Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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Degeneration of Representations of Algebras and Quivers

Master's thesis in Mathematical Sciences
Supervisor: Sverre Olaf Smalø
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Abstract

Representations of associative algebras are homomorphisms from the algebra into a matrix algebra. A group action can be defined on the set of representations which corresponds to conjugation of matrices. The orbits under this action define the degeneration order. Results and examples around this order are discussed, along with some curiosities regarding partitions of natural numbers. Coxeter functors and the dual of the transpose are also considered, and it is in particular demonstrated that these two types of functors do not always coincide.

Samandrag

Representasjonar av assosiative algebraar er homomorfiar frå algebraen inn i ein matrisealgebra. Ein gruppeverknad kan definerast på mengda av representasjonar som korresponderar med konjugering av matriser. Banene under denne verknaden definerer degenereringsordninga. Resultat og eksempel rundt denne ordninga vert diskutert, i tillegg til nokon nysgjerrigheiter om partisjonar av naturlege tal. Coxeterfunktorar og det duale av den transponerte vert også teken i tanke og det demonstrerast spesielt at desse to funktortypane ikkje alltid fell saman.

Contents

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1 Degeneration

We begin by defining representations of algebra and the group action on representations which allows us to discuss degeneration. The Zariski topology is also necessary to this end. A group action on representations of quivers will also be defined such that this action coincides with the group action on representations of algebras. The sections 1.1, 1.2, 1.5 and 1.6 are largely based on [12].

1.1 Representations of Associative Algebras

Definition 1.1. Let R be a commutative ring. An **algebra** over R is an R-module Λ with a multiplication $\Lambda \times \Lambda \to \Lambda$ which satisfies the following criteria for all $r \in R$ and $\alpha, \beta, \gamma \in \Lambda$:

1. $\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$
2. $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$
3. $r(\alpha\beta) = (r\alpha)\beta = \alpha(r\beta)$
4. Λ contains an element 1_Λ such that $1_\Lambda \cdot \alpha = \alpha \cdot 1_\Lambda$.

If in addition $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, we call Λ **associative**. $\triangle$

Example 1.1. Let R be a commutative ring, $d \in \mathbb{N}$ and let $M_d(R)$ be the set of $d \times d$ matrices with entries from R. Here is some useful notation for dealing with matrices. If $M \in M_d(R)$ and $i, j \in \mathbb{N}_d := \{1, \ldots, d\}$, then we let $[M]_{ij}$ denote the ij-th entry, that is the entry on the i-th row and j-th column. Additionally, we let $[M]_{i\bullet}$ denote the i-th row and $[M]_{\bullet j}$ denote the j-th column of M. On another note, we can easily show that $M_d(R)$ is an R-algebra. Assuming we already know that matrix multiplication is left and right distributive, compatible with scalars and associative, then these properties coupled with the fact that the identity matrix I_d, defined such that

$$[I_d]_{ij} = \begin{Bmatrix} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{Bmatrix},$$

acts as a multiplicative identity in $M_d(R)$, we obtain that $M_d(R)$ is an R-algebra. $\triangle$

Definition 1.2. Let R be a commutative ring and let Λ and Λ' be R-algebras. An R-algebra **homomorphism** is a function $f : \Lambda \to \Lambda'$ which satisfies the following criteria for all $r \in R$ and $\alpha, \beta \in \Lambda$:

1. $f(r\alpha) = rf(\alpha)$
2. $f(\alpha + \beta) = f(\alpha) + f(\beta)$
3. $f(\alpha\beta) = f(\alpha)f(\beta)$

4. $f(1_\Lambda) = 1_{\Lambda'}$.

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If Λ is an algebra over a commutative ring R, then a Λ-homomorphism $f : \Lambda \to \Lambda$ is called an endomorphism, and the set of Λ-endomorphisms is denoted $\mathrm{End}_R(\Lambda) := \{f : \Lambda \to \Lambda \mid f \text{ is a } \Lambda\text{-homomorphism}\}$. If f is a bijection in addition, then f is called an automorphism on Λ, and we define $\mathrm{Aut}_R(\Lambda) := \{f : \Lambda \to \Lambda \mid f \text{ is a bijective homomorphism on } \Lambda\}$. Bijective algebra homomorphisms are called algebra isomorphisms.

Let R be a commutative ring, Λ an R-algebra and $\lambda \in U(\Lambda)$, where $U(\Lambda)$ is the set of invertible elements of Λ. Define $\phi_\lambda : \Lambda \to \Lambda$ such that $\phi_\lambda(\sigma) = \lambda\sigma\lambda^{-1}$ for all $\sigma \in \Lambda$. In Appendix A we show that ϕ_λ is a Λ-automorphism. We also give it a name.

Definition 1.3. Let R be a be a commutative ring, Λ an R-algebra and $\lambda \in U(\Lambda)$. An **inner automorphism** is a function $\phi_\lambda : \Lambda \to \Lambda; \sigma \mapsto \lambda\sigma\lambda^{-1}\ \forall\sigma \in U(\Lambda)$. We write $\mathrm{Inn}(\Lambda) = \{\phi_\lambda \mid \lambda \in \Lambda\}$ for the set of inner Λ-automorphisms. Δ

For the next lemma we define the center of an algebra Λ over a commutative ring to be the set

$$Z(\Lambda) := \{\lambda \in \Lambda \mid \lambda\sigma = \sigma\lambda\ \forall\sigma \in \Lambda\}.$$

The lemma is based on a similar result for groups given in [6].

Lemma 1.1. Let R be a commutative ring and Λ an R-algebra. Then ${}^{U(\Lambda)}/_{Z(U(\Lambda))}$ and $\mathrm{Inn}(\Lambda)$ are isomorphic as groups $\Leftrightarrow Z(U(\Lambda)) \subseteq Z(\Lambda)$.

Proof. We should verify that ${}^{U(\Lambda)}/_{Z(U(\Lambda))}$ and $\mathrm{Inn}(\Lambda)$ are groups. We already know that $U(\Lambda)$ is a group under multiplication since Λ is a ring, and we show that the center of any group is a subgroup in Appendix B, so ${}^{U(\Lambda)}/_{Z(U(\Lambda))}$ is a group. In Appendix C we show that $\mathrm{Inn}(\Lambda)$ is a group. Then it remains to show that these groups are isomorphic $\Leftrightarrow Z(U(\Lambda)) \subseteq Z(\Lambda)$.

($\Leftarrow$) Suppose $Z(U(\Lambda)) \subseteq Z(\Lambda)$ and define the function

$$\begin{aligned}\Phi : U(\Lambda) &\to \mathrm{Inn}(\Lambda)\\ \lambda &\mapsto \phi_\lambda\end{aligned}$$

for all $\lambda \in U(\Lambda)$. Since given $\lambda, \sigma \in U(\Lambda)$, $\Phi(\lambda\sigma)(\tau) = \phi_{\lambda\sigma}(\tau) = (\lambda\sigma)\tau(\lambda\sigma)^{-1} = \lambda(\sigma\tau\sigma^{-1})\lambda^{-1} = (\phi_\lambda \circ \phi_\sigma)(\tau)\ \forall \tau \in U(\Lambda)$
$\Rightarrow \Phi(\lambda\sigma) = \phi_\lambda \circ \phi_\sigma$, we have that Φ is a group homomorphism. Furthermore, if $\phi \in \mathrm{Inn}(\Lambda)$, then $\exists \lambda \in U(\Lambda)$ such that for every $\sigma \in \Lambda$, we have that $\phi(\sigma) = \lambda\sigma\lambda^{-1} = \phi_\lambda(\sigma) = (\Phi(\lambda))(\sigma) \Rightarrow \phi = \Phi(\lambda) \Rightarrow \phi \in \Phi(U(\Lambda))$, so Φ is onto.
The kernel of Φ consists of every element $\lambda \in \Lambda$ such that $\Phi(\Lambda) = \phi_1$, the identity in $\mathrm{Inn}(\Lambda)$. Let $\lambda \in \ker\Phi$. Then $\lambda\sigma\lambda^{-1} = \phi_\lambda(\sigma) = (\Phi(\lambda))(\sigma) = \phi_1(\sigma) = 1 \cdot \sigma \cdot 1 = \sigma\ \forall \sigma \in \Lambda$, that is $\lambda\sigma\lambda^{-1} = \sigma \Rightarrow \sigma\lambda = \lambda\sigma\ \forall \sigma \in \Lambda \Rightarrow \lambda \in Z(U(\Lambda))$. Thus $\ker\Phi \subseteq Z(U(\Lambda))$.
Now suppose $\lambda \in Z(U(\Lambda))$. Since by assumption $Z(U(\Lambda)) \subseteq Z(\Lambda)$, $\lambda \in Z(\Lambda)$. Then $\forall \sigma \in \Lambda$, we have that $\lambda\sigma = \sigma\lambda \Rightarrow \sigma = \lambda\sigma\lambda^{-1} = \phi_\lambda(\sigma) = (\Phi(\lambda))(\sigma) \Rightarrow \Phi(\lambda) = \phi_1 \Rightarrow \lambda \in \ker\Phi$, so $Z(U(\Lambda)) \subseteq \ker\Phi$.
Thus $Z(U(\Lambda)) = \ker\Phi$, and we have that $U(\Lambda)/\ker\Phi \cong \Phi(U(\Lambda))$, so $U(\Lambda)/Z(U(\Lambda)) \simeq \mathrm{Inn}(\Lambda)$.

($\Rightarrow$) Suppose $U(\Lambda)/Z(U(\Lambda)) \simeq \mathrm{Inn}(\Lambda)$. Then $\ker\Phi = Z(U(\Lambda))$, so if $\lambda \in Z(U(\Lambda))$, then $\lambda\sigma\lambda^{-1} = \phi_\lambda(\sigma) = (\Phi(\lambda))(\sigma) = \phi_1(\sigma) = \sigma \Rightarrow \lambda\sigma = \sigma\lambda\ \forall \sigma \in \Lambda \Rightarrow \lambda \in Z(\Lambda)$.
Thus $Z(U(\Lambda)) \subseteq Z(\Lambda)$.

Hence $U(\Lambda)/Z(U(\Lambda)) \simeq \mathrm{Inn}(\Lambda) \Leftrightarrow Z(U(\Lambda)) \subseteq Z(\Lambda)$. □

Example 1.2. Let R be a commutative ring, $d \in \mathbb{N}$ and consider the R-algebra $M_d(R)$. We shall show that $\mathrm{Gl}_d(R)/U(R)I_d \simeq \mathrm{Inn}(M_d(R))$, where $U(R)I_d := \{rI_d \mid r \in U(R)\}$ denotes the set of $d \times d$ scalar matrices of $U(R)$. Since $\mathrm{Gl}_d(R) = U(M_d(R))$, then if we can show that $Z(\mathrm{Gl}_d(R)) = U(R)I_d$ and $Z(\mathrm{Gl}_d(R)) \subseteq Z(M_d(R))$, we can conclude that $\mathrm{Gl}_d(R)/U(R)I_d$ and $\mathrm{Inn}(M_d(R))$ are isomorphic as a consequence of Lemma 1.1 above.

The proof that $Z(\mathrm{Gl}_d(R)) = U(R)I_d$ is based on a similar proof found in [3].

Let $d = 1$. $M_1(R) = R$, $\mathrm{Gl}_1(R) = U(R)$ and $U(R)I_1 = U(R)$, so $\mathrm{Gl}_1(R)/U(R)I_1 = U(R)/U(R) \simeq \langle 1 \rangle$, the trivial group under multiplication. If $\phi_u \in \mathrm{Inn}(M_1(R)) = \mathrm{Inn}(R)$, then $\phi_u(r) = uru^{-1} = ruu^{-1} = r\ \forall r \in R$ since R is commutative, so $\phi_u = \phi_1 \Rightarrow \mathrm{Inn}(R) \simeq \langle 1 \rangle$. Thus $\mathrm{Gl}_1(R)/U(R)I_1 \simeq \mathrm{Inn}(M_1(R))$.

Let $d > 1$ and suppose $A \in U(R)I_d$, that is $\exists r \in U(R)$ such that $A = rI_d$. Then

$$BA = BrI_d = rBI_d = rI_dB = AB\ \forall B \in \mathrm{Gl}_d(R)$$

$$\Rightarrow A \in Z(\mathrm{Gl}_d(R))$$
$$\Rightarrow U(R)I_d \subseteq Z(\mathrm{Gl}_d(R)).$$

Now suppose $A \in Z(\mathrm{Gl}_d(R))$, $p, q, i, j \in \mathbb{N}_d, p \neq q$ and $r \in R$. Consider the matrix $e_{pq}(r) \in M_d(R)$ defined such that

$$[e_{pq}(r)]_{ij} = \begin{Bmatrix} r \text{ if } i = p \text{ and } j = q \\ 0 \text{ otherwise} \end{Bmatrix}.$$

Define the matrix $E_{pq}(r) := I_d + e_{pq}(r)$. In Appendix D we show that $E_{pq}(r)E_{pq}(-r) = I_d = E_{pq}(-r)E_{pq}(r)$, so $E_{pq}(r) \in \mathrm{Gl}_d(R)$ and in particular we have that $AE_{pq}(r) = E_{pq}A$. Then

$$Ae_{pq}(r) = A(E_{pq}(r) - I_d) = AE_{pq}(r) - AI_d$$
$$= E_{pq}(r)A - I_dA = (E_{pq}(r) - I_d)A = e_{pq}(r)A,$$

that is $Ae_{pq}(r) = e_{pq}(r)A$. Furthermore, if $[A]_{qp} \neq 0$, then

$$[Ae_{pq}(1)]_{qq} = \sum_{k=1}^{d} [A]_{qk}[e_{pq}(1)]_{kq} = [A]_{qp}[e_{pq}(1)]_{pq} = [A]_{qp} \neq 0$$

since $[e_{pq}(1)]_{kq} = 0$ for all $k \in \mathbb{N}_d \setminus \{p\}$, but

$$[e_{pq}(1)A]_{qq} = \sum_{k=1}^{d} [e_{pq}(1)]_{qk}[A]_{kq} = 0$$

since $[e_{pq}(1)]_{qk} = 0 \ \forall k \in \mathbb{N}_d$. Then $[A]_{qp} = 0$, so A is a diagonal matrix. Now let $\pi : \mathbb{N}_d \to \mathbb{N}_d$ be a permutation of $\mathbb{N}_d$, that is a bijection on $\mathbb{N}_d$. We define $P_\pi \in M_d(R)$ to be the permutation matrix which is defined such that $[P_\pi]_{ij} = [I_d]_{\pi(i)j}$. If $M \in M_d(R)$, then

$$[P_\pi M]_{ij} = \sum_{k=1}^{d} [P_\pi]_{ik}[M]_{kj} = [P_\pi]_{i\pi(i)}[M]_{\pi(i)j} = [M]_{\pi(i)j}$$

$\Rightarrow [P_\pi M]_{i\bullet} = [M]_{\pi(i)\bullet}$ and

$$[MP_\pi]_{ij} = \sum_{k=1}^{d} [M]_{ik}[P_\pi]_{kj} = [M]_{i\pi(j)}[P_\pi]_{\pi(j)j} = [M]_{i\pi(j)}$$

$\Rightarrow [MP_\pi]_{\bullet j} = [M]_{\bullet \pi(j)}$, so a left and right multiplication by P_π represents a permutation given by π of rows and columns, respectively. If π is the permutation which interchanges p and q, that is

$$\pi(i) = \left\{ \begin{matrix} q \text{ if } i = p \\ p \text{ if } i = q \\ i \text{ otherwise} \end{matrix} \right\}.$$

Notice that $[P_\pi^2]_{ij} = [P_\pi]_{\pi(i)j} = [I_d]_{\pi^2(i)j}$ and that

$$\pi^2(i) = (\pi \circ \pi)(i) = \left\{ \begin{matrix} \pi(q) = p \text{ if } i = p \\ \pi(p) = q \text{ if } i = q \\ \pi(i) = i \text{ otherwise} \end{matrix} \right\},$$

which implies that $\pi^2(i) = i \Rightarrow [P_\pi^2]_{ij} = [I_d]_{ij} \Rightarrow P_\pi^2 = I_d$, so $P_\pi \in \mathrm{Gl}_d(R)$. Then $P_\pi A = AP_\pi \Rightarrow P_\pi AP_\pi = P_\pi^2 A = A$. Also, by our calculations above, we can see that $[P_\pi AP_\pi]_{ij} = [AP_\pi]_{\pi(i)j} = [A]_{\pi(i)\pi(j)}$, that is,

$$[P_\pi AP_\pi]_{ij} = \left\{ \begin{matrix} [A]_{qq} \text{ if } i = j = p \\ [A]_{pp} \text{ if } i = j = q \\ [A]_{ii} \text{ if } p \neq i = j \neq q \\ 0 \text{ otherwise} \end{matrix} \right\}.$$

Thus $P_\pi AP_\pi$ is a matrix with nonzero entries along the diagonal. We obtain that $[P_\pi AP_\pi]_{ij} = 0$ when $i \neq j$ from the fact that since A is an invertible diagonal matrix and $P_\pi AP_\pi$ is a permutation of the entries of A, then the entries along the diagonal of $P_\pi AP_\pi$ are the only nonzero ones of $P_\pi AP_\pi$. Since $P_\pi AP_\pi = A$, then $[A]_{pp} = [A]_{qq}$, so the entries along the diagonal of A are all identical, so $A = rI_d$ for some $r \in R$. Moreover, A is invertible, so $\exists A' \in \mathrm{Gl}_d(R)$ such that $I_d = AA' = rI_dA' = rA'$. Then we have that

$$r[A']_{ij} = \left\{ \begin{matrix} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{matrix} \right\}$$

$$\Rightarrow rr' = 1 \text{ for some } r' \in R \Rightarrow r \in U(R) \Rightarrow A \in U(R)I_d$$
$$\Rightarrow Z(\mathrm{Gl}_d(R)) \subseteq U(R)I_d,$$

and since we previously showed that $U(R)I_d \subseteq Z(\mathrm{Gl}_d(R))$, we can conclude that $Z(\mathrm{Gl}_d(R)) = U(R)I_d$.

All that remains to show then is that $Z(\mathrm{Gl}_d(R)) \subseteq Z(M_d(R))$. Let $A \in Z(\mathrm{Gl}_d(R)) = U(R)I_d$. Then $\exists r \in U(R)$ such that $A = rI_d$. Let $B \in M_d(R)$. Then

$$BA = BrI_d = rBI_d = rI_dB = AB \Rightarrow A \in Z(M_d(R))$$
$$\Rightarrow Z(\mathrm{Gl}_d(R)) \subseteq Z(M_d(R)).$$

Hence ${}^{\mathrm{Gl}_d(R)}/_{U(R)I_d} \simeq \mathrm{Inn}(M_d(R))$ for all $d \in \mathbb{N}$. $\triangle$

Definition 1.4. Let R be a commutative ring, Λ an R-algebra and $d \in \mathbb{N}$. A **representation** of Λ with rank d is an R-algebra homomorphism $f : \Lambda \to M_d(R)$. The set of d-dimensional representations of Λ is denoted by

$$\mathrm{rep}_d \Lambda = \{f : \Lambda \to M_d(R) \mid f \text{ is an } R\text{-algebra homomorphism}\}.$$

$\triangle$

We can define an action of ${}^{\mathrm{Gl}_d(R)}/_{U(R)I_d}$ on $\mathrm{rep}_d \Lambda$ by

$$\begin{aligned} \Psi : {}^{\mathrm{Gl}_d(R)}/_{U(R)I_d} \times \mathrm{rep}_d \Lambda &\to \mathrm{rep}_d \Lambda \\ (\overline{A}, f) &\mapsto \overline{\Phi}(\overline{A}) \circ f \end{aligned}$$

for all $f \in \mathrm{rep}_d \Lambda$, where

$$\begin{aligned} \overline{\Phi} : {}^{\mathrm{Gl}_d(R)}/_{U(R)I_d} &\to \mathrm{Inn}(M_d(R)) \\ \overline{A} &\mapsto \phi_A \end{aligned}$$

for all representatives $A \in \mathrm{Gl}_d(R)$ of the cosets $A \cdot U(R)I_d = \overline{A} \in {}^{\mathrm{Gl}_d(R)}/_{U(R)I_d}$. By Example 1.2, we know that $Z(\mathrm{Gl}_d(R)) \subseteq Z(M_d(R))$, which means that $\overline{\Phi}$ is the induced isomorphism of the homomorphism Φ defined in the proof of Lemma 1.1. Therefore $\overline{\Phi}$ is well-defined, so Ψ is well-defined. To simplify notation, we often suppress $U(R)I_d$ and call Ψ an action of $\mathrm{Gl}_d(R)$ on $\mathrm{rep}_d(\Lambda)$. If $f \in \mathrm{rep}_d(\Lambda)$, then we let $\mathrm{Gl}_d(R)f := \{\phi_G \circ f \mid \phi_G \in \mathrm{Inn}(\Lambda)\}$ denote the orbit of f.

Remark 1. Let R be a commutative ring and Λ an R-algebra. Any $m \in \mathrm{rep}_d \Lambda$ defines a Λ-module $M_m := R^d$ where for any $\lambda \in \Lambda$ and $v \in M_m$, scalar multiplication in M_m is defined as $\lambda v := m(\lambda)v$. We have that $m(\lambda) \in M_d(R)$ $\forall \lambda \in \Lambda$, so the scalar multiplication in M_m is actually multiplication with a matrix. $\triangle$

Lemma 1.2. Let R be a commutative ring, Λ an R- module, $d \in \mathbb{N}$ and $m, m' \in \operatorname{rep}_d \Lambda$. Then
$M_m \cong M_{m'} \Leftrightarrow \mathrm{Gl}_d(R)m = \mathrm{Gl}_d(R)m'$.

Proof.

($\Leftarrow$) Suppose $\mathrm{Gl}_d(R)m = \mathrm{Gl}_d(R)m'$. Since $m = \phi_{I_d} \circ m \in \mathrm{Gl}_d(R)m = \mathrm{Gl}_d(R)m'$, meaning $m \in \mathrm{Gl}_d(R)m'$, then $\exists \phi_G \in \mathrm{Inn}(M_d(R))$ such that $m = \phi_G \circ m'$. This means that $m(\lambda) = Gm'(\lambda)G^{-1}$ $\forall \lambda \in \Lambda$. Now define the function $T : M_m \to M_{m'}; v \mapsto G^{-1}v$ for all $v \in M_m$ and let $v, w \in M_m$ and $\lambda \in \Lambda$. Then

1. $T(v + w) = G^{-1}(v + w) = G^{-1}v + G^{-1}w = T(v) + T(w)$, since multiplication by matrices is distributive.
2. $T(\lambda v) = G^{-1}(\lambda v) = G^{-1}m(\lambda)v = G^{-1}(Gm'(\lambda)G^{-1})v$ $= m'(\lambda)G^{-1}v = \lambda(G^{-1}v) = \lambda T(v)$ by the definition of scalar multiplication in M_m and $M_{m'}$.

T is then a Λ-module homomorphism. Moreover, since $G^{-1} \in \mathrm{Gl}_d(R)$, then T is a bijection, so it is an Λ-module isomorphism between M_m and $M_{m'}$. Thus $M_m \cong M_{m'}$.

($\Rightarrow$) Suppose $M_m \cong M_{m'}$. Then $\exists T : M_m \to M_{m'}$ which is an isomorphism, so $\exists G \in \mathrm{Gl}_d(R)$ such that $T(v) = Gv$ for all $v \in M_m$. If $v \in M_m$ and $\lambda \in \Lambda$, then

$$Gm(\lambda)v = T(m(\lambda)v) = T(\lambda v) = \lambda T(v) = m'(\lambda)T(v) = m'(\lambda)Gv$$

$$\Rightarrow Gm(\lambda) = m'(\lambda)G \Rightarrow m'(\lambda) = Gm(\lambda)G^{-1} = \phi_G(m(\lambda))$$
$$\Rightarrow m' = \phi_G \circ m \Leftrightarrow m = \phi_{G^{-1}} \circ m'.$$

Then we have that

- $f \in \mathrm{Gl}_d(R)m \Rightarrow \exists \phi_A \in \mathrm{Inn}(M_d(R))$ such that $f = \phi_A \circ m = \phi_A \circ \phi_{G^{-1}} \circ m' = \phi_{AG^{-1}} \circ m' \Rightarrow f \in \mathrm{Gl}_d(R)m' \Rightarrow \mathrm{Gl}_d(R)m \subseteq \mathrm{Gl}_d(R)m'$.
- $f \in \mathrm{Gl}_d(R)m' \Rightarrow \exists \phi_A \in \mathrm{Inn}(M_d(R))$ such that $f = \phi_A \circ m' = \phi_A \circ \phi_G \circ m = \phi_{AG} \circ m \Rightarrow f \in \mathrm{Gl}_d(R)m \Rightarrow \mathrm{Gl}_d(R)m' \subseteq \mathrm{Gl}_d(R)m$.

Thus $\mathrm{Gl}_d(R)m = \mathrm{Gl}_d(R)m'$.

Hence $M_m \cong M_{m'} \Leftrightarrow \mathrm{Gl}_d(R)m = \mathrm{Gl}_d(R)m'$. $\square$

For a bit of notation, let

- $M_{\mathrm{rep}_d \Lambda} := \{M_m \mid m \in \mathrm{rep}_d \Lambda\}$ be the set consisting of the kind of Λ-modules described in Remark 1.
- $[M] := \{N \in M_{\mathrm{rep}_d \Lambda} \mid M \cong N\}$ be the isomorphism class of $M \in M_{\mathrm{rep}_d \Lambda}$.
- ${}^{M_{\mathrm{rep}_d \Lambda}}/_{\cong} := \{[M] \mid M \in M_{\mathrm{rep}_d \Lambda}\}$ be the set of isomorphism classes in $M_{\mathrm{rep}_d \Lambda}$.
- ${}^{\mathrm{rep}_d \Lambda}/_{\mathrm{Gl}_d(R)} := \{\mathrm{Gl}_d(R)f \mid f \in \mathrm{rep}_d(\Lambda)\}$ be the set of $\mathrm{Gl}_d(R)$-orbits in $\mathrm{rep}_d \Lambda$.

Then a consequence of Lemma 1.2 is that the function

$$
\begin{aligned}
O : {}^{M_{\mathrm{rep}_d \Lambda}}/_{\cong} &\to {}^{\mathrm{rep}_d \Lambda}/_{\mathrm{Gl}_d(R)} \\
[M_m] &\mapsto \mathrm{Gl}_d(R)m
\end{aligned}
$$

is a bijection, since:

- for $m, m' \in \mathrm{rep}_d(\Lambda)$ such that $O\big([M_m]\big) = O\big([M_{m'}]\big)$, we have that $\mathrm{Gl}_d(R)m = O\big([M_m]\big) = O\big([M_{m'}]\big) = \mathrm{Gl}_d(R)m' \Rightarrow M_m \cong M_{m'} \Rightarrow [M_m] = [M_{m'}]$. Thus O is injective.
- If $p \in {}^{\mathrm{rep}_d \Lambda}/_{\mathrm{Gl}_d(R)}$, then $\exists g \in \mathrm{rep}_d(\Lambda)$ such that $p = \mathrm{Gl}_d(R)g$, and $O\big([M_g]\big) = \mathrm{Gl}_d(R)g$, so $p = O\big([M_g]\big)$ for some $g \in \mathrm{rep}_d(\Lambda)$. Thus O is surjective.

Hence there is a bijection between the set of isomorphism classes of Λ-modules that are free and has length d as R-modules and the set of $\mathrm{Gl}_d(R)$-orbits in $\mathrm{rep}_d(\Lambda)$.

1.2 Representations Correspond to Matrix Tuples

We can show that the set of representations $\mathrm{rep}_d(\Lambda)$ are in bijection with a subset of $M_d(R)$. I order to show this, we first need a lemma, and proving the lemma requires us to do some work beforehand.

Let $R\langle X_1, \dots, X_n\rangle$ denote the free R-algebra on $n \in \mathbb{N}$ indeterminates, where R is a commutative ring. Let Γ be some R-algebra and $\gamma = (\gamma_1, \dots, \gamma_n) \in \Gamma^n$. For $I = (i_1, \dots, i_N) \in \mathbb{N}_n^N$, where $N \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, we introduce the notation $\gamma_I := \prod_{j=1}^N \gamma_{i_j}$. If $N = 0$, we say that $\gamma_I := 1$. We then define

$$\langle\gamma\rangle^* := \left\{\gamma_I \;\middle|\; I \in \mathbb{N}_n^N,\ N \in \mathbb{N}_0\right\}.$$

$\langle\gamma\rangle^*$ equipped with the multiplication from Γ becomes what we call a monoid. A monoid is akin to a group because it is a set with a closed binary operation which is associative and admits an identity element, but the elements in a monoid do not necessarily have inverses with respect to the binary operation.

Having defined $\langle\gamma\rangle^*$,we can then write any element $x \in R\langle X_1, \ldots, X_n\rangle$ as $x = \sum_{\omega\in\langle X\rangle^*} r_\omega \omega$ where $r_\omega \in R$ for each $\omega \in \langle X\rangle^*$ and $X = (X_1, \ldots, X_n) \in R\langle X_1, \ldots, X_n\rangle^n$. If we let Γ' be another R-algebra and $\gamma' = (\gamma'_1, \ldots, \gamma'_n) \in (\Gamma')^n$, then we define a function

$$\begin{aligned} \rho_{\gamma',\gamma} : \langle\gamma\rangle^* &\to \langle\gamma'\rangle^* \\ \gamma_I &\mapsto \gamma'_I \end{aligned}$$

for all $I \in \mathbb{N}_n^N$ and $N \in \mathbb{N}_0$. The function $\rho_{\gamma',\gamma}$ is what we call a monoid homomorphism, that is it has the properties $\rho_{\gamma',\gamma}(\lambda\sigma) = \rho_{\gamma',\gamma}(\lambda)\rho_{\gamma',\gamma}(\sigma)$ for all $\lambda, \sigma \in \langle\gamma\rangle^*$ and $\rho_{\gamma',\gamma}(1_\Gamma) = 1_{\Gamma'}$. To prove this, let $\lambda, \sigma \in \langle\gamma\rangle^*$. We can write these elements as $\lambda = \gamma_I$ and $\sigma = \gamma_J$ where $I = (i_1, \ldots, i_{N_1}) \in \mathbb{N}_n^{N_1}$, $J = (j_1, \ldots, j_{N_2}) \in \mathbb{N}_n^{N_2}$ and $N_1, N_2 \in \mathbb{N}_0$. Let $K = (i_1, \ldots, i_{N_1}, j_1, \ldots, j_{N_2})$ and observe the following.

$$\rho_{\gamma',\gamma}(\lambda\sigma) = \rho_{\gamma',\gamma}(\gamma_I\gamma_J) = \rho_{\gamma',\gamma}(\gamma_K) = \gamma'_K$$

$$= \gamma'_I\gamma'_J = \rho_{\gamma',\gamma}(\gamma_I)\,\rho_{\gamma',\gamma}(\gamma_J) = \rho_{\gamma',\gamma}(\lambda)\rho_{\gamma',\gamma}(\sigma).$$

The definitions of monoids and monoid homomorphisms are from [7].

We also have that $\rho_{\gamma',\gamma}(1_\Gamma) = \prod_{j=1}^0 \gamma_{i_j} = \prod_{j=1}^0 \rho(\gamma_{i_j}) = 1_{\Gamma'}$. Thus $\rho_{\gamma',\gamma}$ is a monoid homomorphism. We also define a function

$$\begin{aligned} P_{\gamma',\gamma} : \Gamma &\to \Gamma' \\ \sum_{\omega\in\langle\gamma\rangle^*} r_\omega\omega &\mapsto \sum_{\omega\in\langle\gamma\rangle^*} r_\omega \rho_{\gamma',\gamma}(\omega) \end{aligned}$$

for all $r_\omega \in R$ for each $\omega \in \langle\gamma\rangle^*$. We show that $P_{\gamma',\gamma}$ is an R-algebra, so let $t \in R$ and $\lambda = \sum_{\alpha\in\langle X\rangle^*} r_\alpha\alpha, \sigma = \sum_{\alpha\in\langle X\rangle^*} s_\alpha\alpha \in \Gamma$ for some $r_\alpha, s_\alpha \in R$ $\forall\alpha \in \langle\gamma\rangle^*$. Then

1.

$$P_{\gamma',\gamma}(t\lambda) = P_{\gamma',\gamma}\left(t\sum_{\alpha\in\langle\gamma\rangle^*} r_\alpha\alpha\right) = P_{\gamma',\gamma}\left(\sum_{\alpha\in\langle\gamma\rangle^*} tr_\alpha\alpha\right)$$

$$= \sum_{\alpha\in\langle\gamma\rangle^*} tr_\alpha\rho_{\gamma',\gamma}(\alpha) = t\sum_{\alpha\in\langle\gamma\rangle^*} r_\alpha\rho_{\gamma',\gamma}(\alpha) = tP_{\gamma',\gamma}\left(\sum_{\alpha\in\langle\gamma\rangle^*} r_\alpha\alpha\right)$$

$$= tP_{\gamma',\gamma}(\lambda).$$

2.

$$P_{\gamma',\gamma}(\lambda+\sigma) = \left(\sum_{\alpha\in\langle\gamma\rangle^*} r_\alpha\alpha + \sum_{\alpha\in\langle\gamma\rangle^*} s_\alpha\alpha\right)$$

$$= P_{\gamma',\gamma}\left(\sum_{\alpha\in\langle\gamma\rangle^*} (r_\alpha+s_\alpha)\alpha\right) = \sum_{\alpha\in\langle\gamma\rangle^*} (r_\alpha+s_\alpha)\rho_{\gamma',\gamma}(\alpha)$$

$$= \sum_{\alpha\in\langle\gamma\rangle^*} r_\alpha\rho_{\gamma',\gamma}(\alpha) + \sum_{\alpha\in\langle\gamma\rangle^*} s_\alpha\rho_{\gamma',\gamma}(\alpha)$$

$$= P_{\gamma',\gamma}\left(\sum_{\alpha\in\langle\gamma\rangle^*} r_\alpha\alpha\right) + P_{\gamma',\gamma}\left(\sum_{\alpha\in\langle\gamma\rangle^*} s_\alpha\alpha\right) = P_{\gamma',\gamma}(\lambda) + P_{\gamma',\gamma}(\sigma).$$

3.

$$P_{\gamma',\gamma}(\lambda\sigma) = P_{\gamma',\gamma}\left(\left(\sum_{\alpha\in\langle\gamma\rangle^*} r_\alpha\alpha\right)\left(\sum_{\beta\in\langle\gamma\rangle^*} r_\beta\beta\right)\right)$$

$$= P_{\gamma',\gamma}\left(\sum_{\alpha,\beta\in\langle\gamma\rangle^*} r_\alpha s_\beta\alpha\beta\right) = \sum_{\alpha,\beta\in\langle\gamma\rangle^*} r_\alpha s_\beta\rho_{\gamma',\gamma}(\alpha\beta)$$

$$= \sum_{\alpha,\beta\in\langle\gamma\rangle^*} r_\alpha s_\beta\rho_{\gamma',\gamma}(\alpha)\rho_{\gamma',\gamma}(\beta) = \sum_{\alpha,\beta\in\langle\gamma\rangle^*} r_\alpha\rho_{\gamma',\gamma}(\alpha)s_\beta\rho_{\gamma',\gamma}(\beta)$$

$$= \left(\sum_{\alpha\in\langle\gamma\rangle^*} r_\alpha\rho_{\gamma',\gamma}(\alpha)\right)\left(\sum_{\beta\in\langle\gamma\rangle^*} r_\beta\rho_{\gamma',\gamma}(\beta)\right)$$

$$= P_{\gamma',\gamma}\left(\sum_{\alpha\in\langle\gamma\rangle^*} r_\alpha \alpha\right) P_{\gamma',\gamma}\left(\sum_{\beta\in\langle\gamma\rangle^*} r_\beta \beta\right) = P_{\gamma',\gamma}(\lambda)P_{\gamma',\gamma}(\sigma).$$

Thus $P_{\gamma',\gamma}$ is an R-algebra homomorphism.

Lemma 1.3. Let Λ be an algebra over a commutative ring R. Then $\Lambda \cong {}^{R\langle X_1,\ldots,X_n\rangle}/_{I}$ for some ideal I in $R\langle X_1,\ldots,X_n\rangle$ and a suitable $n \in \mathbb{N}$ $\Leftrightarrow$ Λ is finitely generated.

Proof.

