Jon Wallem Anundsen

Quivers, Gröbner bases, and tensor products

Master's thesis in Mathematical Sciences
Supervisor: Øyvind Solberg
June 2022

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Abstract

We present the theory of Gröbner bases in algebras with a multiplicative basis, and in particular of Gröbner bases in path algebras.

We show how to construct the tensor product $\Lambda \otimes_K \Gamma$ of algebras Λ and Γ as a quotient of a path algebra, when Λ and Γ are given as quotients of path algebras. We extend this to a new result showing how we can construct the tensor product $\Lambda \otimes_\Sigma \Gamma$, where Σ is another algebra. Additionally, we show how we can explicitly describe a Gröbner basis for the case where the tensor product is taken over the field K.

We also investigate how, given a quiver Q and an ideal $I \subseteq KQ$ satisfying the condition

$$J_Q^m \subseteq I$$

for some m, we can find a new quiver Q' and ideal $I' \subseteq KQ'$ such that

$$KQ/I \cong KQ'/I',$$

and such that I' is an admissible ideal, i.e.

$$J_{Q'}^m \subseteq I' \subseteq J_{Q'}^2$$

for some m.

Sammendrag

Vi presenterer teori for Gröbner-basis i algebraer med multiplikativ basis, og spesielt for Gröbner-basis i vei-algebraer.

Vi viser hvordan man kan konstruere tensorproduktet $\Lambda \otimes_K \Gamma$ av algebraer Λ og Γ som en kvotient av an vei-algebra, når Λ og Γ er gitt som kvotienter av vei-algebraer. Vi utvider dette til et nytt resultat som viser hvordan vi kan konstruere tensorproduktet $\Lambda \otimes_\Sigma \Gamma$, der Σ er en annen algebra. I tillegg viser vi hvordan vi eksplisitt kan beskrive en Gröbner-basis for tilfellet der tensorproduktet tas over kroppen K.

Vi undersøker også, gitt et kogger Q og et ideal $I \subseteq KQ$ som tilfredsstiller betingelsen

$$J_Q^m \subseteq I$$

for en eller annen verdi av m, hvordan vi kan finne et nytt kogger Q' og et ideal $I' \subseteq KQ'$ slik at

$$KQ/I \cong KQ'/I',$$

og slik at I' er et tillatelig ideal, det vil si

$$J_{Q'}^m \subseteq I \subseteq J_{Q'}^2$$

for en eller annen verdi av m.

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I would like to thank my advisor Øyvind Solberg for suggesting the topics in this thesis, for all the feedback he has given me, and for the helpful discussions that we have had throughout the year that I've been working on this thesis.

Prerequisites

It is assumed that the reader is familiar with the basics of ring theory in general, and with quivers and path algebras in particular. It should be enough to have taken the course MA3203 Ring Theory at NTNU, or similar.

For Chapter 3, it would benefit the reader to be familiar with tensor products. At NTNU, this is covered in the course MA3204 Homological Algebra. However, we recall the definition of tensor products at the start of Chapter 3, so it may not be strictly necessary to know anything about tensor products beforehand.

Conventions regarding terminology and notation

Throughout this thesis, K always denotes a field.

All rings have a multiplicative identity, and all algebras are associative and unital. All ring homomorphisms and algebra homomorphisms preserve the multiplicative identity unless otherwise noted.

Given a ring Λ and an ideal $I \subseteq \Lambda$, we sometimes use the notation $[\lambda]$ the equivalence class $\lambda + I$. If $\lambda, \mu \in \Lambda$ and $\lambda + I = \mu + I$, then we write

$$\lambda \equiv \mu \pmod{I}.$$

All quivers are finite by definition. When Q is a quiver, we denote the sets of vertices and arrows by Q_0 and Q_1, respectively. Multiplication of paths in a quiver is written from right to left. For example, for arrows $\alpha\colon v_1 \to v_2$ and $\beta\colon v_2 \to v_3$, we have a path $\beta\alpha$ from v_1 to v_3. The source and target of a path p are denoted by $\mathfrak{s}(p)$ and $\mathfrak{t}(p)$, respectively. The length of p is denoted by $\mathfrak{l}(p)$.

Contents

Chapter 1

Introduction

In 1965, Bruno Buchberger introduced the concept of Gröbner bases in his PhD thesis, naming them after his advisor Wolfgang Gröbner [Buc06]. Since then, Gröbner bases have become an important tool in computational commutative algebra and computational algebraic geometry. They were originally defined in the setting of commutative polynomial rings in a finite number of variables over a field, and while this remains the most notable use case for Gröbner bases, they have since been generalized to various other settings. In particular, there is a Gröbner basis theory for path algebras, and this theory is used to facilitate computations with the package QPA (Quivers and Path Algebras) [QPA22] for the programming language GAP [GAP20]. In this thesis, we present this theory of Gröbner bases. We also study tensor products of algebras.

This thesis has three main chapters. In Chapter 2, we present Gröbner basis theory for algebras with a multiplicative basis, with an emphasis on path algebras. We see how Gröbner bases can be used to facilitate computations with quotient algebras, and show how they may be computed. Chapter 3 deals with tensor products of algebras. We prove a result that shows how we can construct the tensor product $\Lambda \otimes_K \Gamma$ as a quotient of a path algebra, when Λ and Γ are given as quotients of path algebras. We then generalize this to obtain a result that shows how we can construct the tensor product $\Lambda \otimes_\Sigma \Gamma$, where Σ is another algebra. Finally, we show that for the case of tensor products over the field K, we can explicitly describe a Gröbner basis, which means that we do not need spend time computing one. In Chapter 4, we consider the following question: Given a quiver Q and an ideal $I \subseteq KQ$ which is "lower-admissible", i.e. which satisfies the lower bound

$$J_Q^m \subseteq I$$

for some m, can we find another quiver Q' and an ideal $I' \subseteq KQ'$ such that

$$KQ/I \cong KQ'/I',$$

and such that I' is an admissible ideal? We answer this question in the affirmative, and we present algorithms to find Q' and I'.

Chapter 2

Gröbner basis theory

In this chapter, we present the theory of Gröbner bases in algebras with a multiplicative basis, with an emphasis on quivers and path algebras. We will see how Gröbner bases lend themselves towards computational use, and in particular how they facilitate computations with quotient algebras.

Section 2.1 introduces the basic concepts of Gröbner basis theory that underpin the rest of the chapter, and indeed the rest of this thesis. In Section 2.2, we show how Gröbner basis theory can be used to define canonical representatives of equivalence classes in a quotient algebra, and how these representatives can be computed with the aid of Gröbner bases. Section 2.3 discusses reduced Gröbner bases, which are in some sense the "best" kind of Gröbner bases. Finally, in Section 2.4, we focus exclusively on path algebras, and show how Gröbner bases may be computed. We also prove a theorem that gives a sufficient condition for the existence of a finite Gröbner basis for an ideal in a path algebra.

It bears mentioning that most of the ideas in this chapter are not original. Unless otherwise stated, all theory in this chapter is based on [Gre99], with the exception of some lemmas used in the proof of correctness for Algorithm 2.3. Content and ideas based on other sources than [Gre99] are cited whenever they appear.

2.1 Admissible orders and Gröbner bases

We will now introduce the basic concepts that are foundational to Gröbner basis theory. Although we are primarily interested in path algebras in this thesis, most of this chapter deals with Gröbner bases in a more general setting, namely in the context of algebras with a multiplicative basis. This allows us to state results that hold not only for path algebras, but also for the classical case of a commutative polynomial ring. Let us start by explaining what is meant by an algebra with a multiplicative basis.

Definition 2.1. Let Λ be a K-algebra. A K-basis $\mathcal{B}$ of Λ is called a **multiplicative basis** if for all $p, q \in \mathcal{B}$, we have that either $pq \in \mathcal{B}$ or $pq = 0$. ■

A notable example of an algebra with a multiplicative basis is the commutative polynomial ring $K[x_1, \ldots, x_n]$, in which the canonical choice of a multiplicative basis $\mathcal{B}$ is the set of all monomials, i.e. all elements of the form

$$x_1^{m_1} \ldots x_n^{m_n}$$

for non-negative integers m_i. This is the context in which Gröbner bases were originally introduced. Another example, and one which will be central to this thesis, is the path algebra KQ of a quiver Q, where the set of all paths in Q forms a multiplicative basis.

Although an algebra might have several different multiplicative bases, we will always assume that if Λ is a path algebra (resp. polynomial ring), then the multiplicative basis $\mathcal{B}$ denotes the set of all paths (resp. monomials).

Gröbner basis theory requires that the elements of a multiplicative basis $\mathcal{B}$ have been given a total order, more specifically an *admissible* order. This is essentially an order that is compatible with the multiplication of basis elements. Before we state the definition, recall that a **well-order** on a set X is a total order $\leq$ on X such that every nonempty subset of X has a smallest element. For $x, y \in X$, we write $x < y$ if $x \leq y$ and $x \neq y$.

Definition 2.2. Let Λ be a K-algebra with multiplicative basis $\mathcal{B}$. An **admissible order** on $\mathcal{B}$ is a well-order $\leq$ on $\mathcal{B}$ such that the following statements hold for all $p, q, r \in \mathcal{B}$:

(i) If $p < q$ and pr and qr are nonzero, then $pr < qr$.

(ii) If $p < q$ and rp and rq are nonzero, then $rp < rq$.

(iii) If pq is nonzero, then $p \leq pq$ and $q \leq pq$.

■

Let us look at some examples of admissible orders.

Example 2.3. Let $\Lambda = K[x_1, \ldots, x_n]$. We define an order on the monomials in the following way:

$$x_1^{m_1} \ldots x_n^{m_n} < x_1^{m'_1} \ldots x_n^{m'_n}$$

if and only if there exists some j such that

$$m_i = m'_i$$

for $i \leq j$, and such that

$$m_j < m'_j.$$

This is known as the **lexicographic order** on the monomials in $K[x_1, \ldots, x_n]$ ■

Example 2.4. Let Q be a quiver, and let $\Lambda = KQ$ be the path algebra of Q over K. Recall that $\mathcal{B}$ then denotes the multiplicative basis consisting of all paths in Q. Let $\leq$ be any total order on $Q_0 \cup Q_1$ such that $v < \alpha$ for all vertices $v \in Q_0$ and arrows $\alpha \in Q_1$. We extend $\leq$ to an order on $\mathcal{B}$ in the following way: If p is a path of length at least 2, then $v < p$ and $\alpha < p$ for all vertices v and arrows α. If q is some other path of length at least 2, then $p < q$ if and only if one of the following holds:

- $\mathfrak{l}(p) < \mathfrak{l}(q)$.
- $p = \alpha_1 \ldots \alpha_r$ and $q = \beta_1 \ldots \beta_r$ for some arrows α_i and β_i, and there exists some $j \in \{ 1, \ldots, r \}$ such that $\alpha_i = \beta_i$ for $1 \leq i < j$ and $\alpha_j < \beta_j$.

It can be seen that this is an admissible order on $\mathcal{B}$, which we call the **left length-lexicographic order**, or just the (left) length-lex order.

For a concrete example, consider the following quiver.

$$Q\colon \begin{tikzcd} v_1 \arrow[r, "\alpha"] & v_2 \arrow[loop above, "\gamma"] \arrow[r, "\beta"] & v_3 \end{tikzcd}$$

Suppose that we order the vertices and arrows of Q as

$$v_1 < v_2 < v_3 < \alpha < \beta < \gamma.$$

Then the paths in Q of length at least 2 have the following order in the left length-lexicographic ordering:

$$\beta\alpha < \beta\gamma < \gamma\alpha < \gamma\gamma < \beta\gamma\alpha < \beta\gamma\gamma < \gamma\gamma\alpha < \gamma\gamma\gamma < \ldots$$

It is also possible to define a *right* length-lex order. This is defined in the same way as the left length-lex ordering, except that arrows are compared from right to left instead of left to right. For example, if Q is the quiver above with the same order on the vertices and arrows, then the paths of length at least two have the following order in the right length-lex ordering:

$$\beta\alpha < \gamma\alpha < \beta\gamma < \gamma\gamma < \beta\gamma\alpha < \gamma\gamma\alpha < \beta\gamma\gamma < \gamma\gamma\gamma < \ldots$$

■

In the case where $\Lambda = KQ$ for a quiver Q, we will often just say that "$\leq$ is an admissible order on Q" when we mean that $\leq$ is an admissible order on the set of paths in Q.

Throughout the rest of this chapter, let Λ be a K-algebra with multiplicative basis $\mathcal{B}$, and assume that we are given some admissible order $\leq$ on $\mathcal{B}$. Although statements about Gröbner basis theory depend on the choice of order, we will often omit reference to the order when it is clear from context (in both this chapter and elsewhere in the thesis). For example, we might simply say that a set G is a Gröbner basis, when what we really mean is that G is a Gröbner basis *with respect to the admissible order* $\leq$.

Before we can state the definition of Gröbner bases, we need a few more preliminary concepts.

Definition 2.5. Let
$$x = \sum_{i=1}^{n} a_i p_i \in \Lambda,$$
where $a_i \in K \setminus \{ 0 \}$ and the basis elements $p_i \in \mathcal{B}$ are distinct. Then we say that the elements $p_1, \ldots, p_n$ **appear in** x. ■

For example, if $\Lambda = K[x, y]$ is the commutative polynomial ring in two variables, then x appears in $xy + x + 1$, but x does not appear in $xy + 1$.

Definition 2.6. Let $x \in \Lambda \setminus \{ 0 \}$. The **tip** of x, denoted $\mathrm{Tip}(x)$, is the largest basis element that appears in x. We denote the coefficient of $\mathrm{Tip}(x)$ in x by $\mathrm{CTip}(x)$.

If $X \subseteq \Lambda$ is a subset, we define
$$\mathrm{Tip}(X) = \{ \mathrm{Tip}(x) \mid x \in X \text{ and } x \neq 0 \} \subseteq \mathcal{B}.$$
We also define
$$\mathrm{NonTip}(X) = \mathcal{B} \setminus \mathrm{Tip}(X) \subseteq \mathcal{B}.$$
■

Example 2.7. Consider the following quiver:

$$Q\colon \begin{tikzcd} v \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

We give Q the (left) length-lex order with $\alpha < \beta$. Assume that $\mathrm{char}(K) = 0$. Then some examples of tips are
$$\mathrm{Tip}(2\alpha\beta + \beta) = \alpha\beta$$

and

$$\mathrm{Tip}(\alpha - \beta) = \beta.$$

For these two elements, the coefficients of the tips are

$$\mathrm{CTip}(2\alpha\beta + \beta) = 2$$

and

$$\mathrm{CTip}(\alpha - \beta) = -1.$$

■

We can now define the concept of Gröbner bases. We will eventually see that a Gröbner basis for an ideal is a generating set that is suitable for doing computations.

Definition 2.8. Let $I \subseteq \Lambda$ be an ideal. We say that a subset G of I is a **Gröbner basis** for I if

$$\langle \mathrm{Tip}(G) \rangle = \langle \mathrm{Tip}(I) \rangle .$$

■

When working with Gröbner bases, it is useful to have a notion of divisibility of basis elements. If p and q are elements of the multiplicative basis $\mathcal{B}$, then we say that p **divides** q (or that q is **divisible** by p) if there exist basis elements $r, s \in \mathcal{B}$ such that $q = rps$. We then write $p \mid q$.

Lemma 2.9. *Let S be a subset of the multiplicative basis $\mathcal{B}$. Let p be an element of $\mathcal{B} \cap \langle S \rangle$. Then p is divisible by some element of S.*

Proof. Since $p \in \langle S \rangle$, the element p can be written as a finite sum of elements of the form $\lambda s \mu$, where $\lambda, \mu \in \Lambda$ and $s \in S$. Since $\lambda, \mu \in \mathrm{Span}(\mathcal{B})$, it follows that

$$p = \sum_{i=1}^{n} a_i q_i s_i r_i$$

for scalars $a_i \in K \setminus \{ 0 \}$, basis elements $q_i, r_i \in \mathcal{B}$, and $s_i \in S$. Moreover, we can assume that $q_i s_i r_i \neq 0$, and that $q_i s_i r_i \neq q_j s_j r_j$ if $i \neq j$. But then we have written p as a linear combination of distinct basis elements $q_i s_i r_i \in \mathcal{B}$ with nonzero coefficients, so since p is an element of the basis $\mathcal{B}$, it follows from linear independence that $n = 1$ and $a_1 = 1$. Thus $p = q_1 s_1 r_1$. □

As a consequence of Lemma 2.9, we have the following alternate characterization of Gröbner bases, which we will use frequently throughout this thesis.

Corollary 2.10. *Let $I \subseteq \Lambda$ be an ideal, and let G be a subset of I. Then G is a Gröbner basis for I if and only if for all $x \in I \setminus \{ 0 \}$, there exists an element $g \in G$ such that* $\mathrm{Tip}(g)$ *divides* $\mathrm{Tip}(x)$.

Proof. First assume that G is a Gröbner basis for I. Let $x \in I \setminus \{ 0 \}$. Then

$$\mathrm{Tip}(x) \in \langle \mathrm{Tip}(I) \rangle = \langle \mathrm{Tip}(G) \rangle ,$$

and hence $\mathrm{Tip}(x) \in \mathcal{B} \cap \langle \mathrm{Tip}(G) \rangle$. Since $\mathrm{Tip}(G)$ is a subset of $\mathcal{B}$, Lemma 2.9 then implies that $\mathrm{Tip}(x)$ is divisible by some element of $\mathrm{Tip}(G)$, as desired.

Conversely, assume that for all $x \in I \setminus \{ 0 \}$, there exists some $g \in G$ such that $\mathrm{Tip}(g)$ divides $\mathrm{Tip}(x)$. Then every element of $\mathrm{Tip}(I)$ is a multiple of some element of $\mathrm{Tip}(G)$, and hence $\mathrm{Tip}(I) \subseteq \langle \mathrm{Tip}(G) \rangle$. But we also have $\mathrm{Tip}(G) \subseteq \mathrm{Tip}(I)$ since G is a subset of I, and hence

$$\langle \mathrm{Tip}(I) \rangle = \langle \mathrm{Tip}(G) \rangle .$$

Hence G is a Gröbner basis for I. □

Note that in our definition of Gröbner bases, we did not assume that G is a generating set for I. That is because this assumption would be redundant, as the following result shows.

Lemma 2.11. *Let $I \subseteq \Lambda$ be an ideal, and let $G \subseteq I$ be a Gröbner basis for I. Then $\langle G \rangle = I$.*

Proof. For the sake of contradiction, suppose that $I \neq \langle G \rangle$. Then, because $\leq$ is a well-order, there must exist some $x \in I \setminus \langle G \rangle$ such that $\mathrm{Tip}(x)$ is minimal with respect to $\leq$. Since G is a Gröbner basis, there exist $p, q \in \mathcal{B}$ and $g \in G$ such that

$$\mathrm{Tip}(x) = p \, \mathrm{Tip}(g) q.$$

Without loss of generality, assume that

$$\mathrm{CTip}(x) = \mathrm{CTip}(g) = 1.$$

Note that $x \neq pgq$ since $x \notin \langle G \rangle$, so $x - pgq \neq 0$. Therefore it has a tip, and

$$\mathrm{Tip}(x - pgq) < \mathrm{Tip}(x).$$

By the minimality of $\mathrm{Tip}(x)$, we have that $x - pgq \in \langle G \rangle$. This implies that $x \in \langle G \rangle$, which is a contradiction. □

We illustrate the concept of Gröbner bases with the following example.

Example 2.12. Let Q be the following quiver:

$$
\begin{tikzcd}
Q\colon & v_1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & v_2
\end{tikzcd}
$$

We give Q the length-lex order with

$$v_1 < v_2 < v_3 < \alpha < \beta.$$

Consider the ideal

$$I = \langle \beta\alpha - \beta, \alpha \rangle \subseteq KQ.$$

Then the set $\{ \beta\alpha - \beta, \alpha \}$ is not a Gröbner basis for I, because we have

$$\beta = \beta\alpha - (\beta\alpha - \beta) \in I,$$

but β is not divisible by any element of the set

$$\operatorname{Tip} \{ \beta\alpha - \beta, \alpha \} = \{ \beta\alpha, \alpha \}.$$

On the other hand, I is also generated by the set $\{ \alpha, \beta \}$, and this set is a Gröbner basis for I. ■

2.2 Normal forms and remainders

In this section, we will see how Gröbner bases can be used to do computations with quotient algebras. Let $I \subseteq \Lambda$ be an ideal, and suppose that we are interested in representing the quotient Λ/I on a computer. Then it is important to be able to test if two elements of Λ/I are equal. That is, given two elements λ and μ of Λ, we need to be able to determine whether

$$\lambda + I = \mu + I.$$

In general, it is not obvious how we can create a computer program that performs such an equality test. We will need to find a way to compute some sort of canonical representative for an element of Λ/I. In order to define such representatives, we require the following result. Recall that for a set $X \subseteq \Lambda$, we denote by $\operatorname{NonTip}(X)$ the set of basis elements in $\mathcal{B}$ that are not contained in the set $\operatorname{Tip}(X)$.

Proposition 2.13. *Let I be an ideal in Λ. Then as vector spaces,*

$$\Lambda = I \oplus \operatorname{Span}(\operatorname{NonTip}(I)).$$

Proof. Let $x \in I \cap \mathrm{Span}(\mathrm{NonTip}(I))$. If $x \neq 0$, then $\mathrm{Tip}(x) \in \mathrm{Tip}(I)$. But this is impossible since $x \in \mathrm{Span}(\mathrm{NonTip}(I))$. Hence we must have $x = 0$, which shows that $I + \mathrm{Span}(\mathrm{NonTip}(I))$ is a direct sum.

For the sake of contradiction, suppose that $\Lambda \neq I + \mathrm{Span}(\mathrm{NonTip}(I))$. Since $\leq$ is a well-order, there must exist some element

$$x \in \Lambda \backslash (I + \mathrm{Span}(\mathrm{NonTip}(I)))$$

such that $\mathrm{Tip}(x)$ is minimal. Without loss of generality, we may assume that $\mathrm{CTip}(x) = 1$. Then we either have $x - \mathrm{Tip}(x) = 0$ or

$$\mathrm{Tip}(x - \mathrm{Tip}(x)) < \mathrm{Tip}(x),$$

so it follows from the minimality of $\mathrm{Tip}(x)$ that

$$x - \mathrm{Tip}(x) \in I + \mathrm{Span}(\mathrm{NonTip}(I)).$$

This implies that $\mathrm{Tip}(x) \in \mathrm{Tip}(I)$, for if $\mathrm{Tip}(x) \in \mathrm{NonTip}(I)$ we would find that

$$x = (x - \mathrm{Tip}(x)) + \mathrm{Tip}(x) \in I + \mathrm{Span}(\mathrm{NonTip}(I)).$$

Since $\mathrm{Tip}(x) \in \mathrm{Tip}(I)$, there exist scalars $a_i \in K \backslash \{ 0 \}$ and basis elements $p_i \in \mathcal{B}$ such that

$$\mathrm{Tip}(x) + \sum_i a_i p_i \in I,$$

where $p_i < \mathrm{Tip}(x)$. By the minimality of $\mathrm{Tip}(x)$, we see that

$$\sum_i a_i p_i \in I + \mathrm{Span}(\mathrm{NonTip}(I)).$$

We then have

$$\mathrm{Tip}(x) = \left(\mathrm{Tip}(x) + \sum_i a_i p_i \right) - \sum_i a_i p_i \in I + \mathrm{Span}(\mathrm{NonTip}(I)).$$

But then both $x - \mathrm{Tip}(x)$ and $\mathrm{Tip}(x)$ are elements of $I + \mathrm{Span}(\mathrm{NonTip}(I))$. It follows that $x \in I + \mathrm{Span}(\mathrm{NonTip}(I))$, which is a contradiction. □

The following definition makes sense in light of Proposition 2.13.

Definition 2.14. Let I be an ideal of Λ, and let $x \in \Lambda$. The **normal form of x modulo I** is the unique element $N(x)$ of $\mathrm{Span}(\mathrm{NonTip}(I))$ such that $x - N(x) \in I$. ■

As a consequence of Proposition 2.13, we have the following result about normal forms.

Corollary 2.15. *Let I be an ideal in Λ, and let $x, y \in \Lambda$.*

(i) $x \equiv y \pmod{I}$ *if and only if* $N(x) = N(y)$.

(ii) $x \equiv N(x) \pmod{I}$.

(iii) There is a vector space isomorphism

$$f\colon \Lambda/I \to \mathrm{Span}(\mathrm{NonTip}(I))$$

given by $f(a + I) = N(a)$.

(iv) $N(xy) = N\big(N(x)N(y)\big)$.

Proof. The first three points follow immediately from Proposition 2.13. Moreover, since $x \equiv N(x) \pmod{I}$ and $y \equiv N(y) \pmod{I}$, we have

$$xy \equiv N(x)N(y) \pmod{I},$$

and hence $N(xy) = N(N(x)N(y))$ by point (i). □

Corollary 2.15 shows that normal forms provide a way to check whether two elements of the quotient algebra Λ/I are equal. Moreover, point (iii) shows that normal forms are compatible with the vector space structure of the quotient, while point (iv) shows that they are at least somewhat compatible with ring multiplication. This suggests that normal forms are a suitable choice of canonical representatives for elements of Λ/I, so if we wish to do computations in Λ/I, we should find a way to compute normal forms modulo I. To this end, we introduce a notion of division and remainders that is analogous to the classical "division algorithm" in the polynomial ring $K[x]$.

Definition 2.16. Let X be a (possibly infinite) subset of Λ, and let $y \in \Lambda \setminus \{ 0 \}$. We say that an element $r \in \Lambda$ is a **remainder of y under division by X**, and write $y \Rightarrow_X r$, if there exist elements $u_i, v_i \in \Lambda$ and $x_i \in X \setminus \{ 0 \}$ such that:

(i) $y = \sum_i u_i x_i v_i + r$.

(ii) For all i, if $u_i x_i v_i \neq 0$, then $\mathrm{Tip}(u_i x_i v_i) \leq \mathrm{Tip}(y)$.

(iii) For all $x \in X \setminus \{ 0 \}$ and all $p \in \mathcal{B}$, if p appears in r, then $\mathrm{Tip}(x_i)$ does not divide p.

If $y = 0$, then we say that $y \Rightarrow_X r$ if and only if $r = 0$. ■

Remark 2.17. If the remainder r in Definition 2.16 is zero, then point (iii) is redundant. In particular, it follows that if $X \subseteq Y$ are subsets of Λ, and if $y \Rightarrow_X 0$, then we also have $y \Rightarrow_Y 0$. ■

Note that if $\Lambda = K[x]$ is the polynomial ring in one variable and $X = \{g(x)\}$ is a set containing a single (nonzero) polynomial $g(x)$, then the notion of remainders defined in Definition 2.16 coincides with the usual notion of polynomial remainders. That is, it can be shown that for a polynomial $f(x) \in K[x]$, we have $f(x) \Rightarrow_{\{g(x)\}} r(x)$ if and only if there exists some polynomial $q(x)$ such that

$$f(x) = q(x)g(x) + r(x),$$

and such that either $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$.

We illustrate the concept of remainders with some examples. Note that both of these examples show that remainders need not be unique.

Example 2.18. Let $\Lambda = K[x]$ be the polynomial ring in one variable. We give the monomials in Λ the only possible admissible ordering, i.e. the one where

$$x^m < x^n$$

if and only if $m < n$. Let $g_1(x) = x$ and $g_2(x) = x^2 + 1$, and consider the set

$$X = \{\, g_1(x), g_2(x) \,\} .$$

Let $f(x) = x^3 + x^2 + 1$. Then we have

$$f(x) = x^2 g_1(x) + g_2(x) + 0.$$

Writing $f(x)$ in terms of $g_1(x)$ and $g_2(x)$ in this way satisfies Definition 2.16, so $f(x)$ has remainder 0 under division by X. However, we also have

$$f(x) = x^2 g_1(x) + x g_1(x) + 1,$$

so 1 is also a remainder of $f(x)$ under division by X. ■

Example 2.19. Consider the following quiver.

$$Q\colon \begin{tikzcd} v_1 \arrow[loop left, "\gamma"] \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & v_2 \arrow[loop right, "\delta"] \end{tikzcd}$$

Let $\Lambda = KQ$. We give Q the left length-lex order with

$$v_1 < v_2 < \alpha < \beta < \gamma < \delta.$$

Let $x_1 = \alpha\gamma$ and $x_2 = \delta\alpha + \beta$, and consider the set

$$X = \{ x_1, x_2 \} .$$

Let $y = \delta\alpha\gamma + \beta\gamma + \alpha\gamma$. Then we have

$$y = x_2\gamma + x_1 + 0,$$

and hence y has remainder 0 under division by X. We also have

$$y = \delta x_1 + x_1 + \beta\gamma,$$

so y also has remainder $\beta\gamma$ under division by X. ■

Lemma 2.20. *Let X be a subset of Λ, and let $y, r \in \Lambda$. Suppose that $r \neq 0$ and $y \Rightarrow_X r$. Then*

$$\mathrm{Tip}(r) \leq \mathrm{Tip}(y).$$

Proof. Let $u_i, v_i \in \Lambda$ and $x_i \in X \setminus \{ 0 \}$ be elements such that

$$y = \sum_i u_i x_i v_i + r, \tag{2.1}$$

and such that the other conditions in Definition 2.16 are satisfied. Suppose, for the sake of contradiction, that $\mathrm{Tip}(r) > \mathrm{Tip}(y)$. Then by (2.1), $\mathrm{Tip}(r)$ must cancel with some of the terms in $\sum_i u_i x_i v_i$, so there exists at least one index i such that $\mathrm{Tip}(r)$ appears in $u_i x_i v_i$. But then

$$\mathrm{Tip}(u_i x_i v_i) \geq \mathrm{Tip}(r),$$

which contradicts our assumption that

$$\mathrm{Tip}(u_i x_i v_i) \leq \mathrm{Tip}(y).$$

□

Algorithm 2.1, REMAINDER(y, X), lets us compute remainders, assuming that we're dividing by a finite set. Note that many of the variables in the algorithm, such as $m_{i,j}$, are not actually needed for the algorithm to be correct, but they do make it easier to reason about the algorithm.

Algorithm 2.1: REMAINDER(y, X)

Input: An element $y \in \Lambda$, a finite set $X = \{ x_1, \ldots, x_n \} \subseteq \Lambda \setminus \{ 0 \}$

Output: A remainder of y under division by X

1 **for** $i = 1$ **to** n **do**
2     $m_{i,0} \leftarrow 0$;
3 **end**
4 $r_0 \leftarrow 0$;
5 $z_0 \leftarrow y$;
6 $j \leftarrow 0$;
7 **while** $z_j \neq 0$ **do**
8   DIVISION_OCCURRED $\leftarrow$ False;
9   **for** $i = 1$ **to** n **do**
10   **if** $z_j \neq 0$ *and* $\mathrm{Tip}(z_j) = u \,\mathrm{Tip}(x_i) v$ *for some* $u, v \in \mathcal{B}$ **then**
11             $j \leftarrow j + 1$;
12             $m_{i,j} \leftarrow m_{i,j-1} + 1$;
13             $u_{i,m_{i,j}} \leftarrow \frac{\mathrm{CTip}(z_{j-1})}{\mathrm{CTip}(x_i)} u$;
14             $v_{i,m_{i,j}} \leftarrow v$;
15             $z_j \leftarrow z_{j-1} - u_{i,m_{i,j}} x_i v_{i,m_{i,j}}$;
16             $r_j \leftarrow r_{j-1}$;
17   DIVISION_OCCURRED $\leftarrow$ True;
18   **end**
19   **end**
20   **if** DIVISION_OCCURRED = False **then**
21         $j \leftarrow j + 1$;
22         $m_{i,j} \leftarrow m_{i,j-1}$;
23         $z_j \leftarrow z_{j-1} - \mathrm{CTip}(z_{j-1}) \,\mathrm{Tip}(z_{j-1})$;
24         $r_j \leftarrow r_{j-1} + \mathrm{CTip}(z_{j-1}) \,\mathrm{Tip}(z_{j-1})$;
25   **end**
26 **end**
27 $r \leftarrow r_j$;
28 **return** r;

Proposition 2.21. *Algorithm 2.1 produces a correct result. That is, given y and X as in the input to the algorithm, the algorithm terminates after a finite number of steps, and outputs a remainder r of y under division by X.*

Proof. If $y = 0$, then the algorithm terminates shortly after it reaches line 7, and clearly produces a correct result. So assume that $y \neq 0$. First note that there is a descending chain

$$\operatorname{Tip}(z_0) > \operatorname{Tip}(z_1) > \operatorname{Tip}(z_2) > \dots$$

of elements of the multiplicative basis $\mathcal{B}$. Since $\leq$ is a well-order, this chain must eventually end, i.e. $z_j = 0$ for some j, so the algorithm terminates.

Let us now prove that when the algorithm terminates, it produces a correct result. We proceed by induction on the index j. Let $P_1(j)$, $P_2(j)$, and $P_3(j)$ denote the following three statements.

$P_1(j)$: $y = \sum_{i=1}^{n} \sum_{k=1}^{m_{i,j}} u_{i,k} x_i v_{i,k} + r_j + z_j$.

$P_2(j)$: $\operatorname{Tip}(u_{i,k} x_i v_{i,k}) \leq \operatorname{Tip}(y)$ for all i and $1 \leq k \leq m_{i,j}$.

$P_3(j)$: If $p \in \mathcal{B}$ appears in r, then $\operatorname{Tip}(x_i)$ does not divide p for any i.

Clearly the statements $P_1(0)$, $P_2(0)$, and $P_3(0)$ are true. For the inductive step, suppose that $j > 0$, and assume that $P_1(j-1)$, $P_2(j-1)$, and $P_3(j-1)$ are true. Let us check that the statements $P_1(j)$ through $P_3(j)$ are then also true.

First assume that $\operatorname{Tip}(z_{j-1})$ is not divisible by the tip of any element of X, so that no divisions will occur during the for loop (line 7 to line 19). Hence we have

$$z_j = z_{j-1} - \operatorname{CTip}(z_{j-1}) \operatorname{Tip}(z_{j-1})$$

and

$$r_j = r_{j-1} + \operatorname{CTip}(z_{j-1}) \operatorname{Tip}(z_{j-1}).$$

Moreover, we have $m_{i,j} = m_{i,j-1}$. Thus the truth of $P_1(j)$ and $P_2(j)$ follows immediately from the inductive assumption. We also see that $P_3(j)$ follows from $P_3(j-1)$, because the only basis element that appears in r_j but not in r_{j-1} is $\operatorname{Tip}(z_{j-1})$, and we assumed that $\operatorname{Tip}(z_{j-1})$ is not divisible by any element of X.

Now assume that $\operatorname{Tip}(z_{j-1})$ is divisible by the tip of some element of X. Let i be the index such that the algorithm divides $\operatorname{Tip}(z_{j-1})$ by $\operatorname{Tip}(x_i)$, where $\mathrm{Tip}(z_{j-1}) = u\,\mathrm{Tip}(x_i)v$. Then we have

$$\begin{aligned}
&\sum_{l=1}^{n}\sum_{k=1}^{m_{l,j}} u_{l,k}x_l v_{l,k} + r_j + z_j \\
&\stackrel{*}{=} \left(\left(\sum_{l=1}^{n}\sum_{k=1}^{m_{l,j-1}} u_{l,k}x_l v_{l,k}\right) + u_{i,m_{i,j}}x_i v_{i,m_{i,j}}\right) + r_{j-1} + \left(z_{j-1} - u_{i,m_{i,j}}x_i v_{i,m_{i,j}}\right) \\
&= \sum_{l=1}^{n}\sum_{k=1}^{m_{l,j-1}} u_{l,k}x_l v_{l,k} + r_{j-1} + z_{j-1} \\
&\stackrel{\dagger}{=} y,
\end{aligned}$$

where the equality marked with $*$ is due to the fact that $m_{i,j} = m_{i,j-1} + 1$ and $m_{l,j} = m_{l,j-1}$ for $l \neq i$, while the equality marked with $\dagger$ follows from the inductive assumption. Thus we see that $P_1(j)$ is true. We also see that $P_2(j)$ is true because

$$\mathrm{Tip}(u_{i,m_{i,j}}x_i v_{i,m_{i,j}}) = \mathrm{Tip}(z_j) < \mathrm{Tip}(z_{j-1}) < \ldots < \mathrm{Tip}(z_0) = \mathrm{Tip}(y),$$

while $P_3(j)$ is true because $r_j = r_{j-1}$.

By induction, we see that the statements $P_1(j)$, $P_2(j)$, and $P_3(j)$ hold for all values of the index j. But when the algorithm terminates, the element z_j is zero, and then these three statements are just the definition of remainders. Hence we see that, at the end of the algorithm, r_j is a remainder of y under division by X. $\square$

Remark 2.22. Algorithm 2.1 assumes that we are performing division by a finite set. However, we can also use the algorithm to perform division by certain infinite sets. Suppose that $X \subseteq \Lambda\setminus\{\,0\,\}$ is an infinite set such that for any basis element $p \in \mathcal{B}$, the set

$$\{\, x \in X \mid \mathrm{Tip}(x) \leq p \,\}$$

is finite. Given an element $y \in \Lambda$, let $x_1, \ldots, x_n \in X$ be those elements of X whose tips are less than or equal to $\mathrm{Tip}(y)$. Let r be a remainder of y under division by the finite set $\{\, x_1, \ldots, x_n \,\}$. Then for any $x \in X\setminus\{\, x_1, \ldots, x_n \,\}$, we have

$$\mathrm{Tip}(x) > \mathrm{Tip}(y) \geq \mathrm{Tip}(r).$$

It follows that $\mathrm{Tip}(x) \nmid p$ whenever $p \in \mathcal{B}$ is a basis element that appears in r. Hence r is also a remainder of y under division by X. Thus we can perform division of y by the infinite set X by applying Algorithm 2.1 to y and the finite set $\{\, x_1, \ldots, x_n \,\}$.

Also note that the proof of Proposition 2.21 does not actually depend on the assumption that the set X is finite, except of course for the fact that the algorithm REMAINDER(y, X) could not be implemented on a computer (and hence would not deserve to be called an "algorithm") if X were infinite. In particular, if $X \subseteq \Lambda$ is an arbitrary infinite subset, then Proposition 2.21 shows that there exists a remainder of y under division by X. ■

As shown in Example 2.18 and Example 2.19, remainders need not be unique. However, when we are dividing by a Gröbner basis, there is only one remainder.

