Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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Concerning some aspects of twisted generating functions

Master's thesis in Mathematical Sciences
Supervisor: Gereon Quick
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Abstract

The nearby Lagrangian conjecture predicts that any closed exact Lagrangian L in the cotangent bundle of a closed manifold M is Hamiltonian isotopic to the zero section. Via a theorem of Sikorav [4], the conjecture predicts that L admits a *generating function*. As shown by Giroux [11] and Latour [18], having a generating function is equivalent to the stable Gauss map of L being nullhomotopic.

A weaker class of objects, called *twisted generating functions*, were introduced in the recent paper [TGNL] by M. Abouzaid, S. Courte, S. Guillermo and T. Kragh. The authors prove an existence result for such twisted functions, and perform a doubling trick to be able to do Morse theory in this context. As shown by Abouzaid, Kragh, Fukaya, Seidel, Smith, Nadler and more recently Guillermo, the map induced by projecting in the cotangent bundle gives a homotopy equivalence $\pi : L \to M$. A Morse theoretic consequence of this equivalence is extracted, and this is combined with Böksted's theorem in algebraic K-theory to conclude that the stable Gauss map is trivial on homotopy groups.

This thesis aims to provide additional details, background material and alternative viewpoints to the the results in [TGNL]. The thesis has three main parts:

1. Properties of the classifying space $B(F, Q)$ associated to a topological monoid Q acting on a space F.
2. An alternative proof of the homotopy lifting property of transversal Lagrangians under symplectic reduction.
3. Details on the Morse theory of the twisted generating function and a spectral sequence argument.

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Contents

1 Introduction

The nearby Lagrangian conjecture is an important conjecture in symplectic topology which has recently inspired new techniques in the field. Recall that for any smooth n-manifold M, the cotangent bundle T^*M has a natural symplectic structure given by the derivative of the Liouville form λ. An immersion $l : L \to T^*M$ of another smooth n-manifold L is called Lagrangian if $l^*\lambda$ is a closed 1-form, and exact Lagrangian if $l^*\lambda$ is exact. When both M and L are closed manifolds, an exact Lagrangian embedding $l : L \to T^*M$ is called nearby Lagrangian. The nearby Lagrangian conjecture postulates that any nearby Lagrangian embedding is Hamiltonian isotopic to the zero section M_0 in T^*M. The status of this conjecture is far from settled. In fact it has only been shown when M is S^1, S^2, $\mathbb{R}P^2$ or T^2 [5], [2], [16]. The techniques used in these cases are uniquely 2-dimensional (Riemann mapping) or 4-dimensional (positivity of intersections), so they bear no hope of solving the conjecture for arbitrary M. The best general result so far states that restricting the bundle projection $\pi_M : T^*M \to M$ to the image of L gives a simple homotopy equivalence $\pi : L \to M$. This has both been shown using pseudoholomorphic techniques [1], [9], and more recently using microlocal sheaves [12]. Both of these proofs take the approach that to study the symplectic topology of T^*M, we should study some associated triangulated category. In the pseudoholomorphic approach, the associated category is some version of a derived Fukaya category $\mathbf{D}\mathsf{Fuk}(T^*M)$, and in the microlocal case one studies the derived category of sheaves $\mathbf{D}^b(R_{M\times\mathbb{R}})$. These two categories are related by the singular support functor which is almost an equivalence of categories [26].

The present thesis is meant as a companion piece to the recent paper [TGNL] by M. Abouzaid, S. Courte, S. Guillermo and T. Kragh, in which generating functions are used to study nearby Lagrangians. A generating function for an exact Lagrangian $L \in T^*M$ is a function $f : M \times \mathbb{R}^k \to \mathbb{R}$ such that the fiberwise critical set embeds into the cotangent bundle

$$\begin{aligned} \Sigma_f = \{(x,v) \in M \times \mathbb{R}^k \mid \partial_v f(x,v) = 0\} &\longrightarrow T^*M \\ (x,v) &\longmapsto (x, \partial_x f(x,v)) \end{aligned} \tag{1.1}$$

with image L. These objects are related to sheaves by taking the fiberwise homology of the sublevel sets of f (see [26, p. 9.1.2]), but seem to be more rigid than the corresponding sheaves. A fundamental motivation for studying generating functions for nearby Lagrangians is a classical result of Sikorav [4] which states that having a generating function is invariant under Hamiltonian isotopy. The zero section clearly admits a generating function, so the nearby Lagrangian conjecture predicts that any nearby Lagrangian also admits a generating function.

We will not cover all of [TGNL] in this thesis, but instead focus our energy on three areas where we feel that additional details and background material would be enlightening, or where we have found alternative viewpoints. These three areas define the three main sections of this thesis, and can be summed up as follows.

Section 2: The construction and properties of the simplicial space $B(-,Q)$ expanding on sections A.2 and A.3 of [TGNL].

Section 3: The proof of the homotopy lifting property for transversal Lagrangians under symplectic reduction, following sections 2.1 and 2.2 of [TGNL].

Section 4: The homology computation and related arguments from section 3.3 of [TGNL].

Each section has a distinct "flavour" due to the different techniques and inspirations at play. While we will provide some of the relevant context connecting these areas, many details that can be found in [TGNL] will be omitted. This structure might make for a somewhat disjointed experience if this thesis is read in isolation, which is why it should rather be read alongside with [TGNL]. In this introduction we will give some motivation to the study of twisted generating functions, before we give a roadmap of the three sections.

A starting point for the study of generating functions in [TGNL] is the following classical result proven independently by E. Giroux [11] and F. Latour [18].

Theorem 1.1 (Giroux–Latour)**.** *A nearby Lagrangian embedding $L \to T^*M$ admits a generating function if and only if the stable Gauss map $L \to \Lambda_0(\mathbb{C}^\infty)$*[1] *is nullhomotopic.*

The key to proving this theorem is a fibration result, which is essentially a linear version of the aforementioned result of Sikorav, and which we prove as Proposition 3.26. It states that we have a fibration (up to stabilization)

$$\mathcal{Q} \longrightarrow \Lambda^V(E) \longrightarrow \Lambda_0(E). \tag{1.2}$$

The two rightmost terms are Grassmanians consisting of linear Lagrangian subspaces of a symplectic vector space E, while $\mathcal{Q}$ is a topological monoid of quadratic forms. Roughly speaking, a lift to the space $\Lambda^V(E)$ is the linear information required to construct a generating function.

A key idea in [TGNL] is that the fibration (1.2) is in some sense $\mathcal{Q}$-equivariant. If we could mod out the action of $\mathcal{Q}$, we could turn this fibration into a homotopy equivalence, so that all maps, not only nullhomotopic maps, may be lifted. This requires coming up with a particular model for the homotopy quotient $\Lambda^V(E)/\mathcal{Q}$. The model we will use is the simplicial space $B(\Lambda^V(E), \mathcal{Q})$, a generalization of the bar construction used to construct classifying spaces BG of topological groups. The construction and properties of this model will be explained in section 2. Our contributions to this part of the material is mainly gathering up different results from the literature and giving accessible, rigorous proofs. The material corresponds to appendix A of [TGNL], and the techniques are mainly simplicial. For sake of convenience we recall some standard facts and establish some notation concerning simplicial sets and spaces in subsection 2.1.

In section 3 we will define the notion of twisted generating function. As we will see, the definition is closely related to the information contained in a map $M \to |B(\Lambda^V(\mathbb{C}^\infty), \mathcal{Q})|$. The properties of the bar construction proved in section 2 will allow us to show that the fibration (1.2) induces a homotopy equivalence

$$|B\left(\Lambda_\infty^{i\mathbb{R}^\infty}(\mathbb{C}^\infty), \mathcal{Q}\right)| \simeq \Lambda_0(\mathbb{C}^\infty).$$

[1]While U/O is a more standard notation for the stable Lagrangian Grassmanian, we use $\Lambda_0(\mathbb{C}^\infty)$ to stay consistent with later notation.

This has the following consequence, which can be seen as an adaptation of Theorem 1.1 to twisted generating functions.

Theorem 1.2. *An exact Lagrangian immersion $L \to T^*M$ admits a twisted generating function if and only if the stable Gauss map factors through the projection $\pi : L \to M$ up to homotopy.*

We will, as mentioned, reprove the fibration (1.2). Our proof is based on the proof given in [TGNL], but uses direct manipulation of Lagrangians and symplectic relations rather than quadratic forms. We will also provide some more details on the various homotopy equivalences involved, as well as provide a different proof for lemma 2.22 of [TGNL]. The techniques of this section are rooted in (symplectic) linear algebra.

Section 4 is concerned with the Morse homological consequenes of having a generating function. Using the fact that $\pi : L \to M$ is a homotopy equivalence for any nearby Lagrangian, it is clear that nearby Lagrangians admit twisted generating functions. This result, however, is not very sharp, since many immersed exact Lagrangians also admit twisted generating functions, see for instance figure 2. One way to exploit the hypothesis that L is embedded is the doubling trick, which is explained in section 3 of [TGNL]. This allows for the construction of a more rigid kind of generating function to which we can apply Morse theory. In particular we perform a computation involving a spectral sequence, resembling the one appearing in [9]. The main result is

Theorem 1.3. *For a nearby Lagrangian $L \to T^*M$, the s-double of L admits a twisted generating function f such that for all $x \in X$, 0 is a regular value of the function f_x and for all sufficiently large a, the homology of the fiberwise sublevel set is*

$$H_*(\{f_x \leq 0\}, \{f_x \leq -a\}) = \mathbb{Z}[d].$$

We provide a substantial amount of details omitted in [TGNL] and as mentioned we argue using spectral sequences and local systems rather than derived sheaves and derived functors. We consider our proof of Theorem 1.3 as one of the main original contributions of this thesis.

2 Classifying spaces and monoids

The theory of principal G-bundles for a topological group G is well known. As one can show, a principal G-bundle on a space X is determined by an open cover U_i of X, and a Čech cocycle of clutching functions $U_{ij} \to G$. The space BG is a classifying space for such cocycles in the sense that equivalence classes of cocycles are in one to one correspondence with homotopy classes of maps to BG. The punchline is that studying the topology of BG can teach us a lot about G-bundles. The goal of this section is to recover parts of this theory when we replace the topological group G with a topological monoid Q. We begin with specifying some notation and recalling some standard facts from the theory of simplicial spaces and sets, before we move on to constructing our classifying spaces.

2.1 Simplicial sets and spaces

Simplicial objects are objects that are built up from smaller objects fitting together like triangles and tetrahedra. To make this precise, we define the simplex category.

Definition 2.1. The simplex category Δ is the category whose objects are the ordered sets

$$[n] = \{0 < 1 < ... < n-1 < n\}$$

and whose morphisms are nondecreasing maps $\alpha : [n] \to [m]$.

There are two especially important classes of morphisms in Δ, namely the faces and degeneracies.

Definition 2.2. The ith face map is the unique nondecreasing injection

$$\partial_i^n : [n-1] \longrightarrow [n]$$

whose image does not contain i. The ith degeneracy is the unique nondecreasing surjection

$$s_i^n : [n+1] \longrightarrow [n]$$

hitting i twice.

To increase readability, we will omit the superscript. One can check explicitly (and tediously) that these maps satisfy the simplicial identities

$$\begin{aligned}
\partial_i \partial_j &= \partial_{j-1}\partial_i && \text{for } i < j \\
s_i s_j &= s_{j+1} s_i && \text{for } i \leq j \\
\partial_i s_j &= s_{j+1}\partial_i && \text{for } i < j \\
\partial_i s_i &= \partial_{i+1} s_i = id && \\
\partial_i s_j &= s_j \partial_{i-1} && \text{for } i > j+1.
\end{aligned} \tag{2.1}$$

Remark 2.3. The face and degeneracy maps are special in the sense that all other morphisms can be obtained by composing maps of these two types. We will not prove this here, but remark that the proof consists of factoring any nondecreasing map α as $\alpha = e \circ m$ for some surjection e and some injection m. Then one can write e as a product of s_i's and m as a product of ∂_i's.

Definition 2.4. A simplicial object E in a category C is a contravariant functor $E : \Delta \to C$. We will often write E_n for $E([n])$ and $\alpha^* : E_n \to E_m$ for $E(\alpha)$ where $\alpha : [m] \to [n]$ is a nondecreasing map.

This thesis will be mainly concerned with the cases where $C = \mathsf{Set}$ or $C = \mathsf{Top}$. Simplicial objects in these categories will be referred to as simplicial sets and simplicial spaces respectively. We will always view simplicial sets as special cases of simplicial spaces by interpreting a set as a discrete space.

Example 2.5. For a simplicial space X, we call X_0 the space of vertices of X. To access the ith vertex of an n-simplex we pull back by the map $v_i^n : [0] \to [n], v_i(0) = i$. As we can see, any n-simplex σ has $n+1$ (not necessarily distinct) vertices $v_0^*\sigma, v_1^*\sigma, ..., v_n^*\sigma$.

We call X_1 the edges of X. For $0 \leq i \leq j \leq n$, we have edge maps $e_{i,j}^n : [1] \to [n]$ sending 0 to i and 1 to j. The pullback with this map sends a simplex to the edge "going from the ith vertex to the jth vertex." Note that the edge inherits an orientation from the order on $[n]$.

Together, the edges and vertices of a simplicial set will form a directed graph. In many cases we will use this directed graph as a visualization of a simplicial set. In fact, the structures we wish to classify will turn out to only depend on what happens at the levels X_0, X_1 and X_2, so very little is lost in such visualizations.

One could equivalently define a simplicial object in C as a collection of objects indexed by the natural numbers, with specified face and degeneracy morphisms in C satisfying an analogue of the identities (2.1). In view of remark 2.3, any nondecreasing map can be described as a composition of the faces and degeneracies, so functionality on all maps is equivalent to functoriality on faces and degeneracies.

Definition 2.6. A simplicial map of simplicial spaces is a natural transformation $f : X \to Y$. In other words, we have for each $n \in \mathbb{N}$ a continuous map $f_n : X_n \to Y_n$, such that for each nondecreasing $\alpha : [n] \to [m]$ the following diagram commutes.

$$\begin{tikzcd} X_m \arrow[r, "f_m"] \arrow[d, "X(\alpha)"'] & Y_m \arrow[d, "Y(\alpha)"] \\ X_n \arrow[r, "f_n"'] & Y_n \end{tikzcd}$$

Our main purpose will be to use simplicial spaces and sets as combinatorial models for topological spaces. The idea of a geometric realization will make this explicit. First, we need to define what a geometric n-simplex is.

