi_{\mathcal{K}}^*(A)) \end{array}$$

If we observe that $\pi_{\mathcal{K}}(k \times id_{\mathcal{X}}) = id_{\mathcal{X}}$, it is clear that the composition sends A to the constant diagram of shape $\mathcal{K}$ in $\mathbb{D}(\mathcal{X})$.

ii) We know that $\pi_{\mathcal{K}}^*$ admits adjoints by Der iii), and if $dia_{\mathcal{K},\mathcal{X}}$ is an equivalence it will also admit adjoints. By i), this means that the diagonal functor $dia_{\mathcal{K},\mathcal{X}} \circ \pi_{\mathcal{K}}^* = \Delta_{\mathcal{K}}$ will admit adjoints.

□

Theorem 3.19. $\mathbb{D}$ *is a derivator. Given a small category* $\mathcal{X}$*, the category* $\mathbb{D}(\mathcal{X})$ *admits products and coproducts.*

Proof. Let $\mathcal{S}$ be a discrete category, which can also be considered as a set by ignoring the identity morphisms. By lemma 3.18, if we let $\mathcal{K} = \mathcal{S}$, it is enough to show that $dia_{\mathcal{S},\mathcal{X}}$ is an equivalence. This is true by the following commuting square where the three equivalences come from Der i), and the two known facts $\mathcal{S} \times \mathcal{X} \cong \sqcup_{s\in\mathcal{S}}\mathcal{X}_s$ and $\sqcap_{s\in\mathcal{S}}\mathcal{X}_s \cong \mathcal{X}^{\mathcal{S}}$.

$$\begin{array}{ccc} \mathbb{D}(\mathcal{S}\times\mathcal{X}) & \xrightarrow{dia_{\mathcal{S},\mathcal{X}}} & \mathbb{D}(\mathcal{X})^{\mathcal{S}} \\ \cong\downarrow & & \uparrow\cong \\ \mathbb{D}(\sqcup_{s\in\mathcal{S}}\mathcal{X}_s) & \xrightarrow[\cong]{} & \sqcap_{s\in\mathcal{S}}\mathbb{D}(\mathcal{X})_s \end{array}$$

□

Corollary 3.20. *The category* $\mathbb{D}(\mathcal{X})$ *admits initial and terminal objects.*

Proof. Similarly to the proof of the theorem, by setting $\mathcal{K} = \emptyset$ in lemma 3.18 the resulting functor $dia_{\emptyset,\mathcal{X}} = Id_{\mathbb{1}}$ is an equivalence. □

Definition 3.21. Let $\mathcal{K}$ and $\mathcal{X}$ be small categories. Then the *shifted derivator* $\mathbb{D}^{\mathcal{K}}$ is given by

$$\mathbb{D}^{\mathcal{K}}(\mathcal{X}) = \mathbb{D}(\mathcal{K}\times\mathcal{X})$$

Proposition 3.22. Let $\mathbb{D}$ be a pre-derivator and $\mathcal{K} \in \mathcal{C}at$. Then $\mathbb{D}$ is a derivator if and only if $\mathbb{D}^{\mathcal{K}}$ is a derivator.

Proof. (⇐) If $\mathbb{D}^{\mathcal{K}}$ is a derivator, then in particular $\mathbb{D}^{\mathbb{1}} \cong \mathbb{D}$ is a derivator.
(⇒) We assume $\mathbb{D}$ is a derivator.

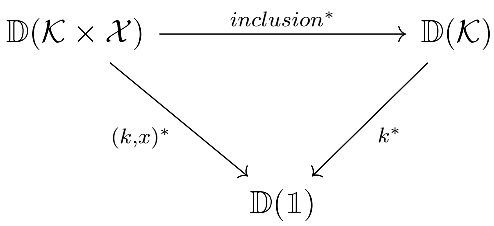
[Der i)]

$$\begin{aligned}\mathbb{D}^{\mathcal{K}}(\mathcal{X}_1 \sqcup \mathcal{X}_2) &= \mathbb{D}(\mathcal{K} \times (\mathcal{X}_1 \sqcup \mathcal{X}_2)) \\ &= \mathbb{D}(\mathcal{K} \times \mathcal{X}_1 \sqcup \mathcal{K} \times \mathcal{X}_2) \\ &= \mathbb{D}(\mathcal{K} \times \mathcal{X}_1) \times \mathbb{D}(\mathcal{K} \times \mathcal{X}_2) \\ &= \mathbb{D}^{\mathcal{K}}(\mathcal{X}_1) \times \mathbb{D}^{\mathcal{K}}(\mathcal{X}_2)\end{aligned}$$

and

$\mathbb{D}^{\mathcal{K}}(\emptyset) = \mathbb{D}(\mathcal{K} \times \emptyset) = \mathbb{D}(\emptyset) = \mathbb{1}$

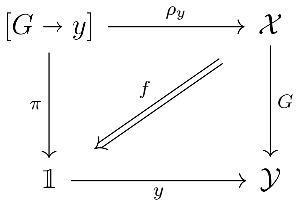
[Der ii)] Let $f \in \mathbb{D}^{\mathcal{K}}(\mathcal{X})$. Consider the commuting triangle.

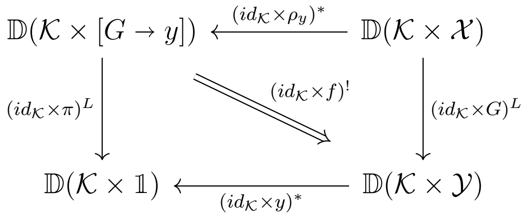


f is an isomorphism in $\mathbb{D}(\mathcal{K} \times \mathcal{X})$ if and only if $f_{(k,x)}$ is an isomorphism in $\mathbb{D}(\mathbb{1})$ for all $(k, x) \in \mathcal{K} \times \mathcal{X}$ if and only if f_x is an isomorphism in $\mathbb{D}(\mathcal{K}) = \mathbb{D}^{\mathcal{K}}(\mathbb{1})$ for all $x \in \mathcal{X}$.

[Der iii)] A functor $\mathcal{X} \xrightarrow{G} \mathcal{Y}$ induces $\mathbb{D}^{\mathcal{K}}(\mathcal{Y}) = \mathbb{D}(\mathcal{K} \times \mathcal{Y}) \xrightarrow{id_{\mathcal{K}} \times G} \mathbb{D}(\mathcal{K} \times \mathcal{X}) = \mathbb{D}^{\mathcal{K}}(\mathcal{X})$, which admits adjoints because $\mathbb{D}$ is a derivator.

[Der iv)] The goal is to show that $(id_{\mathcal{K}} \times f)^!$ from the following diagram is a natural isomorphism.





The transformation $(id_{\mathcal{K}} \times f)^!$ is induced by $id_{\mathcal{K}} \times f$ in the following diagram where we have pasted a slice square to the left.

$$\begin{array}{ccccc}
[(id_{\mathcal{K}} \times \pi) \to k] & \xrightarrow{\rho_k} & \mathcal{K} \times [G \to y] & \xrightarrow{id_{\mathcal{K}} \times \rho_y} & \mathcal{K} \times \mathcal{X} \\
\Big\downarrow \pi & & \Big\downarrow id_{\mathcal{K}} \times \pi & \overset{id_{\mathcal{K}} \times f}{\swarrow\!\!\!\swarrow} & \Big\downarrow id_{\mathcal{K}} \times G \\
\mathbb{1} & \xrightarrow[k]{} & \mathcal{K} \times \mathbb{1} & \xrightarrow[id_{\mathcal{K}} \times y]{} & \mathcal{K} \times \mathcal{Y}
\end{array}$$

By lemma 3.13, it is enough to show that the left and outer square induce Beck-Chevalley isomorphisms. The left square is by construction a slice square, so by Der iv) it will induce an isomorphism. Tracing what the functors do to objects, it can be seen that the outer square is the same as the slice square

$$\begin{array}{ccc}
[(id_{\mathcal{K}} \times G) \to (k, y)] & \xrightarrow{\rho_{k,y}} & \mathcal{K} \times \mathcal{X} \\
\Big\downarrow \pi & & \Big\downarrow id_{\mathcal{K}} \times G \\
\mathbb{1} & \xrightarrow[(k,y)]{} & \mathcal{K} \times \mathcal{Y}
\end{array}$$

which also will induce an isomorphism by Der iv).

□

4 Pointed derivators

In this chapter we introduce the notion of a pointed derivator, requiring a zero object in the underlying category. This enables us to prove lemma 4.9, which we think of as an extension by zero. At the end of section 4.1 we highlight a subtle observation, which will be used liberally throughout. We define cartesian and cocartesian squares in order to introduce the fiber, cone, suspension and loop functors in section 4.2. These will be revisited in theorem 6.9, the main result of the thesis. The last section shows how the (co)cartesian squares can be calculated pointwise.

4.1 Extension by zero

Definition 4.1. A derivator $\mathbb{D}$ is *pointed* if the underlying category $\mathbb{D}(\mathbb{1})$ has a zero object, that is, an object which is both initial and terminal.

Example 4.2. The represented derivator is pointed if and only if $\mathcal{C}$ admits a zero object, as $\mathbb{D}_{rep}(\mathbb{1}) \cong \mathcal{C}$ is the underlying category.

Example 4.3. The homotopy derivator $\mathbb{D}_{\mathcal{A}}$ is pointed due to the fact that $\mathcal{A}$ is abelian ([Gro19] example 3.18).

Lemma 4.4. *Let $\mathbb{D}$ be a pointed derivator and $\mathcal{K}$ a small category. Then,*

i) $\mathbb{D}$ is pointed if and only if $\mathbb{D}^{op}$ is pointed.

ii) $\mathbb{D}$ is pointed if and only if $\mathbb{D}^{\mathcal{K}}$ is pointed.

Proof. i) **(⇒)** If $\mathbb{D}$ is pointed, then $\mathbb{D}(\mathbb{1})$ has a zero object, hence $(\mathbb{D}(\mathbb{1}))^{op}$ has a zero object. Since $\mathbb{D}^{op}(\mathbb{1}) = (\mathbb{D}(\mathbb{1}^{op}))^{op} = (\mathbb{D}(\mathbb{1}))^{op}$ we are done.
(⇐) We assume $\mathbb{D}^{op}$ is pointed, and by (⇒) its opposite must be pointed, which is $(\mathbb{D}^{op})^{op} = \mathbb{D}$.

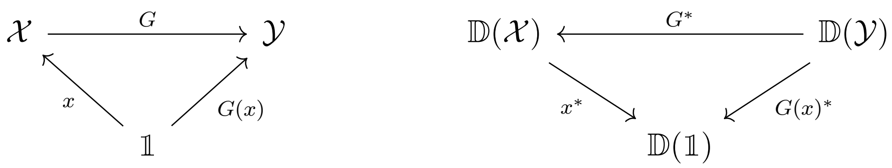
ii) **(⇒)** By corollary 3.20 we know $\mathbb{D}^{\mathcal{K}}(\mathbb{1}) \cong \mathbb{D}(\mathcal{K})$ has an initial and terminal object, say i and t. The morphism between them, $i \xrightarrow{f} t$, is an isomorphism in $\mathbb{D}(\mathcal{K})$ if and only if $i_k \xrightarrow{f_k} t_k$ is an isomorphism in $\mathbb{D}(\mathbb{1})$ for all $k \in \mathcal{K}$. As k^* is both left and right adjoint, we know by observation 1.12 that i_k and t_k are initial and terminal in $\mathbb{D}(\mathbb{1})$, respectively. In addition, $\mathbb{D}$ is pointed, hence must both i_k and t_k be the zero element in $\mathbb{D}(\mathbb{1})$. This implies f_k is an isomorphism for all k and hence f is an

isomorphism.
($\Leftarrow$) If $\mathbb{D}^{\mathcal{K}}$ is pointed for all $\mathcal{K} \in \mathcal{C}at$, then in particular $\mathbb{D}^{\mathbb{1}} \cong \mathbb{D}$ is pointed.

□

Lemma 4.5. *Let $\mathcal{X} \xrightarrow{G} \mathcal{Y}$ be a functor. Then G^*, G^L and G^R preserve zero.*

Proof. Let $x \in \mathcal{X}$ and consider the induced commuting triangle on the right.