($\Leftarrow$) Suppose Λ is finitely generated, that is Λ can be generated by n elements for some $n \in \mathbb{N}$, say $\lambda_1,\ldots,\lambda_n \in \Lambda$. Let $\lambda = (\lambda_1,\ldots,\lambda_n)$ and consider the R-algebra homomorphism $P_{\lambda,X}$. Since Λ is generated by $\lambda_1,\ldots,\lambda_n$, if we let $r_\omega \in R$ for each $\omega \in \langle X\rangle^*$ and define $s_\gamma = \sum_{\omega\in\rho^{-1}_{\lambda,X}(\gamma)} r_\omega$ for each $\gamma \in \rho_{\lambda,X}(\langle X\rangle^*) = \langle\lambda\rangle^*$, then we can write any element $\sigma \in \Lambda$ as

$$\sigma = \sum_{\gamma\in\langle\lambda\rangle^*} s_\gamma\gamma = \sum_{\gamma\in\langle\lambda\rangle^*}\left(\sum_{\omega\in\rho^{-1}_{\lambda,X}(\gamma)} r_\omega\right)\gamma = \sum_{\omega\in\langle X\rangle^*} r_\omega \rho_{\lambda,X}(\omega)$$

$$= P_{\lambda,X}\left(\sum_{\omega\in\langle X\rangle^*} r_\omega\omega\right).$$

Then $P_{\lambda,X}$ is surjective and we have an induced isomorphism

$$\overline{P_{\lambda,X}} : {}^{R\langle X_1,\ldots,X_n\rangle}/_{\ker P_{\lambda,X}} \to \Lambda$$
$$x + \ker P_{\lambda,X} \mapsto P_{\lambda,X}(x)$$

for all representatives $x \in R\langle X_1,\ldots,X_n\rangle$ of the cosets $x + \ker P_{\lambda,X}$. Thus ${}^{R\langle X_1,\ldots,X_n\rangle}/_{I} \cong \Lambda$ for some ideal I in $R\langle X_1,\ldots,X_n\rangle$.

($\Rightarrow$) Suppose that there is some ideal I in $R\langle X_1,\ldots,X_n\rangle$ such that

$${}^{R\langle X_1,\ldots,X_n\rangle}/_{I} \cong \Lambda$$

where $n \in \mathbb{N}$ and let g be an isomorphism from ${}^{R\langle X_1,\ldots,X_n\rangle}/_I$ to Λ. We have a surjective quotient map

$$\begin{aligned} q : R\langle X_1, \ldots, X_n\rangle &\to {}^{R\langle X_1,\ldots,X_n\rangle}/_I \\ x &\mapsto x + I \end{aligned}$$

for all $x \in R\langle X_1, \ldots, X_n\rangle$, which is an R-algebra homomorphism. $f := g \circ q$ is then a surjective R-algebra homomorphism. Let $I = (i_1, \ldots, i_N) \in \mathbb{N}_n^N$ for some $N \in \mathbb{N}_0$. Since f is an R-algebra homomorphism, $f(X_I) = \prod_{j=1}^N f(X_{i_j}) =: f(X)_I$ and

$$f(\langle X\rangle^*) = \left\{ f(X)_I \;\middle|\; I \in \mathbb{N}_n^N,\ N \in \mathbb{N}_0 \right\}.$$

Then $f(x) = \sum_{\omega\in\langle X\rangle^*} r_\omega f(\omega)$ is a linear combination of elements in $f(\langle X\rangle^*)$ for all $x = \sum_{\omega\in\langle X\rangle^*} r_\omega \omega \in R\langle X_1, \ldots, X_n\rangle$ where $r_\omega \in R$ for each $\omega \in \langle X\rangle^*$, and since f is surjective, every element in $\Lambda = \operatorname{Im} f$ is a linear combination of words in $f(\langle X\rangle^*)$. Λ is then generated by $f(X_1), \ldots, f(X_n)$. Thus Λ is finitely generated.

Hence $\Lambda \cong {}^{R\langle X_1,\ldots,X_n\rangle}/_I$ for some ideal I in $R\langle X_1, \ldots, X_n\rangle$ and $n \in \mathbb{N}$ if and only if Λ is finitely generated. □

For the following result, define

$$\Xi(I) := \{A \in M_d(R)^n \mid P_{A,X}(\lambda) = 0\ \forall\lambda \in I\}$$

for any ideal I in $R\langle X_1, \ldots, X_n\rangle$.

Proposition 1.1. Let R be a commutative ring, $d \in \mathbb{N}$, Λ a finitely generated R-algebra and I an ideal in $R\langle X_1, \ldots, X_n\rangle$ for some $n \in \mathbb{N}$ such that ${}^{R\langle X_1,\ldots,X_n\rangle}/_I \cong \Lambda$. Then
$\operatorname{rep}_d \Lambda$ is in bijection with $\Xi(I)$.

Proof. Let $g : {}^{R\langle X_1,\ldots,X_n\rangle}/_I \tilde{\to} \Lambda$ be an R-algebra isomorphism and

$$\begin{aligned} q : R\langle X_1, \ldots, X_n\rangle &\to {}^{R\langle X_1,\ldots,X_n\rangle}/_I \\ x &\mapsto x + I \end{aligned}$$

$\forall x \in R\langle X_1, \ldots, X_n\rangle$. Define a function

$$\begin{aligned} \Theta : \operatorname{rep}_d \Lambda &\to \Xi(I) \\ f &\mapsto \big(f \circ g \circ q(X_1), \ldots, f \circ g \circ q(X_n)\big) \end{aligned}$$

for all $f \in \operatorname{rep}_d \Lambda$. We would like to show that Θ is an isomorphism, and we do so by first showing that it is well-defined, then bijective and lastly that Θ is an R-algebra homomorphism.

1. If $J \in \mathbb{N}_n^N$ for some $N \in \mathbb{N}_0$ and $f \in \operatorname{rep}_d \Lambda$, then

$$P_{\Theta(f),X}(X_J) = \rho_{\Theta(f),X}(X_J) = \Theta(f)_J = fgq(X)_J = fgq(X_J)$$

since f, g and q are R-algebra homomorphisms. Here concatenation of functions is assumed to signify function composition. We then have that

$$P_{\Theta(f),X}(\omega) = f \circ g \circ q(\omega)$$

for all $\omega \in \langle X \rangle^*$, so if $\lambda \in I$ such that $\lambda = \sum_{\omega \in \langle X \rangle^*} r_\omega \omega$ where $r_\omega \in R$ for each $\omega \in \langle X \rangle^*$, then

$$P_{\Theta(f),X}(\lambda) = P_{\Theta(f),X}\left(\sum_{\omega \in \langle X \rangle^*} r_\omega \omega\right) = \sum_{\omega \in \langle X \rangle^*} r_\omega \rho_{\Theta(f),X}(\omega)$$

$$= \sum_{\omega \in \langle X \rangle^*} r_\omega f \circ g \circ q(\omega) = f \circ g \circ q\left(\sum_{\omega \in \langle X \rangle^*} r_\omega \omega\right) = f \circ g(0) = 0.$$

This means that Θ is well-defined.

2. We prove that Θ is a bijection by first showing it is injective and then that it is surjective.

   (a) To show injectivity, assume $f_1, f_2 \in \operatorname{rep}_d \Lambda$ such that $\Theta(f_1) = \Theta(f_2)$. Then

$$(f_1 gq(X_1), \dots, f_1 gq(X_n)) = (f_2 gq(X_1), \dots, f_2 gq(X_n))$$

$$\Leftrightarrow f_1(g \circ q(X_i)) = f_1(g \circ q(X_i)) \ \forall i \in \mathbb{N}_n.$$

   Since both f_1 and f_2 are R-algebra homomorphisms, we then have that $f_1(\omega) = f_2(\omega)$ for all $\omega \in \langle g \circ q(X) \rangle^*$, where

$$g \circ q(X) = (g \circ q(X_1), \dots, g \circ q(X_n)),$$

and consequently that

$$f_1\left(\sum_{\omega\in\langle g\circ q(X)\rangle^*} r_\omega\omega\right) = f_2\left(\sum_{\omega\in\langle g\circ q(X)\rangle^*} r_\omega\omega\right)$$

for all $r_\omega \in R$, $\omega \in \langle g\circ q(X)\rangle^*$. Furthermore, by the proof of Lemma 1.3, we have that Λ is generated by $g\circ q(X_1),\ldots,g\circ q(X_n)$, so every element $\lambda \in \Lambda$ can be written as $\lambda = \sum_{\omega\in\langle g\circ q(X)\rangle^*} r_\omega\omega$ for some $r_\omega \in R$ for all $\omega \in \langle g\circ q(X)\rangle^*$. Then $f_1(\lambda) = f_2(\lambda)\ \forall\lambda \in \Lambda$, so $f_1 = f_2$. Thus Θ is injective.

(b) To prove surjectivity, let $A = (A_1,\ldots,A_n) \in \Xi(I)$. We want to show that $\exists f \in \operatorname{rep}_d\Lambda$ such that $\Theta(f) = A$, that is $f\circ g\circ q(X_i) = A_i\ \forall i \in \mathbb{N}_n$. Consider the R-algebra homomorphism $P_{A,g\circ q(X)}$. We have that $P_{A,g\circ q(X)}(g\circ q(X_i)) = \rho_{A,g\circ q(X)}(X_i) = A_i$ for all $i \in \mathbb{N}_n$, so $\Theta(P_{A,g\circ q(X)}) = A$. Then $\exists f \in \operatorname{rep}_d\Lambda$ such that $\Theta(f) = A$ for every $A \in M_d(R)^n$. Thus Θ is surjective.

Hence Θ is a bijection. $\square$

1.3 Representations of Quivers

In this section we introduce concepts related to representations of quivers. We also give a group action on quiver representations and show that it coincides with the action of $\mathrm{Gl}_d(R)$ on algebra representations.

A quiver $\Gamma = (\Gamma_0,\Gamma_1)$ consists of a finite set Γ_0 of vertices and a set Γ_1 of edges. Each edge $\alpha \in \Gamma_1$ has a starting point $s(\alpha)$ and an end point $e(\alpha)$.

Let k be a field and Γ a quiver. A representation (V,f) of Γ consists of a family of k-vector spaces that contains a vector space $V(i)$ for each vertex $i \in \Gamma_0$ and a family f of k-linear transformations that contains a transformation f_α for each $\alpha \in \Gamma_1$.

Let (V,f) and (W,g) be representations of a common quiver Γ. A homomorphism $h : (V,f) \to (W,g)$ between representations consists of k-linear transformations $h(i) : V(i) \to W(i)$ for each $i \in \Gamma_0$ such that for every

$\alpha : i \to j$ in Γ_1, where $\alpha : i \to j$ means $s(\alpha) = i$ and $e(\alpha) = j$, the diagram

$$\begin{array}{ccc} V(i) & \xrightarrow{h(i)} & W(i) \\ \downarrow{\scriptstyle f_\alpha} & & \downarrow{\scriptstyle g_\alpha} \\ V(j) & \xrightarrow{h(j)} & W(j) \end{array}$$

commutes. If $h(i)$ is an isomorphism for every $i \in \Gamma_0$, then we say that h is an isomorphism.

For any quiver Γ and field k we can construct a category $\operatorname{rep}\Gamma$ whose objects are representations of Γ over k. The set of morphisms $\operatorname{Hom}_{\operatorname{rep}\Gamma}(V, W)$ between any two objects V and W consists of the homomorphisms between V and W. Define for any $d \in \mathbb{N}$ the full subcategory $\operatorname{rep}_d \Gamma$ of $\operatorname{rep}\Gamma$ whose objects (V, f) satisfy the equation $\sum_{i\in\Gamma_0} \dim_k V(i) = d$. Let $m = |\Gamma_0|$, $D = (d_1, \dots, d_m) \in \mathbb{N}^m$ and define the full subcategory $\operatorname{rep}_D \Gamma$ of $\operatorname{rep}\Gamma$ whose objects are the representations (V, f) where $\dim_k V(i) = d_i$ for each $i \in \Gamma_0$. If $\sum_{i\in\Gamma_0} d_i = d$, then $\operatorname{rep}_D \Gamma$ is a full subcategory of $\operatorname{rep}_d \Gamma$.

Let $D(d_1, \dots, d_m) \in \mathbb{N}^m$ and consider the cartesian product $M_D(k) := \prod_{i=1}^m M_{d_i}(k)$. Define addition, multiplication and scalar multiplication on $M_D(k)$ as component-wise addition, multiplication and scalar multiplication on matrices. This defines a k-algebra structure on $M_D(k)$, as we show in Appendix E. Then $U(M_D(k)) = \mathrm{Gl}_D(k) := \prod_{i=1}^m \mathrm{Gl}_{d_i}(k)$.

Choose a basis for every vector space in every representation in $\operatorname{rep}_D \Gamma$. Define then a group action of $\mathrm{Gl}_D(k)$ on the objects of $\operatorname{rep}_D \Gamma$, denoted $\mathrm{Ob}(\operatorname{rep}_D \Gamma)$, as

$$\begin{aligned} \mathrm{Gl}_D(k) \times \mathrm{Ob}(\operatorname{rep}_D \Gamma) &\to \mathrm{Ob}(\operatorname{rep}_D \Gamma) \\ (A, (V, f)) &\mapsto (W, g) \end{aligned}$$

for all representations (V, f) in $\operatorname{rep}_D \Gamma$ and $A = (A_1, \dots, A_m) \in \mathrm{Gl}_D(k)$. The representation (W, g) is defined such that $W(i) = V(i)$ for all $i \in \Gamma_0$ and for every edge $\alpha : i \to j \in \Gamma_1$ we have that g_α is the unique linear transformation $W(i) \to W(j)$ such that the diagram

$$\begin{array}{ccc} V(i) & \xleftarrow{A_i^{-1}} & W(i) \\ \downarrow{\scriptstyle f_\alpha} & & \downarrow{\scriptstyle g_\alpha} \\ V(j) & \xrightarrow{A_j} & W(j) \end{array}$$

commutes, that is $g_\alpha = A_j f_\alpha A_i^{-1}$. Then $A_i g_\alpha = f_\alpha A_j$, so A defines a bijective homomorphism between (V, f) and (W, g), which means (V, f) and (W, g) are isomorphic. Since there is a bijective correspondence between representations of Γ and $k\Gamma$-modules, we can say that two representations are in the same $\mathrm{Gl}_d(k)$-orbit if and only if they are in the same $\mathrm{Gl}_D(k)$-orbit, where d is the rank of each of the modules the two representations correspond to.

If k is a finite field, say $\mathrm{GF}(q)$ for some prime power $q \in \mathbb{N}$, then we can find the size of the Gl_d-orbits. We have that for the group action of any finite group G on a set X, the size of the orbit of an element x is $\frac{|G|}{|G_x|}$ where G_x is the stabilizer subgroup of x. In the case of representations of quivers we have $G = \mathrm{Gl}_D(k)$ and $X = \mathrm{Ob}(\mathrm{rep}_D\,\Gamma)$, so if $x = (V, f)$, then $G_x = \{A \in \mathrm{Gl}_d(k) \mid A \cdot (V, f) = (V, f)\}$. We have that $A \cdot (V, f) = (V, f)$ if and only if A describes an isomorphism from (V, f) to itself. We call such isomorphisms for automorphisms on (V, f), and we denote $\mathrm{Aut}(V, f) = \{\text{automorphisms on } (V, f)\}$. Then $G_x = \mathrm{Aut}(x)$, so if Gx denotes the orbit of x, then

$$|Gx| = \frac{|\mathrm{Gl}_D(k)|}{|\mathrm{Aut}(V, f)|}.$$

1.4 The Zariski Topology

The Zariski is the last puzzle piece needed to define degeneration on $\mathrm{Gl}_d(K)$-orbits. We define affine spaces and the Zariski topology and state some properties of these concepts.

The following definition is based on the one found in [13].

Definition 1.5. Let V be a vector space over a field K. An affine space is a nonempty set A together with an addition $A \times V \to A$ which satisfies the following criteria for all $p \in A$.

1. $(p + a) + b = p + (a + b)\ \forall\ a, b \in V$.
2. Given $q \in A$, $\exists!\ a \in V$ such that $q = p + a$.

Δ

It is not uncommon to include a third condition to the affine addition in A, namely that $p = p + 0_V$ for all $p \in A$. This condition is however implied by the two others. We first have by (2) that $p = p + a$ for some unique $a \in V$.

Then $p = p + a = (p + a) + a = p + (a + a)$, so $p = p + a$ and $p = p + (a + a)$, but by the uniqueness of a, we have that $a = a + a \Rightarrow a = 0_V \Rightarrow p = p + 0_V$.

An example of an affine space is $\mathbb{A}^n := K^n$ with standard vector addition, where K is an algebraically closed field and $n \in \mathbb{N}$. $\mathbb{A}^n$ is also a vector space and all vector spaces are in fact affine spaces.

Next we have affine algebraic sets. These are defined to be the algebraic sets of $\mathbb{A}^n$. That is, affine algebraic sets are on the form

$$V(S) = \{x \in \mathbb{A}^n \mid f(x) = 0 \; \forall f \in S\}$$

where $S \subseteq K[X_1, \ldots, X_n]$. This definition of affine algebraic sets, the following properties and the subsequent definition of the Zariski topology are based on [10].

Lemma 1.4. The following statements about algebraic sets are true.

1. $V(S) = V((S))$ for any subset $S \subseteq K[X_1, \ldots, X_n]$. (S) denotes the ideal generated by S.

2. $V(A) \cup V(B) = V(AB)$ for ideals A, B in $K[X_1, \ldots, X_n]$.

3. $\bigcap_{i \in I} V(A_i) = V(\sum_{i \in I} A_i)$ for ideals A_i in $K[X_1, \ldots, X_n]$, $i \in I$, where I is some set of indices.

Proof.

1. Let S be a subset of $K[X_1, \ldots, X_n]$. $S \subseteq (S)$ implies that if $x \in V((S))$, i.e. if $f(x) = 0$ for all $f \in (S)$, then $g(x) = 0$ for all $g \in V(S)$. Thus

$$V((S)) \subseteq V(S).$$

   Next, if $a_i \in K[X_1, \ldots, X_n]$ and $s_i \in S$ for $i \in \mathbb{N}_m, m \in \mathbb{N}$, and $x \in V(S)$, then $s_i(x) = 0$, implying $\left(\sum_{i=1}^m a_i s_i\right)(x) = 0$. Since every element in (S) is on the form $\sum_{i=1}^m a_i s_i$, then $f(x) = 0$ for all $f \in V((S))$ $\Rightarrow$

$$V(S) \subseteq V((S)).$$

   Hence $V(S) = V((S))$.

2. Let A, B be ideals in $K[X_1, \ldots, X_n]$. Suppose $x \in V(A) \cup V(B)$. Then $f(x) = 0$ for all $f \in A$ or $g(x) = 0$ for all $g \in B$. If $f_1, \ldots, f_m \in A$

and $g_1, \dots, g_m \in B$, $m \in \mathbb{N}$, then $\left(\sum_{i=1}^m f_i g_i\right)(x) = 0$ since $f_i(x) = 0$ or $g_i(x) = 0$ for each $i \in \mathbb{N}_m$. Then $x \in V(AB)$, and

$$V(A) \cup V(B) \subseteq V(AB).$$

Suppose $x \in V(AB)$. We can assume that there exists a $f \in A$ such that $f(x) \neq 0$ because if $f(x) = 0\ \forall f \in A$, $x \in V(A) \cup V(B)$ and we are done. For every $g \in B$ we have that $(fg)(x) = f(x)g(x) = 0$. Since K is an integral domain, we then have that $f(x) = 0$ or $g(x) = 0$, but $f(x) \neq 0$, so $g(x) = 0$. Since g is an arbitrary element in B, then $g(x) = 0\ \forall g \in B$. Then $x \in V(B) \Rightarrow x \in V(A) \cup V(B) \Rightarrow$

$$V(AB) \subseteq V(A) \cup V(B).$$

Hence $V(A) \cup V(B) = V(AB)$.

3. Let A_i be ideals in $K[X_1, \dots, X_n]$ for all i in an indexing set I. Suppose $x \in \bigcap_{i \in I} V(A_i) \Rightarrow x \in V(A_i)$ for every $i \in I$. Let $f_i \in A_i\ \forall i \in I$. Since $f_i(x) = 0\ \forall \in I$, then $\left(\sum_{i \in I} f_i\right)(x) = 0$, so $x \in V\left(\sum_{i \in I} A_i\right) \Rightarrow$

$$\bigcap_{i \in I} V(A_i) \subseteq V\left(\sum_{i \in I} A_i\right).$$

Suppose $x \in V\left(\sum_{i \in I} A_i\right)$. Suppose $f_i \in A_i$ for each $i \in I$. Then $\sum_{i \in I} f_i \in \sum_{i \in I} A_i$. Let $j \in I$. The sum $f_j + \sum_{i \in I}(-f_i) = \sum_{i \in I \setminus \{j\}}(-f_i)$ is also in $\sum_{i \in I} A_i$ since $f_j - f_j = 0 \in A_j$ and $-f_i \in A_i\ \forall i \in I \setminus \{j\}$. $\sum_{i \in I} f_i + \sum_{i \in I \setminus \{j\}}(-f_i) = f_j$, so $f_j \in \sum_{i \in I} A_i$. Then $f_j(x) = 0$, so $x \in V(A_j)\ \forall j \in I \Rightarrow x \in \bigcap_{i \in I} V(A_i) \Rightarrow$

$$V\left(\sum_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} V(A_i).$$

Hence $V\left(\sum_{i \in I} A_i\right) = \bigcap_{i \in I} V(A_i)$.

□

With algebraic sets in our arsenal, we move on to the Zariski topology.

Definition 1.6. The **Zariski topology** is defined such that the closed sets are the algebraic sets of $\mathbb{A}^n$. △

This does really define a topology:

- if $f = 0$, then $f(x) = 0 \; \forall x \in \mathbb{A}^n \Rightarrow V((0)) = \mathbb{A}^n \Rightarrow \mathbb{A}^n$ is closed.
- if $f(x) = 0$ for some $f \in K[X_1, \ldots, X_n]$ and $x \in \mathbb{A}^n$, then $(f+1)(x) = 1$. Then, for every $x \in \mathbb{A}^n$, there always exists a function g such that $g(x) \neq 0$, so $V(K[X_1, \ldots, X_n]) = \varnothing \Rightarrow \varnothing$ is closed.
- by Lemma 1.4, we see that finite unions and arbitrary intersections of algebraic sets are again algebraic sets, which in the context of the Zariski topology means that finite unions and arbitrary intersections of closed sets are closed.

Let S be a subset of $\mathbb{A}^n$. For the sake of convenience, if A is a subset of $K[X_1, \ldots, X_n]$, we write $A(S) = 0$ if $f(x) = 0$ for all $f \in A$ and $x \in S$. The closure is

$$\overline{S} = \bigcap_{\substack{A \text{ ideal} \\ A(S)=0}} V(A) = V\left(\sum_{\substack{A \text{ ideal} \\ A(S)=0}} A \right).$$

We can show that show that

$$\sum_{\substack{A \text{ ideal} \\ A(S)=0}} A = \{f \in K[X_1, \ldots, X_n] \mid f(x) = 0 \; \forall x \in S\}.$$

1. Suppose $f \in \sum_{\substack{A \text{ ideal} \\ A(S)=0}} A$. Then $f(x)$ for each $x \in S$
$$\Rightarrow f \in \{f \in K[X_1, \ldots, X_n] \mid f(x) = 0 \; \forall x \in S\}.$$
2. Suppose $f \in \{f \in K[X_1, \ldots, X_n] \mid f(x) = 0 \; \forall x \in S\}$. We have that $f \in (f)$, which is the ideal generated by f, and $(f) \subseteq \sum_{\substack{A \text{ ideal} \\ A(S)=0}} A$. Then $f \in \sum_{\substack{A \text{ ideal} \\ A(S)=0}} A$.

Then $\overline{S} = \{x \in \mathbb{A}^n \mid f(x) = 0 \; \forall f \in K[X_1, \ldots, X_n]$ such that $f(S) = 0\}$.

1.5 Degeneration

We now have the necessary concepts and results needed for defining degeneration on $\mathrm{Gl}_d(K)$-orbits for algebraically closed fields K. Afterwards we state a result about how the degeneration order relates to certain short exact sequences and how this result can be used as a way to expand the definition of degeneration. We also give a second order $\leq_{\mathrm{ext}}$, and discuss how this order and degeneration are associated. Lastly we present an example of degeneration on certain representations of the Cronecker quiver.

Remark 2. Let Λ be an algebra over a algebraically closed field K which is generated by $n \in \mathbb{N}$ elements, $d \in \mathbb{N}$ and let Θ be the group isomorphism described in the proof of Lemma 1.1. Then we have the following facts about the $\mathrm{Gl}_d(K)$-orbits in $\mathrm{rep}_d(\Lambda)$:

1. $\mathrm{Gl}_d(K)m$ is open in its Zariski closure $\overline{\mathrm{Gl}_d(K)m} := \{m' \in \mathrm{rep}_d(\Lambda) \mid p(\Theta(m')) = 0$ where p is a polynomial in nd^2 variables such that $p(\Theta(f)) = 0\ \forall f \in \mathrm{Gl}_d(R)m\}$.
2. If $m \in \mathrm{rep}_d(\Lambda)$, then $\overline{\mathrm{Gl}_d(K)m}$ is a union of orbits.
3. $\dim\left(\overline{\mathrm{Gl}_d(K)m} \setminus \mathrm{Gl}_d(K)m\right) < \dim\left(\overline{\mathrm{Gl}_d(K)m}\right)$ for all $m \in \mathrm{rep}_d(\Lambda)$, where dimension is referring to Krull dimension of the variety. Moreover, the following formula for the dimension holds: $\dim(\mathrm{Gl}_d(K)m) = d^2 - \dim\left(\mathrm{End}_\Lambda(M_m)\right)$.

Δ

For the following definition, recall the function O such that $O\big([M_m]\big) = \mathrm{Gl}_d(K)m$ for all $m \in \mathrm{rep}_d(\Lambda)$.

Definition 1.7. Let Λ be a finite-dimensional algebra over an algebraically closed field K, $d \in \mathbb{N}$ and let $M, N \in M_{\mathrm{rep}_d \Lambda}$. Then we say that $[M]$ **degenerates to** $[N]$, denoted $[M] \leq_{\deg} [N]$, if and only if the orbit corresponding to $[N]$ is included in the closure of the orbit corresponding to $[M]$, that is $O([N]) \subseteq \overline{O([M])}$. Δ

Remark 3. We sometimes drop the equivalence class notation and say M degenerates to N, or $M \leq_{\deg} N$ for some $M, N \in M_{\mathrm{rep}_d \Lambda}$, but this really means that $[M] \leq_{\deg} [N]$. Δ

Theorem 1.5. Let K be an algebraically closed field, Λ a finite-dimensional K-algebra and M, N Λ-modules that are finite-dimensional as K-modules. Then $M \leq_{\text{deg}} N$ if and only if there exists an A which is finite-dimensional as a K-module such that the sequence $0 \to A \to A \oplus M \to N \to 0$ is short exact.

We can then extend the definition of degeneration to include Λ-modules that have finite length as R-modules where Λ is any algebra over a commutative ring R. For any such Λ-modules M, N, we say that $M \leq_{\text{deg}} N$ if there is an Λ-module X that has finite length as an R-module such that the sequence $0 \to X \to X \oplus M \to N \to 0$ is exact.

Remark 4. The degeneration relation $\leq_{\text{deg}}$ defines a partial order on isomorphism classes of Λ-modules that have finite length as R-modules.

1. Let M be an Λ-module that has finite length as an R-module. Then for any Λ-module X that has finite length as an R-module we have that the sequence $0 \longrightarrow X \xrightarrow{\binom{\mathrm{id}_X}{0}} X \oplus M \xrightarrow{(0\ \mathrm{id}_M)} M \longrightarrow 0$ is short exact. Thus $M \leq_{\text{deg}} M$.

2. We prove antisymmetry later on.

3. Let L, M, N be Λ-modules that have finite length as R-modules. There there are Λ-modules X, Y that have finite length as R-modules such that the sequences

$$0 \longrightarrow X \xrightarrow{\binom{f_1}{f_2}} X \oplus L \xrightarrow{(g_1\ g_2)} M \longrightarrow 0$$

and

$$0 \longrightarrow Y \xrightarrow{\binom{f_1'}{f_2'}} Y \oplus M \xrightarrow{(g_1'\ g_2')} N \longrightarrow 0$$

commute. We have that the map

$$Y \oplus X \xrightarrow{\begin{pmatrix} \mathrm{id}_Y & 0 \\ 0 & f_1 \\ 0 & f_2 \end{pmatrix}} Y \oplus X \oplus L$$

is injective and the maps

$$Y \oplus X \oplus L \xrightarrow{\begin{pmatrix} f_1' & 0 & 0 \\ f_2' & g_1 & g_2 \end{pmatrix}} Y \oplus M \quad \text{and} \quad Y \oplus M \xrightarrow{(g_1'\ g_2')} N$$

are surjevtive. Then the composition

$$\begin{pmatrix} g_1' & g_2' \end{pmatrix} \begin{pmatrix} f_1' & 0 & 0 \\ f_2' & g_1 & g_2 \end{pmatrix} = \begin{pmatrix} g_1'f_1' + g_2'f_2' & g_2g_1' & g_2g_2' \end{pmatrix}$$

$$= \begin{pmatrix} 0 & g_2g_1' & g_2g_2' \end{pmatrix} : Y \oplus X \oplus L \to N$$

is surjective. Thus we get that the sequence

$$0 \longrightarrow Y \oplus X \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & f_1 \\ 0 & f_2 \end{smallmatrix}\right)} Y \oplus X \oplus L \xrightarrow{\left(\begin{smallmatrix} 0 & g_2g_1' & g_2g_2' \end{smallmatrix}\right)} N \longrightarrow 0$$

is short exact since

$$\begin{pmatrix} 0 & g_2g_1' & g_2g_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & f_1 \\ 0 & f_2 \end{pmatrix} = \begin{pmatrix} 0 & g_2'(g_1f_1 + g_2f_2) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & g_2' \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

Then $L \leq_{\text{deg}} N$.

$\triangle$

Definition 1.8. Let M, N be Λ-modules that have finite length as R-modules. We say that $[M] \leq_{\text{ext}} [N]$ if there exist Λ-modules A, B that have finite length as R-modules such that $N \cong A \oplus B$ and $0 \to A \to M \to B \to 0$ is a short exact sequence. $\triangle$

Remark 5. We sometimes write $M \leq_{\text{ext}} N$, which means $[M] \leq_{\text{ext}} [N]$. $\triangle$

Theorem 1.6. Let M, N be Λ-modules that have finite length as R-modules. If $M \leq_{\text{ext}} N$, then $M \leq_{\text{deg}} N$.

Proof. Suppose M, N are Λ-modules that have finite length as R-modules such that $M \leq_{ext} N$. Then there are Λ-modules A, B that have finite length as R-modules such that $N \cong A \oplus B$ and we have an exact sequence

$$0 \longrightarrow A \xrightarrow{f} M \xrightarrow{g} B \longrightarrow 0 .$$

Then the sequence

$$0 \longrightarrow A \xrightarrow{\left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right)} A \oplus M \xrightarrow{\left(\begin{smallmatrix} \mathrm{id}_A & 0 \\ 0 & g \end{smallmatrix}\right)} A \oplus B \longrightarrow 0$$

is exact since $\left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right) : A \to A \oplus M$ is injective, $\left(\begin{smallmatrix} \mathrm{id}_A & 0 \\ 0 & g \end{smallmatrix}\right) : A \oplus M \to A \oplus B$ is surjective and $\left(\begin{smallmatrix} \mathrm{id}_A & 0 \\ 0 & g \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right)$. Thus $M \leq_{\mathrm{deg}} A \oplus B \cong N$. $\square$

Corollary 1.6.1. Let M, N be Λ-modules that have finite length as R-modules. If $M \leq_{\mathrm{ext}} N$, then $T \oplus M \leq_{\mathrm{deg}} T \oplus N$ for any Λ-module T that has finite length as a R-module.

Proof. Suppose M, N, T are Λ-modules which have finite length as R-modules and let $N \cong A \oplus B$ such that $0 \to A \xrightarrow{f} M \xrightarrow{g} B \to 0$ is short exact for suitable Λ-homomorphisms f and g. Then

$$0 \longrightarrow T \oplus A \xrightarrow{\left(\begin{smallmatrix} \mathrm{id}_T & 0 \\ 0 & f \end{smallmatrix}\right)} T \oplus M \xrightarrow{\left(\begin{smallmatrix} 0 & g \end{smallmatrix}\right)} B \longrightarrow 0$$

is short exact, so $T \oplus M \leq_{\mathrm{deg}} T \oplus A \oplus B \cong T \oplus N$ by Theorem 1.6. $\square$

Example 1.3. Let Γ be the quiver
$$\begin{tikzcd} 1 \arrow[d, "\alpha"] \\ 2 \arrow[loop below, "\beta"] \end{tikzcd}$$
. We want to construct two representations of Γ. For one representation (V, f) we want that $f_\beta^2 = 0$ and $f_\beta f_\alpha \neq 0$. For the other representation (W, g) we want that $g_\beta^2 = 0$ and $g_\beta g_\alpha = 0$. Then let $(V, f) =$
$$\begin{tikzcd} k \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \\ k^2 \arrow[loop below, "\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)"] \end{tikzcd}$$
and $(W, f) =$
$$\begin{tikzcd} k \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \\ k^2 \arrow[loop below, "\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)"] \end{tikzcd}$$
. We can show that $(V, f) \leq_{\mathrm{deg}} (W, g)$.

$$\begin{tikzcd}[ampersand replacement=\&]
 \& \begin{array}{c} 0 \\ \downarrow \\ k^2 \\ \circlearrowright \\ \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right) \end{array} \arrow[dr, "{(0,\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right))}"] \& \\
\begin{array}{c} 0 \\ \downarrow \\ k^2 \\ \circlearrowright \\ \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right) \end{array} \arrow[ur, "{(0,\left(\begin{smallmatrix}0 & -1\\ 0 & 0\end{smallmatrix}\right))}"] \arrow[dr, swap, "{(0,\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right))}"] \& \oplus \& \begin{array}{c} k \\ \downarrow \left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right) \\ k^2 \\ \circlearrowright \\ \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right) \end{array} \\
 \& \begin{array}{c} k \\ \downarrow \left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right) \\ k^2 \\ \circlearrowright \\ \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right) \end{array} \arrow[ur, swap, "{(1,\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right))}"] \&
\end{tikzcd}$$

The commutative diagram above implies the existence of a short exact sequence

$$0 \longrightarrow X \longrightarrow X \oplus (V,f) \longrightarrow (W,g) \longrightarrow 0$$

where $X = \begin{array}{c} 0 \\ \downarrow \\ k^2 \\ \circlearrowright \\ \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right) \end{array}$. Thus $(V,f) \leq_{\deg} (W,g)$. Another observation is that (V,f) and (W,g) are indecomposable. Since $f_\alpha \neq 0$ and $g_\alpha \neq 0$, then we cannot have decompositions of the form $\begin{array}{c} 0 \\ \downarrow \\ k^2 \\ \circlearrowright \end{array} \oplus \begin{array}{c} k \\ \downarrow \\ 0 \\ \circlearrowright \end{array}$. The other possible sort of decomposition would be $\begin{array}{c} 0 \\ \downarrow \\ k \\ \circlearrowright \end{array} \oplus \begin{array}{c} k \\ \downarrow \\ k \\ \circlearrowright \end{array}$, but $f_\beta \neq 0$, $g_\beta \neq 0$ and

$f_\beta^2 = g_\beta^2 = 0$, which rules out this type of decomposition. Then we have proper degeneration between indecomposables, so there exist modules M, N such that

$$M \leq_{\text{deg}} N \not\Rightarrow M \leq_{\text{ext}} N.$$

$\triangle$

Example 1.4. Consider the Cronecker quiver $\Gamma = \begin{array}{c} 1 \\ \downdownarrows \\ 2 \end{array}$, $k = \mathrm{GF}(q)$ for a prime power $q \in \mathbb{N}$. The number of representations with dimension vector $(2,2)$ is

$$|\mathrm{Ob}(\mathrm{rep}_{(2,2)}\,\Gamma)| = |M_2(\mathrm{GF}(q))^2| = q^8.$$

The group acting on $\mathrm{Ob}(\mathrm{rep}_{(2,2)}\,\Gamma)$ is $G = \mathrm{Gl}_2(k) \times \mathrm{Gl}_2(k)$ and we have that $|G| = (q^2-1)(q^2-q)\cdot(q^2-1)(q^2-q)$. Let us find the G-orbits of $\mathrm{Ob}(\mathrm{rep}_{(2,2)}\,\Gamma)$. If $x \in \mathrm{rep}_{(2,2)}\,\Gamma$, then the size of its orbit is $|Gx| = \frac{(q^2-1)(q^2-q)\cdot(q^2-1)(q^2-q)}{|\mathrm{Aut}(x)|}$.