Definition 2.7. The geometric n-simplex is the topological space

$$\Delta^n = \left\{ (t_0, t_1, ..., t_n) \subset [0,1]^n \;\middle|\; \sum_{i=0}^{n} t_i = 1 \right\}$$

with the subspace topology. For any map $\alpha : [n] \to [m]$, there is a natural continuous map

$$\alpha_* : \Delta^n \longrightarrow \Delta^m$$
$$(t_0, ..., t_n) \longmapsto (s_0, ..., s_m)$$

where

$$s_j = \sum_{i \in \alpha^{-1}(j)} t_i.$$

The idea is now to turn a simplicial space into a topological space by associating each "abstract" n-simplex $\sigma \in E_n$ to a geometric n-simplex, and to use the face and degeneracy maps to glue these together.

Definition 2.8. For a simplicial space E, the geometric realization is the topological space

$$|E| = \left(\coprod_{n \in \mathbb{N}} E_n \times \Delta^n \right) \Big/ \sim,$$

where $(\alpha^* \sigma, t) \sim (\sigma, \alpha_* t)$ for all nondecreasing α. We denote the quotient map of this equivalence relation by r. The n-skeleton of the geometric realization is the image of the zero through n-simplices under this quotient map, namely

$$|E|_n = r \left(\coprod_{k=0}^{n} E_n \times \Delta^n \right).$$

Note that $E_n \times \Delta^n$ is equipped with the product topology, and that this definition works equally well for simplicial sets when we interpret a set as a discrete topological space. A simplicial map $f : X \to Y$ gives rise to a continuous map $|f| : |X| \to |Y|$ defined by $(\sigma, t) \to (f^* \sigma, t)$ which glues correctly due to the naturality condition on f.

While the degeneracies of a simplicial set give rise to a richer combinatorial structure which will be useful later, it can sometimes (especially when dealing with geometric realizations) be easier to get rid of them.

Definition 2.9. A simplex $\sigma \in E_n$ is called degenerate if it is in the image of f^* for some $f : [n] \to [m]$ with $m < n$. A simplex is called nondegenerate if it is not degenerate. We denote the space of nondegenerate n-simplices in E by E_n^{nd}.

This gives rise to a useful description of the underlying set of the geometric realization of a simplicial space.

Lemma 2.10. *There is a bijection*

$$\coprod_{n \in \mathbb{N}} E_n^{nd} \times \mathrm{int}(\Delta^n) \simeq |E|.$$

Proof. Any simplex factors through a unique maximal nondegenerate simplex. Points in $E_n^{nd} \times \mathrm{int}(\Delta^n)$ are not identified with any other points of this form under the quotient since the points in $\mathrm{int}(\Delta^n)$ are not in the image of any f_* where $f : [m] \to [n]$, $m < n$. □

We now give som illustrative examples of simplicial spaces and their geometric realizations.

Example 2.11. The combinatorial n-simplex is the simplicial set $\Delta[n] = \mathrm{Hom}_{\Delta}(-,[n])$. In other words,

$$\Delta[n]_k = \{f : [k] \longrightarrow [n] \text{ nondecreasing}\}$$

with morphisms given by precomposing. We leave it as an exercise to show that $|\Delta[n]| \simeq \Delta^n$, but remark that $\Delta[n]^{nd}_n = \{id_{[n]}\}$, and that any simplex clearly factors through this one.

Example 2.12. For a topological space X, we define a simplicial set $\mathrm{Sing}\,X$ by setting

$$\mathrm{Sing}\,X_n = \mathrm{Hom}_{\mathsf{Top}}(\Delta^n, X),$$

and letting $f : [m] \to [n]$ give rise to the map

$$f^* : (\Delta[n] \xrightarrow{\sigma} X) \longmapsto (\Delta[m] \xrightarrow{f_*} \Delta[n] \xrightarrow{\sigma} X).$$

A standard part of the theory is the fact that up to homotopy the functor Sing is adjoint to the geometric realization functor. In particular, the natural map

$$\begin{aligned} |\,\mathrm{Sing}\,X| &\longrightarrow X \\ (\sigma, t) &\longmapsto \sigma(t) \end{aligned}$$

is a weak homotopy equivalence [19, Proposition 16.2].

Example 2.13. Any topological space X can be turned in to a simplicial space by considering the constant functor valued at X. We call this the constant simplicial space at X, and denote it (somewhat abusively) by X. It has $X_n = X$ for all n, and all face and degeneracies given by the identity on X. The only nondegenerate simplices are $X_0 = X$, so it is easily seen that the geometric realization is just X.

Definition 2.14. A bisimplicial object in C is a contravariant functor

$$X : \Delta \times \Delta \longrightarrow C.$$

That is, for each $(n, m) \in \mathbb{N}^2$, an object $X_{n,m}$ in C, and for each pair of nondecreasing morphisms $f : [n] \to [k]$, $g : [m] \to [l]$ a morphism

$$(f, g)^* : X_{k,l} \longrightarrow X_{n.m}.$$

For a bisimplicial space X, there are three a priori different ways to take the geometric realization, corresponding to the following three simplicial spaces

$$\begin{aligned} \mathrm{Diag}\,X_n &= X_{n,n} & & \\ X^I_n &= |X_{\bullet,n}| & \qquad (X_{\bullet,n})_m &= X_{m,n} \\ X^{II}_n &= |X_{n,\bullet}| & \qquad (X_{n,\bullet})_m &= X_{n,m}. \end{aligned}$$

Fortunately taking geometric realizations of these three amount to the same thing, as made precise by the following result [23, p. 86].

Proposition 2.15. *For a bisimplicial space X, there are functorial homeomorphisms*

$$|\operatorname{Diag} X| \simeq |X^I| \simeq |X^{II}|.$$

We now give some examples of why this is useful.

Example 2.16. For two simplicial spaces X and Y, we define their product as the diagonal of the bisimplicial space $(X \otimes Y)_{n,m} = X_n \times Y_m$. We leave it as an exercise to check that this serves as a product in the category of simplicial spaces. Using Proposition 2.15 we see that

$$|X \times Y| = |X| \times |Y|.$$

Example 2.17. To replace a simplicial space X with a simplicial set, we first define the bisimplicial set Sing X by

$$\operatorname{Sing} X_{n,m} = (\operatorname{Sing} X_m)_n \operatorname{Hom}_{\mathsf{Top}}(\Delta^n, X_m),$$

and with maps given in an obvious way. Developing the geometric realization with m fixed yields $|Sing(X_m)| \sim X_m$, like in 2.12. One can ask if this levelwise weak equivalence is enough to combine with Proposition 2.15 to conclude that $|\operatorname{Diag}\operatorname{Sing} E| \to |E|$ is at least a weak equivalence. The following results (combined with CW-approximation) will show that under some mild conditions this holds.

Unsurprisingly we want to turn the weak equivalences we have encountered so far into homotopy equivalences. To achieve this, we work in the category of spaces homotopy equivalent to a CW-complex. Since it will make no difference to us, we do not distinguish between spaces that are actual CW-complexes and spaces that are just homotopy equivalent to CW-complexes; we refer to both as CW-complexes. It would be useful to have conditions for when the realization of a simplicial space is a CW-complex. Note that while the realization of a simplicial set must always be a CW-complex, the same cannot be true in general for simplicial spaces since we can take the constant simplicial space at some pathological space. One useful notion in this regard is that of a good simplicial space as defined in [24].

Definition 2.18. We call a simplicial space E good if for all $n \in \mathbb{N}$ and $0 \leq i \leq n-1$, the map $s_i : E_{n-1} \to E_n$ is a cofibration.

Lemma 2.19. *Let A and B be good simplicial spaces, and $f : A \to B$ a simplicial map such that all $f_n : A_n \to B_n$ are homotopy equivalences. Then the induced map on realizations is a homotopy equivalence $f : |A| \to |B|$.*

Proof. See [24, Appendix A]. □

We now use this to give a sufficient condition for E to have the homotopy type of a CW-complex.

Corollary 2.20. *Let E be a good simplicial space such that all E_n are CW-complexes. Then $|E|$ is a CW-complex.*

Proof. Consider the simplicial space $(SingE)^I_n = |Sing(E_n)|$. Since the inclusion of $\partial_i\Delta_n \hookrightarrow \Delta_n$ is a cofibration, this simplical space is good. Since E_n is a CW-complex, each component of the simplicial map $|\operatorname{Sing}(E_n)| \to E_n$ from Example 2.17 is a homotopy equivalence. Since both spaces are good, Lemma 2.19 and 2.15 yields a homotopy equivalence $|\operatorname{Diag}\operatorname{Sing} E| \simeq |E|$. Since $\operatorname{Diag}\operatorname{Sing} E$ is a simplicial set, its realization is a CW-complex, and so is $|E|$.

□

Another useful construction on simplicial sets is the cone, which is inspired by the familiar cone construction $CX = X \times [0,1]/X \times \{1\}$ on topological spaces. In fact we will see that there is homeomorphism $|CX| \to C|X|$.

Definition 2.21. The cone of a simplicial set X is given by

$$(CX)_n = \{(\sigma, k) \mid k = 0, ..., n, \ \ \sigma \in X_{n-k}\}$$

$$\begin{aligned}
\partial_i(\sigma, k) &= (\sigma, k-1) && \text{for } \ i < k \\
\partial_i(\sigma, k) &= (\partial_{i-k}, k) && \text{for } \ i \geq k \\
s_i(\sigma, k) &= (\sigma, k+1) && \text{for } \ i < k \\
s_i(\sigma, k) &= (s_{i-k}\sigma, k) && \text{for } \ i \geq k
\end{aligned}$$

Note that for the definition to make sense we will define $X_{-1} =*$ and for every vertex $v \in X_0$, $\partial_0 v =*$. Then the "tip" of the cone will be the vertex $(*, 1)$. This could of course be avoided by adding special definitions for $(CX)_0$, but this would arguably make notation less clear. The intuition here is that we are adding a single new vertex $(*, 1)$, and then using the number k to keep track of how many of the verticies degenerate to this vertex. It is not hard to see that the nondegenerate simplices are

$$(CX)^{Nd}_n = \left\{(\sigma, k) \mid k = 0, 1 \ \ \sigma \in X^{Nd}_{n-k}\right\}.$$

In particular, a nondegenerate n-simplex in CX is either a nondegenerate simplex in X, or it is "spanned" by a nondegenerate $n-1$-simplex and the tip $(*, 1)$.

Lemma 2.22. *$|CX|$ is contractible.*

Proof. We omit a proof, but remark that this can be seen using simplicial homotopy, or directly in the geometric realization. □

A particularly well behaved class of simplicial sets, inspired by the combinatorial simplices, is the class of directed simplicial sets.

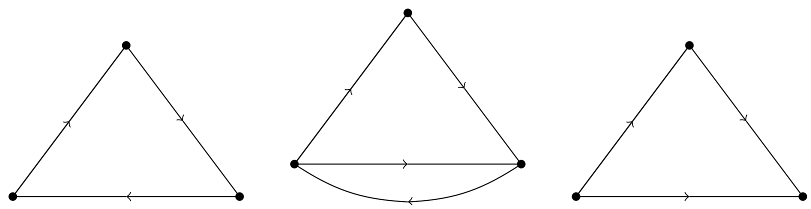
Definition 2.23. A simplicial set Z is called directed if the set Z_0 of verticies is equipped with a partial order $\leq$ such that for any simplex σ,

$$v_0^*\sigma \leq v_1^*\sigma \leq ... \leq v_n^*\sigma \tag{2.2}$$

and such that for all $n \in \mathbb{N}$, the map

$$\begin{aligned}
Z_n &\longrightarrow Z_0^{n+1} \\
\sigma &\longmapsto (v_0^*\sigma, v_1^*\sigma, ..., v_n^*\sigma)
\end{aligned}$$

is injective.



(a) Not directed (b) Not directed (c) Directed

Figure 1: Three (non)examples of directedness at the Z_1 level.

One of the nice things about directed simplicial sets is that a simplex $\sigma \in Z_n$ is nondegenerate if and only if

$$v_0^*\sigma < v_1^*\sigma < ... < v_n^*\sigma.$$

To see this, just assume that $v_i^*\sigma = v_{i+1}^*\sigma$. Then σ and $s_i\partial_{i+1}\sigma$ have the same verticies, so by injectivity they must be equal.

2.2 Classifying simplicial cocycles

Our construction of BQ will be based on the simplicial methods introduced in the previous subsection. We will start by coming up with a classifying object for *simplicial* cocycles, and then later translate these into Čech cocycles. We start by making the notion of a cocycle valued in a monoid precise.

Definition 2.24. A topological monoid is a topological space Q together with a continuous, associative binary operation $- \cdot - : Q \times Q \to Q$ with a two sided unit $e \in Q$. We call Q discrete if Q is a discrete topological space. We call Q commutative if $q_1 \cdot q_2 = q_2 \cdot q_1$ for all $q_1, q_2 \in Q$.

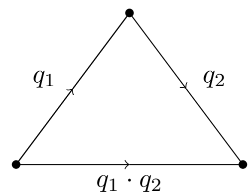
Definition 2.25. For a simplicial space E, and a topological monoid Q, a map $h : E_1 \to Q$ is called a 1-cocycle if for any $\sigma \in E_2$

$$h \circ e_{0,2}(\sigma) = h \circ e_{0,1}(\sigma) \cdot h \circ e_{1,2}(\sigma),$$

and for any $v \in E_0$

$$h \circ s_0(v) = e.$$

The following picture illustrates the graph of some 2-simplex. We will use labels on the edges to indicate the value of the edge under h.



We see that the first condition applied to a degenerate 2-simplex implies that

$$(h \circ s_0 \circ v_0) \cdot h = h = h \cdot (h \circ s_0 \circ v_1).$$

This shows that there is no conflict between the two requirements. The second requirement is mostly important to get uniqueness later. Note that there could be several elements in Q satisfying the above equation. We could equivalently have defined a cocycle as a map from the nondegenerate 1-simplices, and later extended to all E_1 by sending degenerate edges to e.

Lemma 2.26. *If $h : E_1 \to Q$ is a 1-cocycle, then for any $0 \leq j \leq k \leq n$*

$$h \cdot e^n_{j,k} = \prod_{j<i\leq k} h \circ e^n_{i-1,i}$$

as maps $E_n : \to Q$. Note that we take the empty product to equal the unit e.