$\mathbb{D}(\mathbb{1})$ has a zero object by definition, and by lemma 4.4 ii) both $\mathbb{D}(\mathcal{Y})$ and $\mathbb{D}(\mathcal{X})$ also have zero objects. $G(x)^*$ and x^* are both left and right adjoints, implying $G(x)^*(0_{\mathcal{Y}}) = 0_{\mathbb{1}}$ and $x^*(0_{\mathcal{X}}) = 0_{\mathbb{1}}$, by observation 1.12.

Consider the unique morphism $G^*(0_{\mathcal{Y}}) \to 0_{\mathcal{X}}$. This is an isomorphism if and only if $x^*G^*(0_{\mathcal{Y}}) \to x^*0_{\mathcal{X}}$ is an isomorphism in $\mathbb{D}(\mathbb{1})$, which is true because by the two equations above. Hence G^* preserves zero, and again by observation 1.12 G^L and G^R preserve zero. □

Lemma 4.6. *Let $\mathcal{X}, \mathcal{Y}$ be small categories and $\mathcal{X} \xrightarrow{G} \mathcal{Y}$ a fully faithful functor. For $B \in \mathbb{D}(\mathcal{Y})$, then*

i) the following are equivalent

1) $B \in essIm(G^R)$.

2) the adjunction unit $Id \xrightarrow{\eta} G^RG^*$ induces an isomorphism $B \to G^RG^*(B)$.

3) the adjunction unit $Id \xrightarrow{\eta} G^RG^*$ induces an isomorphism $B_y \to G^RG^*(B)_y$ for all $y \in \mathcal{Y} - G(\mathcal{X})$.

ii) the following are equivalent

1) $B \in essIm(G^L)$.

2) the adjunction counit $G^LG^* \xrightarrow{\epsilon} Id$ induces an isomorphism $G^LG^*(B) \to B$.

3) the adjunction counit $G^LG^* \xrightarrow{\epsilon} Id$ induces an isomorphism $G^LG^*(B)_y \to B_y$ for all $y \in \mathcal{Y} - G(\mathcal{X})$.

Proof. We prove ii). By lemma 3.15 we have that G^L is fully faithful.

[1) $\iff$ 2)]

$$\begin{aligned} & B \in essIm(G^L) \\ \iff & B \cong G^L(A) \text{ for some } A \in \mathbb{D}(\mathcal{X}) \\ \iff & G^*(B) \cong G^*G^L(A) \cong A \quad \text{using lemma 1.14} \\ \iff & G^LG^*(B) \cong G^L(A) \cong B \end{aligned}$$

[2) $\iff$ 3)] By Der ii) it is clear that 2) $\Rightarrow$ 3). It remains to show that if $y \in G(\mathcal{X})$ then we have that $G^LG^*(B)_y \to B_y$ is an isomorphism. Therefore, let $y = G(x)$ for some $x \in \mathcal{X}$, then

$$B_y \cong G^*(B)_x \cong G^*G^LG^*(B)_x \cong G^LG^*(B)_y$$

where the first and last steps are given by the commuting triangle

$$\begin{array}{ccc} \mathbb{D}(\mathcal{X}) & \xleftarrow{\quad G^* \quad} & \mathbb{D}(\mathcal{Y}) \\ & {\scriptstyle x^*}\searrow \quad \swarrow{\scriptstyle y^*} & \\ & \mathbb{D}(\mathbb{1}) & \end{array}$$

and the middle step is by lemma 1.14. $\square$

It is time to introduce some useful categories. Let $[1]$ be the category with two objects and a morphism between them, $0 \to 1$.This is often called the *arrow category*. We denote by $\square$ the category $[1] \times [1]$ which is the following poset considered as a category.

$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \downarrow \\ (0,1) & \longrightarrow & (1,1) \end{array}$$

In addition, there are full subcategories $\ulcorner$ and $\lrcorner$, together with the canonical inclusions $\ulcorner \xrightarrow{i_\ulcorner} \square$ and $\lrcorner \xrightarrow{i_\lrcorner} \square$.

$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \\ (0,1) & & \end{array} \qquad\qquad \begin{array}{ccc} & & (1,0) \\ & & \downarrow \\ (0,1) & \longrightarrow & (1,1) \end{array}$$

Definition 4.7. Let $\mathcal{X} \xrightarrow{G} \mathcal{Y}$ be a fully faithful, then

i) G is a *sieve* if whenever there is a morphism $y \to G(x)$ it implies $y \in G(\mathcal{X})$.

ii) G is a *cosieve* if whenever there is a morphism $G(x) \to y$ it implies $y \in G(\mathcal{X})$.

There is an immediate example of a sieve and cosieve.

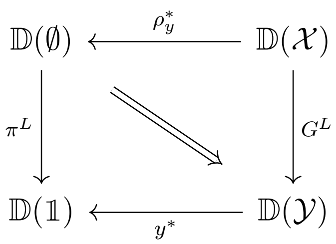
Example 4.8. The functor $\ulcorner \xrightarrow{i_\ulcorner} \square$ is a sieve, and $\lrcorner \xrightarrow{i_\lrcorner} \square$ is a cosieve.

Lemma 4.9. *$\mathbb{D}$ is a pointed derivator and $\mathcal{X} \xrightarrow{G} \mathcal{Y}$ is a functor.*

i) If G is a sieve, then G^R is fully faithful and $B \in \mathbb{D}(\mathcal{Y})$ lies in the essential image of G^R if and only if $B_y \cong 0$ for all $y \in \mathcal{Y} - G(\mathcal{X})$.

ii) If G is a cosieve, then G^L is fully faithful and $B \in \mathbb{D}(\mathcal{Y})$ lies in the essential image of G^L if and only if $B_y \cong 0$ for all $y \in \mathcal{Y} - G(\mathcal{X})$.

Proof. We prove ii). Since G is a cosieve it is fully faithful, and by lemma 3.15, G^L is also fully faithful. By the previous lemma we need to show that $G^L G^*(B)_y \xrightarrow{\epsilon_y} B_y$ is an isomorphism for all $y \in \mathcal{Y} - G(\mathcal{X})$ if and only if $B_y \cong 0$ for all $y \in \mathcal{Y} - G(\mathcal{X})$. In aid of this, let $y \in \mathcal{Y} - G(\mathcal{X})$. In this case the slice category $[G \to y]$ is the empty set and by Der iv) we have $y^* G^L G^*(B) \cong \pi^L \rho_y^* G^*(B)$

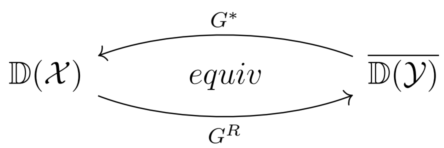


giving us that

$$B_y \cong G^L G^*(B)_y \cong \pi^L \rho_y^* G^*(B) \cong \pi^L(0) \cong 0$$

where the last step by the fact that adjoints preserve zero (lemma 4.5). $\square$

Remark 4.10. Let $\overline{\mathbb{D}(\mathcal{Y})}$ be the full subcategory of $\mathbb{D}(\mathcal{Y})$ such that all objects are in the essential image of G^R. Then G^R will be dense and fully faithful, and hence an equivalence.



Example 4.11. Let $[1] \xrightarrow{h} \ulcorner$ be the functor identifying the horizontal morphism and let $A \in \mathbb{D}([1])$ with underlying shape $a \xrightarrow{b} f$. Then h is a sieve and by lemma 4.9 we have that $h^R(A)$ has the following underlying shape. This is the reason it might be appropriate to think of the lemma as an *extension by zero*.

$$
\begin{array}{ccccc}
a \xrightarrow{f} b & \xmapsto{h^R} & a & \xrightarrow{f} & b \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}
$$

In the example, we see that $h^R(A)_{(0,0)} \cong A_{(0,0)}$ and $h^R(A)_{(1,0)} \cong A_{(1,0)}$. The subtle reason for this is mainly Der iv). Consider the following where we have applied the derivator on a slice diagram in order to understand $(0,0)^* h^R(A)$.

$$
\begin{array}{ccc}
\mathbb{D}([(0,0) \to h]) & \xleftarrow{\rho^*_{(0,0)}} & \mathbb{D}([1]) \\
{\scriptstyle \pi^R}\big\downarrow & \overset{\cong}{\Leftarrow} & \big\downarrow{\scriptstyle h^R} \\
\mathbb{D}(\mathbb{1}) & \xleftarrow[(0,0)^*]{} & \mathbb{D}(\ulcorner)
\end{array}
$$

The slice category $[(0,0) \to h]$ has the initial object $((0,0), id)$. By the dual of observation 1.13, and observation 1.22, $(\pi^*, (0,0))^*$ is an adjoint pair, meaning $\pi^R \cong (0,0)^*$. In addition, $[(0,0) \to h] \cong [1]$, hence $\rho_{(0,0)} \cong Id$. This implies

$$
(0,0)^* h^R(A) \cong \pi^R \rho_{(0,0)}(A) \cong (0,0)^* \rho_{(0,0)}(A) \cong (0,0)^*(A)
$$

The case for $(1,0)$ is simpler as the slice category will be equivalent to $\mathbb{1}$. These arguments hold for many similar cases and will be used implicitly multiple times throughout the thesis.

4.2 Fiber, cone, suspension and loop

Definition 4.12. Let $\mathbb{D}$ be a derivator and $B \in \mathbb{D}(\square)$.

i) B is said to be *cocartesian* if B is in the essential image of $i_{\ulcorner}^{L}$.

ii) B is said to be *cartesian* if B is in the essential image of $i_{\lrcorner}^{R}$.

Consider the following compositions where h and v identify the horizontal and vertical morphisms, respectively.

$$[1] \xrightarrow{h} \ulcorner \xrightarrow{i_{\ulcorner}} \square \xleftarrow{v} [1]$$

$$[1] \xrightarrow{v} \lrcorner \xrightarrow{i_{\lrcorner}} \square \xleftarrow{h} [1]$$

Definition 4.13. $\mathbb{D}$ is a pointed derivator. We define

i) the *cone functor* as the composition

$$Cone : \mathbb{D}([1]) \xrightarrow{h^R} \mathbb{D}(\ulcorner) \xrightarrow{i_{\ulcorner}^{L}} \mathbb{D}(\square) \xrightarrow{v^*} \mathbb{D}([1])$$

ii) the *fiber functor* as the composition

$$Fiber : \mathbb{D}([1]) \xrightarrow{v^L} \mathbb{D}(\lrcorner) \xrightarrow{i_{\lrcorner}^{R}} \mathbb{D}(\square) \xrightarrow{h^*} \mathbb{D}([1])$$

As seen in the previous example, h is a sieve and h^R is therefore an extension by zero. Let $A \in \mathbb{D}([1])$ have the underlying shape $a \xrightarrow{f} b$. Then $i_{\ulcorner}^{L} h^R(A)$ is cocartesian by definition and will have an underlying shape of

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow{\scriptstyle cone(f)} \\ 0 & \longrightarrow & C(f) \end{array}$$

where $C(f)$ is notation for $i_{\ulcorner}^{L} h^R(A)_{(1,1)}$. Likewise, the squares in $\mathbb{D}(\square)$ that lie in the essential image of $i_{\lrcorner}^{R} v^L$ will be cartesian and have an underlying shape of

$$\begin{array}{ccc} F(f) & \xrightarrow{fiber(f)} & a \\ \downarrow & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & b \end{array}$$

Example 4.14. The cone functor is a generalisation of a cokernel. Let $\mathbb{D} = \mathbb{D}_{rep}$ and let $A \in \mathbb{D}([1]) = \mathcal{C}^{[1]}$ have the underlying shape $a \xrightarrow{f} b$ as above. We know $h^R(A)$ is the extension by zero and want to see what $i_{\ulcorner}^L$ does to this diagram, in particular at the corner $(1,1)$. Consider the following slice square and the induced square using $\mathbb{D}_{rep}$.

$$\begin{array}{ccc} [i_{\ulcorner} \to (1,1)] & \xrightarrow{\rho_{(1,1)}} & \ulcorner \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle i_{\ulcorner}} \\ \mathbb{1} & \xrightarrow[(1,1)]{} & \square \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathcal{C}^{[i_{\ulcorner} \to (1,1)]} & \xleftarrow{Id} & \mathcal{C}^{\ulcorner} \\ {\scriptstyle \pi^L}\downarrow & & \downarrow{\scriptstyle i_{\ulcorner}^L} \\ \mathcal{C} & \xleftarrow[(1,1)^*]{} & \mathcal{C}^{\square} \end{array}$$

Here we are using that $[i_{\ulcorner} \to (1,1)] \cong \ulcorner$. So for $h^R(A)$ in $\mathcal{C}^{\ulcorner}$, Der iv) gives us that $(1,1)^* i_{\ulcorner}^L(A) \cong \pi^L(A)$. As we saw in example 2.8, this is the colimit of A. And the colimit of the diagram

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \\ 0 & & \end{array}$$

is the cokernel of f. Similarly, the fiber is a generalisation of the kernel.