Through the following calculations we try to find representatives for all the G-orbits of $\mathrm{Ob}(\mathrm{rep}_{(2,2)}\,\Gamma)$.

1. $x_1(a) = \begin{array}{c} k^2 \\ {\scriptstyle I_2}\downdownarrows{\scriptstyle J(a)} \\ k^2 \end{array}$ where $J(a)$ is the matrix of Jordan Canonical form with $a \in k$ along its diagonal. We also define $x_1(\infty) = \begin{array}{c} k^2 \\ {\scriptstyle J(0)}\downdownarrows{\scriptstyle I_2} \\ k^2 \end{array}$. There is one J for each element in k, and any two J's correspond to non-isomorphic representations. We can interpret $x_1(\infty)$ as being the "point at infinity". Then there are $q+1$ orbits represented by x_1. As for the size of these orbits, we have that $\mathrm{Aut}(x_1) = k[X]/(X^2)$ which has size $q^2 - q$, so $|Gx_1| = \frac{((q^2-1)(q^2-q))^2}{q^2-q} = (q^2-1)^2(q^2-q)$.

2. $x_2(M) = \begin{array}{c} k^2 \\ {\scriptstyle I_2}\downdownarrows{\scriptstyle M} \\ k^2 \end{array}$, where M is a matrix whose characteristic polynomial is irreducible in $k[X]$. The formula for the number of irreducible

polynomials over GF(q) of degree n is $\frac{1}{n}\sum_{d|n}\mu(\frac{n}{d})q^d$. Here $\mu(1) = 1$ and $\mu(r) = (-1)^s$ where s is the number of factors of r. This formula is from [4]. Since the characteristic polynomial of M has degree 2, then the number of orbits is $\frac{q^2-q}{2}$. Furthermore, $\mathrm{Aut}(x_2) = \mathrm{GF}(q^2)^* \Rightarrow |\mathrm{Aut}(x_2)| = q^2 - 1 \Rightarrow |Gx_2| = (q^2-1)(q^2-q)^2$.

3. $x_3 = \begin{array}{c} k \\ {\scriptstyle \binom{1}{0}}\downdownarrows{\scriptstyle \binom{0}{1}} \\ k^2 \end{array} \oplus \begin{array}{c} k \\ \downdownarrows \\ 0 \end{array}$. There is only one orbit represented by x_3. We have that $\mathrm{End}(V \oplus W) \cong \begin{pmatrix} \mathrm{End}(V) & \mathrm{Hom}(W,V) \\ \mathrm{Hom}(V,W) & \mathrm{End}(W) \end{pmatrix}$, so $\mathrm{End}(x_3) \cong \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$. If an endomorphism $a = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}$ on x_3 is bijective, then all entries along its diagonal are nonzero. Thus $\mathrm{Aut}(x_3) = \begin{pmatrix} k^* & 0 \\ k & k^* \end{pmatrix}$, so $|\mathrm{Aut}(x_3)| \cong |k| \cdot |k^*|^2 = q(q-1)^2$ and $|Gx_3| = q(q^2-1)^2$.

4. $x_4 = \begin{array}{c} k^2 \\ {\scriptstyle (1\ 0)}\downdownarrows{\scriptstyle (0\ 1)} \\ k \end{array} \oplus \begin{array}{c} 0 \\ \downdownarrows \\ k \end{array}$. There is only one orbit belonging to this representation. We have that $\mathrm{End}(x_4) \cong \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ and any automorphism on x_4 must have nonzero entries along its diagonal, so $\mathrm{Aut}(x_4) \cong \begin{pmatrix} k^* & k \\ 0 & k^* \end{pmatrix}$. Then $|Gx_4| = q(q^2-1)^2$.

5. $x_5(a,b,c,d) = \begin{array}{c} k \\ {\scriptstyle a}\downdownarrows{\scriptstyle b} \\ k \end{array} \oplus \begin{array}{c} k \\ {\scriptstyle c}\downdownarrows{\scriptstyle d} \\ k \end{array}$ where $a,b,c,d \in k$ such that we do not have that both a and b equal are 0, or that both and c and d equal 0, and we have that $\begin{array}{c} k \\ {\scriptstyle a}\downdownarrows{\scriptstyle b} \\ k \end{array} \cong \begin{array}{c} k \\ {\scriptstyle c}\downdownarrows{\scriptstyle d} \\ k \end{array}$. For every $\alpha \in k^*$ and $\beta \in k$, $\begin{array}{c} k \\ {\scriptstyle \alpha}\downdownarrows{\scriptstyle \beta} \\ k \end{array} \cong \begin{array}{c} k \\ {\scriptstyle 1}\downdownarrows{\scriptstyle \gamma} \\ k \end{array}$ for some $\gamma \in k$. Then every isomorphism class of representations on the form $\begin{array}{c} k \\ {\scriptstyle \alpha}\downdownarrows{\scriptstyle \beta} \\ k \end{array}$ can be represented by $\begin{array}{c} k \\ {\scriptstyle 1}\downdownarrows{\scriptstyle \gamma} \\ k \end{array}$ for some $\gamma \in k$. Then there are $|k| = q$ isomorphism classes of representations on this form and thus there are also q orbits represented by x_5 where $a, c \in k^*$

and $b, d \in k$. If $\alpha = 0$ and $\beta \in k^*$, then $\begin{array}{c} k \\ {\scriptstyle\alpha}\downdownarrows{\scriptstyle\beta} \\ k \end{array} \cong \begin{array}{c} k \\ {\scriptstyle 0}\downdownarrows{\scriptstyle 1} \\ k \end{array}$, and $\begin{array}{c} k \\ {\scriptstyle 0}\downdownarrows{\scriptstyle 1} \\ k \end{array}$ represents a unique isomorphism class, which means that there is one orbit represented by x_5 where $a = c = 0$ and $b, d \in k^*$. Thus there are $q + 1$ orbits in total which are represented by x_5. We can represent q orbits by $x_5(\lambda) := \begin{array}{c} k \\ {\scriptstyle 1}\downdownarrows{\scriptstyle\lambda} \\ k \end{array} \oplus \begin{array}{c} k \\ {\scriptstyle 1}\downdownarrows{\scriptstyle\lambda} \\ k \end{array}$ for $\lambda \in k$ and the last orbit by

$$x_5(\infty) := \begin{array}{c} k \\ {\scriptstyle 0}\downdownarrows{\scriptstyle 1} \\ k \end{array} \oplus \begin{array}{c} k \\ {\scriptstyle 0}\downdownarrows{\scriptstyle 1} \\ k \end{array}.$$

We have that $\mathrm{End}(x_5) \cong \left(\begin{smallmatrix} k & k \\ k & k \end{smallmatrix}\right)$, so $\mathrm{Aut}(x_5) \cong \mathrm{Gl}_2(k)$ and $|\mathrm{Aut}(x_5)| = (q^2 - 1)(q^2 - q)$. Then $|Gx_5| = (q^2 - 1)(q^2 - q)$.

6. $x_6(a, b, c, d) = \begin{array}{c} k \\ {\scriptstyle a}\downdownarrows{\scriptstyle b} \\ k \end{array} \oplus \begin{array}{c} k \\ {\scriptstyle c}\downdownarrows{\scriptstyle d} \\ k \end{array}$ where $a, b, c, d \in k$ such that we do not have that both a and b equal 0, or that both and c and d equal 0, and we have that $\begin{array}{c} k \\ {\scriptstyle a}\downdownarrows{\scriptstyle b} \\ k \end{array} \ncong \begin{array}{c} k \\ {\scriptstyle c}\downdownarrows{\scriptstyle d} \\ k \end{array}$. There are $q + 1$ isomorphism classes of representations on the form of $\begin{array}{c} k \\ {\scriptstyle\alpha}\downdownarrows{\scriptstyle\beta} \\ k \end{array}$ where $\alpha, \beta \in k$ and not both α and β are zero, so since any x_6 is a direct sum of two such representations, then there are $\binom{q+1}{2} = \frac{q^2+q}{2}$ orbits represented by x_6. We can represent some of these orbits by $x_6(\lambda, \lambda') := \begin{array}{c} k \\ {\scriptstyle 1}\downdownarrows{\scriptstyle\lambda} \\ k \end{array} \oplus \begin{array}{c} k \\ {\scriptstyle 1}\downdownarrows{\scriptstyle\lambda'} \\ k \end{array}$ for varying $\lambda, \lambda' \in k$ such that $\lambda \neq \lambda'$. It might seem like allowing any λ and λ' such that $\lambda \neq \lambda'$ would mean there are $q^2 - q$ orbits represented by $x_6(\lambda, \lambda')$, but since

$x_6(\lambda, \lambda') \cong x_6(\lambda', \lambda)$, then there are actually $\frac{q^2-q}{2} = \binom{q}{2}$ such orbits. The other q orbits can be be represented by $x_6(\lambda, \infty) := \begin{array}{c} k \\ {\scriptstyle 1}\downdownarrows{\scriptstyle \lambda} \\ k \end{array} \oplus \begin{array}{c} k \\ {\scriptstyle 0}\downdownarrows{\scriptstyle 1} \\ k \end{array}$.

Let $f \in \operatorname{Hom}\left(\begin{array}{c} k \\ {\scriptstyle a}\downdownarrows{\scriptstyle b} \\ k \end{array}, \begin{array}{c} k \\ {\scriptstyle c}\downdownarrows{\scriptstyle d} \\ k \end{array} \right)$. Then there exist $f_1, f_2 \in k$ such that the diagram $\begin{array}{ccc} k & \xrightarrow{f_1} & k \\ {\scriptstyle a}\downdownarrows{\scriptstyle b} & & {\scriptstyle c}\downdownarrows{\scriptstyle d} \\ k & \xrightarrow{f_2} & k \end{array}$ commutes, that is $f_2 a = c f_1$ and $f_2 c = d f_1$. Since the representations $\begin{array}{c} k \\ {\scriptstyle a}\downdownarrows{\scriptstyle b} \\ k \end{array}$ and $\begin{array}{c} k \\ {\scriptstyle c}\downdownarrows{\scriptstyle d} \\ k \end{array}$ are supposed to be non-isomorphic, then $f_1 = 0$ or $f_2 = 0$. Suppose $f_1 = 0$ and $f_2 \neq 0$, then $a = b = 0$. If $f_1 \neq 0$ and $f_2 = 0$, then $c = d = 0$. This contradicts the initial assumption that not both a and b can equal zero or that not both c and d can equal zero. Thus $f_1 = f_2 = 0 \Rightarrow f = 0$, so $\operatorname{Hom}\left(\begin{array}{c} k \\ {\scriptstyle a}\downdownarrows{\scriptstyle b} \\ k \end{array}, \begin{array}{c} k \\ {\scriptstyle c}\downdownarrows{\scriptstyle d} \\ k \end{array} \right) = 0$. We then have that $\operatorname{End}(x_6) \cong \left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)$ and

$$\begin{aligned} \operatorname{Aut}(x_6) = \begin{pmatrix} k^* & 0 \\ 0 & k^* \end{pmatrix} &\Rightarrow |\operatorname{Aut}(x_6)| = |k^*|^2 = (q-1)^2 \\ &\Rightarrow |Gx_6| = ((q^2-q)(q+1))^2. \end{aligned}$$

7. $x_7 = \begin{array}{c} k \\ {\scriptstyle a}\downdownarrows{\scriptstyle b} \\ k \end{array} \oplus \begin{array}{c} k \\ \downdownarrows \\ 0 \end{array} \oplus \begin{array}{c} 0 \\ \downdownarrows \\ k \end{array}$ where not both $a \in k$ and $b \in k$ are zero. There are $q+1$ isomorphism classes of representations of the type $\begin{array}{c} k \\ {\scriptstyle a}\downdownarrows{\scriptstyle b} \\ k \end{array}$, so x_7 represents $q+1$ orbits. We can represent q of these orbits

by $x_7(\lambda) := \begin{array}{c} k \\ {\scriptstyle 1}\Big\downarrow\Big\downarrow{\scriptstyle \lambda} \\ k \end{array} \oplus \begin{array}{c} k \\ \Big\downarrow\Big\downarrow \\ 0 \end{array} \oplus \begin{array}{c} 0 \\ \Big\downarrow\Big\downarrow \\ k \end{array}$ where $\lambda \in k$. The last orbit can be represented by $x_7(\infty) := \begin{array}{c} k \\ {\scriptstyle 0}\Big\downarrow\Big\downarrow{\scriptstyle 1} \\ k \end{array} \oplus \begin{array}{c} k \\ \Big\downarrow\Big\downarrow \\ 0 \end{array} \oplus \begin{array}{c} 0 \\ \Big\downarrow\Big\downarrow \\ k \end{array}$.

We have that $\mathrm{End}(x_7) \cong \left(\begin{smallmatrix} k & 0 & k \\ k & k & 0 \\ 0 & 0 & k \end{smallmatrix}\right)$. Invertible elements in $\mathrm{End}(x_7)$ must have nonzero entries along their diagonals, so

$$\mathrm{Aut}(x_7) \cong \begin{pmatrix} k^* & 0 & k \\ k & k^* & 0 \\ 0 & 0 & k^* \end{pmatrix}.$$

Then $|\mathrm{Aut}(x_7)| = q^2(q-1)^3$ and $|Gx_7| = (q^2-1)(q+1)$.

8. $x_8 = \begin{array}{c} k^2 \\ {\scriptstyle \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)}\Big\downarrow\Big\downarrow{\scriptstyle \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)} \\ k^2 \end{array} = \left(\begin{array}{c} k \\ \Big\downarrow\Big\downarrow \\ 0 \end{array}\right)^2 \oplus \left(\begin{array}{c} 0 \\ \Big\downarrow\Big\downarrow \\ k \end{array}\right)^2$. There are no variables in this expression, so x_8 only represents one orbit.

Let $M_1, M_2 \in M_2(k)$. The diagram $\begin{array}{ccc} k^2 & \xrightarrow{M_1} & k^2 \\ {\scriptstyle \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)}\Big\downarrow\Big\downarrow{\scriptstyle \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)} & & {\scriptstyle \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)}\Big\downarrow\Big\downarrow{\scriptstyle \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)} \\ k^2 & \xrightarrow{M_2} & k^2 \end{array}$ commutes since $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \cdot M_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) = M_2 \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then

$$M = (M_1, M_2) \in \mathrm{End}(x_8) \Rightarrow \mathrm{End}(x_8) \cong M_2(k) \times M_2(k)$$

$$\Rightarrow \mathrm{Aut}(x_8) \cong \mathrm{Gl}_2(k) \times \mathrm{Gl}_2(k) \Rightarrow |\mathrm{Aut}(x_8)| = ((q^2-1)(q^2-q))^2$$

$$\Rightarrow |Gx_8| = 1.$$

We collect these findings in the following table.

Type	Characterization	Orbits	Orbit Size
x_1	2×2 Jordan matrix	$q+1$	$(q^2-1)^2(q^2-q)$
x_2	irreducible characteristic polynomial	$\dfrac{q^2-q}{2}$	$(q^2-1)(q^2-q)^2$
x_3	$\begin{array}{c} k \\ \downdownarrows \\ k^2 \end{array} \oplus \begin{array}{c} k \\ \downdownarrows \\ 0 \end{array}$	1	$q(q^2-1)^2$
x_4	$\begin{array}{c} k^2 \\ \downdownarrows \\ k \end{array} \oplus \begin{array}{c} 0 \\ \downdownarrows \\ k \end{array}$	1	$q(q^2-1)^2$
x_5	$\begin{array}{c} k \\ \downdownarrows \\ k \end{array} \cong \begin{array}{c} k \\ \downdownarrows \\ k \end{array}$	$q+1$	$(q^2-1)(q^2-q)$
x_6	$\begin{array}{c} k \\ \downdownarrows \\ k \end{array} \ncong \begin{array}{c} k \\ \downdownarrows \\ k \end{array}$	$\dfrac{q^2+q}{2}$	$(q^2-q)^2(q+1)^2$
x_7	$\begin{array}{c} k \\ \downdownarrows \\ k \end{array} \oplus \begin{array}{c} k \\ \downdownarrows \\ 0 \end{array} \oplus \begin{array}{c} 0 \\ \downdownarrows \\ k \end{array}$	$q+1$	$(q^2-1)(q+1)$
x_8	$\begin{array}{c} k \\ {\scriptstyle 0}\downdownarrows{\scriptstyle 0} \\ k \end{array} \oplus \begin{array}{c} k \\ {\scriptstyle 0}\downdownarrows{\scriptstyle 0} \\ k \end{array}$	1	1

The sum

$$\sum_{i=1}^{8} |\{\text{orbits of type } x_i\}| \cdot |\text{any orbit of type } x_i|$$

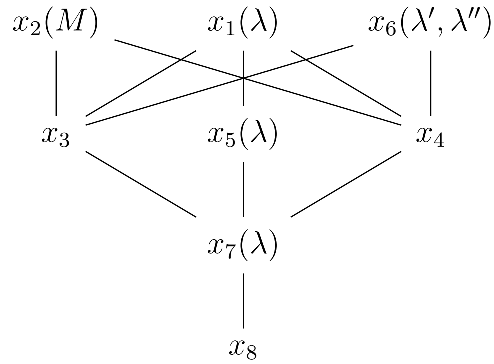
should equal the total number of representations of the quiver Γ with dimension vector $(2,2)$ over k, which is q^8. We can then compute this sum as an assurance that we have not missed any representations.

$$\begin{gathered}
(q+1)(q^2-1)^2(q^2-q) \\
+\frac{q^2-q}{2}(q^2-1)(q^2-q)^2 \\
+1\cdot q(q^2-1)^2 \\
+1\cdot q(q^2-1)^2 \\
+(q+1)(q^2-1)(q^2-q) \\
+\frac{q^2+q}{2}(q^2-q)^2(q+1)^2 \\
+(q+1)(q^2-1)(q+1) \\
+1\cdot 1
\end{gathered}$$

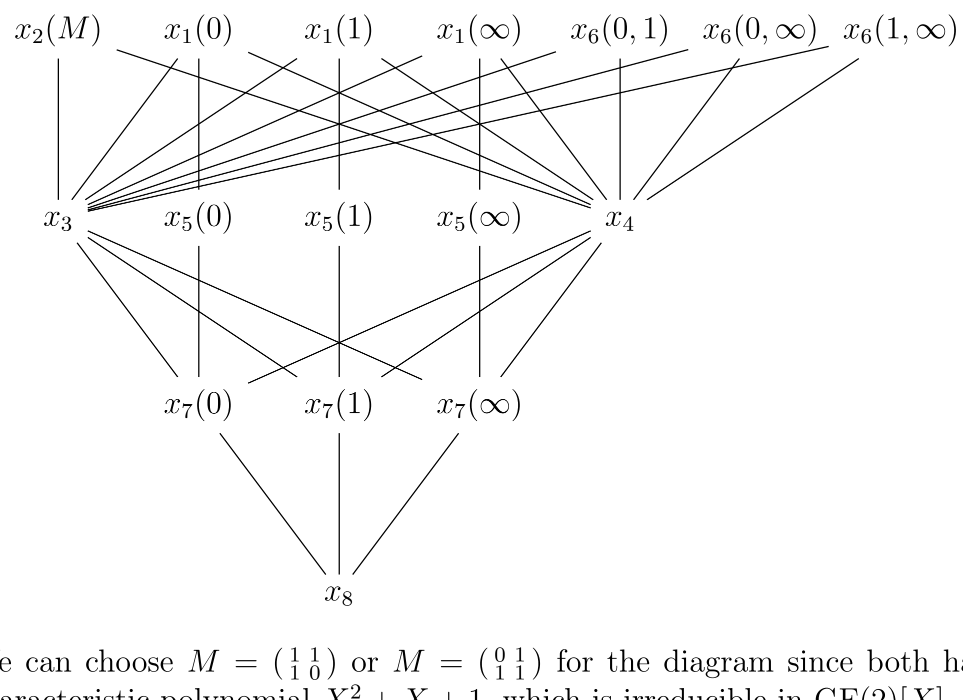
$$\begin{array}{lcccccccc}
= & & q^7 & & -3q^5 & & +3q^3 & -q & \\
& +\frac{1}{2}q^8 & -\frac{3}{2}q^7 & +q^6 & +q^5 & -\frac{3}{2}q^4 & +\frac{1}{2}q^3 & & \\
& & & & +q^5 & & -2q^3 & +q & \\
& & & & +q^5 & & -2q^3 & +q & \\
& & & & +q^5 & & -2q^3 & +q & \\
& +\frac{1}{2}q^8 & +\frac{1}{2}q^7 & -q^6 & -q^5 & +\frac{1}{2}q^4 & +\frac{1}{2}q^3 & & \\
& & & & & +q^4 & +2q^3 & -2q & -1 \\
& & & & & & & & +1
\end{array}$$

$$\begin{aligned}
&= 1\cdot q^8+0\cdot q^7+0\cdot q^6+0\cdot q^5+0\cdot q^4+0\cdot q^3+0\cdot q^2+0\cdot q+0 \\
&= q^8.
\end{aligned}$$

Below is a Hasse diagram which depicts part of, if not the entire degeneration order on $\mathrm{rep}_k^{(2,2)}\,\Gamma$. The edges in the diagram stand for degeneration in such a way that if an edge connects two vertices x and y where x is above y in the diagram, which could look like $\begin{array}{c} x \\ | \\ y \end{array}$, then $x \leq_{\deg} y$.



The degenerations depicted above exist for every $\lambda, \lambda', \lambda'' \in k \cup \{\infty\}$ where $\lambda' \neq \lambda''$ and $M \in M_2(k)$ whose characteristic polynomial is irreducible. If $k = \mathrm{GF}(2)$, then we can make the following degeneration diagram which includes every orbit.



We can choose $M = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ for the diagram since both have characteristic polynomial $X^2 + X + 1$, which is irreducible in $\mathrm{GF}(2)[X]$.

Through the following calculations we show that the edges in the Hasse diagram above actually correspond to degeneration. The first points prove the degenerations $x_i \leq_{\deg} x_j$ between the first two rows in the degeneration diagram by constructing a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & A_1 \arrow[r, "f_1"] \arrow[d, shift right, "a_1"'] \arrow[d, shift left, "a_2"] & B_1 \arrow[r, "g_1"] \arrow[d, shift right, "b_1"'] \arrow[d, shift left, "b_2"] & C_1 \arrow[r] \arrow[d, shift right, "c_1"'] \arrow[d, shift left, "c_2"] & 0 \arrow[d, shift right] \arrow[d, shift left] \\
0 \arrow[r] & A_2 \arrow[r, "f_2"'] & B_2 \arrow[r, "g_2"'] & C_2 \arrow[r] & 0
\end{tikzcd}$$

where $x_i \cong B$ and $x_j \cong A \oplus C$ for $A = \begin{tikzcd} A_1 \arrow[d, shift right, "a_1"'] \arrow[d, shift left, "a_2"] \\ A_2 \end{tikzcd}$, $B = \begin{tikzcd} B_1 \arrow[d, shift right, "b_1"'] \arrow[d, shift left, "b_2"] \\ B_2 \end{tikzcd}$, $C = \begin{tikzcd} C_1 \arrow[d, shift right, "c_1"'] \arrow[d, shift left, "c_2"] \\ C_2 \end{tikzcd}$, $f = (f_1, f_2)$ and $g = (g_1, g_2)$. If the rows

$$0 \longrightarrow A_1 \xrightarrow{f_1} B_1 \xrightarrow{g_1} C_1 \longrightarrow 0$$

and

$$0 \longrightarrow A_2 \xrightarrow{f_2} B_2 \xrightarrow{g_2} C_2 \longrightarrow 0$$

are short exact, then the sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is short exact. Leveraging Theorem 1.6 then gives $B \leq_{\deg} A \oplus C$, or equivalently $x_i \leq_{\deg} x_j$.

- $\underline{x_1(\lambda) \leq_{\deg} x_3 \text{ for all } \lambda \in k \cup \{\infty\}}$:
  If $\lambda \in k$, then

  $$x_1(\lambda) = \begin{tikzcd} k^2 \arrow[d, shift right, "I_2"'] \arrow[d, shift left, "J(\lambda)"] \\ k^2 \end{tikzcd} = \begin{tikzcd} k^2 \arrow[d, shift right, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix} \lambda & 0 \\ 1 & \lambda \end{smallmatrix}\right)"] \\ k^2 \end{tikzcd},$$

  and we have that

  $$x_3 = \begin{tikzcd} k \arrow[d, shift right, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \\ k^2 \end{tikzcd} \oplus \begin{tikzcd} k \arrow[d, shift right] \arrow[d, shift left] \\ 0 \end{tikzcd}.$$

The diagram

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[d, shift right, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] & k^2 \arrow[r, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] \arrow[d, shift right, "\left(\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix}\lambda & 0\\1 & \lambda\end{smallmatrix}\right)"] & k \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[d, shift right] \arrow[d, shift left] \\
0 \arrow[r] & k^2 \arrow[r, "\left(\begin{smallmatrix}1 & \lambda\\0 & 1\end{smallmatrix}\right)"'] & k^2 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd},
$$

commutes and has short exact rows.

For $\lambda = \infty$, we have that

$$
x_1(\infty) = \begin{tikzcd} k^2 \arrow[d, shift right, "J(0)"'] \arrow[d, shift left, "I_2"] \\ k^2 \end{tikzcd} = \begin{tikzcd} k^2 \arrow[d, shift right, "\left(\begin{smallmatrix}0 & 0\\1 & 0\end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right)"] \\ k^2 \end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[d, shift right, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] & k^2 \arrow[r, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] \arrow[d, shift right, "\left(\begin{smallmatrix}0 & 0\\1 & 0\end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right)"] & k \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[d, shift right] \arrow[d, shift left] \\
0 \arrow[r] & k^2 \arrow[r, "\left(\begin{smallmatrix}0 & 1\\1 & 0\end{smallmatrix}\right)"'] & k^2 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
$$

commutes and has short exact rows.

- <u>$x_1(\lambda) \leq_{\deg} x_4$ for all $\lambda \in k \cup \{\infty\}$:</u>

  We have that

$$
x_4 = \begin{tikzcd} k^2 \arrow[d, shift right, "\left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] \\ k \end{tikzcd} \oplus \begin{tikzcd} 0 \arrow[d, shift right] \arrow[d, shift left] \\ k \end{tikzcd} \cong \begin{tikzcd} 0 \arrow[d, shift right] \arrow[d, shift left] \\ k \end{tikzcd} \oplus \begin{tikzcd} k^2 \arrow[d, shift right, "\left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] \\ k \end{tikzcd}
$$

  by definition. For $\lambda \in k$ we have the following commutative diagram with short exact rows.

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k^2 \arrow[r, "\left(\begin{smallmatrix}0 & 1\\1 & \lambda\end{smallmatrix}\right)"] \arrow[d, shift right, "\left(\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix}\lambda & 0\\1 & \lambda\end{smallmatrix}\right)"] & k^2 \arrow[r] \arrow[d, shift right, "\left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] & 0 \arrow[d, shift right] \arrow[d, shift left] \\
0 \arrow[r] & k \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] & k^2 \arrow[r, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"'] & k \arrow[r] & 0
\end{tikzcd}
$$

For $\lambda = \infty$, we have the following commutative diagram with short exact rows.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k^2 \arrow[r, "\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)"] \arrow[d, shift right, "\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)"] & k^2 \arrow[r] \arrow[d, shift right, "(1\ 0)"'] \arrow[d, shift left, "(0\ 1)"] & 0 \arrow[d, shift right] \arrow[d, shift left] \\
0 \arrow[r] & k \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] & k^2 \arrow[r, "(0\ 1)"'] & k \arrow[r] & 0
\end{tikzcd}$$

- <u>$x_1(\lambda) \leq_{\deg} x_5(\lambda)$ for all $\lambda \in k \cup \{\infty\}$:</u>

  For $\lambda \in k$, we have that

$$x_5(\lambda) = \begin{tikzcd} k \arrow[d, shift right, "1"'] \arrow[d, shift left, "\lambda"] \\ k \end{tikzcd} \oplus \begin{tikzcd} k \arrow[d, shift right, "1"'] \arrow[d, shift left, "\lambda"] \\ k \end{tikzcd}.$$

  The diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k \arrow[r, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\lambda"] & k^2 \arrow[r, "(1\ 0)"] \arrow[d, shift right, "\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix}\lambda&0\\1&\lambda\end{smallmatrix}\right)"] & k \arrow[r] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\lambda"] & 0 \arrow[d, shift right] \arrow[d, shift left] \\
0 \arrow[r] & k \arrow[r, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] & k^2 \arrow[r, "(1\ 0)"'] & k \arrow[r] & 0
\end{tikzcd}$$

  commutes and has short exact rows.

  If $\lambda = \infty$, then

$$x_5(\lambda) = x_5(\infty) = \begin{tikzcd} k \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] \\ k \end{tikzcd} \oplus \begin{tikzcd} k \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] \\ k \end{tikzcd}.$$

  Then we have the following commutative diagram with short exact rows.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k \arrow[r, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] & k^2 \arrow[r, "(1\ 0)"] \arrow[d, shift right, "\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)"] & k \arrow[r] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] & 0 \arrow[d, shift right] \arrow[d, shift left] \\
0 \arrow[r] & k \arrow[r, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] & k^2 \arrow[r, "(1\ 0)"'] & k \arrow[r] & 0
\end{tikzcd}$$

- <u>$x_2(M) \leq_{\deg} x_3 \ \forall M \in M_2(k)$, irreducible characteristic polynomial:</u>

  Let $f(X) = X^2 - aX - b$ be irreducible in $k[X]$ for some $a, b \in k$. We have that $f(X) = X(X-a) - b = \det\left(\begin{smallmatrix} x & -b \\ -1 & x-a \end{smallmatrix}\right) = \det\left(X\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 & b \\ 1 & a \end{smallmatrix}\right)\right)$. Then f is the characteristic polynomial of $\left(\begin{smallmatrix} 0 & b \\ 1 & a \end{smallmatrix}\right)$, and we get the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, shift right, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] & k^2 \arrow[r, "(0\ 1)"] \arrow[d, shift right, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix} 0 & b \\ 1 & a \end{smallmatrix}\right)"] & k \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[d, shift right] \arrow[d, shift left] \\
0 \arrow[r] & k^2 \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] & k^2 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}.$$

- <u>$x_2(M) \leq_{\deg} x_4 \ \forall M \in M_2(k)$, irreducible characteristic polynomial:</u>

  The following commutative diagram has exact rows.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k^2 \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & b \end{smallmatrix}\right)"] \arrow[d, shift right, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix} 0 & b \\ 1 & a \end{smallmatrix}\right)"] & k^2 \arrow[r] \arrow[d, shift right, "(1\ 0)"'] \arrow[d, shift left, "(0\ 1)"] & 0 \arrow[d, shift right] \arrow[d, shift left] \\
0 \arrow[r] & k \arrow[r, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"'] & k^2 \arrow[r, "(1\ 0)"'] & k \arrow[r] & 0
\end{tikzcd}$$

- <u>$x_6(\lambda', \lambda'') \leq_{\deg} x_3$ for all $\lambda', \lambda'' \in k \cup \{\infty\}$ such that $\lambda' \neq \lambda''$:</u>

  If $\lambda', \lambda'' \in k$ such that $\lambda' \neq \lambda''$, then

$$x_6(\lambda, \lambda') = \begin{tikzcd} k \arrow[d, shift right, "1"'] \arrow[d, shift left, "\lambda"] \\ k \end{tikzcd} \oplus \begin{tikzcd} k \arrow[d, shift right, "1"'] \arrow[d, shift left, "\lambda'"] \\ k \end{tikzcd} = \begin{tikzcd} k^2 \arrow[d, shift right, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda' \end{smallmatrix}\right)"] \\ k^2 \end{tikzcd}.$$

  This gives us the following commutative diagram with short exact rows.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k \arrow[r, "\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)"] \arrow[d, shift right, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] & k^2 \arrow[r, "(1\ -1)"] \arrow[d, shift right, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix} \lambda' & 0 \\ 0 & \lambda'' \end{smallmatrix}\right)"] & k \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[d, shift right] \arrow[d, shift left] \\
0 \arrow[r] & k^2 \arrow[r, "\left(\begin{smallmatrix} 1 & \lambda' \\ 1 & \lambda'' \end{smallmatrix}\right)"'] & k^2 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

If $\lambda' \in k$ and $\lambda'' = \infty$, then

$$x_6(\lambda', \lambda'') = x_6(\lambda, \infty) = \begin{tikzcd} k \arrow[d, shift right, "1"'] \arrow[d, shift left, "\lambda"] \\ k \end{tikzcd} \oplus \begin{tikzcd} k \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] \\ k \end{tikzcd} = \begin{tikzcd} k^2 \arrow[d, shift right, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \\ k^2 \end{tikzcd}.$$

Then we have the following commutative diagram with short exact rows.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k \arrow[r, "\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)"] \arrow[d, shift right, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] & k^2 \arrow[r, "(1\ -1)"] \arrow[d, shift right, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix} \lambda' & 0 \\ 0 & 1 \end{smallmatrix}\right)"] & k \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[d, shift right] \arrow[d, shift left] \\ 0 \arrow[r] & k^2 \arrow[r, "\left(\begin{smallmatrix} 1 & \lambda' \\ 0 & 1 \end{smallmatrix}\right)"'] & k^2 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

- <u>$x_6(\lambda', \lambda'') \leq_{\deg} x_4$ for all $\lambda', \lambda'' \in k \cup \{\infty\}$ such that $\lambda' \neq \lambda''$:</u>

  For λ', λ'' such that $\lambda' \neq \lambda''$, the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k^2 \arrow[r, "\left(\begin{smallmatrix} 1 & 1 \\ \lambda' & \lambda'' \end{smallmatrix}\right)"] \arrow[d, shift right, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix} \lambda' & 0 \\ 0 & \lambda'' \end{smallmatrix}\right)"] & k^2 \arrow[r] \arrow[d, shift right, "(1\ 0)"'] \arrow[d, shift left, "(0\ 1)"] & 0 \arrow[d, shift right] \arrow[d, shift left] \\ 0 \arrow[r] & k \arrow[r, "\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)"'] & k^2 \arrow[r, "(1\ 1)"'] & k \arrow[r] & 0 \end{tikzcd}$$

  commutes and has short exact rows.

  For $\lambda' \in k$ and $\lambda'' = \infty$, the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k^2 \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ \lambda' & 1 \end{smallmatrix}\right)"] \arrow[d, shift right, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"'] \arrow[d, shift left, "\left(\begin{smallmatrix} \lambda' & 0 \\ 0 & 1 \end{smallmatrix}\right)"] & k^2 \arrow[r] \arrow[d, shift right, "(1\ 0)"'] \arrow[d, shift left, "(0\ 1)"] & 0 \arrow[d, shift right] \arrow[d, shift left] \\ 0 \arrow[r] & k \arrow[r, "\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)"'] & k^2 \arrow[r, "(1\ 1)"'] & k \arrow[r] & 0 \end{tikzcd}$$

  commutes and has short exact rows.

The following points prove the last degenerations $x_i \leq_{\deg} x_j$ in the degeneration diagram by constructing commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & A_1 \arrow[r, "f_1"] \arrow[d, shift right, "a_1"'] \arrow[d, shift left, "a_2"] & B_1 \arrow[r, "g_1"] \arrow[d, shift right, "b_1"'] \arrow[d, shift left, "b_2"] & C_1 \arrow[r] \arrow[d, shift right, "c_1"'] \arrow[d, shift left, "c_2"] & 0 \arrow[d, shift right] \arrow[d, shift left] \\ 0 \arrow[r] & A_2 \arrow[r, "f_2"'] & B_2 \arrow[r, "g_2"'] & C_2 \arrow[r] & 0 \end{tikzcd}$$

with exact rows, where $x_i \cong T \oplus B$ and $x_j \cong T \oplus A \oplus C$ for $A = \begin{array}{c} A_1 \\ {\scriptstyle a_1}\downdownarrows{\scriptstyle a_2} \\ A_2 \end{array}$,

$B = \begin{array}{c} B_1 \\ {\scriptstyle b_1}\downdownarrows{\scriptstyle b_2} \\ B_2 \end{array}$, $C = \begin{array}{c} C_1 \\ {\scriptstyle c_1}\downdownarrows{\scriptstyle c_2} \\ C_2 \end{array}$, some representation T of Γ, $f = (f_1, f_2)$ and $g = (g_1, g_2)$. The sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is then short exact, so by Corollary 1.6.1 we have that $T \oplus B \leq_{\deg} T \oplus A \oplus C$, which is equivalent to $x_i \leq_{\deg} x_j$.