Proof. We will prove this by induction on n. The base case is $n = 2$ where the equation is just the cocycle condition (note also that this takes care of the degenerate edges). We will further assume that $j = 0$ and $k = n$, since if not, we could factor through E_{k-j} by the injective map $f : [k-j] \to [n]$ with image $j, j+1, ..., k-1, k$. This satisfies

$$\begin{aligned} e^n_{j,k} &= f^* \circ e^{k-j}_{0,k-j} \\ e^n_{j+i-1,j+i} &= f^* \circ e^{k-j}_{i-1,i}, \end{aligned}$$

so the result for $n' = k - j$ already takes care of this case. We therefore consider the map

$$\begin{aligned} g : [2] &\longrightarrow [n+1] \\ 0 &\longmapsto 0 \\ 1 &\longmapsto n \\ 2 &\longmapsto n+1. \end{aligned}$$

One can now check the following factorizations

$$\begin{aligned} e^{n+1}_{0,n+1} &= e^2_{0,2} \circ g^* \\ e^{n+1}_{0,n} &= e^2_{0,1} \circ g^* \\ e^{n+1}_{n,n+1} &= e^2_{1,2} \circ g^*. \end{aligned}$$

Applying the coycle condition then yields

$$
\begin{aligned}
h \circ e^{n+1}_{0,n+1} &= h \circ e^2_{0,2} \circ g^* \\
&= ((h \circ e^2_{0,1}) \cdot (h \circ e^2_{1,2})) \circ g^* \\
&= (h \circ e^{n+1}_{0,n}) \cdot (h \circ e^{n+1}_{n,n+1}) \\
&= \prod_{0<i\leq n} h \circ e^{n+1}_{i-1,i} \cdot h \circ e^{n+1}_{n,n+1} = \prod_{0<i\leq n+1} h \circ e^{n+1}_{i-1,i},
\end{aligned}
$$

which completes the induction step. □

We now want to come up with a simplicial space BQ such that simplicial maps $E \to BQ$ are in one to one correspondence with 1-cocycles $h\colon E_1 \to Q$. Clearly we should let $BQ_1 = Q$, but how should we define BQ_n? Lemma 2.26 shows that the value of a cocycle h on the edges of an n-simplex σ is determined by the value of h on the n consecutive edges $e_{i-1,i}\sigma$. The idea is to let an n-simplex in BQ also be uniquely determined by n elements in Q corresponding to the n-consecutive edges. The maps will be chosen in a way that enforces the cocycle condition.

Definition 2.27. For any topological monoid Q, let BQ be the simplicial space with objects

$$(BQ)_n = Q^n$$

and maps

$$
\begin{aligned}
\alpha^*\colon Q^n &\longrightarrow Q^n \\
(q_1, ..., q_n) &\longmapsto (p_1, ..., p_m),
\end{aligned}
$$

where

$$p_k = \prod_{\alpha(k-1)<j\leq\alpha(k)} q_j.$$

(Again, empty products equal e.) Equivalently we could define the faces and degeneracies

$$
\begin{aligned}
\partial_k(q_1, ..., q_n) &= (q_1, ..., q_{k-1}, q_k \cdot q_{k+1}, q_{k+2}, ...q_n) && 1 \leq k \leq n-1 \\
\partial_0(q_1, ..., q_n) &= (q_2, ..., q_n) \\
\partial_n(q_1, ..., q_n) &= (q_1, ..., q_{n-1}) \\
s_k(q_1, ..., q_n) &= (q_1, ..., q_k, e, q_{k+1}, ..., q_n) && 0 \leq k \leq n.
\end{aligned}
$$

We leave it as an exercise to check that these two definitions agree, and that they satisfy the simplicial identities.

The upshot is that this definition accomplishes exactly what we wanted it to.

Proposition 2.28. *A 1-cocycle $h\colon E_1 \to Q$ extends uniquely to a simplicial map $H\colon E \to BQ$.*

Proof. For existence, define the extension

$$H_n = (H_n^1, ..., H_n^n) : E_n \longrightarrow Q^n$$

by $H_n^k = h \circ e_{k-1,k}^n$. To show that this map is simplicial, consider any nondecreasing $\alpha : [n] \to [m]$. We need to show that the following diagram commutes.

$$\begin{tikzcd} E_m \arrow[r, "H_m"] \arrow[d, "\alpha^*"'] & Q^m \arrow[d, "\alpha^*"] \\ E_n \arrow[r, "H_n"'] & Q^n \end{tikzcd}$$

One can check explicitly that $e_{k-1,k}^n \circ \alpha^* = e_{\alpha(k-1),\alpha(k)}^m$, so by Lemma 2.26,

$$\begin{aligned} H_n^k \circ \alpha^* &= h \circ e_{k-1,k}^n \circ \alpha^* \\ &= h \circ e_{\alpha(k-1),\alpha(k)}^m \\ &= \prod_{\alpha(k-1)<j\leq\alpha(k)} h \circ e_{j-1,j}^m . \end{aligned}$$

By definition of $\alpha^* : Q^m \to Q^n$, we see that this is precisely the kth component of $\alpha^* \circ H_m$, so the diagram commutes.

For uniqueness, assume we have a simplicial map $G : E \to BQ$ with $G_1 = h$. Then for each $0 \leq k \leq n$, the following diagram commutes.

$$\begin{tikzcd} E_n \arrow[r, "G_n"] \arrow[d, "e_{k-1,k}^n"'] & Q^n \arrow[d, "e_{k-1,k}^n"] \\ E_1 \arrow[r, "h"'] & Q \end{tikzcd}$$

One can easily check that in BQ, the map $e_{k-1,k}^n$ corresponds to projection to the kth factor. Hence $G_n^k = h \circ e_{k-1,k}^n = H_n^k$, so $H = G$. □

2.3 Associated bundles

We are now ready to move on to associated bundles. We start with an example from the theory of G-bundles.

Example 2.29. Let $F \to E \to X$ be a bundle with structure group G. This means that G acts continuously on the fiber F. The clutching functions of E determine a principal G-bundle P, and we can equivalently write E as $P \times_G F$. The principal bundle P is defined by an open cover U_i of X, and clutching functions $g_{ij} : U_{ij} \to G$. Over each U_i, a local section of E is just a map $f_i : U_i \to F$. A collection of such local sections can be glued into a global section if and only if we have

$$f_j(x) = f_i(x) \cdot g_{ij}(x) \ \ \forall x \in U_{ij}.$$

Even though we have not actually defined total spaces and sections for Q-bundles, we want to generalize the above by coming up with a classifying space for such sections.

Definition 2.30. Denote a continuous right-action of Q on a space F by.

$$\begin{aligned} F \times Q &\longrightarrow F \\ (x, q) &\longmapsto x \cdot q, \end{aligned}$$

A Q-twisted map to F from a simplicial space E is given by a map

$$f : E_0 \longrightarrow F$$

and a 1-cocycle

$$h : E_1 \longrightarrow Q,$$

such that for any edge $\sigma \in E_1$,

$$f \circ v_1(\sigma) = f \circ v_0(\sigma) \cdot h(\sigma).$$

The following picture shows the values on the graph of a 2-simplex.

$$\begin{tikzcd} & x \cdot q_1 \arrow[dr, "q_2"] & \\ x \arrow[ur, "q_1"] \arrow[rr, "q_1 \cdot q_2"'] & & x \cdot q_1 \cdot q_2 \end{tikzcd}$$

The construction of the classifying space for such structures is very similar to the construction of BQ, except that we use the vertices to keep track of where in F we are. Since the value at vertices with inbound edges are determined by the above relations, we only need to keep track of a single point in F which will correspond to the value of f at the 0th vertex.

Definition 2.31. For any topological monoid Q, and right Q space F, let $B(F, Q)$ be the simplicial space with objects

$$B(F, Q)_n = F \times Q^n,$$

and maps

$$\begin{aligned} \alpha^* : F \times Q^n &\longrightarrow F \times Q^m \\ (x; q_1, ..., q_n) &\longmapsto (x'; p_1, ..., p_m), \end{aligned}$$

where

$$p_k = \prod_{\alpha(k-1) < j \leq \alpha(k)} q_j \quad \text{and} \quad x' = x \cdot \prod_{1 \leq j \leq \alpha(0)} q_j.$$

Equivalently we could define the faces and degenracies

$$\partial_k(x; q_1, ..., q_n) = (x; q_1, ..., q_{k-1}, q_k \cdot q_{k+1}, q_{k+2}, ...q_n) \qquad 1 \le k \le n-1$$
$$\partial_0(x; q_1, ..., q_n) = (x \cdot q_1; q_2, ..., q_n)$$
$$\partial_n(x; q_1, ..., q_n) = (x; q_1, ..., q_{n-1})$$
$$s_k(x; q_1, ..., q_n) = (x; q_1, ..., q_k, e, q_{k+1}, ..., q_n).$$

We leave it as an exercise to check that these two definitions agree, and that they satisfy the simplicial identities. It is also an instructive exercise to see how these identities play out in the above picture of a 2-simplex.

Remark 2.32. When the space F is discrete, we can think of $B(F, Q)$ is as the nerve of a certain category. Define $C(F, Q)$ to be the category whose objects are elements of F, and whose morphisms are given by

$$\mathrm{Hom}_{C(F,Q)}(x, y) = \{q \in Q \mid y = x \cdot q\}.$$

Then $B(F, Q)$ is precisely the *nerve* of $C(F, Q)$, and $|B(F, Q)|$ is the classifying space of $C(F, Q)$.

As for BQ we get the desired classification property pretty much by construction.

Proposition 2.33. *A Q-twisted map to F*

$$f : E_0 \longrightarrow F$$
$$h : E_1 \longrightarrow Q$$

determines a unique simplicial map $H : E \to B(F, Q)$ with $H_0 = f$, $H_1 = f \times h$.

Proof. We define a map

$$H_m = (H_m^0, H_m^1, ..., H_m^m) : E_m \longrightarrow F \times Q^m$$

by

$$H_m^k = \begin{cases} h \circ e_{k-1,k}^m & 1 \le k \le m \\ f \circ v_0 & k = 0 \end{cases}.$$

For any nondecreasing $\alpha : [n] \to [m]$, we need to show that the following diagram commutes.

$$\begin{array}{ccc} E_m & \xrightarrow{H_m} & F \times Q^m \\ {\scriptstyle \alpha^*}\downarrow & & \downarrow{\scriptstyle \alpha^*} \\ E_n & \xrightarrow[H_n]{} & F \times Q^n \end{array}$$

On the n-copies of Q the proof that this commutes is exactly the same as the proof for BQ. For the F factor, we have

$$(f \circ v_0) \cdot \left(\prod_{1 \le k \le \alpha(0)} (h \circ e_{j-1,j}) \right) = (f \circ v_0) \cdot (h \circ e_{0,\alpha(0)}^m) = f \circ v_{\alpha(0)},$$

where we have used both Lemma 2.26 and the twisting condition. Unwinding the definitions, this shows that the diagram commutes. As in the proof of Proposition 2.28, uniqueness can be checked by considering the commutative diagrams obtained by taking $\alpha = e^n_{k-1,k}$ and $\alpha = v^n_0$. □

We are now ready to state and prove some results concerning the homotopy theory of $|B(F,Q)|$. In many ways these result generalize appropriate results for classifying spaces for groups. The first result is a good pointer that in many ways, $|B(F,Q)|$ is a homotopy model for the quotient of F by the action of Q.

Lemma 2.34. *The space $|B(Q,Q)|$ is contractible.*

Proof. The idea for this proof will be to define a Q-twisted map to Q on $B(Q,Q) \times \Delta[1]$, so that when we extend this bundle as in Lemma 2.33. We then get a simplicial map $B(Q,Q) \times \Delta[1] \to B(Q,Q)$ which restricts to the identity and a constant map on opposite ends of $\Delta[1]$. In view of Example 2.16, passing to geometric realizations this map will provide a contraction of $|B(Q,Q)|$.

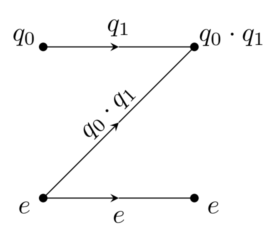
We have $(B(Q,Q) \times \Delta[1])_i = B(Q,Q)_i \times \Delta[1]_i$. We only need to specify values at $i = 0$ and $i = 1$. $\Delta[1]_0$ consists of the two vertices 0 and 1, which are explicitly the obvious constant maps $[0] \to [1]$. $\Delta[1]_1$ consists of three edges, namely two degenerate edges $s_0(0)$ and $s_0(1)$ coming from the constant maps $[1] \to [1]$, and the nondegenerate edge Δ_1 coming from $id : [1] \to [1]$. We now define our twisted cocycle by

$$\begin{aligned} f : Q \times \Delta[1]_0 &\longrightarrow Q \\ (q_0, 0) &\longmapsto e \\ (q_0, 1) &\longmapsto q_0 \end{aligned}$$

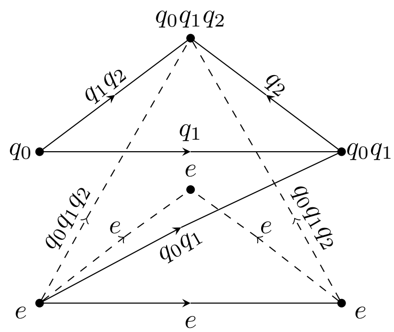
and

$$\begin{aligned} h : Q^2 \times \Delta[1]_1 &\longrightarrow Q \\ (q_0, q_1, s_0(0)) &\longmapsto e \\ (q_0, q_1, s_0(1)) &\longmapsto q_1 \\ (q_0, q_1, \Delta_1) &\longmapsto q \cdot q_1. \end{aligned}$$

The following picture illustrates the above definition at the subset $\{(q_0, q_1)\} \times \Delta[1]_1$.



As we can see from this graph, our definition satisfies the twisting condition. The two "missing edges" in this picture are covered by the definition applied to the degenerate edges $s_0 \circ v_0(q_0, q_1) = (q_0, e)$ and $s_0 \circ v_1(q_0, q_1) = (q_0 \cdot q_1, e)$. To check the cocycle condition, we similarly consider $\{(q_0, q_1, q_2)\} \times \Delta[1]_2$. Here we have to deal with the four possible maps $[2] \to [1]$. The following picture shows the graphs of these four 2-cells.



Each of the rectangular sides of the triangular prism is constructed from the definition of f and h at each of the edges of the 2-cell. As is apparent from the picture, the cocycle condition is satisfied on all four triangles. We omit a symbolic proof of this, but remark that this technique is inspired by the definition of simplicial homotopy. For a more detailed reference on simplicial homotopy, we refer to [19].

By Proposition 2.33 we can uniquely extend this cocycle to a simplicial map $B(Q, Q) \times \Delta[1] \to BQ$. It should be clear that restricting this map to 1 gives the identity on $B(Q, Q)$, while restricting to 0 gives a constant map. $\square$

The analogue of $B(F, Q)$ for groups is the associated bundle $EG \times_G F$. It is naturally a fiber bundle

$$F \to EG \times_G G \to BG.$$

This also explains the above result since

$$EG \times_G G \cong EG \simeq * \, .$$

With monoids, we do not have enough structure to get a fiber bundle, not even a fibration. However, under certain assumptions we get the weaker notion of a quasifibration.

Definition 2.35. A quasifibration $p\colon E \to B$ is a continuous map to a path connected B such that for all $b \in B$, $x_0 \in p^{-1}(b)$ and $i \in \mathbb{N}$, the induced homomorphism

$$p_*\colon \pi_i(E, p^{-1}(b), x_0) \longrightarrow \pi_i(B, b)$$

is an isomorphism.