Proposition 2.23. *Let $I \subseteq \Lambda$ be an ideal, and let G be a Gröbner basis for I. Let y be an element of Λ. Then the remainder of y under division by G is unique. In fact, the remainder is the normal form of y modulo I.*

Proof. If $y = 0$, then the only remainder of y is 0, which is also the normal form of y. Assume that $y \neq 0$.

Let $r \in \Lambda$ be a remainder of y under division by G. Then there exist elements $u_i, v_i \in \Lambda$ and $g_i \in G$ such that

$$y = \sum_i u_i g_i v_i + r,$$

and such that the other conditions in Definition 2.16 are satisfied. We first show that $r \in \mathrm{Span}(\mathrm{NonTip}(I))$. Suppose $p \in \mathcal{B}$ is a basis element that appears in r. Then by the definition of remainders, we have assumed that p is not divisible by $\mathrm{Tip}(g)$ for any element $g \in G$. But since G is a Gröbner basis for I, this must mean that $p \notin \mathrm{Tip}(I)$, i.e. $p \in \mathrm{NonTip}(I)$. Hence $r \in \mathrm{Span}(\mathrm{NonTip}(I))$.

Now observe that

$$y - r = \sum_i u_i g_i v_i,$$

which is an element of I because each g_i lies in I. Since the normal form of y is the unique element

$$N(y) \in \mathrm{Span}(\mathrm{NonTip}(I))$$

such that $y - N(y) \in I$, it follows that $r = N(y)$. □

Combining Proposition 2.23 with Algorithm 2.1, we get the following important corollary.

Corollary 2.24. *Let $I \subseteq \Lambda$ be an ideal, and suppose that $G \subseteq I$ is a finite Gröbner basis. Then there is an algorithm to compute normal forms modulo I.*

Recall that for an ideal $I \subseteq \Lambda$ and for $x, y \in \Lambda$, we have $N(x) = N(y)$ if and only if $x + I = y + I$. In particular, Corollary 2.24 shows that there is an algorithm to determine whether two elements of Λ/I are equal, at least if we are given a finite Gröbner basis for I. This solves the problem of how to represent Λ/I on a computer, which was outlined in the beginning of this section. This is one of the main reasons why Gröbner bases are useful.

Of course, none of this is of very much use unless we can actually find a (finite) Gröbner basis for an ideal. We will come back to this problem in Section 2.4.

2.3 Reduced Gröbner bases

We will now discuss reduced Gröbner bases. We will see that every ideal has a unique reduced Gröbner basis, which is in some sense the "best" possible Gröbner basis for the ideal, in part because it satisfies a minimality condition. We start with the following definition.

Definition 2.25. Let $I \subseteq \Lambda$ be an ideal, and let $G \subseteq I$ be a Gröbner basis for I. We say that G is a **reduced Gröbner basis** if the following three conditions hold:

(i) $0 \notin G$.

(ii) If $g \in G$, then $\mathrm{CTip}(g) = 1$.

(iii) If $g, g' \in G$ are distinct elements and $p \in \mathcal{B}$ is a basis element which appears in g, then $\mathrm{Tip}(g')$ does not divide p.

■

The following lemma will be useful when proving the uniqueness of reduced Gröbner bases.

Lemma 2.26. *Let $I \subseteq \Lambda$ be an ideal, and suppose $G \subseteq I$ is a reduced Gröbner basis for I. Then*

$$g - \mathrm{Tip}(g) \in \mathrm{Span}(\mathrm{NonTip}(I))$$

for all $g \in G$.

Proof. Let $g \in G$, and let $x = g - \mathrm{Tip}(g)$. For the sake of contradiction, suppose that $x \notin \mathrm{Span}(\mathrm{NonTip}(I))$. Then there must be some basis element $p \in \mathcal{B}$ such that $p \in \mathrm{Tip}(I)$ and p appears in x. Since G is a Gröbner basis for I, there exist elements $q, r \in \mathcal{B}$ and $g' \in G$ such that $p = q\,\mathrm{Tip}(g')r$. We then have that $g \neq g'$, because

$$\begin{aligned} \mathrm{Tip}(g') &\leq q\,\mathrm{Tip}(g')r \\ &= p \\ &\leq \mathrm{Tip}(x) \\ &< \mathrm{Tip}(g). \end{aligned}$$

The first inequality holds because $\leq$ is an admissible order, and the last inequality holds because $\mathrm{CTip}(g) = 1$. But this contradicts item (iii) in Definition 2.25, because $\mathrm{Tip}(g')$ divides p, and p appears in g. Therefore it must be the case that $x \in \mathrm{Span}(\mathrm{NonTip}(I))$. $\square$

The next result will help us prove the existence of a reduced Gröbner basis. We first introduce some terminology. A **monomial** is an element of $\mathcal{B}$. An ideal is called a **monomial ideal** if it is generated by a set of monomials.

Proposition 2.27. *Let $I \subseteq \Lambda$ be a monomial ideal. Then there exists a smallest monomial generating set of I.*

Proof. If $I = 0$, then $\emptyset$ is the smallest monomial generating set. So assume that $I \neq 0$. (In particular, this implies that a generating set of I cannot be empty.)

Let $S = I \cap \mathcal{B}$ be the set of all monomials in I. Consider the set

$$M = \{\, p \in S \mid \text{if } q \in S \text{ divides } p, \text{ then } q = p \,\}.$$

We claim that M is the smallest monomial generating set of I. To see that this is true, we first show that M generates I. Let C be some monomial generating set of I, and let c be an element of C. If $c \notin M$, then c is properly divisible by some element of S, so we can write $c = p_1 s_1 q_1$ for some $s_1 \in S \setminus \{\, c \,\}$ and $p_1, q_1 \in \mathcal{B}$. Observe that $c > s_1$ since $\leq$ is an admissible order. If we also have $s_1 \notin M$, we can continue this process, and we get a descending chain of elements of S:

$$c > s_1 > s_2 > \ldots$$

Since $\leq$ is a well-order, this chain must eventually terminate; that is, the process cannot be continued indefinitely. Thus we eventually get

$$c = (p_1 p_2 \ldots p_n) m (q_n q_{n-1} \ldots q_1) \in \langle M \rangle,$$

where $p_i, q_i \in \mathcal{B}$ and $m \in M$. This shows that $C \subseteq \langle M \rangle$, and it follows that $I = \langle M \rangle$.

Next we show that any monomial generating set of I contains M. So let $C \subseteq I$ be a monomial generating set, and let m be any element of M. Since $m \in \langle C \rangle$, it follows from Lemma 2.9 that there exists some $c \in C$ which divides m. But since $c \in S$ and $m \in M$, the definition of the set M implies that $c = m$. In particular we see that $m \in C$, and hence $M \subseteq C$. □

We can now prove the following result, which allows us to speak of *the* reduced Gröbner basis for an ideal (with respect to a given admissible order).

Proposition 2.28. *Let $I \subseteq \Lambda$ be an ideal. Then there exists a unique reduced Gröbner basis for I.*

Proof. We first prove the existence of a reduced Gröbner basis. Let T be the smallest monomial generating set of $\langle \mathrm{Tip}(I) \rangle$, which exists by Proposition 2.27. Consider the set

$$G = \{\, t - N(t) \mid t \in T \,\} \subseteq I,$$

where $N(t)$ is the normal form of t modulo I. We claim that G is a reduced Gröbner basis for I. To see that G is a Gröbner basis, let $t \in T$, and let

$$g = t - N(t) \in G.$$

Since $\mathrm{Tip}(I)$ is a monomial generating set for $\langle \mathrm{Tip}(I) \rangle$, we have $T \subseteq \mathrm{Tip}(I)$, and in particular $t \in \mathrm{Tip}(I)$. It follows that $g \neq 0$, because $N(t) \notin \mathrm{Tip}(I)$. Moreover, we have $\mathrm{Tip}(g) \in \mathrm{Tip}(I)$ and $N(t) \in \mathrm{Span}(\mathrm{NonTip}(I))$, and hence $\mathrm{Tip}(g)$ does not appear in $N(t)$. It follows that $\mathrm{Tip}(g) = t$, and hence $\mathrm{Tip}(G) = T$. Since T is a generating set for $\langle \mathrm{Tip}(I) \rangle$, we see that G is a Gröbner basis.

To see that G is reduced, note that we saw in the previous paragraph that $0 \notin G$. We also have $\mathrm{CTip}(g) = 1$ for all $g \in G$, since $\mathrm{Tip}(t - N(t)) = t$ for all $t \in T$. Finally, suppose that $g, g' \in G$ are distinct elements. Then $g = t - N(t)$ and $g' = t' - N(t')$ for some distinct elements $t, t' \in T$. From the construction of the minimal monomial generating set T in the proof of Proposition 2.27,[1] it is clear that t' cannot divide t. Moreover, t' does not divide any basis element which appears in $N(t)$, since this would imply that $N(t) \notin \mathrm{Span}(\mathrm{NonTip}(I))$. It follows that G is a reduced Gröbner basis.

[1]Note that we used the variable name M instead of T in the proof of Proposition 2.27.

We now prove the uniqueness of G. Let H be some reduced Gröbner basis for I. For any $h \in H$, we have

$$\mathrm{Tip}(h) - h \in \mathrm{Span}(\mathrm{NonTip}(I))$$

by Lemma 2.26. Then since $h \in I$ and

$$\mathrm{Tip}(h) = (\mathrm{Tip}(h) - h) + h,$$

it follows that $N(\mathrm{Tip}(h)) = \mathrm{Tip}(h) - h$. In particular,

$$\mathrm{Tip}(h) - N(\mathrm{Tip}(h)) = h.$$

This shows that the set

$$\{\, x - N(x) \mid x \in \mathrm{Tip}(H) \,\}$$

is equal to H. But $\langle \mathrm{Tip}(H) \rangle = I$ since H is a Gröbner basis, so $T \subseteq \mathrm{Tip}(H)$. Then we have

$$\{\, t - N(t) \mid t \in T \,\} \subseteq \{\, x - N(x) \mid x \in \mathrm{Tip}(H) \,\},$$

or in other words, $G \subseteq H$.

To see that we also have $H \subseteq G$, suppose that there exists some element $h \in H \backslash G$. Since G is a Gröbner basis, there exists an element $g \in G$ such that $\mathrm{Tip}(g)$ divides $\mathrm{Tip}(h)$. But this contradicts our assumption that H is a reduced Gröbner basis, because $g \in H$ and $g \neq h$. Therefore $H = G$. □

Remark 2.29. The proof of Proposition 2.28 suggests an algorithm for computing the reduced Gröbner basis for an ideal I, assuming that we already know a finite Gröbner basis G for I. We proceed as follows: We first compute the set

$$T = \{\, t \in \mathrm{Tip}(G) \mid \text{if } t' \text{ divides } t \text{ for } t' \in \mathrm{Tip}(G), \text{ then } t' = t \,\}.$$

For each $t \in T$, we then use Algorithm 2.1 to compute the normal form $N(t)$ of t modulo I, which is possible because we know a finite Gröbner basis for I. Then the set

$$\{\, t - N(t) \mid t \in T \,\}$$

is the reduced Gröbner basis of I. ■

One of the reasons why reduced Gröbner bases are nice is the following minimality condition.

Proposition 2.30 ([Lea06, Proposition 2.11])**.** *Let $I \subseteq \Lambda$ be an ideal, and let G be the reduced Gröbner basis of I. Let $H \subseteq I$ be any subset. Then H is a Gröbner basis for the ideal I if and only if* $\operatorname{Tip}(G) \subseteq \operatorname{Tip}(H)$*.*

Proof. First assume that $\operatorname{Tip}(G) \subseteq \operatorname{Tip}(H)$. Since G is a Gröbner basis for I, it follows that

$$\langle \operatorname{Tip}(H) \rangle = \langle \operatorname{Tip}(I) \rangle .$$

In other words, H is a Gröbner basis for I.

Conversely, assume that H is a Gröbner basis for I. Then $\operatorname{Tip}(H)$ is a monomial generating set for $\langle \operatorname{Tip}(I) \rangle$, so

$$T \subseteq \operatorname{Tip}(H),$$

where T is the smallest monomial generating set for $\langle \operatorname{Tip}(I) \rangle$. But by the proof of Proposition 2.28, T is equal to $\operatorname{Tip}(G)$, so we see that

$$\operatorname{Tip}(G) \subseteq \operatorname{Tip}(H).$$

□

Corollary 2.31 ([Lea06])**.** *Let $I \subseteq \Lambda$ be an ideal, and let G be the reduced Gröbner basis of I. Then G has the smallest cardinality of all Gröbner bases for I.*

Proof. Let H be any Gröbner basis for I. Then by Proposition 2.30,

$$\operatorname{Tip}(G) \subseteq \operatorname{Tip}(H),$$

and hence

$$|\operatorname{Tip}(G)| \leq |\operatorname{Tip}(H)|.$$

Note that if $g, g' \in G$ are distinct elements, then $\operatorname{Tip}(g) \neq \operatorname{Tip}(g')$ by Definition 2.25. It follows that $|G| = |\operatorname{Tip}(G)|$. Moreover, the map

$$\begin{aligned} H \setminus \{ 0 \} &\to \operatorname{Tip}(H) \\ h &\mapsto \operatorname{Tip}(h) \end{aligned}$$

is surjective, and hence $|\operatorname{Tip}(H)| \leq |H|$. Thus we see that

$$|G| = |\operatorname{Tip}(G)| \leq |\operatorname{Tip}(H)| \leq |H|,$$

as desired. □

In particular, an ideal I has a finite Gröbner basis if and only if the reduced Gröbner basis of I is finite.

2.4 Gröbner bases in path algebras

Up until this point, we have been assuming that Λ is some K-algebra with multiplicative basis $\mathcal{B}$ and admissible order $\leq$. This was sufficient for our purposes, as none of the theory we have dealt with so far required any additional assumptions. However, much of the content that we will cover in this section requires a somewhat different treatment for path algebras than for the classical case of commutative polynomial rings. We therefore narrow our scope to focus only on path algebras. Throughout the rest of this chapter, assume that $\Lambda = KQ$ is the path algebra of some quiver Q.

2.4.1 Tip reduced and uniform sets

When working with Gröbner bases, and with generating sets for ideals more generally, it is often convenient to impose some additional technical conditions on the sets we're working with. The first condition is that of uniformity.

Definition 2.32. Let

$$x = \sum_{i=1}^{n} a_i p_i \in \Lambda,$$

where $a_i \in K \setminus \{ 0 \}$ and where the $p_i \in \mathcal{B}$ are distinct paths. Then x is said to be **uniform** if there exist vertices v and w in Q such that every path p_i starts at v and ends at w.

A subset $X \subseteq \Lambda$ is called **uniform** if every individual element of X is uniform. ■

Example 2.33. Consider the following quiver.

$$\begin{tikzcd} Q\colon\ v_1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & v_2 \arrow[r, "\gamma"] & v_3 \end{tikzcd}$$

Then $\alpha - \beta$ is uniform, while the elements $\alpha + v_1$ and $\alpha + \gamma$ are not uniform. ■

The following result shows that we can always replace a Gröbner basis with a uniform Gröbner basis.

Lemma 2.34. *Let G be a generating set for an ideal I in Λ. Then the set*

$$H = \{ vgw \mid g \in G,\ v \text{ and } w \text{ are vertices in } Q \}$$

is a uniform generating set for I. Moreover, if G is a Gröbner basis for I, then so is H.

Proof. The set H is clearly uniform. Moreover, every element $g \in G$ can be written as
$$g = \sum_{v,w \in Q_0} vgw \in \langle H \rangle ,$$
which shows that $I \subseteq \langle H \rangle$ since G generates I. Because $H \subseteq I$, we also have $\langle H \rangle \subseteq I$. Hence $\langle H \rangle = I$.

Now suppose that G is a Gröbner basis. If $g \in G \setminus \{ 0 \}$, then there exist vertices v and w in Q such that $\mathrm{Tip}(g)$ is a path from w to v. Then $vgw \in H$, and
$$\mathrm{Tip}(g) = v \,\mathrm{Tip}(g) w = \mathrm{Tip}(vgw) \in \mathrm{Tip}(H).$$
This shows that
$$\mathrm{Tip}(G) \subseteq \mathrm{Tip}(H).$$
Since G is a Gröbner basis, this implies that H is also a Gröbner basis. □

The next technical condition we will consider is the notion of a tip reduced set.

Definition 2.35. A subset $X \subseteq \Lambda$ is said to be **tip reduced** if for all $x, y \in X \setminus \{ 0 \}$ with $x \neq y$, we have that $\mathrm{Tip}(x)$ does not divide $\mathrm{Tip}(y)$. ■

A notable example of a set that is both tip reduced and uniform is the reduced Gröbner basis of an ideal, as the following proposition shows.

Proposition 2.36 ([Lea06, Proposition 2.13])**.** *Let $I \subseteq \Lambda$ be any ideal, and let G be the reduced Gröbner basis of I. Then G is tip reduced and uniform.*

Proof. The fact that G is tip reduced follows immediately from Definition 2.25.

For the sake of contradiction, assume that G is not uniform. Let $g \in G$ be some non-uniform element. Then there exist vertices v and w in Q such that $v \,\mathrm{Tip}(g) w = 0$, but $vgw \neq 0$. Because $vgw \in I$ and G is a Gröbner basis for I, there exists some $g' \in G$ such that $\mathrm{Tip}(g') \mid \mathrm{Tip}(vgw)$. Note that
$$\mathrm{Tip}(g') \leq \mathrm{Tip}(vgw) < \mathrm{Tip}(g),$$
and hence $g' \neq g$. But this contradicts the fact that G is a reduced Gröbner basis. □

Algorithm 2.2, TipReduce(G), allows us to replace any finite generating set for an ideal with a generating set that is tip reduced.

Algorithm 2.2: TipReduce(G)

Input: A finite subset $G = \{ g_1, \ldots, g_n \} \subseteq \Lambda$
Output: A finite tip reduced subset $H \subseteq \Lambda$ that generates the same ideal as G

```
for i = 1 to n do
    h_i ← g_i;
end
H_0 ← { h_1, ..., h_n };
k ← 0;
do
    MODIFIED ← False;
    for i = 1 to n with h_i ≠ 0 do
        for j = 1 to n with j ≠ i do
            if h_j ≠ 0 and Tip(h_j) = p Tip(h_i) q for paths p, q then
                x ← h_j − (CTip(h_j) / CTip(h_i)) p h_i q;
                k ← k + 1;
                H_k ← { h_1, ..., h_{j−1}, x, h_{j+1}, ..., h_n };
                h_j ← x;
                MODIFIED ← True;
            end
        end
    end
while MODIFIED = True;
H ← H_k;
return H;
```

Proposition 2.37. *The algorithm* TipReduce(G) *(Algorithm 2.2) produces a correct result. Moreover, if G is uniform, then so is H; and if G is a Gröbner basis, then so is H.*

Proof. We first show that the algorithm terminates. During every iteration of the do-while loop (except the last iteration, if there is one), there is at least one value of the index k such that some element $h_j \in H_k$ is replaced with an element whose tip is strictly smaller. Since $\leq$ is a well-order and the set H_k is finite (and because its size stays constant as k increases), this can only happen a finite number of times. Hence the algorithm terminates.

When the do-while loop terminates, there do not exist any distinct elements $h_i, h_j \in H_k$ such that $\mathrm{Tip}(h_i)$ divides $\mathrm{Tip}(h_j)$. In other words, the returned set H is tip reduced.

Next we show that $\langle G \rangle = \langle H_k \rangle$ for all k. This is clearly true for $k = 0$. So let $k \geq 0$, and assume, inductively, that $\langle H_k \rangle = \langle G \rangle$. On line 11, we define

$$x = h_j - \frac{\mathrm{CTip}(h_j)}{\mathrm{CTip}(h_i)} p h_i q.$$

Since h_i and h_j are elements of H_k, x is an element of $\langle H_k \rangle = \langle G \rangle$. Hence $\langle H_{k+1} \rangle \subseteq \langle G \rangle$. To see that $\langle G \rangle \subseteq \langle H_{k+1} \rangle$, observe that

$$h_j = x + \frac{\mathrm{CTip}(h_j)}{\mathrm{CTip}(h_i)} p h_i q.$$

This is an element of $\langle H_{k+1} \rangle$ because x and h_i are elements of H_{k+1}. Hence $\langle H_{k+1} \rangle$ contains the set

$$\{\, h_1, \ldots, h_n \,\},$$

which is a generating set for $\langle G \rangle$ by our inductive assumption. This shows that $\langle G \rangle = \langle H_{k+1} \rangle$.

Now suppose that the set G is uniform. In order to show that the set H produced by the algorithm is uniform, it is enough to show that the element

$$x = h_j - \frac{\mathrm{CTip}(h_j)}{\mathrm{CTip}(h_i)} p h_i q$$

defined on line 11 is uniform. Clearly $p h_i q$ is uniform. Moreover, we may assume (inductively) that h_j is uniform. Since $\mathrm{Tip}(h_j) = p \,\mathrm{Tip}(h_i) q$, the element x must also be uniform.

Finally, suppose that G is a Gröbner basis. Since h_j is replaced with x on line 13, the element $\mathrm{Tip}(h_j) \in \mathrm{Tip}(H_k)$ might not be contained in $\mathrm{Tip}(H_{k+1})$. However, $\mathrm{Tip}(h_i)$ is still an element of $\mathrm{Tip}(H_{k+1})$, so we see that

$$\mathrm{Tip}(h_j) \in \langle \mathrm{Tip}(H_{k+1}) \rangle,$$

because $\mathrm{Tip}(h_j) = p \,\mathrm{Tip}(h_i) q$. By induction, it follows that H is a Gröbner basis. □

We see that any finite generating set (or Gröbner basis) can be replaced with a tip reduced uniform generating set (or Gröbner basis) by applying Lemma 2.34 and Algorithm 2.2, in that order. Thus it is not unreasonable to assume that the sets we're working with are tip reduced and uniform. Such sets are often more convenient to work with, one reason being that there exists a criterion for determining if such a set is a Gröbner basis, as we will see in the next subsection.

2.4.2 Overlap relations

In the theory of Gröbner bases in commutative polynomial rings, there is a concept called S-polynomials, which are used in the computation of Gröbner bases. When working with path algebras, the analogous concept is that of overlap relations, which we now define.

Definition 2.38. Let f and g be elements of $\Lambda \setminus \{ 0 \}$ (not necessarily distinct) and suppose that there exist paths p and q in Q such that the following conditions hold:

(i) $\operatorname{Tip}(f)p = q \operatorname{Tip}(g) \neq 0$.

(ii) $\operatorname{Tip}(f)$ does not divide q and $\operatorname{Tip}(g)$ does not divide p.

Then we say that f and g **have a** (p, q)**-overlap**, and their **overlap relation** is

$$\mathfrak{o}(f, g, p, q) = \frac{fp}{\operatorname{CTip}(f)} - \frac{qg}{\operatorname{CTip}(g)}.$$

■

Remark 2.39. If $f \in \Lambda \setminus \{ 0 \}$ is any nonzero element, then there exist vertices v and w such that

$$\operatorname{Tip}(f)v = w \operatorname{Tip}(f).$$

If $\operatorname{Tip}(f)$ is not a vertex, then this technically satisfies the definition of an overlap. However, this is a rather trivial and uninteresting overlap, so in this thesis we will always ignore this particular kind of overlap. ■

Note that if condition (i) in Definition 2.38 is satisfied, then condition (ii) is equivalent to assuming that $\mathfrak{l}(p) < \mathfrak{l}(\operatorname{Tip}(f))$ and $\mathfrak{l}(q) < \mathfrak{l}(\operatorname{Tip}(g))$. This ensures that there really is an "overlap", in the sense that if

$$\operatorname{Tip}(f) = \alpha_1 \dots \alpha_m$$

and

$$\operatorname{Tip}(g) = \beta_1 \dots \beta_n$$

for arrows α_i and β_j, then we must have

$$\alpha_{\mathfrak{l}(q)+1} \dots \alpha_m = \beta_1 \dots \beta_{n-\mathfrak{l}(p)}.$$

We illustrate this with an example.

Example 2.40. We consider the following quiver.

$$Q\colon\ \begin{tikzcd} v_1 \arrow[r, bend left, "\alpha"] & v_2 \arrow[l, bend left, "\beta"] \arrow[r, "\gamma"] & v_3 \end{tikzcd}$$

We give Q the left length-lex order with

$$v_1 < v_2 < v_3 < \alpha < \beta < \gamma.$$

Assume that $\operatorname{char}(K) = 0$. Let $f = \gamma\alpha\beta + \gamma$ and $g = 2\alpha\beta\alpha + \alpha$. The path $\gamma\alpha\beta$ contains a factor $\alpha\beta$ on the right, while $\alpha\beta\alpha$ contains a factor of $\alpha\beta$ on the left. Hence f and g have an overlap, or more precisely an (α, γ)-overlap. The overlap relation is

$$\begin{aligned} \mathfrak{o}(f, g, \alpha, \gamma) &= f\alpha - \frac{1}{2}\gamma g \\ &= \gamma\alpha\beta\alpha + \gamma\alpha - \gamma\alpha\beta\alpha - \frac{1}{2}\gamma\alpha \\ &= \frac{1}{2}\gamma\alpha. \end{aligned}$$

Now let $h = \alpha\beta\alpha\beta\alpha - \alpha$. This element overlaps with itself in two ways: there is both a $(\beta\alpha\beta\alpha, \alpha\beta\alpha\beta)$-overlap and a $(\beta\alpha, \alpha\beta)$-overlap. In both cases the overlap relation is 0. ■

We will now see how overlap relations can be used to check if a given generating set is a Gröbner basis.

Theorem 2.41. *Let $G \subseteq \Lambda$ be a (possibly infinite) tip reduced uniform set. Then the following are equivalent.*

(i) G is a Gröbner basis for the ideal $\langle G \rangle$.

(ii) For all elements $g_1, g_2 \in G \setminus \{ 0 \}$ and all paths p and q in Q, if g_1 and g_2 have a (p, q)-overlap, then

$$\mathfrak{o}(g_1, g_2, p, q) \Rightarrow_G 0.$$

Proof. First assume that G is a Gröbner basis. For any $g_1, g_2 \in G$ with a (p, q)-overlap, the overlap relation is

$$\mathfrak{o}(g_1, g_2, p, q) = \frac{g_1 p}{\operatorname{CTip}(g_1)} - \frac{q g_2}{\operatorname{CTip}(g_2)}.$$

This is an element of $\langle G \rangle$. As mentioned in Remark 2.22, there exists some element $r \in \Lambda$ such that

$$\mathfrak{o}(g_1, g_2, p, q) \Rightarrow_G r.$$

But since G is a Gröbner basis, r must be the normal form (modulo the ideal $\langle G \rangle$) of the overlap relation, by Proposition 2.23. Since the overlap relation is an element of $\langle G \rangle$, its normal form is zero, and hence $r = 0$, as desired.

Conversely, assume that all overlap relations of elements of G have remainder zero under division by G. For the sake of contradiction, assume that G is not a Gröbner basis. Then there exists some element $x \in \langle G \rangle \setminus \{ 0 \}$ such that $\mathrm{Tip}(x) \notin \langle \mathrm{Tip}(G) \rangle$. Since x is an element of $\langle G \rangle$, we may write

$$x = \sum_{i,j} a_{ij} p_{ij} g_i q_{ij} \tag{2.2}$$

for scalars $a_{ij} \in K \setminus \{ 0 \}$, paths p_{ij} and q_{ij} in Q, and distinct elements $g_i \in G$. We may assume that $p_{ij} g_i q_{ij} \neq 0$. Each g_i may be written as a K-linear combination of distinct paths γ_{ik}:

$$g_i = \sum_k b_{ik} \gamma_{ik},$$

for scalars $b_{ik} \in K \setminus \{ 0 \}$. We then get

$$x = \sum_{i,j,k} a_{ij} b_{ik} p_{ij} \gamma_{ik} q_{ij}. \tag{2.3}$$

Let p^* be the largest of the paths $p_{ij} \gamma_{ik} q_{ij}$ with respect to the ordering $\leq$, and write $p^* = p_{ij} \gamma_{ik} q_{ij}$ for some fixed i, j, and k. Since g_i is uniform, we have that $p_{ij} \,\mathrm{Tip}(g_i) q_{ij} \neq 0$, and hence

$$p_{ij} \gamma_{ik} q_{ij} \leq p_{ij} \,\mathrm{Tip}(g_i) q_{ij}.$$

It follows that $\gamma_{ik} = \mathrm{Tip}(g_i)$, and in particular, $p^* \in \langle \mathrm{Tip}(G) \rangle$. But we have assumed that $\mathrm{Tip}(x) \notin \langle \mathrm{Tip}(G) \rangle$, so $p^* \neq \mathrm{Tip}(x)$. But then $\mathrm{Tip}(x) < p^*$, so the occurrences of p^* in the right-hand side of (2.3) must cancel each other out. Thus p^* appears at least twice in (2.3), so there exist indices i, i', j, j' with $(i, j) \neq (i', j')$ such that

$$p^* = p_{ij} \,\mathrm{Tip}(g_i) q_{ij} = p_{i'j'} \,\mathrm{Tip}(g_{i'}) q_{i'j'}.$$

Write $p = p_{ij}$, $g = g_i$, $q = q_{ij}$, $a = a_{ij}$, $p' = p_{i'j'}$, $g' = g_{i'j'}$, $q' = q_{i'j'}$, and $a' = a_{i'j'}$.

Out of all the possible ways to write x as in (2.2), we may assume that we have chosen one such that p^* is minimal with respect to the well-order $\leq$. Moreover, once we have chosen a minimal value of p^*, we may assume that the number of appearances of p^* in the right-hand side of (2.3) is minimal. Then we have the following possible cases.

1. $\underline{\mathfrak{l}(p) = \mathfrak{l}(p')}$: It follows that $p = p'$, and hence $\mathrm{Tip}(g)$ contains $\mathrm{Tip}(g')$ as a subpath, or vice versa. But then one of them divides the other, so since G is tip reduced, it follows that $g = g'$ and $i = i'$. We then also have that $q = q'$. But then we can rewrite (2.2) by replacing the two terms $apgq$ and $a'p'g'q'$ with the one term $(a + a')pgq$. This produces a version of (2.3) in which the number of occurrences of p^* is one less than before (or possibly two less, if $a + a' = 0$[2]), which contradicts our assumption about p^*.

2. $\underline{\mathfrak{l}(p) < \mathfrak{l}(p')}$: Either $\mathfrak{l}(q) = \mathfrak{l}(q')$, $\mathfrak{l}(q) < \mathfrak{l}(q')$, or $\mathfrak{l}(q) > \mathfrak{l}(q')$.

   2.1. $\underline{\mathfrak{l}(q) = \mathfrak{l}(q')}$: Same as case 1.

   2.2. $\underline{\mathfrak{l}(q) < \mathfrak{l}(q')}$: Then $\mathrm{Tip}(g)$ contains $\mathrm{Tip}(g')$ as a subpath. Since G is tip reduced, we then get $g = g'$. But this is impossible, because

   $$\mathfrak{l}(p^*) = \mathfrak{l}(p) + \mathfrak{l}(q) + \mathfrak{l}(\mathrm{Tip}(g)) = \mathfrak{l}(p') + \mathfrak{l}(q') + \mathfrak{l}(\mathrm{Tip}(g')),$$

   which contradicts the inequality $\mathfrak{l}(p) + \mathfrak{l}(q) < \mathfrak{l}(p') + \mathfrak{l}(q')$.

   2.3. $\underline{\mathfrak{l}(q) > \mathfrak{l}(q')}$: Either $\mathfrak{l}(p') < \mathfrak{l}(p\,\mathrm{Tip}(g))$ or $\mathfrak{l}(p') \geq \mathfrak{l}(p\,\mathrm{Tip}(g))$.

      2.3.1. $\underline{\mathfrak{l}(p') < \mathfrak{l}(p\,\mathrm{Tip}(g))}$: Since we are assuming that $\mathfrak{l}(p) < \mathfrak{l}(p')$ and $\mathfrak{l}(q) > \mathfrak{l}(q')$, there exist paths r and s in Q such that $q = rq'$, $p' = ps$, and $\mathrm{Tip}(g)r = s\,\mathrm{Tip}(g')$. Notice that

      $$\begin{aligned} \mathfrak{l}(\mathrm{Tip}(g)) - \mathfrak{l}(s) &= \mathfrak{l}(\mathrm{Tip}(g)) - \big(\mathfrak{l}(p') - \mathfrak{l}(p)\big) \\ &= \big(\mathfrak{l}(\mathrm{Tip}(g)) + \mathfrak{l}(p)\big) - \mathfrak{l}(p') \\ &= \mathfrak{l}(p\,\mathrm{Tip}(g)) - \mathfrak{l}(p') \\ &> 0. \end{aligned}$$

      It follows that $\mathrm{Tip}(g)$ contains s as a proper subpath. In particular, we have $\mathrm{Tip}(g) \nmid s$, and a similar argument shows that

[2] If $a + a' = 0$ and p^* appears exactly twice in (2.3), then p^* would no longer appear at all in (2.3) after we rewrite (2.2), which might seem like a problem. However, this is actually impossible, because we have assumed that p^* is minimal with respect to $\leq$.

$\mathrm{Tip}(g') \nmid r$. Hence by Definition 2.38, g and g' have an (r,s)-overlap, so we can consider the overlap relation $\mathfrak{o}(g,g',r,s)$. Then

$$\begin{aligned}
&pgq \\
&= pgq - \frac{\mathrm{CTip}(g)}{\mathrm{CTip}(g')}p'g'q' + \frac{\mathrm{CTip}(g)}{\mathrm{CTip}(g')}p'g'q' \\
&= pgrq' - \frac{\mathrm{CTip}(g)}{\mathrm{CTip}(g')}psg'q' + \frac{\mathrm{CTip}(g)}{\mathrm{CTip}(g')}p'g'q' \\
&= \mathrm{CTip}(g)p\left(\frac{gr}{\mathrm{CTip}(g)} - \frac{sg'}{\mathrm{CTip}(g')}\right)q' + \frac{\mathrm{CTip}(g)}{\mathrm{CTip}(g')}p'g'q' \\
&= \mathrm{CTip}(g)p\mathfrak{o}(g,g',r,s)q' + \frac{\mathrm{CTip}(g)}{\mathrm{CTip}(g')}p'g'q'. \qquad (2.4)
\end{aligned}$$

By assumption, $\mathfrak{o}(g,g',r,s)$ has remainder 0 under division by G. It follows from the definition of remainders that $\mathfrak{o}(g,g',r,s)$ can be written as a K-linear combination

$$\mathfrak{o}(g,g',r,s) = \sum_l c_l\tilde{p}_l\tilde{g}_l\tilde{q}_l, \qquad (2.5)$$

for paths $\tilde{p}_l$ and $\tilde{q}_l$ in Q and elements $\tilde{g}_l \in G$, where

$$\mathrm{Tip}(\tilde{p}_l\tilde{g}_l\tilde{q}_l) \leq \mathrm{Tip}(\mathfrak{o}(g,g',r,s)) < \mathrm{Tip}(g)r = s\,\mathrm{Tip}(g').$$

By using (2.4) and (2.5), we get

$$\begin{aligned}
&apgq + a'p'g'q' \\
&= a\left(\mathrm{CTip}(g)p\mathfrak{o}(g,g',r,s)q' + \frac{\mathrm{CTip}(g)}{\mathrm{CTip}(g')}p'g'q'\right) + a'p'g'q' \\
&= a\left(\mathrm{CTip}(g)p\left(\sum_l c_l\tilde{p}_l\tilde{g}_l\tilde{q}_l\right)q' + \frac{\mathrm{CTip}(g)}{\mathrm{CTip}(g')}p'g'q'\right) + a'p'g'q' \\
&= a''p'g'q' + \sum_l c'_l p\tilde{p}_l\tilde{g}_l\tilde{q}_l q',
\end{aligned}$$

where $a'' = a\frac{\mathrm{CTip}(g)}{\mathrm{CTip}(g')} + a'$ and $c'_l = a\,\mathrm{CTip}(g)c_l$. Hence we can rewrite (2.2) by replacing the two terms $apgq$ and $a'p'g'q'$ with the single term $a''p'g'q'$, plus several terms of the form $c'_l p\tilde{p}_l\tilde{g}_l\tilde{q}_l q'$. Note that whenever $c'_l p\tilde{p}_l\tilde{g}_l\tilde{q}_l q' \neq 0$, we have

$$\mathrm{Tip}(c'_l p\tilde{p}_l\tilde{g}_l\tilde{q}_l q') = p\,\mathrm{Tip}(\tilde{p}_l\tilde{g}_l\tilde{q}_l)q' < p(s\,\mathrm{Tip}(g'))q' = p^*,$$

so p^* does not appear in $c_l' p\tilde{p}_l\tilde{g}_l\tilde{q}_l q'$. It follows that by rewriting (2.2) in this way, we decrease the number of occurrences of p^* in (2.3). But this contradicts our assumption about p^*.[3]

2.3.2. $\underline{\mathfrak{l}(p') \geq \mathfrak{l}(p\,\mathrm{Tip}(g))}$: Then there exists some path r in Q such that $p' = p\,\mathrm{Tip}(g)r$ and $q = r\,\mathrm{Tip}(g')q'$. Write

$$g = c\,\mathrm{Tip}(g) + \sum_i c_i p_i$$

and

$$g' = d\,\mathrm{Tip}(g') + \sum_i d_i p_i'$$

for scalars c, c_i, d, d_i and paths $p_i < \mathrm{Tip}(g)$, $p_i' < \mathrm{Tip}(g')$. Then we have that

$$\begin{aligned}
pgq &= pgr\,\mathrm{Tip}(g')q' \\
&= pgr\left(\frac{1}{d}g' - \frac{1}{d}g' + \mathrm{Tip}(g')\right)q' \\
&= \frac{1}{d}pgrg'q' - pgr\left(\frac{1}{d}g' - \mathrm{Tip}(g')\right)q' \\
&= \frac{1}{d}p\left(c\,\mathrm{Tip}(g) + \sum_i c_i p_i\right)rg'q' - pgr\left(\sum_i \frac{d_i}{d}p_i'\right)q' \\
&= \frac{c}{d}p\,\mathrm{Tip}(g)rg'q' + \sum_i \frac{c_i}{d}pp_irg'q' - \sum_i \frac{d_i}{d}pgrp_i'q' \\
&= \frac{c}{d}p'g'q' + \sum_i \frac{c_i}{d}pp_irg'q' - \sum_i \frac{d_i}{d}pgrp_i'q'.
\end{aligned}$$

Thus we can rewrite (2.2) by replacing the two terms $apgq$ and $a'p'g'q'$ with the one term $(a' + a\frac{c}{d})p'g'q'$, plus a K-linear combination of elements of the form $pp_irg'q'$ or $pgrp_i'q'$. This decreases the number of occurrences of p^* in (2.3), because $\mathrm{Tip}(pp_irg'q')$ and $\mathrm{Tip}(pgrp_i'q')$ are strictly smaller than p^* with respect to the admissible order $\leq$. But this contradicts our assumption about p^*.