- <u>$x_5(\lambda) \leq_{\deg} x_7(\lambda)$ for all $\lambda \in k \cup \{\infty\}$:</u>

  If $\lambda \in k$, then

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & 0 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & k \arrow[r, "1"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\lambda"] & k \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & 0 \arrow[d, shift left] \arrow[d, shift right] \\
0 \arrow[r] & k \arrow[r, "1"'] & k \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

  is commutative and has exact rows. We get

$$x_5(\lambda) = \begin{array}{c} k \\ {\scriptstyle 1}\downdownarrows{\scriptstyle \lambda} \\ k \end{array} \oplus \begin{array}{c} k \\ {\scriptstyle 1}\downdownarrows{\scriptstyle \lambda} \\ k \end{array} \leq_{\deg} \begin{array}{c} k \\ {\scriptstyle 1}\downdownarrows{\scriptstyle \lambda} \\ k \end{array} \oplus \begin{array}{c} 0 \\ \downdownarrows \\ k \end{array} \oplus \begin{array}{c} k \\ \downdownarrows \\ 0 \end{array} \cong x_7(\lambda).$$

  Then $x_5(\lambda) \leq_{\deg} x_7(\lambda)$ for all $\lambda \in k$.

  Let $\lambda = \infty$. The following commutative diagram has exact rows.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & 0 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & k \arrow[r, "1"] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] & k \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & 0 \arrow[d, shift left] \arrow[d, shift right] \\
0 \arrow[r] & k \arrow[r, "1"'] & k \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

  We then have that

$$x_5(\infty) = \begin{array}{c} k \\ {\scriptstyle 0}\downdownarrows{\scriptstyle 1} \\ k \end{array} \oplus \begin{array}{c} k \\ {\scriptstyle 0}\downdownarrows{\scriptstyle 1} \\ k \end{array} \leq_{\deg} \begin{array}{c} k \\ {\scriptstyle 0}\downdownarrows{\scriptstyle 1} \\ k \end{array} \oplus \begin{array}{c} 0 \\ \downdownarrows \\ k \end{array} \oplus \begin{array}{c} k \\ \downdownarrows \\ 0 \end{array} \cong x_7(\infty).$$

  Then $x_5(\infty) \leq_{\deg} x_7(\infty)$.

- $x_3 \leq_{\deg} x_7(\lambda)$ for all $\lambda \in k \cup \{\infty\}$:
  For $\lambda \in k$ we have that

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k \arrow[r, "1"] \arrow[d, shift right, swap, "\binom{1}{0}"] \arrow[d, shift left, "\binom{0}{1}"] & k \arrow[r] \arrow[d, shift right, swap, "1"] \arrow[d, shift left, "\lambda"] & 0 \arrow[d, shift right] \arrow[d, shift left] \\
0 \arrow[r] & k \arrow[r, swap, "\binom{\lambda}{-1}"] & k^2 \arrow[r, swap, "(1\ \lambda)"] & k \arrow[r] & 0
\end{tikzcd}$$

is commutative and has short exact rows. Then

$$\begin{tikzcd}[column sep=small]
 & k \arrow[dd, shift right] \arrow[dd, shift left] & & k \arrow[dd, shift right, swap, "\binom{1}{0}"] \arrow[dd, shift left, "\binom{0}{1}"] & & k \arrow[dd, shift right] \arrow[dd, shift left] & & 0 \arrow[dd, shift right] \arrow[dd, shift left] & & k \arrow[dd, shift right, swap, "1"] \arrow[dd, shift left, "\lambda"] & \\
x_3 \cong & & \oplus & & \leq_{\deg} & & \oplus & & \oplus & & \cong x_7(\lambda), \\
 & 0 & & k^2 & & 0 & & k & & k &
\end{tikzcd}$$

so $x_3 \leq_{\deg} x_7(\lambda)$ for $\lambda \in k$.
Let $\lambda = \infty$. The diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[r, "\binom{0}{1}"] \arrow[d, shift right] \arrow[d, shift left] & k \arrow[r, "1"] \arrow[d, shift right, swap, "\binom{1}{0}"] \arrow[d, shift left, "\binom{0}{1}"] & k \arrow[r] \arrow[d, shift right, swap, "0"] \arrow[d, shift left, "1"] & 0 \arrow[d, shift right] \arrow[d, shift left] \\
0 \arrow[r] & k \arrow[r, swap, "\binom{1}{0}"] & k^2 \arrow[r, swap, "(0\ 1)"] & k \arrow[r] & 0
\end{tikzcd}$$

is commutative and has exact rows. Then

$$\begin{tikzcd}[column sep=small]
 & k \arrow[dd, shift right] \arrow[dd, shift left] & & k \arrow[dd, shift right, swap, "\binom{1}{0}"] \arrow[dd, shift left, "\binom{0}{1}"] & & k \arrow[dd, shift right] \arrow[dd, shift left] & & 0 \arrow[dd, shift right] \arrow[dd, shift left] & & k \arrow[dd, shift right, swap, "0"] \arrow[dd, shift left, "1"] & \\
x_3 \cong & & \oplus & & \leq_{\deg} & & \oplus & & \oplus & & \cong x_7(\infty), \\
 & 0 & & k^2 & & 0 & & k & & k &
\end{tikzcd}$$

so $x_3(\infty) \leq_{\deg} x_7(\infty)$.

- $x_4 \leq_{\deg} x_7(\lambda)$ for all $\lambda \in k \cup \{\infty\}$:
  Let $\lambda \in k$. Then

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k \arrow[r, "\binom{1}{\lambda}"] \arrow[d, shift right, swap, "1"] \arrow[d, shift left, "\lambda"] & k^2 \arrow[r, "(\lambda\ -1)"] \arrow[d, shift right, swap, "(1\ 0)"] \arrow[d, shift left, "(0\ 1)"] & k \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[d, shift right] \arrow[d, shift left] \\
0 \arrow[r] & k \arrow[r, swap, "1"] & k \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

is commutative and has short exact rows. We obtain that

$$x_4 \cong \begin{tikzcd} 0 \arrow[d, shift right] \arrow[d, shift left] \\ k \end{tikzcd} \oplus \begin{tikzcd} k^2 \arrow[d, shift right, "(1\ 0)"'] \arrow[d, shift left, "(0\ 1)"] \\ k \end{tikzcd} \leq_{\deg} \begin{tikzcd} 0 \arrow[d, shift right] \arrow[d, shift left] \\ k \end{tikzcd} \oplus \begin{tikzcd} k \arrow[d, shift right, "1"'] \arrow[d, shift left, "\lambda"] \\ k \end{tikzcd} \oplus \begin{tikzcd} k \arrow[d, shift right] \arrow[d, shift left] \\ 0 \end{tikzcd} \cong x_7(\lambda).$$

Then $x_4 \leq_{\deg} x_7(\lambda)$ for all $\lambda \in k$.

Let $\lambda = \infty$. The diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k \arrow[r, "\binom{0}{1}"] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] & k^2 \arrow[r, "(1\ 0)"] \arrow[d, shift right, "(1\ 0)"'] \arrow[d, shift left, "(0\ 1)"] & k \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[d, shift right] \arrow[d, shift left] \\ 0 \arrow[r] & k \arrow[r, "1"'] & k \arrow[r] & k \arrow[r] & 0 \end{tikzcd}$$

is commutative and has exact rows. Then

$$x_4 \cong \begin{tikzcd} 0 \arrow[d, shift right] \arrow[d, shift left] \\ k \end{tikzcd} \oplus \begin{tikzcd} k^2 \arrow[d, shift right, "(1\ 0)"'] \arrow[d, shift left, "(0\ 1)"] \\ k \end{tikzcd} \leq_{\deg} \begin{tikzcd} 0 \arrow[d, shift right] \arrow[d, shift left] \\ k \end{tikzcd} \oplus \begin{tikzcd} k \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] \\ k \end{tikzcd} \oplus \begin{tikzcd} k \arrow[d, shift right] \arrow[d, shift left] \\ 0 \end{tikzcd} \cong x_7(\infty).$$

Thus $x_4 \leq_{\deg} x_7(\infty)$.

- $\underline{x_7(\lambda) \leq_{\deg} x_8 \text{ for all } \lambda \in k \cup \{\infty\}:}$
  If $\lambda \in k$, then

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & k \arrow[r, "1"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\lambda"] & k \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \arrow[d, shift right] \arrow[d, shift left] \\ 0 \arrow[r] & k \arrow[r, "1"'] & k \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

is commutative and has exact rows. We have that

$$x_7(\lambda) \cong \begin{tikzcd} 0 \arrow[d, shift right] \arrow[d, shift left] \\ k \end{tikzcd} \oplus \begin{tikzcd} k \arrow[d, shift right] \arrow[d, shift left] \\ 0 \end{tikzcd} \oplus \begin{tikzcd} k \arrow[d, shift right, "1"'] \arrow[d, shift left, "\lambda"] \\ k \end{tikzcd} \leq_{\deg} \begin{tikzcd} 0 \arrow[d, shift right] \arrow[d, shift left] \\ k \end{tikzcd} \oplus \begin{tikzcd} k \arrow[d, shift right] \arrow[d, shift left] \\ 0 \end{tikzcd} \oplus \begin{tikzcd} 0 \arrow[d, shift right] \arrow[d, shift left] \\ k \end{tikzcd} \oplus \begin{tikzcd} k \arrow[d, shift right] \arrow[d, shift left] \\ 0 \end{tikzcd} \cong x_8.$$

Then $x_7(\lambda) \leq_{\deg} x_8$ for all $\lambda \in k$.

Let $\lambda = \infty$. Then

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & 0 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & k \arrow[r, "1"] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] & k \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & 0 \arrow[d, shift left] \arrow[d, shift right] \\
0 \arrow[r] & k \arrow[r, "1"'] & k \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

is commutative and has exact rows. We get that

$$x_7(\infty) \cong \begin{array}{c}0\\ \downdownarrows\\ k\end{array} \oplus \begin{array}{c}k\\ \downdownarrows\\ 0\end{array} \oplus \begin{array}{c}k\\ {}_0\!\downdownarrows_1\\ k\end{array} \leq_{\text{deg}} \begin{array}{c}0\\ \downdownarrows\\ k\end{array} \oplus \begin{array}{c}k\\ \downdownarrows\\ 0\end{array} \oplus \begin{array}{c}0\\ \downdownarrows\\ k\end{array} \oplus \begin{array}{c}k\\ \downdownarrows\\ 0\end{array} \cong x_8.$$

Then $x_7(\infty) \leq_{\text{deg}} x_8$.

Δ

1.6 Additional Orders

There are two more orders on modules that we wish to discuss, namely virtual degeneration and the hom order. We use these orders in particular to show that degeneration is a partial orderand we get a chain of implications from the ext order to the hom order. In this section we assume Λ is an algebra over a commutative ring R.

Definition 1.9. Let M and N be Λ-modules that have finite length as R-modules. We say that M **virtually degenerates to** N, or $M \leq_{vdeg} N$, if there exists a Λ-module B that has finite length as an R-module such that $M \oplus B \leq_{\text{deg}} N \oplus B$. Δ

Theorem 1.7. Let M, N be Λ-modules that have finite length as R-modules. If $M \leq_{\text{deg}} N$, then $M \leq_{\text{vdeg}} N$.

Proof. Suppose $M \leq_{\text{deg}} N$. Corollary 1.6.1 implies $M \oplus A \leq_{\text{deg}} N \oplus A$ for any Λ-module A that has finite length as an R-module. In particular there exists such a module, so $M \leq_{\text{vdeg}} N$. $\square$

Let A be a Λ-module that has finite length as a R-module. We let $l(A)$ denote the length of A as an R-module.

Definition 1.10. The **hom order** on Λ-modules that has finite length as R-modules is denoted by $\leq_{\text{hom}}$ and is defined as the following. $M \leq_{\text{hom}} N$ if $l(\text{Hom}_\Lambda(X, M)) \leq l(\text{Hom}_\Lambda(X, N))$ for every Λ-module X that has finite length as an R-module. Δ

Theorem 1.8. Let M, N be Λ-modules that have finite length as R-modules. If $M \leq_{\text{vdeg}} N$, then $M \leq_{\text{hom}} N$.

Proof. Suppose $M \leq_{\text{vdeg}} N$. The there exist Λ-modules A, B that have finite length as R-modules such that the sequence

$$0 \longrightarrow A \longrightarrow A \oplus B \oplus M \longrightarrow B \oplus N \longrightarrow 0$$

is exact. Let X be a Λ-module that has finite length as an R-module. Applying the left exact covariant functor $\text{Hom}_\Lambda(X, \ )$ gives an exact sequence

$$0 \longrightarrow \text{Hom}_\Lambda(X, A) \longrightarrow \text{Hom}_\Lambda(X, A \oplus B \oplus M) \longrightarrow \text{Hom}_\Lambda(X, B \oplus N)\ .$$

If we denote $\text{Hom}_\Lambda(X, \ )$ by $[X, \ ]$, then we get the following inequality.

$$\begin{aligned} l[X, A \oplus B \oplus M] \leq l[X, A] + l[X, B \oplus N] \\ \Rightarrow l([X, A \oplus B] \oplus [X, M]) \leq l([X, A] \oplus [X, B] \oplus [X, N]) \\ \Rightarrow l[X, A \oplus B] + l[X, M] \leq l[X, A \oplus B] + l[X, N] \\ \Rightarrow l[X, M] \leq l[X, N] \\ \Leftrightarrow l(\text{Hom}_\Lambda(X, M)) \leq l(\text{Hom}_\Lambda(X, N)). \end{aligned}$$

□

Now we can show antisymmetry of the degeneration order. Suppose M, N are Λ-modules that have finite length as R-modules such that $M \leq_{\text{deg}} N$ and $N \leq_{\text{deg}} M$. Then $M \leq_{\text{hom}} N$ and $N \leq_{\text{hom}} M$, so in particular $l(\text{Hom}_\Lambda(M, M)) = l(\text{Hom}_\Lambda(M, N))$. We also have a Λ-module X that has finite length as an Λ-module with finite length as an R-module such that the following sequence is short exact.

$$0 \longrightarrow X \longrightarrow X \oplus N \longrightarrow M \longrightarrow 0$$

This sequence splits due to the equality $l(\text{Hom}_\Lambda(M, M)) = l(\text{Hom}_\Lambda(M, N))$, which means $X \oplus N \cong X \oplus M$. The Krull-Schmidt theorem then gives that

$M \cong N$. Since we already showed reflexivity and transitivity, then we can conclude that degeneration is a partial order on Λ-modules that have finite length as R-modules.

The last theorem also gives rise to the following sequence of implications for any Λ-modules M, N that have finite length as R-modules.

$$M \leq_{\text{ext}} N \Rightarrow M \leq_{\text{deg}} N \Rightarrow M \leq_{\text{vdeg}} N \Rightarrow M \leq_{\text{hom}} N$$

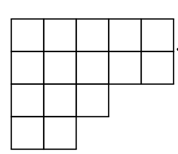
2 Partitions of Natural Numbers

Definition 2.1. Let $n \in \mathbb{N}$. A **partition** of n is a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ such that $\sum_{i=1}^{n} \alpha_i = n$ and $\alpha_i \geq \alpha_{i+1}$ $\forall i \in \mathbb{N}_{n-1}$. We call α_i the i-th part of α for each $i \in \mathbb{N}_n$, and we say that the number of non-zero parts of α is the number of parts of α.

- If $\alpha_i > \alpha_{i+1}$ $\forall i \in \mathbb{N}_{n-1}$, then we say that α is a strict partition. Another name for such an α is a partition with distinct parts.
- The length of α is defined as $l(\alpha) = |i \in \mathbb{N}_n | \alpha_i > 0|$, which is the number of nonzero parts of α.
- The set of all partitions of n is denoted $\mathcal{P}_n$.
- The set of all strict partitions of n is denoted $\widehat{\mathcal{P}_n}$.

$\triangle$

One way to depict partitions of natural numbers is by drawing Young diagrams. The Young diagram of $\alpha \in \mathcal{P}_n$ consists of n squares arranged in $l(\alpha)$ rows where row i is comprised by α_i squares for each i in $\mathbb{N}_d$. For example, the Young diagram of $(5, 5, 3, 2, 0, \ldots, 0)$ is

.

2.1 Counting Partitions with Power Series

In [8], a couple of formulas are given, such as $\prod_{t=1}^{\infty}(1+x^t) = \sum_{n=0}^{\infty}|\hat{\mathcal{P}}_n|x^n$ or $\prod_{t=1}^{\infty}\sum_{i=0}^{\infty}x^{ti} = \sum_{n=0}^{\infty}|\mathcal{P}_n|x^n$, which relate the number of certain types of partitions to infinite products. The first part of this section attempts to explain this relationship and expand on it.

Example 2.1.

- Consider the infinite product

$$\prod_{t=1}^{\infty}(1+x^t).$$

If we write the product vertically as

$$\begin{gathered} \vdots \\ \cdot\left(1+x^3\right) \\ \cdot\left(1+x^2\right) \\ \cdot\,(1+x) \end{gathered}$$

and change the numbers in the exponents for boxes, resulting in the expression

$$\begin{gathered} \vdots \\ \cdot\,(1+x^{\ydiagram{3}}) \\ \cdot\,(1+x^{\ydiagram{2}}) \\ \cdot\,(1+x^{\ydiagram{1}})\,, \end{gathered}$$

then multiplying out the expression gives a sum where each term corresponds to a partition. For instance, a term corresponding to the partition $\ydiagram{5,3,2,1}$ can be obtained by multiplying the boxed entries in the expression as shown below.

$$
\begin{gathered}
\vdots \\
\cdot \left(\boxed{1} + x^{\ydiagram{6}} \right) \\
\cdot \left(1 + \boxed{x^{\ydiagram{5}}} \right) \\
\cdot \left(\boxed{1} + x^{\ydiagram{4}} \right) \\
\cdot \left(1 + \boxed{x^{\ydiagram{3}}} \right) \\
\cdot \left(1 + \boxed{x^{\ydiagram{2}}} \right) \\
\cdot \left(1 + \boxed{x^{\ydiagram{1}}} \right) \\
\rightsquigarrow x^{\ydiagram{5,3,2,1}}
\end{gathered}
$$

Not all partitions can be obtained this way. One example is the partition $\ydiagram{3,3,1}$, as the part $\ydiagram{3}$ occurs twice in $\ydiagram{3,3,1}$, but its corresponding entry $x^{\ydiagram{3}}$ only shows up once in the product $\prod_{t=1}^{\infty}(1+x^t)$. In fact, no partition with parts that occur more than once can be obtained the way we obtained $\ydiagram{5,3,2,1}$. Then, since every part is represented exactly once in the expression, it might be reasonable to believe that all the strict partitions and only those can be obtained from multiplying out $\prod_{t=1}^{\infty}(1+x^t)$.

n	$\alpha \in \widehat{\mathcal{P}_n}$	$\lvert\widehat{\mathcal{P}_n}\rvert$
0	(the trivial partition)	1
1	$\ydiagram{1}$	1
2	$\ydiagram{2}$	1
3	$\ydiagram{3}$ $\ydiagram{2,1}$	2
4	$\ydiagram{4}$ $\ydiagram{3,1}$	2
5	$\ydiagram{5}$ $\ydiagram{4,1}$ $\ydiagram{3,2}$	3
6	$\ydiagram{6}$ $\ydiagram{5,1}$ $\ydiagram{4,2}$ $\ydiagram{3,2,1}$	4

$$
\begin{aligned}
&\prod_{t=1}^{\infty}(1+x^t)\\
&= 1\\
&+ x\\
&+ x^2\\
&+ x^3 + x^2x\\
&+ x^4 + x^3x\\
&+ x^5 + x^4x + x^3x^2\\
&+ x^6 + x^5x + x^4x^2 + x^3x^2x\\
&+ \cdots\\
&= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + \cdots .
\end{aligned}
$$

We see from this that all the strict partitions of n can be obtained from $\prod_{t=1}^{\infty}(1+x^t)$ for the first few $n \in \mathbb{N}$.

- Let us look at another infinite product,

$$\prod_{t=1}^{\infty}\sum_{i=0}^{\infty} x^{ti} = \prod_{t=1}^{\infty}\frac{1}{1-x^t},$$

write vertically and switch the numbers in the exponents for boxes like we did in the previous example. This looks like

$$
\begin{gathered}
\vdots\\
(1 + x^3 + x^6 + x^9 + \cdots)\\
(1 + x^2 + x^4 + x^6 + \cdots)\\
(1 + x + x^2 + x^3 + \cdots),
\end{gathered}
$$

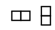
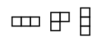
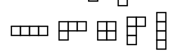
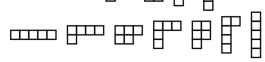
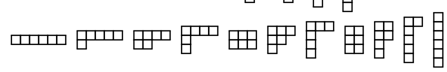
and then

$$
\begin{gathered}
\vdots \\
\left(1 + x^{\ydiagram{3}} + x^{\ydiagram{3,3}} + x^{\ydiagram{3,3,3}} + \cdots\right) \\
\left(1 + x^{\ydiagram{2}} + x^{\ydiagram{2,2}} + x^{\ydiagram{2,2,2}} + \cdots\right) \\
\left(1 + x^{\ydiagram{1}} + x^{\ydiagram{1,1}} + x^{\ydiagram{1,1,1}} + \cdots\right).
\end{gathered}
$$

If we multiply out this product, we get a sum whose terms correspond to partitions like in the previous example, but we can obtain many more partitions from $\prod_{t=1}^{\infty}\sum_{i=0}^{\infty} x^{ti}$ than $\prod_{t=1}^{\infty}(1+x^t)$. Firstly, the same strict partitions represented in $\prod_{t=1}^{\infty}(1+x^t)$ are also represented in $\prod_{t=1}^{\infty}\sum_{i=0}^{\infty} x^{ti}$ since 1 and x^t are terms in $\sum_{i=0}^{\infty} x^{ti}$ for all $t \in \mathbb{N}$. Secondly, if we for instance multiply the boxed entries in

$$
\begin{gathered}
\vdots \\
\left(\boxed{1} + x^{\ydiagram{4}} + x^{\ydiagram{4,4}} + x^{\ydiagram{4,4,4}} + \cdots\right) \\
\left(1 + x^{\ydiagram{3}} + \boxed{x^{\ydiagram{3,3}}} + x^{\ydiagram{3,3,3}} + \cdots\right) \\
\left(\boxed{1} + x^{\ydiagram{2}} + x^{\ydiagram{2,2}} + x^{\ydiagram{2,2,2}} + \cdots\right) \\
\left(1 + \boxed{x^{\ydiagram{1}}} + x^{\ydiagram{1,1}} + x^{\ydiagram{1,1,1}} + \cdots\right) \\
\rightsquigarrow x^{\ydiagram{3,3,1}},
\end{gathered}
$$

the partition $\ydiagram{3,3,1}$ is obtained and this partition could not be obtained from $\prod_{t=1}^{\infty}(1+x^t)$. Then $\prod_{t=1}^{\infty}\sum_{i=0}^{\infty} x^{ti}$ yields all the strict partitions which could be obtained from $\prod_{t=1}^{\infty}(1+x^t)$. Actually, any part which is repeated any number of times is represented in $\prod_{t=1}^{\infty}\sum_{i=0}^{\infty} x^{ti}$ and that might mean that any partition can be obtained from $\prod_{t=1}^{\infty}\sum_{i=0}^{\infty} x^{ti}$. If that is the case, then writing $\prod_{t=1}^{\infty}\sum_{i=0}^{\infty} x^{ti} = \sum_{n=0}^{\infty} a_n x^n$ means $a_n = |\mathcal{P}_n|$, the total number of partitions of n, for all $n \in \mathbb{N}_0$. We can easily test this for a few $n \in \mathbb{N}_0$.

n	$\alpha \in \mathcal{P}_n$	$\lvert\mathcal{P}_n\rvert$
0	(the trivial partition)	1
1		1
2		2
3		3
4		5
5		7
6		11

Finding the first coefficients of $\sum_{n=0}^{\infty} a_n x^n$ yields

$$\prod_{t=1}^{\infty}\sum_{i=0}^{\infty} x^{ti} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 \cdots ,$$

which coincides with the number of partitions of every $n \in \{0\} \cup \mathbb{N}_6$.

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For any subset $T \subseteq \mathbb{N}$ and sequence $s = (s_0, s_1, s_2, \dots)$ where $s_0 = 1$ and $s_1, s_2, \dots \in \mathbb{Z}$, we denote $\mathbb{N}_0^{|T|} = \mathbb{N}_0 \times \overset{|T|}{\cdots} \times \mathbb{N}_0$, so we have that

$$\prod_{t\in T}\sum_{i=0}^{\infty} s_i x^{it}$$

$$= (s_0+s_1x^{t_1}+s_2x^{2t_1}+\cdots)(s_0+s_1x^{t_2}+s_2x^{2t_2}+\cdots)(s_0+s_1x^{t_3}+s_2x^{2t_3}+\cdots)\cdots$$

$$= \sum_{(i_t)_{t\in T}\in\mathbb{N}_0^{|T|}} \left(s_{i_{t_1}} x^{i_{t_1}t_1} \cdot s_{i_{t_2}} x^{i_{t_2}t_2} \cdot s_{i_{t_3}} x^{i_{t_3}t_3} \cdots\right)$$

$$= \sum_{(i_t)_{t\in T}\in\mathbb{N}_0^{|T|}} \prod_{t\in T} s_{i_t} x^{i_t t} = \sum_{n=0}^{\infty} \left(\sum_{\substack{(i_t)_{t\in T}\in\mathbb{N}_0^{|T|} \\ \sum_{t\in T} i_t t = n}} \prod_{t\in T} s_{i_t} \right) x^n.$$

Setting $r_{s,T}(n) := \sum_{\substack{(i_t)_{t\in T}\in\mathbb{N}_0^{|T|} \\ \sum_{t\in T} i_t t = n}} \prod_{t\in T} s_{i_t}$, gives the equality

$$\prod_{t\in T}\sum_{i=0}^{\infty} s_i x^{it} = \sum_{n=0}^{\infty} r_{s,T}(n) x^n.$$

We can show that there is a bijection between the set of sequences $(i_t)_{t\in T}$ which are such that $\sum_{t\in T} i_t t = n$, where $i_t \in \mathbb{N}_0$ for all $t \in T$, and the set $\mathcal{P}_n$ of all partitions of $n \in \mathbb{N}$ whose parts are in T. To help us see this, we choose an indexing on the set T, that is we say $T = \{t_1, t_2, \dots\}$ where $t_{j+1} > t_j$ for every applicable j. Given a sequence $(i_t)_{t\in T}$ such that $\sum_{t\in T} i_t t = n$, we let $J = \max\left\{j \in \{1, 2, \dots, |T|\} \;\middle|\; i_{t_j} > 0\right\}$, which is the largest index in $(i_t)_{t\in T}$ such that its corresponding number i_J in the sequence is nonzero. Note that T might not be finite, so we could have $|T| = \infty$, but since $\sum_{t\in T} i_t t = n$, then J is always finite as long as n is finite. We construct a partition

$$\left(t_J, \overset{i_{t_J}}{\cdots}, t_J, t_{J-1}, \overset{i_{t_{J-1}}}{\cdots}, t_{J-1}, \dots, t_1, \overset{i_{t_1}}{\cdots}, t_1, 0, \dots, 0\right) \in \mathcal{P}_n$$

which corresponds to $(i_t)_{t\in T}$. This suggests that there is a function ς from the set of sequences $(i_t)_{t\in T} \in \mathbb{N}_0^{|T|}$ such that $\sum_{t\in T} t i_t = n$ to the set of partitions of n whose parts are elements in T, and ς is such that

$$\varsigma\left((i_t)_{t\in T}\right) = \left(t_J, \overset{i_{t_J}}{\cdots}, t_J, t_{J-1}, \overset{i_{t_{J-1}}}{\cdots}, t_{J-1}, \dots, t_1, \overset{i_{t_1}}{\cdots}, t_1, 0, \dots, 0\right).$$

We get the following facts about ς.

1. Any sequence $(i_t)_{t\in T} \in \mathbb{N}_0^{|T|}$ such that $\sum_{t\in T} t i_t = n$ admits a unique partition of n under ς. This is because if two sequences $(a_t)_{t\in T}, (b_t)_{t\in T} \in \mathbb{N}_0^{|T|}$ are such that $\varsigma\left((a_t)_{t\in T}\right) = \varsigma\left((b_t)_{t\in T}\right)$, that is
$$\left(t_{J_1}, \overset{a_{t_{J_1}}}{\cdots}, t_{J_1}, t_{J_1-1}, \overset{a_{t_{J_1-1}}}{\cdots}, t_{J_1-1}, \dots, t_1, \overset{a_{t_1}}{\cdots}, t_1, 0, \dots, 0\right)$$
$$= \left(t_{J_2}, \overset{b_{t_{J_2}}}{\cdots}, t_{J_2}, t_{J_2-1}, \overset{b_{t_{J_2-1}}}{\cdots}, t_{J_2-1}, \dots, t_1, \overset{b_{t_1}}{\cdots}, t_1, 0, \dots, 0\right)$$
where we have that $J_1 = \max\left\{j \in \{1, 2, \dots, |T|\} \;\middle|\; a_{t_j} > 0\right\}$ and $J_2 = \max\left\{j \in \{1, 2, \dots, |T|\} \;\middle|\; b_{t_j} > 0\right\}$, then $J_1 = J_2$ and $a_{t_j} = b_{t_j}$ $\forall j \in \mathbb{N}_{J_1}$ and we obtain that $(a_t)_{t\in T} = (b_t)_{t\in T}$ since $a_{t_j} = b_{t_j} = 0$ for $j \in \{J_1 + 1, J_1 + 2, \dots, |T|\}$.

2. Every partition of n whose parts are elements in T is the image of a sequence in $\mathbb{N}_0^{|T|}$ under ς, since if $\alpha = (\alpha_1, \dots, \alpha_n)$ is such a partition, then
$$\alpha = \left(t_J, \overset{i_{t_J}}{\cdots}, t_J, t_{J-1}, \overset{i_{t_{J-1}}}{\cdots}, t_{J-1}, \dots, t_1, \overset{i_{t_1}}{\cdots}, t_1, 0, \dots, 0\right)$$

where $i_{t_j} = |\{i \in \mathbb{N}_n \mid \alpha_i = t_j\}| \; \forall j \in \{1, 2, \ldots, |T|\}$ and $J \in \mathbb{N}_{|T|}$ such that $t_J = \alpha_1$. By setting $i_{t_j} = 0 \; \forall j \in \{J+1, J+2, \ldots, |T|\}$, we get that $\sum_{t\in T} t i_t = \sum_{j=1}^{J} t_j i_{t_j} = \sum_{i=1}^{n} \alpha_i = n$, and the partition α is then admitted by the sequence $(i_t)_{t\in T} \in \mathbb{N}_0^{|T|}$.

Thus ς is a bijection.

Again let s be a sequence $(s_0, s_1, \ldots)$ with $s_0 = 1$ and $s_1, s_2, \cdots \in \mathbb{Z}$, and write $\pi_{\alpha,s} = \prod_{j=1}^{J} s_{i_{t_j}}$ for any

$$\alpha = \left(t_J, \overset{i_{t_J}}{\ldots}, t_J, t_{J-1}, \overset{i_{t_{J-1}}}{\ldots}, t_{J-1}, \ldots, t_1, \overset{i_{t_1}}{\ldots}, t_1, 0, \ldots, 0\right) \in \mathcal{P}_{n,T},$$

where $\mathcal{P}_{n,T}$ is the set of all partitions of n whose parts are elements in T. This means that every part t_j of α is assigned the element $s_{i_{t_j}}$ in s, where i_{t_j} is the number of times t_j occurs in α. The $s_{i_{t_j}}$ are then multiplied together. For instance, if $s = (s_i)_{i\in\mathbb{N}_0}$, then $\pi_{\alpha,s} = 1$ for all $\alpha \in \mathcal{P}_{n,T}$, so $r_{s,T}(n) = |\mathcal{P}_{n,T}|$. From this definition of $\pi_{\alpha,s}$ we can see that

$$r_{s,T}(n) = \sum_{\alpha\in\mathcal{P}_{n,T}} \pi_{\alpha,s},$$

so

$$\prod_{t\in T} \sum_{i=0}^{\infty} s_i x^{it} = \sum_{n=0}^{\infty} r_{s,T}(n) = \sum_{n=0}^{\infty} \sum_{\alpha\in\mathcal{P}_{n,T}} \pi_{\alpha,s}.$$

Thus we have a connection between infinite products and sums over partitions. We illustrate what we just achieved with some examples.

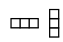
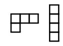
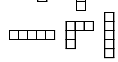
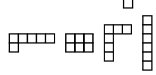
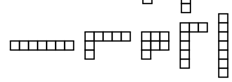
Example 2.2.

- Suppose we want to count the number of partitions with odd parts where every part occurs exactly an odd number of times. Then we can let T be the set of odd numbers, and $s = (1, 1, 0, 1, 0, 1, 0, \ldots)$, that is $s_i = 1$ if $i = 0$ or i is odd and $s_i = 0$ otherwise. $r_{s,T}(n)$ sums over partitions with parts from T, which in this case are partitions with only odd parts. For any such partition

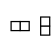
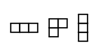
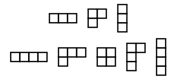
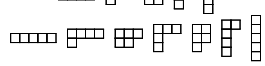
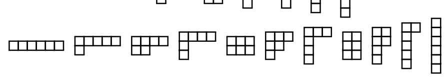
  $$\alpha = \left(t_J, \overset{i_{t_J}}{\ldots}, t_J, t_{J-1}, \overset{i_{t_{J-1}}}{\ldots}, t_{J-1}, \ldots, t_1, \overset{i_{t_1}}{\ldots}, t_1, 0, \ldots, 0\right) \in \mathcal{P}_{n,T},$$

  its corresponding summand $\pi_{\alpha,s} \left(= \prod_{j=1}^{J} s_{i_{t_j}}\right) = 0$ if at least one i_{t_j} is an even number, that is the part t_j occurs an even number of times in

α, and $\pi_{\alpha,s} = 1$ if i_{t_j} is odd for all $j \in \mathbb{N}_J$, which is equivalent to every part t_j occuring an odd number of times in α. Thus α increases $r_{s,T}(n)$ by 1 if and only if the parts in α occur an odd number of times, and the α's permitted are those with only odd parts, so $r_{s,T}(n)$ does really equal the number of partitions with only odd parts whose parts have an odd number of occurences.

n	$\alpha \in \mathcal{P}_{n,T}$	$\pi_{\alpha,s}$	$r_{s,T}(n)$
0	(the trivial partition)	1	1
1		1	1
2		0	0
3		$1, 1$	2
4		$1, 0$	1
5		$1, 0, 1$	2
6		$1, 0, 1, 0$	2
7		$1, 0, 0, 0, 1$	2

- Let $T = \mathbb{N}$ and $s = (1, 1, 2, 0, 0, 0, \dots)$. Then $r_{s,t}(n)$ sums over all partitions and the parts in any partition α can occur at most twice. If a part occurs more than twice, then $\pi_{\alpha,s} = 0$, and if all parts appear at most twice, we have that $\pi_{\alpha,s} = 2^k$ where k is the number of parts that appear twice in α.

n	$\alpha \in \mathcal{P}_{n,T}$	$\pi_{\alpha,s}$	$r_{s,T}(n)$
0	(the trivial partition)	1	1
1		1	1
2		$1, 2$	3
3		$1, 1, 0$	1
4		$1, 1, 2, 2, 0$	6
5		$1, 1, 1, 2, 2, 0, 0$	7
6		$1, 1, 1, 2, 2, 1, 0, 0, 4, 0, 0$	12

- Let T be the set of prime numbers and $s = (1, -1, 1, -1, 1, -1, 1, \dots)$. Then $\mathcal{P}_{n,T}$ is the set of partitions whose parts are primes. For any $\alpha \in \mathcal{P}_{n,T}$, let $t \in T$ be a part which occurs i_t times in α. The part t

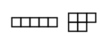
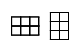
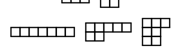
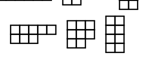
corresponds to the factor s_{i_t} in $\pi_{\alpha,s}$. If i_t is odd, then $s_{i_t} = -1$, and if i_t is even, then $s_{i_t} = 1$. Thus $s_{i_t} = (-1)^{i_t}$. Writing

$$\alpha = \left(t_J, \overset{i_{t_J}}{\ldots}, t_J, t_{J-1}, \overset{i_{t_{J-1}}}{\ldots}, t_{J-1}, \ldots, t_1, \overset{i_{t_1}}{\ldots}, t_1, 0, \ldots, 0\right),$$

we obtain that

$$\pi_{\alpha,s} = \prod_{j=1}^{J} s_{i_{t_j}} = \prod_{j=1}^{J}(-1)^{i_{t_j}} = (-1)^{\sum_{j=1}^{J} i_{t_j}} = (-1)^{l(\alpha)}.$$

Then $\pi_{\alpha,s} = -1$ if the length of α is odd and $\pi_{\alpha,s} = 1$ if α has even length.

n	$\alpha \in \mathcal{P}_{n,T}$	$\pi_{\alpha,s}$	$r_{s,T}(n)$
0	(the trivial partition)	1	1
1		0	0
2		-1	-1
3		-1	-1
4		1	1
5		$-1, 1$	0
6		$1, -1$	0
7		$-1, 1, -1$	-1
8		$1, -1, 1$	1

- If $T = \mathbb{N}$ and $s = (1)_{i\in\mathbb{N}_0}$, then $\mathcal{P}_{n,T} = \mathcal{P}_n$, so $r_{s,T}(n)$ sums over all partitions of n, and for each partition $\alpha \in \mathcal{P}_n$ we have that $\pi_{\alpha,s} = 1$, so $r_{s,T}(n) = \sum_{\alpha\in\mathcal{P}_n} 1 = |\mathcal{P}_n|$, which is the number of partitions of $n \in \mathbb{N}$. The generating function is

$$\sum_{n=0}^{\infty} r_{s,T}(n)x^n = \prod_{t=1}^{\infty}\sum_{i=0}^{\infty} s_i x^{ti} = \prod_{t=1}^{\infty}\sum_{i=0}^{\infty} x^{ti} = \prod_{t=1}^{\infty}\frac{1}{1-x^t}.$$

- Let $T = \mathbb{N}$ and $s = (1, 1, 0, 0, 0, \ldots)$. Then $r_{s,T}(n)$ sums over all partitions of n. If all parts in any $\alpha \in \mathcal{P}_n$ only occur once, then $\pi_{\alpha,s} = 1$, and if at least one part appears more than once, then $\pi_{\alpha,s} = 0$. Thus $r_{s,T}(n)$ equals the number partitions of $n \in \mathbb{N}$ whose parts only occur once, which are the strict partitions of n. The generating function is

$$\sum_{n=0}^{\infty} r_{s,T}(n)x^n = \prod_{t=1}^{\infty}\sum_{i=0}^{\infty} s_i x^{ti} = \prod_{t=1}^{\infty}(1 + x^t).$$

- If T is the set of odd natural numbers and $s = (1)_{i\in\mathbb{N}_0}$, then $r_{s,T}(n)$ sums over the partitions of n whose parts are odd and $\pi_{\alpha,s} = 1$ for all partitions $\alpha \in \mathcal{P}_{n,T}$. Then $r_{s,T}(n)$ equals the number of partitions whose only positive parts are odd and the generating function is

$$\sum_{n=0}^{\infty} r_{s,T}(n)x^n = \prod_{t\in T}\sum_{i=0}^{\infty} s_i x^{it}$$

$$= \prod_{t=1}^{\infty}\sum_{i=1}^{\infty} x^{(2t-1)i} = \prod_{t=1}^{\infty}\frac{1}{1-x^{2t-1}} = \prod_{t=1}^{\infty}\frac{1-x^{2t}}{(1-x^{2t-1})(1-x^{2t})}$$

$$= \frac{1-x^2}{(1-x)(1-x^2)}\frac{1-x^4}{(1-x^3)(1-x^4)}\frac{1-x^6}{(1-x^5)(1-x^6)}\cdots$$

$$= \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\cdots = \prod_{t=1}^{\infty}\frac{1-x^{2t}}{1-x^t} = \prod_{t=1}^{\infty}(1+x^t)$$

since $(1+x^t)(1-x^t) = 1-x^{2t} \Rightarrow \frac{1-x^{2t}}{1-x^t} = 1+x^t\ \forall t \in \mathbb{N}$. Thus the number of partitions of any $n \in \mathbb{N}$ with only odd positive parts equals the number of strict partitions of n. This fact and the reasoning for it is a special case of the proof given in [5].