Lemma 4.15. $\mathbb{D}$ *is a pointed derivator. Then there is an adjoint pair*

$$(cone, fiber) : \mathbb{D}([1]) \rightleftarrows \mathbb{D}([1])$$

Proof. We consider the full subcategories $\overline{\mathbb{D}(\ulcorner)}$, $\overline{\mathbb{D}(\square)}$ and $\overline{\mathbb{D}(\ulcorner)}$, all consisting of objects B such that $B_{(0,1)} \cong 0$. By construction of $\overline{\mathbb{D}(\ulcorner)}$ and $\overline{\mathbb{D}(\ulcorner)}$, h^R and v^L will be dense and therefore equivalences (similar to remark 4.10). The two inner pairs of functors are adjoint by definition.

$$\mathbb{D}([1]) \underset{h^*}{\overset{h^R}{\rightleftarrows}} \overline{\mathbb{D}(\ulcorner)} \underset{i_{\ulcorner}^*}{\overset{i_{\ulcorner}^L}{\rightleftarrows}} \overline{\mathbb{D}(\square)} \underset{i_{\lrcorner}^*}{\overset{i_{\lrcorner}^R}{\leftrightarrows}} \overline{\mathbb{D}(\lrcorner)} \underset{v^*}{\overset{v^L}{\leftrightarrows}} \mathbb{D}([1])$$

$\square$

Consider the following composition where $(0,0)$ and $(1,1)$ identify the respective corners.

$$\mathbb{1} \xrightarrow{(0,0)} \ulcorner \xrightarrow{i_\ulcorner} \square \xleftarrow{(1,1)} \mathbb{1}$$

$$\mathbb{1} \xrightarrow{(1,1)} \lrcorner \xrightarrow{i_\lrcorner} \square \xleftarrow{(0,0)} \mathbb{1}$$

Definition 4.16. $\mathbb{D}$ is a pointed derivator. We define

i) the *suspension functor* as the composition

$$\Sigma : \mathbb{D}(\mathbb{1}) \xrightarrow{(0,0)^R} \mathbb{D}(\ulcorner) \xrightarrow{i_\ulcorner^L} \mathbb{D}(\square) \xrightarrow{(1,1)^*} \mathbb{D}(\mathbb{1})$$

ii) the *loop functor* as the composition

$$\Omega : \mathbb{D}(\mathbb{1}) \xrightarrow{(1,1)^L} \mathbb{D}(\lrcorner) \xrightarrow{i_\lrcorner^R} \mathbb{D}(\square) \xrightarrow{(0,0)^*} \mathbb{D}(\mathbb{1})$$

Again, we notice that $(0,0)$ is a sieve and therefore $(0,0)^R$ is an extension by zero. Any $A \in \mathbb{D}(\square)$ which is in the essential image of $i_\ulcorner^L(0,0)^R$ is cocartesian and will have an underlying shape of

$$\begin{array}{ccc} a & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma a \end{array}$$

Likewise, there will be a cartesian square $A \in \mathbb{D}(\square)$ with an underlying shape of

$$\begin{array}{ccc} \Omega a & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & a \end{array}$$

Example 4.17. For the represented derivator, this is a special case of example 4.14, so Σa is the cokernel of 0, namely 0. Hence, the functor $\Sigma \cong 0 : \mathcal{C} \to \mathcal{C}$.

Lemma 4.18. $\mathbb{D}$ *is a pointed derivator. Then there is an adjoint pair*

$$(\Sigma, \Omega) : \mathbb{D}(\mathbb{1}) \rightleftarrows \mathbb{D}(\mathbb{1})$$

Proof. Analogous to lemma 4.15. $\square$

4.3 Pointwise cartesian and cocartesian

This section aims to prove that if there is a (co)cartesian square B in a category $\mathbb{D}(\mathcal{K}\times\square)$, then that is the same as the underlying pointwise squares B_k being (co)cartesian in $\mathbb{D}(\square)$ for all $k \in \mathcal{K}$.

Lemma 4.19. *Let $\mathbb{D}$ be a pointed derivator, $\mathbb{1} \xrightarrow{i} \mathcal{X}$ is the functor identifying the initial object i in $\mathcal{X}$, and $\mathbb{1} \xrightarrow{0} [1]$ is a particular such functor.*

i) $B \in \mathbb{D}(\mathcal{X})$ is in the essential image of i^L if and only if $B_i \cong B_x$ for all $x \in \mathcal{X}$.

ii) Let $[1] \times \mathbb{1} \xrightarrow{Id\times 0} [1] \times [1] \cong \square$ be the functor identifying the top horizontal morphism, and let $B \in \mathbb{D}(\square)$. Then B is in the essential image of $(Id \times 0)^L$ if and only if the vertical morphisms $B_{(0,0)} \to B_{(0,1)}$ and $B_{(1,0)} \to B_{(1,1)}$ are isomorphisms.

iii) Let $B \in \mathbb{D}(\square)$. Assume $B_{(0,0)} \to B_{(0,1)}$ is an isomorphism. Then B is cocartesian if and only if $B_{(1,0)} \to B_{(1,1)}$ is an isomorphism.

Proof. i) Notice first that the slice category $[i \to x]$ only has a single element because i is initial. Hence the following square commutes using Der iv) and that $[i \to x] \cong \mathbb{1}$.

$$\begin{array}{ccc} \mathbb{D}(\mathbb{1}) & \xleftarrow{Id} & \mathbb{D}(\mathbb{1}) \\ \Big\downarrow{\scriptstyle Id} & & \Big\downarrow{\scriptstyle i^L} \\ \mathbb{D}(\mathbb{1}) & \xleftarrow[x^*]{} & \mathbb{D}(\mathcal{X}) \end{array}$$

This means $B \cong i^L A$ for some $A \in \mathbb{D}(\mathbb{1})$ if and only if $B_x \cong x^*B \cong x^*i^LA \cong A$ for all $x \in \mathcal{X}$ including i. Hence, $B \in essIm(i^L)$ if and only if $B_i \cong A \cong B_x$.

ii) An important observation is that $[1] \times \mathbb{1} \xrightarrow{Id\times 0} [1] \times [1]$ is the same functor as $\mathbb{D}^{[1]}(\mathbb{1}) \xrightarrow{0^L} \mathbb{D}^{[1]}([1])$, where 0^L is notation for $(\mathbb{D}^{[1]}(0))^L$. By i) $B \in essIm(0^L)$ if and only if $B_0 \cong B_1 \in \mathbb{D}^{[1]}(\mathbb{1}) \cong \mathbb{D}([1])$. Looking at this in the underlying category $\mathbb{D}(\mathbb{1})$ using the functors $0^*, 1^* : \mathbb{D}([1]) \to \mathbb{D}(\mathbb{1})$, this means $B_{(0,0)} \to B_{(0,1)}$ and $B_{(1,0)} \to B_{(1,1)}$ are isomorphisms.

iii) ($\Leftarrow$) We observe that $[1] \xrightarrow{Id\times 0} \square$ is the composition

$$[1] \xrightarrow{h} \ulcorner \xrightarrow{i_\ulcorner} \square$$

where h identifies the horizontal morphism. By the assumption and ii) we have that $B \in essIm((Id \times 0)^L)$, implying $B \in essIm((i_\ulcorner h)^L)$ which means B is cocartesian.
(⇒) We assume B is cocartesian, so let $B \cong i_\ulcorner^L A$ for some $A \in \mathbb{D}(\ulcorner)$. If we can show that $A \in essIm(h^L)$, then $B \in essIm((Id \times 0)^L)$ and by ii) we will be done. Showing that $A \in essIm(h^L)$ is the same as showing $h^L h^* A \cong A$ by 4.6 ii). By assumption, $A_{(0,0)} \cong A_{(0,1)}$, hence the underlying shape of A is

$$\begin{array}{ccc} A_{(0,0)} & \longrightarrow & A_{(1,0)} \\ {\scriptstyle\cong}\big\downarrow & & \\ A_{(0,1)} & & \end{array}$$

Using h^* on A we restrict to the morphism $A_{(0,0)} \to A_{(1,0)}$, and using h^L on h^*A gives us exactly the same underlying shape as above, meaning $h^L h^* A \cong A$.

Above there are three subtle uses of the discussion after example 4.11. These are that the left vertical morphism in A is an isomorphism because it is in B, that h^*A has underlying shape $A_{(0,0)} \to A_{(1,0)}$, and that the left vertical morphism in $h^L h^* A$ is an isomorphism. The confused reader is encouraged to write out the corresponding slice categories and squares, and hopefully see that they reduce to similar cases as the mentioned discussion.

□

Lemma 4.20. *$\mathbb{D}$ is a derivator. Let $\mathcal{X}, \mathcal{X}', \mathcal{Y}, \mathcal{Y}'$ be small categories, and let $\mathcal{X} \xrightarrow{U} \mathcal{X}'$, $\mathcal{Y} \xrightarrow{V} \mathcal{Y}'$ be functors. Then, given the squares*

$$\begin{array}{ccc} \mathcal{Y} \times \mathcal{X} & \xrightarrow{V \times Id_{\mathcal{X}}} & \mathcal{Y}' \times \mathcal{X} \\ {\scriptstyle Id_{\mathcal{Y}} \times U}\big\downarrow & \overset{Id_1}{\swarrow} & \big\downarrow{\scriptstyle Id_{\mathcal{Y}'} \times U} \\ \mathcal{Y} \times \mathcal{X}' & \xrightarrow[V \times Id_{\mathcal{X}'}]{} & \mathcal{Y}' \times \mathcal{X}' \end{array} \qquad \begin{array}{ccc} \mathcal{Y} \times \mathcal{X} & \xrightarrow{V \times Id_{\mathcal{X}}} & \mathcal{Y}' \times \mathcal{X} \\ {\scriptstyle Id_{\mathcal{Y}} \times U}\big\downarrow & \overset{Id_2}{\nearrow} & \big\downarrow{\scriptstyle Id_{\mathcal{Y}'} \times U} \\ \mathcal{Y} \times \mathcal{X}' & \xrightarrow[V \times Id_{\mathcal{X}'}]{} & \mathcal{Y}' \times \mathcal{X}' \end{array}$$

the induced Beck-Chevalley transformations, $Id_1^!$ and $Id_2^!$, are isomorphisms.

$$\begin{array}{ccc} \mathbb{D}(\mathcal{Y} \times \mathcal{X}) & \xleftarrow{(V \times Id_{\mathcal{X}})^*} & \mathbb{D}(\mathcal{Y}' \times \mathcal{X}) \\ {\scriptstyle (Id_{\mathcal{Y}} \times U)^L}\big\downarrow & \overset{Id_1^!}{\searrow} & \big\downarrow{\scriptstyle (Id_{\mathcal{Y}'} \times U)^L} \\ \mathbb{D}(\mathcal{Y} \times \mathcal{X}') & \xleftarrow[(V \times Id_{\mathcal{X}'})^*]{} & \mathbb{D}(\mathcal{Y}' \times \mathcal{X}') \end{array} \qquad \begin{array}{ccc} \mathbb{D}(\mathcal{Y} \times \mathcal{X}) & \xleftarrow{(V \times Id_{\mathcal{X}})^*} & \mathbb{D}(\mathcal{Y}' \times \mathcal{X}) \\ {\scriptstyle (Id_{\mathcal{Y}} \times U)^R}\big\downarrow & \overset{Id_2^!}{\nwarrow} & \big\downarrow{\scriptstyle (Id_{\mathcal{Y}'} \times U)^R} \\ \mathbb{D}(\mathcal{Y} \times \mathcal{X}') & \xleftarrow[(V \times Id_{\mathcal{X}'})^*]{} & \mathbb{D}(\mathcal{Y}' \times \mathcal{X}') \end{array}$$

Proof. We show that $Id^!$ is an isomorphism. Consider the following diagram.