All fibrations are quasifibrations, but not conversely. However one can show that a quasifibration gives rise to a long exact sequence in homotopy groups which is equivalent to the regular fibration sequence when the map is an actual fibration. Relative homotopy groups have some Mayer–Vietoris like properties which allows one to prove the following lemma.

Lemma 2.36. *A map $p\colon E \to B$ is a quasifibration if and only if any of the following conditions are satisfied*

a) *B can be decomposed as a union of open sets V_1 and V_2 such that the restrictions of p to $p^{-1}(V_1)$, $p^{-1}(V_2)$ and $p^{-1}(V_1 \cap V_2)$ are all quasifibrations.*

b) *B can be decomposed as the union of an increasing sequence of subspaces $B_1 \subset B_2 \subset ...$ with the property that any compact subspace $K \in B$ is contained in some B_n and such that the restriction of p to each $p^{-1}(B_n)$ is a quasifibration.*

c) *There is a deformation F_t of E into a subspace E' covering a deformation f_t of B into a subspace B' such that p restricted to E' is a quasifibration and such that the restriction $F_1 : p^{-1}(b) \to p^{-1}(f_1(b))$ is a weak equivalence.*

Proof. See [15, Lemma 4K.3]. □

The following lemma highlights one of the difficulties of working with monoids rather than groups. For a group G acting on F, any $g \in G$ induces a homeomorphism $F \xrightarrow{\cdot g} F$ since a continuous inverse is given by acting with g^{-1}. This is not necessarily the case for monoid actions, which is why we need to include extra assumption that $F \xrightarrow{\cdot q} F$ is a homotopy equivalence.

Lemma 2.37. *Let $T \to E \xrightarrow{p} F$ be a Q-equivariant quasifibration of right Q-spaces E and F. If for all $q \in Q$, acting with q induces weak equivalences on E and F, then the induced map*

$$|B(E,Q)| \longrightarrow |B(F,Q)|$$

is a quasifibration with fiber T.

Proof. This proof is a very modest generalization of [14, Lemma D.1]. To identify the fiber, we decompose the geometric realizations as in Lemma 2.10. In these coordinates, the induced map is

$$(e, q_1, ..., q_n, t_0, ..., t_n) \longmapsto (p(e), q_1, ..., q_n, t_0, ..., t_n) \tag{2.3}$$

so the fiber over any point is homeomorphic to T. To prove the map is a quasifibration, we will use Lemma 2.36. To begin the proof, decompose $|B(Q,F)|$ as the increasing union of skeleta $|B(Q,f)|_0 \subset |B(Q,F)|_1 \subset ....$ By 2.36 b), it suffices to show that each $|B(E,Q)|_n \to |B(F,Q)|_n$ is a quasifibration. Assume for induction that we have a quasifibration at the $n-1$-skeleta. If we fix an ε-neighborhood U of $\partial\Delta^n$ in Δ^n, we can set

$$V_1 = r(B(F,Q)_n \times U)$$
$$V_2 = r(B(F,Q)_n \times \mathrm{int}(\Delta^n)).$$

Then V_1 and V_2 are open by def of the quotient topology, and $V_1 \cup V_2 = |B(F,Q)|_n$. Over V_2 and $V_1 \cap V_2$, the induced map is just $p \times id$ as in (2.3), so here our map is a quasifibration by assumption. By 2.36.*a*) it now suffices to prove that p is a quasifibration over V_2. For sufficiently small ε, this neighborhood, and a corresponding neighborhood of $|B(E,Q)|_{n-1}$ in $|B(E,Q)|_n$ deform onto the respective $n-1$-skeleta, over which p

is a quasifibration by the induction hypothesis. Therefore it suffices by 2.36 c) to prove that these retractions induce a weak equivalence

$$F_1 : p^{-1}(b) \longrightarrow p^{-1}(f_1(b)).$$

In our representation, the retraction happens only in the simplicial coordinates $t_0, ..., t_n$. At f_1, at least one of these are zero. This point is identified with $(\alpha^*(f, q_1, ..q_n), t'_0, ..., t'_k)$ for a unique nondecreasing $\alpha : [k] \to [n]$ such that all $t'_i \neq 0$. Importantly, the same α applies in $|B(E, Q)|$, so for any $e \in p^{-1}(b)$

$$F_1(e, q_0, ..., q_n, t_0, ..., t_n) = (\alpha^*(e, q_0, ..., q_n), t'_0, ..., t'_k).$$

Since the effect of α on the q_k's is the same in $|B(E, Q)|$ and $|B(F, Q)|$, and the effect of α on e and f is acting on the right with a fixed $q \in Q$, we have up to canonical identification that

$$F_1 = \cdot q : p^{-1}(f) \longrightarrow p^{-1}(b \cdot q).$$

Now consider the map of quasifibration sequences

$$\begin{tikzcd} p^{-1}(b) \arrow[r, hook] \arrow[d, "\cdot q"] & E \arrow[r, "p"] \arrow[d, "\cdot q"] & F \arrow[d, "\cdot q"] \\ p^{-1}(b \cdot q) \arrow[r, hook] & E \arrow[r, "p"] & F \end{tikzcd}$$

Since the two right hand maps are weak equivalences by assumption, applying the five lemma to the long exact sequence of homotopy groups shows that F_1 is a weak equivalence. □

We now give some immediate consequences.

Corollary 2.38. *Let F be a right Q space. If for all $q \in Q$, the map $\cdot q : Q \to Q$ is a weak equivalence, then there is a quasifibration*

$$F \longrightarrow |B(F, Q)| \longrightarrow |BQ|.$$

Proof. It is easily seen from the definition that $|BQ|$ is just $|B(*, Q)|$, and interpreting $*$ as a trivial Q space, Lemma 2.37 applied to the equivariant fibration $F \to *$ gives the result. □

Even for CW-complexes, the notion of a quasifibration is strictly weaker than a Serre fibration. However, knowing that the spaces involved are CW-complexes allows us to conclude that each fiber is homotopy equivalent to the homotopy fibre. As we will soon show, this can be quite useful. We begin by stating sufficient conditions for $|B(F, Q)|$ to be a CW-complex. All the spaces appearing in subsequent sections will satisfy the hypothesis of this lemma, therefore we will also assume this hypothesis throughout the remainder of this section.

Lemma 2.39. *If F is a CW-complex and Q is a locally finite CW-complex, then $|B(F, Q)|$ is also a CW-complex.*

Proof. The goal is to satisfy the hypothesis of Corollary 2.20. Since Q is locally finite, all the products $F \times Q^n$ are CW-complexes. It remains to show that all the maps

$$F \times Q^{n-1} \xrightarrow{s_i} F \times Q^n \tag{2.4}$$

$$(x, q_1, ..., q_{n-1}) \longmapsto (x, q_1, ..., q_{i-1}, e, q_i, ..., q_{n-1}) \tag{2.5}$$

are closed cofibrations. The closed part is obvious since $\{e\}$ is closed. Since Q is a CW-complex, the inclusion point e is a neighbourhood deformation retract (NDR). Hence there is a neighbourhood N and a homotopy $X \times I \to X$ whose restriction to N takes id_N to the constant map at e. Extending this homotopy trivially on the other factors gives an obvious deformation retraction

$$F \times Q^{i-1} \times N \times Q^{n-i-1} \longrightarrow F \times Q^{i-1} \times \{e\} \times Q^{n-i-1}.$$

This shows that s_i is equivalent to the inclusion of a NDR, so a cofibration. □

Corollary 2.40. *Let F and F' be right Q-spaces. If $F \to F'$ is a Q-equivariant homotopy equivalence, then the induced map $|B(F, Q)| \to |B(F', Q)|$ is a homotopy equivalence.*

Proof. A homotopy equivalence is just a fibration with contractible fiber. By Lemma 2.37 we get a quasifibration

$$* \longrightarrow B(F, Q) \longrightarrow B(F', Q).$$

Passing to the exact sequence of homotopy groups, we see that the induced map gives isomorphism on all homotopy groups, so by Whitehead's theorem it is a homotopy equivalence. □

We also prove the following two technical lemmas concerned with the interaction of certain categorical constructions and the classifying space construction.

Lemma 2.41. *Let Q be a topological monoid. Let $F = \operatorname{colim}\big(F_1 \to F_2 \to ...\big)$ be a colimit of right Q-spaces where each map in the colimit is Q-equivariant. If all F_k and Q are compactly generated Hausdorff spaces, then $|B(F, Q)| \cong \operatorname{colim}\big(|B(F_1, Q)| \to |B(F_2, Q)| \to ...\big)$.*

Proof. The proof is purely formal. Since the product functor is a left adjoint in the category of CGHD spaces, it preserves colimits. Hence we have for all n that

$$F \times Q^n \times \Delta^n \cong \operatorname{colim}\big(F_1 \times Q^n \times \Delta^n \longrightarrow F_2 \times Q^n \times \Delta^n \longrightarrow ...\big).$$

To construct the geometric realization from these spaces, all we need is to take pushouts. The pushouts are over attatching maps coming from simplicial relations. The Q-equivariance of the maps $F_k \to F_{k+1}$ implies that thesee colimits commute. □

Lemma 2.42. *Given a cospan of topological spaces $D \xrightarrow{g} \Lambda \xleftarrow{\rho} F$ where F is a right Q-space and ρ is Q-invariant, we have a natural Q-action on the pullback g^*F and a canonical isomorphism*

$$g^*|B(F, Q)| \cong |B(g^*F, Q)|$$

Proof. Recall that the pullback can be constructed as

$$g^*F = \{(x, f) \subset D \times F \mid g(x) = \rho(f)\}$$

with the obvious projections to F and D. Now we define a right action of Q on g^*F by setting $(x.f)\cdot q = (x, f \cdot q)$. This is well defined since ρ is invariant. It is now easily seen that the projection to F is equivariant with respect to the actions, while the projection to D is invariant. This means that they descend to maps making the following diagram commute

$$\begin{array}{ccc} |B(g^*F, Q)| & \longrightarrow & |B(F, Q)| \\ \downarrow & & \downarrow{\scriptstyle \rho} \\ D & \xrightarrow[g]{} & \Lambda \end{array}$$

So we see that $|B(g^*F, Q)|$ comes with the required maps to satisfy the universal property of $g^*|B(F, Q)|$. It remains to show that any other such diagram factors through the above. This should be apparent by considering the disjoint union composition of the geometric realizations from Lemma 2.10, and we omit this part of the proof.

□

2.4 Monoid maps and group completions

This section is not essential for the following. Its main purpose was to verify the claim $B\mathbb{Z} \simeq B\mathbb{N}$ made in Lemma 2.22 of [TGNL]. While we prove this in the present section, we ended up using an alternative proof for Lemma 2.22 which does not rely on this result. We chose to include this section anyways since it provides some intuition.

For discrete groups G it is well known that $|BG|$ is an Eilenberg–MacLane space of type $K(G, 1)$, i.e. a CW-complex with all homotopy groups trivial except for $\pi_1(K(G, 1)) = G$. From a monoid M, one can construct a group completion M^* with a canonical map $\phi\colon M \to M^*$. It is natural to ask what the relationship between $|BM|$ and $|BM^*|$ is. It was in fact conjectured that these should be homotopy equivalent. It turns out that the map is at least a 1-equivalence, but also that any finite homotopy type can be realized as $|BM|$ for some discrete monoid M. For instance there exists a five element monoid P with $|BP| \simeq S^2$, which can clearly not be a $K(P^*, 1)$ [8]. However the equivalence $|BM| \simeq |BM^*|$ will hold in some favourable cases which we will explore in this subsection.

Consider a map $f\colon N \to M$ of topological monoids. By first applying f and then multiplying, we get an action of M on N. Our first goal will be to prove that (under certain homotopy equivalence conditions as before) we have a quasifibration

$$|B(N, M)| \longrightarrow |BM| \longrightarrow |BN|.$$

To prove this result we will exploit the fact that N acts on M both on the left and on the right. These two actions commute by associativity. To exploit both these actions we will need to define a version of $B(F, Q)$ for left actions. To distinguish between left and right we temporarily denote $F \mathbin{/\!/} Q :\ = B(F, Q)$.

Definition 2.43. For a left Q space F, let $Q \mathbin{\backslash\!\backslash} F$ be the simplicial space given by $Q \mathbin{\backslash\!\backslash} F_n = Q^n \times F$ with maps

$$\begin{aligned}
&\partial_k(q_1, ..., q_n; x) = (q_1, ..., q_{k-1}, q_k \cdot q_{k+1}, q_{k+2}, ...q_n; x) \qquad 1 \leq k \leq n-1\\
&\partial_0(q_1, ..., q_n; x) = (q_2, ..., q_n; x)\\
&\partial_n(q_1, ..., q_n; x) = (x; q_1, ..., q_{n-1}; q_n \cdot x)\\
&s_k(q_1, ..., q_n; x) = (q_1, ..., q_k, e, q_{k+1}, ..., q_n; x).
\end{aligned}$$

We can immediately see that with this definition we have $Q \mathbin{\backslash\!\backslash} * \simeq BQ$. It is also clear that most of our results about $|B(F, Q)|$ have direct analogues in for $Q \mathbin{\backslash\!\backslash} F$. For instance, $|Q \mathbin{\backslash\!\backslash} Q|$ is contractible, and under obvious assumptions we have a quasifibration $T \to |Q \mathbin{\backslash\!\backslash} E| \to |Q \mathbin{\backslash\!\backslash} F|$.

If a space F has commuting actions of M on the left and N on the right, we get an induced left action of M on $|F /\!/ N|$ defined at the simplicial level by

$$\begin{aligned}
F \times N^k &\xrightarrow{m\cdot} F \times N^k\\
(f, n_1, ..., n_k) &\longrightarrow (m \cdot f, n_1, ..., n_k)
\end{aligned}$$

for all $m \in M$. Likewise we get an induced right action of N on $|M \mathbin{\backslash\!\backslash} F||$.

Lemma 2.44. *The spaces $|M \mathbin{\backslash\!\backslash} |F /\!/ N||$ and $||M \mathbin{\backslash\!\backslash} F| /\!/ N|$ are naturally homeomorphic.*

Proof. These can be seen as the two of the different realizations of the bisimplicial space

$$(M \mathbin{\backslash\!\backslash} F /\!/ N)_m, n = M^m \times F \times N^n$$

with the obvious maps. This is well defined since the actions of M and N commute. Proposition 2.15 then gives the result. □

One application of this is the following

Proposition 2.45. *If $f : M \to N$ is a continuous map of monoids and for all $m \in M$, the action map $|B(N, M)| \xrightarrow{m\cdot} |B(N, M)|$ is a homotopy equivalence, then we have a quasifibration*

$$|B(N, M)| \longrightarrow |BM| \xrightarrow{|Bf|} |BN|.$$

If $|B(N, M)|$ is contractible, $|Bf|$ is a homotopy equivalence.