- Suppose $T = \mathbb{N}$ and $s = (1, -1, 0, 0, 0, \dots)$. Then $r_{s,T}(n)$ sums over $\mathcal{P}_n$. We also have that for any $\alpha \in \mathcal{P}_n$, $\pi_{\alpha,s} = 0$ if any part in α occurs more than once, $\pi_{\alpha,s} = 1$ if all parts in α only occur once and the number of part that occur only once is even, and $\pi_{\alpha,s} = -1$ if if all parts in α only occur once and the number of part that occur only once is odd. In other words, if α is not a strict partition, then $\pi_{\alpha,s} = 0$, if α is strict and $l(\alpha)$ is even, then $\pi_{\alpha,s} = 1$, and if α is strict and $l(\alpha)$ is odd, then $\pi_{\alpha,s} = -1$. Thus, if α is strict, then $\pi_{\alpha,s} = (-1)^{l(\alpha)}$. Then $r_{s,T}(n)$ is the number of strict partitions of $n \in \mathbb{N}$ with even length minus the number of strict partitions of n with odd length. The generating function is

$$\prod_{t=1}^{\infty}\sum_{i=0}^{\infty} s_i x^{ti} = \prod_{t=1}^{\infty}(1-x^t).$$

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Let $K = \mathrm{GF}(q)$, the finite field of order q for q a prime power and let $n \in \mathbb{N}$. We have that $|\operatorname{Gl}_n(K)| = (q^n - 1)(q^n - q)\cdots(q^n - q^{n-1})$ and $|M_n(K)| = q^{n^2}$. Then

$$\frac{|\operatorname{Gl}_n(K)|}{|M_n(K)|} = \frac{\prod_{t=0}^{n-1}(q^n - q^t)}{q^{n^2}} = \frac{\prod_{t=0}^{n-1} q^n \left(1 - \frac{1}{q^{n-t}}\right)}{q^{n^2}}$$

$$= \frac{q^{n^2} \prod_{t=1}^{n} \left(1 - \frac{1}{q^t}\right)}{q^{n^2}} = \prod_{t=1}^{n} \left(1 - \frac{1}{q^t}\right).$$

Taking successive field extensions of $\mathrm{GF}(q)$ approaches

$$\lim_{i\to\infty} \frac{|\operatorname{Gl}_n(\mathrm{GF}(q^i))|}{|M_n(\mathrm{GF}(q^i))|} = \lim_{i\to\infty} \prod_{t=1}^{n} \left(1 - \frac{1}{(q^i)^t}\right) = \prod_{t=1}^{n} (1 - 0) = 1.$$

If we instead take the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{|\operatorname{Gl}_n(K)|}{|M_n(K)|} = \lim_{n\to\infty} \prod_{t=1}^{n} \left(1 - \frac{1}{q^t}\right) = \prod_{t=1}^{\infty} \left(1 - \frac{1}{q^t}\right),$$

which we can explore further through the next theorem. the theorem is found in [9] and the proof we give is based on the proof from the same article.

Theorem 2.1.

$$\prod_{t=1}^{\infty} (1 - x^t) = \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{k(3k-1)}{2}}.$$

Proof. The idea of this proof will be to construct a function f which maps a partition with length m onto a partition with either length $m-1$ or $m+1$. We can show that if n is not equal to $\frac{k(3k-1)}{2}$ for any $k \in \mathbb{Z}$, then f is a bijection and f^2 is the identity. This will imply that every strict partition of n with odd length can be paired with a unique strict partition with even length, and vice versa, which means that there are exactly as many strict partitions with odd length as there are strict partitions with even length. We already showed that

$$\prod_{t=1}^{\infty}(1 - x^t) = \sum_{n=0}^{\infty} r(n)x^n$$

where $r(n)$ for $n \in \mathbb{N}$ is the number of strict partitions of n with even length minus the number of strict partitions of n with odd length, and $r(0) = 1$. Thus $r(n) = 0$ for $n \neq \frac{k(3k-1)}{2}$ $\forall k \in \mathbb{Z}$.

We will also show that if $\exists k \in \mathbb{Z}$ such that $n = \frac{k(3k-1)}{2}$, then the number of strict partitions of even length is either exactly one more than the number of strict partitions of odd length if k is even, and exactly one less if k is odd. Thus $r(n) = (-1)^k$ if $n = \frac{k(3k-1)}{2}$ for some $k \in \mathbb{Z}$.

Provided the claims above are true, we can then conclude that

$$\prod_{t=1}^{\infty} \left(1 - x^t\right) = \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{k(3k-1)}{2}}.$$

We now prove the claims, and we start by specifying the function f.

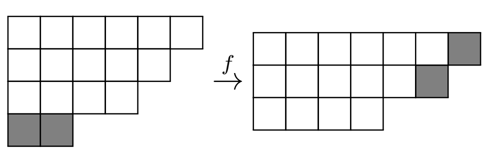
Let α be a strict partition of $n \in \mathbb{N}$. and let d_α be equal to the number $\max\{i \in \mathbb{N}_n \mid \alpha_i = \alpha_1 - i + 1\}$. We have that the sequence of parts $(\alpha_1, \ldots, \alpha_{d_\alpha})$ is such that each part not on one of the ends of the sequence is preceded by a part which is one larger and followed by a part which is one smaller. We call this sequence the first diagonal of α, so d_α is then the length of the first diagonal. Let f be a function from the set of strict partitions to the set of general partitions defined such that if $\alpha_{l(\alpha)} \leq d_\alpha$, then

$$f(\alpha_1, \ldots, \alpha_n) = (\alpha_1 + 1, \ldots, \alpha_{\alpha_{l(\alpha)}} + 1, \alpha_{\alpha_{l(\alpha)}+1}, \ldots, \alpha_{l(\alpha)-1}, 0, \ldots, 0) \in \mathbb{N}_0^n,$$

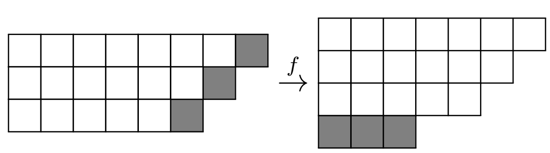
and if $\alpha_{l(\alpha)} > d_\alpha$, then

$$f(\alpha_1, \ldots, \alpha_n) = (\alpha_1 - 1, \ldots, \alpha_{d_\alpha} - 1, \alpha_{d_\alpha+1}, \ldots, \alpha_{l(\alpha)}, d_\alpha, 0, \ldots, 0) \in \mathbb{N}_0^n.$$

In terms of Young diagrams, if $\alpha_{l(\alpha)} \leq d_\alpha$, then f takes the last part of α and moves it onto the first diagonal.



If $\alpha_{l(\alpha)} > d_\alpha$, then f takes the first diagonal and moves it underneath the last part.



From here we look at four different cases. For now we specifically avoid the cases where $l(\alpha) = d_\alpha$ and either $\alpha_{l(\alpha)} = d_\alpha$ or $\alpha_{l(\alpha)} = d_\alpha + 1$. The reason for this choice will become apparent later in the proof.

1. If $l(\alpha) > d_\alpha$ and $\alpha_{l(\alpha)} \leq d_\alpha$, or $l(\alpha) = d_\alpha$ and $\alpha_{l(\alpha)} < d_\alpha$, then

$$f(\alpha) = (\alpha_1 + 1, \dots, \alpha_{\alpha_{l(\alpha)}} + 1, \alpha_{\alpha_{l(\alpha)}+1}, \dots, \alpha_{l(\alpha)-1}, 0, \dots, 0).$$

If $l(f(\alpha)) > d_{f(\alpha)}$, then

$$f(\alpha)_{l(f(\alpha))} = f(\alpha)_{l(\alpha)-1} \geq \alpha_{l(\alpha)-1} > \alpha_{l(\alpha)} = d_{f(\alpha)} \Rightarrow f(\alpha)_{l(f(\alpha))} > d_{f(\alpha)},$$

so

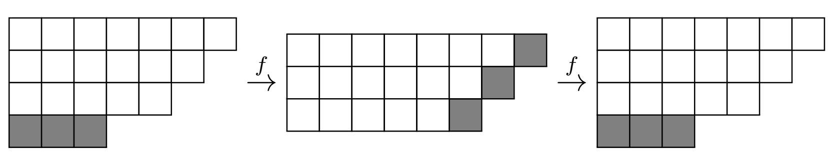
$$f(f(\alpha))$$
$$=$$
$$\big(f(\alpha)_1 - 1, \dots, f(\alpha)_{d_{f(\alpha)}} - 1, f(\alpha)_{d_{f(\alpha)}+1}, \dots, f(\alpha)_{l(f(\alpha))}, d_{f(\alpha)}, 0, \dots, 0\big)$$
$$= (\alpha_1, \dots, \alpha_{\alpha_{l(\alpha)}}, \alpha_{\alpha_{l(\alpha)}+1}, \dots, \alpha_{l(\alpha)-1}, \alpha_{l(\alpha)}, 0, \dots, 0) = \alpha.$$

If $l(f(\alpha)) = d_{f(\alpha)}$, then

$$f(\alpha)_{l(f(\alpha))} = f(\alpha)_{l(\alpha)-1} = \alpha_{l(\alpha)-1} + 1 > \alpha_{l(\alpha)} + 1 = d_{f(\alpha)} + 1$$
$$\Rightarrow f(\alpha)_{l(f(\alpha))} > d_{f(\alpha)},$$

so

$$f(f(\alpha)) = \big(f(\alpha)_1 - 1, \dots, f(\alpha)_{d_{f(\alpha)}} - 1, d_{f(\alpha)}, 0, \dots, 0\big)$$
$$= (\alpha_1, \dots, \alpha_{l(\alpha)-1}, \alpha_{l(\alpha)}, 0, \dots, 0) = \alpha.$$



In either case, $f^2(\alpha) = \alpha$. Thus f is bijective on this type of partition if we restrict the target of f and is its own inverse. This means that for every strict partition α such that $l(\alpha) > d_\alpha$ and $\alpha_{l(\alpha)} \leq d_\alpha$, or $l(\alpha) = d_\alpha$ and $\alpha_{l(\alpha)} < d_\alpha$, there exists a unique strict partition β of n such that $l(\beta) = l(\alpha) - 1$.

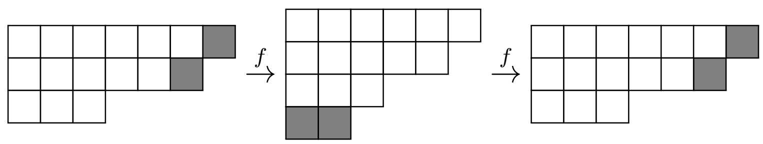
2. If $l(\alpha) > d_\alpha$ and $\alpha_{l(\alpha)} > d_\alpha$, or $l(\alpha) = d_\alpha$ and $\alpha_{l(\alpha)} > d_\alpha + 1$, then

$$f(\alpha) = (\alpha_1 - 1, \dots, \alpha_{d_\alpha} - 1, \alpha_{d_\alpha+1}, \dots, \alpha_{l(\alpha)}, d_\alpha, 0, \dots, 0).$$

If $l(f(\alpha)) > d_{f(\alpha)}$, then $f(\alpha)_{l(f(\alpha))} = d_\alpha \leq d_{f(\alpha)}$. If $l(f(\alpha)) = d_{f(\alpha)}$, then $f(\alpha)_{l(f(\alpha))} = d_\alpha < d_{f(\alpha)}$. In either case, $f(f(\alpha))$

$$= \big(f(\alpha)_1 + 1, \dots, f(\alpha)_{d_\alpha} + 1, f(\alpha)_{d_\alpha+1}, \dots, f(\alpha)_{l(f(\alpha))-1}, 0, \dots, 0\big)$$

$$= (\alpha_1, \dots, \alpha_{d_\alpha}, \alpha_{d_\alpha+1}, \dots, \alpha_{l(\alpha)}, 0, \dots, 0) = \alpha.$$



Then $f^2(\alpha) = \alpha$, so f is a bijection on this type of partition too, and is its own inverse. Thus for each strict partition α such that $l(\alpha) > d_\alpha$ and $\alpha_{l(\alpha)} > d_\alpha$, or $l(\alpha) = d_\alpha$ and $\alpha_{l(\alpha)} > d_\alpha + 1$, there exist a unique partition β of n such that $l(\beta) = l(\alpha) + 1$.

Thus, if no strict partition α of $n \in \mathbb{N}$ respects $l(\alpha) = d_\alpha$ and either $\alpha_{l(\alpha)} = d_\alpha$ or $\alpha_{l(\alpha)} = d_\alpha + 1$, then the number of strict partitions of n with odd length equals the number of strict partitions of n with even length. Hence the coefficient of x^n in the series expansion of $\prod_{t=1}^{\infty}(1 - x^t)$ is 0.

Now we check what happens if $l(\alpha) = d_\alpha$ and $\alpha_{l(\alpha)} = d_\alpha$, or $l(\alpha) = d_\alpha$ and $\alpha_{l(\alpha)} = d_\alpha + 1$.

3. If $n = 1$, then the only strict partition, and in fact the only partition altogether, is $\alpha = (1)$. We have that $l(\alpha) = d_\alpha = 1$ and $\alpha_{l(\alpha)} = d_\alpha = 1$. Since α is the only partition of 1, it does not correspond to any other partition under f. Also, $n = 1 = \frac{2}{2} = \frac{1(3\cdot 1-1)}{2} = \frac{k(3k-1)}{2}$ for $k = 1$.

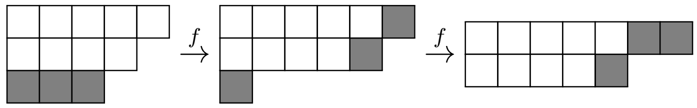
Let $n \in \mathbb{N} \setminus \{1\}$. If α is a strict partition with $l(\alpha) = d_\alpha$ and $\alpha_{l(\alpha)} = d_\alpha$, then

$$f(\alpha) = (\alpha_1 + 1, \dots, \alpha_{l(\alpha)-1} + 1, 1, 0, \dots, 0),$$

but now $f(\alpha)_{l(f(\alpha))} = 1 \leq d_{f(\alpha)}$, so $f(f(\alpha))$

$$= \big(f(\alpha)_1 + 1, f(\alpha)_2, \dots, f(\alpha)_{l(\alpha)-1}, 0, \dots, 0\big)$$

$$= \big(\alpha_1 + 2, \alpha_2 + 1, \dots, \alpha_{l(\alpha)-1} + 1, 0, \dots, 0\big) \neq \alpha.$$



Then we cannot say that α corresponds to a unique β like we did before. If n is such that there exists a partition such that $l(\alpha) = d_\alpha$ and $\alpha_{l(\alpha)} = d_\alpha$, then $n = \sum_{i=0}^{l(\alpha)-1}(l(\alpha)+i) = \frac{l(\alpha)(3l(\alpha)-1)}{2} = \frac{k(3k-1)}{2}$ for $k = l(\alpha)$.

Conversely, if $n = \frac{k(3k-1)}{2}$ for some $k \in \mathbb{N}$, then we can write $n = \sum_{i=0}^{k-1}(k+i)$ and construct the strict partition

$$\alpha = (2k-1, 2k-2, \dots, k, 0, \dots, 0).$$

We have that $l(\alpha) = d_\alpha = k$ and $\alpha_{l(\alpha)} = d_\alpha = k$.

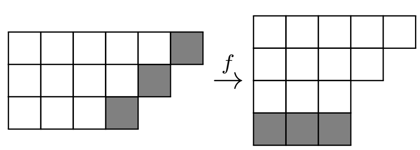
Thus there exists a strict partition α of n such that $l(\alpha) = d_\alpha$ and $\alpha_{l(\alpha)} = d_\alpha$ if and only if $n = \frac{k(3k-1)}{2}$ for some $k \in \mathbb{N}$.

4. If $n = 2$, then there is only one strict partition, namely $\alpha = (2, 0)$, for which we have that $l(\alpha) = d_\alpha = 1$ and $\alpha_{l(\alpha)} = d_\alpha + 1 = 2$. Since α is the only strict partition, α does not correspond to any other strict partition under f. We can also write $n = 2 = \frac{4}{2} = \frac{(-1)(3(-1)-1)}{2} = \frac{k(3k-1)}{2}$ for $k = -1$.

   Let $n \in \mathbb{N} \setminus \{2\}$. If α is a strict partition with $l(\alpha) = d_\alpha$ and $\alpha_{l(\alpha)} = d_\alpha + 1$, then

$$f(\alpha) = (\alpha_1 - 1, \dots, \alpha_{l(\alpha)} - 1, l(\alpha), 0, \dots, 0)$$
$$= (\alpha_1 - 1, \dots, d_\alpha, d_\alpha, 0, \dots, 0),$$

   which is not a strict partition.



   Then this α does not correspond to any strict partition under f. Moreover, we have that

$$n = \sum_{i=1}^{l(\alpha)} l(\alpha)+i = \frac{l(\alpha)(3l(\alpha)+1)}{2} = \frac{-l(\alpha)(3(-l(\alpha))-1)}{2} = \frac{k(3k-1)}{2}$$

for $k = -l(\alpha)$. On the other hand, if we can write $n = \frac{k(3k-1)}{2}$ for some $k \in \mathbb{Z} \setminus \mathbb{N}_0$, then $n = \frac{(-k)(3(-k)+1)}{2} = \sum_{i=1}^{-k} -k + i$. We can then define the partition $\alpha = (-2k, -2k-1, \ldots, -k+1, 0, \ldots, 0)$. We have that $l(\alpha) = d_\alpha = -k$ and $\alpha_{l(\alpha)} = d_\alpha + 1 = -k+1$. Thus there exists a strict partition α of n such that $l(\alpha) = d_\alpha$ and $\alpha_{l(\alpha)} = d_\alpha + 1$ if and only if $n = \frac{k(3k-1)}{2}$ for some $k \in \mathbb{Z} \setminus \mathbb{N}_0$.

If we collect everything we have proved up to this point, we get that for any $n \in \mathbb{N}$, there exists a strict partition of n which does not correspond to any other strict partition if and only if $n = \frac{k(3k-1)}{2}$ for some $k \in \mathbb{Z} \setminus \{0\}$.

Next we show that there is at most one partition of any n which does not correspond to any other partition under f. To see this, suppose that there are two partitions α and β of n such that $l(\alpha) = d_\alpha$, $\alpha_{l(\alpha)} = d_\alpha$, $f(f(\alpha)) \neq \alpha$, $l(\beta) = d_\beta$, $\beta_{l(\beta)} = d_\beta$ and $f(f(\beta)) \neq \beta$. Then $n = \frac{k_1(3k_1-1)}{2}$ and $n = \frac{k_2(3k_2-1)}{2}$ for some $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$. We have that

$$\frac{k_1(3k_1-1)}{2} = \frac{k_2(3k_2-1)}{2} \Rightarrow 3k_1^2 - k_1 = 3k_2^2 - k_2 \Rightarrow 3k_1^2 - k_1 - (3k_2^2 - k_2) = 0$$

$$\Rightarrow k_1 = \frac{1 \pm \sqrt{1 + 4 \cdot 3(3k_2^2 - k_2)}}{2 \cdot 3} = \frac{1 \pm \sqrt{36k_2^2 - 12k_2 + 1}}{6} = \frac{1 \pm (6k_2 - 1)}{6}$$

$$\Rightarrow k_1 = \frac{1 + 6k_2 - 1}{6} = \frac{6k_2}{6} = k_2,$$

or

$$k_1 = \frac{1 - (6k_2 - 1)}{6} = \frac{-6k_2 + 2}{6} = -k_2 + \frac{1}{3}.$$

Suppose that $k_1 = -k_2 + \frac{1}{3}$ and notice that for all $k_2 \in \mathbb{Z}$, $k_1 = -k_2 + \frac{1}{3} \notin \mathbb{Z}$, which contradicts the requirement that $k_1 \in \mathbb{Z}$. Then we must have that $k_1 = k_2$.

From this argument we can see that if $n = \frac{k(3k-1)}{2}$ for some $k \in \mathbb{Z} \setminus \{0\}$, then there is exactly one exceptional strict partition α of n. Since $k = l(\alpha)$ or $k = -l(\alpha)$, α has an odd number of parts if k is odd and α has an even number of parts if k is even. We have that

$$\prod_{t=1}^{\infty}(1 - x^t) = \sum_{n=0}^{\infty} r(n)x^n$$

where $r(n)$ for $n \in \mathbb{N}$ is the number of partitions with an even number of parts minus the number of partitions with an odd number of parts and $r(0) = 1$.

Then, if $n = \frac{k(3k-1)}{2}$, $r(n) = 1$ when k is even and $r(n) = -1$ when k is odd, so $r\left(\frac{k(3k-1)}{2}\right) = (-1)^k$. If there is no $k \in \mathbb{Z}$ such that $n = \frac{k(3k-1)}{2}$, then $r(n) = 0$. Thus

$$\prod_{t=1}^{\infty}\left(1 - x^t\right) = \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{k(3k-1)}{2}}.$$

$\square$

From this theorem we then obtain the equality

$$\lim_{n\to\infty} \frac{|\mathrm{Gl}_n(K)|}{|M_n(K)|} = \sum_{k=-\infty}^{\infty} (-1)^k q^{-\frac{k(3k-1)}{2}}$$

where $K = \mathrm{GF}(q)$ for a prime power $Q \in \mathbb{N}$. By calculating $\frac{k(3k-1)}{2}$ for a few k, we get

k	0	1	-1	2	-2	3	-3	4	-4	5	-5	6	-6
$\frac{k(3k-1)}{2}$	0	1	2	5	7	12	15	22	26	35	40	51	57

which we can use to find the first couple of terms in the series expansion of $\lim_{n\to\infty} \frac{|\mathrm{Gl}_n(K)|}{|M_n(K)|}$.

$$\lim_{n\to\infty} \frac{|\mathrm{Gl}_n(K)|}{|M_n(K)|}$$

$$= 1 - \frac{1}{q} - \frac{1}{q^2} + \frac{1}{q^5} + \frac{1}{q^7} - \frac{1}{q^{12}} - \frac{1}{q^{17}} + \frac{1}{q^{22}} + \frac{1}{q^{26}} - \frac{1}{q^{35}} - \frac{1}{q^{40}} + \frac{1}{q^{51}} + \frac{1}{q^{57}} - \cdots$$

$$= (0.q-2\, q-1\, 0\, 0\, 1\, 0\, 0\, q-1\, q-1\, q-1\, q-1\, q-2\, q-1\, q-1\, q-1\, 0\, 0 \ldots)_q,$$

where $a = (b)_q$ denotes that a is the base q representation of b for any $a \in \mathbb{R}$. For $q = 2$ we get

$$\lim_{n\to\infty} \frac{|\mathrm{Gl}_n(\mathrm{GF}(2))|}{|M_n(\mathrm{GF}(2))|} = (0.0100100111101110000010\ldots)_2 < (0.1)_2 = \frac{1}{2}.$$

If q is any prime power, then

$$\frac{q-2}{q} = (0.q-2)_q < \lim_{n\to\infty} \frac{|\mathrm{Gl}_n(\mathrm{GF}(q))|}{|M_n(\mathrm{GF}(q))|}$$

$$= (0.q-2\, q-1\, 0\, 0\, 1\, 0 \ldots)_q < (0.q-1)_q = \frac{q-1}{q}.$$

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver where $m = |\Gamma_0| = |\Gamma_1|$, that is the number of vertices equals the number of edges. Consider representations of Γ over $K = \mathrm{GF}(q)$ with dimension vector $D = (n, n, \ldots, n) \in \mathbb{N}^m$. The size of each orbit is less or equal to $\frac{|\mathrm{Gl}_n(K)^m|}{q-1}$, so

$$|M_n(K)^m| = |\mathrm{Ob}(\mathrm{rep}_D \Gamma)| = \sum_{\substack{\rho \text{ orbit of} \\ \text{objects in} \, \mathrm{rep}_D \Gamma}} |\rho|,$$

$$\leq \sum_{\substack{\rho \text{ orbit of} \\ \text{objects in} \, \mathrm{rep}_D \Gamma}} \frac{|\mathrm{Gl}_n(K)^m|}{q-1} = |\{\text{orbits of objects in } \mathrm{rep}_D \Gamma\}| \cdot \frac{|\mathrm{Gl}_n(K)^m|}{q-1}$$

$$\Rightarrow |\{\text{orbits of objects in } \mathrm{rep}_D \Gamma\}| \geq (q-1)\frac{|M_n(K)^m|}{|\mathrm{Gl}_n(K)^m|}.$$

By taking the limit as $n \to \infty$, we get that the number of orbits is greater than or equal to $(q-1)\left(\lim_{n\to\infty} \frac{|\mathrm{Gl}_n(K)^m|}{|M_n(K)^m|}\right)^{-1} > \frac{q(q-1)}{q-1} = q$.

2.2 Degeneration over Principal Ideal Domains

There is a nice correspondence between degeneration of certain modules over principal ideal domains and that which is called the dominant order on partitions. Many of the following concepts and ideas are based on 3.3 in [11].

Definition 2.2. Let $n \in \mathbb{N}$. **The dominant order** on the set of partitions of n is defined such that if α and β are partitions of n, then we say that α **is dominated by** β, or equivalently that β **dominates** α if $\sum_{i=1}^k \alpha_i \leq \sum_{i=1}^k \beta_i$ for all $k \in \mathbb{N}_n$. We write $\alpha \leq_{\mathrm{dom}} \beta$ as a shorthand. $\triangle$

It is fairly straight-forward to see that the dominant order is a partial order for all $n \in \mathbb{N}$.

1. Let α be a partition of n. $\sum_{i=1}^k \alpha_i = \sum_{i=1}^k \alpha_i \; \forall k \in \mathbb{N}_n \Rightarrow \alpha \leq_{\mathrm{dom}} \alpha$, which shows reflexivity.

2. Let α and β be partitions of n and suppose $\alpha \leq_{\mathrm{dom}} \beta$ and $\beta \leq_{\mathrm{dom}} \alpha$. This implies that $\sum_{i=1}^k \alpha_i = \sum_{i=1}^k \beta_i$. First of all, we have that $\alpha_1 =$

$\sum_{i=1}^{1} \alpha_i = \sum_{i=1}^{1} \beta_i = \beta_1$. Secondly, suppose $r \in \mathbb{N}_{n-1}$. By assumption, we have that $\sum_{i=1}^{r} \alpha_i = \sum_{i=1}^{r} \beta_i$ and $\sum_{i=1}^{r+1} i = \sum_{i=1}^{r+1} \beta_i$. Then

$$\alpha_{r+1} = \sum_{i=1}^{r+1} \alpha_i - \sum_{i=1}^{r} \alpha_i = \sum_{i=1}^{r+1} \beta_i - \sum_{i=1}^{r} \beta_i = \beta_{r+1}.$$

Thus $\alpha = \beta$, which proves antisymmetry.

3. Suppose α, β and γ are partitions of n such that $\alpha \leq_{\text{dom}} \beta$ and $\beta \leq_{\text{dom}} \gamma$. Then $\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i \leq \sum_{i=1}^{k} \gamma_i \ \forall k \in \mathbb{N}_n \Rightarrow \alpha \leq_{\text{dom}} \gamma$, which shows transitivity.

Let M be a finitely generated module over a principal ideal domain R. By the Structure Theorem for Finitely Generated Modules over Principle Ideal Domains, we have a following R-module isomorphism

$$M \cong \bigoplus_{i=1}^{m} R/(q_i)$$

for some $q_1, \ldots, q_m \in R$ and $m \in \mathbb{N}$. Since R is a PID, then any generating element $q \in R$ of an ideal (q) in R can be written as $q = p^r$, a power of an irreducible element $p \in R$, where $r \in \mathbb{N}$. Then $M \cong \bigoplus_{i=1}^{m} R/(p_i^{r_i})$ for some irreducible elements $p_1, \ldots, p_m \in R$ and integers $r_1, \ldots, r_m \in \mathbb{N}$.

For the following definition, we have that $\text{mod}_d R$ is the category of all R-modules of length $d \in \mathbb{N}$ where R is any ring.

Definition 2.3. Let R be a principal ideal domain and p an irreducible element in R. Define $\mathcal{M}_d(p)$ to be the full subcategory of $\text{mod}_d R$ such that $M \in \text{Ob}(\mathcal{M}_d(p))$ if and only if $M \cong \bigoplus_{i=1}^{d} R/(p^{\alpha_i})$ where $(\alpha_1, \ldots, \alpha_d) \in \mathcal{P}_d$. $\triangle$

Let $M \in \mathcal{M}_d(p)$ and assume $M \cong \bigoplus_{i=1}^{d} R/(p^{\alpha_i})$ where $\alpha_1 \geq \cdots \geq \alpha_d$ and $\sum_{i=1}^{d} \alpha_i = d$. Define the partition $\alpha_M = (\alpha_1, \ldots, \alpha_d) \in \mathcal{P}_d$.

Next, define the conjugate partition α' of a partition α of $n \in \mathbb{N}$ to be such that $\alpha'_i = |\{j \in \mathbb{N}_n | \alpha_j \geq i\}|$ for each $i \in \mathbb{N}_n$. If we draw a Young diagram of α, then α'_i corresponds to the i-th column in the diagram for every $i \in \mathbb{N}_d$.

Lemma 2.2. If $M \in \mathcal{M}_d(p)$, then $l\,\text{Hom}\left(R/(p^i), M\right) = \sum_{k=1}^{i} \alpha'_k$ for all $i \in \mathbb{N}_d$.

Proof. If $M \in \mathcal{M}_d(p)$, then we have that $M \cong \bigoplus_{k=1}^{d} {}^{R}\!/\!{}_{(p^{\alpha_k})}$ for some $\alpha_M = (\alpha_1, \dots, \alpha_d) \in \mathcal{P}_d$ and

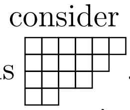
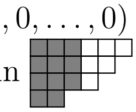
$$\operatorname{Hom}\left({}^{R}\!/\!{}_{(p^i)}, {}^{R}\!/\!{}_{(p^j)}\right) \cong {}^{R}\!/\!{}_{\left(p^{\min\{i,j\}}\right)}$$

$$\Rightarrow \operatorname{Hom}\left({}^{R}\!/\!{}_{(p^i)}, M\right) \cong \operatorname{Hom}\left({}^{R}\!/\!{}_{(p^i)}, \bigoplus_{k=1}^{d} {}^{R}\!/\!{}_{(p^{\alpha_k})}\right)$$

$$\cong \bigoplus_{k=1}^{d} \operatorname{Hom}\left({}^{R}\!/\!{}_{(p^i)}, {}^{R}\!/\!{}_{(p^{\alpha_k})}\right) \cong \bigoplus_{k=1}^{d} {}^{R}\!/\!{}_{\left(p^{\min\{i,\alpha_k\}}\right)}$$

$$\Rightarrow l \operatorname{Hom}\left({}^{R}\!/\!{}_{(p^i)}, M\right) = \sum_{k=1}^{d} \min\{i, \alpha_k\}$$

$$\overset{(1)}{=} \sum_{k=1}^{i} |\{j \in \mathbb{N}_d \mid \alpha_j \geq k\}| = \sum_{k=1}^{i} \alpha'_k.$$

Equality (1) can be obtained by drawing the Young diagram of α_M and counting only i boxes in each row that consists of at least i and all the boxes in every row that consists of less than i boxes. The resulting number should be the as what we get if we count the boxes in the first i columns. For example, let $i = 3$ and consider the partition $\alpha = (6, 5, 4, 2, 0, \dots, 0) \in \mathcal{P}_{17}$, whose Young diagram is . We count the grey boxes in  in two ways. If we count them row-wise, then we get

$$\sum_{k=1}^{17} \min\{3, \alpha_k\} = 3 + 3 + 3 + 2 + 0 + \cdots + 0 = 11.$$

If we count the grey boxes column-wise, then we get

$$\sum_{k=1}^{3} |\{j \in \mathbb{N}_4 \mid \alpha_j \geq k\}| = \sum_{k=1}^{3} \alpha'_k = 4 + 4 + 3 = 11.$$

Equation (1) clearly holds for this example.