3. $\underline{\mathfrak{l}(p') < \mathfrak{l}(p)}$: Same as case 2.

□

[3] If $a'' = 0$, then it might be the case that p^* no longer appears in (2.3). But this again leads to a contradiction of the minimality of p^*.

Note that when proving the implication (i) $\Rightarrow$ (ii) in Theorem 2.41, we did not need to use the assumption that G is tip reduced and uniform. Moreover, our proof for this implication did not actually rely on the definition of the overlap relation, but would have worked for any element of $\langle G \rangle$. In other words, if G is an arbitrary Gröbner basis and y is an element of $\langle G \rangle$, then $y \Rightarrow_G 0$.

The following examples show how the implication (ii) $\Rightarrow$ (i) in Theorem 2.41 may fail without the assumption that the set G is tip reduced and uniform.

Example 2.42. Let Q denote the following quiver.

$$Q\colon\ v_1 \xrightarrow{\alpha} v_2$$

We give Q an admissible order by declaring

$$v_1 < v_2 < \alpha.$$

Let $G = \{\, \alpha + v_1 \,\}$. Then G is tip reduced, but not uniform. There are no overlaps between elements of G, so the assumption that all overlap relations have remainder zero is vacuously true. But G is not a Gröbner basis, because

$$v_1 = v_1(\alpha + v_1) \in \langle G \rangle\,,$$

but v_1 is not divisible by α. ■

Example 2.43. Let Q be the following quiver.

$$\begin{tikzcd} Q\colon\ v_1 \arrow[r, "\alpha"] \arrow[rrr, bend right, "\delta"'] & v_2 \arrow[r, "\beta"] & v_3 \arrow[r, "\gamma"] & v_4 \end{tikzcd}$$

We give Q the left length-lex order with

$$v_1 < v_2 < v_3 < v_4 < \alpha < \beta < \gamma < \delta.$$

Let $G = \{\, \gamma\beta\alpha + \delta, \beta \,\}$. The set G is uniform, but not tip reduced. There are no overlaps between elements of G. However, G is not a Gröbner basis, because

$$\delta = (\gamma\beta\alpha + \delta) - \gamma\beta\alpha \in \langle G \rangle\,,$$

but δ is not divisible by $\gamma\beta\alpha$ or β. ■

2.4.3 Computation and existence of finite Gröbner bases

We will now see how we can compute Gröbner bases. This is done by using a variant of Buchberger's algorithm. The original version of Buchberger's algorithm, which was introduced by Bruno Buchberger in his PhD thesis, computes Gröbner bases in commutative polynomial rings. Algorithm 2.3 is an adaptation of this algorithm to the non-commutative setting of path algebras. Recall that if S is a finite subset of Λ, then $\textsc{TipReduce}(S)$ denotes the tip reduced set that is obtained by applying Algorithm 2.2 to S. Also, for an element $y \in \Lambda$, we denote by $\textsc{Remainder}(y, S)$ the remainder of y under division by S produced by Algorithm 2.1.

Algorithm 2.3: Buchberger's algorithm for path algebras

Input: A finite, tip reduced, uniform subset $\{ f_1, \dots, f_n \} \subseteq \Lambda$
Output: A finite Gröbner basis G for the ideal $\langle f_1, \dots, f_n \rangle$, if one exists

$G_0 \leftarrow \{ f_1, \dots, f_n \}$;
$l \leftarrow 0$;
do
    MODIFIED $\leftarrow$ False;
    $X \leftarrow \emptyset$;
    for $g, h \in G_l$ **do**
        for *all paths* p, q *such that* g *and* h *have a* (p, q)*-overlap* **do**
            $r \leftarrow \textsc{Remainder}(\mathfrak{o}(g, h, p, q), G_l)$;
            if $r \neq 0$ **then**
                $X \leftarrow X \cup \{ r \}$;
                MODIFIED $\leftarrow$ True;
            end
        end
    end
    if MODIFIED $=$ True **then**
        $G_{l+1} \leftarrow \textsc{TipReduce}(G_l \cup X)$;
        $l \leftarrow l + 1$;
    end
while MODIFIED $=$ True;
$G \leftarrow G_l$;
return G;

The algorithm above is based on an algorithm presented in [Gre99]. However, there are two details in Algorithm 2.3 which were omitted in [Gre99], possibly by accident. Firstly, the algorithm in [Gre99] does not assume that the input set $\{ f_1, \dots, f_n \}$ is tip reduced. Secondly, it does not perform the tip reduction step on line 16 of Algorithm 2.3, and instead does the equivalent of letting $G_{l+1} = G_l \cup X$. Before proving the correctness of Algorithm 2.3, we explain why these two details are necessary.

If the input set $\{ f_1, \dots, f_n \}$ were uniform but not tip reduced, and if there were no overlaps between its elements, then Algorithm 2.3 would terminate immediately, and would return the set $\{ f_1, \dots, f_n \}$. However, Example 2.44 shows that this set would not necessarily be a Gröbner basis. It is therefore necessary to make sure that the input to the algorithm is tip reduced.

The following example shows that the tip reduction step on line 16 is necessary.

Example 2.44. Let Q denote the following quiver.

$$\begin{tikzcd} Q\colon\ v_1 \arrow[r, "\alpha_1"] & v_2 \arrow[r, "\alpha_2"] \arrow[rr, bend right, "\beta_1"'] & v_3 \arrow[r, "\alpha_3"] \arrow[rr, bend left, "\beta_2"] & v_4 \arrow[r, "\alpha_4"] & v_5 \arrow[r, "\alpha_5"] & v_6 \end{tikzcd}$$

We give Q the left length-lex order with

$$v_1 < v_2 < v_3 < v_4 < v_5 < v_6 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5 < \beta_1 < \beta_2.$$

We consider the following set of relations in KQ:

$$G_0 = \{ \alpha_4\alpha_3 - \beta_2, \alpha_3\alpha_2 - \beta_1, \alpha_5\beta_2\alpha_2\alpha_1 \}.$$

The set G_0 is tip reduced and uniform. Let us see what happens if we apply Algorithm 2.3 to G_0, but omit the tip reduction step on line 16. The only overlap relation between elements of G_0 is

$$\begin{aligned} \mathfrak{o}(\alpha_4\alpha_3 - \beta_2, \alpha_3\alpha_2 - \beta_1, \alpha_2, \alpha_4) &= (\alpha_4\alpha_3 - \beta_2)\alpha_2 - \alpha_4(\alpha_3\alpha_2 - \beta_1) \\ &= -\beta_2\alpha_2 + \alpha_4\beta_1. \end{aligned}$$

Neither $\beta_2\alpha_2$ nor $\alpha_4\beta_1$ is divisible by any element of $\mathrm{Tip}(G_0)$, so the overlap relation is its own remainder under division by G_0. We therefore add the overlap relation to our set of relations, and since we do not perform a tip reduction step, we obtain the set

$$G_1 = \{ \alpha_4\alpha_3 - \beta_2, \alpha_3\alpha_2 - \beta_1, \alpha_5\beta_2\alpha_2\alpha_1, -\beta_2\alpha_2 + \alpha_4\beta_1 \}.$$

Note that $\mathrm{Tip}(-\beta_2\alpha_2 + \alpha_4\beta_1) = \beta_2\alpha_2$, which divides $\alpha_5\beta_2\alpha_2\alpha_1$. Hence G_1 is not tip reduced.

The only overlap between elements of G_1 is the (α_2, α_4)-overlap between $\alpha_4\alpha_3 - \beta_2$ and $\alpha_3\alpha_2 - \beta_1$. Since the corresponding overlap relation is an element of G_1, we have

$$\mathfrak{o}(\alpha_4\alpha_3 - \beta_2, \alpha_3\alpha_2 - \beta_1, \alpha_2, \alpha_4) \Rightarrow_{G_l} 0.$$

Therefore we do not add any more elements to the generating set G_l, and the algorithm terminates. However, G_1 is not a Gröbner basis, because

$$\alpha_5\alpha_4\beta_1\alpha_1 = \alpha_5\beta_2\alpha_2\alpha_1 + \alpha_5(-\beta_2\alpha_2 + \alpha_4\beta_1)\alpha_1 \in \langle G_1 \rangle\,,$$

but $\alpha_5\alpha_4\beta_1\alpha_1$ is not divisible by any element of

$$\mathrm{Tip}(G_1) = \{\ \alpha_4\alpha_3, \alpha_3\alpha_2, \alpha_5\beta_2\alpha_2\alpha_1, \beta_2\alpha_2\ \}\,.$$

This shows that Algorithm 2.3 could produce an incorrect result if we were to omit the tip reduction step. ■

Let us now prove that Algorithm 2.3 does what it is supposed to. We will need a few technical lemmas.

Lemma 2.45. *Let $\{\ f_1, \ldots, f_n\ \} \subseteq \Lambda$ be a tip reduced uniform subset. Then at any time during the execution of Algorithm 2.3, the set G_l is a tip reduced uniform generating set for the ideal $I = \langle f_1, \ldots, f_n \rangle$.*

Proof.

The set G_l is clearly tip reduced. Moreover, if $g, h \in G_l$ are elements that have a (p, q)-overlap such that $\mathfrak{o}(g, h, p, q) \Rightarrow_{G_l} r$ for some $r \in \Lambda$, then r is an element of $\langle G_l \rangle$. It follows that the set $G_l \cup X$ in Algorithm 2.3 generates the same ideal as G_l, and hence so does G_{l+1}, by Proposition 2.37. By induction, we see that for every l, the set G_l is a generating set for I.

It remains to be shown that G_l is a uniform set for all l. This is clearly true when $l = 0$. So suppose that $l \geq 1$, and that G_l is uniform. The tip reduction algorithm preserves uniform sets by Proposition 2.37, so in order to show that G_{l+1} is uniform, it suffices to show that if $g, h \in G_l$ have a (p, q)-overlap, then the element r computed on line 8 is uniform. Recall that the overlap relation is given by

$$\mathfrak{o}(g, h, p, q) = \frac{gp}{\mathrm{CTip}(g)} - \frac{qh}{\mathrm{CTip}(h)}.$$

Since g and h are both uniform, and because $\mathrm{Tip}(g)p = q\,\mathrm{Tip}(h)$, it follows that the overlap relation is also uniform. Hence r is also uniform, because the algorithm Remainder (Algorithm 2.1) produces a uniform remainder whenever the input element is uniform. □

The following lemma will allow us to express an element of G_{l+1} in terms of elements of $G_l \cup X$, where G_l, G_{l+1}, and X are the sets defined in Algorithm 2.3. This will be useful in the proof of Lemma 2.47.

Lemma 2.46. *Let G be a finite uniform subset of Λ, and let H be the tip reduced generating set for $\langle G \rangle$ that is obtained when applying the tip reduction algorithm (Algorithm 2.2) to G. Let $g \neq 0$ be an element of G. Then there exist elements $y_j \in H$ (not necessarily distinct), scalars $c_j \in K$, and paths u_j and v_j in Q such that the following conditions hold.*

(i) $g = \sum_{j=1}^{t} c_j u_j y_j v_j$.

(ii) $\mathrm{Tip}(u_j y_j v_j) \leq \mathrm{Tip}(g)$.

(iii) There is only one index j such that $\mathrm{Tip}(u_j y_j v_j) = \mathrm{Tip}(g)$.

Proof.
Recall that Algorithm 2.2 produces a finite sequence $H_0, H_1, H_2, \ldots$ of generating sets for the ideal $\langle G \rangle$, and that the last set in this sequence is the tip reduced set H.

In order to prove the lemma, we will prove that for each value of the index k in the algorithm, there exist elements $y_j \in H_k$, scalars c_j, and paths u_j and v_j such that

$$g = \sum_{j=1}^{t} c_j u_j y_j v_j,$$

and such that conditions (ii) and (iii) are also satisfied. We first consider the case $k = 0$. Since g is uniform, there exist vertices u and v such that $g = ugv$. Hence we see that our claim is true for $k = 0$, since g is an element of $H_0 = G$.

Now let $k \geq 1$, and assume inductively that the claim above is true for all smaller values of k. Then there exist elements $y_j \in H_{k-1}$, scalars c_j, and paths u_j and v_j such that $g = \sum_{j=1}^{t} c_j u_j y_j v_j$, and such that conditions (ii) and (iii) are satisfied. If each of the elements $y_j \in H_{k-1}$ is also an element of H_k, then we are done. So assume that $y_j \notin H_k$ for some j. Since the set H_k was obtained by replacing one element of H_{k-1}, there exists exactly one element $h \in H_{k-1}$ such that $h \notin H_k$. Recall from Algorithm 2.2 that there must exist some element $h' \in H_{k-1} \setminus \{ h \}$ and paths p and q in Q such that $\mathrm{Tip}(h) = p\,\mathrm{Tip}(h')q$, and such that if $\tilde{h}$ is the element

$$\tilde{h} = h - \frac{\mathrm{CTip}(h)}{\mathrm{CTip}(h')} p h' q,$$

then $\tilde{h} \in H_k$. Without loss of generality, we may assume that there exists some index i such that

$$y_1 = y_2 = \ldots = y_i = h,$$

and such that $y_j \neq h$ for $j > i$. Then we may write g in the following way:

$$\begin{aligned}
g &= \sum_{j=1}^{t} c_j u_j y_j v_j \\
&= \sum_{j=1}^{i} c_j u_j h v_j + \sum_{j=i+1}^{t} c_j u_j y_j v_j \\
&= \sum_{j=1}^{i} c_j u_j \left(\tilde{h} + \frac{\mathrm{CTip}(h)}{\mathrm{CTip}(h')} p h' q \right) v_j + \sum_{j=i+1}^{t} c_j u_j y_j v_j \\
&= \sum_{j=1}^{i} c_j u_j \tilde{h} v_j + \sum_{j=1}^{i} c'_j (u_j p) h' (q v_j) + \sum_{j=i+1}^{t} c_j u_j y_j v_j,
\end{aligned}$$

where $c'_j = c_j \frac{\mathrm{CTip}(h)}{\mathrm{CTip}(h')}$. Since $\tilde{h}$, h', and y_j are elements of H_k (for $j > i$), we see that this new expression for g has the desired form.

In order to complete the proof, we must show that conditions (ii) and (iii) are satisfied. For $i + 1 \leq j \leq t$, we know that

$$\mathrm{Tip}(u_j y_j v_j) \leq \mathrm{Tip}(g),$$

by the inductive assumption. Moreover, for $1 \leq j \leq i$, we have

$$\mathrm{Tip}(u_j p h' q v_j) = u_j \, \mathrm{Tip}(p h' q) v_j = u_j \, \mathrm{Tip}(h) v_j \leq \mathrm{Tip}(g)$$

and

$$\mathrm{Tip}(u_j \tilde{h} v_j) = u_j \, \mathrm{Tip}(\tilde{h}) v_j < u_j \, \mathrm{Tip}(h) v_j \leq \mathrm{Tip}(g).$$

This shows that condition (ii) is satisfied. Moreover, because

$$\mathrm{Tip}(u_j \tilde{h} v_j) < \mathrm{Tip}(g)$$

and

$$\mathrm{Tip}(u_j p h' q v_j) = \mathrm{Tip}(u_j h v_j),$$

we have not increased the number of terms whose tip is equal to $\mathrm{Tip}(g)$. Hence there is still only one such term, by the inductive assumption. This shows that condition (iii) is satisfied. $\square$

The following lemma is the key to proving that Algorithm 2.3 terminates if a finite Gröbner basis exists. The statement of the lemma is based on a brief comment preceding [Gre99, Proposition 2.8], where it was stated that the lemma could be proven by modifying the proof of Theorem 2.41. For the sake of completeness, I have developed a complete proof. However, this proof turned out to be quite long and technical, so we delay it to an appendix.

Lemma 2.47. *Let $\{ f_1, \dots, f_n \} \subseteq \Lambda$ be a tip reduced uniform subset, and let $I = \langle f_1, \dots, f_n \rangle$. Let x be an element of the reduced Gröbner basis of I. Then during the execution of Algorithm 2.3, there exists some l such that* $\mathrm{Tip}(x) \in \mathrm{Tip}(G_l)$.

Proof. See Appendix A. □

We can now prove that Algorithm 2.3 is correct, and in particular that it terminates if a finite Gröbner basis exists.

Theorem 2.48. *Let $\{ f_1, \dots, f_n \} \subseteq \Lambda$ be a finite, tip reduced, uniform subset, and let $I = \langle f_1, \dots, f_n \rangle \subseteq \Lambda$. Suppose I has a finite Gröbner basis. Then Algorithm 2.3 terminates, and produces a finite Gröbner basis for I.*

Proof. Let H be the reduced Gröbner basis of I. Since we are assuming that I has a finite Gröbner basis, H is finite by Corollary 2.31. Hence we may write

$$H = \{ x_1, \dots, x_m \} .$$

For $1 \leq i \leq m$, Lemma 2.47 tells us that there must exist some l_i such that $\mathrm{Tip}(x_i) \in \mathrm{Tip}(G_{l_i})$. It may be shown $\langle \mathrm{Tip}(G_l) \rangle \subseteq \langle \mathrm{Tip}(G_{l+1}) \rangle$ for all l, so we see that if we let $L = \max \{ l_1, \dots, l_m \}$, then $\mathrm{Tip}(H) \subseteq \langle \mathrm{Tip}(G_L) \rangle$. But then G_L is a Gröbner basis for I, so whenever elements $g, h \in G_L$ have a (p, q)-overlap, we know that $\mathfrak{o}(g, h, p, q) \Rightarrow_{G_L} 0$, and this remainder is unique by Proposition 2.23. Hence the do-while loop in the algorithm terminates, and the algorithm outputs the finite Gröbner basis G_L. □

Algorithm 2.3 computes a finite Gröbner basis for an ideal, if one exists. In the setting of commutative polynomial rings over a field, it can be shown that all ideals have a finite Gröbner basis. However, this is not true for ideals in path algebras, as the following example shows.

Example 2.49. Let $\Lambda = K \langle x, y \rangle$ be the free algebra on two generators, which we may interpret as the path algebra of a quiver with one vertex and two loops. We give the monomials in Λ some arbitrary admissible order. Consider the set

$$G = \{\, xx, xyx, xyyx, xyyyx, \dots \,\}.$$

It can be seen that G is the reduced Gröbner basis for the ideal $\langle G \rangle$. But the reduced Gröbner basis has minimal cardinality among all Gröbner bases, so this means that all Gröbner bases for $\langle G \rangle$ are infinite. ■

In Example 2.49, the ideal $\langle G \rangle$ was given by an infinite generating set, so it is perhaps not so surprising that it did not have a finite Gröbner basis. However, the following example shows that even a finitely generated ideal might only have an infinite Gröbner basis.

Example 2.50 ([Arn10])**.** Let $\Lambda = K\langle x, y \rangle$ be the free algebra on two generators, and give the monomials in Λ an arbitrary admissible order where $x > y$. Let I be the ideal in Λ generated by $xx - xy$. Hence by construction, I is a finitely generated ideal. We claim that the set

$$G = \{\, xx - xy, xyx - xyy, xyyx - xyyy, \dots \}$$

is the reduced Gröbner basis for I.

We first show that G is the reduced Gröbner basis for some ideal. Let

$$g_n = xy^{n-1}x - xy^n.$$

Note that G is precisely the set of elements of the form g_n for $n \geq 1$. Also observe that the tip of g_n is $xy^{n-1}x$, because $x > y$. It follows that the only overlap relations between elements of G are those of the form

$$\mathfrak{o}(g_i, g_j, y^{j-1}x, xy^{i-1})$$

for $i, j \geq 1$. Then a straightforward computation shows that

$$\mathfrak{o}(g_i, g_j, y^{j-1}x, xy^{i-1}) = g_i y^j - g_{i+j}, \tag{2.6}$$

and moreover, the tips of $g_i y^j$ and g_{i+j} are both less than or equal to the tip of the overlap relation. This shows that

$$\mathfrak{o}(g_i, g_j, y^{j-1}x, xy^{i-1}) \Rightarrow_G 0.$$

Then it follows from Theorem 2.41 that G is a Gröbner basis for $\langle G \rangle$. Additionally, since $\operatorname{Tip}(g_i) = xy^{i-1}x$, which does not divide $xy^{j-1}x$ or xy^j for $j \neq i$, we see that G is a reduced Gröbner basis.

Next we show that $I = \langle G \rangle$. Note that $I \subseteq \langle G \rangle$, because G contains the generator $xx - xy$. To show that $\langle G \rangle \subseteq I$, it suffices show that $g_n \in I$ for all integers $n \geq 1$. We proceed by induction on n. The base case $n = 1$ is satisfied since $g_1 = xx - xy$, so assume that $n \geq 1$ is some fixed integer such that $g_n \in I$. By using (2.6) with $i = n$ and $j = 1$, we see that

$$g_{n+1} = g_n y - \mathfrak{o}(g_n, g_1, x, xy^{n-1}),$$

which is an element of I because $g_1 \in I$ and $g_n \in I$. This shows that $I = \langle G \rangle$.

We see that G is the reduced Gröbner basis of the ideal I, and consequently all Gröbner bases of I are infinite. ■

The fact that finite Gröbner bases are not guaranteed to exist even for finitely generated ideals is rather unfortunate. Thankfully, it turns out that a finite Gröbner basis for an ideal I always exists if Λ/I is finite-dimensional, as the following result shows.

Theorem 2.51. *Let $I \subseteq \Lambda$ be an ideal such that the quotient algebra Λ/I is finite-dimensional. Then I has a finite Gröbner basis.*

Proof. By the construction of the reduced Gröbner basis given in the proof of Proposition 2.28, it suffices to show that the ideal $\langle \mathrm{Tip}(I) \rangle$ is generated by a finite set of paths. Consider the set

$$X = (Q_0 \cup \{\, \alpha p \mid \alpha \in Q_1, p \in \mathrm{NonTip}(I) \,\}) \cap \mathrm{Tip}(I),$$

where Q_0 and Q_1 denote the set of vertices in Q and the set of arrows in Q, respectively. We are assuming that Λ/I is finite-dimensional, so $\mathrm{NonTip}(I)$ is a finite set by Proposition 2.13. Since Q_0 and Q_1 are also finite, we see that the set $X \subseteq \mathrm{Tip}(I)$ is a finite set of paths.

We claim that X generates $\langle \mathrm{Tip}(I) \rangle$. Let t be an element of the minimal monomial generating set of $\langle \mathrm{Tip}(I) \rangle$. If t is a vertex, then $t \in Q_0 \cap \mathrm{Tip}(I)$, and hence $t \in X$. If t is not a vertex, then there exists an arrow α and a (possibly trivial) path p such that $t = \alpha p$. Since t is a minimal monomial generator of $\langle \mathrm{Tip}(I) \rangle$, the path p cannot be an element of $\mathrm{Tip}(I)$. In other words, we see that $p \in \mathrm{NonTip}(I)$, and hence $t \in X$. This shows that X generates $\langle \mathrm{Tip}(I) \rangle$. □

Chapter 3

Tensor products of algebras

In this chapter, we study tensor products of algebras. One reason why one might be interested in this topic is bimodules. If M is a Λ-Γ-bimodule over K-algebras Λ and Γ, then M can be turned into a left module over $\Lambda \otimes_{\mathbb{Z}} \Gamma^{\mathrm{op}}$ in a natural way. If in addition M is K-symmetric (that is, if the left K-module structure that M inherits from Λ coincides with the right K-module structure inherited from Γ), then M is also a $\Lambda \otimes_K \Gamma^{\mathrm{op}}$-module. Thus we can understand bimodules by studying left modules over tensor products. We will not pursue this approach to bimodules in further detail in this thesis, and mention it only as one possible source of motivation.

In Section 3.1, we briefly explain how the tensor product of algebras is itself an algebra. Then in Section 3.2, we show how tensor products of algebras can be constructed using quivers. We prove a theorem (Theorem 3.13) that shows how the tensor product $\Lambda \otimes_K \Gamma$ can be realized as a quotient of a path algebra when Λ and Γ are themselves quotients of path algebras. We then generalize this to a new result (Theorem 3.18) that allows us to compute the tensor product $\Lambda \otimes_\Sigma \Gamma$, where Σ is another algebra. In Section 3.3, we apply Gröbner basis theory to the computation of tensor products. We prove a result (Theorem 3.26) that shows how we can explicitly describe a Gröbner basis for the ideal by which $\Lambda \otimes_K \Gamma$ is a quotient. Unfortunately, we find that this result does not extend in a nice way to the tensor product $\Lambda \otimes_\Sigma \Gamma$.

3.1 The ring structure of the tensor product

We begin this chapter by studying how the tensor product of algebras (or of rings in general) may be given a ring structure. Let us first recall the definition of the tensor product of modules over a ring.

Definition 3.1. Let R be a ring (not necessarily commutative), let M be a right R-module, and let N be a left R-module. Let A be an abelian group. Then a map $\phi\colon M \times N \to A$ is called **R-balanced** if the following conditions hold.

(i) ϕ is $\mathbb{Z}$-bilinear, i.e.

$$\phi(m, n + n') = \phi(m, n) + \phi(m, n')$$

and

$$\phi(m + m', n) = \phi(m, n) + \phi(m', n)$$

for all $m, m' \in M$ and $n, n' \in N$.

(ii) $\phi(mr, n) = \phi(m, rn)$ for all $m \in M$, $n \in N$, and $r \in R$.

■

Definition 3.2. Let R be a ring, let M be a right R-module, and let N be a left R-module. Then a **tensor product** of M and N over R is an abelian group $M \otimes_R N$ along with an R-balanced map $\phi\colon M \times N \to M \otimes_R N$, such that for every abelian group A and every R-balanced map $\psi\colon M \times N \to A$, there exists a unique group homomorphism $f\colon M \otimes_R N \to A$ making the following diagram commute:

$$\begin{tikzcd} M \times N \arrow[r, "\phi"] \arrow[dr, "\psi"'] & M \otimes_R N \arrow[d, dashed, "\exists! f"] \\ & A \end{tikzcd}$$

■

It can be shown that a tensor product $M \otimes_R N$ always exists, and it is unique up to isomorphism. Hence we may speak of *the* tensor product of M and N. Given ϕ as in the definition above and for module elements $m \in M$ and $n \in N$, we typically denote the element $\phi(m, n)$ by $m \otimes n$. Such elements are often called **elementary tensors**. Not all elements of $M \otimes_R N$ have this form, but every element can be written as a finite sum of elementary tensors.

In general, the tensor product $M \otimes_R N$ is only an abelian group. However, if N is an R-bimodule, then we may turn the tensor product into a right R-module by defining

$$(m \otimes n)r = m \otimes (nr)$$

for elementary tensors $m \otimes n$ and ring elements $r \in R$. This ring action on elementary tensors can be shown to be well defined by using the universal property of the tensor product, and it extends uniquely to a module structure on $M \otimes_R N$. Similarly, if M is an R-bimodule, then the tensor product can be given a left R-module structure.

In this thesis, we will primarily be interested in tensor products of rings, and in particular of K-algebras. Let A, B, and C be rings, and assume that A and B are C-bimodules, so that we may form the tensor product $A \otimes_C B$. We wish to turn the tensor product into a ring by defining

$$(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$$

for elementary tensors $a \otimes b$ and $a' \otimes b'$. The following lemma shows that we can do this, as long as we impose an additional condition on the C-module structures of A and B; we essentially need to assume that A and B are C-algebras, although we should probably not use the term "algebras", since C is not necessarily commutative.

Lemma 3.3. *Let A, B, and C be rings, and let $f\colon C \to A$ and $g\colon C \to B$ be ring homomorphisms such that* $\operatorname{Im}(f) \subseteq Z(A)$ *and* $\operatorname{Im}(g) \subseteq Z(B)$. *View A and B as C-bimodules via the homomorphisms f and g, respectively. Then the tensor product $A \otimes_C B$ is a ring, with multiplication given by*

$$\left(\sum_i a_i \otimes b_i\right)\left(\sum_j a'_j \otimes b'_j\right) = \sum_{i,j} (a_i a'_j) \otimes (b_i b'_j)$$

for $a_i, a'_j \in A$ and $b_i, b'_j \in B$. Moreover, if A, B, and C are algebras over a commutative ring R, and if f and g are R-linear, then $A \otimes_C B$ is an R-algebra.

Proof. It is straightforward to verify that the ring axioms hold, so we omit this step. However, we need to check that multiplication is well defined. Fix elements $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$. Consider the map

$$\begin{aligned} \psi\colon A \times B &\to A \otimes_C B \\ (a, b) &\mapsto \sum_i a_i a \otimes b_i b. \end{aligned}$$

The map ψ is clearly $\mathbb{Z}$-bilinear. To see that ψ is C-balanced, we must show that $\psi(ac, b) = \psi(a, cb)$ for all $a \in A$, $b \in B$, and $c \in C$. Note that, since the C-module structures of A and B are given by ring homomorphisms, we have $a_i(ac) = (a_i a)c$ and $c(b_i b) = (cb_i)b$. Moreover, because the image of g is contained in the centre of B, we have $c(b_i b) = b_i(cb)$. Hence we see that

$$\begin{aligned}
\psi(ac, b) &= \sum_i a_i(ac) \otimes b_i b \\
&= \sum_i (a_i a)c \otimes b_i b \\
&= \sum_i a_i a \otimes c(b_i b) \\
&= \sum_i a_i a \otimes b_i(cb) \\
&= \psi(a, cb).
\end{aligned}$$

Thus ψ is C-balanced, so by the universal property of the tensor product, there exists a unique group homomorphism $h\colon A \otimes_C B \to A \otimes_C B$ such that $h(a \otimes b) = \psi(a, b)$ for all elementary tensors $a \otimes b$. Now suppose that $a'_j, a''_k \in A$ and $b'_j, b''_k \in B$ are elements such that $\sum_j a'_j \otimes b'_j = \sum_k a''_k \otimes b''_k$. Then we have

$$\begin{aligned}
\sum_{i,j} a_i a'_j \otimes b_i b'_j &= h\left(\sum_j a'_j \otimes b'_j\right) \\
&= h\left(\sum_k a''_k \otimes b''_k\right) \\
&= \sum_{i,k} a_i a''_k \otimes b_i b''_k.
\end{aligned}$$

This shows that the multiplication in $A \otimes_C B$ is well defined with respect to the second operand. Similarly, it can be shown that multiplication is well defined with respect to the first operand.

Finally, suppose that A, B, and C are algebras over a commutative ring R, and suppose that f and g are algebra homomorphisms. Let $\phi\colon R \to A \otimes_C B$ be the function given by

$$\phi(r) = f(r \cdot 1_C) \otimes 1_B,$$

or equivalently by

$$\phi(r) = 1_A \otimes g(r \cdot 1_C),$$

for $r \in R$. Then ϕ can be seen to be a ring homomorphism, and $\mathrm{Im}(\phi)$ is contained in $Z(A \otimes_C B)$ because $\mathrm{Im}(f) \subseteq Z(A)$. Hence $A \otimes_C B$ is an R-algebra. $\square$

Remark 3.4. When proving that $A \otimes_C B$ is an R-algebra, we did not actually need to use the assumption that the maps f and g are R-linear. However, this assumption ensures that the R-module structure of $A \otimes_C B$ is compatible with the R-module structures of A and B, in the sense that

$$r(a \otimes b) = (ra) \otimes b = a \otimes (rb).$$

This would not necessarily be true if the maps were not R-linear. ■

If we do not assume that the images of f and g are contained in the centres of A and B, respectively, then the multiplication in Lemma 3.3 will typically not be well defined. We illustrate this with an example.

Example 3.5 (Inspired by [Wof21]). Consider the following two quivers.

$$Q\colon\ v_1 \xrightarrow{\alpha} v_2$$

$$R\colon\ v_1 \qquad\qquad v_2$$

We may view KR as a subring of KQ. However, KR is not contained in the centre of KQ, because $v_1\alpha = 0$ while $\alpha v_1 = \alpha$. We consider the tensor product $KQ \otimes_{KR} KR$, which we may identify with KQ via the identification $x \otimes 1 = x$ for $x \in KQ$. For the sake of contradiction, assume that the multiplication in Lemma 3.3 is well defined. Then we have

$$\begin{aligned}
\alpha &= \alpha \otimes 1 \\
&= \alpha v_1 \otimes 1 \\
&= \alpha \otimes v_1 \\
&= (1 \otimes v_1)(\alpha \otimes 1) \\
&= (v_1 \otimes 1)(\alpha \otimes 1) \\
&= v_1\alpha \otimes 1 \\
&= 0,
\end{aligned}$$

which is a contradiction.

■

3.2 Constructing the tensor product of path algebras

We will now see how we can construct the tensor product of (quotients of) path algebras. Throughout this chapter, let (Q, ρ) and (R, σ) be quivers with relations (not necessarily admissible, or even finite), and let $\Lambda = KQ/\langle\rho\rangle$ and $\Gamma = KR/\langle\sigma\rangle$. We will first consider how we can construct the tensor product of Λ and Γ over the field K, before generalizing to tensor products over an algebra.

3.2.1 Tensor products over a field

We want to realize $\Lambda \otimes_K \Gamma$ as a quotient of a path algebra KP for some quiver P. The first thing we need to do is to find out what this quiver P should look like. Note that an arbitrary element of $\Lambda \otimes_K \Gamma$ can be written as a K-linear combination of elements of the form $[p] \otimes [q]$ for paths p in Q and q in R. The element $[p] \otimes [q]$ can in turn be written as a product of elements that have the form $[v] \otimes [w]$, $[v] \otimes [\beta]$, or $[\alpha] \otimes [w]$ for vertices v, w and arrows α, β. This is similar to the fact that in a path algebra, every element can be written as a linear combination of products of vertices and arrows. This motivates our definition of product quivers.

Definition 3.6 ([Les94])**.** The **product quiver** of Q and R is the quiver $Q \times R$, with vertex set

$$(Q \times R)_0 = Q_0 \times R_0,$$

and arrow set

$$(Q \times R)_1 = (Q_1 \times R_0) \cup (Q_0 \times R_1).$$

A pair $(\alpha, w) \in Q_1 \times R_0$ is regarded as an arrow from $(\mathfrak{s}(\alpha), w)$ to $(\mathfrak{t}(\alpha), w)$, and similarly for $(v, \beta) \in Q_0 \times R_1$. ■

Note that although Definition 3.6 cites [Les94] (as that is the oldest source for the concept of product quivers that I am personally aware of), the notation and terminology in said source is somewhat different from the one used in our definition. A source with similar notation and terminology to that used in this thesis can be found in [Her08].

Notation. Instead of using pair notation as in Definition 3.6, we use the following notation: if $v \in Q_0, w \in R_0$ are vertices and $\alpha \in Q_1, \beta \in R_1$ are arrows, we denote the vertex (v, w) by $v \times w$, and similarly we denote the arrows (α, w) and (v, β) by $\alpha \times w$ and $v \times \beta$, respectively.

Given a sequence of composable arrows $\alpha_1, \alpha_2, \ldots, \alpha_n$ in Q, let p be the path $\alpha_1\alpha_2 \ldots \alpha_n$. Then for any vertex w in R, we define

$$p \times w = (\alpha_1 \times w)(\alpha_2 \times w) \ldots (\alpha_n \times w).$$

More generally, if
$$x = \sum_i a_i p_i \in KQ$$
is an arbitrary K-linear combination of paths from Q, and if w is any vertex in R, we define
$$x \times w = \sum_i a_i(p_i \times w) \in K(Q \times R).$$
If v is a vertex in Q and $y \in KR$ is an arbitrary element, we define $v \times y$ in an analogous manner.

Note that we do *not* define $x \times y$ for arbitrary elements $x \in KQ$ and $y \in KR$, since this is ambiguous. For example, if α and β are arrows in Q and R, respectively, then it is not clear if $\alpha \times \beta$ should denote the path
$$\big(\alpha \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(\alpha) \times \beta\big),$$
or if it should instead denote the path
$$\big(\mathfrak{t}(\alpha) \times \beta\big)\big(\alpha \times \mathfrak{s}(\beta)\big).$$
■

We illustrate the notion of product quivers with an example.

Example 3.7. Let Q and R denote the following quivers.

$$Q\colon \; \begin{tikzcd} v_1 \arrow[r, "\alpha_1"] & v_2 \arrow[loop right, "\alpha_2"] \end{tikzcd}$$

$$R\colon \; \begin{tikzcd} w_1 \arrow[r, "\beta_1"] & w_2 \arrow[r, "\beta_2"] & w_3 \end{tikzcd}$$

Then the product quiver $Q \times R$ is the following quiver:

$$Q \times R\colon \; \begin{tikzcd} v_1 \times w_1 \arrow[r, "v_1 \times \beta_1"] \arrow[d, "\alpha_1 \times w_1"'] & v_1 \times w_2 \arrow[r, "v_1 \times \beta_2"] \arrow[d, "\alpha_1 \times w_2"] & v_1 \times w_3 \arrow[d, "\alpha_1 \times w_3"] \\ v_2 \times w_1 \arrow[r, "v_2 \times \beta_1"'] \arrow[loop below, "\alpha_2 \times w_1"] & v_2 \times w_2 \arrow[r, "v_2 \times \beta_2"'] \arrow[loop below, "\alpha_2 \times w_2"] & v_2 \times w_3 \arrow[loop below, "\alpha \times w_3"] \end{tikzcd}$$

An example of a path in $Q \times R$ is
$$(v_2 \times \beta_2\beta_1)(\alpha_2^2\alpha_1 \times w_1) = (v_2 \times \beta_2)(v_2 \times \beta_1)(\alpha_2 \times w_1)^2(\alpha_1 \times w_1).$$

We can cross a vertex in Q with an arbitrary element of KR, and vice versa. For example, we have

$$v_2 \times (3\beta_2 - 5\beta_2\beta_1) = 3(v_2 \times \beta_2) - 5(v_2 \times \beta_2)(v_2 \times \beta_1).$$

■

Remark 3.8. If p and p' are composable paths in Q and $w \in R_0$ is a vertex, then it follows immediately from the definition of $(pp') \times w$ that

$$(pp') \times w = (p \times w)(p' \times w).$$

In fact, it is also true (and easy to verify) that

$$(xx') \times w = (x \times w)(x' \times w)$$

for arbitrary elements $x, x' \in KQ$. Similarly,

$$v \times (yy') = (v \times y)(v \times y')$$

for all vertices $v \in Q_0$ and elements $y, y' \in KR$. ■

We wish to express the tensor product $\Lambda \otimes_K \Gamma$ as a quotient of the path algebra $K(Q \times R)$, where $[\alpha] \otimes [w]$ should be the image of $\alpha \times w$ for an arrow α and a vertex w, and similarly for $[v] \otimes [\beta]$. Note that in $\Lambda \otimes_K \Gamma$, we have

$$[\alpha] \otimes [\beta] = \big([\alpha] \otimes [\mathfrak{t}(\beta)]\big)\big([\mathfrak{s}(\alpha)] \otimes [\beta]\big) = \big([\mathfrak{t}(\alpha)] \otimes [\beta]\big)\big([\alpha] \otimes [\mathfrak{s}(\beta)]\big).$$

We therefore need to impose relations on $K(Q \times R)$ that represent the above equality. This leads us to the following definition.