$$\begin{array}{ccccc}
\mathcal{Y}\times[U\to x'] & \xrightarrow{Id\times\rho_{x'}} & \mathcal{Y}\times\mathcal{X} & \xrightarrow{V\times Id_{\mathcal{X}}} & \mathcal{Y}'\times\mathcal{X} \\
{\scriptstyle Id\times\pi}\downarrow & \swarrow & \downarrow{\scriptstyle Id_{\mathcal{Y}}\times U} & \overset{Id_1}{\swarrow} & \downarrow{\scriptstyle Id_{\mathcal{Y}'}\times U} \\
\mathcal{Y}\times\mathbb{1} & \xrightarrow[Id\times x']{} & \mathcal{Y}\times\mathcal{X}' & \xrightarrow[V\times Id_{\mathcal{X}'}]{} & \mathcal{Y}'\times\mathcal{X}'
\end{array}$$

Using the derivator $\mathbb{D}^{\mathcal{Y}}$ and Der iv), we have that the induced Beck-Chevalley transformation in the left square is an isomorphism. By lemma 3.13, if the Beck-Chevalley transformation associated to the outer square is an isomorphism, we are done. To this end, consider the diagram with the same outer square.

$$\begin{array}{ccccc}
\mathcal{Y}\times[U\to x'] & \xrightarrow{V\times Id_{slice}} & \mathcal{Y}'\times[U\to x'] & \xrightarrow{Id_{\mathcal{Y}'}\times\rho_{x'}} & \mathcal{Y}'\times\mathcal{X} \\
{\scriptstyle Id_{\mathcal{Y}}\times\pi}\downarrow & \swarrow & {\scriptstyle Id_{\mathcal{Y}}\times\pi}\downarrow & \swarrow & \downarrow{\scriptstyle Id_{\mathcal{Y}'}\times U} \\
\mathcal{Y}\times\mathbb{1} & \xrightarrow[V\times Id_{\mathbb{1}}]{} & \mathcal{Y}'\times\mathbb{1} & \xrightarrow[Id_{\mathcal{Y}'}\times x']{} & \mathcal{Y}'\times\mathcal{X}'
\end{array}$$

The right square Beck-Chevalley transformation is an isomorphism by similar reasoning as above, only using $\mathbb{D}^{\mathcal{Y}'}$ instead. Therefore, if the left square Beck-Chevalley transformation is an isomorphism, we are done. Consider the following diagram where $[U \to x']$ is denoted as $\mathcal{S}$ for simplicity.

$$\begin{array}{ccccccc}
\mathcal{S} & \xrightarrow{(y,id)\times Id_{slice}} & [Id_{\mathcal{Y}}\to y]\times\mathcal{S} & \xrightarrow{\rho_y\times Id_{slice}} & \mathcal{Y}\times\mathcal{S} & \xrightarrow{V\times Id_{slice}} & \mathcal{Y}'\times\mathcal{S} \\
{\scriptstyle \pi}\downarrow & \swarrow & {\scriptstyle \pi}\downarrow & \swarrow & {\scriptstyle Id_{\mathcal{Y}}\times\pi}\downarrow & \swarrow & \downarrow{\scriptstyle Id_{\mathcal{Y}'}\times\pi} \\
\mathbb{1} & \xrightarrow[Id_{\mathbb{1}}]{} & \mathbb{1} & \xrightarrow[y]{} & \mathcal{Y}\times\mathbb{1} & \xrightarrow[V\times Id_{\mathbb{1}}]{} & \mathcal{Y}'\times\mathbb{1}
\end{array} \qquad (4.1)$$

The middle square is a slice square by realising that $[Id_{\mathcal{Y}} \to y] \times \mathcal{S} \cong [(Id_{\mathcal{Y}} \times \pi) \to y]$, meaning that the associated Beck-Chevalley transformation is an isomorphism. The left square also induces an isomorphism by using that (y, id) is a terminal object in $[Id_{\mathcal{Y}} \to y]$ together with observation 1.13 and lemma 3.14. This means that if the outer square Beck-Chevalley transformation is an isomorphism, we are done. The outer square is the same as the following square.

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{V(y)\times Id_{slice}} & \mathcal{Y}'\times\mathcal{S} \\ {\scriptstyle\pi}\Big\downarrow & \Swarrow & \Big\downarrow{\scriptstyle Id_{\mathcal{Y}'}\times\pi} \\ \mathbb{1} & \xrightarrow[V(y)]{} & \mathcal{Y}'\times\mathbb{1} \end{array}$$

which can be extended to the following square, denoting $V(y)$ as y'.

$$\begin{array}{ccccc} \mathcal{S} & \xrightarrow{(y',id)\times Id_{slice}} & [Id_{\mathcal{Y}'}\to y']\times\mathcal{S} & \xrightarrow{\rho_{y'}\times Id_{slice}} & \mathcal{Y}'\times\mathcal{S} \\ {\scriptstyle\pi}\Big\downarrow & \Swarrow & \Big\downarrow{\scriptstyle\pi} & \Swarrow & \Big\downarrow{\scriptstyle Id_{\mathcal{Y}'}\times\pi} \\ \mathbb{1} & \xrightarrow[Id]{} & \mathbb{1} & \xrightarrow[y']{} & \mathcal{Y}'\times\mathbb{1} \end{array}$$

Comparing this diagram to diagram 4.1, we realise that we have almost exactly the same squares as the two leftmost squares in 4.1. Therefore, by the same reasoning as above, both squares induce natural isomorphisms, and by one last reference to lemma 3.13 we are done.

□

Lemma 4.21. *$\mathbb{D}$ is a pointed derivator. Let $\mathcal{X}, \mathcal{X}', \mathcal{Y}$ be small categories, and $\mathcal{X} \xrightarrow{U} \mathcal{X}'$ a fully faithful functor. Consider the diagram where $\mathbb{D}$ has been applied to the left square.*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{Id\times y} & \mathcal{X}\times\mathcal{Y} \\ {\scriptstyle U}\Big\downarrow & & \Big\downarrow{\scriptstyle U\times Id} \\ \mathcal{X}' & \xrightarrow[Id\times y]{} & \mathcal{X}'\times\mathcal{Y} \end{array} \quad\rightsquigarrow\quad \begin{array}{ccc} \mathbb{D}(\mathcal{X}) & \xleftarrow{y^*} & \mathbb{D}(\mathcal{X}\times\mathcal{Y}) \\ {\scriptstyle U^L}\Big\downarrow\Big\downarrow{\scriptstyle U^R} & & {\scriptstyle (U\times Id)^L}\Big\downarrow\Big\downarrow{\scriptstyle (U\times Id)^R} \\ \mathbb{D}(\mathcal{X}') & \xleftarrow[y^*]{} & \mathbb{D}(\mathcal{X}'\times\mathcal{Y}) \end{array}$$

Let $B \in \mathbb{D}(\mathcal{X}'\times\mathcal{Y})$. Then

i) B is in the essential image of $(U\times Id)^L$ if and only if B_y is in the essential image of U^L for all $y \in \mathcal{Y}$.

ii) B is in the essential image of $(U\times Id)^R$ if and only if B_y is in the essential image of U^R for all $y \in \mathcal{Y}$.

Proof. We show i). U is fully faithful, hence $U \times Id$ is fully faithful. By lemma 4.6, B is in the essential image of $(U \times Id)^L$ if and only if the counit $(U \times Id)^L(U \times Id)^*B \to B$ is an isomorphism. This counit is the same as the induced Beck-Chevalley transformation from the right square in the diagram

$$
\begin{array}{ccccc}
\mathcal{X} & \xrightarrow{Id \times y} & \mathcal{X} \times \mathcal{Y} & \xrightarrow{(U \times Id)} & \mathcal{X}' \times \mathcal{Y} \\
\downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U \times Id} & & \downarrow{\scriptstyle Id} \\
\mathcal{X}' & \xrightarrow[Id \times y]{} & \mathcal{X}' \times \mathcal{Y} & \xrightarrow[Id]{} & \mathcal{X}' \times \mathcal{Y}
\end{array}
$$

The left square induces an isomorphism (by lemma 4.20), implying that B lies in the essential image of $(U \times Id)^L$ if and only if the outer square induces an isomorphism, by lemma 3.13.

The counit $U^L U^* \to Id$ is the Beck-Chevalley transformation associated to the following left square, so by similar reasoning we see that B_y is in the essential image of U^L for all $y \in \mathcal{Y}$ if and only if the outer square induces an isomorphism.

$$
\begin{array}{ccccc}
\mathcal{X} & \xrightarrow{U} & \mathcal{X}' & \xrightarrow{Id \times y} & \mathcal{X}' \times \mathcal{Y} \\
\downarrow{\scriptstyle U} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id} \\
\mathcal{X}' & \xrightarrow[Id]{} & \mathcal{X}' & \xrightarrow[Id \times y]{} & \mathcal{X}' \times \mathcal{Y}
\end{array}
$$

As the two outer squares agree, we are done. □

The previous lemma immediately provides the following result.

Corollary 4.22. $\mathbb{D}$ *is a pointed derivator. Let* $\mathcal{K}$ *be a small category and* $B \in \mathbb{D}^{\mathcal{K}}(\square)$. *Then*

i) B *is cocartesian if and only if* B_k *is cocartesian for all* $k \in \mathcal{K}$.

ii) B *is cartesian if and only if* B_k *is cartesian for all* $k \in \mathcal{K}$.

5 Stable derivators

In the last chapter we introduced cartesian and cocartesian squares. Stable derivators require that these two notions coincide. In section 5.1 we explore related properties, in particular the two out of three-property in the context of derivators and the characterisation of stable derivators. Along the way we prove lemma 5.6 and 5.9, both dealing with the nuances of generating (co)cartesian squares. The most important statement of the chapter is found in section 5.2, that is, stable derivators give rise to additive categories.

5.1 Definitions and properties

Definition 5.1. A derivator $\mathbb{D}$ is said to be *stable* if it is pointed and every $B \in \mathbb{D}(\square)$ is cartesian if and only if it is cocartesian. Squares that are both cartesian and cocartesian are called *bicartesian*.

Example 5.2. It can be shown that a derivator $\mathbb{D}$ is stable if and only if the adjunction $(\Sigma, \Omega) : \mathbb{D}(\mathbb{1}) \rightleftarrows \mathbb{D}(\mathbb{1})$ is an equivalence (see [GPS13] theorem 7.1). For the represented derivator, $\Sigma \cong 0$ (example 4.17), hence it is stable if and only if $\mathcal{C} \cong \mathbb{1}$.

Example 5.3. With some constraint on $\mathcal{A}$, the homotopy derivator $\mathbb{D}_{\mathcal{A}}$ is stable. See [Gro19, ex. 3.30] for details.

Lemma 5.4. *$\mathbb{D}$ is a derivator. $\mathbb{D}$ is stable if and only if $\mathbb{D}^{op}$ is stable.*

Proof. By lemma 4.4 i), we only need to check that cartesian and cocartesian squares are bicartesian.

$$\begin{aligned}
&A \text{ cocartesian in } \mathbb{D}^{op}(\square) \\
\iff\ & A \in essIm \text{ of } \mathbb{D}^{op}(\ulcorner) \xrightarrow{(\mathbb{D}^{op}(i_\ulcorner))^L} \mathbb{D}^{op}(\square) \\
\iff\ & A \in essIm \text{ of } \mathbb{D}(\ulcorner^{op})^{op} \xrightarrow{((i_\ulcorner^{op})^R)^{op}} \mathbb{D}(\square^{op})^{op} \\
\iff\ & A \in essIm \text{ of } \mathbb{D}(\ulcorner^{op}) \xrightarrow{(i_\ulcorner^{op})^R} \mathbb{D}(\square^{op}) \\
\iff\ & A \in essIm \text{ of } \mathbb{D}(\lrcorner) \xrightarrow{i_\lrcorner^R} \mathbb{D}(\square) \\
\iff\ & A \text{ cartesian in } \mathbb{D}(\square)
\end{aligned}$$

If we assume $\mathbb{D}$ is stable, we get that A is cocartesian in $\mathbb{D}^{op}(\square)$ if and only if A is cartesian in $\mathbb{D}(\Box)$ if and only if A is cocartesian in $\mathbb{D}(\Box)$ if and only if A is cartesian in $\mathbb{D}^{op}(\Box)$. Similarly for the opposite implication.