To prove the general case, let $c \in \mathbb{N}_d \backslash \mathbb{N}_{l(\alpha_M)-1} = \{l(\alpha_M), l(\alpha_M)+1, \dots, d\}$ such that

$$\beta := (\min\{i, \alpha_1\}, \dots, \min\{i, \alpha_c\}) \in \mathcal{P}_c.$$

We have that $\beta' = (|\{j \in \mathbb{N}_c \mid \beta_j \geq 1\}|, \ldots, |\{j \in \mathbb{N}_c \mid \beta_j \geq c\}|) \in \mathcal{P}_c$ by the definition of conjugate partitions, so

$$\sum_{k=1}^{d} \min\{i, \alpha_k\} = \sum_{k=1}^{c} \min\{i, \alpha_k\} = \sum_{k=1}^{c} \beta_k = c$$

$$= \sum_{k=1}^{c} \beta'_k = \sum_{k=1}^{c} |\{j \in \mathbb{N}_c \mid \beta_j \geq k\}| = \sum_{k=1}^{c} |\{j \in \mathbb{N}_c \mid \min\{i, \alpha_j\} \geq k\}|$$

$$\overset{(2)}{=} \sum_{k=1}^{c} |\{j \in \mathbb{N}_d \mid \min\{i, \alpha_j\} \geq k\}|$$

$$\overset{(3)}{=} \sum_{k=1}^{i} |\{j \in \mathbb{N}_d \mid \min\{i, \alpha_j\} \geq k\}|$$

$$\overset{(4)}{=} \sum_{k=1}^{i} |\{j \in \mathbb{N}_d \mid \alpha_j \geq k\}|.$$

(2) Equality (2) holds because if $j \in \mathbb{N}_d \setminus \mathbb{N}_c$, then

$$\alpha_j = 0 \Rightarrow \min\{i, \alpha_j\} = 0,$$

so

$$|\{j \in \mathbb{N}_c \mid \min\{i, \alpha_j\} \geq k\}| = |\{j \in \mathbb{N}_d \mid \min\{i, \alpha_j\} \geq k\}|$$

for all $k \in \mathbb{N}_c$.

(3) We obtain (3) because if $c \geq k > i$, then $\min\{i, \alpha_j\} \leq i < k$ for all $j \in \mathbb{N}_d$

$$\Rightarrow |\{j \in \mathbb{N}_d \mid \min\{i, \alpha_j\} \geq k\}| = 0$$

for all $k \in \mathbb{N}_d \setminus \mathbb{N}_c$.

(4) Let $k \in \mathbb{N}_i$. Since $\alpha_j \geq \min\{i, \alpha_j\}$ for all $j \in \mathbb{N}_d$, then

$$|\{j \in \mathbb{N}_d \mid \min\{i, \alpha_j\} \geq k\}| \leq |\{j \in \mathbb{N}_d \mid \alpha_j \geq k\}|.$$

We have that if $\alpha_j \geq k$, then $\min\{i, \alpha_j\} \geq k$ since $k \leq i$, so

$$|\{j \in \mathbb{N}_d \mid \min\{i, \alpha_j\} \geq k\}| \geq |\{j \in \mathbb{N}_d \mid \alpha_j \geq k\}|.$$

Then
$$|\{j \in \mathbb{N}_d \mid \min\{i, \alpha_j\} \geq k\}| = |\{j \in \mathbb{N}_d \mid \alpha_j \geq k\}|$$
if $\alpha_j \geq k$. If $\alpha_j < k$, then $\min\{i, \alpha_j\} < k$, so
$$|\{j \in \mathbb{N}_d \mid \min\{i, \alpha_j\} \geq k\}| = |\{j \in \mathbb{N}_d \mid \alpha_j \geq k\}| = 0.$$
Thus
$$|\{j \in \mathbb{N}_d \mid \min\{i, \alpha_j\} \geq k\}| = |\{j \in \mathbb{N}_d \mid \alpha_j \geq k\}|$$
for all $k \in \mathbb{N}_i$.

$\square$

Lemma 2.3. Let $M, N \in \mathcal{M}_d(p)$. If $\alpha'_M \leq_{\text{dom}} \alpha'_N$ minimally, then $M \leq_{\text{deg}} N$.

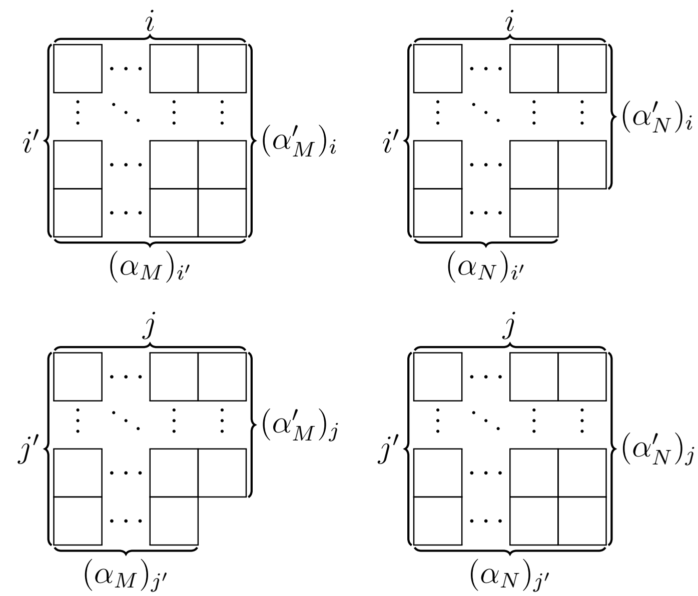
Proof. Let $M, N \in \mathcal{M}_d(p)$ and suppose $\alpha'_M \leq_{\text{dom}} \alpha'_N$ minimally. Then there are $i, j \in \mathbb{N}_d$ such that $i > j$ and
$$(\alpha'_N)_k = \left\{\begin{matrix} (\alpha'_M)_i - 1 & \text{if } k = i \\ (\alpha'_M)_j + 1 & \text{if } k = j \\ (\alpha'_M)_k & \text{otherwise} \end{matrix}\right\}.$$
This can be seen as obtaining α_N from α_M by

- shortening $(\alpha'_M)_i$, the i-th column in the Young diagram of α_M, by one.
- lengthening the j-th column $(\alpha'_M)_j$ by one.

If we write $i' = (\alpha'_M)_i$ and $j' = (\alpha'_N)_j$, then this is equivalent to

- shortening the row with index $(\alpha'_M)_i = i'$, which is $(\alpha_M)_{i'}$, by one.
- lengthening the row with index $(\alpha'_M)_j + 1 = (\alpha'_N)_j = j'$, which is $(\alpha_M)_{j'}$, by one.

Visually, it looks like the following.



Then we have that

$$(\alpha_N)_k = \left\{ \begin{array}{ll} (\alpha_M)_{i'} + 1 & \text{if } k = i' \\ (\alpha_M)_{j'} - 1 & \text{if } k = j' \\ (\alpha_M)_k & \text{otherwise} \end{array} \right\}.$$

Then

$$N \cong {}^{R}\!/\!\left(p^{(\alpha_M)_1}\right) \oplus \cdots \oplus {}^{R}\!/\!\left(p^{(\alpha_M)_{i'}+1}\right) \oplus \cdots \oplus {}^{R}\!/\!\left(p^{(\alpha_M)_{j'}-1}\right) \oplus \cdots \oplus {}^{R}\!/\!\left(p^{(\alpha_M)_d}\right)$$

$$\cong {}^{R}\!/\!\left(p^{(\alpha_M)_{i'}+1}\right) \oplus {}^{R}\!/\!\left(p^{(\alpha_M)_{j'}-1}\right) \oplus \bigoplus_{k \in \mathbb{N}_d \setminus \{i', j'\}} {}^{R}\!/\!\left(p^{(\alpha_M)_k}\right).$$

Suppose $r, s \in \mathbb{N}_d$ such that $r < s$. Consider the sequence

$$0 \longrightarrow {}^{R}\!/\!\left(p^{s-1}\right) \xrightarrow{\binom{\overline{1}}{\overline{p}}} {}^{R}\!/\!\left(p^{r}\right) \oplus {}^{R}\!/\!\left(p^{s}\right) \xrightarrow{(-\overline{p}\ \overline{1})} {}^{R}\!/\!\left(p^{r+1}\right) \longrightarrow 0\ .$$

For $v \in R$, $\overline{v}$ denotes the homomorphism $\overline{v} : {}^{R}\!/\!\left(p^{k}\right) \to {}^{R}\!/\!\left(p^{l}\right)$ for $k, l \in \mathbb{N}$ such that $\overline{v}\left(x + \left(p^k\right)\right) = vx + \left(p^l\right)$ for all $x \in R$. Given $a, b \in R$, then $\binom{\overline{1}}{\overline{p}}(a + (p^{s-1})) = \begin{pmatrix} a+(p^r) \\ pa+(p^s) \end{pmatrix}$ and $(\,-\overline{p}\ \overline{1}\,)\begin{pmatrix} a+(p^r) \\ b+(p^s) \end{pmatrix} = (-pa + b) + (p^{r+1})$. Let

$a \in R$ such that $a \notin (p^{s-1})$. If $a = a'p^t$ for some $a' \in R$ and $t \in \mathbb{N}_0$, then $t < s-1$. We have that $pa = a'p^{t+1}$, and since $t+1 < s$, then $pa \notin (p^s)$. Then $\overline{pa} \neq \overline{0}$, and $\ker \left(\begin{smallmatrix} \overline{1} \\ \overline{p} \end{smallmatrix}\right) = 0$, so $\left(\begin{smallmatrix} \overline{1} \\ \overline{p} \end{smallmatrix}\right)$ is injective.

Let $b \in R$. If $b \in (p^{r+1})$, then $b + (p^{r+1}) = 0 + (p^{r+1}) = \left(\begin{smallmatrix} -\overline{p} & \overline{1} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0+(p^r) \\ 0+(p^s) \end{smallmatrix}\right)$. If $b \notin (p^{r+1})$, then $b \notin (p^s)$ since $r+1 \leq s$. Then $b + (p^{r+1}) = \left(\begin{smallmatrix} -\overline{p} & \overline{1} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0+(p^r) \\ b+(p^s) \end{smallmatrix}\right)$, so $\left(\begin{smallmatrix} -\overline{p} & \overline{1} \end{smallmatrix}\right)$ is surjective.

We also have that $\left(\begin{smallmatrix} -\overline{p} & \overline{1} \end{smallmatrix}\right)\left(\begin{smallmatrix} \overline{1} \\ \overline{p} \end{smallmatrix}\right) = -\overline{p} + \overline{p} = \overline{0}$. Then the sequence above is short exact. In particular we get that

$$0 \longrightarrow {}^{R}\!\big/\!\left(p^{(\alpha_M)_{j'}-1}\right) \xrightarrow{\left(\begin{smallmatrix} \overline{1} \\ \overline{p} \end{smallmatrix}\right)} {}^{R}\!\big/\!\left(p^{(\alpha_M)_{i'}}\right) \oplus {}^{R}\!\big/\!\left(p^{(\alpha_M)_{j'}}\right) \xrightarrow{\left(\begin{smallmatrix} -\overline{p} & \overline{1} \end{smallmatrix}\right)} {}^{R}\!\big/\!\left(p^{(\alpha_M)_{i'}+1}\right) \longrightarrow 0$$

is short exact.

Suppose A, B, C are R modules and suppose $f : A \to B$ and $g : B \to C$ are R-homomorphisms. If the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is short exact, then the sequence

$$0 \longrightarrow A \oplus T \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)} B \oplus T \xrightarrow{\left(\begin{smallmatrix} g & 0 \end{smallmatrix}\right)} C \longrightarrow 0$$

is also short exact for any R-module T. Then

$$\begin{array}{c}
0 \\
\big\downarrow \\
{}^{R}\!\big/\!\left(p^{(\alpha_M)_{j'}-1}\right) \oplus \bigoplus_{k \in \mathbb{N}_d \setminus \{i', j'\}} {}^{R}\!\big/\!\left(p^{(\alpha_M)_k}\right) \\
\big\downarrow \left(\begin{smallmatrix} \overline{1} & 0 \\ \overline{p} & 0 \\ 0 & I_{d-2} \end{smallmatrix}\right) \\
{}^{R}\!\big/\!\left(p^{(\alpha_M)_{i'}}\right) \oplus {}^{R}\!\big/\!\left(p^{(\alpha_M)_{j'}}\right) \oplus \bigoplus_{k \in \mathbb{N}_d \setminus \{i', j'\}} {}^{R}\!\big/\!\left(p^{(\alpha_M)_k}\right) \\
\big\downarrow \left(\begin{smallmatrix} -\overline{p} & \overline{1} & 0 \end{smallmatrix}\right) \\
{}^{R}\!\big/\!\left(p^{(\alpha_M)_{i'}+1}\right) \\
\big\downarrow \\
0
\end{array}$$

is short exact

$$\Rightarrow {}^{R}\!/\!\left(p^{(\alpha_M)_{i'}}\right) \oplus {}^{R}\!/\!\left(p^{(\alpha_M)_{j'}}\right) \oplus \bigoplus_{k \in \mathbb{N}_d \setminus \{i',j'\}} {}^{R}\!/\!\left(p^{(\alpha_M)_k}\right)$$

$$\leq_{\deg} {}^{R}\!/\!\left(p^{(\alpha_M)_{i'}+1}\right) \oplus {}^{R}\!/\!\left(p^{(\alpha_M)_{j'}-1}\right) \oplus \bigoplus_{k \in \mathbb{N}_d \setminus \{i',j'\}} {}^{R}\!/\!\left(p^{(\alpha_M)_k}\right)$$

by Theorem 1.6. Since

$$ {}^{R}\!/\!\left(p^{(\alpha_M)_{i'}}\right) \oplus {}^{R}\!/\!\left(p^{(\alpha_M)_{j'}}\right) \oplus \bigoplus_{k \in \mathbb{N}_d \setminus \{i',j'\}} {}^{R}\!/\!\left(p^{(\alpha_M)_k}\right)$$

$$\cong \bigoplus_{k \in \mathbb{N}_d} {}^{R}\!/\!\left(p^{(\alpha_M)_k}\right) \cong M$$

and

$$ {}^{R}\!/\!\left(p^{(\alpha_M)_{i'}}\right) \oplus {}^{R}\!/\!\left(p^{(\alpha_M)_{j'}}\right) \oplus \bigoplus_{k \in \mathbb{N}_d \setminus \{i',j'\}} {}^{R}\!/\!\left(p^{(\alpha_M)_k}\right) \cong N,$$

then $M \leq_{\deg} N$. □

Theorem 2.4. Let $M, N \in \mathcal{M}_d(p)$. Then $M \leq_{\deg} N \Leftrightarrow \alpha'_M \leq_{\mathrm{dom}} \alpha'_N$.

Proof. Assume $M, N \in \mathcal{M}_d(p)$.

$(\Rightarrow)$

$$M \leq_{\deg} N$$
$$\Rightarrow M \leq_{\mathrm{hom}} N$$
$$\Leftrightarrow l\,\mathrm{Hom}\,(X, M) \leq l\,\mathrm{Hom}\,(X, N) \text{ for all } R\text{-modules } X \text{ of finite length}$$
$$\Rightarrow l\,\mathrm{Hom}\left({}^{R}\!/\!(p^i), M\right) \leq l\,\mathrm{Hom}\left({}^{R}\!/\!(p^i), N\right) \ \forall i \in \mathbb{N}_d$$
$$\overset{\text{Lemma 2.2}}{\Leftrightarrow} \sum_{k=1}^{i} (\alpha'_M)_k \leq \sum_{k=1}^{i} (\alpha'_N)_k \ \forall i \in \mathbb{N}_d$$
$$\Leftrightarrow \alpha'_M \leq_{\mathrm{dom}} \alpha'_N.$$

Thus $M \leq_{\deg} N \Rightarrow \alpha'_M \leq_{\mathrm{dom}} \alpha'_N$.

$(\Leftarrow)$ Suppose $\alpha'_M \leq_{\mathrm{dom}} \alpha'_N$. There are only finitely many isomorphism classes of objects in $\mathcal{M}_d(p)$, so there exist $M_0, M_1, \ldots, M_n \in \mathcal{M}_d(p)$ for some $n \in \mathbb{N}$ such that $\alpha'_M = \alpha'_{M_0} \leq_{\mathrm{dom}} \alpha'_{M_1} \leq_{\mathrm{dom}} \cdots \leq_{\mathrm{dom}} \alpha'_{M_n} = \alpha'_N$ where $\alpha'_{M_{i-1}} \leq_{\mathrm{dom}} \alpha'_{M_i}$ minimally for every $i \in \mathbb{N}_n$. By Lemma 2.3 we have that $M \cong M_0 \leq_{\deg} M_1 \leq_{\deg} \cdots \leq_{\deg} M_n \cong N \Rightarrow M \leq_{\deg} N$.

□

3 The Dual of the Transpose and Coxeter Functors

In this last section we will discuss a few functors on categories of representations and modules.

3.1 Coxeter Functors

The definitions and most of the calculations in this section are from [2]. The example at the end is an exception.

Let Γ be a finite connected graph. An orientation σ on Γ gives every edge a direction. This means an orientation consist of two functions $s_\sigma, t_\sigma : \Gamma_1 \to \Gamma_0$, where we say that $s_\sigma(\alpha)$ is the starting point and that $e_\sigma(\alpha)$ is the end point of $\alpha \in \Gamma_1$. It is implied here that α already connects $s_\sigma(\alpha)$ and $e_\sigma(\alpha)$ regardless of orientation.

This definition of graphs with orientations is equivalent with the definition of quivers. The reason for introducing this new notion is because this section will discuss changing the orientation on a graph.

Let i be any vertex in Γ. We say that i is $(-)$-accessible if $e_\sigma(\alpha) \neq i$ $\forall \alpha \in \Gamma_1$ and we say that i is $(+)$-accessible if $s_\sigma(\alpha) \neq i$ $\forall \alpha \in \Gamma_1$.

Denote by Γ^i the subset of Γ_1 which consists of the edges α such that $s_\sigma(\alpha) = i$ or $e_\sigma(\alpha) = i$. Let $\kappa_i\sigma$ denote the orientation that reverses the direction of the edges in Γ^i and leaves all other edges with the same direction they had under σ.

Let k be a field and (V, f) an object in the category $\mathrm{rep}(\Gamma, \sigma)$ of representations over k.

1. Suppose i is a $(+)$-accessible vertex in Γ. Define a representation (W, g) of $(\Gamma, \kappa_i\sigma)$ where $W(j) = V(j)$ for all $j \neq i$ in Γ_0 and if we write $\Gamma^i = \{\alpha_1, \ldots, \alpha_m\}$, then $W(i)$ is the kernel of the map from $\bigoplus_{t=1}^m V(s_\sigma(\alpha_t))$ to $V(i)$ which is given as the matrix $\begin{pmatrix} f_{\alpha_1} & \cdots & f_{\alpha_m} \end{pmatrix}$.

$$W(i) = \ker \left(\bigoplus_{t=1}^m V(s_\sigma(\alpha_t)) \xrightarrow{(\,f_{\alpha_1} \,\cdots\, f_{\alpha_m}\,)} V(i) \right).$$

   The maps are defined such that $g_\alpha = f_\alpha$ for $\alpha \notin \Gamma^i$ and for each $\alpha_t \in \Gamma^i$ we define g_{α_t} as the natural inclusion from $W(i)$ into $\bigoplus_{t=1}^m V(s_\sigma(\alpha_t))$

composed with the projection from this direct sum onto $V(s_\sigma(\alpha_t)) = W(s_\sigma(\alpha_t))$. We then have that

$$\begin{pmatrix} f_{\alpha_1} & \cdots & f_{\alpha_m} \end{pmatrix} \begin{pmatrix} g_{\alpha_1} \\ \vdots \\ g_{\alpha_m} \end{pmatrix} = 0.$$

We refer to the representation (W, g) by the notation $C_i^+(V, f)$.

2. Suppose i is a $(-)$-accessible vertex in Γ. Define a representation $C_i^-(V, f) = (W, g)$ of $(\Gamma, \kappa_i\sigma)$ where $W(j) = V(j)$ for all $j \neq i$ in Γ_0 and $g_\alpha = f_\alpha \; \forall \alpha \notin \Gamma^i$. We define $W(i)$ to be the cokernel of the map from $V(i)$ to $\bigoplus_{t=1}^m V(e_\sigma(\alpha_t))$ which is given by the matrix $\begin{pmatrix} f_{\alpha_1} \\ \vdots \\ f_{\alpha_m} \end{pmatrix}$.

$$W(i) = \mathrm{Coker}\left(V(i) \xrightarrow{\begin{pmatrix} f_{\alpha_1} \\ \vdots \\ f_{\alpha_m} \end{pmatrix}} \bigoplus_{t=1}^m V(e_\sigma(\alpha_t)) \right).$$

For $\alpha \in \Gamma^i$ we define the map $g_\alpha : W(e_{\sigma(\alpha)}) \to W(i)$ as the composition of the natural inclusion from $W(e_\sigma(\alpha)) = V(e_\sigma(\alpha))$ into $\bigoplus_{t=1}^m V(e_\sigma(\alpha_t))$ and the projection from this direct sum onto $W(i)$. Then

$$\begin{pmatrix} g_{\alpha_1} & \cdots & g_{\alpha_m} \end{pmatrix} \begin{pmatrix} f_{\alpha_1} \\ \vdots \\ f_{\alpha_m} \end{pmatrix} = 0.$$

For a representation Λ of (Γ, σ), where we write $\Lambda(i)$ for the vector space at vertex i and Λ_α for the linear transformation at edge α, and a $(+)$-accessible vertex i, we have that $C_i^- C_i^+ \Lambda$ has $C_i^- C_i^+ \Lambda(j) = \Lambda(j)$ for $j \neq i$ and

$$C_i^- C_i^+ \Lambda(i) = \bigoplus_{t \in 1}^m \Lambda(s_\sigma(\alpha_t)) / C_i^+ \Lambda(i) \cong \mathrm{Im}\begin{pmatrix} \Lambda_{\alpha_1} & \cdots & \Lambda_{\alpha_m} \end{pmatrix},$$

where $(\alpha_1, \ldots, \alpha_m) = \Gamma^i$. $C_i^- C_i^+ \Lambda(i)$ is then 0 if $\Lambda_\alpha = 0$ for each $\alpha \in \Gamma^i$. This means that if Λ is a representation where $\Lambda(j) = 0$ for $j \neq i$, then $C_i^+ C_i^- \Lambda(i) = 0$, so $C_i^+ C_i^- \Lambda$ is the zero representation.

If i is instead a $(-)$-accessible vertex, then we have $C_i^+C_i^-\Lambda(j) = \Lambda(j)$ for $j \neq i$ and

$$C_i^+C_i^-\Lambda(i) = \ker\left(C^-\Lambda_{\alpha_1} \ \cdots \ C^-\Lambda_{\alpha_m} \right) = {}^{\Lambda(i)}\big/_{\bigcap_{t=1}^m \ker(\Lambda_{\alpha_t})}.$$

Let Λ be a representation such that $\Lambda(j) = 0$ for $j \neq i$. Then $\Lambda_\alpha = 0 \Rightarrow \ker(\Lambda_\alpha) = \Lambda(i)$ for each $\alpha \in \Gamma^i \Rightarrow \Lambda(i) = \bigcap_{t=1}^m \ker(\Lambda_{\alpha_t}) \Rightarrow C_i^-C_i^+\Lambda(i) = 0 \Rightarrow C_i^-C_i^+\Lambda = 0$.

There is a natural way to construct functors from C^- and C^+. Suppose a vertex i is $(-)$-accessible and let $h : V \to W$ be a homomorphism between two representations V and W of (Γ, σ). We define $C_i^-(h)$ such that $C_i^-h(j) = h(j)$ for $j \neq i$ and $C_i^-h(i)$ is the unique linear transformation which makes the following diagram commute.

$$\begin{tikzcd}
V(i) \arrow[r, "\begin{pmatrix} V_{\alpha_1} \\ \vdots \\ V_{\alpha_m} \end{pmatrix}"] \arrow[d, "h(i)"] & \bigoplus_{t=1}^m V(e_\sigma\alpha_t) \arrow[r, "\left((C_i^-V)_{\alpha_1} \ \cdots \ (C_i^-V)_{\alpha_m} \right)"] \arrow[d, "\begin{pmatrix} h(e_\sigma\alpha_1) & 0 & \cdots & 0 \\ 0 & h(e_\sigma\alpha_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h(e_\sigma\alpha_m) \end{pmatrix}"] & C_i^-V(i) \arrow[d, "C_i^-h(i)"] \\
W(i) \arrow[r, "\begin{pmatrix} W_{\alpha_1} \\ \vdots \\ W_{\alpha_m} \end{pmatrix}"'] & \bigoplus_{t=1}^m W(e_\sigma\alpha_t) \arrow[r, "\left((C_i^-W)_{\alpha_1} \ \cdots \ (C_i^-W)_{\alpha_m} \right)"'] & C_i^-W(i)
\end{tikzcd}$$

For the identity homomorphism $\mathrm{id}_V : V \to V$ we get that $C_i^-\mathrm{id}_V(i) = \mathrm{id}_{C_i^-V(i)}$ and if we have another homomorphism $h' : W \to U$ between W and a representation U, then $C_i^-(h' \circ h)(i) = (C_i^-h')(i) \circ (C_i^-h)(i)$ since $(h' \circ h)(i) = h'(i) \circ h(i)$ and

$$\begin{pmatrix} h'(e_\sigma\alpha_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & h'(e_\sigma\alpha_m) \end{pmatrix} \begin{pmatrix} h(e_\sigma\alpha_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & h(e_\sigma\alpha_m) \end{pmatrix}$$

$$= \begin{pmatrix} h'(e_\sigma\alpha_1) \circ h(e_\sigma\alpha_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & h'(e_\sigma\alpha_m) \circ h(e_\sigma\alpha_m) \end{pmatrix}$$

$$= \begin{pmatrix} (h' \circ h)(e_\sigma \alpha_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (h' \circ h)(e_\sigma \alpha_m) \end{pmatrix}.$$

Then $C_i^- : \mathrm{rep}(\Gamma, \sigma) \to \mathrm{rep}(\Gamma, \kappa_i \sigma)$ is a functor.

If i is instead $(+)$-accessible, then we define $C_i^+ h(j) = h(j)$ for $j \neq i$ and $C_i^+ h(i)$ to be the unique linear transformation which makes the diagram

$$\begin{tikzcd}
C_i^+ V(i) \arrow[r, "\begin{pmatrix} (C_i^+ V)_{\alpha_1} \\ \vdots \\ (C_i^+ V)_{\alpha_m} \end{pmatrix}"] \arrow[d, "C_i^+ h(i)"] & \bigoplus_{t=1}^m V(s_\sigma \alpha_t) \arrow[r, "\begin{pmatrix} V_{\alpha_1} & \cdots & V_{\alpha_m} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} h(s_\sigma \alpha_1) & 0 & \cdots & 0 \\ 0 & h(s_\sigma \alpha_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h(s_\sigma \alpha_m) \end{pmatrix}"] & V(i) \arrow[d, "h(i)"] \\
C_i^+ W(i) \arrow[r, swap, "\begin{pmatrix} (C_i^+ W)_{\alpha_1} \\ \vdots \\ (C_i^+ W)_{\alpha_m} \end{pmatrix}"] & \bigoplus_{t=1}^m W(s_\sigma \alpha_t) \arrow[r, swap, "\begin{pmatrix} W_{\alpha_1} & \cdots & W_{\alpha_m} \end{pmatrix}"] & W(i)
\end{tikzcd}$$

commute. Then $C_i^+ : \mathrm{rep}(\Gamma, \sigma) \to \mathrm{rep}(\Gamma, \kappa_i \sigma)$ is functor.

Suppose (Γ, σ) has no oriented cycles. Then we can identify each vertex in Γ with a natural number such that $\Gamma_0 = \mathbb{N}_n$ for some $n \in \mathbb{N}$ and $s_\sigma(\alpha) < e_\sigma(\alpha)\ \forall \alpha \in \Gamma_1$. Any vertex $i \in \mathbb{N}_n$ is $(-)$-accessible under the orientation $\kappa_{i-1} \cdots \kappa_2 \kappa_1 \sigma$ and $(+)$-accessible under $\kappa_{i+1} \kappa_{i+2} \cdots \kappa_n \sigma$.

Definition 3.1. Let (Γ, σ) be a oriented graph with no oriented cycles and n vertices. Number the vertices so that $s_\sigma(\alpha) < e_\sigma(\alpha)\ \forall \alpha \in \Gamma_1$. We define $\mathrm{Cox}^+ = C_1^+ C_2^+ \cdots C_n^+$ and $\mathrm{Cox}^- = C_n^- \cdots C_2^- C_1^-$. Δ

We note that if (V, f) is a representation of (Γ, σ), then $\mathrm{Cox}^+(V, f)$ and $\mathrm{Cox}^-(V, f)$ are also representations of this graph with the original orientation σ. This is because the direction of every edge $\alpha \in \Gamma_1$ is reversed exactly twice in the compositions $C_1^+ C_2^+ \cdots C_n^+$ and $C_n^- \cdots C_2^- C_1^-$, once by $C_{s_\sigma(\alpha)}^+$ and $C_{s_\sigma(\alpha)}^-$, and the other time by $C_{e_\sigma(\alpha)}^+$ and $C_{e_\sigma(\alpha)}^-$, respectively.

Another important observation is that Cox^+ and Cox^- do not depend on the specific numbering of the vertices.

Since Cox^+ and Cox^- are compositions of fuctors, then Cox^+ and Cox^- are also functors.

Example 3.1. Let (Γ, σ) be the oriented graph $\begin{tikzcd} & 1 \arrow[dl] \arrow[dd] \\ 2 \arrow[dr] & \\ & 3 \end{tikzcd}$. Let k be a field. Consider representations with dimension vector (n, n, n). Suppose

$$\Lambda = \begin{tikzcd} & k^n \arrow[dl, "I_n"'] \arrow[dd, "M"] \\ k^n \arrow[dr, "I_n"'] & \\ & k^n \end{tikzcd}$$

where M is a $n \times n$ matrix. Then

$$C_3^+(\Lambda) = \begin{tikzcd} & k^n \arrow[dl, "I_n"'] \\ k^n & \\ & k^n \arrow[ul, "-M"] \arrow[uu, "I_n"'] \end{tikzcd}, C_2^+C_3^+(\Lambda) = \begin{tikzcd} & k^n \\ k^n \arrow[ur, "M"] \arrow[dr, "I_n"'] & \\ & k^n \arrow[uu, "I_n"'] \end{tikzcd}, \mathrm{Cox}^+(\Lambda) = \begin{tikzcd} & k^n \arrow[dl, "I_n"'] \arrow[dd, "-M"] \\ k^n \arrow[dr, "I_n"'] & \\ & k^n \end{tikzcd}.$$

If

$$\Lambda = \begin{tikzcd} & k^n \arrow[dl, "I_n"'] \arrow[dd, "I_n"] \\ k^n \arrow[dr, "M"'] & \\ & k^n \end{tikzcd},$$

then

$$C_3^+(\Lambda) = \begin{tikzcd} & k^n \arrow[dl, "I_n"'] \\ k^n & \\ & k^n \arrow[ul, "I_n"] \arrow[uu, "-M"'] \end{tikzcd}, C_2^+C_3^+(\Lambda) = \begin{tikzcd} & k^n \\ k^n \arrow[ur, "-I_n"] \arrow[dr, "I_n"'] & \\ & k^n \arrow[uu, "-M"'] \end{tikzcd}, \mathrm{Cox}^+(\Lambda) = \begin{tikzcd} & k^n \arrow[dl, "-M"'] \arrow[dd, "I_n"] \\ k^n \arrow[dr, "I_n"'] & \\ & k^n \end{tikzcd}.$$

If

$$\Lambda = \begin{tikzcd} & k^n \arrow[dl, "M"'] \arrow[dd, "I_n"] \\ k^n \arrow[dr, "I_n"'] & \\ & k^n \end{tikzcd},$$

then

$$C_3^+(\Lambda) = \begin{tikzcd} & k^n \arrow[dl, "M"'] \\ k^n & \\ & k^n \arrow[ul, "I_n"] \arrow[uu, "-I_n"'] \end{tikzcd}, C_2^+C_3^+(\Lambda) = \begin{tikzcd} & k^n \\ k^n \arrow[ur, "I_n"] \arrow[dr, "-M"'] & \\ & k^n \arrow[uu, "-I_n"'] \end{tikzcd}, \mathrm{Cox}^+(\Lambda) = \begin{tikzcd} & k^n \arrow[dl, "I_n"'] \arrow[dd, "I_n"] \\ k^n \arrow[dr, "-M"'] & \\ & k^n \end{tikzcd}.$$

For any matrix we have a homomorphism ϕ given by the following diagram.

$$\begin{tikzcd} & k^n \arrow[dl, "A"'] \arrow[dd, "I_n"] \arrow[rr, bend left, "\phi(1)"] & & k^n \arrow[dl, "I_n"'] \arrow[dd, "I_n"] \\ k^n \arrow[dr, "I_n"'] \arrow[rr, bend left, "\phi(2)"] & & k^n \arrow[dr, "A"'] & \\ & k^n \arrow[rr, bend right, "\phi(3)"'] & & k^n \end{tikzcd}$$

Then

$$\begin{aligned} I_n\phi(1) = \phi(2)A, I_n\phi(1) = \phi(3)I_n, A\phi(2) = \phi(3)I_n \\ \Rightarrow \phi(2)A = \phi(1) = \phi(3) = A\phi(2). \end{aligned}$$

This means that $\phi = (AB, B, AB)$ where B is a $n \times n$ matrix such that $AB = BA$, so $\mathrm{Hom}\left(\begin{tikzcd} & k^n \arrow[dl, "I_n"'] \arrow[dd, "I_n"] \\ k^n \arrow[dr, "A"'] & \\ & k^n \end{tikzcd}, \begin{tikzcd} & k^n \arrow[dl, "A"'] \arrow[dd, "I_n"] \\ k^n \arrow[dr, "I_n"'] & \\ & k^n \end{tikzcd} \right) \cong \{B \in M_n(k) \mid AB = BA\}$. If we choose B to be invertible, $B = I_n$ for example, then ϕ is an isomorphism if and only if A is invertible as well. We can in similar fashion show that $\mathrm{Hom}\left(\begin{tikzcd} & k^n \arrow[dl, "A"'] \arrow[dd, "I_n"] \\ k^n \arrow[dr, "I_n"'] & \\ & k^n \end{tikzcd}, \begin{tikzcd} & k^n \arrow[dl, "I_n"'] \arrow[dd, "I_n"] \\ k^n \arrow[dr, "A"'] & \\ & k^n \end{tikzcd} \right) \cong \{B \in M_n(k) \mid AB = BA\}$. Suppose A

is invertible. The map ψ given by the following diagram.

$$
\begin{tikzcd}
 & k^n \arrow[dl, "A"'] \arrow[dd, "I_n"] \arrow[rr, bend left, "\psi(1)"] & & k^n \arrow[dl, "I_n"'] \arrow[dd, "A^{-1}"] \\
k^n \arrow[dr, "I_n"'] \arrow[rr, bend left, "\psi(2)"] & & k^n \arrow[dr, "I_n"'] & \\
 & k^n \arrow[rr, bend right, "\psi(3)"'] & & k^n
\end{tikzcd}
$$

This means that

$$I_n\psi(1) = \psi(2)A, A^{-1}\psi(1) = \psi(3)I_n, I_n\psi(2) = \psi(3)I_n$$
$$\Rightarrow \psi(1)A^{-1} = \psi(2) = \psi(3) = A^{-1}\psi(1).$$

Then $\psi = (B, A^{-1}B, A^{-1}B)$ for some $n \times n$ matrix B which commutes with A. If B is invertible, then ψ is an isomorphism. Assuming M is invertible

$$
\begin{tikzcd}
 & k^n \arrow[dl, "M"'] \arrow[dd, "I_n"] \\
k^n \arrow[dr, "I_n"'] & \\
 & k^n
\end{tikzcd}
\cong
\begin{tikzcd}
 & k^n \arrow[dl, "I_n"'] \arrow[dd, "I_n"] \\
k^n \arrow[dr, "M"'] & \\
 & k^n
\end{tikzcd}
\cong
\begin{tikzcd}
 & k^n \arrow[dl, "I_n"'] \arrow[dd, "M^{-1}"] \\
k^n \arrow[dr, "I_n"'] & \\
 & k^n
\end{tikzcd}
$$

and

$$
\mathrm{Cox}^+\left(
\begin{tikzcd}
 & k^n \arrow[dl, "M"'] \arrow[dd, "I_n"] \\
k^n \arrow[dr, "I_n"'] & \\
 & k^n
\end{tikzcd}
\right)
\cong \mathrm{Cox}^+\left(
\begin{tikzcd}
 & k^n \arrow[dl, "I_n"'] \arrow[dd, "I_n"] \\
k^n \arrow[dr, "M"'] & \\
 & k^n
\end{tikzcd}
\right)
\cong \mathrm{Cox}^+\left(
\begin{tikzcd}
 & k^n \arrow[dl, "I_n"'] \arrow[dd, "M^{-1}"] \\
k^n \arrow[dr, "I_n"'] & \\
 & k^n
\end{tikzcd}.
\right)
$$

Let $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 = \sum_{i=0}^{n} a_i X^i$ with $a_n = 1$ be an irreducible polynomial in $k[X]$ with coefficients $a_0, \ldots, a_{n-1}$ in k. We can choose M to be its companion matrix

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & -a_0 \\
1 & 0 & \cdots & 0 & -a_1 \\
0 & 1 & \cdots & 0 & -a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -a_{n-1}
\end{pmatrix}.
$$

Then

$$M^{-1} = \begin{pmatrix} -a_1a_0^{-1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-2}a_0^{-1} & 0 & \cdots & 1 & 0 \\ -a_{n-1}a_0^{-1} & 0 & \cdots & 0 & 1 \\ -a_0^{-1} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

which has characteristic polynomial

$$\det \begin{pmatrix} X + a_1a_0^{-1} & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2}a_0^{-1} & 0 & \cdots & -1 & 0 \\ a_{n-1}a_0^{-1} & 0 & \cdots & X & -1 \\ a_0^{-1} & 0 & \cdots & 0 & X \end{pmatrix}$$

$$= X^n + a_2a_0^{-1}X^{n-1} + a_3a_0^{-1}X^{n-2} + \cdots + a_{n-1}a_0^{-1}X + a_0^{-1}$$

$$= a_0^{-1}\sum_{i=0}^{n} a_{n-i}X^i =: f^*(X).$$

We have that f^* is irreducible. To prove this, suppose $\lambda \in k$ is a root of f^*. Since $\lambda \neq 0$, then

$$f(\lambda^{-1}) = \sum_{i=0}^{n} a_i(\lambda^{-1})^i = \sum_{i=0}^{n} a_{n-i}\lambda^{-i} = (\lambda^{-1})^n \sum_{i=0}^{n} a_{n-i}\lambda^{n-i}$$

$$= a_0\lambda^{-n}f^*(\lambda) = a_0\lambda^{-n}\cdot 0 = 0.,$$

which seems to imply that λ^{-1} is a root of f, but that contradicts the assumption that f is irreducible in $k[X]$. Then f^* is irreducible.