Definition 3.9 ([Les94])**.** Let α and β be arrows in Q and R, respectively. The **commutativity relation** of α and β is the element

$$\mathrm{Com}(\alpha, \beta) = \big(\alpha \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(\alpha) \times \beta\big) - \big(\mathfrak{t}(\alpha) \times \beta\big)\big(\alpha \times \mathfrak{s}(\beta)\big)$$

in $K(Q \times R)$. We let $\mathrm{Com}(Q, R)$ denote the set

$$\mathrm{Com}(Q, R) = \{\, \mathrm{Com}(\alpha, \beta) \mid \alpha \in Q_1, \beta \in R_1 \,\}$$

of all commutativity relations in $K(Q \times R)$. ■

Note that although the concept of commutativity relations is taken from [Les94], the term "commutativity relation" itself is from [Ska11].

In the tensor product $\Lambda \otimes_K \Gamma$, we don't just have the equality

$$\big([\alpha] \otimes [\mathfrak{t}(\beta)]\big)\big([\mathfrak{s}(\alpha)] \otimes [\beta]\big) = \big([\mathfrak{t}(\alpha)] \otimes [\beta]\big)\big([\alpha] \otimes [\mathfrak{s}(\beta)]\big)$$

for arrows α and β, we also have

$$\big([p] \otimes [\mathfrak{t}(q)]\big)\big([\mathfrak{s}(p)] \otimes [q]\big) = \big([\mathfrak{t}(p)] \otimes [q]\big)\big([p] \otimes [\mathfrak{s}(q)]\big)$$

for arbitrary paths p and q. The following lemma shows that this behavior is already captured by our definition of commutativity relations.

Lemma 3.10. *Let p and q be paths in Q and R, respectively. Then*

$$\big(p \times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p) \times q\big) \equiv \big(\mathfrak{t}(p) \times q\big)\big(p \times \mathfrak{s}(q)\big) \pmod{\langle \mathrm{Com}(Q, R)\rangle}.$$

Proof. If either p or q is a vertex, then the two sides of the congruence are actually equal, and there is nothing to prove. So assume that p and q are paths of length at least 1.

We proceed by induction on $n = \max\{\mathfrak{l}(p), \mathfrak{l}(q)\}$. We have just seen that the base case $n = 0$ holds, so assume that $n > 0$, and that the result is true for all paths of length less than n. Since we are assuming that p and q are not vertices, there exist arrows $\alpha \in Q_1, \beta \in R_1$ and paths p' in Q and q' in R such that $p = \alpha p'$ and $q = \beta q'$. Then we get

$$\begin{aligned}
\big(p \times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p) \times q\big) &= \big(\alpha p' \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(p') \times \beta q'\big) \\
&= \big(\alpha \times \mathfrak{t}(\beta)\big)\big(p' \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(p') \times \beta\big)\big(\mathfrak{s}(p') \times q'\big) \\
&\overset{*}{\equiv} \big(\alpha \times \mathfrak{t}(\beta)\big)\big(\mathfrak{t}(p') \times \beta\big)\big(p' \times \mathfrak{s}(\beta)\big)\big(\mathfrak{s}(p') \times q'\big) \\
&\overset{\dagger}{\equiv} \big(\mathfrak{t}(\alpha) \times \beta\big)\big(\alpha \times \mathfrak{s}(\beta)\big)\big(\mathfrak{t}(p') \times q'\big)\big(p' \times \mathfrak{s}(q')\big) \\
&\overset{\ddagger}{\equiv} \big(\mathfrak{t}(\alpha) \times \beta\big)\big(\mathfrak{t}(\alpha) \times q'\big)\big(\alpha \times \mathfrak{s}(q')\big)\big(p' \times \mathfrak{s}(q')\big) \\
&= \big(\mathfrak{t}(\alpha) \times \beta q'\big)\big(\alpha p' \times \mathfrak{s}(q')\big) \\
&= \big(\mathfrak{t}(p) \times q\big)\big(p \times \mathfrak{s}(q)\big) \pmod{\langle \mathrm{Com}(Q, R)\rangle},
\end{aligned}$$

where the marked congruences are true for the following reasons:

* If $n = 1$, then p' is a vertex and the congruence is actually an equality. Otherwise, β and p' both have length strictly less than n, and then the congruence follows from the inductive assumption.

† The congruence

$$\big(\alpha \times \mathfrak{t}(\beta)\big)\big(\mathfrak{t}(p') \times \beta\big) \equiv \big(\mathfrak{t}(\alpha) \times \beta\big)\big(\alpha \times \mathfrak{s}(\beta)\big)$$

follows from the definition of $\mathrm{Com}(Q, R)$ and the fact that $\mathfrak{t}(p') = \mathfrak{s}(\alpha)$. The congruence

$$\big(p' \times \mathfrak{s}(\beta)\big)\big(\mathfrak{s}(p') \times q'\big) \equiv \big(\mathfrak{t}(p') \times q'\big)\big(p' \times \mathfrak{s}(q')\big)$$

follows the inductive assumption along with the fact that $\mathfrak{s}(\beta) = \mathfrak{t}(q')$.

‡ Similar to $*$, using the fact that $\mathfrak{s}(\beta) = \mathfrak{t}(q')$ and $\mathfrak{t}(p') = \mathfrak{s}(\alpha)$.

□

Since $\Lambda = KQ/\langle\rho\rangle$, we have

$$[x] \otimes [\mu] = 0 \tag{3.1}$$

in $\Lambda \otimes_K \Gamma$ for all relations $x \in \rho$ and elements $\mu \in KR$. Similarly, we have

$$[\lambda] \otimes [y] = 0 \tag{3.2}$$

for all $\lambda \in KQ$ and $y \in \sigma$. We need to introduce relations in $K(Q \times R)$ that represent these equalities. Note that

$$[x] \otimes [\mu] = ([x] \otimes 1)(1 \otimes [\mu]) = \left(\sum_{w \in R_0} [x] \otimes [w]\right)(1 \otimes [\mu]),$$

so in order to represent (3.1), we only need relations for the case where μ is a vertex in R. Similarly, to represent the equality (3.2), we only need to consider the case where λ is a vertex in Q. This leads us to our definition of inclusion sets. This concept is taken from [Les94], but the terminology and notation are from [Ska11].

Definition 3.11 ([Les94])**.** Let $X \subseteq KQ$ be an arbitrary subset. Then the **inclusion set** of X in $K(Q \times R)$ is the set

$$\mathrm{Inc}_1(X) = \{\, x \times w \mid x \in X, w \in R_0 \,\}.$$

Similarly, for a subset $Y \subseteq KR$, we define

$$\mathrm{Inc}_2(Y) = \{\, v \times y \mid y \in Y, v \in Q_0 \,\}.$$

■

The following simple result is often convenient.

Lemma 3.12. *Let w be a vertex in R, and let $x, x' \in KQ$ be elements such that*

$$x \equiv x' \pmod{\langle \rho \rangle}.$$

Then

$$x \times w \equiv x' \times w \pmod{\langle \mathrm{Inc}_1(\rho) \rangle}.$$

Similarly, if v is a vertex in Q and $y, y' \in KR$ are elements such that

$$y \equiv y' \pmod{\langle \sigma \rangle},$$

then

$$v \times y \equiv v \times y' \pmod{\langle \mathrm{Inc}_2(\sigma) \rangle}.$$

Proof. We only prove the first statement; the second statement follows from a similar argument.

By assumption, $x - x'$ is an element of $\langle \rho \rangle$. Hence there exist elements $z_i \in \rho$ and $\lambda_i, \mu_i \in KQ$ such that $x - x' = \sum_i \lambda_i z_i \mu_i$. Then

$$x \times w - x' \times w = \sum_i \big(\lambda_i z_i \mu_i \times w\big) = \sum_i (\lambda_i \times w)(z_i \times w)(\mu_i \times w),$$

which is an element of the ideal $\langle \mathrm{Inc}_1(\rho) \rangle$ because $z_i \times w \in \mathrm{Inc}_1(\rho)$. □

We can now state and prove the following theorem, which shows how we can realize the tensor product $\Lambda \otimes_K \Gamma$ as a quotient of $K(Q \times R)$. Because the tensor product is taken over the field K, we use the notation I_K for the ideal such that $K(Q \times R)/I_K \cong \Lambda \otimes_K \Gamma$.

Theorem 3.13 ([Les94, Lemma 1.3])**.** *Let I_K be the ideal*

$$I_K = \langle \mathrm{Com}(Q, R), \mathrm{Inc}_1(\rho), \mathrm{Inc}_2(\sigma) \rangle$$

in $K(Q \times R)$. Then the tensor product of Λ and Γ over K is

$$\Lambda \otimes_K \Gamma \cong K(Q \times R)/I_K.$$

Proof.

Let $\phi\colon K(Q \times R) \to \Lambda \otimes_K \Gamma$ be the unique K-linear map that satisfies the following criteria:

- $\phi(v \times w) = [v] \otimes [w]$ for all vertices v in Q and w in R.
- $\phi(\alpha \times w) = [\alpha] \otimes [w]$ for any arrow α in Q and any vertex w in R.

- $\phi(v \times \beta) = [v] \otimes [\beta]$ for any vertex v in Q and any arrow β in R.
- If $\gamma_1, \dots, \gamma_n$ is a sequence of composable arrows in the product quiver $Q \times R$, then
$$\phi(\gamma_1 \dots \gamma_n) = \phi(\gamma_1) \dots \phi(\gamma_n).$$

It can be seen that ϕ is a homomorphism of K-algebras. Moreover, ϕ is surjective, because its image contains all elements of the form $[v] \otimes [w]$, $[\alpha] \otimes [w]$, or $[v] \otimes [\beta]$, and the set of such elements generates $\Lambda \otimes_K \Gamma$ as an algebra.

We claim that $I_K \subseteq \operatorname{Ker} \phi$. Let $\alpha\colon v \to v'$ and $\beta\colon w \to w'$ be arrows in Q and R, respectively. Then the commutativity relation of α and β is contained in the kernel:

$$\begin{aligned}
&\phi((\alpha \times w')(v \times \beta) - (v' \times \beta)(\alpha \times w)) \\
&= ([\alpha] \otimes [w'])([v] \otimes [\beta]) - ([v'] \otimes [\beta])([\alpha] \otimes [w]) \\
&= [\alpha v] \otimes [w'\beta] - [v'\alpha] \otimes [\beta w] \\
&= [\alpha] \otimes [\beta] - [\alpha] \otimes [\beta] \\
&= 0.
\end{aligned}$$

This shows that $\operatorname{Com}(Q, R) \subseteq \operatorname{Ker} \phi$. The fact that $\operatorname{Inc}_1(\rho)$ and $\operatorname{Inc}_2(\sigma)$ are contained in the kernel is an immediate consequence of the fact that $\Lambda = KQ/\langle \rho \rangle$ and $\Gamma = KR/\langle \sigma \rangle$. Hence $I_K \subseteq \operatorname{Ker} \phi$.

Since $I_K \subseteq \operatorname{Ker} \phi$, the map ϕ induces an algebra homomorphism

$$\overline{\phi}\colon K(Q \times R)/I_K \to \Lambda \otimes_K \Gamma,$$

and this homomorphism is surjective. To complete the proof, we will show that $\overline{\phi}$ is an isomorphism by constructing an inverse map.

We will find an inverse of $\overline{\phi}$ by using the universal property of the tensor product. We therefore need to find a K-bilinear map from $\Lambda \times \Gamma$ to $K(Q \times R)/I_K$. We first consider the map

$$\begin{gathered}
\psi\colon KQ \times KR \to K(Q \times R) \\
\left(\sum_{i=1}^{m} a_i p_i, \sum_{j=1}^{n} b_j q_j \right) \mapsto \sum_{i,j} a_i b_j (p_i \times \mathfrak{t}(q_j))(\mathfrak{s}(p_i) \times q_j),
\end{gathered}$$

where $a_i, b_j \in K$ are scalars, and p_i, q_j are paths in Q and R, respectively. The map ψ is clearly K-bilinear. Moreover, suppose that x is an element of the ideal $\langle \rho \rangle$. Then we can write

$$x = \sum_i a_i p_i x_i p_i'$$

for some scalars a_i, paths p_i and p'_i in Q, and elements $x_i \in \rho$. For each i, write

$$x_i = \sum_j b_{ij} p''_{ij},$$

where b_{ij} is a scalar and p''_{ij} is a path. Then for any path q in R, we have

$$\begin{aligned}
\psi(x, q) &= \psi\left(\sum_{i,j} a_i b_{ij} p_i p''_{ij} p'_i, q\right) \\
&= \sum_{i,j} a_i b_{ij} \big(p_i p''_{ij} p'_i \times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p'_i) \times q\big) \\
&= \sum_i a_i \big(p_i \times \mathfrak{t}(q)\big) \left(\sum_j b_{ij} p''_{ij} \times \mathfrak{t}(q)\right) \big(p'_i \times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p'_i) \times q\big) \\
&= \sum_i a_i \big(p_i \times \mathfrak{t}(q)\big)\big(x_i \times \mathfrak{t}(q)\big)\big(p'_i \times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p'_i) \times q\big).
\end{aligned}$$

Since $x_i \times \mathfrak{t}(q) \in \mathrm{Inc}_1(\rho)$, we see that $\psi(x, q) \in \langle \mathrm{Inc}_1(\rho) \rangle \subseteq I_K$. Since ψ is K-bilinear, it follows that ψ maps $\langle \rho \rangle \times KR$ into I_K. A similar argument shows that ψ also maps $KQ \times \langle \sigma \rangle$ into I_K.

The argument above shows that ψ induces a (well defined) K-bilinear map:

$$\begin{aligned}
\overline{\psi} \colon (KQ/\langle \rho \rangle) \times (KR/\langle \sigma \rangle) &\to K(Q \times R)/I_K \\
(x + \langle \rho \rangle, y + \langle \sigma \rangle) &\mapsto \psi(x, y) + I_K.
\end{aligned}$$

Now the universal property of the tensor product tells us that there is a (unique) K-linear map

$$h \colon \Lambda \otimes_K \Gamma \to K(Q \times R)/I_K$$

such that $h(x \otimes y) = \overline{\psi}(x, y)$ for all $x \in \Lambda, y \in \Gamma$. We will see that h is the inverse of $\overline{\phi}$.

We claim that h is a homomorphism of K-algebras. To prove this claim, it suffices to show that h is multiplicative on elementary tensors of the form $[p] \otimes [q]$, where p and q are paths. So let p and p' be paths in Q, and let q and q' be paths in R. Without loss of generality, assume that $\mathfrak{s}(p) = \mathfrak{t}(p')$ and $\mathfrak{s}(q) = \mathfrak{t}(q')$. Then

$$\begin{aligned}
h\big(([p] \otimes [q])([p'] \otimes [q'])\big) &= h([pp'] \otimes [qq']) \\
&= \overline{\psi}([pp'], [qq']) \\
&= \big(pp' \times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p') \times qq'\big) + I_K.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
h\big([p]\otimes[q]\big)h\big([p']\otimes[q']\big) &= \overline{\psi}([p],[q])\overline{\psi}([p'],[q']) \\
&= \big(p\times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p)\times q\big)\big(p'\times \mathfrak{t}(q')\big)\big(\mathfrak{s}(p')\times q'\big)+I_K \\
&= \big(p\times \mathfrak{t}(q)\big)\big(\mathfrak{t}(p')\times q\big)\big(p'\times \mathfrak{s}(q)\big)\big(\mathfrak{s}(p')\times q'\big)+I_K \\
&\overset{*}{=} \big(p\times \mathfrak{t}(q)\big)\big(p'\times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p')\times q\big)\big(\mathfrak{s}(p')\times q'\big)+I_K \\
&= \big(pp'\times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p')\times qq'\big)+I_K,
\end{aligned}$$

where the marked equality follows from Lemma 3.10. This shows that

$$h\big(([p]\otimes[q])([p']\otimes[q'])\big) = h\big([p]\otimes[q]\big)h\big([p']\otimes[q']\big),$$

as desired.

We can now show that h is the inverse of $\overline{\phi}$. We claim that

$$h\overline{\phi}(z+I_k) = z+I_K \tag{3.3}$$

for all $z\in K(Q\times R)$. To see that this is true, consider a vertex $v\times w$ in $Q\times R$. Then we have

$$\begin{aligned}
h\overline{\phi}\big(v\times w+I_K\big) &= h([v]\otimes[w]) \\
&= \psi(v,w)+I_K \\
&= \big(v\times \mathfrak{t}(w)\big)\big(\mathfrak{s}(v)\times w\big)+I_K \\
&= (v\times w)(v\times w)+I_K \\
&= v\times w+I_K.
\end{aligned}$$

Moreover, for an arrow of the form $\alpha\times w$ in $Q\times R$, we have

$$\begin{aligned}
h\overline{\phi}\big((\alpha\times w)+I_K\big) &= h([\alpha]\otimes[w]) \\
&= \big(\alpha\times \mathfrak{t}(w)\big)\big(\mathfrak{s}(\alpha)\times w\big)+I_K \\
&= \big(\alpha\times w\big)\big(\mathfrak{s}(\alpha)\times w\big)+I_K \\
&= \alpha\times w+I_K.
\end{aligned}$$

Similarly, we have that

$$h\overline{\phi}\big(v\times\beta+I_K\big) = v\times\beta+I_K$$

for an arrow $v\times\beta$. This shows that (3.3) holds as long as z is either a vertex or an arrow. Because $h\circ\overline{\phi}$ is a homomorphism of K-algebras, it follows that (3.3) holds for all $z\in K(Q\times R)$, since an arbitrary element z is a K-linear combination of products of arrows and vertices. Hence $h\circ\overline{\phi}$ is equal to the identity on $K(Q\times R)/I_K$, so h is a left inverse of $\overline{\phi}$. Since $\overline{\phi}$ is surjective, it follows that h is also the right inverse of $\overline{\phi}$, and hence h and $\overline{\phi}$ are mutually inverse isomorphisms. $\square$

Let us look at an example computation of tensor products using Theorem 3.13.

Example 3.14. Suppose Q and R are the following quivers.

$$\begin{tikzcd} Q\colon & v \arrow[loop left, "\alpha_1"] \arrow[loop right, "\alpha_2"] \end{tikzcd}$$

$$\begin{tikzcd} R\colon & w_1 \arrow[r, "\beta"] & w_2 \end{tikzcd}$$

Let $\rho = \{ \alpha_1^3, \alpha_2^3 \}$ and $\sigma = \emptyset$. The product quiver $Q \times R$ is the following quiver.

$$\begin{tikzcd} Q \times R\colon & v \times w_1 \arrow[loop above, "\alpha_1 \times w_1"] \arrow[loop below, "\alpha_2 \times w_1"] \arrow[r, "v \times \beta"] & v \times w_2 \arrow[loop above, "\alpha_1 \times w_2"] \arrow[loop below, "\alpha_2 \times w_2"] \end{tikzcd}$$

Let us find the relations in $K(Q \times R)$ that give the tensor product of

$$\Lambda = KQ / \langle \rho \rangle$$

and

$$\Gamma = KR / \langle \sigma \rangle$$

over K. The set of commutativity relations is

$$\begin{aligned} \mathrm{Com}(Q, R) = \{&(\alpha_1 \times w_2)(v \times \beta) - (v \times \beta)(\alpha_1 \times w_1), \\ &(\alpha_2 \times w_2)(v \times \beta) - (v \times \beta)(\alpha_2 \times w_1)\}. \end{aligned}$$

The inclusion set of ρ in $K(Q \times R)$ is

$$\mathrm{Inc}_1(\rho) = \{ \alpha_1^3 \times w_1, \alpha_1^3 \times w_2, \alpha_2^3 \times w_1, \alpha_2^3 \times w_2 \},$$

and the inclusion set of σ is

$$\mathrm{Inc}_2(\sigma) = \emptyset.$$

Thus the tensor product of Λ and Γ is

$$\Lambda \otimes_K \Gamma \cong K(Q \times R) / \langle X \rangle,$$

where X is the set

$$\begin{aligned} X = \{&(\alpha_1 \times w_2)(v \times \beta) - (v \times \beta)(\alpha_1 \times w_1), \\ &(\alpha_2 \times w_2)(v \times \beta) - (v \times \beta)(\alpha_2 \times w_1), \\ &\alpha_1^3 \times w_1,\ \alpha_1^3 \times w_2,\ \alpha_2^3 \times w_1,\ \alpha_2^3 \times w_2\}. \end{aligned}$$

■

3.2.2 Tensor products over an algebra

In Section 3.2.1, we saw how we can construct the tensor product $\Lambda \otimes_K \Gamma$. But what if we want to compute the tensor product of Λ and Γ over some other algebra Σ? For example, it is conceivable that by computing $\Lambda \otimes_\Sigma \Gamma$, we could learn something interesting about the original algebras Λ and Γ that we can't easily learn by studying $\Lambda \otimes_K \Gamma$.[1]

We will use the construction of $\Lambda \otimes_K \Gamma$ in Theorem 3.13 as a starting point for our construction of the tensor product over an algebra. Recall that we have defined the algebras $\Lambda = KQ/\langle\rho\rangle$ and $\Gamma = KR/\langle\sigma\rangle$. Throughout the rest of this section, let S be a quiver, and let $\Sigma = KS$. Suppose we are given algebra homomorphisms

$$f\colon \Sigma \to \Lambda$$

and

$$g\colon \Sigma \to \Gamma$$

such that $\mathrm{Im}(f) \subseteq Z(\Lambda)$ and $\mathrm{Im}(g) \subseteq Z(\Gamma)$. These homomorphisms allow us to view Λ and Γ as Σ-bimodules, and by Lemma 3.3, the tensor product $\Lambda \otimes_\Sigma \Gamma$ is a K-algebra.[2]

Observe that in the tensor product over Σ, we must have

$$xf(s) \otimes y = x \otimes g(s)y \tag{3.4}$$

for all $x \in \Lambda$, $y \in \Gamma$, and $s \in \Sigma$. Hence we need to introduce relations in $K(Q \times R)$ that represent this equality. Note that we have the following:

$$xf(s) \otimes y = (x \otimes y)(f(s) \otimes 1) = (x \otimes y) \sum_{v \in Q_0, w \in R_0} [v]f(s) \otimes [w]$$
$$x \otimes g(s)y = (x \otimes y)(1 \otimes g(s)) = (x \otimes y) \sum_{v \in Q_0, w \in R_0} [v] \otimes g(s)[w]$$

In order to represent (3.4), it is therefore enough to add relations for the case where x and y are cosets of vertices in Q and R, respectively. This leads us to our definition of balancing relations. However, we first need some additional

[1] This possibility was part of the original motivation for studying tensor products over an algebra when my advisor suggested the topic to me. In the end, I didn't have time to investigate this possibility in depth, but it remains a potential topic for future research.

[2] We could also consider the case where $\Sigma = KS/\langle\tau\rangle$ for a relation set τ. However, if Λ and Γ are $KS/\langle\tau\rangle$-modules, then they are also KS-modules, and the tensor products $\Lambda \otimes_{KS} \Gamma$ and $\Lambda \otimes_{KS/\langle\tau\rangle} \Gamma$ are isomorphic. It therefore suffices to consider the case $\Sigma = KS$.

setup. For an element $s \in \Sigma$, we pick some fixed, arbitrary representative $\tilde{f}(s) \in KQ$ of the equivalence class $f(s) \in KQ/\langle\rho\rangle$. This defines a function

$$\tilde{f} \colon \Sigma \to KQ.$$

(This function is typically not a homomorphism). Similarly, we define a function

$$\tilde{g} \colon \Sigma \to KR.$$

Definition 3.15. A **Σ-balancing relation** (with respect to our choice of $\tilde{f}$ and $\tilde{g}$) is an element in $K(Q \times R)$ of the form

$$v\tilde{f}(s) \times w - v \times \tilde{g}(s)w,$$

where $s \in \Sigma$ is either a vertex or an arrow, and where v and w are vertices in Q and R, respectively.

We let $\mathrm{Bal}_\Sigma(Q,R)$ denote the set of Σ-balancing relations (with respect to $\tilde{f}$ and $\tilde{g}$) in $K(Q \times R)$. ■

Remark 3.16. If $\Sigma \cong K$, then there is only one vertex in S, and no arrows. Hence all balancing relations have the form

$$v\tilde{f}(1_K) \times w - v \times \tilde{g}(1_K)w.$$

Since $f(1_K) = 1_\Lambda$ and $g(1_K) = 1_\Gamma$, it would be natural to choose representatives in such a way that $\tilde{f}(1_K) = 1_{KQ}$ and $\tilde{g}(1_K) = 1_{KR}$. Then we find that all balancing relations are zero. ■

The following result justifies Definition 3.15's assumption that s is a vertex or an arrow.

Lemma 3.17. *Let v and w be vertices in Q and R, respectively, and let $s \in \Sigma$ be an arbitrary element (not necessarily a vertex or an arrow). Then*

$$v\tilde{f}(s) \times w \equiv v \times \tilde{g}(s)w \pmod{\langle \mathrm{Bal}_\Sigma(Q,R), \mathrm{Inc}_1(\rho), \mathrm{Inc}_2(\sigma)\rangle}.$$

Proof. We first prove the case where s is a path p in the quiver S. If p is a vertex, then the statement of the lemma is clearly true, by the definition of $\mathrm{Bal}_\Sigma(Q,R)$. So suppose that p is a nontrivial path, i.e. that $p = \gamma_1 \dots \gamma_n$ for some composable arrows γ_i in S. Since $\mathrm{Im}(f)$ is contained in the centre of Λ, we have that

$$[v]f(\gamma_i) = [v^2]f(\gamma_i) = [v]([v]f(\gamma_i)) = [v]f(\gamma_i)[v].$$

Consequently, we get that

$$v\tilde{f}(\gamma_i) \equiv v\tilde{f}(\gamma_i)v \pmod{\langle \rho \rangle},$$

and by a similar argument

$$\tilde{g}(\gamma_i)w \equiv w\tilde{g}(\gamma_i)w \pmod{\langle \sigma \rangle}.$$

By repeatedly applying this observation, we get the following congruences:

$$\begin{aligned} v\tilde{f}(s) &\equiv v\tilde{f}(\gamma_1)\dots\tilde{f}(\gamma_n) \equiv v\tilde{f}(\gamma_1)v\dots v\tilde{f}(\gamma_n) \pmod{\langle \rho \rangle} \\ \tilde{g}(s)w &\equiv \tilde{g}(\gamma_1)\dots\tilde{g}(\gamma_n)w \equiv \tilde{g}(\gamma_1)w\dots w\tilde{g}(\gamma_n)w \pmod{\langle \sigma \rangle} \end{aligned} \tag{3.5}$$

This gives us the following:

$$\begin{aligned} v\tilde{f}(s) \times w &\overset{*}{\equiv} \big(v\tilde{f}(\gamma_1)v\dots v\tilde{f}(\gamma_n)\big) \times w \\ &= \big(v\tilde{f}(\gamma_1) \times w\big)\dots\big(v\tilde{f}(\gamma_n) \times w\big) \\ &\overset{\dagger}{\equiv} \big(v \times \tilde{g}(\gamma_1)w\big)\dots\big(v \times \tilde{g}(\gamma_n)w\big) \\ &= v \times \big(\tilde{g}(\gamma_1)w\dots w\tilde{g}(\gamma_n)w\big) \\ &\overset{*}{\equiv} v \times \tilde{g}(s)w \pmod{\langle \mathrm{Bal}_\Sigma(Q,R), \mathrm{Inc}_1(\rho), \mathrm{Inc}_2(\sigma) \rangle}, \end{aligned}$$

where the congruences marked with $*$ follow from (3.5) and Lemma 3.12, and the congruence marked with $\dagger$ follows from the definition of balancing relations. This proves the case where s is path.

We now prove the statement for an arbitrary element of Σ. Let

$$s = \sum_i a_i p_i \in \Sigma,$$

where each p_i is a path in the quiver S. Then

$$\begin{aligned} v\tilde{f}(s) \times w &\equiv \sum_i a_i(v\tilde{f}(p_i) \times w) \\ &\equiv \sum_i a_i(v \times \tilde{g}(p_i)w) \\ &\equiv v \times \tilde{g}(s)w \pmod{\langle \mathrm{Bal}_\Sigma(Q,R), \mathrm{Inc}_1(\rho), \mathrm{Inc}_2(\sigma) \rangle}. \end{aligned}$$

□

We are now ready to prove the following generalization of Theorem 3.13. Since we are taking the tensor product of Λ and Γ over the algebra Σ, we denote by I_Σ the ideal such that $K(Q \times R)/I_\Sigma \cong \Lambda \otimes_\Sigma \Gamma$.

Theorem 3.18. *Let I_Σ be the ideal*

$$I_\Sigma = \langle \mathrm{Com}(Q,R), \mathrm{Inc}_1(\rho), \mathrm{Inc}_2(\sigma), \mathrm{Bal}_\Sigma(Q,R)\rangle$$

in $K(Q \times R)$. Then the tensor product of Λ and Γ over Σ is

$$\Lambda \otimes_\Sigma \Gamma \cong K(Q \times R)/I_\Sigma.$$

Proof. Recall that in the proof of Theorem 3.13, we saw that there was a surjective algebra homomorphism

$$\phi\colon K(Q \times R) \to \Lambda \otimes_K \Gamma$$

given by $\phi(v \times w) = [v] \otimes [w]$ for vertices $v \times w$, and similarly for arrows $v \times \beta$ and $\alpha \times w$. Note that the canonical map $\Lambda \times \Gamma \to \Lambda \otimes_\Sigma \Gamma$ is K-bilinear, and hence there exists a K-linear map

$$t\colon \Lambda \otimes_K \Gamma \to \Lambda \otimes_\Sigma \Gamma$$

given by $t(\lambda \otimes_K \gamma) = \lambda \otimes_\Sigma \gamma$ for elementary tensors $\lambda \otimes_K \gamma$ in $\Lambda \otimes_K \Gamma$. It is clear that t is a surjective algebra homomorphism, and hence the composition

$$K(Q \times R) \xrightarrow{\phi} \Lambda \otimes_K \Gamma \xrightarrow{t} \Lambda \otimes_\Sigma \Gamma$$

is also a surjective algebra homomorphism.

We claim that $I_\Sigma \subseteq \mathrm{Ker}(t \circ \phi)$. In the proof of Theorem 3.13, we saw that the sets $\mathrm{Com}(Q,R)$, $\mathrm{Inc}_1(\rho)$, and $\mathrm{Inc}_2(\sigma)$ are contained in $\mathrm{Ker}\,\phi \subseteq \mathrm{Ker}(t \circ \phi)$, so it suffices to show that $\mathrm{Bal}_\Sigma(Q,R) \subseteq \mathrm{Ker}(t \circ \phi)$. Suppose that v and w are vertices in Q and R, respectively, and let $s \in \Sigma$ be either a vertex or an arrow. Then

$$\begin{aligned} t\phi((v\tilde{f}(s) \times w) - (v \times \tilde{g}(s)w)) &= [v\tilde{f}(s)] \otimes [w] - [v] \otimes [\tilde{g}(s)w] \\ &= [v\tilde{f}(s)] \otimes [w] - [v] \otimes (g(s)[w]) \\ &= [v\tilde{f}(s)] \otimes [w] - ([v]f(s)) \otimes [w] \\ &= [v\tilde{f}(s)] \otimes [w] - [v\tilde{f}(s)] \otimes [w] \\ &= 0, \end{aligned}$$

which shows that $\mathrm{Bal}_\Sigma(Q,R) \subseteq \mathrm{Ker}(t \circ \phi)$.

Since $I_\Sigma \subseteq \mathrm{Ker}(t \circ \phi)$, there is an induced map

$$\overline{t \circ \phi}\colon K(Q \times R)/I_\Sigma \to \Lambda \otimes_\Sigma \Gamma,$$

and this map is a surjective algebra homomorphism. We will show that $\overline{t \circ \phi}$ is an isomorphism by constructing an inverse map.

As in Theorem 3.13, we let I_K denote the ideal

$$I_K = \langle \mathrm{Com}(Q, R), \mathrm{Inc}_1(\rho), \mathrm{Inc}_2(\sigma) \rangle$$

in $K(Q \times R)$. Recall from the proof of Theorem 3.13 that there is a (well defined) K-bilinear map

$$\overline{\psi}\colon \Lambda \times \Gamma \to K(Q \times R)/I_K$$

given by $\overline{\psi}([p], [q]) = (p \times \mathfrak{t}(q))(\mathfrak{s}(p) \times q) + I_K$ for paths p and q in Q and R, respectively. Consider the composition

$$\Lambda \times \Gamma \xrightarrow{\overline{\psi}} K(Q \times R)/I_K \xrightarrow{\pi} K(Q \times R)/I_\Sigma,$$

where π is the algebra homomorphism given by $\pi(z + I_K) = z + I_\Sigma$. Note that $\pi \circ \overline{\psi}$ is K-bilinear. Let us verify that it is also Σ-balanced, i.e. that

$$\pi\overline{\psi}(s\lambda, \gamma) = \pi\overline{\psi}(\lambda, s\gamma)$$

for $\lambda \in \Lambda$, $\gamma \in \Gamma$, and $s \in \Sigma$. It suffices to check the special case where $\lambda = [p]$ and $\gamma = [q]$ for paths p and q. For an element $s \in \Sigma$, we have

$$\begin{aligned}
\pi\overline{\psi}(s[p], [q]) &= \pi\overline{\psi}\left([\tilde{f}(s)p], [q]\right) \\
&= \big(\tilde{f}(s)p \times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p) \times q\big) + I_\Sigma \\
&\overset{*}{=} \big(p\tilde{f}(s) \times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p) \times q\big) + I_\Sigma \\
&= \big(p \times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p)\tilde{f}(s) \times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p) \times q\big) + I_\Sigma \\
&\overset{\dagger}{=} \big(p \times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p) \times \tilde{g}(s)\mathfrak{t}(q)\big)\big(\mathfrak{s}(p) \times q\big) + I_\Sigma \\
&= \big(p \times \mathfrak{t}(q)\big)\big(\mathfrak{s}(p) \times \tilde{g}(s)q\big) + I_\Sigma \\
&= \pi\overline{\psi}\left([p], [\tilde{g}(s)q]\right) \\
&= \pi\overline{\psi}\left([p], s[q]\right),
\end{aligned}$$

where the equality marked with $*$ is due to the fact that $f(s)$ is contained in the centre of Λ, while the equality marked with $\dagger$ follows from Lemma 3.17. This shows that $\pi \circ \overline{\psi}$ is a Σ-balanced map. Then by the universal property of the tensor product, there exists a unique group homomorphism

$$h\colon \Lambda \otimes_\Sigma \Gamma \to K(Q \times R)/I_\Sigma$$

such that $h(x \otimes y) = \pi\overline{\psi}(x, y)$ for $x \in \Lambda$ and $y \in \Gamma$. Then h is the inverse of $\overline{t \circ \phi}$, by a similar argument to the one given in the proof of Theorem 3.13. $\square$

Let us use Theorem 3.18 to compute a tensor product.

Example 3.19. Let Q, R, and S denote the following quivers.

$$Q\colon \begin{tikzcd} & v_2 \arrow[dr, "\alpha_2"] & \\ v_1 \arrow[ur, "\alpha_1"] & & v_3 \arrow[ll, "\alpha_3"] \end{tikzcd}$$

$$R\colon \begin{tikzcd} w_1 \arrow[r, "\beta"] & w_2 \end{tikzcd}$$

$$S\colon \begin{tikzcd} u \arrow[loop right, "\gamma"] \end{tikzcd}$$

We let $\rho = \{ \alpha_3\alpha_2\alpha_1 \}$ and $\sigma = \emptyset$, and set $\Lambda = KQ/\langle\rho\rangle$, $\Gamma = KR/\langle\sigma\rangle$, and $\Sigma = KS$. We let $f\colon \Sigma \to \Lambda$ be the unique algebra homomorphism such that

$$f(\gamma) = [1 + \alpha_1\alpha_3\alpha_2 + \alpha_2\alpha_1\alpha_3],$$

and we let $g\colon \Sigma \to \Gamma$ be the unique homomorphism such that $g(\gamma) = 0$. Note that $\mathrm{Im}(f) \subseteq Z(\Lambda)$ and $\mathrm{Im}(g) \subseteq Z(\Gamma)$. Then the tensor product of Λ and Γ over Σ is

$$\Lambda \otimes_\Sigma \Gamma \cong K(Q \times R)/\langle \mathrm{Com}(Q,R) \cup \mathrm{Inc}_1(\rho) \cup \mathrm{Bal}_\Sigma(Q,R)\rangle,$$

where we have omitted $\mathrm{Inc}_2(\sigma)$ because it is empty. The product quiver of Q and R is the following quiver.

$$Q \times R\colon \begin{tikzcd} v_1 \times w_1 \arrow[dr, "\alpha_1\times w_1"] \arrow[ddd, bend right, "v_1\times\beta"'] & & v_3 \times w_1 \arrow[ll, "\alpha_3\times w_1"'] \arrow[ddd, bend left, "v_3\times\beta"] \\ & v_2 \times w_1 \arrow[ur, "\alpha_2\times w_1"] \arrow[d, "v_2\times\beta"] & \\ & v_2 \times w_2 \arrow[dr, "\alpha_2\times w_2"] & \\ v_1 \times w_2 \arrow[ur, "\alpha_1\times w_2"] & & v_3 \times w_2 \arrow[ll, "\alpha_3\times w_2"] \end{tikzcd}$$

It can be seen that, for a natural choice of the functions $\tilde{f}$ and $\tilde{g}$, the set of (nonzero) balancing relations is

$$\begin{aligned}\mathrm{Bal}_\Sigma(Q,R) = \{&v_1 \times w_1,\ v_2 \times w_1 + \alpha_1\alpha_3\alpha_2 \times w_1,\ v_3 \times w_1 + \alpha_2\alpha_1\alpha_3 \times w_1 \\ &v_1 \times w_2,\ v_2 \times w_2 + \alpha_1\alpha_3\alpha_2 \times w_2,\ v_3 \times w_2 + \alpha_2\alpha_1\alpha_3 \times w_2\}.\end{aligned}$$

Since the vertex $v_1 \times w_1$ is an element of the set $\mathrm{Bal}_\Sigma(Q, R)$, we see that $[v_1] \otimes [w_1] = 0$ in $\Lambda \otimes_\Sigma \Gamma$. But since $\alpha_1\alpha_3\alpha_2$ is a path that moves through the vertex v_1, we have

$$[\alpha_1\alpha_3\alpha_2] \otimes [w_1] = ([\alpha_1] \otimes [w_1])([v_1] \otimes [w_1])([\alpha_3\alpha_2] \otimes [w_1]) = 0.$$

Since the element $v_2 \times w_1 + \alpha_1\alpha_3\alpha_2 \times w_1$ is a balancing relation, we now find that

$$[v_2] \otimes [w_1] = -[\alpha_1\alpha_3\alpha_2] \otimes [w_1] = 0.$$

Similarly, we also have $[v_3] \otimes [w_1] = 0$, and a similar argument shows that

$$[v_1] \otimes [w_2] = [v_2] \otimes [w_2] = [v_3] \otimes [w_2] = 0.$$

But this means that all the vertices of $Q \times R$ have image zero in the tensor product $\Lambda \otimes_\Sigma \Gamma$. It follows that

$$\Lambda \otimes_\Sigma \Gamma = 0.$$

■

Recall that if P is a quiver, then an ideal $I \subseteq KP$ is called **admissible** if there exists some integer m such that

$$J_P^m \subseteq I \subseteq J_P^2,$$

where J_P is the ideal in KP generated by the arrows of P. In Example 3.19, we saw that even if the ideals $\langle\rho\rangle \subseteq KQ$ and $\langle\sigma\rangle \subseteq KR$ are admissible, the ideal I_Σ might not be admissible, since in Example 3.19 we had $I_\Sigma = K(Q \times R)$. However, we will see that the ideal I_Σ satisfies a weaker condition, namely that of being "lower-admissible", a term we now define.