Proof. Consider the following commutative diagram.

$$\begin{array}{ccccc}
 & & |BM| & \longrightarrow & |BN| \\
 & & \downarrow & & \downarrow \\
 & & ||N \mathbin{\backslash\!\backslash} N| /\!/ M| & \longrightarrow & ||N \mathbin{\backslash\!\backslash} N| /\!/ N| \\
 & & \downarrow & & \downarrow \\
 & & |N \mathbin{\backslash\!\backslash} |N /\!/ M|| & \longrightarrow & |N \mathbin{\backslash\!\backslash} |N /\!/ N|| \\
 & & \downarrow & & \downarrow \\
|N /\!/ M| & \longrightarrow & |N \mathbin{\backslash\!\backslash} |N /\!/ M|| & \longrightarrow & |BN|
\end{array}$$

All the vertical maps are homotopy equivalences, and the bottom row is a quasifibration by Lemma 2.37. In the case where $|N \mathbin{/\mkern-6mu/} M|$ is contractible, we get a levelwise homotopy equivalence in the third row. □

To use this result we will now develop sufficient conditions for $|B(M, N)|$ to be contractible. Our motivation for this will be to show that the inclusion $\mathbb{N} \to \mathbb{Z}$ induces equivalence $|B\mathbb{N}| \to |B\mathbb{Z}|$. We will use a couple monoid theoretic definitions.

Definition 2.46. A monoid M is called left cancellative if for all $m, m_1, m_2 \in M$

$$m \cdot m_1 = m \cdot m_2 \implies m_1 = m_2,$$

and right cancellative if for all $m, m_1, m_2 \in M$

$$m_1 \cdot m = m_2 \cdot m \implies m_1 = m_2.$$

Definition 2.47. A submonoid $N \subset M$ is called filtering if for any $m_1, m_2 \in M$ there exists $m \in M$ and $n_1, n_2 \in N$ such that

$$m_1 = m \cdot n_1 \text{ and } m_2 = m \cdot n_2.$$

The goal of the next lemmas will be to prove that when $N \subset M$ is a filtering submonoid of a left cancellative discrete monoid, $|B(M, N)|$ is contractible. One way to prove this is to note that these conditions are exactly the ones required for $\mathcal{C}(M, N)$ to be a filtered category. It is a standard result of the classifying spaces of categories that this is contractible. Here we present a less categorical proof of the same fact building on the theory we already have.

Lemma 2.48. *Let E be a simplicial set with E_0 finite and let M be a left cancellative monoid with a filtering submonoid N. Then any simplicial map*

$$f : E \longrightarrow B(M, N) \tag{2.6}$$

extends to a simplicial map

$$\tilde{f} : CX \longrightarrow B(M, N).$$

Proof. In this proof we will repeatedly use Proposition 2.33, and work with N-twisted M-bundles on E instead of simplicial maps. The moral of the proof is that the filtering property of $N \subset M$ will allow us to connect more and more vertices to the tip of the cone, while left cancellation in M will ensure that we do not create any higher dimensional holes during this process.

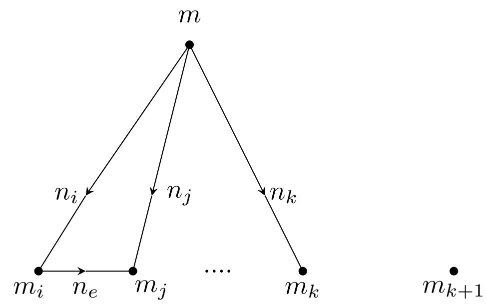
We will induct on the vertices $E_0 = \{x_0, ..., x_N\}$, so we set

$$E_n^K = \{\sigma \in E^n \mid v_i(\sigma) \in \{x_0, ..., x_k\} \ \forall i = 0, ..., n\},$$

and assume we have $f_k : CE^k \to B(M, N)$ extending $f|_{E^k}$. We denote

$$\begin{aligned} m_i &= f(x_i) \in M && \text{for } i = 1, ..., k+1 \\ m &= f_k((*, 1)) \in M && \\ (m, n_i) &= f_k((x_i, 1)) \in M \times N && \text{for } i = 1, ..., k. \end{aligned}$$

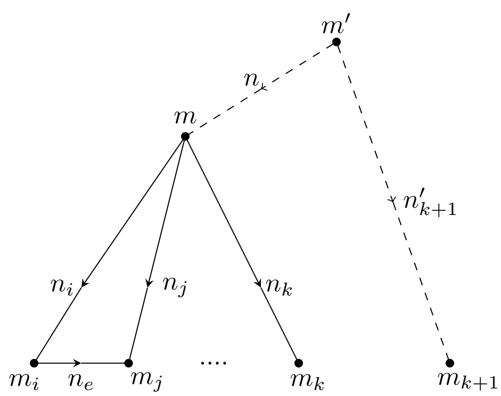
Now we know from our characterizations of the simplicial structure on $B(M, N)$ that the map f_k can only be simplicial if $m_i = m \cdot n_i$ for all $i = 1, ..., k$. Moreover if $e \in E_1^k$ is any edge from x_i to x_j with $f(e) = (m_i, n_e)$ then the cocycle condition on the 2-cell $(e, 1)$ implies that $n_j = n_i \cdot n$. We summarize our information in the following picture.



Now since N is filtering, we know that there exists $m' \in M$, $n, n'_{k+1} \in N$ such that $m = m' \cdot n$ and $m_{k+1} = m \cdot n'_{k+1}$. Using this data, we define a new extension

$$\begin{aligned} f_{k+1}((*, 1)) &= m' \\ f_{k+1}((m_i, 1)) &= (m', n \cdot n_i) \qquad\qquad \text{for } \ i = 1, ..., k \\ f_{k+1}((m_{k+1}, 1)) &= (m', n'_{k+1}) \end{aligned}$$

summarized in the following picture.



It is now obvious from our construction that f_{k+1} satisfies the twisting condition. The cocycle condition is clearly still satisfied on any $(\sigma, 0)$ for $\sigma \in E^2$ since we are extending f. For 2-cells of the form $(e, 1)$ as above, we have

$$n_j = n_i \cdot n_e \implies n \cdot n_j = n \cdot n_i \cdot n_e,$$

so the cocycle condition still holds. It remains to check this when e is some edge going to or from x_{k+1}. If $f(e) = (m_{k+1}, n_e)$ we have

$$m' \cdot n \cdot n_i = m_i = m_{k+1} \cdot n_e = m' \cdot n'_{k+1} \cdot n_e,$$

so by cancelling m' we get $n = n'_{k+1} \cdot n_e$ which is exactly what we need. A similar argument shows the case when e goes from x_i to x_{k+1}. This concludes the induction step. The base case is trivial, so since E_0 was finite this concludes the proof. $\square$

Proposition 2.49. *If N is a filtering submonoid of the discrete left cancellative monoid M, the inclusion induces a homotopy equivalence $|BN| \to |BM|$.*

Proof. If we can show that $\pi_n(|B(M,N)|) = 0$ for all n, the result will follow from Lemma 2.45 and Whitehead. To that end let $[f] \in \pi_n|B(M,N)|$. Since the monoids are discrete, $B(N,M)$ is a simplicial set, and so simplicial approximation implies that f is homotopic to $|F|$ for some simplicial map $F\colon E \to B(N,M)$, where E is a simplicial set with $|E| \simeq S^n$. Now by compactness of S^n, we may assume E_0 is finite, so by Lemma 2.48 we can extend F to a simplicial map $\tilde{F}\colon CE \to B(M,N)$. By Lemma 2.22, the geometric realization of this map gives a homotopy $|F| \sim *$. $\square$

Example 2.50. There are several different settings in which $N \subset M$ turns out to be a filtering submonoid. One example is to take N to be left cancellative and commutative, and to let M be the group completion. One can show that since N is cancellative, the group completion map $\phi\colon N \to N^*$ is injective, so it is equivalent to the inclusion of a submonoid. To show that it is filtering, just write any pair of elements $n_1, n_2 \in N^*$ as

$$n_1 = n_1^+ - n_1^- \text{ and } n_2 = n_2^+ - n_2^-$$

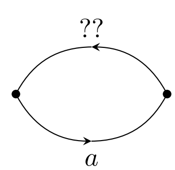
for $n_i^\pm \in N$, and take $m = -n_1^- - n_2^-$. This gives

$$n_1 = m + n_1^+ + n_2^- \text{ and } n_2 = n_2^+ + n_1^-.$$

In particular this shows that $\mathbb{N}$ is a filtering submonoid of the left cancellative $\mathbb{Z}$, so we get the desired equivalence $|B\mathbb{N}| \to |B\mathbb{Z}|$.

2.5 MV maps

So far we have been working with cocycles and twists on simplicial spaces. This does not quite match the definition of an actual principal bundle, where we need an open cover and a Čech 1-cocycle. The purpose of this subsection is to introduce a tool to translate between these two concepts. We will state the relevant definitions and results from [TGNL, Appendix A], but we leave out all proofs. One thing to note is that we really need a procedure to create *directed* simplicial sets. To see why, consider the following attempt to define a nontrivial $\mathbb{N}$-cocycle.



Note that the cocycle condition applied to the degenerate 2-simplex above is $a + b = 0$. For $a, b \in \mathbb{N}$ this is only possible if $a = b = 0$ which is very boring.

With this in mind we define

Definition 2.51. A directed open cover of a topological space X is given by a partially ordered index set $(I, \leq)$ and an open cover $(U_i)_{i \in I}$, such that for all $x \in X$ the set

$$I_x = \{ i \in I \mid x \in U_i \}$$

is finite and totally ordered by $\leq$, and such that each U_i is non empty.

Directed open covers can be pulled back over continuous maps if we take care to discard any indices giving empty preimages. In particular we can restrict the cover to any subspace of X. For any nondecreasing sequence $i_0 \leq i_1 \leq \dots \leq i_n$, we denote

$$U_{i_0, i_1, \dots, i_n} = \bigcap_{k=0}^{n} U_{i_k}.$$

From a directed open cover we construct a directed simplicial set as follows.

Definition 2.52. Let $(I, \leq)$ index the directed open cover $(U_i)_{i \in I}$. The Mayer–Vietoris blow up of (U_i) is a simplicial space which we denote by $MV(U_\bullet)$. Its n-simplicies are given by

$$MV(U_\bullet)_n = \coprod_{i_0 \leq i_1 \leq \dots \leq i_n} U_{i_0, \dots, i_n},$$

and any nondecreasing $\alpha : [n] \to [m]$ induces α^* via the inclusions

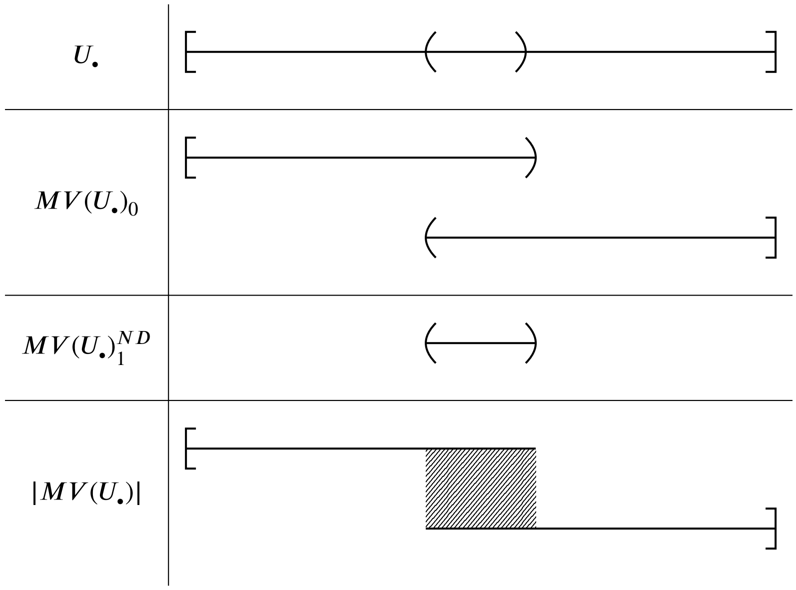
$$U_{i_0, \dots, i_m} \hookrightarrow U_{i_{\alpha(0)}, i_{\alpha(1)}, \dots, i_{\alpha(n)}}.$$

In particular the face map ∂_k^n is given by the inclusions

$$U_{i_0, \dots, i_k, \dots, i_n} \hookrightarrow U_{i_0, \dots, \hat{i}_k, \dots, i_n}.$$

To see that $MV(U_\bullet)$ is ordered, we label any n-simplex as $(x, i_0, \dots, i_n)$ where $i_0 \leq \dots \leq i_n$ and $x \in U_{i_0, \dots, i_n}$. The vertex set then inherits the order from I by setting $(x, i) \leq (y, j)$ if and only if $i \leq j$. Now the vertex maps are $v_k(x, i_0, \dots, i_n) = (x, i_k)$, so (2.2) is satisfied by definition. It should also be easy to see that a simplex is uniquely determined by its vertices.

Example 2.53. The following table illustrates the $MV(U_\bullet)$ construction for a simple open cover of $X = [0, 1]$.



There is an obvious map $|MV(U_\bullet)| \to X$ given by the inclusions $U_{i_0,\ldots,i_n} \hookrightarrow X$. (One can think of this as a simplicial map to the constant simplicial space at X). In [TGNL] the following result concerning this map is shown.

Lemma 2.54. *For any directed open cover $(U_i)_{i\in I}$ of X, the natural map $|MV(U_\bullet)| \to X$ is a homotopy equivalence.*

With these definitions it should be apparent that a Čech cocycle on an open cover $U_\bullet$ of X is the same as a simplicial cocycle on $MV(U_\bullet)$. With this in mind, we make the following definition.

Definition 2.55. Let S be a simplicial space. An MV-map from X to S is specified by a directed open cover $U_\bullet$ of X and a simplicial map $MV(U_\bullet) \to S$.

The topologically meaningful information connected to an MV map is contained in the homotopy class of the geometric realizations. It is therefore clear that we can at least factor out the following equivalence.

Definition 2.56. Two MV-maps $f : MV(U_\bullet) \to S$ and $g : MV(V_\bullet) \to S$ are equivalent if there exists an MV-map $F : MV(W_\bullet) \to S$ over $X \times [0,1]$ that restricts to f and g on opposite ends of $[0,1]$.

In fact, this is precisely the equivalence relation we want, as stated in the following result which can be extracted from [TGNL, Corollary A.11].

Proposition 2.57. *Let X be a space and S a simplicial space. Then Geometric realization induces a bijection*

$$\left\{\begin{array}{l}\text{Equivalence classes of MV-}\\ \text{maps over } X \text{ to } S\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{Homotopy classes of maps}\\ X \longrightarrow |S|\end{array}\right\}.$$

In particular, we are interested in combining this with Proposition 2.33.