Now suppose $k = \mathrm{GF}(q)$ where $q \in \mathbb{N}$ is a prime power and suppose $f \in \mathrm{GF}(q)[X]$ is primitive with respect to the field extension $\mathrm{GF}(q^m)$, which means there exists a root $\omega \in \mathrm{GF}(q^m)$ of f which multiplicatively generates $U\,\mathrm{GF}(q^m) = \mathrm{GF}(q^m) \setminus \{0\}$ and that f is the minimal polynomial of ω. First off, we know that ω^{-1} is a root of f^*, and if $(\omega^{-1})^r = 1$, then $\omega^r = \omega^r(\omega^{-1})^r = 1$, so ω^{-1} generates $U\,\mathrm{GF}(q^m)$. Secondly, suppose $\sum_{i=0}^{n'} b_iX^i$ with $n' < n$ is an irreducible polynomial in $k[X]$ with ω^{-1} as a root. Then ω is a root of $b_0^{-1}\sum_{i=0}^{n'} b_{n-1}X^i$, which is irreducible in $k[X]$, but that contradicts the assumption that f is the minimal polynomial of ω. Thus f^* is primitive.

On another note, we have that

$$-M = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & a_{n-1} \end{pmatrix},$$

which has characteristic polynomial

$$\det \begin{pmatrix} X & 0 & \cdots & 0 & -a_0 \\ 1 & X & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & X - a_{n-1} \end{pmatrix}$$

$$= X^n - a_{n-1}X^{n-1} + a_{n-2}X^{n-2} - \cdots + (-1)^n a_0 = \sum_{i=0}^{n} (-1)^{n-i} a_i X^i$$

where $a_n = 1$. We can show that this polynomial is irreducible. Suppose $\tilde{f}(X) = \sum_{i=0}^{n} (-1)^{n-i} a_i X^i$ has a root $\lambda \in k$. If n is even, then

$$f(-\lambda) = (-\lambda)^n + a_{n-1}(-\lambda)^{n-1} + \cdots + a_1(-\lambda) + a_0$$

$$= \lambda^n - a_{n-1}\lambda^{n-1} + \cdots - a_1\lambda + a_0 = \tilde{f}(\lambda) = 0.$$

If n is odd, then

$$f(-\lambda) = -\lambda^n + a_{n-1}\lambda^{n-1} - \cdots - a_1\lambda + a_0$$

$$= -(\lambda^n - a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda - a_0) = -\tilde{f}(\lambda) = 0.$$

In either case we have that $-\lambda$ is a root of f, but this contradicts the assumption that f is irreducible. Then $\tilde{f}$ has to be irreducible in $k[X]$. Suppose again that $f \in \mathrm{GF}(q)[X]$ is a primitive polynomial with respect to the field extension $\mathrm{GF}(q^m)$. This means there is some $\omega \in \mathrm{GF}(q^m)$ which generates $U\,\mathrm{GF}(q^m) = \mathrm{GF}(q^m) \setminus \{0\}$ multiplicatively and has f as its minimal polynomial. We can assert that $\tilde{f}$ is the minimal polynomial of $-\omega$ because if some other irreducible polynomial $x^{n'} + b_{n'-1}x^{n'-1} + \cdots + b_0$ in $k[X]$ with degree $n' < n$ had $-\omega$ as its root, then ω would be a root of

$x^{n'} - b_{n'-1}x^{n'-1} + \cdots + (-1)^{n'}b_0$, which would imply f is not primitive. Since $-\omega$ is a root of $\tilde{f}$, then $\tilde{f}$ is primitive if $-\omega$ generates $U\,\mathrm{GF}(q^m)$. This is obviously the case whenever $\operatorname{char} k = 2$, as $-\omega = \omega$ when $\operatorname{char} k = 2$. In general, if the order of $-\overline{X}$ is $q^m - 1$ in ${}^{k[X]}/_{f(X)}$, then $\tilde{f}$ is primitive.

For example, choose $k = \mathrm{GF}(3)$ and $n = 2$. We first check that X^2+2X+2 is primitive. For $\overline{X} = X + (X^2 + 2X + 2)$ in ${}^{\mathrm{GF}(3)[X]}/_{(X^2+2X+2)}$, we have that

$$\overline{X}^2 = \overline{X} + \overline{1} \Rightarrow \overline{X}^4 = \overline{X}^2 + 2\overline{X} + \overline{1} = -\overline{1} \Rightarrow \overline{X}^8 = \overline{1}$$

$\Rightarrow X^2 + 2X + 2$ is primitive. For $-\overline{X}$ we have

$$(-\overline{X})^2 = \overline{X} + \overline{1} \Rightarrow (-\overline{X})^4 = -\overline{1} \Rightarrow (-\overline{X})^8 = \overline{1}$$

$\Rightarrow X^2 + X + 2$ is primitive.

We can actually be sure that if we assume q is odd and $q^m \equiv 1 \pmod 4$, then $-\omega$ also generates $U\,\mathrm{GF}(q^m)$. This is because $\frac{q^m-1}{2}$ is even in that case, which implies $(-\omega)^{\frac{q^m-1}{2}} = \omega^{\frac{q^m-1}{2}} = -1$ since $(-\omega)^r = (-1)^r\omega^r\ \forall r \in \mathbb{N}$, so the order of $-\omega$ has to be $q^m - 1$. This means $-\omega$ generates $U\,\mathrm{GF}(q^m) \Rightarrow \tilde{f}$ is primitive.

If q is odd and $q^m \not\equiv 1 \pmod 4$, then $q^m \equiv 3 \pmod 4 \Rightarrow \frac{q^m-1}{2}$ is odd $\Rightarrow (-\omega)^{\frac{q^m-1}{2}} = -\omega^{\frac{q^m-1}{2}} = -(-1) = 1 \Rightarrow$ the order of $-\omega$ is $\frac{q^m-1}{2} < q^m - 1$ $\Rightarrow -\omega$ does not generate $U\,\mathrm{GF}(q^m)$. Since this is the case for all roots of $\tilde{f}$, then $\tilde{f}$ is not primitive.

We can get even more specific about this. If $x \equiv 1 \pmod 4$, then $x^r \equiv 1 \pmod 4$ for all $r \in \mathbb{N}$. If $x \equiv 3 \pmod 4$, then $x^r \equiv 3 \pmod 4$ when r is odd and $x \equiv 1 \pmod 4$ when r is even. Thus $\tilde{f}$ is primitive if and only if one of the following criteria are satisfied:

1. $\operatorname{char} \mathrm{GF}(q) = 2$,
2. $q \equiv 1 \pmod 4$,
3. $q \equiv 3 \pmod 4$ and m is even.

If we identify the isomorphism class represented by the representation

$$\begin{tikzcd} & k^n \arrow[dl, "I_n"'] \arrow[dd, "M"] \\ k^n \arrow[dr, "I_n"'] & \\ & k^n \end{tikzcd}$$

with the characteristic polynomial of M, then for this type of representation we can view Cox^+ as a map on the set of monic irreducible polynomials of degree n.

Suppose

$$M = J_n(\lambda) := \begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & \cdots & 1 & \lambda \end{pmatrix}$$

for some $\lambda \in k$, which is a matrix on Jordan Canonical form. Then

$$-M = \begin{pmatrix} -\lambda & 0 & \cdots & 0 & 0 \\ -1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & \cdots & -1 & -\lambda \end{pmatrix}$$

and the Jordan Canonical form of this matrix is

$$J_n(-\lambda) = \begin{pmatrix} -\lambda & 0 & \cdots & 0 & 0 \\ 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & \cdots & 1 & -\lambda \end{pmatrix}$$

since $-M$ has one eigenvalue $-\lambda$ with multiplicity n as its characteristic polynomial is $(X-\lambda)^n$. If $\lambda \neq 0$, then

$$M^{-1} = \begin{pmatrix} \lambda^{-1} & 0 & \cdots & 0 & 0 \\ -1 & \lambda^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda^{-1} & 0 \\ 0 & 0 & \cdots & -1 & \lambda^{-1} \end{pmatrix},$$

whose Jordan Canonical matrix is

$$J_n(\lambda^{-1}) = \begin{pmatrix} \lambda^{-1} & 0 & \cdots & 0 & 0 \\ 1 & \lambda^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda^{-1} & 0 \\ 0 & 0 & \cdots & 1 & \lambda^{-1} \end{pmatrix}.$$

Then we have that

$$\begin{tikzcd}[row sep=small] & k^n \arrow[dl, "I_n"'] \arrow[dd, "J_n(\lambda)"] \\ k^n \arrow[dr, "I_n"'] & \\ & k^n \end{tikzcd} \cong \begin{tikzcd}[row sep=small] & k^n \arrow[dl, "J_n(\lambda^{-1})"'] \arrow[dd, "I_n"] \\ k^n \arrow[dr, "I_n"'] & \\ & k^n \end{tikzcd} \cong \begin{tikzcd}[row sep=small] & k^n \arrow[dl, "I_n"'] \arrow[dd, "I_n"] \\ k^n \arrow[dr, "J_n(\lambda^{-1})"'] & \\ & k^n \end{tikzcd}.$$

If we identify the representation $\begin{tikzcd}[row sep=small] & k^n \arrow[dl, "I_n"'] \arrow[dd, "J_n(\lambda)"] \\ k^n \arrow[dr, "I_n"'] & \\ & k^n \end{tikzcd}$ with its one eigenvalue λ, for each $\lambda \in k$, $\begin{tikzcd}[row sep=small] & k^n \arrow[dl, "J_n(0)"'] \arrow[dd, "I_n"] \\ k^n \arrow[dr, "I_n"'] & \\ & k^n \end{tikzcd}$ with a symbol ∞_1 and $\begin{tikzcd}[row sep=small] & k^n \arrow[dl, "I_n"'] \arrow[dd, "I_n"] \\ k^n \arrow[dr, "J_n(0)"'] & \\ & k^n \end{tikzcd}$ with another symbol ∞_2, then Cox^+ on this type of representation can be seen as the bijection on $k \cup \{\infty_1, \infty_2\}$ which sends $\lambda \in k$ to $-\lambda$, ∞_1 to ∞_2 and ∞_2 to ∞_1. $\triangle$

3.2 The Dual of the Transpose

This section is based on II.3, II.4 and IV.1 in [1]. We define the dual and the transpose, then we give some properties of the dual and the dual of the transpose. The transpose does not always describe a functor, so we give some criteria for whenthis is the case. Lastly we consider an example where we compare the dual of the transpose and Cox^+ on some representations of quivers.

We begin by describing the dual. Let R be a commutative artin ring. Then R has only finitely many isomorphism classes of simple submodules $S_1, S_2, \ldots, S_n$. Let $I(S_i)$ be the injective envelope of S_i for each i and let $J = \bigoplus_{i=1}^n I(S_i)$, which is the injective envelope of $\bigoplus_{i=1}^n S_i$. We have that the functor $D : \operatorname{mod} R \to \operatorname{mod} R$ such that $D = \operatorname{Hom}_{\operatorname{mod} R}(\ , J)$ is a duality which induces a duality $D : \operatorname{mod} \Lambda \to \operatorname{mod}(\Lambda^{\mathrm{op}})$ where Λ is an artin R-algebra.

Let C be in $\operatorname{mod} \Lambda$ and let $P_1 \xrightarrow{f} P_0 \longrightarrow C \longrightarrow 0$ be a minimal projective resolution of C. Then $C \cong \operatorname{Coker} f$. Applying the duality $(\ )^* = \operatorname{Hom}_{\operatorname{mod}\Lambda}(\ , \Lambda)$ on f gives the morphism $f^* : P_0^* \to P_1^*$. We define the transpose of C as $\operatorname{Tr} C = \operatorname{Coker} f^*$. This transformation does not induce a duality $\operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{\mathrm{op}}$, and there is in general not even a functor which maps C to $\operatorname{Tr} C$. It is often still useful to consider this map.

Let C be an object in $\operatorname{mod}\Lambda$. Then we have a decomposition $C \cong C_{\mathscr{P}} \oplus C'$, which is unique up to isomorphism, where $C_{\mathscr{P}}$ has no projective summands and C' is projective. Let $\operatorname{mod}_{\mathscr{P}}$ denote the subcategory of $\operatorname{mod}\Lambda$ every object C satisfies $C = C_{\mathscr{P}}$ and $\operatorname{Hom}_{\operatorname{mod}_{\mathscr{P}}\Lambda}(A, B) = \operatorname{Hom}_{\operatorname{mod}\Lambda}(A, B)$ for all objects A, B in $\operatorname{mod}_{\mathscr{P}}\Lambda$.

If C is an indecomposable and non-projective object in $\operatorname{mod}\Lambda$, and we have a minimal projective resolution $P_1 \xrightarrow{f} P_0 \longrightarrow C \longrightarrow 0$ of C, then f is indecomposable an indecomposable map that is not an isomorphism. Thus $f^* : P_0^* \to P_1^*$ is also an indecomposable map which is not an isomorphism, which implies $\operatorname{Coker} f^* = \operatorname{Tr} C$ is indecomposable. We can also see that $P_0 \xrightarrow{f'} P_1 \longrightarrow \operatorname{Tr} C \longrightarrow 0$ is a minimal projective resolution of $\operatorname{Tr} C$ if C is not projective. If $C = P$ is projective, then $0 \to P \to P \to 0$ is a minimal projective resolution of C, but $P^* \to 0 \to 0 \to 0$ is not a minimal projective resolution of $\operatorname{Tr} P = 0$. From these arguments we obtain the following properties of the transpose.

Proposition 3.1.

1. $\operatorname{Tr}(\bigoplus_{i=1}^{n} A_i) \cong \bigoplus_{i=1}^{n} \operatorname{Tr} A_i$ where $A_1, \ldots, A_n$ are objects in $\operatorname{mod}\Lambda$.
2. $\operatorname{Tr} A = 0$ if and only A is projective.
3. $\operatorname{Tr}\operatorname{Tr} A \cong A_{\mathscr{P}}$ for all objects A in $\operatorname{mod}\Lambda$.
4. Let A and B be objects in $\operatorname{mod}_{\mathscr{P}}\Lambda$. Then $\operatorname{Tr} A \cong \operatorname{Tr} B$ if and only if $A \cong B$.
5. $\operatorname{Tr} : \operatorname{mod}\Lambda \to \operatorname{mod}(\Lambda^{\mathrm{op}})$ induces a bijection between the isomorphism classes of indecomposable objects in $\operatorname{mod}_{\mathscr{P}}\Lambda$ and the isomorphism classes of indecomposable objects in $\operatorname{mod}_{\mathscr{P}}(\Lambda^{\mathrm{op}})$.

We now define similar notions for injective modules as we did for projectives above. Let C be an object in $\operatorname{mod}\Lambda$. There is a decomposition $C \cong C_{\mathscr{I}} \oplus C'$, which is unique up to isomorphism, where $C_{\mathscr{I}}$ has no nonzero injective summands and C' is injective. Denote by $\operatorname{mod}_{\mathscr{I}}\Lambda$ the full subcategory of $\operatorname{mod}\Lambda$ where $C = C_{\mathscr{I}}$ for each object C. We now list some properties of $D\operatorname{Tr} : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$ which is the composition of the maps

$\operatorname{Tr} : \operatorname{mod}\Lambda \to \operatorname{mod}(\Lambda^{\mathrm{op}})$ and $D : \operatorname{mod}(\Lambda^{\mathrm{op}}) \to \operatorname{mod}\Lambda$, and $\operatorname{Tr} D : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$ which is the compostition of the maps $D : \operatorname{mod}\Lambda \to \operatorname{mod}(\Lambda^{\mathrm{op}})$ and $\operatorname{Tr} : \operatorname{mod}(\Lambda^{\mathrm{op}}) \to \operatorname{mod}\Lambda$. These properties are derived from Proposition 3.1.

Proposition 3.2.

1. $D\operatorname{Tr}(\bigoplus_{i=1}^n A_i) \cong \bigoplus_{i=1}^n D\operatorname{Tr} A_i$ where $A_1, \ldots, A_n$ are objects in $\operatorname{mod}\Lambda$.
2. $D\operatorname{Tr} A = 0$ if and only if A is projective.
3. $D\operatorname{Tr} A$ is an object in $\operatorname{mod}_{\mathscr{I}}\Lambda$ for all objects A in $\operatorname{mod}\Lambda$.
4. $(\operatorname{Tr} D)(D\operatorname{Tr})A \cong A_{\mathscr{P}}$ for all objects A in $\operatorname{mod}\Lambda$.
5. If A and B are objects in $\operatorname{mod}_{\mathscr{P}}\Lambda$, then $D\operatorname{Tr} A \cong D\operatorname{Tr} B$ if and only if $A \cong B$.
6. $D\operatorname{Tr} : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$ induces a bijection between the isomorphism classes of indecomposable objects in $\operatorname{mod}_{\mathscr{P}}\Lambda$ and the isomorphism classes of indecomposable objects in $\operatorname{mod}_{\mathscr{I}}\Lambda$ with $\operatorname{Tr} D$ as inverse.

The transpose might not be a functor as a map $\operatorname{Tr} : \operatorname{mod}\Lambda \to \operatorname{mod}(\Lambda^{\mathrm{op}})$, but we can turn it into a functor by defining it on an appropriate factor category. We proceed by discussing factor categories.

A relation $\mathscr{R}$ on an R-category $\mathscr{A}$ consists of Λ-submodules $\mathscr{R}(A,B) \subseteq \operatorname{Hom}_{\mathscr{A}}(A,B)$ such that if $\otimes_R$ denotes the composition map $\operatorname{Hom}_{\mathscr{A}}(A,B) \otimes_R \operatorname{Hom}_{\mathscr{A}}(B,C) \to \operatorname{Hom}_{\mathscr{A}}(A,C)$, then

1. $\operatorname{Im}\big(\mathscr{R}(A,B) \otimes_R \operatorname{Hom}_{\mathscr{A}}(B,C) \to \operatorname{Hom}_{\mathscr{A}}(A,C)\big) \subseteq \mathscr{R}(A,C)$,
2. $\operatorname{Im}\big(\operatorname{Hom}_{\mathscr{A}}(A,B) \otimes_R \mathscr{R}(B,C) \to \operatorname{Hom}_{\mathscr{A}}(A,C)\big) \subseteq \mathscr{R}(A,C)$.

The factor category ${}^{\mathscr{A}}\!/\!_{\mathscr{R}}$ is defined as the category where $\operatorname{Ob}({}^{\mathscr{A}}\!/\!_{\mathscr{R}}) = \operatorname{Ob}\mathscr{A}$, $\operatorname{Hom}_{\mathscr{A}/\mathscr{R}}(A,B) = {}^{\operatorname{Hom}_{\mathscr{A}}(A,B)}\!/\!_{\mathscr{R}(A,B)}$ and composition is such that

$$(g + \mathscr{R}(B,C))(f + \mathscr{R}(A,B)) = gf + \mathscr{R}(A,C)$$

for all $A, B, C \in \operatorname{Ob}({}^{\mathscr{A}}\!/\!_{\mathscr{R}})$, $f \in \operatorname{Hom}_{\mathscr{A}}(A,B)$ and $g \in \operatorname{Hom}_{\mathscr{A}}(B,C)$.

Let $\mathscr{P}(\Lambda)$ denote the category of finitely generated projective Λ-modules. The morphism category of $\mathscr{P}(\Lambda)$ is an R-category $\operatorname{Morph}\mathscr{P}(\Lambda)$ where the objects are the morphisms $f : P_1 \to P_2$ in $\mathscr{P}(\Lambda)$ and the morphisms between two objects $f : P_1 \to P_2$ and $f' : P_1' \to P_2'$ are pairs (g_1, g_2) of maps $g_1 : P_1 \to P_1'$ and $g_2 : P_2 \to P_2'$ such that the diagram

$$\begin{tikzcd}
P_1 \arrow[r, "f"] \arrow[d, "g_1"] & P_2 \arrow[d, "g_2"] \\
P_1' \arrow[r, "f'"] & P_2'
\end{tikzcd}$$

commutes. Addition and composition on the morphism sets in $\operatorname{Morph} \mathscr{P}(\Lambda)$ are defined component-wise.

The R-functor $\operatorname{Coker} : \operatorname{Morph} \mathscr{P}(\Lambda) \to \operatorname{mod}(\Lambda)$ is defined such that $\operatorname{Coker}(f : P_1 \to P_2) = \operatorname{Coker} f$ for every object f in $\operatorname{Morph} \mathscr{P}(\Lambda)$, and $\operatorname{Coker}(g_1, g_2)$ is the unique morphism $\operatorname{Coker} f \to \operatorname{Coker} f'$ that makes the diagram

$$\begin{tikzcd}
P_1 \arrow[r, "f"] \arrow[d, "g_1"] & P_2 \arrow[r] \arrow[d, "g_2"] & \operatorname{Coker} f \arrow[d, "\operatorname{Coker}(g_1, g_2)"] \\
P_1' \arrow[r, "f'"] & P_2' \arrow[r] & \operatorname{Coker} f'
\end{tikzcd}$$

commute. The functor Coker is full and dense, and $\operatorname{Coker}(g_1, g_2) = 0$ if and only if $g_2 = f'h$ for some $h : P_2 \to P_1'$, that is

$$\begin{tikzcd}
 & P_2 \arrow[dl, "h"'] \arrow[d, "g_2"] \\
P_1' \arrow[r, "f'"] & P_2'
\end{tikzcd}$$

commutes. From this we can define a relation $\mathscr{R}$ on $\operatorname{Morph} \mathscr{P}(\Lambda)$ such that $\mathscr{R}(f, f')$ consists of the morphisms (g_2, g_2) between f and f' that satisfy $g_2 = f'h$ for some $h : P_2 \to P_1'$. Then $\operatorname{Coker} : \operatorname{Morph} \mathscr{P}(\Lambda) \to \operatorname{mod} \Lambda$ induces an equivalence of categories between ${}^{\operatorname{Morph} \mathscr{P}(\Lambda)}/_{\mathscr{R}}$ and $\operatorname{mod} \Lambda$.

We have a duality $T = \operatorname{Hom}_{\operatorname{mod} \Lambda}(\ , \Lambda)|_{\mathscr{P}(\Lambda)} : \mathscr{P}(\Lambda) \to \mathscr{P}(\Lambda^{\mathrm{op}})$ defined such that $P \mapsto \operatorname{Hom}(P, \Lambda)$, which induces a duality $T : \operatorname{Morph} \mathscr{P}(\Lambda) \to \operatorname{Morph} \mathscr{P}(\Lambda^{\mathrm{op}})$ that maps an object $f : P_1 \to P_2$ to $f^* : P_2^* \to P_1^*$. If (g_1, g_2) is in $\mathscr{R}(f, f')$, then there is an $h : P_2 \to P_1'$ such that $g_2 = f'h$ we have the diagram

$$\begin{array}{ccc} P_2'^* & \xrightarrow{f'^*} & P_1'^* \\ \Big\downarrow g_2^* & \swarrow h^* & \Big\downarrow g_1^* \\ P_2 & \xrightarrow{f^*} & P_1 \end{array}$$

with $g_2^* = h^* f'^*$. To have $(g_2^*, g_1^*) \in \mathscr{R}(f'^*, f^*)$ we need $g_1^* = f^* h^*$, but we see from the diagram above that this is not necessarily the case. We want a relation with this property, that is we want a relation $\mathscr{P}$ on $\operatorname{Morph} \mathscr{P}(\Lambda)$ such that $(g_2^*, g_1^*) \in \mathscr{P}(f'^*, f^*)$ if $(g_1, g_2) \in \mathscr{P}(f, f')$. The smallest such relation that also contains $\mathscr{R}$ is generated by the following maps. For $f : P_1 \to P_2$ and $f' : P_1' \to P_2'$ we have that $(g_1, g_2) : f \to f'$ is in $\mathscr{P}(f, f')$ if and only if there is some object $h : P_2 \to P_1'$ in $\operatorname{Morph} \mathscr{P}(\Lambda)$ such that $g_1 = hf$ or $g_2 = f'h$. We then get that the duality $T : \operatorname{Morph} \mathscr{P}(\Lambda) \to \operatorname{Morph} \mathscr{P}(\Lambda^{\mathrm{op}})$ induces a duality $\operatorname{Tr} : \operatorname{Morph} \mathscr{P}(\Lambda) \to \operatorname{Morph} \mathscr{P}(\Lambda^{\mathrm{op}})$ with inverse duality $\operatorname{Tr} : \operatorname{Morph} \mathscr{P}(\Lambda^{\mathrm{op}}) \to \operatorname{Morph} \mathscr{P}(\Lambda)$. We have the following result.

Lemma 3.1. Let (g_1, g_2) be a morphism between two objects $f : P_1 \to P_2$ and $f' : P_1' \to P_2'$ in $\operatorname{Morph} \mathscr{P}(\Lambda)$. Then (g_1, g_2) is in $\mathscr{P}(f, f')$ if and only if there exists some $h : P_2 \to P_1'$ such that $f'hf = g_2 f$.

Proof. Let $f : P_1 \to P_2$ and $f' : P_1' \to P_2'$ be objects in $\operatorname{Morph} \mathscr{P}(\Lambda)$ with a morphism $(g_1, g_2) : f \to f'$.

($\Rightarrow$) Suppose $\exists h : P_2 \to P_1'$ such that $g_2 = f'h$ or $g_1 = hf$. If $g_2 = f'h$, then $g_2 f = f'hf$. If $g_1 = hf$, then since $f'g_1 = g_2 f$, we have that $g_2 f = f'g_1 = f'hf$. In either case we have $f'hf = g_2 f$.

($\Leftarrow$) Assume there is an object $h : P_2 \to P_1'$ such that $g_2 f = f'hf$. Then $f'hf = f'g_1$, so $(g_1, f'h)$ is a morphism between f and f'. We also have that $(g_1, f'h) \in \mathscr{P}(f, f')$ due to the simple reason that $f'h = f'h$. Furthermore $(g_1, g_2) - (g_1, f'h) = (0, g_2 - f'h) \in \mathscr{P}(f, f')$ because $0 = 0f$, so since $(g_1, g_2) = (0, g_2 - f'h) + (g_1, f'h)$ and $\operatorname{Morph} \mathscr{P}(\Lambda)$ is preadditive, then (g_1, g_2) is in $\mathscr{P}(f, f')$.

□

We now wish to transfer these considerations from $\operatorname{Morph} \mathscr{P}(\Lambda)$ to $\operatorname{mod} \Lambda$.

Consider a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "f"] \arrow[d, "g_1"] & P_2 \arrow[r, "\epsilon"] \arrow[d, "g_2"] & \operatorname{Coker} f \arrow[r] \arrow[d, "\operatorname{Coker}(g_1,g_2)"] & 0 \\
0 \arrow[r] & P_1' \arrow[r, "f'"] & P_2' \arrow[r, "\epsilon'"] & \operatorname{Coker} f' \arrow[r] & 0
\end{tikzcd}$$

with short exact rows. Then (g_1, g_2) is in $\mathscr{P}(f, f')$ if and only if there is some $t : \operatorname{Coker} f \to P_2'$ such that $\operatorname{Coker}(g_1, g_2) = \epsilon t$. The image of $\mathscr{P}$ under the functor $\operatorname{Coker} : \operatorname{Morph} \mathscr{P}(\Lambda) \to \operatorname{mod} \Lambda$ consists of morphisms $A \to B$ that can be written as the composition $A \to P \to B$ where P is a projective object in $\operatorname{mod} \Lambda$.

Additionally, since $\mathscr{P}$ contains the relation $\mathscr{R}$ on $\operatorname{Morph} \mathscr{P}(\Lambda)$ and there exists an equivalence between ${}^{\operatorname{Morph} \mathscr{P}(\Lambda)}/_{\mathscr{R}}$ and $\operatorname{mod} \Lambda$, then the image of $\mathscr{P}$ under the full and dense functor $\operatorname{Coker} : \operatorname{Morph} \mathscr{P}(\Lambda) \to \operatorname{mod} \Lambda$ is a relation on $\operatorname{mod} \Lambda$. We denote this image by $\mathscr{P}$.

We say that a morphism $f : A \to B$ in $\operatorname{mod} \Lambda$ factors through a projective module if $f = hg$ with $g : A \to P$ and $h : P \to B$ where P is a projective module. We denote ${}^{\operatorname{Hom}_{\operatorname{mod} \Lambda}(A,B)}/_{\mathscr{P}(A,B)}$ by $\underline{\operatorname{Hom}}_{\operatorname{mod} \Lambda}(A, B)$ and the factor category ${}^{\operatorname{mod} \Lambda}/_{\mathscr{P}}$ by $\underline{\operatorname{mod}}\Lambda$.

Since $\operatorname{Coker} : \operatorname{Morph} \mathscr{P}(\Lambda) \to \operatorname{mod} \Lambda$ is full and dense, it induces an equivalence $\operatorname{Coker} : {}^{\operatorname{Morph} \mathscr{P}(\Lambda)}/_{\mathscr{P}} \to \underline{\operatorname{mod}}\Lambda$. The duality $\operatorname{Tr} : {}^{\operatorname{Morph} \mathscr{P}(\Lambda)}/_{\mathscr{P}} \to {}^{\operatorname{Morph} \mathscr{P}(\Lambda^{\mathrm{op}})}/_{\mathscr{P}}$ then induces a duality $\operatorname{Tr} : \underline{\operatorname{mod}}\Lambda \to \underline{\operatorname{mod}}\Lambda^{\mathrm{op}}$.

Let C be an object in $\operatorname{mod} \Lambda$. Then we have a decomposition $C \cong C_{\mathscr{P}} \oplus C'$, which is unique up to isomorphism, where $C_{\mathscr{P}}$ has no projective summands and C' is projective. Let $\operatorname{mod}_{\mathscr{P}}$ denote the subcategory of $\operatorname{mod} \Lambda$ every object C satisfies $C = C_{\mathscr{P}}$ and $\operatorname{Hom}_{\operatorname{mod}_{\mathscr{P}} \Lambda}(A, B) = \operatorname{Hom}_{\operatorname{mod} \Lambda}(A, B)$ for all objects A, B in $\operatorname{mod}_{\mathscr{P}} \Lambda$. The relation $\mathscr{P}$ on $\operatorname{mod} \Lambda$ induces a relation on $\operatorname{mod}_{\mathscr{P}} \Lambda$, which we also denote by $\mathscr{P}$, and we denote the category ${}^{\operatorname{mod}_{\mathscr{P}} \Lambda}/_{\mathscr{P}}$ by $\underline{\operatorname{mod}}_{\mathscr{P}}\Lambda$. The inclusion $\operatorname{mod}_{\mathscr{P}} \Lambda \to \operatorname{mod} \Lambda$ then induces an equivalence of categories $\underline{\operatorname{mod}}_{\mathscr{P}}\Lambda \to \underline{\operatorname{mod}}\Lambda$ and we also get a duality $\operatorname{Tr} : \underline{\operatorname{mod}}_{\mathscr{P}}\Lambda \to \underline{\operatorname{mod}}_{\mathscr{P}}\Lambda^{\mathrm{op}}$.

Now we want to consider $D \operatorname{Tr}$, the dual of the transpose. We can see that if $D : \operatorname{mod} \Lambda \to \operatorname{mod}(\Lambda^{\mathrm{op}})$ denotes the duality such that $X \mapsto \operatorname{Hom}_{\operatorname{mod} R}(X, J)$ where J is the direct sum of the injective envelopes of every simple non-isomorphic R-module, then for A, B in $\operatorname{mod} \Lambda$, the morphism $f : A \to B$ is in $\mathscr{P}(A, B)$ if and only if there is an injective module I and morphisms $g : D(B) \to I$ such that $D(f) : D(B) \to D(A)$ equals hg. This motivates the notion of categories modulo injectives.

We say that a morphism $f : A \to B$ factors through an injective module if $f = hg$ with $g : A \to I$ and $h : I \to B$ where I is an injective module. We define the relation $\mathscr{I}$ on $\mathrm{mod}\,\Lambda$ such that the elements of $\mathscr{I}(A, B)$ are the morphisms in $\mathrm{Hom}_{\mathrm{mod}\,\Lambda}(A, B)$ that factor through an injective module. Denote the sets of morphisms ${}^{\mathrm{Hom}_{\mathrm{mod}\,\Lambda}(A,B)}/_{\mathscr{I}(A,B)}$ by $\overline{\mathrm{Hom}}_{\mathrm{mod}\,\Lambda}(A, B)$ and denote the factor category ${}^{\mathrm{mod}\,\Lambda}/_{\mathscr{I}}$ by $\overline{\mathrm{mod}}\Lambda$. We have that the duality $D : \underline{\mathrm{mod}\,\Lambda} \to \mathrm{mod}\,\Lambda^{\mathrm{op}}$ induces $D : \underline{\mathrm{mod}}\Lambda \to \overline{\mathrm{mod}}(\Lambda^{\mathrm{op}})$ and $D\,\mathrm{Tr} : \underline{\mathrm{mod}}\Lambda \to \overline{\mathrm{mod}}\Lambda$ is an equivalence of categories with inverse equivalence $\mathrm{Tr}\,D : \overline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda$. The relation $\mathscr{I}$ on $\mathrm{mod}\,\Lambda$ induces a relation on $\mathrm{mod}_{\mathscr{I}}\,\Lambda$ which we also denote by $\mathscr{I}$. Denote ${}^{\mathrm{mod}_{\mathscr{I}}\,\Lambda}/_{\mathscr{I}}$ by $\overline{\mathrm{mod}}_{\mathscr{I}}\Lambda$. Then the inclusion $\mathrm{mod}_{\mathscr{I}}\,\Lambda \to \overline{\mathrm{mod}\,\Lambda}$ induces an equivalence of categories $\overline{\mathrm{mod}}_{\mathscr{I}}\Lambda \to \overline{\mathrm{mod}}\Lambda$ since C in $\overline{\mathrm{mod}}\Lambda$ is the zero object if and only if C is injective.