Definition 3.20. Let P be any quiver, and let $I \subseteq KP$ be an ideal. We say that I is **lower-admissible** if there exists some integer m such that $J_P^m \subseteq I$. ■

Proposition 3.21. *Suppose that the ideals $\langle\rho\rangle \subseteq KQ$ and $\langle\sigma\rangle \subseteq KR$ are admissible. Then the ideal*

$$I_K = \langle \mathrm{Com}(Q, R), \mathrm{Inc}_1(\rho), \mathrm{Inc}_2(\sigma)\rangle$$

is admissible, while the ideal

$$I_\Sigma = \langle \mathrm{Com}(Q, R), \mathrm{Inc}_1(\rho), \mathrm{Inc}_2(\sigma), \mathrm{Bal}_\Sigma(Q, R)\rangle$$

is lower-admissible.

Proof. Since ρ and σ are assumed to generate admissible ideals, there exist integers m_1 and m_2 such that $J_Q^{m_1} \subseteq \langle \rho \rangle \subseteq J_Q^2$ and $J_R^{m_2} \subseteq \langle \sigma \rangle \subseteq J_R^2$. Let $m = \max\{m_1, m_2\}$, and let p be a path in $Q \times R$ of length at least $2m$. Then there exist paths q in Q and r in R such that

$$p + I_K = [q] \otimes [r] = ([q] \otimes [\mathfrak{t}(r)])([\mathfrak{s}(q)] \otimes [r])$$

in the tensor product $\Lambda \otimes_K \Gamma$, which we identify with $K(Q \times R)/I_K$. In other words, we have

$$p \equiv \big(q \times \mathfrak{t}(r)\big)\big(\mathfrak{s}(q) \times r\big) \pmod{I_K}.$$

Since $\mathfrak{l}(p) = \mathfrak{l}(q) + \mathfrak{l}(r)$ and $\mathfrak{l}(p) \geq 2m$, we see that $\mathfrak{l}(q) \geq m$ or $\mathfrak{l}(r) \geq m$. Without loss of generality, assume that $\mathfrak{l}(q) \geq m$. Then $q \in J_Q^m \subseteq \langle \rho \rangle$, so

$$q \times \mathfrak{t}(r) \in \langle \mathrm{Inc}_1(\rho) \rangle \subseteq I_K,$$

and hence

$$p \equiv 0 \pmod{I_K}.$$

In other words, $p \in I_K$. This shows that $J_{Q \times R}^{2m} \subseteq I_K$, and hence I_K is lower-admissible. To see that the upper bound $I_K \subseteq J_{Q \times R}^2$ is also satisfied, note that $\mathrm{Com}(Q, R)$ is clearly contained in $J_{Q \times R}^2$, and so are $\mathrm{Inc}_1(\rho)$ and $\mathrm{Inc}_2(\sigma)$, because $\rho \subseteq J_Q^2$ and $\sigma \subseteq J_R^2$. Hence I_K is an admissible ideal. Moreover, the ideal I_Σ is lower-admissible since $I_K \subseteq I_\Sigma$. $\square$

In Chapter 4, we will see how we can replace $Q \times R$ and I_Σ with another quiver P and an ideal $I \subseteq KP$ such that $K(Q \times R)/I_\Sigma \cong KP/I$, and such that I is an admissible ideal.

3.3 A Gröbner basis for the tensor product

In Theorem 3.13 and Theorem 3.18, we saw how we could find ideals I_K and I_Σ in the path algebra $K(Q \times R)$, such that

$$\Lambda \otimes_K \Gamma \cong K(Q \times R)/I_K$$

and

$$\Lambda \otimes_\Sigma \Gamma \cong K(Q \times R)/I_\Sigma,$$

where $\Lambda = KQ/\langle \rho \rangle$ and $\Gamma = KR/\langle \sigma \rangle$. It is of interest to represent the algebras $\Lambda \otimes_K \Gamma$ and $\Lambda \otimes_\Sigma \Gamma$ on a computer, for instance using the package QPA (Quivers and Path Algebras) for the programming language GAP. In order to do this, we need to find Gröbner bases for I_K and I_Σ. Of course, we could simply use an algorithm, such as Buchberger's algorithm (Algorithm

2.3), to compute Gröbner bases. However, this is potentially time-consuming, especially because the quiver $Q \times R$ and the generating sets for I_K and I_Σ can easily become quite large. It would therefore be nice if we could explicitly describe Gröbner bases for I_K and I_Σ, assuming that we are given Gröbner bases for $\langle \rho \rangle$ and $\langle \sigma \rangle$. In this section, we show that this can be done for the ideal I_K. Unfortunately, it seems that this is not possible for the ideal I_Σ, as we will see.

In order to find a Gröbner basis, we must first choose some admissible order on the set of paths in $Q \times R$. Throughout this section, assume that the paths in Q and R are ordered according to the left length-lexicographic ordering, as in Example 2.4.[3] Let $v, v' \in Q_0$ and $w, w' \in R_0$ be vertices, and let $\alpha, \alpha' \in Q_1$ and $\beta, \beta' \in R_1$ be arrows. We order the vertices and arrows in $Q \times R$ as follows:

- $v \times w < v' \times w'$ if $v < v'$, or if $v = v'$ and $w < w'$.
- Every vertex is smaller than every arrow. In other words, $v \times w < \alpha \times w'$ and $v \times w < v' \times \beta$.
- $v \times \beta < v' \times \beta'$ if $v < v'$, or if $v = v'$ and $\beta < \beta'$.
- $v \times \beta < \alpha \times w$.
- $\alpha \times w < \alpha' \times w'$ if $\alpha < \alpha'$, or if $\alpha = \alpha'$ and $w < w'$.

We extend this to an admissible order on the set of paths in $Q \times R$ by using the left length-lexicographic order.

Example 3.22. Let Q and R denote the following quivers.

$$Q\colon \begin{tikzcd} v_1 \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_2"'] & v_2 \end{tikzcd}$$

$$R\colon \begin{tikzcd} w_1 \arrow[r, "\beta"] & w_2 \end{tikzcd}$$

The product quiver of Q and R is the following quiver.

$$Q \times R\colon \begin{tikzcd} v_1 \times w_1 \arrow[r, "v_1\times\beta"] \arrow[d, bend right, "\alpha_1\times w_1"'] \arrow[d, bend left, "\alpha_2\times w_1"] & v_1 \times w_2 \arrow[d, bend right, "\alpha_1\times w_2"'] \arrow[d, bend left, "\alpha_2\times w_2"] \\ v_2 \times w_1 \arrow[r, "v_2\times\beta"'] & v_2 \times w_2 \end{tikzcd}$$

[3]One reason to use the length-lex order is the fact that this is the only order supported by QPA.

Suppose that the order on the vertices and arrows of Q is given by

$$v_1 < v_2 < \alpha_1 < \alpha_2,$$

and that the order on the vertices and arrows of R is given by

$$w_1 < w_2 < \beta.$$

Then the order on the vertices in $Q \times R$ is

$$v_1 \times w_1 < v_1 \times w_2 < v_2 \times w_1 < v_2 \times w_2,$$

while the order on the arrows in $Q \times R$ is

$$v_1 \times \beta < v_2 \times \beta < \alpha_1 \times w_1 < \alpha_1 \times w_2 < \alpha_2 \times w_1 < \alpha_2 \times w_2.$$

■

Let us start by finding a Gröbner basis for the ideal I_K, which we used to construct $\Lambda \otimes_K \Gamma$.

3.3.1 A Gröbner basis for the tensor product over a field

Recall that the ideal $I_K \subseteq K(Q \times R)$ in Theorem 3.13 is given by

$$I_K = \langle \mathrm{Com}(Q, R), \mathrm{Inc}_1(\rho), \mathrm{Inc}_2(\sigma) \rangle .$$

Let G and H be Gröbner bases for the ideals $\langle \rho \rangle \subseteq KQ$ and $\langle \sigma \rangle \subseteq KR$, respectively. Since $\langle G \rangle = \langle \rho \rangle$ and $\langle H \rangle = \langle \sigma \rangle$, it follows that the set

$$X = \mathrm{Com}(Q, R) \cup \mathrm{Inc}_1(G) \cup \mathrm{Inc}_2(H)$$

is a generating set for I_K. It is natural to ask if X is a Gröbner basis. We will see that the answer is affirmative, as long as we assume that $\mathrm{Tip}(G) \subseteq J_Q^2$ and $\mathrm{Tip}(H) \subseteq J_R^2$, where J_Q and J_R denote the ideals generated by the arrows in Q and R, respectively. Note that this will always be true if G and H generate admissible ideals.

We will prove that the set X is a Gröbner basis by reducing to the case where G and H are reduced Gröbner bases and then using Theorem 2.41. However, before we can apply Theorem 2.41, we must check that X satisfies the hypothesis of said theorem. This brings us to the following lemma.

Lemma 3.23. *Suppose $G \subseteq KQ$ and $H \subseteq KR$ are tip reduced uniform sets such that $\mathrm{Tip}(G) \subseteq J_Q^2$ and $\mathrm{Tip}(H) \subseteq J_R^2$. Then the set*

$$X = \mathrm{Com}(Q, R) \cup \mathrm{Inc}_1(G) \cup \mathrm{Inc}_2(H)$$

is tip reduced and uniform.

Proof. The set $\mathrm{Com}(Q, R)$ of commutativity relations is uniform, because if $\alpha \in Q_1$ and $\beta \in R_1$ are arrows, then $\mathrm{Com}(\alpha, \beta)$ is a linear combination of paths from $\mathfrak{s}(\alpha) \times \mathfrak{s}(\beta)$ to $\mathfrak{t}(\alpha) \times \mathfrak{t}(\beta)$. The inclusion sets $\mathrm{Inc}_1(G)$ and $\mathrm{Inc}_2(H)$ are also uniform, because G and H are uniform. It follows that X is a uniform set.

Let $x, y \in X \setminus \{ 0 \}$ be some nonzero elements. In order to show that X is tip reduced, we must show that if $\mathrm{Tip}(x) \mid \mathrm{Tip}(y)$, then $x = y$. So assume that $\mathrm{Tip}(x) \mid \mathrm{Tip}(y)$. We consider the following cases.

1. $\underline{x, y \in \mathrm{Com}(Q, R)}$: Let $\alpha, \alpha' \in Q_1$ and $\beta, \beta' \in R_1$ be arrows such that $\underline{x = \mathrm{Com}(\alpha, \beta)}$ and $y = \mathrm{Com}(\alpha', \beta')$. Recall that commutativity relations have the form
$$\mathrm{Com}(\alpha, \beta) = \big(\alpha \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(\alpha) \times \beta\big) - \big(\mathfrak{t}(\alpha) \times \beta\big)\big(\alpha \times \mathfrak{s}(\beta)\big).$$
From our definition of the order on $Q \times R$, it follows that
$$\mathrm{Tip}(\mathrm{Com}(\alpha, \beta)) = \big(\alpha \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(\alpha) \times \beta\big).$$
Since we are assuming that $\mathrm{Tip}(x) \mid \mathrm{Tip}(y)$, we see that
$$\big(\alpha \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(\alpha) \times \beta\big) \mid \big(\alpha' \times \mathfrak{t}(\beta')\big)\big(\mathfrak{s}(\alpha') \times \beta'\big).$$
But this is clearly only possible if $\alpha = \alpha'$ and $\beta = \beta'$. Hence $x = y$.

2. $\underline{x, y \in \mathrm{Inc}_1(G)}$: Then there exist elements $g, g' \in G$ and vertices $w, w' \in$ $\underline{R_0}$ such that $x = g \times w$ and $y = g' \times w'$. Note that
$$\mathrm{Tip}(g \times w) = \mathrm{Tip}(g) \times w.$$
Since $\mathrm{Tip}(x) \mid \mathrm{Tip}(y)$, it follows that
$$\big(\,\mathrm{Tip}(g) \times w\big) \mid \big(\,\mathrm{Tip}(g') \times w'\big).$$
This is only possible if $w = w'$ and $\mathrm{Tip}(g) \mid \mathrm{Tip}(g')$. Since we are assuming that G is a tip reduced set, this implies that $g = g'$. Hence $x = y$.

3. $\underline{x, y \in \mathrm{Inc}_2(H)}$: Similar to case 2.

4. $\underline{x \in \mathrm{Com}(Q, R), y \in \mathrm{Inc}_1(G)}$: Let $\alpha \in Q_1$ and $\beta \in R_1$ be arrows such $\underline{\text{that } x = \mathrm{Com}(\alpha, \beta)}$, and let $g \in G$ be an element and $w \in R_0$ a vertex such that $y = g \times w$. Then
$$\big(\alpha \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(\alpha) \times \beta\big) \mid \big(\,\mathrm{Tip}(g) \times w\big),$$

and in particular,

$$\big(\mathfrak{s}(\alpha) \times \beta\big) \mid \big(\operatorname{Tip}(g) \times w\big).$$

But this is impossible.

5. $\underline{x \in \operatorname{Com}(Q,R), y \in \operatorname{Inc}_2(H)}$: Similar to case 4.

6. $\underline{x \in \operatorname{Inc}_1(G), y \in \operatorname{Com}(Q,R)}$: Let $g \in G$, $w \in R_0$, $\alpha \in Q_1$, and $\beta \in R_1$ such that $x = g \times w$ and $y = \operatorname{Com}(\alpha, \beta)$. Then

$$\big(\operatorname{Tip}(g) \times w\big) \mid \big(\alpha \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(\alpha) \times \beta\big).$$

It follows that $\operatorname{Tip}(g) \in \{\, \alpha, \mathfrak{t}(\alpha), \mathfrak{s}(\alpha) \,\}$. But this contradicts our assumption that $\operatorname{Tip}(G) \subseteq J_Q^2$.

7. $\underline{x \in \operatorname{Inc}_1(H), y \in \operatorname{Com}(Q,R)}$: Similar to case 6.

8. $\underline{x \in \operatorname{Inc}_1(G), y \in \operatorname{Inc}_2(H)}$: Then there exist elements $g \in G, h \in H$ and vertices $v \in Q_0, w \in R_0$ such that $x = g \times w$ and $y = v \times h$. Then

$$\big(\operatorname{Tip}(g) \times w\big) \mid \big(v \times \operatorname{Tip}(h)\big).$$

This is only possible if $\operatorname{Tip}(g) = v$ and $\operatorname{Tip}(h) = w$, which contradicts our assumption that $\operatorname{Tip}(G) \subseteq J_Q^2$ and $\operatorname{Tip}(H) \subseteq J_R^2$.

9. $\underline{x \in \operatorname{Inc}_2(H), x \in \operatorname{Inc}_1(G)}$: Similar to case 8.

□

Before we can prove that the set X is a Gröbner basis, we will need a few more lemmas.

Lemma 3.24. *Suppose $G \subseteq KQ$ is a subset, $y \in KQ$ is any element, and $w \in R_0$ is a vertex. If y has remainder 0 under division by G in KQ, then $y \times w$ has remainder 0 under division by $\operatorname{Inc}_1(G)$ in $K(Q \times R)$.*

Similarly, suppose $H \subseteq KR$ is a subset, $x \in KR$ is an element, and $v \in Q_0$ is a vertex. If x has remainder 0 under division by H in KR, then $v \times x$ has remainder zero under division by $\operatorname{Inc}_2(H)$ in $K(Q \times R)$.

Proof. We only prove the first statement, as the second one follows from a similar argument.

Assume that $y \Rightarrow_G 0$. Then by the definition of remainders (Definition 2.16), there exist elements $g_i \in G$ and $u_i, u_i' \in KQ$ such that

$$y = \sum_i u_i g_i u_i',$$

and such that $\mathrm{Tip}(u_i g_i u_i') \le \mathrm{Tip}(y)$. Then we also have that

$$y \times w = \sum_i (u_i \times w)(g_i \times w)(u_i' \times w).$$

Note that $g_i \times w \in \mathrm{Inc}_1(G)$, and that

$$\begin{aligned}\mathrm{Tip}((u_i \times w)(g_i \times w)(u_i' \times w)) &= \mathrm{Tip}(u_i g_i u_i') \times w \\ &\le \mathrm{Tip}(y) \times w \\ &= \mathrm{Tip}(y \times w).\end{aligned}$$

Hence $y \times w$ has remainder 0 under division by $\mathrm{Inc}_1(G)$. $\square$

We also need the following technical result.

Lemma 3.25. *Let α be an arrow in Q, and let*

$$p = \beta_1 \dots \beta_n$$

be a nontrivial path in R, where each β_i is an arrow. Then the following equality holds:[4]

$$\begin{aligned}\big(\alpha \times \mathfrak{t}(p)\big)\big(\mathfrak{s}(\alpha) \times p\big) = \big(\mathfrak{t}(\alpha) \times p\big)\big(\alpha \times \mathfrak{s}(p)\big) \; + \\ \sum_{j=0}^{n-1} \big(\mathfrak{t}(\alpha) \times \beta_1 \dots \beta_j\big) \operatorname{Com}(\alpha, \beta_{j+1}) \big(\mathfrak{s}(\alpha) \times \beta_{j+2} \dots \beta_n\big).\end{aligned}$$

Moreover, the equality still holds if p is a vertex, as long as we define the sum $\sum_{j=0}^{n-1}(\cdots)$ to equal zero in this case.

Similarly, if β is an arrow in R and if q is a path in Q that is either a vertex or a nontrivial path of the form

$$q = \alpha_1 \dots \alpha_n$$

for arrows α_i, then the following equality holds:

$$\begin{aligned}\big(q \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(q) \times \beta\big) = \big(\mathfrak{t}(q) \times \beta\big)\big(q \times \mathfrak{s}(\beta)\big) \; + \\ \sum_{j=0}^{n-1} \big(\alpha_1 \dots \alpha_j \times \mathfrak{t}(\beta)\big) \operatorname{Com}(\alpha_{j+1}, \beta) \big(\alpha_{j+2} \dots \alpha_n \times \mathfrak{s}(\beta)\big).\end{aligned}$$

[4] If $j = 0$, then we define $\mathfrak{t}(\alpha) \times \beta_1 \dots \beta_j$ to be equal to 1. Similarly, if $j = n - 1$, then $\mathfrak{s}(\alpha) \times \beta_{j+2} \dots \beta_n$ is also equal to 1.

Proof. We only prove the first equality; the second one follows from a similar argument.

We proceed by induction on $n = \mathfrak{l}(p)$. The equality clearly holds if $n = 0$, i.e. if p is a vertex.

If $n = 1$, then we have

$$\begin{aligned}
&\big(\alpha \times \mathfrak{t}(p)\big)\big(\mathfrak{s}(\alpha) \times p\big) \\
&= \big(\alpha \times \mathfrak{t}(\beta_1)\big)\big(\mathfrak{s}(\alpha) \times \beta_1\big) \\
&= \big(\mathfrak{t}(\alpha) \times \beta_1\big)\big(\alpha \times \mathfrak{s}(\beta_1)\big) + \mathrm{Com}(\alpha, \beta_1) \\
&= \big(\mathfrak{t}(\alpha) \times \beta_1\big)\big(\alpha \times \mathfrak{s}(\beta_1)\big) + \sum_{j=0}^{0} \mathrm{Com}(\alpha, \beta_{j+1}),
\end{aligned}$$

as desired.

Now suppose that $n > 1$, and assume that the lemma holds for all non-trivial paths that are strictly shorter than p. Let $p' = \beta_1 \dots \beta_{n-1}$. Then by applying the inductive assumption to p', we see that

$$\begin{aligned}
&\big(\alpha \times \mathfrak{t}(p)\big)\big(\mathfrak{s}(\alpha) \times p\big) \\
&= \big(\alpha \times \mathfrak{t}(p')\big)\big(\mathfrak{s}(\alpha) \times p'\big)\big(\mathfrak{s}(\alpha) \times \beta_n\big) \\
&= \Bigg(\big(\mathfrak{t}(\alpha) \times p'\big)\big(\alpha \times \mathfrak{s}(p')\big) \\
&+ \sum_{j=0}^{n-2} \big(\mathfrak{t}(\alpha) \times \beta_1 \dots \beta_j\big)\, \mathrm{Com}(\alpha, \beta_{j+1})\big(\mathfrak{s}(\alpha) \times \beta_{j+2} \dots \beta_{n-1}\big)\Bigg)\big(\mathfrak{s}(\alpha) \times \beta_n\big) \\
&= \big(\mathfrak{t}(\alpha) \times p'\big)\big(\alpha \times \mathfrak{t}(\beta_n)\big)\big(\mathfrak{s}(\alpha) \times \beta_n\big) \\
&+ \sum_{j=0}^{n-2} \big(\mathfrak{t}(\alpha) \times \beta_1 \dots \beta_j\big)\, \mathrm{Com}(\alpha, \beta_{j+1})\big(\mathfrak{s}(\alpha) \times \beta_{j+2} \dots \beta_{n-1}\beta_n\big). \qquad (3.6)
\end{aligned}$$

Also observe that

$$\begin{aligned}
&\big(\mathfrak{t}(\alpha) \times p'\big)\big(\alpha \times \mathfrak{t}(\beta_n)\big)\big(\mathfrak{s}(\alpha) \times \beta_n\big) \\
&= \big(\mathfrak{t}(\alpha) \times p'\big)\Big(\big(\mathfrak{t}(\alpha) \times \beta_n\big)\big(\alpha \times \mathfrak{s}(\beta_n)\big) + \mathrm{Com}(\alpha, \beta_n)\Big) \\
&= \big(\mathfrak{t}(\alpha) \times p'\beta_n\big)\big(\alpha \times \mathfrak{s}(\beta_n)\big) + \big(\mathfrak{t}(\alpha) \times p'\big)\, \mathrm{Com}(\alpha, \beta_n) \\
&= \big(\mathfrak{t}(\alpha) \times p\big)\big(\alpha \times \mathfrak{s}(\beta_n)\big) + \big(\mathfrak{t}(\alpha) \times \beta_1 \dots \beta_{n-1}\big)\, \mathrm{Com}(\alpha, \beta_n). \qquad (3.7)
\end{aligned}$$

Combining (3.6) and (3.7), we then get the following:

$$\begin{aligned}
&\big(\alpha \times \mathfrak{t}(p)\big)\big(\mathfrak{s}(\alpha) \times p\big)\\
&= \big(\mathfrak{t}(\alpha) \times p\big)\big(\alpha \times \mathfrak{s}(\beta_n)\big) + \big(\mathfrak{t}(\alpha) \times \beta_1 \dots \beta_{n-1}\big) \operatorname{Com}(\alpha, \beta_n)\\
&+ \sum_{j=0}^{n-2} \big(\mathfrak{t}(\alpha) \times \beta_1 \dots \beta_j\big) \operatorname{Com}(\alpha, \beta_{j+1})\big(\mathfrak{s}(\alpha) \times \beta_{j+2} \dots \beta_n\big)\\
&= \big(\mathfrak{t}(\alpha) \times p\big)\big(\alpha \times \mathfrak{s}(\beta_n)\big)\\
&+ \sum_{j=0}^{n-1} \big(\mathfrak{t}(\alpha) \times \beta_1 \dots \beta_j\big) \operatorname{Com}(\alpha, \beta_{j+1})\big(\mathfrak{s}(\alpha) \times \beta_{j+2} \dots \beta_n\big),
\end{aligned}$$

which is what we wanted to show. □

We are now ready to prove the following theorem, which is based on a theorem from Helene Tuft Bjørshol's bachelor thesis [Bjø21]. We present an alternative proof to the one given by Bjørshol. The idea behind our proof is somewhat similar to the one in Bjørshol's thesis, in that we will show that all overlap relations between elements $x, y \in X$ have remainder zero, which we will prove by considering different cases based on whether x and y are elements of $\operatorname{Com}(Q, R)$, $\operatorname{Inc}_1(G)$, or $\operatorname{Inc}_2(H)$. However, the arguments given in each case are quite different from the arguments given in [Bjø21], unless otherwise noted.

Theorem 3.26 ([Bjø21, Theorem 5.4])**.** *Suppose that G and H are Gröbner bases for the ideals $\langle\rho\rangle \subseteq KQ$ and $\langle\sigma\rangle \subseteq KR$, respectively, such that $\operatorname{Tip}(G) \subseteq J_Q^2$ and $\operatorname{Tip}(H) \subseteq J_R^2$. Then the set*

$$X = \operatorname{Com}(Q, R) \cup \operatorname{Inc}_1(G) \cup \operatorname{Inc}_2(H)$$

is a Gröbner basis in $K(Q \times R)$.

Proof. Let G' and H' be the reduced Gröbner bases for $\langle G\rangle$ and $\langle H\rangle$, respectively. Then $\operatorname{Tip}(G') \subseteq \operatorname{Tip}(G)$ and $\operatorname{Tip}(H') \subseteq \operatorname{Tip}(H)$ by Proposition 2.30. Let X' denote the set

$$X' = \operatorname{Com}(Q, R) \cup \operatorname{Inc}_1(G') \cup \operatorname{Inc}_2(H').$$

Then we have

$$\begin{aligned}
\operatorname{Tip}(X') &= \operatorname{Tip}(\operatorname{Com}(Q, R)) \cup \operatorname{Tip}(\operatorname{Inc}_1(G')) \cup \operatorname{Tip}(\operatorname{Inc}_2(H'))\\
&= \operatorname{Tip}(\operatorname{Com}(Q, R)) \cup \operatorname{Inc}_1(\operatorname{Tip}(G')) \cup \operatorname{Inc}_2(\operatorname{Tip}(H'))\\
&\subseteq \operatorname{Tip}(\operatorname{Com}(Q, R)) \cup \operatorname{Inc}_1(\operatorname{Tip}(G)) \cup \operatorname{Inc}_2(\operatorname{Tip}(H))\\
&= \operatorname{Tip}(\operatorname{Com}(Q, R)) \cup \operatorname{Tip}(\operatorname{Inc}_1(G)) \cup \operatorname{Tip}(\operatorname{Inc}_2(H))\\
&= \operatorname{Tip}(X).
\end{aligned}$$

Since $\mathrm{Tip}(X') \subseteq \mathrm{Tip}(X)$, and because X' and X generate the same ideal, it suffices to show that X' is a Gröbner basis. We may therefore assume throughout the rest of the proof that G and H are reduced Gröbner bases. Then G and H are tip reduced and uniform by Proposition 2.36, which in particular implies that the set X is tip reduced and uniform by Lemma 3.23.

Suppose $x, y \in X$ are elements that have a (p, q)-overlap for some paths p and q in $Q \times R$. Since X is tip reduced and uniform, it suffices to check that

$$\mathfrak{o}(x, y, p, q) \Rightarrow_X 0,$$

as it then follows from Theorem 2.41 that X is a Gröbner basis. We have the following possible cases.

1. $\underline{x, y \in \mathrm{Com}(Q, R)}$: Then $x = \mathrm{Com}(\alpha, \beta)$ and $y = \mathrm{Com}(\alpha', \beta')$ for some arrows $\alpha, \alpha' \in Q_1$ and $\alpha', \beta' \in R_1$. Since there is an overlap, we have

$$\mathrm{Tip}(x)p = q\,\mathrm{Tip}(y),$$

or in other words

$$\big(\alpha \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(\alpha) \times \beta\big)p = q\big(\alpha' \times \mathfrak{t}(\beta')\big)\big(\mathfrak{s}(\alpha') \times \beta'\big).$$

From the definition of overlaps, it follows that p and q are arrows in the product quiver $Q \times R$. But this implies that $\mathfrak{s}(\alpha) \times \beta = \alpha' \times \mathfrak{t}(\beta')$, which is impossible. Hence such an overlap cannot exist.

2. $\underline{x \in \mathrm{Com}(Q, R), y \in \mathrm{Inc}_1(G)}$: Then $x = \mathrm{Com}(\alpha, \beta)$ for arrows $\alpha \in Q_1$ and $\beta \in R_1$, and $y = g \times w$ for an element $g \in G$ and a vertex $w \in R_0$. By assumption, we have

$$\big(\alpha \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(\alpha) \times \beta\big)p = q(\mathrm{Tip}(g) \times w).$$

From the definition of overlaps, it follows that q must be an arrow, and in particular $q = \alpha \times \mathfrak{t}(\beta)$. Then we get

$$\big(\mathfrak{s}(\alpha) \times \beta\big)p = \mathrm{Tip}(g) \times w.$$

But this is impossible, since $\mathrm{Tip}(g) \times w$ is a product of arrows which are all of the form $\gamma \times w$ for $\gamma \in Q_1$, and $\mathfrak{s}(\alpha) \times \beta$ does not have this form. Hence such an overlap cannot exist.

3. $\underline{x \in \mathrm{Inc}_2(H), y \in \mathrm{Com}(Q, R)}$: Similar to case 2.

4. $\underline{x \in \mathrm{Inc}_1(G), y \in \mathrm{Inc}_2(H)}$: Then $x = g \times w$ and $y = v \times h$. We see that $\mathrm{Tip}(x) = \mathrm{Tip}(g) \times w$ and $\mathrm{Tip}(y) = v \times \mathrm{Tip}(h)$. But then $\mathrm{Tip}(x)$ and $\mathrm{Tip}(y)$ have no nontrivial subpaths in common, so such an overlap cannot exist.

5. $\underline{x \in \mathrm{Inc}_2(H), y \in \mathrm{Inc}_1(G)}$: Similar to case 4.

6. $\underline{x, y \in \mathrm{Inc}_1(G)}$: Our argument for this case is based on the one given in [Bjø21]. There exist elements $g, g' \in G$ and vertices $w, w' \in R_0$ such that $x = g \times w$ and $y = g' \times w'$. By assumption, we have

$$(\mathrm{Tip}(g) \times w)p = q(\mathrm{Tip}(g') \times w').$$

We may write

$$\mathrm{Tip}(g) = \alpha_1 \dots \alpha_m$$

and

$$\mathrm{Tip}(g') = \alpha'_1 \dots \alpha'_n$$

for some arrows α_i, α'_i. Then it follows from the definition of overlaps that

$$q = (\alpha_1 \dots \alpha_k) \times w$$

and

$$p = (\alpha'_l \dots \alpha'_n) \times w'$$

for some integers $1 \leq k < m$ and $1 < l \leq n$. Moreover, the vertices w and w' must be equal. Letting $\hat{q} = \alpha_1 \dots \alpha_k$ and $\hat{p} = \alpha'_l \dots \alpha'_n$, we see that

$$\mathrm{Tip}(g)\hat{p} = \hat{q}\,\mathrm{Tip}(g').$$

This satisfies the definition of an overlap, because $\mathrm{Tip}(g) \nmid \hat{q}$ and $\mathrm{Tip}(g') \nmid \hat{p}$. Hence an overlap in $\mathrm{Inc}_1(G) \subseteq K(Q \times R)$ gives rise to an overlap in $G \subseteq KQ$.

Let $a = \mathrm{CTip}(g)$ and $b = \mathrm{CTip}(g')$. Then the overlap relation in $K(Q \times R)$ is

$$\begin{aligned}
\mathfrak{o}(x, y, p, q) &= a^{-1}xp - b^{-1}qy \\
&= a^{-1}\big(g \times w\big)\big(\hat{p} \times w\big) - b^{-1}\big(\hat{q} \times w\big)\big(g' \times w\big) \\
&= \big(a^{-1}g\hat{p} - b^{-1}\hat{q}g'\big) \times w \\
&= \mathfrak{o}(g, g', \hat{p}, \hat{q}) \times w.
\end{aligned}$$

By assumption, G is a Gröbner basis, so

$$\mathfrak{o}(g, g', \hat{p}, \hat{q}) \Rightarrow_G 0$$

in KQ. It follows from Lemma 3.24 that

$$\mathfrak{o}(x, y, p, q) \Rightarrow_{\mathrm{Inc}_1(G)} 0$$

in $K(Q \times R)$, and hence

$$\mathfrak{o}(x, y, p, q) \Rightarrow_X 0.$$

7. $\underline{x, y \in \mathrm{Inc}_2(H)}$: Similar to case 6.

8. $\underline{x \in \mathrm{Com}(Q, R), y \in \mathrm{Inc}_2(H)}$: Then $x = \mathrm{Com}(\alpha, \beta)$ for arrows $\alpha \in Q_1$ and $\beta \in R_1$, and $y = v \times h$ for an element $h \in H$ and a vertex $v \in R_0$. By assumption, we have

$$\big(\alpha \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(\alpha) \times \beta\big)p = q(v \times \mathrm{Tip}(h)).$$

From the definition of overlaps, it follows that q must be an arrow, and in particular $q = \alpha \times \mathfrak{t}(\beta)$. Then we get

$$\big(\mathfrak{s}(\alpha) \times \beta\big)p = v \times \mathrm{Tip}(h).$$

Hence p must be a path of the form

$$p = (v \times \beta_1) \dots (v \times \beta_n)$$

for arrows β_i in R, where $n \geq 1$, and moreover, the vertex v must be equal to $\mathfrak{s}(\alpha)$. It follows that

$$\mathrm{Tip}(h) = \beta\beta_1 \dots \beta_n.$$

Write

$$h = a\,\mathrm{Tip}(h) + \sum_{i=1}^{t} b_i r_i$$

for scalars $a, b_i \in K \setminus \{ 0 \}$ and distinct paths $r_i < \mathrm{Tip}(h)$ in R. Then the overlap relation is

$$
\begin{aligned}
&\mathfrak{o}(x,y,p,q)\\
&= 1\cdot xp - a^{-1}qy\\
&= \operatorname{Tip}(x)p - q\operatorname{Tip}(y) - \big(\mathfrak{t}(\alpha)\times\beta\big)\big(\alpha\times\mathfrak{s}(\beta)\big)p - a^{-1}q\sum_i b_i(v\times r_i)\\
&= 0 - \big(\mathfrak{t}(\alpha)\times\beta\big)\big(\alpha\times\mathfrak{s}(\beta)\big)p - a^{-1}q\sum_i b_i(v\times r_i)\\
&= -\big(\mathfrak{t}(\alpha)\times\beta\big)\big(\alpha\times\mathfrak{t}(\beta_1)\big)\big(\mathfrak{s}(\alpha)\times\hat{p}\big)\\
&\quad -\sum_i a^{-1}b_i\big(\alpha\times\mathfrak{t}(\beta)\big)\big(\mathfrak{s}(\alpha)\times r_i\big)\\
&= -\big(\mathfrak{t}(\alpha)\times\beta\big)\big(\alpha\times\mathfrak{t}(\beta_1)\big)\big(\mathfrak{s}(\alpha)\times\hat{p}\big)\\
&\quad -\sum_i a^{-1}b_i\big(\alpha\times\mathfrak{t}(r_i)\big)\big(\mathfrak{s}(\alpha)\times r_i\big),
\end{aligned}
\tag{3.8}
$$

where $\hat{p}=\beta_1\dots\beta_n$. Note that the last equality uses the fact that $\mathfrak{t}(\beta)=\mathfrak{t}(r_i)$, which is true because h is uniform. In order to show that $\mathfrak{o}(x,y,p,q)\Rightarrow_X 0$, we will rewrite (3.8) in a way that satisfies Definition 2.16.