Definition 2.58. Let Q be a topological monoid acting on a space F. A Q bundle on a space X is given by a directed open cover $(U_i)_{i\in I}$ and for each $i < j$, a continuous map $q_{ij} : U_{i,j} \to Q$ satisfying the cocycle condition

$$q_{ik}(x) = q_{ij}(x) \cdot q_{jk}(x) \qquad \text{for all } \ i < j < k \in I \ \text{ and } \ x \in U_{i,j,k}.$$

A Q-twisted map to F over X consits of a Q-bundle (U_i, q_{ij}) as above, and in addition, for each $i \in I$, a map $f_i : U_i \to F$ satisfying the twisting condition

$$f_j(x) = f_i(x) \cdot q_{ij}(x) \qquad \text{for all } \ i < j \in I \ \text{ and } \ x \in U_{ij}.$$

For both these structures we define equivalence similarly to equivalence of MV-maps. Note that results about Q-twisted maps to F automatically carry over to results about Q-bundles by taking F to be a point.

Remark 2.59. Allowing some informal yet suggestive notation, we can now see how this definition relates to example 2.29. The cocycle q_{ij} defines something like a principal Q bundle which we could denote $Q \to P \to X$. The twisting condition implies that the maps f_i glue together to a section of $F \to P \times_Q F \to X$.

Corollary 2.60. *There is a bijection*

$$\left\{ \begin{matrix} \textit{Equivalence classes of } Q\text{-} \\ \textit{twisted maps to } F \textit{ over } X \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \textit{Homotopy classes of maps} \\ X \longrightarrow |B(F,Q)| \end{matrix} \right\}.$$

Proof. Unwinding the definitions, a Q-twisted maps to F over X is precisely the same as a *simplicial* Q-tiwsted map to F over $MV(U_\bullet)$ for some directed open cover $U_\bullet$. The result then follows immediately from Proposition 2.33 and Proposition 2.57. □

One way to obtain an equivalent MV-map is by refining the open cover. This is a useful tool since it will give us the flexibility to find sufficiently well behaved MV-maps within a specified equivalence class.

Definition 2.61. A refinement of a directed open cover $(U_i)_{i\in I}$ is another directed open cover $(V_j)_{j\in J}$ and a map $\gamma : J \to I$ such that $V_j \subset U_{\gamma(j)}$ for all $j \in J$, and such that the restriction $\gamma_x : J_x \to I_x$ is nondecreasing for all $x \in X$.

The proof of the following lemma paraphrases [TGNL, Lemma A.10].

Lemma 2.62. *If $f : MV(U_\bullet) \to S$ is an MV-map over X, and $V_\bullet$ is a refinement of $U_\bullet$, then the map defined on $MV(V_\bullet)$ by including $V_{j_0,\ldots,j_n} \hookrightarrow U_{\gamma(j_1),\ldots\gamma(j_n)}$ and then applying f is equivalent to f.*

The following two results are concerned with applying refinements to find well behaved MV-maps. The proofs can be found in [TGNL, Appendix A].

Lemma 2.63. *If X is a smooth manifold, any directed open cover refines to a directed cover indexed by a totally ordered I.*

Lemma 2.64. *If X is s smooth manifold, and F and Q are smooth with Q acting smoothly on F, then any Q-twisted map to F over X is equivalent to a smooth one.*

3 A twisted Giroux–Latour theorem

This section is concerned with section 2 of [TGNL]. The goal is to use the theory of classifying spaces to generalize Theorem 1.1. We will start by setting up some relevant definitions, then go on to give an outline of Giroux' proof from [11]. We will then briefly explain some of the differences between [TGNL] and [11], which will stake out the course for the rest of this section. We will focus our energy on giving an alternative proof of the homotopy lifting property of symplectic reduction, and on utilizing the results from section 2. Most of the material in this section is concerned with a careful study of the underlying linear algebra. A standard reference on linear symplectic geometry is [21].

3.1 Generating functions

Exact Lagrangians in T^*M can also be seen as Legendrians in the 1-jet bundle $J^1(M)$, which we now explain.

Definition 3.1. Let M be a smooth n-manifold. The 1-jet bundle of M is the space $J^1(M) = T^*M \times \mathbb{R}$. It is equipped with a standard contact structure given by the form $\alpha = p_1^*\lambda - p_2^*\mathrm{d}z$ where $\lambda = p\mathrm{d}q$ is the Liouville form on T^*M, and where z is the natural coordinate on $\mathbb{R}$.

Definition 3.2. Let N and M be smooth n-manifolds. An immersion $\varphi \times z\colon L \to T^*M \times \mathbb{R} = J^1(M)$ is called Legendrian if $(\varphi \times z)^*\alpha = 0$.

Remark 3.3. If $\varphi\colon L \to T^*M$ is an exact Lagrangian, we have by definition that $\varphi^*\lambda = \mathrm{d}f$ for some function $f \to \mathbb{R}$. It is easily seen that $\varphi \times f\colon L \to J^1(M)$ is Legendrian. Dually, if we have some Legendrian $\varphi \times z\colon L \to J^1(M)$, forgetting the z factor gives an exact Lagrangian $\varphi\colon L \to T^*M$.

Example 3.4. For any smooth function $f\colon M \to \mathbb{R}$, the 1-jet graph of f is the embedding $j^1 f\colon M \to J^1(M)$ defined by $x \mapsto (x, \mathrm{d}_x f, f(x))$. This is easily seen to be Legendrian. This construction is the first example of a generating function of a Legendrian immersion. To allow for more interesting Legendrians we will allow functions that take in a fixed number of external variables, namely functions defined on an open subset in $M \times \mathbb{R}^k$.

Definition 3.5. A generating function is a triple (n, U, f) where n is a natural number, $U \subset M \times \mathbb{R}^n$ is an open subset, and $f\colon U \to \mathbb{R}$ is a smooth function such that $\mathrm{d}f \pitchfork T^*M \times \mathbb{R}^n$. Note that we interpret $\mathbb{R}^n$ as the zero section in $T^*\mathbb{R}^n \simeq \mathbb{C}^n$, so the latter condition can also be seen as the derivative with respect to $v \in \mathbb{R}^n$ vanishing transversely. The intersection $\mathrm{d}f \cap T^*M \times \mathbb{R}^n$ is the singular set

$$\Sigma_f = \{(x, v) \in U \mid \partial_v(x, v) = 0\} \tag{3.1}$$

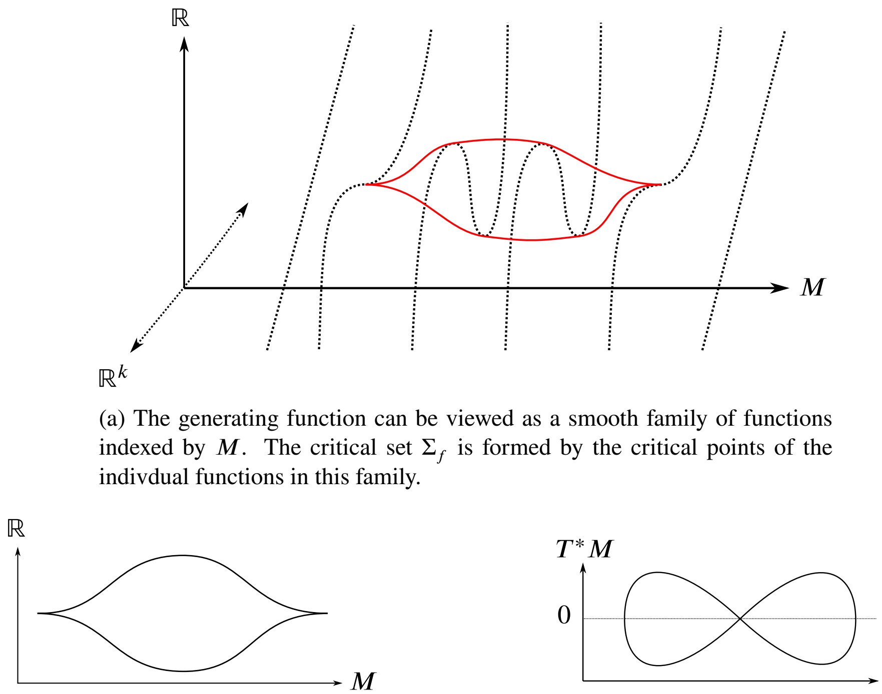
which comes with a natural Legendrian immersion

$$\begin{aligned} i_f \times f\colon \Sigma_f &\longrightarrow T^*M \times \mathbb{R} \\ (x, v) &\longmapsto (x, \partial_x f(x, v), f(x, v)). \end{aligned}$$

We say that the Legendrian immersion $\varphi \times z : L \to J^1(M)$ admits the generating function (n, U, f) if $\varphi \times z$ factors through $i_f \times f$ via a diffeomorphism $\psi : L \to \Sigma_f$.

Example 3.6. When $n = 0$, the singular set is all of M, and $i_f \times f$ agrees with $j^1 f$. In other words, $(0, M, f)$ is a generating function for $j^1 f$

Example 3.7. In low dimension, we can visualize a generating function as a Cerf diagram. Figure 2 uses this technique to show a generating function for a Legendrian unknot in $J^1(\mathbb{R}) = \mathbb{R}^3$.



(a) The generating function can be viewed as a smooth family of functions indexed by M. The critical set Σ_f is formed by the critical points of the indivdual functions in this family.

(b) The front projection of the Legendrian is the Cerf diagram of the family.

(c) The Lagrangian picture is the derivative of the front.

Figure 2: A generating function for the Legendrian unknot.

The Cerf diagram 2b is defined by tracking only the critical values over the points of M. It is equal to the image of the Legendrian under the front projection $J^1(M) \to \mathbb{R}$. The Lagrangian 2c is the image of the Legendrian under the projection $J^1(M) \to T^*M$. Since the contact form looks like $\mathrm{d}z - \lambda$, the Lagrangian being the derivative of the front is precisely the condition for being Legendrian.

Note how the cusps in the Cerf diagram 2b correspond to birth/death of critical points in the family of functions 2a. Note also that at such a birt/death event, the critical point of higher critical value must have higher index than the one of lower critical value.

Example 3.8. Quadratic forms are functions with very simple singular sets. These will allow us to construct new generating functions from existing ones. Let (n, U, f)

be a generating function, and let q be a nondegenerate quadratic form on $\mathbb{R}^k$ (or more generally, let q be a map from M to the space of nondegenerate quadratic forms on $\mathbb{R}^k$). We then claim that $(n+k, U\times\mathbb{R}^k, f\oplus q)$ is also a generating function for $i_f \times f : \Sigma_f \to J^1(M)$. To see this, we use coordinates $(x, v, u) \in U\times\mathbb{R}^k$, and note that the only critical point of a nondegenerate quadratic form is 0, where the value is also 0. Hence

$$\Sigma_{f\oplus q} = \Sigma_f \times \{0\}.$$

The map $(x, v) \mapsto (x, v, 0)$ gives the required diffeomorphism, and the fact that $f(x, v) = f(x, v) + q(0) = f \oplus q(x, v, 0)$ implies that $i_f \times f$ factors through this map.

3.2 The Stable Gauss map

We denote the projection $T^*M \to M$ by π_M and the composition by $\pi = \pi_M \circ \varphi$ By definition, the bundle TT^*M is a symplectic vector bundle. If we were working with embeddings, we could consider $\mathrm{d}\phi(TL) \subset TT^*M|_{\varphi(L)}$ as a subbundle. For immersions we need to work with the pullback bundle $E = \phi^*TT^*M$ instead. This symplectic bundle has a Lagrangian Grassmanian bundle $\Lambda_0(E)$ defined by the fiber

$$\Lambda_0(\mathbb{C}^n) = \{\text{Lagrangian vector subspaces of}\mathbb{C}^n\} = U(n)/O(n)$$

and the same underlying principal $U(n)$-bundle as E. This bundle comes with two sections:

- The vertical section $V(x) = \ker(\mathrm{d}_{\varphi(x)}\pi : T_{\varphi(x)}T^*M \to T_{\pi(x)}M)$.
- The Gauss section $G_\varphi(x) = \mathrm{im}(\mathrm{d}_x\varphi : T_xL \to T_{\pi(x)}M)$.

We now want to construct a map which measures the *stable* difference between these two sections. To define stabilization, we will first trivialize the bundle E. We omit the specifics of this stabilization, but summarize the salient facts in the following lemma.

Lemma 3.9. *For any symplectic vector bundle E over a compact base L, there exists a trivial symplectic bundle $L\times\mathbb{C}^n$ with a morphism $E \to L\times\mathbb{C}^n$ which renders E as a symplectic subbundle and a direct summand. Moreover there is a map of Lagrangian Grassmanians*

$$\Lambda_0(E) \longrightarrow \Lambda_0(L\times\mathbb{C}_n) \simeq L\times\Lambda_0(\mathbb{C}^n)$$

which maps the vertical subbundle of E to the vertical of $L\times\mathbb{C}^n$.

Proof. We omit the proof, which can be found in [TGNL]. □

To stabilize, we wish to pass to the colimit $\Lambda_0(\mathbb{C}^\infty)$ which we now define.

Definition 3.10. If $L \subset \mathbb{C}^n$ is Lagrangian, then so is $\mathbb{R}\oplus L \subset \mathbb{C}^{n+1}$. This defines a smooth map of Lagrangian Grassmanians, and we set

$$\Lambda_0(\mathbb{C}^\infty) = \operatorname{colim}\left(\Lambda_0(\mathbb{C}^1)\xrightarrow{\mathbb{R}\oplus-}\Lambda_0(\mathbb{C}^2)\xrightarrow{\mathbb{R}\oplus}\cdots\right).$$

The stable Gauss map is now defined by the composition

$$g_\varphi : L\xrightarrow{G_\varphi}\Lambda_0(E)\longrightarrow L\times\Lambda_0(\mathbb{C}^n)\xrightarrow{pr_2}\Lambda_0(\mathbb{C}^n)\longrightarrow\Lambda_0(\mathbb{C}^\infty). \tag{3.2}$$

3.3 Symplectic reduction and stabilization

We give definitions of symplectic reduction and show how this is related to generating functions.

Definition 3.11. Let (E,ω) be a symplectic vector space. Recall that a (linear)subspace F is called coisotropic if $F^{\omega} \subset F$, where F^{ω} denotes the symplectic compliment. A standard construction of symplectic geometry is the symplectic reduction over F, defined as the quotient F/F^{ω}. The symplectic form ω descends to a symplectic form on F/F^{ω}, and if $L \subset E$ is Lagrangian, then

$$\rho_F(L) = \frac{(L \cap F) + F^{\omega}}{F^{\omega}} \subset F/F^{\omega}$$

is also Lagrangian.