Example 3.2. Let $\Gamma_{2,1}$ be the quiver
$$\begin{tikzcd} & 1 \arrow[dl, "\beta"'] \arrow[dd, "\alpha"] \\ 2 \arrow[dr, "\gamma"'] & \\ & 3 \end{tikzcd}$$
. Let k be a field and $\Lambda_{2,1} =$
$$\begin{tikzcd} & k \arrow[dl, "1"'] \arrow[dd, "1"] \\ k \arrow[dr, "1"'] & \\ & k \end{tikzcd}$$
for some nonzero. A minimal projective resolution of $\Lambda_{2,1}$ is

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} \Lambda_{2,1} \ ,$$

with

$$P_0 = \begin{tikzcd} & k \arrow[dl, "1"'] \arrow[dd, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \\ k \arrow[dr, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] & \\ & k^2 \end{tikzcd}, P_1 = \begin{tikzcd} & 0 \arrow[dl] \arrow[dd] \\ 0 \arrow[dr] & \\ & k \end{tikzcd},$$

$$p_0 = (1, 1, (\begin{smallmatrix}1 & 1\end{smallmatrix})) \text{ and } p_1 = (0, 0, \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)) .$$

Then the projective resolution is the following.

$$\begin{tikzcd}
 & 0 \arrow[dl] \arrow[dd] \arrow[rr, bend left] & & k \arrow[dl, "1"'] \arrow[dd, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[rr, bend left, "1"] & & k \arrow[dl, "1"'] \arrow[dd, "1"] \\
0 \arrow[dr] \arrow[rr, bend left] & & k \arrow[dr, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] \arrow[rr, bend left, "1"] & & k \arrow[dr, "1"'] & \\
 & k \arrow[rr, bend right, "\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)"'] & & k^2 \arrow[rr, bend right, "(\begin{smallmatrix}1 & 1\end{smallmatrix})"'] & & k
\end{tikzcd}$$

The respective duals of P_0 and P_1 are

$$P_0^* = \begin{tikzcd} & k \\ 0 \arrow[ur] & \\ & 0 \arrow[ul] \arrow[uu] \end{tikzcd} \quad \text{and } P_1^* = \begin{tikzcd} & k^2 \\ k \arrow[ur, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] & \\ & k. \arrow[ul, "1"] \arrow[uu, swap, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \end{tikzcd}$$

We get that $p_1^* = (\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right), 0, 0)$ and from the diagram

$$\begin{tikzcd}
 & k & & k^2 \arrow[ll, bend right, swap, "(1\ 1)"] & & k \arrow[ll, bend right, swap, "\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)"] \\
k \arrow[ur, "1"] & & k \arrow[ll, bend right, swap, "1"] \arrow[ur, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] & & 0 \arrow[ll, bend right] \arrow[ur] & \\
 & k \arrow[ul, "1"] \arrow[uu, swap, "1"] & & k \arrow[ll, bend left, "1"] \arrow[ul, "1"] \arrow[uu, swap, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & & 0 \arrow[ll, bend left] \arrow[ul] \arrow[uu]
\end{tikzcd}$$

we can see that

$$\mathrm{Tr}(\Lambda_{2,1}) = \mathrm{Coker}(p_1^*) = \begin{tikzcd} & k \\ k \arrow[ur, "1"] & \\ & k \arrow[ul, "1"] \arrow[uu, "1"] \end{tikzcd} ,$$

so

$$D\,\mathrm{Tr}(\Lambda_{2,1}) = \begin{tikzcd} & k \arrow[dl, swap, "1"] \arrow[dd, "1"] \\ k \arrow[dr, swap, "1"] & \\ & k \end{tikzcd} = \Lambda_{2,1}.$$

On another note, since

$$\ker\left(k^2 \xrightarrow{(1\ 1)} k \right) = k \xrightarrow{\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)} k^2$$

and

$$\ker\left(k^2 \xrightarrow{(1\ -1)} k \right) = k \xrightarrow{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)} k^2 ,$$

then

$$C_3^+(\Lambda_{2,1}) = \begin{tikzcd} & k \arrow[dl, swap, "1"] \\ k & \\ & k \arrow[ul, "-1"] \arrow[uu, swap, "1"] \end{tikzcd} \Rightarrow C_2^+C_3^+(\Lambda_{2,1}) = \begin{tikzcd} & k \\ k \arrow[ur, "1"] \arrow[dr, swap, "1"] & \\ & k \arrow[uu, swap, "1"] \end{tikzcd}$$

$$\Rightarrow \operatorname{Cox}^+(\Lambda_{2,1}) = C_1^+C_2^+C_3^+(\Lambda_{2,1}) = \begin{tikzcd} & k \arrow[dl, "1"'] \arrow[dd, "-1"] \\ k \arrow[dr, "1"'] & \\ & k \end{tikzcd} .$$

Suppose $\phi : \Lambda_{2,1} \to \operatorname{Cox}^+(\Lambda_{2,1})$ is a homomorphism between representations. If the diagram

$$\begin{tikzcd} & k \arrow[dl, "1"'] \arrow[dd, "1"] \arrow[rrr, bend left, "\phi(1)"] & & & k \arrow[dl, "1"'] \arrow[dd, "-1"] \\ k \arrow[dr, "1"'] \arrow[rrr, bend left, "\phi(2)"] & & & k \arrow[dr, "1"'] & \\ & k \arrow[rrr, bend right, "\phi(3)"'] & & & k \end{tikzcd}$$

is to commute, then

$$\phi(1) = \phi(2),\ \phi(2) = \phi(3) \text{ and } -\phi(1) = \phi(3)$$

$$\Rightarrow \phi(1) = \phi(2) = \phi(3) = 0,$$

so $\operatorname{Hom}\left(\Lambda_{2,1}, \operatorname{Cox}^+(\Lambda_{2,1})\right) = 0$. In particular we get that $\Lambda_{2,1}$ and $\operatorname{Cox}^+(\Lambda_{2,1})$ are not isomorphic. Thus, since $D\operatorname{Tr}(\Lambda_{2,1}) = \Lambda_{2,1}$, then $D\operatorname{Tr}(\Lambda_{2,1}) \ncong \operatorname{Cox}(\Lambda_{2,1})$.

More generally, suppose

$$\Gamma_{m,n} = \begin{tikzcd} & 1 \arrow[dl, "\alpha_1"'] \arrow[dr, "\beta_1"] & \\ 2 \arrow[d, "\alpha_2"'] & & m+1 \arrow[d, "\beta_2"] \\ \vdots \arrow[d, "\alpha_{m-1}"'] & & \vdots \arrow[d, "\beta_{n-1}"] \\ m \arrow[dr, "\alpha_m"'] & & m+n-1 \arrow[dl, "\beta_n"] \\ & m+n & \end{tikzcd}$$

and

$$
\Lambda_{m,n} = \begin{tikzcd}
& k \arrow[dl, "1"'] \arrow[dr, "1"] & \\
k \arrow[d, "1"'] & & k \arrow[d, "1"] \\
\vdots \arrow[d, "1"'] & & \vdots \arrow[d, "1"] \\
k \arrow[dr, "1"'] & & k \arrow[dl, "1"] \\
& k &
\end{tikzcd}
$$

is a representation of $\Gamma_{m,n}$. We claim that $\mathrm{Cox}^{+}(\Lambda_{m,n}) \cong D\,\mathrm{Tr}(\Lambda_{m,n})$ if and only if $m+n$ is even. To prove this claim, we first show that $D\,\mathrm{Tr}(\Lambda_{m,n}) \cong \Lambda_{m,n}$. The diagram

$$
\begin{tikzcd}
& 0 \arrow[dl] \arrow[dr] \arrow[rrr, bend left] & & & k \arrow[dl, "1"'] \arrow[dr, "1"] \arrow[rrr, bend left, "1"] & & & k \arrow[dl, "1"'] \arrow[dr, "1"] & \\
0 \arrow[d] \arrow[rrrrr, bend left] & & 0 \arrow[d] \arrow[r, bend left] & k \arrow[d, "1"'] \arrow[rrrrr, bend left, "1"] & & k \arrow[d, "1"] \arrow[r, bend left, "1"] & k \arrow[d, "1"'] & & k \arrow[d, "1"] \\
\vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d, "1"'] & & \vdots \arrow[d, "1"] & \vdots \arrow[d, "1"'] & & \vdots \arrow[d, "1"] \\
0 \arrow[dr] \arrow[rrrrr, bend right] & & 0 \arrow[dl] \arrow[r, bend right] & k \arrow[dr, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] \arrow[rrrrr, bend right, "1"'] & & k \arrow[dl, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"] \arrow[r, bend right, "1"'] & k \arrow[dr, "1"'] & & k \arrow[dl, "1"] \\
& k \arrow[rrr, bend right, "\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)"'] & & & k^2 \arrow[rrr, bend right, "(1\ 1)"'] & & & k &
\end{tikzcd}
$$

is a minimal projective resolution of $\Lambda_{m,n}$, which we can write as

$$
P_1 \xrightarrow{\ f\ } P_0 \longrightarrow \Lambda_{m,n} \longrightarrow 0 \ .
$$

Taking the duality $\mathrm{Hom}_{\mathrm{mod}\, k\Gamma_{m,n}}(\ , k\Gamma_{m,n})$ where $k\Gamma_{m,n}$ denotes the path algebra on $\Gamma_{m,n}$, we get

$$
0 \longleftarrow \mathrm{Tr}\,\Lambda_{m,n} \longleftarrow P_1^* \xleftarrow[f^*]{} P_0^* \ .
$$

We can express this as a diagram

$$\begin{tikzcd}
& k & & & k^2 \arrow[lll, bend right, "(1\ 1)"'] & & & k \arrow[lll, bend right, "\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)"'] & \\
k \arrow[ur, "1"] & & k \arrow[ul, "1"'] & k \arrow[ur, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[lll, bend right, "1"'] & & k \arrow[ul, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] \arrow[lll, bend right, "1"'] & 0 \arrow[ur] \arrow[lll, bend right] & & 0 \arrow[ul] \arrow[lll, bend right] \\
\vdots \arrow[u, "1"] & & \vdots \arrow[u, "1"'] & \vdots \arrow[u, "1"] & & \vdots \arrow[u, "1"'] & \vdots \arrow[u] & & \vdots \arrow[u] \\
k \arrow[u, "1"] & & k \arrow[u, "1"'] & k \arrow[u, "1"] \arrow[lll, bend left, "1"] & & k \arrow[u, "1"'] \arrow[lll, bend left, "1"] & 0 \arrow[u] \arrow[lll, bend left] & & 0 \arrow[u] \arrow[lll, bend left] \\
& k \arrow[ul, "1"] \arrow[ur, "1"'] & & & k \arrow[ul, "1"] \arrow[ur, "1"'] \arrow[lll, bend left, "1"] & & & 0 \arrow[ul] \arrow[ur] \arrow[lll, bend left] &
\end{tikzcd}$$

and obtain that

$$\operatorname{Tr}\Lambda_{m,n} = \begin{tikzcd}
& k & \\
k \arrow[ur, "1"] & & k \arrow[ul, "1"'] \\
\vdots \arrow[u, "1"] & & \vdots \arrow[u, "1"'] \\
k \arrow[u, "1"] & & k \arrow[u, "1"'] \\
& k \arrow[ul, "1"] \arrow[ur, "1"'] &
\end{tikzcd}$$

which means $D\operatorname{Tr}\Lambda_{m,n} \cong \Lambda_{m,n}$.

Now we show that for the representations

$$\Sigma_l^+ = \ k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k \xleftarrow{1} k$$

and

$$\Sigma_l^- = \ k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k \xleftarrow{-1} k$$

of the graph A_{l+1} with orientation

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow l \leftarrow l+1\ ,$$

we have that if l is odd, then

$$C_2^+ \cdots C_l^+(\Sigma_l^+) = \ k \xleftarrow{1} k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k \xrightarrow{1} k$$

and

$$C_2^+ \cdots C_l^+(\Sigma_l^-) = \ k \xleftarrow{-1} k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k \xrightarrow{1} k \ ,$$

and if l is even, then

$$C_2^+ \cdots C_l^+(\Sigma_l^+) = \ k \xleftarrow{-1} k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k \xrightarrow{1} k$$

and

$$C_2^+ \cdots C_l^+(\Sigma_l^-) = \ k \xleftarrow{1} k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k \xrightarrow{1} k \ .$$

Let $P(h)$ be the statement that

$$C_{l-h}^+ \cdots C_l^+(\Sigma_l^+) = \ k \xrightarrow{1} \cdots \xrightarrow{1} \underset{l-h-1}{k} \xleftarrow{(-1)^{h+1}} \underset{l-h}{k} \xrightarrow{1} \cdots \xrightarrow{1} k$$

and

$$C_{l-h}^+ \cdots C_l^+(\Sigma_l^-) = \ k \xrightarrow{1} \cdots \xrightarrow{1} \underset{l-h-1}{k} \xleftarrow{(-1)^{h}} \underset{l-h}{k} \xrightarrow{1} \cdots \xrightarrow{1} k$$

where $h \in \mathbb{N} \setminus \{1\}$. We show that $P(h)$ is true for all $l \in \mathbb{N} \setminus \{1\}$ and $h \in \{0\} \cup \mathbb{N}_{l-2}$. Suppose $h = 0$. Then

$$C_{l-h}^+ \cdots C_l^+(\Sigma_l^+) = C_l^+(\Sigma_l^+) = \ k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k \xleftarrow{-1} k \xrightarrow{1} k$$

and

$$C_{l-h}^+ \cdots C_l^+(\Sigma_l^-) = C_l^+(\Sigma_l^-) = \ k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k \xleftarrow{1} k \xrightarrow{1} k \ .$$

Assume $P(h)$ is true for some $h \in \{0\} \cup \mathbb{N}_{l-3}$, that is

$$C_{l-h}^+ \cdots C_l^+(\Sigma_l^+) = \ k \xrightarrow{1} \cdots \xrightarrow{1} \underset{l-h-1}{k} \xleftarrow{(-1)^{h+1}} \underset{l-h}{k} \xrightarrow{1} \cdots \xrightarrow{1} k$$

and

$$C_{l-h}^+ \cdots C_l^+(\Sigma_l^-) = \ k \xrightarrow{1} \cdots \xrightarrow{1} \underset{l-h-1}{k} \xleftarrow{(-1)^{h}} \underset{l-h}{k} \xrightarrow{1} \cdots \xrightarrow{1} k \ .$$

Then

$$C^+_{l-(h+1)} \cdots C^+_l(\Sigma^+_l) = \; k \xrightarrow{1} \cdots \xleftarrow{(-1)^{h+2}} \underset{l-h-1}{k} \xrightarrow{1} \underset{l-h}{k} \xrightarrow{1} \cdots \xrightarrow{1} k$$

since

$$\ker\left(k^2 \xrightarrow{\begin{pmatrix} 1 & (-1)^{h+1} \end{pmatrix}} k \right) = \; k \xrightarrow{\begin{pmatrix} (-1)^{h+2} \\ 1 \end{pmatrix}} k^2 \; ,$$

and

$$C^+_{l-(h+1)} \cdots C^+_l(\Sigma^-_l) = \; k \xrightarrow{1} \cdots \xleftarrow{(-1)^{h+1}} \underset{l-h-1}{k} \xrightarrow{1} \underset{l-h}{k} \xrightarrow{1} \cdots \xrightarrow{1} k$$

because

$$\ker\left(k^2 \xrightarrow{\begin{pmatrix} 1 & (-1)^{h} \end{pmatrix}} k \right) = \; k \xrightarrow{\begin{pmatrix} (-1)^{h+1} \\ 1 \end{pmatrix}} k^2 \; .$$

By induction on h we get that $P(h)$ is true for all $h \in \{0\} \cup \mathbb{N}_{l-3}$.

Let $h = l-2$. If l is odd, then $(-1)^h = -1$ and $(-1)^{h+1} = 1$, so

$$C^+_2 \cdots C^+_l(\Sigma^+_l) = \; k \xleftarrow{1} k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k \xrightarrow{1} k$$

and

$$C^+_2 \cdots C^+_l(\Sigma^-_l) = \; k \xleftarrow{-1} k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k \xrightarrow{1} k \; .$$

If l is even, then $(-1)^h = 1$ and $(-1)^{h+1}$, so

$$C^+_2 \cdots C^+_l(\Sigma^+_l) = \; k \xleftarrow{-1} k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k \xrightarrow{1} k$$

and

$$C^+_2 \cdots C^+_l(\Sigma^-_l) = \; k \xleftarrow{1} k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k \xrightarrow{1} k \; .$$

Now we have that

$$C^+_{m+n}(\Lambda_{m,n}) = \begin{tikzcd} & k \arrow[dl, "1"'] \arrow[dr, "1"] & \\ k \arrow[d, "1"'] & & k \arrow[d, "1"] \\ \vdots \arrow[d, "1"'] & & \vdots \arrow[d, "1"] \\ k & & k \\ & k \arrow[ul, "-1"] \arrow[ur, "1"'] & \end{tikzcd} .$$

Then Σ_m^- and Σ_n^+ correspond to the arrows on the left and right parts of the diagram of $C_{m+n}(\Lambda_{m,n})$, respectively. If both m and n are odd, then

$$C_2^+ \cdots C_{m+n}^+(\Lambda_{m,n}) = \begin{tikzcd} & k & \\ k \arrow[ur, "-1"] \arrow[d, "1"'] & & k \arrow[ul, "1"'] \arrow[d, "1"] \\ \vdots \arrow[d, "1"'] & & \vdots \arrow[d, "1"] \\ k \arrow[dr, "1"'] & & k \arrow[dl, "1"] \\ & k & \end{tikzcd} .$$

If both m and n are even, then

$$C_2^+ \cdots C_{m+n}^+(\Lambda_{m,n}) = \begin{tikzcd} & k & \\ k \arrow[ur, "1"] \arrow[d, "1"'] & & k \arrow[ul, "-1"'] \arrow[d, "1"] \\ \vdots \arrow[d, "1"'] & & \vdots \arrow[d, "1"] \\ k \arrow[dr, "1"'] & & k \arrow[dl, "1"] \\ & k & \end{tikzcd} .$$

In both cases we get that

$$\mathrm{Cox}^+(\Lambda_{m,n}) = C_1^+ \cdots C_{m+n}^+(\Lambda_{m,n}) = \begin{tikzcd}
& k \arrow[dl, "1"'] \arrow[dr, "1"] & \\
k \arrow[d, "1"'] & & k \arrow[d, "1"] \\
\vdots \arrow[d, "1"'] & & \vdots \arrow[d, "1"] \\
k \arrow[dr, "1"'] & & k \arrow[dl, "1"] \\
& k &
\end{tikzcd}.$$

Then $\mathrm{Cox}^+(\Lambda_{m,n}) \cong \Lambda_{m,n}$ if $m+n$ is even. If m is odd and n is even, then

$$C_2^+ \cdots C_{m+n}^+(\Lambda_{m,n}) = \begin{tikzcd}
& k & \\
k \arrow[ur, "-1"] \arrow[d, "1"'] & & k \arrow[ul, "-1"'] \arrow[d, "1"] \\
\vdots \arrow[d, "1"'] & & \vdots \arrow[d, "1"] \\
k \arrow[dr, "1"'] & & k \arrow[dl, "1"] \\
& k &
\end{tikzcd}.$$

If m is even and n is odd, then

$$C_2^+ \cdots C_{m+n}^+(\Lambda_{m,n}) = \begin{tikzcd}
& k & \\
k \arrow[ur, "1"] \arrow[d, "1"'] & & k \arrow[ul, "1"'] \arrow[d, "1"] \\
\vdots \arrow[d, "1"'] & & \vdots \arrow[d, "1"] \\
k \arrow[dr, "1"'] & & k \arrow[dl, "1"] \\
& k &
\end{tikzcd}.$$

In both of these cases we have that

$$\operatorname{Cox}^+(\Lambda_{m,n}) = C_1^+ \cdots C_{m+n}^+(\Lambda_{m,n}) = \begin{tikzcd} & k \arrow[dl, "-1"'] \arrow[dr, "1"] & \\ k \arrow[d, "1"'] & & k \arrow[d, "1"] \\ \vdots \arrow[d, "1"'] & & \vdots \arrow[d, "1"] \\ k \arrow[dr, "1"'] & & k \arrow[dl, "1"] \\ & k & \end{tikzcd} .$$

Then $\operatorname{Cox}^+(\Lambda_{m,n}) \ncong \Lambda_{m,n}$ if $m+n$ is odd. Lastly, since $D\operatorname{Tr}(\Lambda_{m,n}) \cong \Lambda_{m,n}$, then we get that $\operatorname{Cox}^+(\Lambda_{m,n}) \cong D\operatorname{Tr}(\Lambda_{m,n})$ if and only if $m+n$ is even. $\triangle$

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A ϕ_λ is an R-algebra automorphism

Let R be a commutative ring, Λ an R-algebra, $\lambda \in U(\Lambda)$ and define the function

$$\begin{aligned} \phi_\lambda : \Lambda &\to \Lambda \\ \sigma &\mapsto \lambda\sigma\lambda^{-1} \end{aligned}$$

for all $\sigma \in \Lambda$.

Claim. ϕ_λ is a Λ-automorphism for all $\lambda \in U(\Lambda)$.

Proof. Suppose $\alpha, \beta \in \Lambda$ and $r \in R$. We show that ϕ_λ is a Λ-homomorphism.

1. ϕ_λ is compatible with scalar multiplication since

$$\phi_\lambda(r\alpha) = \lambda r\alpha\lambda^{-1} = r\lambda\alpha\lambda^{-1} = r\phi_\lambda(\alpha).$$

2. ϕ_λ is compatible with addition since

$$\phi_\lambda(\alpha + \beta) = \lambda(\alpha + \beta)\lambda^{-1} = \lambda\alpha\lambda^{-1} + \lambda\beta\lambda^{-1} = \phi_\lambda(\alpha) + \phi_\lambda(\beta).$$

3. ϕ_λ is compatible with multiplication since

$$\phi_\lambda(\alpha\beta) = \lambda\alpha\beta\lambda^{-1} = \lambda\alpha\lambda^{-1}\lambda\beta\lambda^{-1} = \phi_\lambda(\alpha)\phi_\lambda(\beta).$$

Thus ϕ_λ is a Λ-homomorphism.

Now we show that ϕ_λ is bijective.

1. ϕ_λ is injective since

$$\begin{aligned} \phi_\lambda(\alpha) = \phi_\lambda(\beta) \Rightarrow \lambda\alpha\lambda^{-1} = \lambda\beta\lambda^{-1} &\Rightarrow \lambda^{-1}(\lambda\alpha\lambda^{-1})\lambda = \lambda^{-1}(\lambda\beta\lambda^{-1})\lambda \\ &\Rightarrow (\lambda^{-1}\lambda)\alpha(\lambda^{-1}\lambda) = (\lambda^{-1}\lambda)\beta(\lambda^{-1}\lambda) \Rightarrow \alpha = \beta. \end{aligned}$$

2. ϕ_λ is surjective since

$$\alpha \in \Lambda \Rightarrow \lambda^{-1}\alpha\lambda \in \Lambda \Rightarrow \exists\gamma \in \Lambda : \gamma = \lambda^{-1}\alpha\lambda \Rightarrow \alpha = \lambda\gamma\lambda^{-1} = \phi_\lambda(\gamma).$$

We have shown that ϕ_λ is a injective and surjective Λ-homomorphism. Hence it is a Λ-automorphism.

$\square$

B $Z(G)$ is a subgroup of G

Let G be a group and let $Z(G) = \{g \in G \mid gh = hg \ \forall h \in G\}$ denote the center of G.

Claim. $Z(G)$ is a subgroup of G.

Proof. First observe that $Z(G) \subseteq G$. Suppose $a, b \in Z(G)$.

1. $Z(G)$ is closed under the group operation of G since
$$(ab)g = agb = g(ab) \ \forall g \in G \Rightarrow ab \in Z(G).$$
2. e_G, the identity in G, is contained in $Z(G)$ since
$$eg = g = ge \ \forall g \in G.$$
3. The inverse of every element in $Z(G)$ is contained in $Z(G)$ since
$$g \in Z(G) \Rightarrow g^{-1}h = (h^{-1}g)^{-1} = (gh^{-1})^{-1} = hg^{-1} \ \forall h \in G$$
$$\Rightarrow g^{-1} \in Z(G).$$

Thus $Z(G)$ is a subgroup of G. □

C $\mathrm{Inn}(\Lambda)$ is a group

Let R be a commutative ring and Λ an R-algebra.

Claim. $\mathrm{Inn}(\Lambda)$ is a group under function composition.

Proof. Assume $\alpha, \beta, \gamma \in U(\Lambda)$.

0. $\mathrm{Inn}(\Lambda)$ is closed under function composition since
$$(\phi_\beta \circ \phi_\alpha)(\lambda) = \beta\alpha\lambda\alpha^{-1}\beta^{-1} = (\beta\alpha)\lambda(\beta\alpha)^{-1} = \phi_{\beta\alpha}(\lambda) \ \forall \lambda \in \Lambda$$
$$\Rightarrow \phi_\beta \circ \phi_\alpha = \phi_{\beta\alpha}.$$
1. Composition is associative since
$$((\phi_\gamma \circ \phi_\beta) \circ \phi_\alpha)(\lambda) = (\gamma\beta)\alpha\lambda\alpha^{-1}(\beta^{-1}\gamma^{-1}) = \gamma(\beta\alpha)\lambda(\alpha^{-1}\beta^{-1})\gamma^{-1}$$
$$= (\phi_\gamma \circ (\phi_\beta \circ \phi_\alpha))(\lambda) \ \forall \lambda \in \Gamma$$
$$\Rightarrow (\phi_\gamma \circ \phi_\beta) \circ \phi_\alpha = \phi_\gamma \circ (\phi_\beta \circ \phi_\alpha).$$

2. $\mathrm{Inn}(\Lambda)$ has an identity element since

$$(\phi_{1_\Lambda} \circ \phi_\alpha)(\lambda) = 1_\Lambda(\alpha\lambda\alpha^{-1})1_\Lambda = \alpha\lambda\alpha^{-1}$$
$$= \phi_\alpha(\lambda)$$
$$= \alpha\lambda\alpha^{-1} = \alpha(1_\Lambda \cdot \lambda \cdot 1_\Lambda)\alpha^{-1} = (\phi_\alpha \circ \phi_{1_\Lambda})(\lambda)\ \forall\lambda \in \Lambda$$
$$\Rightarrow \phi_{1_\Lambda} \circ \phi_\alpha = \phi_\alpha = \phi_\alpha \circ \phi_{1_\Lambda}.$$

3. Every element in $\mathrm{Inn}(\Lambda)$ is invertible since

$$(\phi_\alpha \circ \phi_{\alpha^{-1}})(\lambda) = \alpha\alpha^{-1}\lambda\alpha\alpha^{-1} = \lambda$$
$$= \phi_1(\lambda)$$
$$= \lambda = \alpha^{-1}\alpha\lambda\alpha^{-1}\alpha = (\phi_{\alpha^{-1}} \circ \phi_\alpha)(\lambda)\ \forall\lambda \in \Lambda$$
$$\Rightarrow \phi_\alpha \circ \phi_{\alpha^{-1}} = \phi_1 = \phi_{\alpha^{-1}} \circ \phi_\alpha.$$

Thus $\mathrm{Inn}(\Lambda)$ is a group under function composition. □

D $E_{pq}(-r) = (E_{pq}(r))^{-1}$

Let $p, q, i, j \in \{1, \ldots, d\}, p \neq q$ and $r \in R$. Consider the matrix $e_{pq}(r) \in M_d(R)$ defined such that

$$[e_{pq}(r)]_{ij} = \begin{Bmatrix} r \text{ if } i = p \text{ and } j = q \\ 0 \text{ otherwise} \end{Bmatrix}.$$

Define the matrix $E_{pq}(r) := I_d + e_{pq}(r)$.

Claim. $E_{pq}(r)E_{pq}(-r) = I_d = E_{pq}(-r)E_{pq}(r)$.

Proof. We first show that $E_{pq}(r)E_{pq}(-r) = I_d$. First we have that

$$E_{pq}(r)E_{pq}(-r) = I_d^2 + I_d e_{pq}(r) + I_d e_{pq}(-r) + e_{pq}(r)e_{pq}(-r).$$

Notice that $I_d e_{pq}(r) + I_d e_{pq}(-r) = e_{pq}(r) + e_{pq}(-r) = e_{pq}(r - r) = e_{pq}(0) = 0$. If $M \in M_d(R)$, then let $[M]_{i\bullet}$ be the i-th row and $[M]_{\bullet j}$ be the j-th column of M. If $i \neq p$, then $[e_{pq}(r)]_{i\bullet} = \vec{0}$, and if $j \neq q$, then $[e_{pq}(-r)]_{\bullet j} = \vec{0}$. Thus the only potentially nonzero entry in $e_{pq}(r)e_{pq}(-r)$ is then

$$[e_{pq}(r)e_{pq}(-r)]_{pq} = [e_{pq}(r)]_{p\bullet}[e_{pq}(-r)]_{\bullet q} = \sum_{k=1}^{d}[e_{pq}(r)]_{pk}[e_{pq}(-r)]_{kq}.$$

Since $[e_{pq}(r)]_{pj} = 0$ when $j \neq q$, then

$$\sum_{k=1}^{d}[e_{pq}(r)]_{pk}[e_{pq}(-r)]_{kq} = [e_{pq}(r)]_{pq}[e_{pq}(-r)]_{qq}.$$

But $[e_{pq}(-r)]_{qq} = 0$, so $e_{pq}(r)e_{pq}(-r) = 0$. Then $E_{pq}(r)E_{pq}(-r) = I_d$. We show that $E_{pq}(-r)E_{pq}(r) = I_d$ in a similar way.

$$E_{pq}(-r)E_{pq}(r) = I_d^2 + I_d e_{pq}(-r) + I_d e_{pq}(r) + e_{pq}(-r)e_{pq}(r).$$

$I_d e_{pq}(-r) + I_d e_{pq}(r) = e_{pq}(-r) + e_{pq}(r) = e_{pq}(-r + r) = e_{pq}(0) = 0$. If $i \neq p$, then $[e_{pq}(-r)]_{i\bullet} = \vec{0}$, and if $j \neq q$, then $[e_{pq}(r)]_{\bullet j} = \vec{0}$. Thus the only potentially nonzero entry in $e_{pq}(-r)e_{pq}(r)$ is then

$$[e_{pq}(-r)e_{pq}(r)]_{pq} = [e_{pq}(-r)]_{p\bullet}[e_{pq}(r)]_{\bullet q} = \sum_{k=1}^{d}[e_{pq}(-r)]_{pk}[e_{pq}(r)]_{kq}.$$

Since $[e_{pq}(-r)]_{pj} = 0$ when $j \neq q$, then

$$\sum_{k=1}^{d}[e_{pq}(-r)]_{pk}[e_{pq}(r)]_{kq} = [e_{pq}(-r)]_{pq}[e_{pq}(r)]_{qq}.$$

But $[e_{pq}(r)]_{qq} = 0$, so $e_{pq}(-r)e_{pq}(r) = 0$. Then $E_{pq}(-r)E_{pq}(r) = I_d$. Hence $E_{pq}(r)E_{pq}(-r) = I_d = E_{pq}(-r)E_{pq}(r)$. $\square$

E $M_D(k)$ is an associative k-algebra

Let k be a field, $n \in \mathbb{N}$, $D = (d_1, \cdots, d_n) \in \mathbb{N}^n$, $r \in k$ and $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ be elements in $M_D(k) = \prod_{i=1}^{n} M_{d_i}(k)$. Define

- scalar multiplication such that $rA = (rA_1, \dots, A_n)$.

- addition such that $A + B = (A_1 + B_1, \ldots, A_n + B_n)$.
- multiplication such that $AB = (A_1B_1, \ldots, A_nB_n)$.

Claim. $M_D(k)$ is an associative algebra.

Proof. We first show that $M_D(k)$ is an k-vector space. Let $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n)$ and $C = (C_1, \ldots, C_n)$ be elements in $M_D(k)$.

1. Addition is associative since
$$(A + B) + C = (A_1 + B_1, \ldots, A_n + B_n) + (C_1, \ldots, C_n)$$
$$= ((A_1 + B_1) + C_1, \ldots, (A_n + B_n) + C_n)$$
$$= (A_1 + (B_1 + C_1), \ldots, A_n + (B_n + C_n))$$
$$= (A_1, \ldots, A_n) + (B_1 + C_1, \ldots, B_n + C_n) = A + (B + C).$$
2. Define $0_{M_D(k)} = (0_{M_{d_1}(k)}, \ldots, 0_{M_{d_n}(k)})$. This is the additive identity of $M_D(k)$ since
$$A + 0_{M_D(k)} = (A_1 + 0_{M_{d_1}(K)}, \ldots, A_n + 0_{M_{d_n}(k)}) = (A_1, \ldots, A_n)$$
$$= A$$
$$= (A_1, \ldots, A_n) = (0_{M_{d_1}(k)} + A_1, \ldots, 0_{M_{d_n}(k)} + A_n) = 0_{M_D(k)} + A.$$
3. Let $-A = (-A_1, \ldots, -A_n)$. Then any element in $M_D(k)$ has an additive inverse since
$$A + (-A) = (A_1 + (-A_1), \ldots, A_n + (-A_n)) = (0_{M_{d_1}(k)}, \ldots, 0_{M_{d_n}(k)})$$
$$= 0_{M_D(k)}$$
$$= (0_{M_{d_1}(k)}, \ldots, 0_{M_{d_n}(k)}) = ((-A_1) + A_1, \ldots, (-A_n) + A_n) = (-A) + A.$$
4. Addition is commutative since
$$A + B = (A_1 + B_1, \ldots, A_n + B_n) = (B_1 + A_1, \ldots, B_n + A_n) = B + A.$$

Thus $M_D(k)$ is an abelian group.

Now let $r, s \in D$.

1. Scalar multiplication is distributive with addition in $M_D(k)$ since

$$r(A+B) = (r(A_1+B_1),\ldots,r(A_n,B_n)) = (rA_1+rB_1,\ldots,rA_n+rB_n)$$

$$= (rA_1,\ldots,rA_n) + (rB_1,\ldots,rB_n) = rA + rB.$$

2. Scalar multiplication is distributive with addition in k since

$$(r+s)A = ((r+s)A_1,\ldots,(r+s)A_n) = (rA_1+sA_1,\ldots,rA_n+sA_n)$$

$$= (rA_1,\ldots,rA_n) + (sA_1,\ldots,sA_n) = rA + sA.$$

3. Scalar multiplication is compatible with multiplication in k since

$$(rs)A = ((rs)A_1,\ldots,(rs)A_n) = (r(sA_1),\ldots,r(sA_n))$$

$$= r(sA_1,\ldots,sA_n) = r(sA).$$

4. Scalar multiplication is compatible with the multiplicative identity in k since
$$1_kA = (1_kA_1,\ldots,1_kA_n) = (A_1,\ldots,A_n) = A.$$

Thus $M_D(K)$ is an k-vector space.

It is time to show that multiplication behaves the way we want it to.

1. Multiplication is left distributive since

$$A(B+C) = (A_1,\ldots,A_n)(B_1+C_1,\ldots,B_1+C_n)$$

$$= (A_1(B_1+C_1),\ldots,A_n(B_n+C_n)) = (A_1B_1+A_1C_1,\ldots,A_nB_n+A_nC_n)$$
$$= (A_1B_1,\ldots,A_nB_n) + (A_1C_1,\ldots,A_nC_n) = AB + AC.$$

2. Multiplication is right distributive since

$$(A+B)C = (A_1+B_1,\ldots,A_n+B_n)(C_1,\ldots,C_n)$$

$$= ((A_1+B_1)C_1,\ldots,(A_n+B_n)C_n) = (A_1C_1+B_1C_1,\ldots,A_nC_n+B_nC_n)$$
$$= (A_1C_1,\ldots,A_nC_n) + (B_1C_1,\ldots,B_nC_n) = AC + BC.$$

3. Multiplication is compatible with scalars since

$$r(AB) = r(A_1B_1, \ldots, A_nB_n) = (r(A_1B_1), \ldots, r(A_nB_n))$$
$$= ((rA_1)B_1, \ldots, (rA_n)B_n) = (rA_1, \ldots, rA_n)(B_1, \ldots, B_n)$$
$$= (rA)B$$
$$= (rA_1, \ldots, rA_n)(B_1, \ldots, B_n) = ((rA_1)B_1, \ldots, (rA_n)B_n)$$
$$= (A_1(rB_1), \ldots, A_n(rB_n)) = (A_1, \ldots, A_n)(rB_1, \ldots, rB_n)$$
$$= A(rB).$$

4. Multiplication is associative since

$$A(BC) = (A_1, \ldots, A_n)(B_1C_1, \ldots, B_nC_n)$$
$$= (A_1(B_1C_1), \ldots, A_n(B_nC_n)) = ((A_1B_1)C_1, \ldots, (A_nB_n)C_n)$$
$$= (A_1B_1, \ldots, A_nB_n)(C_1, \ldots, C_n) = (AB)C.$$

5. Define $1_{M_D(k)} = (1_{M_{d_1}(k)}, \ldots, 1_{M_{d_n}(k)})$. Then $M_D(k)$ has a multiplicative identity since

$$A1_{M_D(k)} = (A_1 1_{M_{d_1}(k)}, \ldots, A_n 1_{M_{d_n}(k)}) = (A_1, \ldots, A_n)$$
$$= A$$
$$= (A_1, \ldots, A_n) = (1_{M_{d_1}(k)}A_1, \ldots, 1_{M_{d_n}(k)}A_n) = 1_{M_D(k)}A.$$

Hence $M_D(k)$ is an associative algebra. $\square$

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Kunnskap for en bedre verden