Whenever r_i is a nontrivial path, we may write

$$r_i=\gamma_{i,1}\dots\gamma_{i,m_i},$$

where $m_i\geq 1$ and each $\gamma_{i,j}$ is an arrow in R. By applying the first part of Lemma 3.25 to (3.8), we get

$$
\begin{aligned}
&\mathfrak{o}(x,y,p,q)\\
&= -\big(\mathfrak{t}(\alpha)\times\beta\big)\Bigg(\big(\mathfrak{t}(\alpha)\times\hat{p}\big)\big(\alpha\times\mathfrak{s}(\beta_n)\big)\\
&+\sum_{j=0}^{n-1}\big(\mathfrak{t}(\alpha)\times\beta_1\dots\beta_j\big)\operatorname{Com}(\alpha,\beta_{j+1})\big(\mathfrak{s}(\alpha)\times\beta_{j+2}\dots\beta_n\big)\Bigg)\\
&-\sum_i a^{-1}b_i\Bigg(\big(\mathfrak{t}(\alpha)\times r_i\big)\big(\alpha\times\mathfrak{s}(r_i)\big)\\
&+\sum_{j=0}^{m_i-1}\big(\mathfrak{t}(\alpha)\times\gamma_{i,1}\dots\gamma_{i,j}\big)\operatorname{Com}(\alpha,\gamma_{i,j+1})\big(\mathfrak{s}(\alpha)\times\gamma_{i,j+2}\dots\gamma_{i,m_i}\big)\Bigg),
\end{aligned}
\tag{3.9}
$$

where we define sums of the form $\sum_{j=0}^{m_i-1}(\cdots)$ to equal the empty sum, i.e. zero, if r_i is a vertex. For the sake of brevity, we introduce the following notation:

$$\begin{aligned}
u_j &= \mathfrak{t}(\alpha) \times \beta\beta_1 \ldots \beta_j \\
v_j &= \mathfrak{s}(\alpha) \times \beta_{j+2} \ldots \beta_n \\
\varphi_j &= \mathrm{Com}(\alpha, \beta_{j+1}) \\
u'_{ij} &= \mathfrak{t}(\alpha) \times \gamma_{i,1} \ldots \gamma_{i,j} \\
v'_{ij} &= \mathfrak{s}(\alpha) \times \gamma_{i,j+2} \ldots \gamma_{i,m_i} \\
\psi_{ij} &= \mathrm{Com}(\alpha, \gamma_{i,j+1})
\end{aligned}$$

Note that since h is uniform, $\mathfrak{s}(r_i)$ is equal to $\mathfrak{s}(\beta_n)$. By combining this fact with the notation above, we can rewrite (3.9) in the following way:

$$\begin{aligned}
&\mathfrak{o}(x, y, p, q) \\
&= -\left(\mathfrak{t}(\alpha) \times \left(\beta\hat{p} + \sum_i a^{-1} b_i r_i\right)\right)\big(\alpha \times \mathfrak{s}(\beta_n)\big) \\
&\quad - \sum_{j=0}^{n-1} u_j \varphi_j v_j - \sum_i \sum_{j=0}^{m_i-1} a^{-1} b_i u'_{ij} \psi_{ij} v'_{ij} \\
&= -a^{-1}\big(\mathfrak{t}(\alpha) \times h\big)\big(\alpha \times \mathfrak{s}(\beta_n)\big) - \sum_{j=0}^{n-1} u_j \varphi_j v_j - \sum_i \sum_{j=0}^{m_i-1} a^{-1} b_i u'_{ij} \psi_{ij} v'_{ij}.
\end{aligned} \tag{3.10}$$

Since $\mathfrak{t}(\alpha) \times h$, φ_j, and ψ_{ij} are elements of X, we have written $\mathfrak{o}(x, y, p, q)$ in a way that satisfies item (i) in the definition of remainders (Definition 2.16). In order to show that $\mathfrak{o}(x, y, p, q) \Rightarrow_X 0$, it now only remains to check that item (ii) in Definition 2.16 also holds. In other words, we must check that the tips of $\big(\mathfrak{t}(\alpha) \times h\big)\big(\alpha \times \mathfrak{s}(\beta_n)\big)$, $u_j \varphi_j v_j$, and $u'_{ij} \psi_{ij} v'_{ij}$ are all less than or equal to the tip of $\mathfrak{o}(x, y, p, q)$.

We must first determine the tip of $\mathfrak{o}(x, y, p, q)$. Recall that in (3.8), we wrote the overlap relation as a linear combination of paths of the form

$$\big(\mathfrak{t}(\alpha) \times \beta\big)\big(\alpha \times \mathfrak{t}(\beta_1)\big)\big(\mathfrak{s}(\alpha) \times \hat{p}\big)$$

or

$$\big(\alpha \times \mathfrak{t}(r_i)\big)\big(\mathfrak{s}(\alpha) \times r_i\big).$$

Since these paths are distinct and all have nonzero coefficients, the tip of the overlap relation is simply the maximum of the paths:

$$\begin{aligned}\operatorname{Tip}(\mathfrak{o}(x,y,p,q)) = \max\{&\big(\mathfrak{t}(\alpha)\times\beta\big)\big(\alpha\times\mathfrak{t}(\beta_1)\big)\big(\mathfrak{s}(\alpha)\times\hat{p}\big),\\ &\big(\alpha\times\mathfrak{t}(r_1)\big)\big(\mathfrak{s}(\alpha)\times r_1\big),\ldots,\big(\alpha\times\mathfrak{t}(r_t)\big)\big(\mathfrak{s}(\alpha)\times r_t\big)\}.\end{aligned}$$

Now observe that

$$\begin{aligned}\operatorname{Tip}\Big(\big(\mathfrak{t}(\alpha)\times h\big)\big(\alpha\times\mathfrak{s}(\beta_n)\big)\Big) &= \big(\mathfrak{t}(\alpha)\times\operatorname{Tip}(h)\big)\big(\alpha\times\mathfrak{s}(\beta_n)\big)\\ &= \big(\mathfrak{t}(\alpha)\times\beta\big)\big(\mathfrak{t}(\alpha)\times\hat{p}\big)\big(\alpha\times\mathfrak{s}(\beta_n)\big)\\ &\overset{*}{<} \big(\mathfrak{t}(\alpha)\times\beta\big)\big(\alpha\times\mathfrak{t}(\beta_1)\big)\big(\mathfrak{s}(\alpha)\times\hat{p}\big)\\ &\le \operatorname{Tip}(\mathfrak{o}(x,y,p,q)),\end{aligned}$$

where the marked inequality follows from the definition of the left length-lexicographic order, because the two paths being compared have the same length and because $\mathfrak{t}(\alpha)\times\beta_1 < \alpha\times\mathfrak{t}(\beta_1)$. By a similar argument, it can be shown that

$$\begin{aligned}\operatorname{Tip}(u_j\varphi_j v_j) &\le \big(\mathfrak{t}(\alpha)\times\beta\big)\big(\alpha\times\mathfrak{t}(\beta_1)\big)\big(\mathfrak{s}(\alpha)\times\hat{p}\big)\\ &\le \operatorname{Tip}(\mathfrak{o}(x,y,p,q))\end{aligned}$$

and

$$\begin{aligned}\operatorname{Tip}(u'_{ij}\psi_{ij}v'_{ij}) &\le \big(\alpha\times\mathfrak{t}(r_i)\big)\big(\mathfrak{s}(\alpha)\times r_i\big)\\ &\le \operatorname{Tip}(\mathfrak{o}(x,y,p,q)),\end{aligned}$$

whenever r_i is a nontrivial path. We have now shown that

$$\mathfrak{o}(x,y,p,q) \Rightarrow_X 0,$$

as desired.

9. $\underline{x\in\operatorname{Inc}_1(G), y\in\operatorname{Com}(Q,R)}$: This case is very similar to case 8, although this is not immediately obvious. We therefore present a sketch of how the argument given in case 8 can be adapted to the present case.

   There exist an element $g\in G$, a vertex w in R, and arrows α and β such that $x = g\times w$ and $y = \operatorname{Com}(\alpha,\beta)$. Since x and y have a (p,q)-overlap, we have
$$\big(\operatorname{Tip}(g)\times w\big)p = q\big(\alpha\times\mathfrak{t}(\beta)\big)\big(\mathfrak{s}(\alpha)\times\beta\big).$$
   It follows from the definition of overlaps that $p = \mathfrak{s}(\alpha)\times\beta$. Hence we have
$$\operatorname{Tip}(g)\times w = q\big(\alpha\times\mathfrak{t}(\beta)\big),$$

and q is a path of the form

$$q = \big(\alpha_1 \times w\big) \dots \big(\alpha_n \times w\big),$$

for arrows α_i in Q and $n \geq 1$. Moreover, $w = \mathfrak{t}(\beta)$, and

$$\mathrm{Tip}(g) = \alpha_1 \dots \alpha_n \alpha.$$

We write

$$g = a\,\mathrm{Tip}(g) + \sum_{i=1}^{t} b_i r_i$$

for scalars $a, b_i \in K \backslash \{ 0 \}$ and distinct paths $r_i < \mathrm{Tip}(g)$ in Q. Then the overlap relation of x, y, p, and q is

$$\begin{aligned}
&\mathfrak{o}(x,y,p,q)\\
&= a^{-1}xp - 1 \cdot qy\\
&= \ \mathrm{Tip}(x)p - q\,\mathrm{Tip}(y) + \sum_i a^{-1}b_i\big(r_i \times w\big)p + q\big(\mathfrak{t}(\alpha) \times \beta\big)\big(\alpha \times \mathfrak{s}(\beta)\big)\\
&= 0 + \sum_i a^{-1}b_i\big(r_i \times w\big)p + q\big(\mathfrak{t}(\alpha) \times \beta\big)\big(\alpha \times \mathfrak{s}(\beta)\big)\\
&= \ \sum_i a^{-1}b_i\big(r_i \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(\alpha) \times \beta\big) + \big(\hat{q} \times \mathfrak{t}(\beta)\big)\big(\mathfrak{t}(\alpha) \times \beta\big)\big(\alpha \times \mathfrak{s}(\beta)\big)\\
&= \ \sum_i a^{-1}b_i\big(r_i \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(r_i) \times \beta\big) + \big(\hat{q} \times \mathfrak{t}(\beta)\big)\big(\mathfrak{s}(\alpha_n) \times \beta\big)\big(\alpha \times \mathfrak{s}(\beta)\big),
\end{aligned} \tag{3.11}$$

where $\hat{q} = \alpha_1 \dots \alpha_n$. Note that the last equality uses the fact that $\mathfrak{s}(\alpha) = \mathfrak{s}(r_i)$, which is true because g is uniform. Similarly to case 8, we will rewrite this equation in a way that satisfies Definition 2.16. Whenever r_i is a nontrivial path, we can write

$$r_i = \gamma_{i,1} \cdots \gamma_{i,m_i}$$

for some arrows $\gamma_{i,j}$ in Q and an integer $m_i \geq 1$. Then by applying the second part of Lemma 3.25 to (3.11) and using an argument similar to the one we used to show (3.10) in case 8, it can be shown that the following equality holds:

$$
\begin{aligned}
&\mathfrak{o}(x,y,p,q)\\
&= a^{-1}\big(\mathfrak{t}(\alpha_1)\times\beta\big)\big(g\times\mathfrak{s}(\beta)\big)\\
&+\sum_{k=0}^{n-1}\big(\alpha_1\dots\alpha_k\times\mathfrak{t}(\beta)\big)\operatorname{Com}(\alpha_{k+1},\beta)\big(\alpha_{k+2}\dots\alpha_n\alpha\times\mathfrak{s}(\beta)\big)\\
&+\sum_i a^{-1}b_i\sum_{j=0}^{m_i-1}\big(\gamma_{i,1}\dots\gamma_{i,j}\times\mathfrak{t}(\beta)\big)\operatorname{Com}(\gamma_{i,j+1},\beta)\big(\gamma_{i,j+2}\dots\gamma_{i,m_i}\times\mathfrak{s}(\beta)\big)
\end{aligned}
$$

As in case 8, we have now written $\mathfrak{o}(x,y,p,q)$ in a way that satisfies item (i) in Definition 2.16. The fact that item (ii) is also satisfied follows from a similar argument to the one presented in case 8. This shows that

$$\mathfrak{o}(x,y,p,q)\Rightarrow_X 0.$$

□

Remark 3.27. The assumption that $\operatorname{Tip}(G)\subseteq J_Q^2$ and $\operatorname{Tip}(H)\subseteq J_R^2$ is essential to the proof of Theorem 3.26. If this assumption is not satisfied, then X will typically not be a tip reduced set,[5] even in the special case where G and H are reduced Gröbner bases, and hence we cannot use Theorem 2.41. With that being said, I have not been able to find an example where X is not a Gröbner basis, although I admittedly haven't had much time to try to find one. It is therefore plausible that the conclusion of Theorem 3.26 holds even without the assumption that $\operatorname{Tip}(G)\subseteq J_Q^2$ and $\operatorname{Tip}(H)\subseteq J_R^2$. However, this would require a different proof. ■

3.3.2 For the tensor product over an algebra

Recall that in Section 3.2.2, we turned Λ and Γ into modules over an algebra $\Sigma = KS$ (for some quiver S) by using algebra homomorphisms $f\colon\Sigma\to\Lambda$ and $g\colon\Sigma\to\Gamma$ such that $\operatorname{Im}(f)\subseteq Z(\Lambda)$ and $\operatorname{Im}(g)\subseteq Z(\Gamma)$. We then constructed the tensor product $\Lambda\otimes_\Sigma\Gamma$ as a quotient of $K(Q\times R)$.

We have just seen that if $G\subseteq KQ$ and $H\subseteq KR$ are Gröbner bases that satisfy a reasonable technical condition, then the set

$$X=\operatorname{Com}(Q,R)\cup\operatorname{Inc}_1(G)\cup\operatorname{Inc}_2(H)$$

[5] For example, if $\alpha\in\operatorname{Tip}(G)$ and $\beta\in R_0$ are arrows, then $\alpha\times\mathfrak{t}(\beta)$ is an element of $\operatorname{Inc}_1(G)$ whose tip divides the tip of $\operatorname{Com}(\alpha,\beta)$.

is a Gröbner basis for the ideal $I_K \subseteq K(Q \times R)$. With this in mind, we might hope that the set

$$Y = \mathrm{Com}(Q, R) \cup \mathrm{Inc}_1(G) \cup \mathrm{Inc}_2(H) \cup \mathrm{Bal}_\Sigma(Q, R)$$

is a Gröbner basis for the ideal I_Σ. Unfortunately, it turns out that this is not generally true, as the following example shows.

Example 3.28. Let Q, R, and S denote quivers that each have one vertex and a single loop, as shown below.

$$Q\colon\ v \circlearrowleft \alpha \qquad\qquad R\colon\ w \circlearrowleft \beta \qquad\qquad S\colon\ u \circlearrowleft \gamma$$

Let $G = \{\, \alpha^4 \,\} \subseteq KQ$ and $H = \{\, \beta^4 \,\} \subseteq KR$. Then G and H are Gröbner bases with respect to the unique admissible orders on Q and R, respectively. We consider the algebras $\Lambda = KQ/\langle G\rangle$, $\Gamma = KR/\langle H\rangle$, and $\Sigma = KS$. Let $f\colon \Sigma \to \Lambda$ be the algebra homomorphism such that

$$f(\gamma) = [v + \alpha + \alpha^3],$$

and let $g\colon \Sigma \to \Gamma$ be the algebra homomorphism such that

$$g(\gamma) = [w].$$

Let Y be the set

$$Y = \mathrm{Com}(Q, R) \cup \mathrm{Inc}_1(G) \cup \mathrm{Inc}_2(H) \cup \mathrm{Bal}_\Sigma(Q, R).$$

We will see that Y is not a Gröbner basis. Let us first find all the elements of Y. The set of commutativity relations is

$$\mathrm{Com}(Q, R) = \{\, (\alpha \times w)(v \times \beta) - (v \times \beta)(\alpha \times w) \,\},$$

while the inclusion sets are

$$\mathrm{Inc}_1(G) = \left\{\, \alpha^4 \times w \,\right\}$$

and

$$\mathrm{Inc}_2(H) = \left\{\, v \times \beta^4 \,\right\}.$$

The set of balancing relations depends on which representatives $\tilde{f}(s) \in KQ$ and $\tilde{g}(s) \in KR$ we choose for $f(s) \in KQ/\langle G\rangle$ and $g(s) \in KR/\langle H\rangle$, where $s = u$ or $s = \gamma$. It could conceivably happen that Y is a Gröbner basis for some choices of representatives but not for others, so let us consider all possible choices of the functions $\tilde{f}$ and $\tilde{g}$. Since KQ is isomorphic to a polynomial ring with one variable, every element of the ideal $\langle \alpha^4 \rangle$ in KQ has the form $P(\alpha)\alpha^4$ for some polynomial P. Hence any choice of representatives for $f(u)$ and $f(\gamma)$ must have the form

$$\tilde{f}(u) = v + P_1(\alpha)\alpha^4$$

and

$$\tilde{f}(\gamma) = v + \alpha + \alpha^3 + P_2(\alpha)\alpha^4$$

for some polynomials P_1 and P_2. Similarly, we have

$$\tilde{g}(u) = w + P_3(\beta)\beta^4$$

and

$$\tilde{g}(\gamma) = w + P_4(\beta)\beta^4$$

for polynomials P_3 and P_4. Then it can be seen that the set of balancing relations is

$$\begin{aligned} \mathrm{Bal}_\Sigma(Q, R) = \big\{ & P_1(\alpha)\alpha^4 \times w - v \times P_3(\beta)\beta^4, \\ & \big(\alpha + \alpha^3 + P_2(\alpha)\alpha^4\big) \times w - v \times P_4(\beta)\beta^4 \big\}. \end{aligned}$$

Now observe that

$$\begin{aligned} \alpha^2 \times w = & \,(\alpha \times w)\big(\big(\alpha + \alpha^3 + P_2(\alpha)\alpha^4\big) \times w\big) \\ & - ((v + P_2(\alpha)\alpha) \times w)\,(\alpha^4 \times w), \end{aligned}$$

which is an element of $\langle X \rangle$ because

$$\big(\alpha + \alpha^3 + P_2(\alpha)\alpha^4\big) \times w \in \mathrm{Bal}_\Sigma(Q, R)$$

and

$$\alpha^4 \times w \in \mathrm{Inc}_1(G).$$

However, $\alpha^2 \times w$ is clearly not divisible by any element of the set

$$\mathrm{Tip}(\mathrm{Com}(Q, R) \cup \mathrm{Inc}_1(G) \cup \mathrm{Inc}_2(H)),$$

nor is it divisible by any element of

$$\mathrm{Tip}(\mathrm{Bal}_\Sigma(Q, R)),$$

regardless of our choice of the polynomials P_1, P_2, P_3, and P_4. Hence the set Y is not a Gröbner basis, regardless of how we choose representatives when defining the functions $\tilde{f}$ and $\tilde{g}$. ■

It is conceivable that we could find a Gröbner basis for I_Σ by choosing a different generating set than Y (preferably one which can be computed quickly), or by making some additional assumptions about the sets G and H, or about the homomorphisms f and g. However, I have not found any way to do this.

But this does not mean that our only option is to apply a naïve version of Buchberger's algorithm (Algorithm 2.3). After all, the set Y contains X as a subset, and we know that X is a Gröbner basis. Recall that when we use Buchberger's algorithm to compute a Gröbner basis for Y, we repeatedly compute remainders of overlap relations $\mathfrak{o}(x, y, p, q)$. If we happen to know that x and y are elements of X, then we know that $\mathfrak{o}(x, y, p, q) \Rightarrow_X 0$ (and hence also $\mathfrak{o}(x, y, p, q) \Rightarrow_Y 0$), so there is no need to compute a remainder in this particular case. Thus we can exploit the fact that X is a Gröbner basis by skipping some of the computations in Buchberger's algorithm.

Actually, the preceding discussion elides an important detail. Buchberger's algorithm does not simply add more elements to the generating set Y; it also tip reduces the resulting set. Thus we should be worried about the possibility that an overlap relation $\mathfrak{o}(x, y, p, q)$ that has remainder zero under division by some set S does not have remainder zero under division by the tip reduced set $\textsc{TipReduce}(S)$. Thankfully, this is not an issue, as the following lemma shows. Note that in this lemma (and in Algorithm 3.1, as we will see), we make an exception to the notation used so far in this section, as we do *not* assume that G and H are Gröbner bases in the path algebras KQ and KR, respectively.

Lemma 3.29. *Let Q be a quiver with an admissible order $\leq$ on the paths in Q, and let G be a finite set of uniform elements of KQ. Let H be the result of applying the tip reduction algorithm (Algorithm 2.2) to G. Let $y \in KQ$ be an element such that*

$$y \Rightarrow_G 0.$$

Then we also have

$$y \Rightarrow_H 0.$$

Proof. By assumption, y has remainder 0 under division by G. Hence there exist elements $g_1, \ldots, g_n \in G$ and elements $u_i, v_i \in KQ$ such that

$$y = \sum_{i=1}^{n} u_i g_i v_i,$$

and such that $\mathrm{Tip}(u_i g_i v_i) \leq \mathrm{Tip}(y)$. By Lemma 2.46, for each g_i there exist elements $h_{ij} \in H$, scalars c_{ij}, and paths p_{ij} and q_{ij} such that

$$g_i = \sum_j c_{ij} p_{ij} h_{ij} q_{ij},$$

and such that $\mathrm{Tip}(p_{ij} h_{ij} q_{ij}) \leq \mathrm{Tip}(g_i)$. Then we have

$$y = \sum_{i=1}^{n} u_i \left(\sum_j c_{ij} p_{ij} h_i q_{ij} \right) v_i = \sum_{i,j} c_{ij} u_i p_{ij} h_{ij} q_{ij} v_i.$$

Moreover,

$$\mathrm{Tip}(u_i p_{ij} h_{ij} q_{ij} v_i) \leq \mathrm{Tip}(u_i g_i v_i) \leq \mathrm{Tip}(y).$$

By the definition of remainders, this shows that

$$y \Rightarrow_H 0.$$

□

The following algorithm incorporates the modifications to Buchberger's algorithm described in the discussion preceding Lemma 3.29. Aside from some differences in notation, the only significant difference from Algorithm 2.3 is the fact that some overlap relations are skipped. (We also need to tip reduce the generating set at the start, because we are not assuming that the input is tip reduced.) The correctness of the algorithm follows from

Lemma 3.29 along with the correctness of Algorithm 2.3.

Algorithm 3.1: Modified version of Buchberger's algorithm

Input: A quiver Q with an admissible order $\leq$, a finite uniform subset $\{ f_1, \ldots, f_n \} \subseteq KQ$, and a subset $Z \subseteq \{ f_1, \ldots, f_n \}$ such that Z is a Gröbner basis for the ideal $\langle Z \rangle$

Output: A finite Gröbner basis G for the ideal $\langle f_1, \ldots, f_n \rangle$, if one exists

```
1  G ← TipReduce({ f_1, ..., f_n });
2  do
3      MODIFIED ← False;
4      X ← ∅;
5      for g, h ∈ G do
6          if g ∉ Z or h ∉ Z then
7              for all paths p, q such that g and h have a (p, q)-overlap
                do
8                  r ← Remainder(o(g, h, p, q), G);
9                  if r ≠ 0 then
10                     X ← X ∪ { r };
11                     MODIFIED ← True;
12                 end
13             end
14         end
15     end
16     if MODIFIED = True then
17         G ← TipReduce(G ∪ X);
18     end
19 while MODIFIED = True;
20 return G;
```

When executing the algorithm above, we need to check if the elements g and h are contained in the Gröbner basis Z. The most obvious way to do this is to simply search through Z and check if we can find g and h. If we store Z as an array and sort it according to some easily computable total order, then we can perform such a search in time $O(\log(|Z|))$ by using a binary search. However, this would probably not be the most efficient option. Instead, we can store a list of boolean flags $[b_1, \ldots, b_n]$. At the start of the algorithm, we set the flag b_i to True if and only if f_i is an element of Z. We then store G as an ordered list, and whenever the ith element of G is modified during the tip reduction step on line 17, we set the flag b_i to False if $i \leq n$. When checking if g and h are elements of Z, we simply check if their corresponding flags are set to True, which can be done in constant time. This will probably be faster than using a binary search on Z.[6]

An implementation of Algorithm 3.1 can be found in Appendix B, along with the results from performance tests comparing it to an implementation of Algorithm 2.3. The performance tests found that Algorithm 3.1 was more efficient than Algorithm 2.3, at least for the examples that I used in the tests. However, it turns out that it is even more efficient to use another algorithm that uses free algebras. See the appendix for more details.

Once we have found generating sets X and Y for the ideals I_K and I_Σ, respectively, where X is a Gröbner basis and $X \subseteq Y$, we can use Algorithm 3.1 to compute a Gröbner basis for I_Σ.[7] However, there is another possible approach that may be more efficient. In Chapter 4, we will see how, given a quiver P and a lower-admissible ideal $I \subseteq KP$, we can find another quiver P' and an admissible ideal $I' \subseteq KP'$, such that

$$KP/I \cong KP'/I'.$$

If the ideals $\langle\rho\rangle \subseteq KQ$ and $\langle\sigma\rangle \subseteq KR$ are admissible, then we know that I_Σ is lower-admissible, but not necessarily admissible. We can therefore use the methods in Chapter 4 to replace the lower-admissible quotient $K(Q \times R)/I_\Sigma$ with an isomorphic admissible quotient KP'/I'. The quiver P' and the generating set for I' can potentially be a lot smaller than $Q \times R$ and Y, respectively, so it seems likely that it would often be more efficient to compute P' and a Gröbner basis for I' than it would be to compute a Gröbner basis for I_Σ. However, I have not had time to implement the algorithms in Chapter 4, so I have not been able to compare the performance of these two approaches.

[6]The latter approach may occasionally produce false negatives, since the flag b_i may be False even if the ith element of G happens to be an element of Z. However, this seems like it should happen only rarely, so it would probably not have a significant impact on performance.

[7]This requires that the set $\mathrm{Bal}_\Sigma(Q, R)$ is uniform. But this is not a problem, because we can make sure that all balancing relations are uniform by choosing $\tilde{f}$ in such a way that $v\tilde{f}(s) = v\tilde{f}(s)v$ for vertices v and elements $s \in \Sigma$, and similarly for $\tilde{g}$.

Chapter 4

From lower-admissible ideals to admissible ideals

In Chapter 3, we studied the tensor product $\Lambda \otimes_\Sigma \Gamma$ of two algebras Λ and Γ. Given that $\Lambda = KQ/\langle\rho\rangle$ and $\Gamma = KR/\langle\sigma\rangle$ for bound quivers (Q,ρ) and (R,σ), we saw that the tensor product was given by

$$\Lambda \otimes_\Sigma \Gamma \cong K(Q \times R)/I$$

for an ideal I. If the ideals $\langle\rho\rangle$ and $\langle\sigma\rangle$ are admissible, then I is not necessarily an admissible ideal, but by Proposition 3.21, it does satisfy the weaker condition

$$J^m_{Q\times R} \subseteq I$$

for some integer m. Recall that in Definition 3.20, we called such ideals **lower-admissible**.

It is often convenient to work with admissible ideals. With the previous paragraph as motivation, it is therefore natural to ask the following question: Given a quiver Q and a lower-admissible ideal $I \subseteq KQ$, can we find another quiver Q' and an ideal $I' \subseteq KQ'$ such that

$$KQ/I \cong KQ'/I',$$

and such that I' is an admissible ideal? In this chapter, we answer this question in the affirmative. Parts of the chapter are based on [Ska11], which presents an algorithm to find Q' and I' in the special case where we also assume that $I \subseteq J_Q$.

4.1 Lower-admissible to pre-admissible

We will first see how we can replace a lower-admissible quotient KQ/I with an isomorphic quotient KQ'/I', where I' satisfies the condition

$$J_{Q'}^m \subseteq I' \subseteq J_{Q'}$$

for some integer m.

Definition 4.1 ([Ska11])**.** Let Q be a quiver, and let $I \subseteq KQ$ be an ideal. We say that I is **pre-admissible** if there exists some integer m such that

$$J_Q^m \subseteq I \subseteq J_Q.$$

■

Let (Q, ρ) be a quiver with lower-admissible relations. Our strategy for finding a quiver Q' with a pre-admissible relation set $\rho' \subseteq KQ'$ such that $KQ/\langle\rho\rangle \cong KQ'/\langle\rho'\rangle$ is fairly simple. If there is a vertex v in Q and a relation $x \in \rho$ such that v appears in x, then we will simply remove v from Q. For any relation $y \in \rho$, we remove all terms of y that are divisible by v. The reason why this works is due to the following lemma.

Lemma 4.2. *Let Q be a quiver, and let $I \subseteq KQ$ be a lower-admissible ideal. Let v be a vertex in Q, and suppose that there exists some element $x \in I$ such that v appears in x. Then $v \in I$.*

Proof. Without loss of generality, we may assume that $x = vxv$, and we may assume that the coefficient of v in x is 1. Then $v - x \in J_Q$. Moreover, $v - (v - x) = x \in I$, so $v \equiv v - x \pmod{I}$. By assumption, there exists some integer m such that $J_Q^m \subseteq I$. Then we have

$$v = v^m \equiv (v - x)^m \equiv 0 \pmod{I},$$

where the last congruence follows from the fact that $(v - x)^m \in J_Q^m$. Hence $v \in I$. □

As mentioned above, whenever a vertex v appears in some relation $x \in \rho$, we will remove those terms of elements of ρ that are divisible by v. The following notation is therefore convenient.

Definition 4.3 ([Ska11])**.** Let Q be a quiver, and let $x \in KQ$. Let $p_1, \dots, p_n$ be the unique paths and $c_1, \dots, c_n$ the unique nonzero scalars such that

$$x = \sum_{i=1}^{n} c_i p_i.$$

Then we let $\mathrm{Terms}(x)$ denote the set of terms $c_i p_i$ of x:

$$\mathrm{Terms}(x) = \{ c_1 p_1, \dots, c_n p_n \}.$$

Moreover, if $I \subseteq KQ$ is an ideal, then we let $\mathrm{Terms}_I(x)$ denote the set of terms of x that are contained in I, i.e.

$$\mathrm{Terms}_I(x) = \mathrm{Terms}(x) \cap I.$$

■

Using Algorithm 4.1, LowerAdmissibleToPreAdmissible(Q, ρ), we can transform a lower-admissible quotient $KQ/\langle \rho \rangle$ into a pre-admissible quotient $KQ'/\langle \rho' \rangle$. Note that for a set V of vertices in Q, we let $Q \backslash V$ denote the subquiver of Q obtained by removing the vertices in V. More precisely, the vertex set of $Q \backslash V$ is

$$(Q \backslash V)_0 = Q_0 \backslash V,$$

while the arrow set is

$$(Q \backslash V)_1 = \{ \alpha \in Q_1 \mid \mathfrak{s}(\alpha) \notin V \text{ and } \mathfrak{t}(\alpha) \notin V \}.$$

Algorithm 4.1: LowerAdmissibleToPreAdmissible(Q, ρ)

Input: A quiver Q and a finite lower-admissible subset $\rho \subseteq KQ$
Output: A quiver $Q' \subseteq Q$ and a finite pre-admissible subset $\rho' \subseteq KQ'$ such that $KQ/\langle\rho\rangle \cong KQ'/\langle\rho'\rangle$

$V \leftarrow \emptyset$; `// Set of vertices to remove`
for $v \in Q_0$ **do**
  if v *appears in* $x \in \rho$ **then**
    $V \leftarrow V \cup \{ v \}$;
  end
end
$Q' \leftarrow Q \backslash V$;
$\rho' \leftarrow \emptyset$;
for $x \in \rho$ **do**
  $T \leftarrow \mathrm{Terms}(x) \backslash \mathrm{Terms}_{\langle V \rangle}(x)$;
  $y \leftarrow \sum_{t \in T} t$;
  $\rho' \leftarrow \rho' \cup \{ y \}$
end
return (Q', ρ');

In order to prove the correctness of Algorithm 4.1, we will need the following lemma.

Lemma 4.4. *Let Q be a quiver and $\rho \subseteq KQ$ a finite lower-admissible set of relations. Let Q' and ρ' be the output produced when Algorithm 4.1 is applied to Q and ρ. Then*

$$\langle\rho'\rangle = KQ' \cap \langle\rho\rangle\,,$$

where $\langle\rho'\rangle$ denotes the ideal in KQ' generated by ρ', and $\langle\rho\rangle$ denotes the ideal in KQ generated by ρ.

Proof. As in Algorithm 4.1, let V denote the set of vertices $v \in Q_0$ such that v appears in some element of ρ. Note that a path p in Q is contained in Q' if and only if p is not divisible by any elements of V. It follows that the ideal $\langle V\rangle$ is spanned (as a vector space) by precisely those paths in KQ which are not paths in Q'. In particular, we have $KQ = KQ' \oplus \langle V\rangle$ as vector spaces. Hence for any element $\lambda \in KQ$, we can let $\lambda' \in KQ'$ and $\lambda^* \in \langle V\rangle$ denote the unique elements such that $\lambda = \lambda' + \lambda^*$.

Let $z \in KQ' \cap \langle\rho\rangle$. Since $z \in \langle\rho\rangle$, there exist elements $f_i, g_i \in KQ$ and $x_i \in \rho$ such that

$$z = \sum_i f_i x_i g_i.$$

Then we see that

$$\begin{aligned} z &= \sum_i (f_i' + f_i^*)(x_i' + x_i^*)(g_i' + g_i^*) \\ &= \sum_i f_i' x_i' g_i' + w \end{aligned}$$

for some element w of the ideal $\langle V \rangle$. Note that $\sum_i f_i' x_i' g_i' \in KQ'$, and hence $w = z^*$, and $z^* = 0$ because $z \in KQ'$. Moreover, observe that x_i' is the sum of those terms of x_i which are not contained in $\langle V \rangle$, and hence $x_i' \in \rho'$ by the construction of ρ'. Thus we have

$$z = \sum_i f_i' x_i' g_i' \in \langle \rho' \rangle .$$

This shows that $KQ' \cap \langle \rho \rangle \subseteq \langle \rho' \rangle$.

To complete the proof, we show that $\langle \rho' \rangle \subseteq KQ' \cap \langle \rho \rangle$. To prove this, it is enough to show that $\rho' \subseteq \langle \rho \rangle$. Let y be an element of ρ'. Then there exists an element $x \in \rho$ such that $y = x'$. Then we have

$$x - y = x^* \in \langle V \rangle .$$

Now note that every vertex in the set V appears in some element of ρ, and hence $V \subseteq \langle \rho \rangle$ by Lemma 4.2. But then $x - y \in \langle \rho \rangle$, and hence $y \in \langle \rho \rangle$. Thus we see that $\rho' \subseteq \langle \rho \rangle$. □

Proposition 4.5. *Algorithm 4.1 produces a correct result.*

Proof. Let Q be a quiver and $\rho \subseteq KQ$ a lower-admissible set of relations, and let (Q', ρ') be the quiver with relations produced by Algorithm 4.1. Throughout this proof, let $\langle \rho' \rangle$ denote the ideal in KQ' generated by ρ'.

We need to show that $\langle \rho' \rangle$ is a pre-admissible ideal of KQ'. From our construction of ρ' in the algorithm, it is clear that no vertex of Q' can appear in an element of ρ', and hence $\langle \rho' \rangle \subseteq J_{Q'}$. Moreover, the ideal $\langle \rho \rangle \subseteq KQ$ is lower-admissible by assumption, so there exists some m such that $J_Q^m \subseteq \langle \rho \rangle$. It follows that $J_{Q'}^m \subseteq J_Q^m \subseteq \langle \rho \rangle$, and in particular $J_{Q'}^m \subseteq KQ' \cap \langle \rho \rangle$. Hence $J_{Q'}^m \subseteq \langle \rho' \rangle$ by Lemma 4.4, so $\langle \rho' \rangle$ is a pre-admissible ideal.

Lastly, we show that $KQ / \langle \rho \rangle \cong KQ' / \langle \rho' \rangle$. Since Q' is a subquiver of Q, there is an inclusion map

$$\phi \colon KQ' \hookrightarrow KQ,$$

and this map is a non-unital algebra homomorphism. Note that $\langle \rho' \rangle \subseteq \langle \rho \rangle$ by Lemma 4.4, so ϕ descends to a map

$$\overline{\phi} \colon KQ' / \langle \rho' \rangle \to KQ / \langle \rho \rangle .$$

We claim that $\overline{\phi}$ is an algebra isomorphism. Let $v \in Q_0$ be a vertex. If $v \in Q_0'$, then $v + \langle \rho \rangle = \overline{\phi}(v + \langle \rho' \rangle)$. Otherwise, we have that $v \in \langle \rho \rangle$ by Lemma 4.2, and hence $v + \langle \rho \rangle = 0$. In either case, we see that $v + \langle \rho \rangle \in \operatorname{Im} \overline{\phi}$. If $\alpha \in Q_1$ is an arrow, then a similar argument shows that $\alpha + \langle \rho \rangle \in \operatorname{Im} \overline{\phi}$. Since the cosets of the vertices and arrows in Q generate $KQ / \langle \rho \rangle$ as an algebra, we see that $\overline{\phi}$ is surjective. To see that $\overline{\phi}$ is injective, let $\lambda \in KQ'$ be an element such that $\overline{\phi}(\lambda + \langle \rho' \rangle) = 0$, i.e. $\lambda \in \langle \rho \rangle$. Then $\lambda \in KQ' \cap \langle \rho \rangle$, so $\lambda \in \langle \rho' \rangle$ by Lemma 4.4. This shows that $\operatorname{Ker} \overline{\phi} = 0$, and hence $\overline{\phi}$ is an isomorphism. □

Let us apply Algorithm 4.1 to an example.

Example 4.6. We consider the following quiver.

$$Q\colon \quad \begin{tikzcd} & v_3 \arrow[loop above, "\delta"] \arrow[dr, "\gamma"] & & \\ v_1 \arrow[ur, "\beta"] \arrow[rr, "\alpha"] & & v_2 \arrow[r, "\varepsilon"] & v_4 \arrow[loop right, "\zeta"] \end{tikzcd}$$

Let ρ be the following lower-admissible set of relations in KQ.

$$\rho = \left\{ \delta^8, \zeta^8, \alpha - \gamma\beta, \delta^3 + v_3, \zeta^2 + v_4. \right\}$$

The vertices that appear in relations in ρ are v_3 and v_4. We therefore remove these vertices from Q, and are left with the following quiver.

$$Q'\colon \; v_1 \xrightarrow{\quad \alpha \quad} v_2$$

After removing all terms that are divisible by v_3 or v_4, we obtain the relation set $\rho' = \{ \alpha \}$. Thus we see that

$$KQ / \langle \rho \rangle \cong KQ' / \langle \alpha \rangle \cong K^2.$$

■

Let (Q, ρ) be a quiver with lower-admissible relations, and let $\leq$ be an admissible order on the set of paths in Q. Consider the quiver with relations (Q', ρ') produced by Algorithm 4.1. Since Q' is a subquiver of Q, we may restrict $\leq$ to an order on the set of paths in Q', which we also denote by $\leq$. This restricted order is still an admissible order. A question that then naturally comes to mind is the following: If ρ is a Gröbner basis with respect to the order on Q, is ρ' a Gröbner basis with respect to the restricted order on Q'? We might expect that the answer is no, since some elements of $\mathrm{Tip}(\rho)$ may disappear from the quiver when we remove vertices. However, it turns out that the answer is actually yes, as the following result shows.