While the above definition gives a function $\Lambda(E) \to \Lambda(F/F^{\omega})$, the intersection operation is not well behaved enough to get a continuous map. However, we can restrict the domain to the set of *transversal* Lagrangians denoted

$$\Lambda_F(E) = \{L \in \Lambda_0(E) \mid L \pitchfork F\},$$

to get a smooth map $\rho_F : \Lambda_F(E) \to \Lambda_0(F/F^{\omega})$.

Example 3.12. We have actually seen symplectic reduction at work already! At the Linear level it is exactly what happens when we construct a Lagrangian immersion from a generating function. If (n, U, f) is a generating function, then the singular set is precisely the transverse intersection between graph$(\mathrm{d}f)$ and the coisotropic submanifold $T^*M \times \mathbb{R}^n$ in $T^*(M \times \mathbb{R}^k)$. Let $\psi : L \to \Sigma_f$ be the diffeomorphism factoring $\varphi \times z$. At the level of Lagrangian tangent spaces, we have for any $(x,v) \in \Sigma_f$ that

$$T_{(x,v)}\Sigma_f \simeq \mathrm{graph}(\mathrm{d}_{(x,v)}f) \bigcap \left(T_xT^*(M) \times \mathbb{R}^k\right) \subset T_{(x,v)}T^*(M \times \mathbb{R}^k).$$

It is easily seen that the symplectic compliment of $F = T_xT^*M \times \mathbb{R}^k$ is $F^{\omega} = 0 \times \mathbb{R}^k$, hence modding out F^{ω} is precisely the projection to T_xT^*M, which agrees with $\mathrm{d}_{(x,v)}i_f$.

We state and prove two technical lemmas about symplectic reduction. The first describes iterated symplectic reduction, while the second shows how symplectomorphisms interact with reduction.

Lemma 3.13. *Let E be a symplectic vector space, and $V \subset W$ two nested coisotropic subspaces. Then $V' = V/W^{\omega}$ is coisotropic in W/W^{ω} and*

$$\rho_V = \rho_{V'} \circ \rho_W.$$

Proof. The coisotropic part is ok since $V^{\omega} \subset V \implies V^{\omega}/W^{\omega} \subset V/W^{\omega}$, and $(V/W^{\omega})^{\omega} = V^{\omega}/W^{\omega}$ by definition of the symplectic structure on W/W^{ω}. Since $V \subset W \implies W^{\omega} \subset V^{\omega}$, we have that

$$\left(\frac{L \cap W + W^{\omega}}{W^{\omega}} \bigcap \frac{V}{W^{\omega}} + \frac{V^{\omega}}{W^{\omega}}\right) \Big/ \frac{V^{\omega}}{W^{\omega}} = \left(\frac{L \cap V + V^{\omega}}{W^{\omega}}\right) \Big/ \frac{V^{\omega}}{W^{\omega}} = \frac{L \cap V + V^{\omega}}{V^{\omega}}.$$

Unwinding the definitions this is exactly $\rho_V = \rho_{V'} \circ \rho_W$. □

Lemma 3.14. *Let E and E' be symplectic vector spaces, and $\Psi : E \to E'$ a linear symplectomorphism. If $F \subset E$ is coisotropic, then so is $\Psi(F) \subset E'$. Furthermore Ψ induces a symplectomorphism*

$$\Psi' : \frac{F}{F^\omega} \to \frac{\Psi(F)}{\Psi(F)^\omega},$$

satisfying

$$\rho_{\Psi(F)}(\Psi(L)) = \Psi'(\rho_F(L))$$

for all Lagrangians $L \in E$.

Proof. The key fact is that $\Psi(F^\omega) = \Psi(F)^\omega$, which we now show.

$$\begin{aligned}
\Psi(F^\omega) &= \{\Psi(y) \mid \omega(y,x) = 0 \ \forall x \in F\} \\
&= \left\{y' \mid \omega\left(\Psi^{-1}(y'), x\right) = 0 \ \forall x \in F\right\} \\
&= \left\{y' \mid \omega\left(\Psi^{-1}(y'), \Psi^{-1}(x')\right) = 0 \ \forall x' \in \Psi(x)\right\} \\
&= \left\{y' \mid \omega'(x', y') = 0 \ \forall x' \in \Psi(x)\right\} = \Psi(F)^\omega
\end{aligned}$$

Specifically, this means that the kernel of the composition

$$F \xrightarrow{\Psi} \Psi(F) \longrightarrow \frac{\Psi(F)}{\Psi(F)^\omega}$$

is exactly F^ω, so we get an induced symplectomorphism as required. To see the part about reduction of Lagrangians, consider

$$\frac{\Psi\left(L \cap F + F^\omega\right)}{\Psi(F)^\omega} = \frac{\Psi(L) \cap \Psi(F) + \Psi(F)^\omega}{\Psi(F)^\omega} = \rho_{\Psi(F)}(\Psi(L)).$$

By definition of Ψ', the leftmost term computes the image $\Psi'\left(\rho_F(L)\right)$. □

Generating functions are special in that they always produce Lagrangians that are graphical over the zero section in $T^*(M \times \mathbb{R}^n)$. Being graphical is equivalent to being transversal to the vertical subbundle V. This motivates the following definitions.

Definition 3.15. For a symplectic vector space E, and a Lagrangian subspace $V \subset E$, we define $\Lambda_n(E)$ to be the space of Lagrangians in $E \times \mathbb{C}^n$ transverse to the coisotropic $E \times \mathbb{R}^n$. (This explains our insistence to denote the "regular" Lagrangian Grassmanian by Λ_0.) We set

$$\Lambda(E) = \coprod_{n \in \mathbb{N}} \Lambda_n(E).$$

We denote by $\Lambda_n^V(E)$ the subset of $\Lambda_n(E)$ of Lagrangians who are also transverse to $V \times i\mathbb{R}^n$, and set

$$\Lambda^V(E) = \coprod_{n \in \mathbb{N}} \Lambda_n^V(E).$$

Nondegenerate quadratic forms are like generating functions over $M = *$, and they will play an important role in the following sections.

Definition 3.16. Let $\mathcal{Q}_n$ be the smooth manifold of nondegenerate quadratic forms on $\mathbb{R}^n$. We set

$$\mathcal{Q} = \coprod_{n \in \mathbb{N}} \mathcal{Q}_n.$$

The direct sum operation

$$q_1 \oplus q_2(u_1, u_2) = q_1(u_1) + q_2(u_2)$$

turns $\mathcal{Q}$ into a topological monoid. The unit is the unique form on $\mathbb{R}^0 = \{0\}$.

We want to define left and right actions of $\mathcal{Q}$ on $\Lambda(E)$ paralleling Example 3.8. Since q is quadratic, the assignment $u \mapsto \mathrm{d}_u q$ is a linear map $\mathbb{R}^n \to (\mathbb{R}_n)^*$. The standard symplectic structure on $\mathbb{C}^n$ identifies $(\mathbb{R}^n)^*$ with $i\mathbb{R}^n$, and so $\mathrm{graph}(\mathrm{d}q) \in \mathbb{C}^n$ is an n-dimensional linear subspace transverse both to $i\mathbb{R}^n$ and $\mathbb{R}^n$. It is a standard result of symplectic geometry that this graph is Lagrangian, and moreover actually defines an isomorphism

$$\mathrm{graph}(\mathrm{d}-) \colon \mathcal{Q}_n \longrightarrow \Lambda_n^0(0).$$

The idea is now to let q act on $\Lambda(E)$ by direct sum with $\mathrm{graph}(\mathrm{d}q)$. To make sure we get Lagrangians in $E \times \mathbb{C}^k$, and not $\mathbb{C}^k \times E$ we need to be a little careful when acting from the left.

Definition 3.17. For any $n \in \mathbb{N}$, let σ_n denote the linear isomorphism

$$\sigma \colon E \oplus \mathbb{C}^n \longrightarrow \mathbb{C}^n \oplus E.$$

We then define a right action by

$$\begin{aligned} \Lambda_n(E) \times \mathcal{Q}_s &\longrightarrow \Lambda_{n+s}(E) \\ (L, q) &\longmapsto L \oplus \mathrm{graph}(\mathrm{d}q), \end{aligned}$$

and a left action by

$$\begin{aligned} \mathcal{Q}_r \times \Lambda_n(E) &\longrightarrow \Lambda_{n+r}(E) \\ (q, L) &\longmapsto \sigma_{n+r}^{-1}\left(\mathrm{graph}(\mathrm{d}q) \oplus \sigma_n(L)\right). \end{aligned}$$

From the structure of σ it should be easy to see that these actions are compatible. Moreover, it should be clear that σ preserves transversality in such a way that the above actions restrict to

$$\mathcal{Q} \times \Lambda^V(E) \times \mathcal{Q} \longrightarrow \Lambda^V(E).$$

For purposes of stabilization which will soon become apparent, we will fix a single quadratic form, namely the form $h(x_1, x_2) = x_1 \cdot x_2$ on $\mathbb{R}^2$.

Definition 3.18. We denote the k-fold stabilizations of G_φ and V respectively by

$$G_\varphi^k = h^k \cdot G_\varphi \quad \text{and} \quad V^k = h^k \cdot V.$$

3.4 Outline and comparison with the classical result

We are now ready to state the precise version of theorem 1.1 as it appears in [11].

Proposition 3.19 (Giroux—Latour)**.** *Let L and M be closed smooth n-manifolds. An exact Lagrangian immersion $\varphi : L \to T^*M$ admits a generating function if and only if the sections V^k and G_φ^k are homotopic for some k.*

The hypothesis of this theorem is not precisely the one we will use, but Proposition 2.6 of [TGNL] shows that it is equivalent to the hypothesis that the stable Gauss map is nullhomotopic. This result will eventually be encompassed by the results of [TGNL], but we will now give a rough sketch since this will inform our approach in the rest of this section.

Proof sketch. The problem is first reduced to closed Lagrangians in $\mathbb{C}^n$ by embedding M in $\mathbb{R}^n$. This allows reinterpreting the sections G_φ and V as maps. Moreover the Lagrangian Grassmanian is connected, so V can be replaced by any constant map. In our approach this step is replaced by trivialization of the tangent bundle as in Lemma 3.9.

Assume first that (k, U, f) is a generating function for φ. As in Example 3.12, we get a lift of G_φ to $\Lambda^{i\mathbb{R}^{n+k}}(\mathbb{C}^n)$ with respect to symplectic reduction. By definition of stabilization, G_φ^k is also a lift of G_φ. This is summarized in the following diagram which commutes only for maps to $\Lambda_0(\mathbb{C}^n)$.

$$\begin{tikzcd}
 & \Lambda_k^{i\mathbb{R}^n}(\mathbb{C}^n) \arrow[d, hook] \\
 & \Lambda_k(\mathbb{C}^n) \arrow[d, "\rho"] \\
L \arrow[uur, "\mathrm{d}\psi"] \arrow[ur, "G_\varphi^k"'] \arrow[r, "G_\varphi"'] & \Lambda_0(\mathbb{C}^n)
\end{tikzcd} \tag{3.3}$$

The rest of the proof now has two crucial ingredients

1. The map $\rho : \Lambda_k(\mathbb{C}^n) \to \Lambda_0(\mathbb{C}^n)$ is a homotopy equivalence.
2. The inclusion $\Lambda_k^V(\mathbb{C}^n) \hookrightarrow \Lambda_k(\mathbb{C}^n)$ is nullhomotopic.

From these it is easily seen that G_φ^k is nullhomotopic.

For the other direction we assume that $G_k\phi$ is nullhomotopic. The crucial result is now that the vertical composition in (3.3) is a fibration *up to further stabilization*. This means that if we take let $H : L \times I \to \Lambda_k(E)$ be a homotopy of G_ϕ^k with a constant map, we can lift the constant end, and stabilize to some $\Lambda_{k+j}^V(\mathbb{C}^n)$ where we can actually lift the entire homotopy, and in particular the stabilized Gauss map.

The lift of the Gauss map gives us the information we need about tangent spaces. To find an actual manifold with these tangent spaces, we fix an embedding $\psi' : L \to \mathbb{R}^l$, and consider the isotropic embedding

$$\varphi \oplus (\psi' \times 0) \times z : L \longrightarrow (T^*M \oplus T^*\mathbb{R}^k) \times \mathbb{R} = J^1(M \times \mathbb{R}^k).$$

After further stabilization we get enough flexibility to homotope the lifted Gauss map so that it is actually tangent with this embedding. The Weinstein theorem from contact geometry then allows us to integrate to get the graph of our generating function. We will not go into this part of the procedure in this thesis, but refer to steps 2 and 3 of [TGNL, section 2.3]. □

The approach in [TGNL] differs in a couple of ways from this outline. First, it packages the stabilization at the level of spaces, as defined in 3.28, to get a genuine fibration $\Lambda^V_\infty(E) \to \Lambda_0(E)$. Second, it notes that the fiber of this is homotopy equivalent to the space of nondegenerate quadratic forms (again with some stabilization) as shown in 3.27.[2] The idea is then to get rid of this fiber by applying the techniques from section 2, yielding a homotopy equivalence $|B(\Lambda^V_\infty(E), \mathcal{Q})| \simeq \Lambda_0(E)$ in Corollary 3.30. A third difference is, as we have already remarked, that we also pass to the limit $E = \mathbb{C}^\infty$, where the homotopy equivalence actually becomes $|B(\mathbb{N}, \mathcal{Q})| \simeq \Lambda_0(\mathbb{C}^\infty)$.

A fourth difference is that instead of lifting the Gauss map $L \to U/O$ directly, it aims to factor through $\pi : L \to M$ up to homotopy. While Giroux and Latour are only concerned with a *local* generating function defined on a potentially tiny neighbourhood of L, [TGNL] aims to eventually construct a global object where this difference is meaningful. The restrictiveness of this factorization is well suited to the study of nearby Lagrangians, since it is known that when L is nearby Lagrangian, $\pi : L \to M$ is a homotopy equivalence [1].

[2] The proof of this gives part of the proof that ρ from 3.3 is a homotopy equivalence; removing one transversality condition identifies the fiber of ρ with the affine space of (not necessarily nondegenerate) quadratic forms, which is contractible.

3.5 Linear symplectic relations and the homotopy lifting property

The goal of this section is to prove that symplectic reduction is a fibration up to stabilization. The main ingredient of the proof is a fundamental technique in the study generating functions, namely the symplectomorphism

$$\begin{aligned}\Psi : \overline{\mathbb{C}^n} \times \mathbb{C}^n &\longrightarrow T^*\mathbb{R}^{2n} \simeq \mathbb{C}^{2n} \\ (x_1 + iy_1, x_2 + iy_2) &\longmapsto (x_2, y_1, y_2 - y_1, x_1 - x_2).\end{aligned} \tag{3.4}$$

One can check that this is a symplectomorphism, and that it sends the diagonal $\Delta_{\mathbb{C}^n}$ to the zero section $\mathbb{R}^{2n}$. In $\overline{\mathbb{C}^n} \times \mathbb{C}^n$, transversal Lagrangians correspond to graphs of symplectomorphisms, while in $\mathbb{C}^{2n}$, they correspond to generating functions. This means that Ψ gives us a way to associate a generating function to a symplectomorphism. This assignment has some desirable properties that we will demonstrate using the language of symplectic relations. This will give us an alternative version of the proof found in [TGNL].