$\Box$

Lemma 5.5. *$\mathbb{D}$ is a derivator and $\mathcal{K}$ a small category. $\mathbb{D}$ is stable if and only if $\mathbb{D}^{\mathcal{K}}$ is stable.*

Proof. Again, by lemma 4.4 i), we only need to check that cartesian and cocartesian squares are bicartesian. Therefore, let $B \in \mathbb{D}^{\mathcal{K}}(\Box)$. Using corollary 4.22 and the assumption that $\mathbb{D}$ is stable we have

$$\begin{aligned} & B \text{ cocartesian in } \mathbb{D}^{\mathcal{K}}(\Box) \\ \iff & B_k \text{ cocartesian in } \mathbb{D}(\Box) \text{ for all } k \in \mathcal{K} \\ \iff & B_k \text{ cartesian in } \mathbb{D}(\Box) \text{ for all } k \in \mathcal{K} \\ \iff & B \text{ cartesian in } \mathbb{D}^{\mathcal{K}}(\Box) \end{aligned}$$

For the other direction, if $\mathbb{D}^{\mathcal{K}}$ is stable for all $\mathcal{K} \in \mathcal{C}at$, then in particular $\mathbb{D}^{\mathbb{1}} = \mathbb{D}$ is stable.

$\Box$

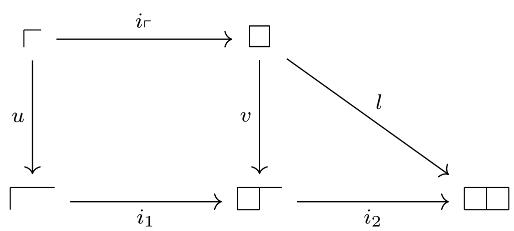
Lemma 5.6. *Let $l, r, o : \Box \to \Box\Box$ be the functors which send a square to the left, right or outer square, respectively. Let $\ulcorner\overline{\ \ } \xrightarrow{i_1} \Box\overline{\ \ } \xrightarrow{i_2} \Box\Box$ and $\ulcorner \xrightarrow{i_\ulcorner^L} \Box$ be the canonical inclusions, $i = i_1 i_2$, and $B \in \mathbb{D}(\Box\Box)$.*

i) $l^*(B) \in essIm(i_\ulcorner^L) \iff i_2^*(B) \in essIm(i_1^L)$

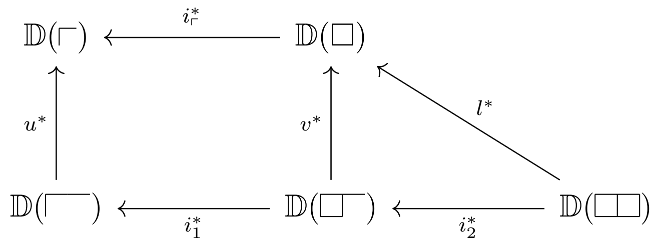
ii) $r^*(B) \in essIm(i_\ulcorner^L) \iff B \in essIm(i_2^L)$

iii) $o^*(B) \in essIm(i_\ulcorner^L) \iff B \in essIm(i^L)$

Proof. i) By lemma 4.6 we need to prove that $i_\ulcorner^L i_\ulcorner^* l^*(B) \to l^*(B)$ is a natural isomorphism if and only if $i_1^L i_1^* i_2^*(B) \to i_2^*(B)$ is a natural isomorphism. Consider the diagram

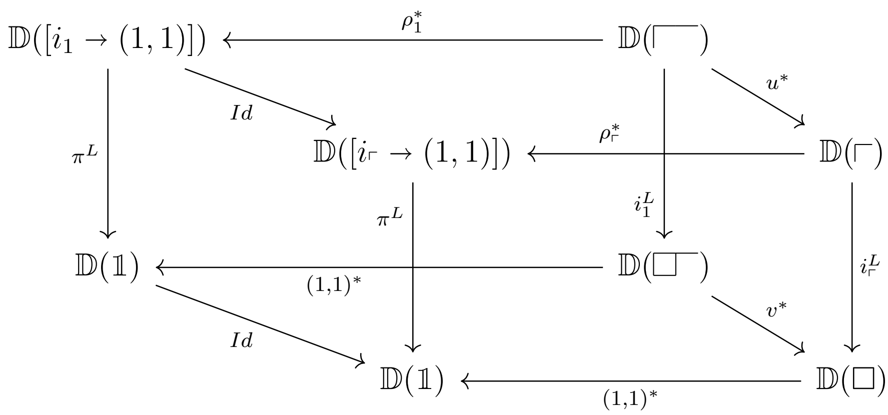


where u, v are inclusions such that everything commutes. Applying the derivator results in a diagram where everything still commutes.



Using this, we can rewrite $i_\ulcorner^L i_\ulcorner^* l^*(B) \to l(B)$ as $i_\ulcorner^L u^* i_1^* i_2^*(B) \to v^* i_2^*(B)$, which is a natural isomorphism if and only if $i_\ulcorner^L u^* i_1^* i_2^*(B)_p \to v^* i_2^*(B)_p$ is a natural isomorphism for all $p \in (\square - i_\ulcorner(\ulcorner)) = (1,1)$.

Now we would like to argue that $(1,1)^* i_\ulcorner^L u^* i_1^* i_2^*(B) = (1,1)^* i_1^L i_1^* i_2^*(B)$. In aid of this, denote $i_1^* i_2^*(B) = C$ and consider the cube induced by slice squares



There is no map from $(2,0)$ to $(1,1)$ in $\overline{\ulcorner\quad}$, so the two slice categories are actually equal, meaning the top square commutes. Using this and Der iv) we can now see that

$$\begin{aligned}
&(1,1)^* i_\ulcorner^L u^* i_1^* i_2^*(B) \\
&= (1,1)^* i_\ulcorner^L u^* C \\
&= \pi^L \rho_\ulcorner^* u^* C \\
&= \pi^L \rho_1^* C \\
&= (1,1)^* i_1^L C \\
&= (1,1)^* i_1^L i_1^* i_2^*(B)
\end{aligned}$$

With this information, $i_{\ulcorner}^{L} u^* i_1^* i_2^*(B)_{(1,1)} \to v^* i_2^*(B)_{(1,1)}$ can be rewritten in the form $i_1^L i_1^* i_2^*(B)_{(1,1)} \to v^* i_2^*(B)_{(1,1)}$. Noticing that $(1,1) = v(1,1)$ implies $v^* i_2^*(B)_{(1,1)} = i_2^*(B)_{(1,1)}$.

In summary, we have shown that $i_{\ulcorner}^{L} i_{\ulcorner}^* l^*(B) \to l^*(B)$ is an isomorphism if and only if $i_1^L i_1^* i_2^*(B)_{(1,1)} \to i_2^*(B)_{(1,1)}$ is an isomorphism, which is if and only if $i_1^L i_1^* i_2^*(B) \to i_2^*(B)$ is an isomorphism, by lemma 4.6.

ii) The idea of this proof is similar to i). However, in this case the goal is to show that $(2,1)^* i_{\ulcorner}^{L} i_{\ulcorner}^* r^*(B) = (2,1)^* i_2^L i_2^*(B)$. Unfortunately, the two corresponding slice categories are not equal, as they were in i). The way to remedy this is to use the following square.

$$\begin{array}{ccc} \mathbb{D}([i_2 \to (2,1)]) & \xrightarrow{j^*} & \mathbb{D}([\ulcorner \to (2,1)]) \\ \big\downarrow{\scriptstyle \pi^L} & & \big\downarrow{\scriptstyle \pi^L} \\ \mathbb{D}(\mathbb{1}) & \xrightarrow[Id]{} & \mathbb{D}(\mathbb{1}) \end{array}$$

The square commutes by 3.14, and the observations that $[i_2 \to (2,1)] \cong \square\overline{\ }$ and $[i_{\ulcorner} \to (2,1)] \cong \ulcorner$ together with the fact that $\ulcorner \xrightarrow{j} \square\overline{\ }$ is a right adjoint to the canonical map $\ulcorner \to \square\overline{\ }$.

iii) Analogous to ii).

□

Proposition 5.7. $\mathbb{D}$ is a pointed derivator. Let the functors be as in lemma 5.6, let $B \in \mathbb{D}(\square\square)$ and assume $l^*(B)$ is cocartesian. Then $r^*(B)$ is cocartesian if and only if $o^*(B)$ is cocartesian.

Proof. We know that $l^*(B)$ is cocartesian if and only if $l^*(B) \in essIm(i_{\ulcorner}^{L})$ if and only if $i_2^*(B) \in essIm(i_1^L)$ by lemma 5.6), which is if and only if $i_1^L i_1^* \to Id$ is a natural isomorphism by lemma 4.6. Using ii) and iii) of lemma 5.6 together with our assumption, we get

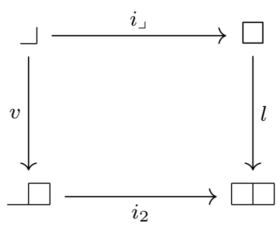
$$
\begin{aligned}
& o^*(B) \text{ cocartesian} \\
& \iff o^*(B) \in essIm(i_{\ulcorner}^L) \\
& \iff B \in essIm(i^L) \\
& \iff i^L i^* B \to B \text{ is a natural isomorphism} \\
& \iff (i_2 i_1)^L (i_2 i_1)^* B \to B \text{ is a natural isomorphism} \\
& \iff i_2^L i_1^L i_1^* i_2^* B \to B \text{ is a natural isomorphism} \\
& \iff i_2^L i_2^* B \to B \text{ is a natural isomorphism} \\
& \iff B \in essIm(i_2^L) \\
& \iff r^*(B) \in essIm(i_{\ulcorner}^L) \\
& \iff r^*(B) cocartesian
\end{aligned}
$$

□

We are rewarded with the two out of three-property in the context of derivators.

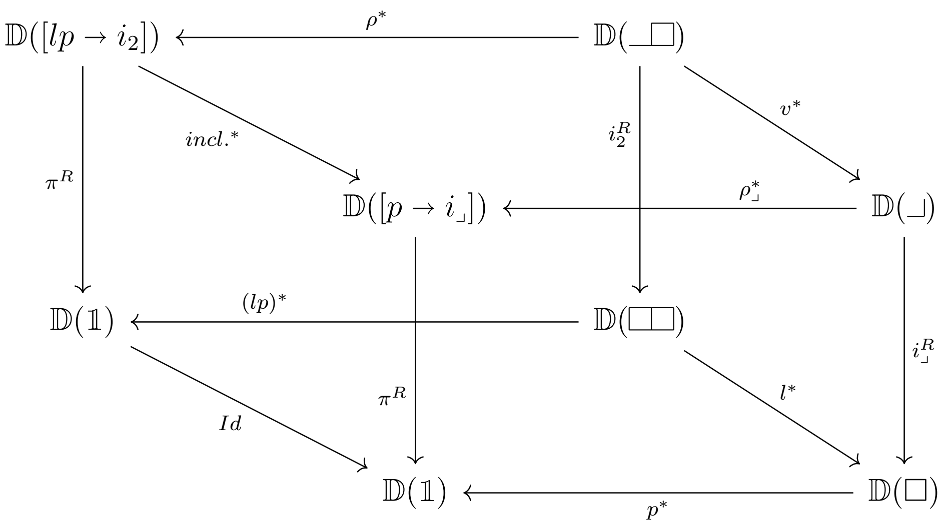
Corollary 5.8. *If any two of $l^*(B), r^*(B)$ and $o^*(B)$ are bicartesian, then so is the third.*

Lemma 5.9. *We have the following square*



where $l, i_2, i_{\lrcorner}$ are as before and v is such that the square commutes. Let $B \in \mathbb{D}(\square\square)$. If B is in the essential image of i_2^R, then $l^*(B)$ is cartesian.

Proof. $B \in essIm(i_2^R)$, so let $B \cong i_2^R(A)$ for some $A \in \mathbb{D}(\lrcorner\square)$. Our goal is to show that $l^* i_2^R(A) \cong i_{\lrcorner}^R v^* A$. By Der ii), this is an isomorphism if and only if $p^* l^* i_2^R(A) \cong p^* i_{\lrcorner}^R v^* A$ for all $p \in \square$. Consider the following diagram.



By Der iv), our goal can be rewritten to proving that $\pi^R\rho^*(A) \cong \pi^R\rho_\lrcorner^* v^*(A)$, which is reduced to showing that $\pi^R\rho^*(A) \cong \pi^R incl^*\rho^*(A)$ because the top square commutes. Hence it is enough to prove that $\pi^R \cong \pi^R incl^*$. This is similar to the corresponding argument in 5.6 ii) when using lemma 3.14. $\square$

Proposition 5.10. $\mathbb{D}$ is a pointed derivator. Then the following are equivalent.

i) $\mathbb{D}$ is stable.

ii) $A \in \mathbb{D}(\square)$ such that $A_{(0,1)} \cong 0$ is cartesian if and only if it is cocartesian.