Proposition 4.7. *Consider (Q, ρ), (Q', ρ'), and $\leq$ as in the preceding paragraph. Suppose that ρ is a Gröbner basis for the ideal $\langle \rho \rangle$ with respect to the order $\leq$. Then ρ' is a Gröbner basis for the ideal $\langle \rho' \rangle \subseteq KQ'$ with respect to the restricted order on Q'.*

Proof. Let $z \neq 0$ be an element of $\langle \rho' \rangle \subseteq KQ'$. Then z is also an element of $\langle \rho \rangle \subseteq KQ$ by Lemma 4.4, so since ρ is a Gröbner basis there exists an element $x \in \rho$ such that $\mathrm{Tip}(x) \mid \mathrm{Tip}(z)$. Let p and q be paths in Q such that $\mathrm{Tip}(z) = p\,\mathrm{Tip}(x)q$. Note that p, q, and $\mathrm{Tip}(x)$ are necessarily paths in the subquiver Q', since $\mathrm{Tip}(z)$ is a path in Q'. Let V denote the set of vertices such that $Q' = Q\backslash V$, and let $y \in \rho'$ be the element such that

$$y = \sum_{t \in T} t,$$

where $T = \mathrm{Terms}(x) \backslash \mathrm{Terms}_{\langle V \rangle}(x)$. Since $\mathrm{Tip}(x)$ is a path in Q', we have

$$\mathrm{CTip}(x)\,\mathrm{Tip}(x) \notin \mathrm{Terms}_{\langle V \rangle}(x),$$

and hence $\mathrm{Tip}(y) = \mathrm{Tip}(x)$. We see that

$$\mathrm{Tip}(z) = p\,\mathrm{Tip}(y)q,$$

so $\mathrm{Tip}(y) \mid \mathrm{Tip}(z)$ in the ring KQ'. This shows that ρ' is a Gröbner basis. $\square$

4.2 Pre-admissible to admissible

Given a quiver Q with lower-admissible relations $\rho \subseteq KQ$, we have seen how we can replace $KQ/\langle \rho \rangle$ with an isomorphic quotient $KQ'/\langle \rho' \rangle$, where ρ' is a pre-admissible relation set. Hence the problem of finding an admissible path algebra quotient that is isomorphic to $KQ/\langle \rho \rangle$ can be reduced to the case where ρ generates a pre-admissible ideal. In this section, we present a solution to this case that is due to Øystein Skartsæterhagen [Ska11].

Before presenting a precise algorithm, we give a less formal overview of the idea behind the solution. Assume that (Q, ρ) is a quiver with uniform pre-admissible relations. If ρ does not generate an admissible ideal, then there exists an arrow α in Q and an element $z \in KQ$ such that $\alpha - z \in \rho$ (up to a scalar multiple), and such that α does not appear in z. We then have

$$\alpha \equiv z \pmod{\langle \rho \rangle}.$$

Thus the arrow α is redundant, in the sense that the element of $KQ/\langle \rho \rangle$ represented by α is also represented by z, so we can remove α from the quiver. If there are relations in ρ that reference α, then we replace those relations by substituting z for α. Having eliminated the arrow α, we then repeat this process until we are left with an admissible set of relations.

However, there is a problem. We want to use the element z as a replacement for α, but what if z has the form

$$z = \sum_i b_i p_i$$

for paths p_i and scalars b_i, where at least one of the paths p_i contains α as a subpath? Then our relation set would still contain references to α, even after we substitute z for α. We solve this problem by repeatedly applying the substitution

$$\alpha \mapsto \sum_i b_i p_i$$

to z itself, until all references to α vanish modulo $\langle \rho \rangle$.

The notion of substitution in the previous paragraphs is made precise by the following definition.

Definition 4.8 ([Ska11])**.** Let Q be a quiver, and let α be an arrow in Q. An element $z \in KQ$ is called **α-uniform** if $z = \mathfrak{t}(\alpha) z \mathfrak{s}(\alpha)$. Given that z is α-uniform, the **substitution map**

$$\mathrm{Subst}_{(\alpha,z)} \colon KQ \to KQ$$

is the unique algebra homomorphism such that $\mathrm{Subst}_{(\alpha,z)}(v) = v$ for all vertices v in Q, and such that

$$\mathrm{Subst}_{(\alpha,z)}(\beta) = \begin{cases} z & \text{if } \beta = \alpha \\ \beta & \text{if } \beta \neq \alpha \end{cases}$$

for all arrows β in Q. ■

Remark 4.9. It is clear that the homomorphism $\mathrm{Subst}_{(\alpha,z)}$ is unique, since the vertices and arrows of Q generate KQ as an algebra. However, it is less clear that such a homomorphism actually exists. To see that it does indeed exist, let $f\colon KQ \to KQ$ be the unique K-linear map such that f agrees with our definition of $\operatorname{Subst}_{(\alpha,z)}$ on the vertices and arrows of Q, and such that

$$f(\beta_1 \dots \beta_n) = f(\beta_1) \dots f(\beta_n)$$

for all arrows $\beta_1, \dots, \beta_n$ for which $\beta_1 \dots \beta_n \neq 0$. Let us verify that f is an algebra homomorphism. It is enough to check that $f(pq) = f(p)f(q)$ for all paths p and q. Note that since z is α-uniform, we have

$$f(p) = \mathfrak{t}(p)f(p)\mathfrak{s}(p), \tag{4.1}$$

regardless of whether p is divisible by α.

First suppose that $pq = 0$. Then we have

$$f(pq) = f(0) = 0.$$

On the other hand, by using (4.1), we see that

$$f(p)f(q) = \mathfrak{t}(p)f(p)\mathfrak{s}(p)\mathfrak{t}(q)f(q)\mathfrak{s}(q) = 0,$$

because $\mathfrak{s}(p) \neq \mathfrak{t}(q)$.

Now suppose that $pq \neq 0$. If p is a vertex, then $p = \mathfrak{t}(q)$, and hence

$$f(pq) = f(q),$$

while

$$f(p)f(q) = f(\mathfrak{t}(q))f(q) = \mathfrak{t}(q)f(q) = f(q),$$

by (4.1). Hence $f(pq) = f(p)f(q)$ if p is a vertex. A similar argument shows that this also holds if q is a vertex. If neither p nor q is a vertex, then the equality $f(pq) = f(p)f(q)$ follows immediately from how we defined f on nontrivial paths.

We see that f is an algebra homomorphism. Hence the homomorphism $\operatorname{Subst}_{(\alpha,z)}$ described in Definition 4.8 really does exist, and it is equal to f. ■

Algorithm 4.2, EliminateArrow(Q, ρ, α, x), eliminates all references to a single arrow α in a relation set ρ, bringing us one step closer to obtaining an admissible relation set. The notation $Q \setminus \{ \alpha \}$ denotes the quiver obtained by removing the arrow α from Q, i.e. the quiver whose vertex set is the same as that of Q, and whose arrow set is $Q_1 \setminus \{ \alpha \}$.

Algorithm 4.2: EliminateArrow(Q, ρ, α, x) ([Ska11, Algorithm 1])

Input: A quiver Q, a finite pre-admissible set $\rho \subseteq KQ$ of uniform relations, an arrow $\alpha \in Q_1$, and a relation $x \in \rho$ such that α appears in x

Output: A quiver $Q' = Q \setminus \{ \alpha \}$ and a finite pre-admissible set $\rho' \subseteq KQ'$ of uniform relations, such that $KQ / \langle \rho \rangle \cong KQ' / \langle \rho' \rangle$

1 $c \leftarrow$ coefficient of α in x;
2 $z_0 \leftarrow \alpha - c^{-1} x$;
3 $i \leftarrow 0$;
4 **while** *there is a term of* z_i *that is divisible by* α **do**
5     $i \leftarrow i + 1$;
6     $\overline{z_i} \leftarrow \mathrm{Subst}_{(\alpha, z_0)}(z_{i-1})$;
7     $T \leftarrow \mathrm{Terms}(\overline{z_i}) \setminus \mathrm{Terms}_{\langle \rho \rangle}(\overline{z_i})$;
8     $z_i \leftarrow \sum_{t \in T} t$;
9 **end**
10 $z \leftarrow z_i$;
11 $Q' \leftarrow Q \setminus \{ \alpha \}$;
12 $\rho' \leftarrow \left\{ \mathrm{Subst}_{(\alpha, z)}(r) \mid r \in \rho \right\}$;
13 **return** (Q', ρ');

Remark 4.10. On line 7 of Algorithm 4.2, we remove those terms of $\overline{z_i}$ which are contained in the ideal $\langle \rho \rangle$. In order to do this, we need to be able to check if an element of KQ is contained in said ideal. We can do this if we first compute a finite Gröbner basis for ρ (with respect to some admissible order), which is possible by Theorem 2.48 and Theorem 2.51 because $KQ / \langle \rho \rangle$ is finite-dimensional.

However, another approach is also possible. Instead of removing those terms of $\overline{z_i}$ which are contained in $\langle \rho \rangle$, we could just remove the terms that are contained in the ideal J_Q^m, where m is some integer such that $J_Q^m \subseteq \langle \rho \rangle$. This would not affect the correctness of the algorithm. Of course, this modified version of the algorithm requires that we know some such value of m.

■

Note that since the relation set ρ is assumed to be uniform, z_0 is an α-uniform element, and hence the substitution map $\mathrm{Subst}_{(\alpha, z_0)}$ is defined. Observe that substitution preserves uniformity (and in particular it preserves

α-uniformity), so the elements $\overline{z_i}$ and z_i are also α-uniform for all i. It follows that the map $\mathrm{Subst}_{(\alpha,z)}$ is also defined, and that ρ' is a uniform set.

Before we prove that Algorithm 4.2 is correct, we consider an example.

Example 4.11. Let Q denote the following quiver.

$$Q\colon \begin{tikzcd}
 & v_2 \arrow[r, "\alpha_5"] \arrow[dr, "\alpha_4"'] & v_5 \arrow[d, "\alpha_6"] & & \\
v_1 \arrow[ur, "\alpha_3"] \arrow[dr, "\alpha_1"'] & & v_4 \arrow[r, "\alpha_7"'] & v_6 \arrow[r, "\alpha_8"'] & v_7 \arrow[ll, bend right, "\alpha_9"'] \\
 & v_3 \arrow[ur, "\alpha_2"'] & & &
\end{tikzcd}$$

We consider the pre-admissible set

$$\rho = \left\{\ \alpha_2\alpha_1 - \alpha_4\alpha_3, \alpha_6\alpha_5 - \alpha_4, \alpha_9\alpha_8\alpha_7\alpha_9 - \alpha_9, (\alpha_9\alpha_8\alpha_7)^3\ \right\}$$

of relations in KQ. The relations preventing ρ from being admissible are $\alpha_6\alpha_5 - \alpha_4$ and $\alpha_9\alpha_8\alpha_7\alpha_9 - \alpha_9$. We start by performing

$$\textsc{EliminateArrow}(Q, \rho, \alpha_4, \alpha_6\alpha_5 - \alpha 4),$$

which gives us the following quiver.

$$Q'\colon \begin{tikzcd}
 & v_2 \arrow[r, "\alpha_5"] & v_5 \arrow[d, "\alpha_6"] & & \\
v_1 \arrow[ur, "\alpha_3"] \arrow[dr, "\alpha_1"'] & & v_4 \arrow[r, "\alpha_7"'] & v_6 \arrow[r, "\alpha_8"'] & v_7 \arrow[ll, bend right, "\alpha_9"'] \\
 & v_3 \arrow[ur, "\alpha_2"'] & & &
\end{tikzcd}$$

Substituting $\alpha_6\alpha_5$ for α_4, we get the set

$$\rho' = \left\{\ \alpha_2\alpha_1 - \alpha_6\alpha_5\alpha_3, \alpha_6\alpha_5 - \alpha_6\alpha_5, \alpha_9\alpha_8\alpha_7\alpha_9 - \alpha_9, (\alpha_9\alpha_8\alpha_7)^3\ \right\}$$

of relations in KQ'. This is still not an admissible set, so we apply the algorithm one more time, this time performing

$$\textsc{EliminateArrow}(Q', \rho', \alpha_9, \alpha_9\alpha_8\alpha_7\alpha_9 - \alpha_9).$$

We then get the following quiver.

$$Q''\colon \begin{tikzcd}
 & v_2 \arrow[r, "\alpha_5"] & v_5 \arrow[d, "\alpha_6"] & & \\
v_1 \arrow[ur, "\alpha_3"] \arrow[dr, "\alpha_1"'] & & v_4 \arrow[r, "\alpha_7"'] & v_6 \arrow[r, "\alpha_8"'] & v_7 \\
 & v_3 \arrow[ur, "\alpha_2"'] & & &
\end{tikzcd}$$

In order to obtain a new set of relations, we must first find a suitable substitute for α_9. We repeatedly perform the substitution $\alpha_9 \mapsto \alpha_9\alpha_8\alpha_7\alpha_9$, and see that

$$\begin{aligned}\alpha_9 &\equiv \alpha_9\alpha_8\alpha_7\alpha_9 \\ &\equiv (\alpha_9\alpha_8\alpha_7\alpha_9)\alpha_8\alpha_7(\alpha_9\alpha_8\alpha_7\alpha_9) \\ &= (\alpha_9\alpha_8\alpha_7)^3\alpha_9 \\ &\equiv 0 \pmod{\langle\rho'\rangle}.\end{aligned}$$

Thus we can substitute 0 for α_9, and we obtain the admissible relation set

$$\rho'' = \{\, \alpha_2\alpha_1 - \alpha_6\alpha_5\alpha_3, \alpha_6\alpha_5 - \alpha_6\alpha_5 \,\}$$

in KQ''. ■

We will now prove that Algorithm 4.2 is correct. In order to do this, we will require some preliminary results. We start by showing that the algorithm terminates.

Lemma 4.12 ([Ska11, Proposition 5.2]). *Algorithm 4.2 terminates.*

Proof. Consider a quiver Q, a set $\rho \subseteq KQ$, an arrow α, and a relation $x \in \rho$ such that the assumptions in Algorithm 4.2 are satisfied. For an element $y \in KQ$, let $L_\alpha(y)$ denote the shortest length of a path appearing in y that is divisible by α. In other words, $L_\alpha(y)$ is the infimum of the set

$$\{\, \mathfrak{l}(p) \mid p \text{ is a path appearing in } y, \text{ and } \alpha \text{ divides } p \,\},$$

where we define $L_\alpha(y) = \infty$ if the set above is empty.

We claim that if p is any nontrivial path, then

$$L_\alpha(\mathrm{Subst}_{(\alpha,z_0)}(p)) > \mathfrak{l}(p). \tag{4.2}$$

We proceed by induction on the length of p. For the base case $\mathfrak{l}(p) = 1$, we must have $p = \beta$ for some arrow β. If $\beta \neq \alpha$, then we have

$$L_\alpha(\mathrm{Subst}_{(\alpha,z_0)}(p)) = L_\alpha(\beta) = \infty.$$

Otherwise, if $p = \alpha$, we have

$$\mathrm{Subst}_{(\alpha,z_0)}(p) = z_0,$$

and hence

$$L_\alpha(\mathrm{Subst}_{(\alpha,z_0)}(p)) = L_\alpha(z_0) \geq 2 > \mathfrak{l}(p),$$

where $L_\alpha(z_0) \geq 2$ because α does not appear in z_0. This shows that the base case for our claim holds.

Now suppose that $\mathfrak{l}(p) \geq 2$, and assume that (4.2) holds for all paths that are strictly shorter than p. Then we can write $p = qq'$ for nontrivial paths q and q'. Now suppose that r is some path that appears in the element

$$\mathrm{Subst}_{(\alpha,z_0)}(p) = \mathrm{Subst}_{(\alpha,z_0)}(q)\,\mathrm{Subst}_{(\alpha,z_0)}(q'),$$

such that $\alpha \mid r$. Then r must have the form $r = ss'$, where s is a path that appears in $\mathrm{Subst}_{(\alpha,z_0)}(q)$ and s' is a path that appears in $\mathrm{Subst}_{(\alpha,z_0)}(q')$. Furthermore, at least one of the paths s and s' must be divisible by α. Without loss of generality, assume that $\alpha \mid s$. Then

$$\begin{aligned}\mathfrak{l}(r) &= \mathfrak{l}(s) + \mathfrak{l}(s') \\ &\geq L_\alpha(\mathrm{Subst}_{(\alpha,z_0)}(q)) + \mathfrak{l}(q'),\end{aligned}$$

where we have used the fact that $\mathfrak{l}(s) \geq \mathfrak{l}(q')$ because $\mathrm{Subst}_{(\alpha,z_0)}(q')$ is a linear combination of paths of length at least $\mathfrak{l}(q')$. Applying the inductive assumption to q', we see that

$$\mathfrak{l}(r) > \mathfrak{l}(q) + \mathfrak{l}(q') = \mathfrak{l}(p),$$

and consequently

$$L_\alpha(\mathrm{Subst}_{(\alpha,z_0)}(p)) > \mathfrak{l}(p),$$

as desired.

Now consider the sequence $z_0, z_1, z_2, \ldots$ produced by the algorithm. If $L_\alpha(z_i) = \infty$, then the algorithm will terminate after the ith iteration of the while loop. Otherwise, there exists at least one path p appearing in z_i such that α divides p. Then by using (4.2), we see that

$$L_\alpha(\mathrm{Subst}_{(\alpha,z_0)}(p)) > \mathfrak{l}(p) \geq L_\alpha(z_i),$$

and hence $L_\alpha(\overline{z_{i+1}}) > L_\alpha(z_i)$. Since we obtain z_{i+1} by removing terms from $\overline{z_{i+1}}$, we also see that

$$L_\alpha(z_{i+1}) \geq L_\alpha(\overline{z_{i+1}}) > L_\alpha(z_i).$$

Since $\langle\rho\rangle$ is assumed to be pre-admissible, there exists some integer m such that $J_Q^m \subseteq \langle\rho\rangle$. By the preceding argumentation, there exists some j such that $L_\alpha(z_j) > m$. Thus if p is a path that appears in z_j and which is divisible by α, then $p \in \langle\rho\rangle$. But by the construction of z_j, none of the paths appearing in z_j are contained in $\langle\rho\rangle$, and hence none of the paths appearing in z_j are divisible by α. This shows that the algorithm terminates after the jth iteration of the while loop. $\square$

Next we show that substitution is compatible with the relation set ρ, in the sense that the map $\mathrm{Subst}_{(\alpha,\lambda)}$ preserves equivalence classes modulo $\langle\rho\rangle$, as long as α and λ belong to the same equivalence class.

Lemma 4.13 ([Ska11, Lemma 5.1])**.** *Let Q be a quiver, and let $I \subseteq KQ$ be an ideal. Let α be an arrow in Q, and let $\lambda \in KQ$ be an α-uniform element. Furthermore, assume that $\alpha \equiv \lambda \pmod{I}$. Then for any element $y \in KQ$, we have*

$$y \equiv \mathrm{Subst}_{(\alpha,\lambda)}(y) \pmod{I}.$$

Proof. Since $\mathrm{Subst}_{(\alpha,\lambda)}$ is an algebra homomorphism, it suffices to check the special case where y is a vertex or an arrow. If y is either a vertex or an arrow different from α, then y is actually equal to $\mathrm{Subst}_{(\alpha,\lambda)}(y)$. Otherwise, we have $y = \alpha$, and hence

$$\mathrm{Subst}_{(\alpha,\lambda)}(y) = \mathrm{Subst}_{(\alpha,\lambda)}(\alpha) = \lambda \equiv \alpha = y \pmod{I}.$$

□

As a consequence of Lemma 4.13, we have the following result.

Corollary 4.14 ([Ska11, Lemma 5.3])**.** *Let (Q, ρ) be a quiver with relations, let α be an arrow in Q, and let $x \in \rho$ be a relation, such that the conditions in Algorithm 4.2 are satisfied. If z is the element defined on line 10 of Algorithm 4.2, then we have*

$$z \equiv \alpha \pmod{\langle\rho\rangle}.$$

Proof. We clearly have

$$z_0 \equiv \alpha \pmod{\langle\rho\rangle}.$$

Moreover, we have

$$z_i \equiv \overline{z_i} \pmod{\langle\rho\rangle}$$

for all i, because we obtain z_i from $\overline{z_i}$ by removing terms that are contained in $\langle\rho\rangle$. We also have

$$\overline{z_i} \equiv z_{i-1} \pmod{\langle\rho\rangle}$$

by Lemma 4.13. By induction, it follows that

$$z \equiv \alpha \pmod{\langle\rho\rangle}.$$

□

Lemma 4.15 ([Ska11, Lemma 5.4]). *Let (Q, ρ) be a quiver with relations, let α be an arrow in Q, and let $x \in \rho$ be a relation, such that the conditions in Algorithm 4.2 are satisfied. Let (Q', ρ') be the quiver with relations produced by the algorithm. Then*

$$\langle \rho' \rangle = KQ' \cap \langle \rho \rangle \,,$$

where $\langle \rho' \rangle$ denotes the ideal in KQ' generated by ρ', and $\langle \rho \rangle$ denotes the ideal in KQ generated by ρ.

Proof. Let $z \in KQ'$ be the element defined on line 10 of the algorithm. Recall that the relation set ρ' is defined as

$$\rho' = \left\{ \operatorname{Subst}_{(\alpha,z)}(r) \mid r \in \rho \right\}.$$

In order to show that $\langle \rho' \rangle \subseteq KQ' \cap \langle \rho \rangle$, it therefore suffices to show that $\operatorname{Subst}_{(\alpha,z)}(r) \in \langle \rho \rangle$ for all $r \in \rho$. But by Corollary 4.14 and Lemma 4.13, we have

$$\operatorname{Subst}_{(\alpha,z)}(r) \equiv r \equiv 0 \pmod{\langle \rho \rangle},$$

i.e. $\operatorname{Subst}_{(\alpha,z)}(r) \in \langle \rho \rangle$. This shows that $\langle \rho' \rangle \subseteq KQ' \cap \langle \rho \rangle$.

Now suppose that $y \in KQ' \cap \langle \rho \rangle$. Since $y \in \langle \rho \rangle$, there exist relations $r_i \in \rho$ and elements $f_i, g_i \in KQ$ such that

$$y = \sum_i f_i r_i g_i.$$

Note that $y = \operatorname{Subst}_{(\alpha,z)}(y)$ because $y \in KQ'$, and hence

$$\begin{aligned} y &= \operatorname{Subst}_{(\alpha,z)} \left(\sum_i f_i r_i g_i \right) \\ &= \sum_i \operatorname{Subst}_{(\alpha,z)}(f_i) \operatorname{Subst}_{(\alpha,z)}(r_i) \operatorname{Subst}_{(\alpha,z)}(g_i). \end{aligned}$$

Note that $\operatorname{Subst}_{(\alpha,z)}(r_i) \in \rho'$, while $\operatorname{Subst}_{(\alpha,z)}(f_i)$ and $\operatorname{Subst}_{(\alpha,z)}(g_i)$ are elements of KQ'. Thus the equality above shows that $y \in \langle \rho' \rangle \subseteq KQ'$, and hence $KQ \cap \langle \rho \rangle \subseteq \langle \rho' \rangle$. □

We now have all the preliminary results we need in order to prove that Algorithm 4.2 does what it is supposed to.

Proposition 4.16 ([Ska11, Proposition 5.5]). *Algorithm 4.2 produces a correct result.*

Proof. Let Q be a quiver, $\rho \subseteq KQ$ a finite pre-admissible set of uniform relations, α an arrow in Q, and $x \in \rho$ a relation such that α appears in x. Let (Q', ρ') be the quiver with relations produced by Algorithm 4.2. We first show that the ideal $\langle \rho' \rangle \subseteq KQ'$ is pre-admissible. Since the ideal $\langle \rho \rangle \subseteq KQ$ is pre-admissible by assumption, there exists some integer m such that

$$J_Q^m \subseteq \langle \rho \rangle \subseteq J_Q.$$

Then we see that $\langle \rho' \rangle \subseteq J_{Q'}$, since our construction of ρ' from ρ did not introduce any vertices into the relations. Let p be a path in Q' of length m. Then $p \in \langle \rho \rangle$, so since we also have $p \in KQ'$, we see that $p \in \langle \rho' \rangle$ by Lemma 4.15. Hence $J_{Q'}^m \subseteq \langle \rho' \rangle$, so $\langle \rho' \rangle$ is pre-admissible.

Next we show that $KQ' / \langle \rho' \rangle \cong KQ / \langle \rho \rangle$. Recall that Q' is the subquiver of Q obtained by removing the arrow α, so there is an inclusion map

$$\phi \colon KQ' \to KQ,$$

and this is a (unital) algebra homomorphism. We have $\langle \rho' \rangle \subseteq \langle \rho \rangle$ by Lemma 4.15, so ϕ induces a map

$$\overline{\phi} \colon KQ' / \langle \rho' \rangle \to KQ / \langle \rho \rangle .$$

We claim that $\overline{\phi}$ is an isomorphism. Using a similar argument to the one given in the proof of Proposition 4.5, it can be shown that $\overline{\phi}$ is injective. To see that ϕ is surjective, let $y \in KQ$ be an arbitrary element. Consider the element $z \in KQ'$ defined on line 10 of the algorithm, and note that

$$z \equiv \alpha \pmod{\langle \rho \rangle}$$

by Corollary 4.14. Let $\tilde{y} = \mathrm{Subst}_{(\alpha,z)}(y) \in KQ'$. Then

$$\tilde{y} \equiv y \pmod{\langle \rho \rangle}$$

by Lemma 4.13, and hence

$$\overline{\phi}(\tilde{y} + \langle \rho' \rangle) = \tilde{y} + \langle \rho \rangle = y + \langle \rho \rangle .$$

Thus $\overline{\phi}$ is surjective, and hence it is an isomorphism. □

Algorithm 4.2 only eliminates a single arrow from a pre-admissible set of relations. By repeating this process until there are no more arrows to remove, we eventually obtain an admissible set of relations. This is described in Algorithm 4.3, PreAdmissibleToAdmissible(Q, ρ).

Algorithm 4.3: PreAdmissibleToAdmissible(Q, ρ) ([Ska11, Algorithm 2])

Input: A quiver Q and a finite pre-admissible set $\rho \subseteq KQ$ of uniform relations

Output: A quiver Q' and a finite admissible set $\rho' \subseteq KQ'$ of uniform relations such that $KQ/\langle\rho\rangle \cong KQ'/\langle\rho'\rangle$

```
Q′ ← Q;
ρ′ ← ρ;
while ρ′ contains an element in which an arrow appears do
    Choose an element x ∈ ρ and an arrow α ∈ Q_1 such that α
      appears in x;
    (Q′, ρ′) ← EliminateArrow(Q′, ρ′, α, x);
end
return (Q′, ρ′);
```

Proposition 4.17 ([Ska11, Proposition 5.8])**.** *Algorithm 4.3 produces a correct result.*

Proof. In every iteration of the while loop, an arrow is removed from the quiver Q'. Since there are only finitely many arrows, this means that the algorithm must terminate. When the algorithm terminates, the set ρ' must generate an admissible ideal, for otherwise there would be more arrows to remove from the quiver Q'. Finally, the fact that ρ' is a uniform set and the fact that $KQ/\langle\rho\rangle \cong KQ'/\langle\rho'\rangle$ both follow from the correctness of Algorithm 4.2. □

It is now clear how we can replace a lower-admissible path algebra quotient $KQ/\langle\rho\rangle$ with an admissible one: we first use Algorithm 4.1 to obtain a pre-admissible quotient, and then we apply Algorithm 4.3 to get an admissible quotient.

In Proposition 4.7, we saw that Algorithm 4.1 preserves Gröbner bases. Our proof of this fact boiled down to showing that if $x \in \rho$ is a relation in the lower admissible set $\rho \subseteq KQ$, and if $\mathrm{Tip}(x)$ is contained in the subquiver Q' produced by the algorithm, then $\mathrm{Tip}(x) \in \mathrm{Tip}(\rho')$. If we could prove that the analogous statement holds for Algorithm 4.2, then it would follow that Algorithm 4.2 and Algorithm 4.3 also preserve Gröbner bases. Unfortunately, the analogous statement does not hold. For let x be an element of a uniform pre-admissible relation set $\rho \subseteq KQ$, and suppose that $\mathrm{Tip}(x)$ is contained in the subquiver Q' produced by Algorithm 4.3. If α is some arrow that is removed from the quiver and replaced with an element z during the execution of the algorithm, then it may happen that

$$\operatorname{Tip}(\operatorname{Subst}_{(\alpha,z)}(x)) > \operatorname{Tip}(x),$$

because some of the lower order terms of x may be replaced by terms that are greater than $\operatorname{Tip}(x)$. It could also happen that $\operatorname{Subst}_{(\alpha,z)}(x) = 0$. As a consequence, $\operatorname{Tip}(x)$ will not necessarily be an element of $\operatorname{Tip}(\rho')$. In particular, Algorithm 4.3 does not preserve Gröbner bases, as the following example shows.

Example 4.18. Consider the following quiver.

$$Q\colon \begin{tikzcd}
& v_2 \arrow[dr, "\beta"] & \\
v_1 \arrow[loop left, "\zeta"] \arrow[ur, "\alpha"] \arrow[rr, "\varepsilon"] \arrow[dr, "\gamma"'] & & v_4 \arrow[loop right, "\eta"] \\
& v_3 \arrow[ur, "\delta"'] &
\end{tikzcd}$$

We equip Q with the left length-lexicographic ordering (Example 2.4), with the vertices ordered numerically and the arrows alphabetically, i.e.

$$v_1 < v_2 < v_3 < v_4$$

and

$$\alpha < \beta < \gamma < \delta < \varepsilon < \zeta < \eta.$$

We let ρ denote the following pre-admissible set of relations in KQ:

$$\rho = \left\{ \beta\alpha\zeta - \varepsilon, \eta\delta\gamma - \eta\varepsilon, \zeta^3, \eta^3, \varepsilon\zeta^2 \right\}.$$

Let us verify that ρ is a Gröbner basis. The set ρ is tip reduced and uniform, so we can use Theorem 2.41. We have the following overlaps between elements of ρ:

- $\beta\alpha\zeta - \epsilon$ and ζ^3 have a $(\zeta^2, \beta\alpha)$-overlap.
- η^3 and $\eta\delta\gamma - \eta\varepsilon$ have a $(\delta\gamma, \eta^2)$-overlap.
- $\varepsilon\zeta^2$ and ζ^3 have both a (ζ, ε)-overlap and a $(\zeta^2, \varepsilon\zeta)$-overlap.
- ζ^3 has both a (ζ, ζ)-overlap and a (ζ^2, ζ^2)-overlap with itself.
- Similarly, η^3 has an (η, η)-overlap and an (η^2, η^2)-overlap with itself.

We must check that all of the overlap relations have remainder zero under division by ρ. The overlap relation of $\beta\alpha\zeta - \varepsilon$ and ζ^3 is

$$\begin{aligned}\mathfrak{o}(\beta\alpha\zeta - \varepsilon, \zeta^3, \zeta^2, \beta\alpha) &= (\beta\alpha\zeta - \varepsilon)\zeta^2 - \beta\alpha\zeta^3 \\ &= -\varepsilon\zeta^2.\end{aligned}$$

This element has remainder zero, because it is simply a scalar multiple of $\varepsilon\zeta^2 \in \rho$. Next, the overlap relation of η^3 and $\eta\delta\gamma - \eta\varepsilon$ is

$$\begin{aligned}\mathfrak{o}(\eta^3, \eta\delta\gamma - \eta\varepsilon, \delta\gamma, \eta^2) &= \eta^3\delta\gamma - \eta^2(\eta\delta\gamma - \eta\varepsilon) \\ &= \eta^3\varepsilon,\end{aligned}$$

which has remainder zero because it is divisible by $\eta^3 \in \rho$. The overlap relations corresponding to the rest of the overlaps are all zero, so we see that ρ is a Gröbner basis.

Let us now see what happens when we replace the pre-admissible quotient $KQ/\langle\rho\rangle$ with an admissible quotient $KQ'/\langle\rho'\rangle$. The relation preventing ρ from being admissible is $\beta\alpha\zeta - \varepsilon$, so we need to remove the arrow ε. We obtain the following quiver.

$$Q'\colon\quad \begin{tikzcd} & v_2 \arrow[dr, "\beta"] & \\ v_1 \arrow[loop left, "\zeta"] \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] & & v_4 \arrow[loop right, "\eta"] \\ & v_3 \arrow[ur, "\delta"'] & \end{tikzcd}$$

We give Q' an admissible order by restricting the order on Q to the set of paths in Q'. By substituting $\beta\alpha\zeta$ for ε, we get the following admissible set of relations in KQ':

$$\rho' = \left\{ \eta\delta\gamma - \eta\beta\alpha\zeta, \zeta^3, \eta^3, \beta\alpha\zeta^3 \right\}.$$

The set of tips of ρ' is

$$\mathrm{Tip}(\rho') = \left\{ \eta\beta\alpha\zeta, \zeta^3, \eta^3, \beta\alpha\zeta^3 \right\}.$$

Consider the element $\eta\delta\gamma\zeta^2$. We have

$$\begin{aligned}\eta\delta\gamma\zeta^2 &= \eta\delta\gamma\zeta^2 - \eta\beta\alpha\zeta^3 + \eta\beta\alpha\zeta^3 \\ &= (\eta\delta\gamma - \eta\beta\alpha\zeta)\zeta^2 + \eta(\beta\alpha\zeta^3).\end{aligned}$$

Since $\eta\delta\gamma - \eta\beta\alpha\zeta$ and $\beta\alpha\zeta^2$ are elements of ρ', the element $\eta\delta\gamma\zeta^2$ is contained in the ideal $\langle\rho'\rangle \subseteq KQ'$. But $\eta\delta\gamma\zeta^2$ is not divisible by any element of $\mathrm{Tip}(\rho')$, so ρ' is not a Gröbner basis. ■

Since Algorithm 4.3 does not preserve Gröbner bases, we will have to find a Gröbner basis for $\rho' \subseteq KQ'$ by some other means. Of course, we could simply use Buchberger's algorithm. However, this means that we would have to compute a Gröbner basis both before and after running Algorithm 4.3, which could be time-consuming. It would therefore be nice if we could devise a modified version of Algorithm 4.3 that preserves Gröbner bases.

As suggested by the discussion preceding Example 4.18, it would be enough to compute a relation set $\rho'' \subseteq KQ'$ such that ρ'' generates the same ideal as ρ', and such that if $x \in \rho$ is an element whose tip is a path in Q', then $\mathrm{Tip}(x) \in \mathrm{Tip}(\rho'')$. However, this approach cannot work, because such a set ρ'' does not exist in general. For example, consider Example 4.18. It can be shown (e.g. by using Buchberger's algorithm) that the set

$$G = \{\ \eta\delta\gamma - \eta\beta\alpha\zeta, \zeta^3, \eta^3, \eta\delta\gamma\zeta^2\ \}$$

is a Gröbner basis for $\langle \rho' \rangle$. Consider the relation

$$x = \eta\delta\gamma - \eta\varepsilon \in \rho.$$

The tip of x is $\eta\delta\gamma$, which is a path in Q'. But this path is not divisible by any element of $\mathrm{Tip}(G)$, so since G is a Gröbner basis, there does not exist any element $y \in \langle \rho \rangle$ such that $\mathrm{Tip}(y) = \mathrm{Tip}(x)$.

Of course, the fact that this specific approach is impossible does not preclude the possibility that there is another way to modify the algorithm so that it preserves Gröbner bases. However, because of time constraints, I was unable to investigate this further in this thesis.

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Appendix A

A long proof

We will now provide a proof of Lemma 2.47, which we used to show that Buchberger's algorithm (Algorithm 2.3) terminates. Recall that $\Lambda = KQ$ is the path algebra of a quiver Q, and that we have fixed some admissible order on Q.

Lemma 2.47. *Let $\{ f_1, \ldots, f_n \} \subseteq \Lambda$ be a tip reduced uniform subset, and let $I = \langle f_1, \ldots, f_n \rangle$. Let x be an element of the reduced Gröbner basis of I. Then during the execution of Algorithm 2.3, there exists some l such that* $\mathrm{Tip}(x) \in \mathrm{Tip}(G_l)$.

Proof.
We adapt the proof of Theorem 2.41. For the sake of contradiction, assume that $\mathrm{Tip}(x) \notin \mathrm{Tip}(G_l)$ for all l. Fix an index l. Since $x \in I$ and G_l is a generating set for I, there exist distinct elements $g_i \in G_l$, paths p_{ij} and q_{ij}, and nonzero scalars a_{ij} such that

$$x = \sum_{i,j} a_{ij} p_{ij} g_i q_{ij}. \tag{A.1}$$

We may assume that $p_{ij} g_i q_{ij} \neq 0$. For each i, we can write

$$g_i = \sum_k b_{ik} \gamma_{ik},$$

where the b_{ik} are nonzero scalars and the γ_{ik} are distinct paths. Then we get

$$x = \sum_{i,j,k} a_{ij} b_{ik} p_{ij} \gamma_{ik} q_{ij}. \tag{A.2}$$

Let p^* be the largest of the paths $p_{ij}\gamma_{ik}q_{ij}$ with respect to the admissible order $\leq$. Out of all possible choices for the index l, and out of all the possible ways to write x as in (A.1), we may assume that we have made our choice in such a way that p^* is minimal with respect to the well-order $\leq$. Moreover, once we have chosen our minimal p^*, we may assume that we have chosen l and our expression for x in (A.1) in such a way that the number of occurrences of p^* in the right-hand side of (A.2) is minimal.

Let i_1, j_1, and k_1 be fixed indices such that $p^* = p_{i_1j_1}\gamma_{i_1k_1}q_{i_1j_1}$. Then $p^* = \mathrm{Tip}(p_{i_1j_1}\gamma_{i_1k_1}q_{i_1j_1})$. Because g_{i_1} is uniform and because we chose p^* to be the largest path appearing in the sum in (A.2), it must be the case that $\gamma_{i_1k_1} = \mathrm{Tip}(g_{i_1})$, and hence $p^* = p_{i_1j_1}\,\mathrm{Tip}(g_{i_1})q_{i_1j_1}$. We claim that $\mathrm{Tip}(x) \neq p^*$. For if $\mathrm{Tip}(x)$ were equal to p^*, then $\mathrm{Tip}(x)$ would be divisible by $\mathrm{Tip}(g_{i_1})$. But g_{i_1} is an element of I and x is an element of the reduced Gröbner basis of I, so this would imply that $\mathrm{Tip}(x) = \mathrm{Tip}(g_{i_1})$, which contradicts our assumption that $\mathrm{Tip}(x) \notin \mathrm{Tip}(G_l)$.

Since $\mathrm{Tip}(x) \neq p^*$ and p^* is the largest of the paths appearing in the right-hand side of (A.2), the appearances of p^* in (A.2) must cancel out. In particular, p^* must appear at least twice in the right-hand side, so there exist indices i_2, j_2, and k_2 such that $(i_1, j_1, k_1) \neq (i_2, j_2, k_2)$ and

$$p_{i_1j_1}\,\mathrm{Tip}(g_{i_1})q_{i_1j_1} = p_{i_2j_2}\gamma_{i_2k_2}q_{i_2j_2}.$$

Note that $p_{i_2j_2}\,\mathrm{Tip}(g_{i_2})q_{i_2j_2} \neq 0$ because g_{i_2} is uniform, and consequently $p_{i_2j_2}\gamma_{i_2k_2}q_{i_2j_2} \leq p_{i_2j_2}\,\mathrm{Tip}(g_{i_2})q_{i_2j_2}$. Hence we have $\gamma_{i_2k_2} = \mathrm{Tip}(g_{i_2})$, and we get

$$p_{i_1j_1}\,\mathrm{Tip}(g_{i_1})q_{i_1j_1} = p_{i_2j_2}\,\mathrm{Tip}(g_{i_2})q_{i_2j_2}.$$

To simplify our notation, we write $p = p_{i_1j_1}$, $g = g_{i_1}$, $q = q_{i_1j_1}$, $a = a_{i_1j_1}$, $p' = p_{i_2j_2}$, $g' = g_{i_2j_2}$, $q' = q_{i_2j_2}$, and $a' = a_{i_2j_2}$.

We are now in a similar situation to the proof of Theorem 2.41. Indeed, most of the cases in that proof carry over to this proof with no modifications needed. The only exception is case 2.3.1, whose proof in Theorem 2.41 used an assumption that may not be satisfied in the present proof. Hence we only need to check that we obtain a contradiction in this case. So assume that $\mathfrak{l}(p) < \mathfrak{l}(p')$, $\mathfrak{l}(q) > \mathfrak{l}(q')$, and $\mathfrak{l}(p') < \mathfrak{l}(p\,\mathrm{Tip}(g))$.

Then there exist paths r and s such that $q = rq'$, $p' = ps$, and $\mathrm{Tip}(g)r = s\,\mathrm{Tip}(g')$. Just as in the proof of Theorem 2.41, we have $\mathrm{Tip}(g) \nmid s$ and $\mathrm{Tip}(g') \nmid r$. Hence there is an (r, s)-overlap of g and g'. By using an argument identical to the one given in the proof of Theorem 2.41, it can be shown that

$$pgq = \mathrm{CTip}(g)p\mathfrak{o}(g, g', r, s)q' + \frac{\mathrm{CTip}(g)}{\mathrm{CTip}(g')}p'g'q'. \tag{A.3}$$

Let $w \in \Lambda$ be the element computed by the algorithm Remainder (Algorithm 2.1) such that

$$\mathfrak{o}(g, g', r, s) \Rightarrow_{G_l} w.$$

Then there exist scalars $c_m \in K \setminus \{ 0 \}$, paths $\tilde{p}_m$ and $\tilde{q}_m$ in Q, and elements $\tilde{g}_m \in G_l$ such that

$$\mathfrak{o}(g, g', r, s) = \sum_m c_m \tilde{p}_m \tilde{g}_m \tilde{q}_m + w, \tag{A.4}$$

where

$$\mathrm{Tip}(\tilde{p}_m \tilde{g}_m \tilde{q}_m) \leq \mathrm{Tip}(\mathfrak{o}(g, g', r, s)),$$

and where $\mathrm{Tip}(y) \nmid \hat{p}$ whenever $y \in G_l \setminus \{ 0 \}$ and $\hat{p}$ is a path that appears in w.

By using (A.3) and (A.4), we get

$$\begin{aligned}
&apgq + a'p'g'q' \\
&= a\left(\mathrm{CTip}(g)p\mathfrak{o}(g, g', r, s)q' + \frac{\mathrm{CTip}(g)}{\mathrm{CTip}(g')}p'g'q'\right) + a'p'g'q' \\
&= a\left(\mathrm{CTip}(g)p\left(\sum_m c_m \tilde{p}_m \tilde{g}_m \tilde{q}_m + w\right)q' + \frac{\mathrm{CTip}(g)}{\mathrm{CTip}(g')}p'g'q'\right) + a'p'g'q' \\
&= a''p'g'q' + \sum_m c'_m p\tilde{p}_m \tilde{g}_m \tilde{q}_m q' + bpwq',
\end{aligned}$$

where $a'' = a\frac{\mathrm{CTip}(g)}{\mathrm{CTip}(g')} + a'$, $c'_m = a\,\mathrm{CTip}(g)c_m$, and $b = a\,\mathrm{CTip}(g)$. Thus we can rewrite (A.1) in the following way:

$$\begin{aligned}
x &= \sum_{i,j} a_{ij} p_{ij} g_i q_{ij} \\
&= \sum_{(i,j)\neq(i_1,j_1),(i_2,j_2)} a_{ij} p_{ij} g_i q_{ij} + a''p'g'q' + \sum_m c'_m p\tilde{p}_m \tilde{g}_m \tilde{q}_m q' + bpwq'. \qquad \text{(A.5)}
\end{aligned}$$

This decreases the number of occurrences of p^* by one.[1] Unlike the proof of Theorem 2.41, however, this does not immediately yield a contradiction, because w is not an element of G_l, so our new expression for x would not have the same form as (A.1). In order to obtain a contradiction, we will use Lemma 2.46.

If $w = 0$, then we get a contradiction just as in the proof of Theorem 2.41. So assume that $w \neq 0$. Then w will be added to the set X during the for loop in Algorithm 2.3. The set G_{l+1} is obtained by tip reducing the set $G_l \cup X$,

[1]This could actually decrease the number of occurrences of p^* by two, if $a'' = 0$. If p^* occurred exactly twice in (A.1), then this would mean that p^* no longer appeared at all. But this would contradict the minimality of p^* with respect to the admissible order.

so by Lemma 2.46, there exist elements $h_t \in G_{l+1}$, scalars d_t, and paths u_t and v_t such that

$$w = \sum_t d_t u_t h_t v_t,$$

where $\mathrm{Tip}(u_t h_t v_t) \leq \mathrm{Tip}(w)$, and such that $\mathrm{Tip}(u_t h_t v_t) = \mathrm{Tip}(w)$ for only one value of t. Similarly, since g_i and $\tilde{g}_m$ are elements of G_l, they can be written as

$$g_i = \sum_{t'} d'_{it'} u'_{it'} h'_{it'} v'_{it'}$$

and

$$\tilde{g}_m = \sum_{t''} \tilde{d}_{mt''} \tilde{u}_{mt''} \tilde{h}_{mt''} \tilde{v}_{mt''},$$

where we again assume that the other conditions in Lemma 2.46 are satisfied. By combining this with (A.5), we obtain the following equality:

$$\begin{aligned} x = &\sum_{(i,j)\neq(i_1,j_1),(i_2,j_2)} \sum_{t'} a_{ij} d'_{it'} p_{ij} u'_{it'} h'_{it'} v'_{it'} q_{ij} \\ &+ \sum_{t'} a'' d'_{i_2t'} p' u'_{i_2t'} h'_{i_2t'} v'_{i_2t'} q' \\ &+ \sum_{m,t''} c'_m \tilde{d}_{mt''} p \tilde{p}_m \tilde{u}_{mt''} \tilde{h}_{mt''} \tilde{v}_{mt''} \tilde{q}_m q' \\ &+ \sum_t b d_t p u_t h_t v_t q'. \end{aligned} \tag{A.6}$$

Since $h'_{it'}$, $\tilde{h}_{mt''}$, and h_t are elements of G_{l+1}, we see that we have found a new way to write x as in (A.1), except that we are using elements of G_{l+1} instead of elements of G_l. We will see that this leads to a contradiction of our minimality assumptions about p^*.

Since w is a remainder of $\mathfrak{o}(g, g', r, s)$, Lemma 2.20 tells us that

$$\mathrm{Tip}(w) \leq \mathrm{Tip}(\mathfrak{o}(g, g', r, s)).$$

Hence we see that

$$\begin{aligned} \mathrm{Tip}(p u_t h_t v_t q') &\leq \mathrm{Tip}(p w q') \\ &\leq p\,\mathrm{Tip}(\mathfrak{o}(g, g', r, s)) q' \\ &< p\,\mathrm{Tip}(g) r q' \\ &= p\,\mathrm{Tip}(g) q \\ &= p^*. \end{aligned}$$

Similarly, using the fact that $\mathrm{Tip}(\tilde{p}_m \tilde{g}_m \tilde{q}_m) \leq \mathrm{Tip}(\mathfrak{o}(g, g', r, s))$, we see that

$$\begin{aligned}\mathrm{Tip}(p\tilde{p}_m \tilde{u}_{mt''} \tilde{h}_{mt''} \tilde{v}_{mt''} \tilde{q}_m q') &\leq \mathrm{Tip}(p\tilde{p}_m \tilde{g}_m \tilde{q}_m q') \\ &< p \,\mathrm{Tip}(g) r q' \\ &= p^*.\end{aligned}$$

This shows that p^* does not appear in the terms

$$p u_t h_t v_t q'$$

or

$$p\tilde{p}_m \tilde{u}_{mt''} \tilde{h}_{mt''} \tilde{v}_{mt''} \tilde{q}_m q'.$$

Moreover, it is straightforward to check that

$$\mathrm{Tip}(p_{ij} u'_{it'} h'_{it'} v'_{it'} q_{ij}) \leq p^*.$$

We see that all paths appearing in the terms in (A.6) are less than or equal to p^*. In other words, p^* is the largest of the paths appearing in (A.6). In order to obtain a contradiction of our minimality assumptions, it is therefore enough to show that the number of occurrences of p^* in (A.6) is strictly smaller than the number of occurrences of p^* in (A.1).

We have seen that p^* does not appear in terms of the form $p u_t h_t v_t q'$ or $p\tilde{p}_m \tilde{u}_{mt''} \tilde{h}_{mt''} \tilde{v}_{mt''} \tilde{q}_m q'$. It therefore suffices to consider terms of the form $p_{ij} u'_{it'} h'_{it'} v'_{it'} q_{ij}$. Recall that by Lemma 2.46, we could choose the elements $u'_{it'}$, $h'_{it'}$, and $v'_{it'}$ in such a way that for any given index i, there is only one value of t' for which

$$\mathrm{Tip}(u'_{it'} h'_{it'} v'_{it'}) = \mathrm{Tip}(g_i),$$

i.e. for which

$$p_{ij} \,\mathrm{Tip}(u'_{it'} h'_{it'} v'_{it'}) q_{ij} = p_{ij} \,\mathrm{Tip}(g_i) q_{ij}$$

for any index j. In particular, if $p_{ij} \,\mathrm{Tip}(g_i) q_{ij} = p^*$, then there is exactly one value of t' such that p^* appears in $p_{ij} u'_{it'} h'_{it'} v'_{it'} q_{ij}$; and if $p_{ij} \,\mathrm{Tip}(g_i) q_{ij} \neq p^*$, then p^* does not appear in $p_{ij} u'_{it'} h'_{it'} v'_{it'} q_{ij}$ for any value of t'. This shows that the number of terms of (A.6) in which p^* appears is equal to the number of occurrences of p^* in (A.5), which is one less than the number of occurrences of p^* in (A.1). But this contradicts our assumption about p^*. $\square$

Appendix B

Implementation of algorithms

This appendix contains implementations of some of the algorithms in this thesis, written in the programming language GAP using the package QPA. Appendix B.1 contains code related to the computation of Gröbner bases, including the modified version of Buchberger's algorithm presented in Chapter 3. Appendix B.2 contains code for computing tensor products of algebras. Appendix B.3 tests comparing the performance of implementations of Algorithm 2.3 and Algorithm 3.1.

The source code displayed in this appendix is also available in the form of PDF attachments. If you are viewing this thesis in a PDF reader, it should be possible to access these attachments by clicking (or possibly double clicking or right clicking) on the following items:

- groebner_basis.g
- tensor_products.g
- timing.g

The source code may eventually also appear on my personal GitHub page, which can be found at https://github.com/jonwanundsen.

B.1 Gröbner basis algorithms

Listing B.1 contains code for computing Gröbner bases, which can also be found in the PDF attachment `groebner_basis.g`. The following are the most important functions found in the code:

- `TipReduceInPlaceWithoutFlags`: Implements TipReduce (Algorithm 2.2).

- `TipReduceInPlaceWithFlags`: Implements TipReduce (Algorithm 2.2). In addition to accepting a list of generators, this function also takes an argument containing a list of k boolean flags, which correspond to the first k generators. If the ith element of the generating list is modified during the tip reduction process, the ith flag is set to `true`.
- `RemainderUnderDivision`: Implements Remainder (Algorithm 2.1).
- `ComputeGroebnerBasisWithoutKnowledge`: Implements Buchberger's algorithm (Algorithm 2.3).
- `ComputeGroebnerBasisWithKnowledge`: Implements the modified version of Buchberger's algorithm from Section 3.3.2 (Algorithm 3.1).

Listing B.1: Implementation of Gröbner basis algorithms in GAP

```
LoadPackage("qpa");

PathDivision := function(path, sub)
    # Returns a list [p, q], where p and q are paths such that
        path = p*sub*q,
    # or returns fail if no such paths exist.
    local index, p, q, path_walk, i;
    if not IsQuiverVertex(sub) then
        # Sub is a nontrivial path.
        index := PositionSublist(WalkOfPath(path), WalkOfPath(
            sub));
        if index = fail then
            return fail;
        else
            path_walk := WalkOfPath(path);
            p := SourceOfPath(path);
            for i in [1..(index - 1)] do
                p := p*path_walk[i];
            od;
            q := TargetOfPath(sub);
            for i in [(index + LengthOfPath(sub))..LengthOfPath(
                path)] do
                q := q*path_walk[i];
            od;
            return [p, q];
        fi;
    else
        # Sub is a vertex.
        if IsQuiverVertex(path) then
            if path = sub then
                return [sub, sub];
            else
```

```
                return fail;
            fi;
        else
            if SourceOfPath(path) = sub then
                return [sub, path];
            else
                path_walk := WalkOfPath(path);
                index := fail;
                p := SourceOfPath(path);
                for i in [1..Length(path_walk)] do
                    p := p*path_walk[i];
                    if TargetOfPath(path_walk[i]) = sub then
                        index := i;
                        break;
                    fi;
                od;
                if index <> fail then
                    q := sub;
                    for i in [(index + 1)..Length(path_walk)] do
                        q := q*path_walk[i];
                    od;
                    return [p, q];
                else
                    return fail;
                fi;
            fi;
        fi;
    fi;
end;

TipReduceInPlaceWithoutFlags := function(generators)
    local modified, i, j, generator1, generator2, pair, c;
    while true do
        modified := false;
        for i in [1..Length(generators)] do
            generator1 := generators[i];
            if not IsZero(generator1) then
                for j in [1..Length(generators)] do
                    generator2 := generators[j];
                    if j <> i and not IsZero(generator2) then
                        # "Divide" the tip of the second
                            generator by the tip
                        # of the first generator:
                        pair := PathDivision(TipMonomial(
                            generator2), TipMonomial(generator1))
                            ;
                        if pair <> fail then
                            c := LeadingCoefficient(generator2)/
                                LeadingCoefficient(generator1);
```

```
                            generators[j] := generator2 - c*(
                                pair[1]*generator1*pair[2]);
                            modified := true;
                        fi;
                    fi;
                od;
            fi;
        od;
        if not modified then
            break;
        fi;
    od;
end;

TipReduceInPlaceWithFlags := function(generators, flags)
    local modified, i, j, generator1, generator2, pair, c;
    while true do
        modified := false;
        for i in [1..Length(generators)] do
            generator1 := generators[i];
            if not IsZero(generator1) then
                for j in [1..Length(generators)] do
                    generator2 := generators[j];
                    if j <> i and not IsZero(generator2) then
                        # "Divide" the tip of the second
                            generator by the tip
                        # of the first generator:
                        pair := PathDivision(TipMonomial(
                            generator2), TipMonomial(generator1))
                            ;
                        if pair <> fail then
                            c := LeadingCoefficient(generator2)/
                                LeadingCoefficient(generator1);
                            generators[j] := generator2 - c*(
                                pair[1]*generator1*pair[2]);
                            if j <= Length(flags) then
                                flags[j] := true;
                            fi;
                            modified := true;
                        fi;
                    fi;
                od;
            fi;
        od;
        if not modified then
            break;
        fi;
    od;
end;
```

```
RemainderUnderDivision := function(y, dividing_set)
    local z, r, x, division_occurred, pair, c;
    r := 0*y; # Sets r to the zero element of the path algebra
        containing y
    z := y;
    while not IsZero(z) do
        division_occurred := false;
        for x in dividing_set do
            if not IsZero(x) then
                pair := PathDivision(TipMonomial(z), TipMonomial
                    (x));
                if pair <> fail then
                    c := LeadingCoefficient(z)/
                        LeadingCoefficient(x);
                    z := z - c*(pair[1]*x*pair[2]);
                    division_occurred := true;
                    if IsZero(z) then
                        break; # Break the for loop because z
                            has become zero
                    fi;
                fi;
            fi;
        od;
        if not division_occurred then
            r := r + LeadingTerm(z);
            z := z - LeadingTerm(z);
        fi;
    od;
    return r;
end;

ComputeOverlaps := function(x, y)
    # This function computes a list of all pairs [p, q] of paths
        p and q
    # such that x and y have a (p, q)-overlap.
    local tip_x, tip_y, i, j, n, path_pairs, arrows_x, arrows_y,
        p, q;
    if IsZero(x) or IsZero(y) then
        return [];
    else
        tip_x := LeadingMonomial(x);
        tip_y := LeadingMonomial(y);
        if IsQuiverVertex(tip_x) or IsQuiverVertex(tip_y) then
            # Overlaps cannot exist if either of the tips is a
                vertex
            return [];
        else
            n := Minimum(LengthOfPath(tip_x), LengthOfPath(tip_y
```

```
            ));
            arrows_x := WalkOfPath(tip_x);
            arrows_y := WalkOfPath(tip_y);
            path_pairs := [];
            for i in [1..n] do
                if arrows_x{[(Length(arrows_x) - i + 1)..Length(
                    arrows_x)]} = arrows_y{[1..i]} then
                    # The last i arrows in the tip of x coincide
                        with the first
                    # i arrows in the tip of y, so there is an
                        overlap.
                    p := TargetOfPath(tip_x);
                    for j in [(i + 1)..Length(arrows_y)] do
                        p := p*arrows_y[j];
                    od;
                    q := SourceOfPath(tip_x);
                    for j in [1..(Length(arrows_x) - i)] do
                        q := q*arrows_x[j];
                    od;
                    Add(path_pairs, [p, q]);
                fi;
            od;
            return path_pairs;
        fi;
    fi;
end;

ComputeOverlapRelations := function(x, y)
    local overlap_relations, overlap, p, q, relation;
    overlap_relations := [];
    for overlap in ComputeOverlaps(x, y) do
        p := overlap[1];
        q := overlap[2];
        relation := (x*p)/LeadingCoefficient(x) - (q*y)/
            LeadingCoefficient(y);
        Add(overlap_relations, relation);
    od;
    return overlap_relations;
end;

ComputeGroebnerBasisWithKnowledge := function(generators, k)
    local groebner_basis, flags, remainders, i, j,
       ith_is_in_grb_basis,
        jth_is_in_grb_basis, overlap_relation, r;
    groebner_basis := ShallowCopy(generators);
    # Note that the variable groebner_basis will not actually be
        a Gröbner basis
    # until the algorithm terminates.
    flags := []; # A list representing which of the first k
```

```
        elements of the
                 # generating set have been modified by tip
                     reduction.
    for i in [1..k] do
        flags[i] := false;
        # The flag is set to false because the corresponding
            element
        # has not yet been modified by tip reduction
    od;
    TipReduceInPlaceWithFlags(groebner_basis, flags);
    while true do
        remainders := [];
        for i in [1..Length(groebner_basis)] do
            # Check if the ith element is among the first k
                elements
            # and has not been modified by tip reduction:
            ith_is_in_grb_basis := (i <= k and not flags[i]);
            for j in [1..Length(groebner_basis)] do
                jth_is_in_grb_basis := (j <= k and not flags[j])
                    ;
                if not ith_is_in_grb_basis or not
                    jth_is_in_grb_basis then
                    # The elements are not both contained in the
                         Gröbner basis
                    # formed by the first k elements of the list
                         given as an
                    # argument to this function, so we need to
                        check overlaps
                    for overlap_relation in
                        ComputeOverlapRelations(groebner_basis[i
                        ], groebner_basis[j]) do
                         r := RemainderUnderDivision(
                             overlap_relation, groebner_basis);
                         if not IsZero(r) then
                             Add(remainders, r);
                         fi;
                    od;
                fi;
            od;
        od;
        if Length(remainders) <> 0 then
            Append(groebner_basis, remainders);
            TipReduceInPlaceWithFlags(groebner_basis, flags);
        else
            break;
        fi;
    od;
    return groebner_basis;
end;
```

```
ComputeGroebnerBasisWithoutKnowledge := function(generators)
    # This function requires that the generating set is uniform.
    local groebner_basis, remainders, i, j, overlap_relation, r;
    groebner_basis := ShallowCopy(generators);
    # Note that the variable groebner_basis will not actually be
        a Gröbner basis
    # until the algorithm terminates.
    TipReduceInPlaceWithoutFlags(groebner_basis);
    while true do
        remainders := [];
        for i in [1..Length(groebner_basis)] do
            for j in [1..Length(groebner_basis)] do
                for overlap_relation in ComputeOverlapRelations(
                    groebner_basis[i], groebner_basis[j]) do
                    r := RemainderUnderDivision(overlap_relation
                        , groebner_basis);
                    if not IsZero(r) then
                        Add(remainders, r);
                    fi;
                od;
            od;
        od;
        if Length(remainders) <> 0 then
            Append(groebner_basis, remainders);
            TipReduceInPlaceWithoutFlags(groebner_basis);
        else
            break;
        fi;
    od;
    return groebner_basis;
end;
```

B.2 Tensor products

Listing B.2 contains an implementation of the construction of the tensor product of algebras discussed in Chapter 3, more specifically in Theorem 3.13 and Theorem 3.18. This code can also be found in the PDF attachment `tensor_products.g`. The most important functions are `TensorProductOverField`, which computes the tensor product $A \otimes_K B$ for K-algebras A and B, and `TensorProductOverAlgebra`, which computes the tensor product $A \otimes_C B$ for K-algebras A, B, and C.

Listing B.2: Implementation of tensor product of algebras in GAP

```
LoadPackage("qpa");
```

```
CrossVertexFirst := function(v, y, product_quiver_path_algebra)
    local y_terms, field, product_quiver, crossed_element, i,
       path,
        coefficient, crossed_path;
    product_quiver := QuiverOfPathAlgebra(
       product_quiver_path_algebra);
    field := LeftActingDomain(product_quiver_path_algebra);
    y_terms := CoefficientsAndMagmaElements(y);
    # y_terms is a list where the entries at odd indices are the
        paths
    # appearing in y, and the entries at even indices are the
       coefficients.
    crossed_element := Zero(product_quiver_path_algebra);
    for i in [1..(Length(y_terms)/2)] do
        path := y_terms[2*i - 1];
        coefficient := y_terms[2*i];
        crossed_path := IncludeInProductQuiver([v, path],
           product_quiver);
        crossed_element := crossed_element + coefficient*
           ElementOfPathAlgebra(
                                  product_quiver_path_algebra,
                                     crossed_path);
    od;
    return crossed_element;
end;

CrossVertexSecond := function(x, w, product_quiver_path_algebra)
    local x_terms, field, product_quiver, crossed_element, i,
       path,
        coefficient, crossed_path;
    product_quiver := QuiverOfPathAlgebra(
       product_quiver_path_algebra);
    field := LeftActingDomain(product_quiver_path_algebra);
    x_terms := CoefficientsAndMagmaElements(x);
    # y_terms is a list where the entries at odd indices are the
        paths
    # appearing in y, and the entries at even indices are the
       coefficients.
    crossed_element := Zero(product_quiver_path_algebra);
    for i in [1..(Length(x_terms)/2)] do
        path := x_terms[2*i - 1];
        coefficient := x_terms[2*i];
        crossed_path := IncludeInProductQuiver([path, w],
           product_quiver);
        crossed_element := crossed_element + coefficient*
           ElementOfPathAlgebra(
                                  product_quiver_path_algebra,
                                     crossed_path);
```

```
    od;
    return crossed_element;
end;

CommutativityRelation := function(a, b,
   product_quiver_path_algebra)
    local Q, path1, path2;
    Q := QuiverOfPathAlgebra(product_quiver_path_algebra);
    path1 := IncludeInProductQuiver([a, SourceOfPath(b)], Q)
            * IncludeInProductQuiver([TargetOfPath(a), b], Q);
    path2 := IncludeInProductQuiver([SourceOfPath(a), b], Q)
            * IncludeInProductQuiver([a, TargetOfPath(b)], Q);
    return ElementOfPathAlgebra(product_quiver_path_algebra,
       path1)
            - ElementOfPathAlgebra(product_quiver_path_algebra,
               path2);
end;

TensorProductOverField_Relations := function(A, B)
    local product_quiver, field, product_pa, tensor_relations,
       Q_A, Q_B, ideal,
        A_grb_basis, B_grb_basis, a, b, vertex, rel,
           tensor_ideal;
    Q_A := QuiverOfPathAlgebra(A);
    Q_B := QuiverOfPathAlgebra(B);
    product_quiver := QuiverProduct(Q_A, Q_B);
    field := LeftActingDomain(A);
    product_pa := PathAlgebra(field, product_quiver);
    tensor_relations := [];
    # Add commutativity relations:
    for a in ArrowsOfQuiver(Q_A) do
        for b in ArrowsOfQuiver(Q_B) do
            Add(tensor_relations, CommutativityRelation(a, b,
               product_pa));
        od;
    od;
    # Get (reduced) Gröbner bases for the ideals by which A and
       B are quotients:
    if IsQuotientOfPathAlgebra(A) then
        ideal := IdealOfQuotient(A);
        A_grb_basis := GroebnerBasisOfIdeal(ideal);
        if not HasIsCompletelyReducedGroebnerBasis(A_grb_basis)
           or not IsCompletelyReducedGroebnerBasis(A_grb_basis)
           then
            A_grb_basis := CompletelyReduceGroebnerBasis(
               A_grb_basis);
        fi;
    else
        A_grb_basis := [];
```

```
    fi;
    if IsQuotientOfPathAlgebra(B) then
        ideal := IdealOfQuotient(B);
        B_grb_basis := GroebnerBasisOfIdeal(ideal);
        if not HasIsCompletelyReducedGroebnerBasis(B_grb_basis)
            or not IsCompletelyReducedGroebnerBasis(B_grb_basis)
            then
             B_grb_basis := CompletelyReduceGroebnerBasis(
                 B_grb_basis);
        fi;
    else
        B_grb_basis := [];
    fi;
    # Include the original relations:
    for vertex in VerticesOfQuiver(Q_A) do
        for rel in B_grb_basis do
            Add(tensor_relations, CrossVertexFirst(vertex, rel,
                product_pa));
        od;
    od;
    for vertex in VerticesOfQuiver(Q_B) do
        for rel in A_grb_basis do
            Add(tensor_relations, CrossVertexSecond(rel, vertex,
                  product_pa));
        od;
    od;
    return [product_pa, tensor_relations];
end;

TensorProductOverField := function(A, B)
    local rels;
    rels := TensorProductOverField_Relations(A, B);
    return rels[1]/rels[2];
end;

BalancingRelation := function(v, w, s, f, g,
    product_quiver_path_algebra)
    # v and w are vertices, s is an element of KS (should be a
        vertex or arrow)
    local image_f, image_g;
    image_f := ImageElm(f, s);
    image_g := ImageElm(g, s);
    if IsElementOfQuotientOfPathAlgebra(image_f) then
        image_f := image_f![1];
    fi;
    if IsElementOfQuotientOfPathAlgebra(image_g) then
        image_g := image_g![1];
    fi;
    return CrossVertexSecond(v*image_f*v, w,
```

```
        product_quiver_path_algebra)
        - CrossVertexFirst(v, w*image_g*w,
            product_quiver_path_algebra);
end;

TensorProductOverAlgebra_Relations := function(A, B, f, g)
    local tensor_over_field, C, product_pa, relations, k, S, v,
        w, s, b;
    C := Source(f);
    if C <> Source(g) then
        ErrorNoReturn("The homomorphisms f and g must have the
            same source");
    fi;
    if Range(f) <> A or Range(g) <> B then
        ErrorNoReturn("The codomain of f must be A, and the
            codomain of g must be B");
    fi;
    if not IsPathAlgebra(C) then
        ErrorNoReturn("The source of f and g must be a path
            algebra");
    fi;
    tensor_over_field := TensorProductOverField_Relations(A, B);
    product_pa := tensor_over_field[1];
    relations := tensor_over_field[2];
    k := Length(relations); # We need to keep track of this for
        the modified
                              # version of Buchberger's algorithm.
    S := QuiverOfPathAlgebra(C);
    for v in VerticesOfQuiver(QuiverOfPathAlgebra(A)) do
        for w in VerticesOfQuiver(QuiverOfPathAlgebra(B)) do
            for s in Concatenation(VerticesOfQuiver(S),
                ArrowsOfQuiver(S)) do
                b := BalancingRelation(v, w, One(C)*s, f, g,
                    product_pa);
                if not IsZero(b) then
                    Add(relations, b);
                fi;
            od;
        od;
    od;
    return [product_pa, relations, k];
end;

TensorProductOverAlgebra := function(A, B, f, g)
    local rels;
    rels := TensorProductOverAlgebra_Relations(A, B, f, g);
    return rels[1]/rels[2];
end;
```

B.3 Performance tests

In this section, we compare the performance of the implementations of different versions of Buchberger's algorithm found in Listing B.1, namely `ComputeGroebnerBasisWithoutKnowledge` (which implements Algorithm 2.3) and `ComputeGroebnerBasisWithKnowledge` (which implements Algorithm 3.1). We also compare these implementations to two functions that are built into QPA. The first of these is `HighLevelGroebnerBasis`, which is an implementation of a standard version of Buchberger's algorithm, similar to Algorithm 2.3. The second is `GBNPGroebnerBasis`, which follows a different approach: Given a quiver Q and a relation set $\rho \subseteq KQ$, it constructs another relation set ρ' in a free algebra $K\langle x_1, \ldots, x_n \rangle$, computes a Gröbner basis for the ideal $\langle \rho' \rangle \subseteq K\langle x_1, \ldots, x_n \rangle$ by using the GAP package GBNP, and then translates the result back to KQ to obtain a Gröbner basis for the ideal $\langle \rho \rangle \subseteq KQ$.

In order to measure the performance of the implementations, we will need some examples. We therefore consider the following quivers.

$$Q\colon \begin{tikzcd} & v_2 \arrow[dr, "\beta"] & \\ v_1 \arrow[ur, "\alpha"] & & v_3 \arrow[ll, "\gamma"] \end{tikzcd}$$

$$S\colon \begin{tikzcd} u \arrow[loop above, "x"] \arrow[loop right, "y"] \arrow[loop left, "z"] \end{tikzcd}$$

$$S'\colon \begin{tikzcd} w \arrow[loop right, "x'"] \end{tikzcd}$$

Let K denote the field of rational numbers. We define the K-algebras $\Lambda = KQ/J_Q^4$, $\Lambda' = KQ/J_Q^8$, $\Sigma = KS$, and $\Sigma' = KS'$. We give Λ a Σ-module structure through the algebra homomorphism $f\colon \Sigma \to \Lambda$ given by

$$f(x) = [\gamma\beta\alpha], \ f(y) = [\alpha\gamma\beta], \ f(z) = [\beta\alpha\gamma].$$

Similarly, we give Λ' a Σ'-module structure by using the algebra homomorphism $f'\colon \Sigma' \to \Lambda'$ given by

$$f'(x') = [\gamma\beta\alpha + \alpha\gamma\beta + \beta\alpha\gamma].$$

Note that the images of f and f' are contained in the centres of Λ and Λ', respectively.

Our performance tests will measure how long it takes to compute the tensor products $\Lambda \otimes_\Sigma \Lambda$ and $\Lambda' \otimes_{\Sigma'} \Lambda'$. More precisely, we will measure how long it takes to compute a Gröbner basis for the ideal

$$I_\Sigma \subseteq K(Q \times Q)$$

from Theorem 3.18, where $K(Q \times Q)/I_\Sigma \cong \Lambda \otimes_\Sigma \Lambda$, and similarly for $I_{\Sigma'}$.

B.3.1 Code

Listing B.3 contains the code used in the performance tests. This code can also be found in the PDF attachment `timing.g`.

Listing B.3: Code for performance tests in GAP

```
LoadPackage("qpa");

Read("groebner_basis.g");
Read("tensor_products.g");


time_high_level := function(path_algebra, relations)
    local start_time, gb;
    start_time := Runtime();
    gb := HighLevelGroebnerBasis(relations, path_algebra);
    return Runtime() - start_time;
end;

time_qpa_implementation := function(path_algebra, relations)
    local start_time, gb;
    start_time := Runtime();
    gb := GBNPGroebnerBasis(relations, path_algebra);
    return Runtime() - start_time;
end;

time_my_naive_implementation := function(path_algebra, relations
   )
    local start_time, gb;
    start_time := Runtime();
    gb := ComputeGroebnerBasisWithoutKnowledge(relations);
    return Runtime() - start_time;
end;

time_my_less_naive_implementation := function(path_algebra,
   relations, k)
    # Still pretty naïve, though.
    local start_time, gb;
    start_time := Runtime();
```

```
    gb := ComputeGroebnerBasisWithKnowledge(relations, k);
    return Runtime() - start_time;
end;

test_examples := function(examples, n)
    # n is the number of runs
    local i, e, tensor, results, name, avg, sum, x, variance,
       std;
    Print("Testing examples with ", n, " iterations\n\n");
    for e in examples do
        Print("Testing example:\n", e, "\n\n");
        results := rec(naive := [], less_naive := [], high_level
            := [], gbnp := []);
        for i in [1 .. n] do
            tensor := TensorProductOverAlgebra_Relations(e.A, e.
               B, e.f, e.g);
            Add(results.naive, time_my_naive_implementation(
               tensor[1], tensor[2]));
            Add(results.less_naive,
               time_my_less_naive_implementation(tensor[1],
               tensor[2], tensor[3]));
            Add(results.high_level, time_high_level(tensor[1],
               tensor[2]));
            Add(results.gbnp, time_qpa_implementation(tensor[1],
                tensor[2]));
        od;
        for name in RecNames(results) do
            avg := Float(Average(results.(name)));
            if n > 1 then
                # Compute the unbiased sample variance:
                sum := 0;
                for x in results.(name) do
                    sum := sum + (x - avg)^2;
                od;
                variance := sum/(n - 1);
                # Then take the square root to get the standard
                    deviation:
                std := Sqrt(variance);
            else
                std := fail;
            fi;
            Print(name, "\n", "Mean: ", avg, " ms\n", "Standard
               deviation: ", std, " ms\n\n");
        od;
        Print("\n\n");
    od;
end;
```

```
EXAMPLES := [];
K := Rationals;
Q := Quiver(3, [ [1, 2, "a"], [2, 3, "b"], [3, 1, "c"] ]);
KQ := PathAlgebra(K, Q);

# Modulo paths of length at least 4:
A := KQ/NthPowerOfArrowIdeal(KQ, 4);
a := A.a;
b := A.b;
c := A.c;
S := Quiver(["u"], [ [1, 1, "x"], [1, 1, "y"], [1, 1, "z"] ]);
KS := PathAlgebra(K, S);
generators := [KS.u, KS.x, KS.y, KS.z];
images := [One(A), a*b*c, b*c*a, c*a*b];
f := AlgebraWithOneHomomorphismByImagesNC(KS, A, generators,
    images);
f!.generators := generators;
f!.genimages := images;
Add(EXAMPLES, rec(A:=A, B:=A, f:=f, g:=f));

# Modulo paths of length at least 8:
A := KQ/NthPowerOfArrowIdeal(KQ, 8);
a := A.a;
b := A.b;
c := A.c;
S := Quiver(["u"], [ [1, 1, "x"] ]);
KS := PathAlgebra(K, S);
generators := [KS.u, KS.x];
images := [One(A), a*b*c + b*c*a + c*a*b];
f := AlgebraWithOneHomomorphismByImagesNC(KS, A, generators,
    images);
f!.generators := generators;
f!.genimages := images;
Add(EXAMPLES, rec(A:=A, B:=A, f:=f, g:=f));

# Three vertices in a cycle; modulo paths of length 4? Or
    something else.
# Or even length 8...
# (For length 3 the centre is just K, so there's not much of
    interest.)

test_examples(EXAMPLES, 1000);
```

B.3.2 Results of performance tests

Table B.1 shows the results I obtained when running the code from Listing B.3. I used version 4.11.0 of GAP, version 1.33 of QPA, and version 1.0.5

of GBNP, and I ran the code on Debian GNU/Linux 11. Each of the numbers in the table was obtained by performing the given computation 1000 times and taking the average. The numbers are rounded to two decimal places.

Table B.1: The results of the performance tests

Tensor product	Time for standard Buchberger (ms)	Time for modified Buchberger (ms)	Time for high level Gröbner basis (ms)	Time for GBNP (ms)
$\Lambda \otimes_{\Sigma} \Lambda$	30.98	30.18	34.77	4.50
$\Lambda' \otimes_{\Sigma'} \Lambda'$	65.91	53.57	68.01	8.06

As we can see from the table, the implementation of the modified version of Buchberger's algorithm (Algorithm 3.1) is more efficient than the implementation of the standard version of the algorithm (Algorithm 2.3), although the difference is quite small when computing $\Lambda \otimes_{\Sigma} \Lambda$. High level Gröbner basis is a bit slower than my own implementation. Finally, it turns out that GBNP is significantly faster than any of the other implementations. I personally found this a bit surprising, since one might expect that the process of translating between a path algebra and a free algebra would add some overhead. Due to time constraints, I was unable to investigate why it is so much faster to use GBNP, although I suspect that the version of Buchberger's algorithm used by GBNP is more efficient than the more general version for path algebras (Algorithm 2.3). It might be possible to improve the performance of GBNP even further by modifying it in the same way that we modified Algorithm 2.3 to obtain Algorithm 3.1, but it is unclear if this would be worth the effort.

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Quivers, Gröbner bases, and tensor products

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