Proof. The direction **i)** $\Rightarrow$ **ii)** is by definition.
ii) $\Rightarrow$ **i)** Consider the inclusions

$$\square \xrightarrow{i_1} \lrcorner\square \xrightarrow{i_2} \square\square$$

which induce

$$\mathbb{D}(\square) \xrightarrow{i_1^L} \mathbb{D}(\lrcorner\square) \xrightarrow{i_2^R} \mathbb{D}(\square\square)$$

where i_1^L is an extension by zero since i_1 is a cosieve. Let $B \in \mathbb{D}(\square\square)$ in the essential image of $i_2^R i_1^L$. Then the underlying shape of B is

$$\begin{array}{ccccc} B_{(0,0)} & \longrightarrow & B_{(0,0)} & \longrightarrow & B_{(0,0)} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B_{(0,0)} & \longrightarrow & B_{(0,0)} \end{array}$$

Recalling the functor r from lemma 5.6, we denote $A = r^*(B)$, and assume it is cartesian. We want to show that A is also cocartesian. By lemma 5.9 we have that $l^*(B)$ is cartesian, implying that the outer square is also cartesian by lemma 5.7. The outer square and the left square, $l^*(B)$, are both cocartesian by ii). This means the right square, A, must also be cocartesian, again by lemma 5.7.

$\square$

5.2 Stable derivators induce additive categories

One way to define a coproduct in a category $\mathcal{X}$, is by the left adjoint of the diagonal functor $\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times \mathcal{X}$, sending an object x to (x, x). Likewise, the product is the right adjoint to the same functor. Comparing this to the functor $\mathbb{D}(\mathbb{1}) \xrightarrow{\pi^*} \mathbb{D}(\mathbb{1}) \times \mathbb{D}(\mathbb{1})$ induced by $\mathbb{1} \times \mathbb{1} \xrightarrow{\pi} \mathbb{1}$, we see that $\Delta = \pi^*$ for $\mathcal{X} = \mathbb{D}(\mathbb{1})$. This implies that the coproduct is π^L, and the product is π^R, leading us to the following definition.

Definition 5.11. Let $\mathbb{1} \times \mathbb{1} \xrightarrow{j} \lrcorner$ and $\mathbb{1} \times \mathbb{1} \xrightarrow{i} \ulcorner$ be functors identifying the corners $(0, 1)$ and $(1, 0)$. For $B \in \mathbb{D}(\ulcorner)$, we say

i) B is a *coproduct cocone* if it is in the essential image of $\mathbb{D}(\mathbb{1} \sqcup \mathbb{1}) \xrightarrow{j^L} \mathbb{D}(\lrcorner)$.

ii) B is a *product cone* if it is in the essential image of $\mathbb{D}(\mathbb{1} \sqcup \mathbb{1}) \xrightarrow{i^R} \mathbb{D}(\ulcorner)$.

Consider the two slice categories

$$\begin{array}{ccc} [j \to (1,1)] & \xrightarrow{\rho_{(1,1)}} & \mathbb{1} \sqcup \mathbb{1} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle j} \\ \mathbb{1} & \xrightarrow[(1,1)]{} & \lrcorner \end{array} \qquad \begin{array}{ccc} [(0,0) \to i] & \xrightarrow{\rho_{(0,0)}} & \mathbb{1} \sqcup \mathbb{1} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle i} \\ \mathbb{1} & \xrightarrow[(0,0)]{} & \ulcorner \end{array}$$

Recalling that $\mathbb{D}(\mathbb{1} \sqcup \mathbb{1}) \cong \mathbb{D}(\mathbb{1}) \times \mathbb{D}(\mathbb{1})$ by Der i), and realising that both slice categories are equivalent to $\mathbb{1} \sqcup \mathbb{1}$, results in the following commuting squares.

$$\begin{array}{ccc} \mathbb{D}(\mathbb{1}) \times \mathbb{D}(\mathbb{1}) & \xleftarrow{Id} & \mathbb{D}(\mathbb{1}) \times \mathbb{D}(\mathbb{1}) \\ {\scriptstyle \pi^L}\downarrow & & \downarrow{\scriptstyle j^L} \\ \mathbb{D}(\mathbb{1}) & \xleftarrow[(1,1)^*]{} & \mathbb{D}(\lrcorner) \end{array} \qquad \begin{array}{ccc} \mathbb{D}(\mathbb{1}) \times \mathbb{D}(\mathbb{1}) & \xleftarrow{Id} & \mathbb{D}(\mathbb{1}) \times \mathbb{D}(\mathbb{1}) \\ {\scriptstyle \pi^R}\downarrow & & \downarrow{\scriptstyle i^R} \\ \mathbb{D}(\mathbb{1}) & \xrightarrow[(0,0)^*]{} & \mathbb{D}(\ulcorner) \end{array}$$

If $B \cong j^L(A)$ for some $A \in \mathbb{D}(\mathbb{1} \times \mathbb{1})$, then $j^L(A)_{(1,1)} \cong (1,1)^* j^L A \cong \pi^L A.$, which, by the discussion above, is the coproduct of A. Likewise, for $B \cong i^R(A)$ we get that $i^R(A)_{(0,0)} \cong \pi^R A$, which is the product.

Lemma 5.12. *Let $i_{\ulcorner}, i_{\lrcorner}, i$ and j be as above, $B \in \mathbb{D}(\square)$, and $B_{(0,0)} \cong 0$. Then*

i) $i_{\lrcorner}^{*}(B)$ is a coproduct cocone if and only if B is cocartesian.

ii) $i_{\ulcorner}^{*}(B)$ is a product cone if and only if B is cartesian.

Proof. We show i). Consider the commuting square

$$\begin{array}{ccc}
\mathbb{D}(\mathbb{1}) \times \mathbb{D}(\mathbb{1}) & \xrightarrow{\quad j^L \quad} & \mathbb{D}(\lrcorner) \\
{\scriptstyle i^L}\big\downarrow & & \big\downarrow{\scriptstyle i_{\lrcorner}^L} \\
\mathbb{D}(\ulcorner) & \xrightarrow[\quad i_{\ulcorner}^L \quad]{} & \mathbb{D}(\square)
\end{array}$$

where both i^L and $i_{\lrcorner}^L$ are extensions by zero by lemma 4.9. Also notice that by lemma 3.15, the functor $i_{\lrcorner}^L$ is fully faithful, hence $i_{\lrcorner}^{*} i_{\lrcorner}^{L} C \cong C$ for all $C \in \mathbb{D}(\lrcorner)$, by lemma 1.14. Hence

$$\begin{aligned}
& B \text{ is cocartesian and } B_{(0,0)} \cong 0 \\
\iff\ & B \in essIm(i_{\ulcorner}^L i^L) \\
\iff\ & B \in essIm(i_{\lrcorner}^L j^L), \text{ say } B \cong i_{\lrcorner}^L j^L A \\
\iff\ & B_{(0,0)} \cong 0 \text{ and } i_{\lrcorner}^{*} B \cong i_{\lrcorner}^{*} i_{\lrcorner}^{L} j^L A \cong j^L A
\end{aligned}$$

□

Definition 5.13. A category $\mathcal{X}$ is *pre-additive* if the following is satisfied.

Add i) $\mathcal{X}$ has a zero object.

Add ii) $\mathcal{X}$ admits products and coproducts.

Add iii) All (co)products are biproducts in $\mathcal{X}$.

Theorem 5.14. *$\mathbb{D}$ is a stable derivator and $\mathcal{X}$ is a small category. Then the category $\mathbb{D}(\mathcal{X})$ is pre-additive.*

Proof. By lemma 5.5 it is enough to show $\mathbb{D}(\mathbb{1})$ is pre-additive.

[Add i] A stable derivator is pointed by definition, so $\mathbb{D}(\mathbb{1})$ has a zero object.

[Add ii] By theorem 3.19.

[Add iii] For two objects $a, b \in \mathbb{D}(\mathbb{1})$, the goal is to show that the product and coproduct of a and b coincide. Therefore, note that the object $(a, b) \in \mathbb{D}(\mathbb{1}) \times \mathbb{D}(\mathbb{1})$ can be identified with an object $(a, b) \in \mathbb{D}(\mathbb{1} \sqcup \mathbb{1})$ by Der i). Let $\mathcal{D}$ denote the poset category $[2] \times [2]$.

$$\begin{array}{ccccc}
(0,0) & \longrightarrow & (1,0) & \longrightarrow & (2,0) \\
\downarrow & & \downarrow & & \downarrow \\
(0,1) & \longrightarrow & (1,1) & \longrightarrow & (2,1) \\
\downarrow & & \downarrow & & \downarrow \\
(0,2) & \longrightarrow & (1,2) & \longrightarrow & (2,2)
\end{array}$$

Consider the full subcategories $\mathbb{1} \sqcup \mathbb{1}$, D_1 and D_2, respectively from left to right,

$$\begin{array}{ccc}
 & & \\
 & & (2,1) \\
(1,2) & &
\end{array}
\quad \subseteq \quad
\begin{array}{ccc}
 & & (2,0) \\
 & & \downarrow \\
 & & (2,1) \\
 & & \\
(0,2) & \to & (1,2)
\end{array}
\quad \subseteq \quad
\begin{array}{ccccc}
 & & & & (2,0) \\
 & & & & \downarrow \\
 & & & & (2,1) \\
 & & & & \downarrow \\
(0,2) & \to & (1,2) & \to & (2,2)
\end{array}$$

together with the inclusions $\mathbb{1} \sqcup \mathbb{1} \xrightarrow{i_1} D_1 \xrightarrow{i_2} D_2 \xrightarrow{i_3} D$. Notice that i_1 and i_3 are cosieves, while i_2 is a sieve. This results in a composition of functors

$$\mathbb{D}(\mathbb{1} \sqcup \mathbb{1}) \xrightarrow{i_1^L} \mathbb{D}(D_1) \xrightarrow{i_2^R} \mathbb{D}(D_2) \xrightarrow{i_3^R} \mathbb{D}(D)$$

where the first two are extensions by zero, using lemma 4.9. Now we can look at the underlying shape of the object $i_3^R i_2^R i_1^L(a, b) \in \mathbb{D}(\mathcal{D})$

$$\begin{array}{ccccc}
d & \longrightarrow & a' & \longrightarrow & 0 \\
\downarrow & 1 & \downarrow & 2 & \downarrow \\
b' & \longrightarrow & c & \longrightarrow & b \\
\downarrow & 3 & \downarrow & 4 & \downarrow \\
0 & \longrightarrow & a & \longrightarrow & 0
\end{array}$$

for some objects $a', b', c, d \in \mathbb{D}(\mathbb{1})$. Adding the objects c, a', b' and d using the functor i_3^R created four cartesian squares. The argument for this is in the same style of the proof of lemma 5.9. By lemma 5.7 and proposition 5.10, any combination of these four squares is bicartesian. Using lemma 4.19 iii) on the outer square, we get that $d \cong 0$. Likewise, the squares $2+4$ and $3+4$ imply that $a' \cong a$ and $b' \cong b$, respectively. Lastly, considering the squares 1 and 4 using lemma 5.12, we get that c is both the product and coproduct of a and b.

□

We will refer the reader to [Gro13, sec.4.1] in regards to proving that the morphism sets in $\mathbb{D}(\mathcal{X})$ are abelian groups, and that composition of morphisms is bilinear. Showing that a morphism has an inverse in this case, is a fairly technical process. However, with this reference, we can now state that $\mathbb{D}(\mathcal{X})$ in fact is an additive category.

For a functor $\mathcal{X} \xrightarrow{G} \mathcal{Y}$, we have shown that the induced functors G^*, G^L, G^R preserve zero in lemma 4.5. A functor is a group homomorphism on morphism sets if it preserves biproducts. Left adjoints preserve coproducts and right adjoints preserve products, hence we can state that G^*, G^L, G^R are additive functors.

6 Triangulated derivators

In order to prove that derivators induce triangulated categories, we must require that they are both stable and strong. Defined in section 6.1, strongness of a derivator will enable us to lift objects and morphisms from underlying categories, so that we can apply appropriate functors to create bicartesian squares. In section 6.2 we finally prove the main result and include a discussion of how this solves the flaw of triangulated categories. Lastly, we prove that the induced functor G^* and its adjoints are exact functors.

6.1 Strong derivators

Definition 6.1. Let $\mathcal{X} \in \mathcal{C}at$. A derivator $\mathbb{D}$ is called *strong* if the partial underlying diagram functor

$$dia_{[1],\mathcal{X}} : \mathbb{D}([1] \times \mathcal{X}) \to \mathbb{D}(\mathcal{X})^{[1]}$$

is full and essentially surjective.

Lemma 6.2. *Let $\mathcal{K} \in \mathcal{C}at$.*

i) $\mathbb{D}$ is strong if and only if $\mathbb{D}^{op}$ is strong.

ii) $\mathbb{D}$ is strong if and only if $\mathbb{D}^{\mathcal{K}}$ is strong.

Proof. i) We show that we can lift an object in $\mathbb{D}^{op}(\mathcal{X})^{[1]} = (\mathbb{D}(\mathcal{X}^{op})^{op})^{[1]}$ to an object in $\mathbb{D}^{op}([1] \times \mathcal{X}) = \mathbb{D}([1] \times \mathcal{X}^{op})^{op}$.

$A \xrightarrow{f^{op}} B$ is an object in $(\mathbb{D}(\mathcal{X}^{op})^{op})^{[1]}$ if and only if $A \xleftarrow{f} B$ is an object in $\mathbb{D}(\mathcal{X}^{op})^{[1]}$. $\mathbb{D}$ is strong, so this can be lifted to an object $A' \xleftarrow{f'} B$ in $\mathbb{D}([1] \times \mathcal{X}^{op})$. Lastly, this is if and only if $A' \xleftarrow{f'^{op}} B'$ is an object in $\mathbb{D}([1] \times \mathcal{X}^{op})^{op}$.

The opposite implication is analogous.

ii) We can lift an object in $\mathbb{D}^{\mathcal{K}}(\mathcal{X})^{[1]} = \mathbb{D}(\mathcal{K} \times \mathcal{X})^{[1]}$ to an object in $\mathbb{D}([1] \times \mathcal{K} \times \mathcal{X}) = \mathbb{D}^{\mathcal{K}}([1] \times \mathcal{X})$ because $\mathbb{D}$ is strong. Conversely, if $\mathbb{D}^{\mathcal{K}}$ is strong, then $\mathbb{D}^{\mathbb{1}} \cong \mathbb{D}$ is strong.

□

Definition 6.3. A derivator $\mathbb{D}$ is *triangulated* if it is stable and strong.

Example 6.4. The represented derivator is strong as $\mathcal{C}^{[1]\times\mathcal{X}} \cong (\mathcal{C}^{\mathcal{X}})^{[1]}$ is an equivalence.

Example 6.5. The derivator of $\mathcal{A}$ is strong ([Gro19, ex. 3.48]).

From example 5.2 we know that the represented derivator rarely is stable. On the other hand, the derivator of $\mathcal{A}$ is both stable and strong, and is therefore a triangulated derivator.

Definition 6.6. A *triangulated category* is an additive category $\mathcal{T}$, together with an auto-equivalence $[1] : \mathcal{T} \to \mathcal{T}$, and a class Δ consisting of sequences, also called *triangles*, of the form $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} a[1]$, such that the following axioms are satisfied.

[T1]
- For any $a \xrightarrow{f} b \in \mathcal{T}$ there is a triangle $a \xrightarrow{f} b \to c \to a[1]$ in Δ.
- For any $a \in \mathcal{T}$ the triangle $a \xrightarrow{id_a} a \to 0 \to a[1]$ is in Δ.
- Δ is closed under isomorphisms.

[T2] For any triangle $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} a[1]$ in Δ, the triangles

$$b \xrightarrow{g} c \xrightarrow{h} a[1] \xrightarrow{-f[1]} b[1]$$

$$c[-1] \xrightarrow{-h[-1]} a \xrightarrow{f} b \xrightarrow{g} c$$

are also in Δ.

[T3] Given the solid part of the diagram

$$\begin{array}{ccccccc} a & \longrightarrow & b & \longrightarrow & c & \longrightarrow & a[1] \\ {\scriptstyle k_1}\downarrow & & {\scriptstyle k_2}\downarrow & & {\scriptstyle k_3}\vdots & & \downarrow{\scriptstyle k_1[1]} \\ a' & \longrightarrow & b' & \longrightarrow & c' & \longrightarrow & a'[1] \end{array}$$

where the left square commutes and the rows are triangles in Δ, there always exists a morphism $c \xrightarrow{k_3} c'$ such that the entire diagram commutes.

[T4] Given the sequence $a \xrightarrow{f} b \xrightarrow{k} u$, there is a commuting diagram

$$\begin{array}{ccccccc} a & \xrightarrow{f} & b & \xrightarrow{g} & c & \longrightarrow & a[1] \\ \| & & \downarrow{\scriptstyle k} & & \downarrow & & \| \\ a & \xrightarrow[kf]{} & u & \longrightarrow & v & \longrightarrow & a[1] \\ & & \downarrow & & \downarrow & & \\ & & w & = & w & & \\ & & \downarrow & & \downarrow & & \\ & & b[1] & \xrightarrow[g[1]]{} & c[1] & & \end{array}$$

where the rows and columns are triangles in Δ.

Remark 6.7. The third object in a triangle is usually called the *cone object*. The fourth axiom is often referred to as the *octahedral axiom*, because if the triangles are drawn as geometric triangles, the fourth axiom creates an octahedral.

6.2 The induced triangulated categories

Remark 6.8. In the following proof we use:

i) $[3]$ is the category $0 \to 1 \to 2 \to 3$.

ii) $\square \xrightarrow{t} \square\overline{\ }$ is an inclusion and sieve, hence t^R is an extension by zero by lemma 4.9.

iii) $\square\overline{\ } \xrightarrow{i_2} \square\square$ is as in lemma 5.6.

Theorem 6.9. *Let $\mathbb{D}$ be a triangulated derivator, and $\mathcal{X}$ a small category. Then $\mathbb{D}(\mathcal{X})$ is a triangulated category.*

Proof. By lemma 5.5 and lemma 6.2, it suffices to show that $\mathbb{D}(\mathbb{1})$ is a triangulated category. In order to prove this, let the auto-equivalence on $\mathbb{D}(\mathbb{1})$ be the suspension functor $\Sigma : \mathbb{D}(\mathbb{1}) \xrightarrow{(0,0)^R} \mathbb{D}(\ulcorner) \xrightarrow{i_\ulcorner^L} \mathbb{D}(\square) \xrightarrow{(1,1)^*} \mathbb{D}(\mathbb{1})$ from section 4.2. This functor also comes to play in the definition of the class Δ.

Recall the cone functor $\mathbb{D}([1]) \xrightarrow{h^R} \mathbb{D}(\ulcorner) \xrightarrow{i_\ulcorner^L} \mathbb{D}(\square) \xrightarrow{v^*} \mathbb{D}([1])$ from the same section. For an object $a \xrightarrow{f} b$ in $\mathbb{D}(\mathbb{1})^{[1]}$ we use that $\mathbb{D}$ is strong and lift this to an object C in $\mathbb{D}([1])$, which has the underlying shape of f. If we use the composition $tria := i_2^L t^R i_\ulcorner^L h^R$ on C it looks like the following on the underlying shapes.

$$
a \xrightarrow{f} b \quad \overset{h^R}{\longmapsto} \quad \begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \\ 0 & & \end{array} \quad \overset{i_\ulcorner^L}{\longmapsto} \quad \begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow{\scriptstyle cone(f)} \\ 0 & \longrightarrow & c \end{array}
$$

$$
\overset{t^R}{\longmapsto} \quad \begin{array}{ccccc} a & \xrightarrow{f} & b & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle cone(f)} & & \\ 0 & \longrightarrow & c & & \end{array} \quad \overset{i_2^L}{\longmapsto} \quad \begin{array}{ccccc} a & \xrightarrow{f} & b & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle cone(f)} & & \downarrow \\ 0 & \longrightarrow & c & \xrightarrow[cone^2(f)]{} & d \end{array}
$$

Consider the last double square. By definition, the left square is cocartesian, and by lemma 5.6 ii), the right square is also cocartesian. The outer square is cocartesian by lemma 5.7, and since $\mathbb{D}$ is stable, all of these are bicartesian. In addition, the corners are zero, so the outer square is in the essential image of $i_{\ulcorner}^{L}(0,0)^{R}$, hence we have an isomorphism $\phi : d \cong \Sigma a$. We pick out the sequence

$$a \xrightarrow{f} b \xrightarrow{cone(f)} c \xrightarrow{\phi cone^2(f)} \Sigma a$$

using the composition $\mathbb{D}(\square\square) \to \mathbb{D}([3]) \xrightarrow{dia_{[3]}} \mathbb{D}(\mathbb{1})^{[3]}$ on the double square. We define Δ as the class consisting of sequences that are isomorphic to the one constructed above.

[T1] By the above discussion, it is clear that any morphism f in $\mathbb{D}(\mathbb{1})$ can be completed to a sequence in Δ. By definition, Δ is closed under isomorphisms. Lastly, using the construction above on id_a for some a in $\mathbb{D}(\mathbb{1})$ results in the sequence

$$a \xrightarrow{id_a} a \xrightarrow{cone(id_a)} 0 \xrightarrow{cone^2(id_a)} \Sigma a$$

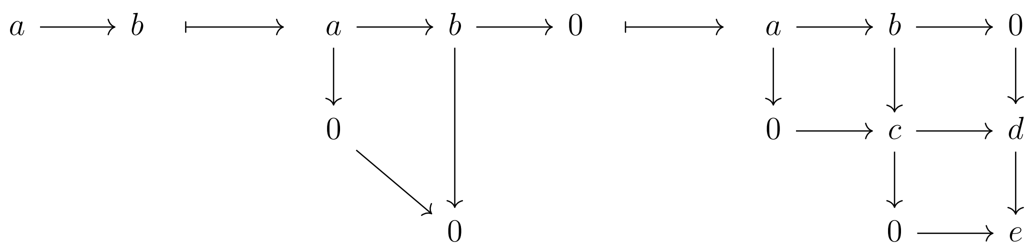
by lemma 4.19 iii).

[T2] Given a sequence $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a$ in Δ, we want to show that

$$b \xrightarrow{g} c \xrightarrow{h} \Sigma a \xrightarrow{-\Sigma f} \Sigma b$$

is in Δ, the other rotation is dual. Let $\mathcal{J}_1$ be the full subcategory $[2] \times [2] - \{(1,1),(2,1),(0,2),(2,2)\}$ and $\mathcal{J}_2$ is $[2] \times [2] - \{(0,2)\}$. If $[1] \xrightarrow{j_1} \mathcal{J}_1 \xrightarrow{j_2} \mathcal{J}_2$ are the canonical inclusions, then they are both sieves and j_1^R is an extension by zero, while j_2^L creates bicartesian squares. The latter is true by the same reasoning as in lemma 5.6.

Therefore, if we lift $a \xrightarrow{f} b$ in $\mathbb{D}(\mathbb{1})^{[1]}$ to an object F in $\mathbb{D}([1])$ using that $\mathbb{D}$ is strong, we can apply the composition $j_2^L j_1^R$ to F and achieve an object in $\mathbb{D}(\mathcal{J}_2)$ with an appropriate underlying shape.

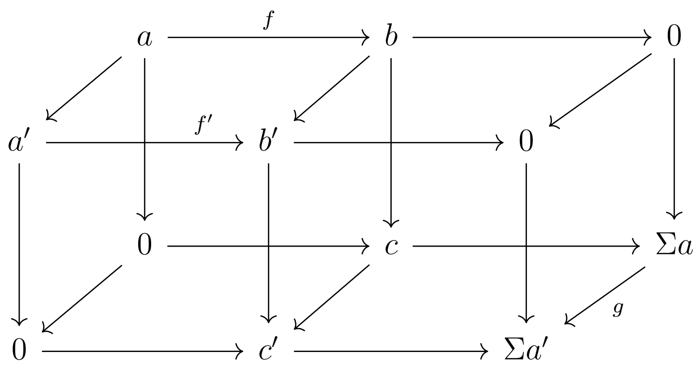


Again, similar to the above discussion regarding the triangles in Δ, we have that the squares are bicartesian, $d \cong \Sigma a$, and $e \cong \Sigma b$. The fact that the morphism between them is $-\Sigma f$ is related to the details of showing that $\mathbb{D}$ admits categories with abelian morphisms sets. The reader is referred to [Gro13, prop. 4.12] for details.

[T3] Consider the diagram

$$\begin{array}{ccccccc} a & \xrightarrow{f} & b & \longrightarrow & c & \longrightarrow & \Sigma a \\ \downarrow & & \downarrow & & & & \downarrow \\ a' & \xrightarrow[f']{} & b' & \longrightarrow & c' & \longrightarrow & \Sigma a' \end{array} \tag{6.1}$$

The leftmost square is a morphism between two objects $(a \xrightarrow{f} b)$ and $(a' \xrightarrow{f'} b')$ in $\mathbb{D}(\mathbb{1})^{[1]}$. As $dia_{[1],\mathbb{1}}$ is full and essentially surjective, we can lift both objects and morphism to objects and morphism $A \xrightarrow{K} A'$ in $\mathbb{D}([1])$. If we now apply the functor $\mathbb{D}([1]) \xrightarrow{tria} \mathbb{D}(\square\square)$ it results in $tria(A) \xrightarrow{tria(K)} tria(A')$ with underlying shape



(6.2)

where everything commutes and the two front and back squares are bicartesian, as seen in the construction of $tria$. Using $dia_{\square\square}$ sends the objects and morphism to their underlying shapes, giving us the morphism $c \to c'$ in $\mathbb{D}(\mathbb{1})$ completing diagram 6.1 such that it commutes.

[T4] There are three parts to proving this axiom. First we argue that we can lift an object in $\mathbb{D}(\mathbb{1})^{\square}$ to $\mathbb{D}(\square)$, and send it to $\mathbb{D}([1] \times \square\square)$ with the $tria$ functor, where it will have the same underlying shape and bicartesian squares as in diagram 6.2. Then we observe a useful result, allowing us to finally show the fourth axiom using the third axiom twice.

As in [T3], we begin with the square in diagram 6.1. This time we view it as a morphism between the objects $(a \to a')$ and $(b \to b')$ in $\mathbb{D}(\mathbb{1})^{[1]}$ and lift it to a morphism $A \xrightarrow{F} B$ in $\mathbb{D}([1])$. That is, an object F in $\mathbb{D}([1])^{[1]}$. Again, this is lifted to an object F' in $\mathbb{D}^{[1]}([1])$, before applying $\mathbb{D}^{[1]}([1]) \xrightarrow{tria} \mathbb{D}^{[1]}(\square\square)$ on it. Keeping track of the underlying shape of $tria(F')$ in $\mathbb{D}^{[1]}(\mathbb{1})$, we end up with

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C & \longrightarrow & \Sigma A \end{array}$$

where the left and right squares are bicartesian, for the same reasons as when constructing the original functor $\mathbb{D}([1]) \xrightarrow{tria} \mathbb{D}(\square\square)$. By corollary 4.22, the pointwise (or underlying) squares will also be bicartesian. In other words, our object $tria(F')$ in $\mathbb{D}^{[1]}(\square\square)$ has exactly the underlying shape in $\mathbb{D}(\mathbb{1})$ as diagram 6.2. This allows us to make a very useful observation.

Consider the right cube in diagram 6.2, where the front and back are bicartesian. By corollary 5.8, the left square is bicartesian if and only if the outer square of the left and front square combined is bicartesian if and only if the outer square of the right and back square combined is bicartesian if and only if the right square is bicartesian. Lastly, this is if and only if g is an isomorphism, by lemma 4.19 iii). We refer to this observation as $\star$.

Finally, we prove [T4]. Given the sequence $a \xrightarrow{f} b \xrightarrow{k} u$, we want to induce the appropriate commuting octahedral diagram. As mentioned, the idea is to use the same construction as in [T3] twice. We begin by viewing the square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \| & & \downarrow k \\ a & \xrightarrow[kf]{} & u \end{array}$$

as an object in $(\mathbb{D}(\mathbb{1})^{[1]})^{[1]}$. Then, as in [T3], we lift it to some object $A \xrightarrow{K} A'$ in $\mathbb{D}([1])^{[1]}$ and apply the functor $\mathbb{D}([1])^{[1]} \xrightarrow{tria} \mathbb{D}([3])^{[1]}$.

$$\begin{array}{ccccccc} a & \xrightarrow{f} & b & \xrightarrow{g} & c & \longrightarrow & \Sigma a \\ \| & & \downarrow k & & \downarrow l & & \| \\ a & \xrightarrow[kf]{} & u & \longrightarrow & v & \longrightarrow & \Sigma a \end{array}$$

Because $\Sigma a \to \Sigma a$ is an isomorphism, by $\star$, the middle square is bicartesian. We repeat the process, only this time on the middle square to get the following.

$$\begin{array}{ccccccc}
a & \xrightarrow{f} & b & \xrightarrow{g} & c & \longrightarrow & \Sigma a \\
\| & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} & & \| \\
a & \xrightarrow[kf]{} & u & \longrightarrow & v & \longrightarrow & \Sigma a \\
 & & \downarrow & & \downarrow{\scriptstyle m} & & \\
 & & w & \xrightarrow[\phi]{} & z & & \\
 & & \downarrow & & \downarrow & & \\
 & & \Sigma b & \xrightarrow[\Sigma g]{} & \Sigma c & &
\end{array}$$

We know the bottom morphism is Σg because $tria$ is a functor such that $3^* \circ tria = \Sigma \circ 0^*$. By $\star$ and the fact that the top middle square is bicartesian, we get that ϕ is an isomorphism. Hence we have

$$\begin{array}{ccccccc}
a & \xrightarrow{f} & b & \xrightarrow{g} & c & \longrightarrow & \Sigma a \\
\| & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} & & \| \\
a & \xrightarrow[kf]{} & u & \longrightarrow & v & \longrightarrow & \Sigma a \\
 & & \downarrow & & \downarrow{\scriptstyle \phi^{-1}m} & & \\
 & & w & = & w & & \\
 & & \downarrow & & \downarrow & & \\
 & & \Sigma b & \xrightarrow[\Sigma g]{} & \Sigma c & &
\end{array}$$

where everything commutes, thus finishing the proof.

□

As advertised in the introduction, one of the issues with triangulated categories is that the cone construction $\mathcal{T}^{[1]} \xrightarrow{Cone} \mathcal{T}$ is not functorial. However, in proving that $\mathbb{D}(\mathbb{1})$ is triangulated, we constructed a functor $\mathbb{D}([1]) \to \mathbb{D}(\mathbb{1})^{[3]}$ which can be extended to the functor $\mathbb{D}([1]) \to \mathbb{D}(\mathbb{1})$ selecting the object c from our sequence, namely the cone of the triangle. This, in addition to the strongness of a triangulated derivator, enables us to establish a replacement for the non-functorial cone construction.

$$\mathbb{D}(\mathbb{1})^{[1]} \xrightarrow{lift} \mathbb{D}([1]) \to \mathbb{D}(\mathbb{1})$$

For the derivator of $\mathcal{A}$, our example of a triangulated derivator, this is exactly utilising the fact that although there is no functor $\mathbf{D}(\mathcal{A})^{[1]} \xrightarrow{Cone} \mathbf{D}(\mathcal{A})$, there is instead a functor $\mathbf{D}(\mathcal{A}^{[1]}) \xrightarrow{Cone} \mathbf{D}(\mathcal{A})$.

Definition 6.10. An *exact functor* is an additive functor $\mathcal{T} \xrightarrow{F} \mathcal{T}'$ between triangulated categories together with a natural isomorphism $F \circ [1]_{\mathcal{T}} \xrightarrow{\alpha} [1]_{\mathcal{T}'} \circ F$, such that for every sequence $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} a[1]$ in $\Delta_{\mathcal{T}}$, the triangle

$$F(a) \xrightarrow{F(f)} F(b) \xrightarrow{F(g)} F(c) \xrightarrow{\alpha_a \circ F(h)} F(a)[1]$$

is in $\Delta_{\mathcal{T}'}$.

We end the chapter with proving that the functors G^*, G^L and G^R are exact functors between the induced triangulated categories.

Proposition 6.11. $\mathbb{D}$ is a triangulated derivator. For the functor $\mathcal{X} \xrightarrow{G} \mathcal{Y}$, the induced G^*, G^L and G^R are exact functors.

Proof. We know G^*, G^L and G^R are additive by the discussion at the end of section 5.2, and that functors that are adjoint to exact functors are also exact. Therefore it suffices to show that G^* preserves triangles. In other words, that given a triangle $a \to b \to c \to \Sigma a$ in $\mathbb{D}(\mathcal{Y})$, the induced $G^*(a) \to G^*(b) \to G^*(c) \to G^*(\Sigma a)$ is a triangle in $\mathbb{D}(\mathcal{X})$. We can assume $a \to b \to c \to \Sigma a$ in $\mathbb{D}(\mathcal{Y})$ came from an object A in $\mathbb{D}(\Box\Box \times \mathcal{Y})$

$$\begin{array}{ccccc} a & \longrightarrow & b & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & c & \longrightarrow & \Sigma a \end{array}$$

as in the construction of triangles in the previous proof. G induces the functor

$$\mathbb{D}(\Box\Box \times \mathcal{Y}) \xrightarrow{\mathbb{D}(\Box\Box \times G)} \mathbb{D}(\Box\Box \times \mathcal{X})$$

If the object $\mathbb{D}(\Box\Box \times G)(A)$ has bicartesian squares, this will result in a triangle in $\mathbb{D}(\mathcal{X})$ with $G^*(\Sigma a) \cong \Sigma G^*(a)$, finishing the proof. It therefore suffices to show that given a bicartesian square A in $\mathbb{D}(\Box \times \mathcal{Y})$, it will result in a bicartesian square in $\mathbb{D}(\Box \times \mathcal{X})$. Since A is bicartesian, it follows from corollary 4.22 that this is equivalent to A_y being bicartesian in $\mathbb{D}(\Box)$ for all $y \in \mathcal{Y}$. A_y is really $\mathbb{D}(\Box \times y)(A)$, and since this is true for all y, then in particular we have that $\mathbb{D}(\Box \times G(x))(A)$ is bicartesian for all $x \in \mathcal{X}$.

We notice $\mathbb{D}(\Box \times G(x)) = \mathbb{D}(\Box \times x) \circ \mathbb{D}(\Box \times G)$ and therefore we have that the object $\mathbb{D}(\Box \times x)(\mathbb{D}(\Box \times G)(A))$ is bicartesian for all $x \in \mathcal{X}$. Again by corollary 4.22, this is equivalent to $\mathbb{D}(\Box \times G)(A)$ being bicartesian in $\mathbb{D}(\Box \times X)$.

$\Box$

Bibliography

[Bra21] Kristoffer Smørås Brakstad. *Triangulated Derivators and their Calabi-Yau Dimension*. June 2021.

[GPS13] Moritz Groth, Kate Ponto and Michael Shulman. *Mayer-Vietoris sequences in stable derivators*. 2013.

[Gro] Alexander Grothendieck. *Les Dérivateurs*.

[Gro13] Moritz Groth. 'Derivators, pointed derivators and stable derivators'. In: *Algebraic & Geometric Topology* 13.1 (Feb. 2013), pp. 313–374.

[Gro19] Moritz Groth. *Higher symmetries in abstract stable homotopy theories*. 2019.

[Kra21] Henning Krause. *Homological Theory of Representations*. Cambridge Studies in Advanced Mathematics, Series Number 195. Cambridge University Press, 2021. ISBN: 1108838898,9781108838894.

[KS74] G. M. Kelly and Ross Street. 'Review of the elements of 2-categories'. In: *Category Seminar*. Ed. by Gregory M. Kelly. Springer Berlin Heidelberg, 1974, pp. 75–103. ISBN: 978-3-540-37270-7.

[KT76] Daniel M. Kan and William P. Thurston. 'Every connected space has the homology of a $K(\pi, 1)$'. In: *Topology* 15.3 (1976), pp. 253–258. ISSN: 0040-9383.

[Lan71] Saunders Mac Lane. *Categories for the working mathematician*. 6th ed. Springer, 1971. ISBN: 9780387900353,0-387-90035-7.

[Opp16] Steffen Oppermann. *Homological Algebra*. Oct. 2016.

[Rie14] Emily Riehl. *Categorical Homotopy Theory*. Cambridge University Press, 2014. ISBN: 9781107048454.

[Ver96] Jean-Louis Verdier. *Des catégories dérivées des catégories abéliennes*. Ed. by Georges Maltisiniotis. Astérisque 239. Société mathématique de France, 1996.

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