Definition 3.20. A relation f "from" Y to X is a subset $f \subset X \times Y$. If $f \subset X \times Y$ and $g \subset Y \times Z$, we define a new relation $f \circ g \subset X \times Z$ by

$$f \circ g = \{(x, z) \in X \times Z \mid \exists y \in Y \text{ s.t. } (x, y) \in f \text{ and } (y, z) \in g\}.$$

Alternatively, we could define composition by taking the intersection $(f \times g) \cap (X \times \Delta_Y \times Z)$ and then projecting this set to $X \times Y$.

Example 3.21. Functions are special cases of relations. To view a function $f : Y \to X$ as a relation, just consider

$$\operatorname{graph}(f) = \{(f(y), y) \mid y \in Y\} \subset X \times Y\}.$$

With these definitions, composition of relations is compatible with composition of functions, namely

$$\operatorname{graph}(f) \circ \operatorname{graph}(g) = \operatorname{graph}(f \circ g).$$

Example 3.22. If we take Z to be a singleton $\star$, any subset $S \subset Y$ defines a relation $S \times \star \subset Y \times \star$. If we have any function $f : Y \to X$, one can check that

$$f(S) \times \star = \operatorname{graph}(f) \circ (S \times \star).$$

In later examples of this phenomenon we will suppress the $\star$.

To generalize structure preserving maps we replace arbitrary subsets with structured subsets.

Definition 3.23. Let E and F be symplectic vector spaces. A linear symplectic relation from F to E is a linear Lagrangian subspace $\psi \subset E \oplus \overline{F}$.

The first thing to note is that this is the right definition to generalize symplectomorphisms since the graph of any such is Lagrangian in $E \oplus \overline{F}$. The second is to note that composing linear symplectic relations $\psi \subset E \oplus \overline{F}$ and $\varphi \subset F \oplus \overline{G}$ is equivalent to the symplectic reduction of $\psi \oplus \varphi \subset E \oplus \overline{F} \oplus F \oplus \overline{G}$ over the coisotropic subspace $E \oplus \Delta_F \oplus G$. Since reduction sends Lagrangians to Lagrangians, this shows that composition is well defined. As before, symplectic reduction is only continuous if we restrict ourselves to Lagrangians transversal to $E \oplus \Delta_F \oplus G$. This will always be the case if either ψ or φ is the graph of a symplectomorphism.

We are now ready to state and prove a crucial lemma, which will be the key to proving the homotopy lifting property. The statement here is almost identical to [11, Lemma II.7].

Lemma 3.24. *For every $n, k \in \mathbb{N}$, there exists an open neighbourhood U_0 of the identity in $U(n)$, and a smooth $\tau_{n,k} : U_0 \times \Lambda_k^V(\mathbb{C}^n) \to \Lambda_{k+2n}^V(\mathbb{C}^n)$ such that the following diagram commutes:*

$$\begin{array}{ccc} U_0 \times \Lambda_k^{i\mathbb{R}^n}(\mathbb{C}^n) & \xrightarrow{\tau_{n,k}} & \Lambda_{k+2n}^{i\mathbb{R}^n}(\mathbb{C}^n) \\ {\scriptstyle id\times\rho}\big\downarrow & & \big\downarrow{\scriptstyle\rho} \\ U_0 \times \Lambda_0(\mathbb{C}^n) & \longrightarrow & \Lambda_0(\mathbb{C}^n). \end{array} \tag{3.5}$$

Where the bottom map is the action $(\theta, L) \mapsto \theta(L)$. Moreover, $\tau_{n,k}$ is equivariant with respect to the right-action of Q on $\Lambda^V(\mathbb{C}^n)$ in the sense that for $q \in Q_j$ we have $\tau_{n.k+j}(\theta, L \cdot q) = \tau_{n,k}(\theta, L) \cdot q$.

Proof. For all n, k, define a symplectomorphism $\Psi_{n,k}$

$$\Psi_{n,k} = id_{\mathbb{C}^n} \oplus \Psi \oplus id_{\mathbb{C}^k} : \mathbb{C}^n \oplus \overline{\mathbb{C}^n} \oplus \mathbb{C}^n \oplus \mathbb{C}^k \longrightarrow \mathbb{C}^n \oplus \mathbb{C}^{2n} \oplus \mathbb{C}^k$$

where Ψ is as in equation (3.4). We claim that we may take

$$\tau_{n,k}(\theta, L) = \Psi_{n,k}\big(\operatorname{graph}(\theta) \oplus L\big).$$

This map is clearly smooth, and the following computation shows that the required diagram commutes. To increase readability, we temporarily denote $\rho_W(L) = \frac{L}{W}$.

$$\begin{aligned} \rho(\tau_{n,k}(\theta, L)) = \frac{\Psi_{n,k}\big(\operatorname{graph}(\theta) \oplus L\big)}{\mathbb{C}^n \oplus \mathbb{R}^{2n+k}} &= \frac{\operatorname{graph}(\theta) \oplus L}{\mathbb{C}^n \oplus \Delta_{\mathbb{C}^n} \oplus \mathbb{R}^k} \\ &= \frac{\operatorname{graph}(\theta) \oplus \rho_{\mathbb{C}^n \oplus \mathbb{R}^k}(L)}{\mathbb{C}^n \oplus \Delta_{\mathbb{C}^n}} = \theta(\rho(L)). \end{aligned}$$

The first equality follows by lemma 3.14 since $\Psi_{n,k}$ is the identity on the first $\mathbb{C}^n$ factor, and since Ψ maps the diagonal to $\mathbb{R}^{2n}$. The second equality follows from Lemma 3.13 applied to $\mathbb{C}^n \oplus \Delta_{\mathbb{C}^n} \oplus \mathbb{R}^k \subset \mathbb{C}^{3n} \oplus \mathbb{R}^k$, and the third follow from example 3.22. It remains to show that the correct transversalities are satisfied. To check transversality with $\mathbb{C}^n \oplus \mathbb{R}^{2n+k}$, we use that

$$\Psi_{n,k}^{-1}(\mathbb{C}^n \oplus \mathbb{R}^{2n+k}) = \mathbb{C}^n \oplus \Delta_{\mathbb{C}^n} \oplus \mathbb{R}^k,$$

and instead check that

$$\big(\operatorname{graph}(\theta) \oplus L\big) \pitchfork \big(\mathbb{C}^n \oplus \Delta_{\mathbb{C}^n} \oplus \mathbb{R}^k\big).$$

This follows from θ being an isomorphism, and from $L \pitchfork (\mathbb{C}^n \oplus \mathbb{R}^k)$. To check transversality with the vertical, we use

$$\Psi_{n,k}^{-1}\big(i\mathbb{R}^{3n+k}\big) = i\mathbb{R}^n \oplus \mathbb{R}^n \oplus i\mathbb{R}^n \oplus i\mathbb{R}^k,$$

which reduces the problem to

$$(\operatorname{graph}(\theta) \oplus L) \pitchfork i\mathbb{R}^n \oplus \mathbb{R}^n \oplus i\mathbb{R}^n \oplus i\mathbb{R}^k \iff \operatorname{graph}(\theta) \pitchfork i\mathbb{R}^n \oplus \mathbb{R}^n \text{ and } L \pitchfork i\mathbb{R}^{n+k}.$$

The L part is always true since $L \in \Lambda_k^{i\mathbb{R}^n}(\mathbb{C}^n)$, but the condition on graph(θ) is not satisfied for all unitary matrices. It is however true if

$$\theta \in U_0 = \{A \in U(n) \mid A(\mathbb{R}^n) \pitchfork i\mathbb{R}^n\},$$

which we take as our open neighbourhood of the identity. □

The increase in number of auxiliary variables from k to $k + 2n$ in this lemma is the part that adds the *up to stabilization* caveat to the fibration statement. For the proof in [11], this lemma is sufficient, but in our approach a bit more care is needed. In particular we need the stabilization to be compatible with the inclusion $U(n) \to U(n+1)$ which we will use to pass to the limit $\mathbb{C}^\infty$. The following lemma shows that we can modify the construction to achieve this.

Lemma 3.25. *For all $n, k \in \mathbb{N}$ there exists a smooth Q-equivariant map $\tau'_{n,k} : \Lambda_k^{i\mathbb{R}^n}(\mathbb{C}^n) \to \Lambda_{k+2n}^{i\mathbb{R}^n}(\mathbb{C}^n)$ making diagram (3.5) commute, and additionally satisfying*

$$\tau'_{n,k}(id, L) = h^n \cdot L \text{ for all } L \in \Lambda_k^{i\mathbb{R}^n}(\mathbb{C}^n).$$

Proof. We will achieve this by defining

$$\tau_{n,k}^{\Phi}(\theta, L) = \Phi \circ \Psi_{n,k}\big(\operatorname{graph}(\theta) \oplus L\big)$$

where Φ is in the group G of symplectomorphisms $\Phi : \mathbb{C}^{3n+k} \to \mathbb{C}^{3n+k}$ satisfying

$$\Phi\left(\mathbb{C}^n \oplus \mathbb{R}^{2n+k}\right) = \mathbb{C}^n \oplus \mathbb{R}^{2n+k} \tag{3.6}$$
$$\Phi\left(i\mathbb{R}^{3n+k}\right) = i\mathbb{R}^{3n+k} \tag{3.7}$$
$$\pi_{\mathbb{C}^n} \circ \psi|_{\mathbb{C}^n \oplus \mathbb{R}^{2n+k}} = \pi_{\mathbb{C}^n}. \tag{3.8}$$

The conditions (3.6) and (3.7) together imply that $\tau_{n,k}^{\Phi}$ satisfy the right transversalities. Condition (3.6) also puts us in the situation described in 3.14, and it should be clear from (3.8) that the induced map on the symplectic quotient is $id_{\mathbb{C}}^n$. This means we are free to replace $\tau_{n,k}$ with $\tau_{n,k}^{\Phi}$ in diagram (3.5). One way to find matrices in G is to symplectisize matrices in $\mathrm{GL}(\mathbb{R}^{3n+k})$ since these by definition satisfy (3.7) and almost (3.6). The symplectization of any $A \in \mathrm{GL}(\mathbb{R}^N)$ is

$$A^\omega = \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^T \end{pmatrix} : \mathbb{R}^N \oplus i\mathbb{R}^N \longrightarrow \mathbb{R}^N \oplus i\mathbb{R}^N.$$

This is the unique symplectomorphism of $\mathbb{C}^N$ covering A. To satisfy ensure condition(3.6), and also satisfy (3.8), we act with matrices of the form

$$A = \begin{pmatrix} 1 & 0 \\ B & C \end{pmatrix} : \mathbb{R}^n \oplus \mathbb{R}^{2n+k} \longrightarrow \mathbb{R}^n \oplus \mathbb{R}^{2n+k}.$$

These have inverse transpose of the form

$$(A^{-1})^T = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} : \mathbb{R}^n \oplus \mathbb{R}^{2n+k} \longrightarrow \mathbb{R}^n \oplus \mathbb{R}^{2n+k}.$$

The 0 in the lower left corner of $(A^{-1})^T$ implies that $A^\omega(i\mathbb{R}^n) \subset \mathbb{C}^n \oplus \mathbb{R}^{2n+k}$. The 1 in the upper left corners imply condition (3.8).

The goal is now to find a smooth map $L \to A(L)$ where $A(L)$ has the above form for all $L \in \Lambda^{i\mathbb{R}^n}(\mathbb{C}^n)$, such that

$$A(L)^\omega(\tau_{n,k}(\theta, L)) = h^n \cdot L.$$

To find such a map, we represent our Lagrangians as graphs of a symmetric matrices in the standard basis. To see how our group acts on these representatives, we consider a Lagrangian $L = (1 + iS)(\mathbb{R}^N)$ for some $N \times N$ real symmetric matrix S, and calculate

$$\begin{aligned} A^\omega(L) &= A^\omega\,(1 + iS)\,(\mathbb{R}^N) \\ &= \left(A + i(A^{-1})^T S\right) A^{-1}(\mathbb{R}^N) \\ &= \left(1 + i(A^{-1})^T S A^{-1}\right)(\mathbb{R}^N). \end{aligned}$$

If we consider only the symmetric matrix representatives, we get the action

$$(A, S) \longmapsto A \cdot S = (A^{-1})^T S A^{-1}. \tag{3.9}$$

The only thing left is to sit down and explicitly compute the symmetric matrices representing $h^n \cdot L$ and $\tau_{n,k}(\theta, L)$. We omit this computation here, but remark that the required map is exactly the one described in [TGNL, equation 2.24]. Note that this map is of the form described above. □

Proposition 3.26. *Let D be a compact smooth manifold, and assume we are given a smooth map $g : D \times [0,1] \to \Lambda_0(E)$. We denote $g_0 = g \circ i_0 : D \to \Lambda_0(E)$. Then there exists $N \in \mathbb{N}$ and a smooth map*

$$\Theta : g_0^*\Lambda^V(E) \times [0,1] \longrightarrow g^*\Lambda^V(E)$$

covering the identity on $D \times [0,1]$, such that;

1. $\Theta(x, L, 0) = h^N \cdot L$.

2. Θ *is right Q-equivariant.*

Proof. We fix an identification $E \simeq \mathbb{C}^n$ such that V corresponds to $i\mathbb{R}^n$. Since $U(n) \to \Lambda_0(\mathbb{C}^n) \simeq U(n)/O(n)$ is a smooth fiber bundle, we may pick a smooth $\theta : D \times [0,1] \to U(n)$ which is the identity over $D \times \{0\}$, and such that $g(x,t) = \theta(x,t)(g_0(x))$. Consider the map

$$\begin{aligned} T : D \times [0,1] \times [0,1] &\longrightarrow U(n) \\ (x,t,s) &\longmapsto \theta(x,t) \cdot \theta(x,s)^{-1}. \end{aligned}$$

Since $T(D \times \Delta) = id$, the preimage of U_0 is an open neighbourhood of $D \times \Delta$. Since D is compact, a basis for the neighbourhoods of $D \times \Delta$ is given by sets of the form $D \times (x-t, x+t) \times (x-s, x+s)$. This means we can find an open cover of $D \times \Delta$ by such sets, all contained in $T^{-1}(U_0)$. By compactness of $D \times \Delta$ we pass to a finite subcover. Reindexing we use this cover to find a partition $0 = t_0 < t_1 < ... < t_M = 1$ such that $T(x,t,t_k) \in U_0$ for all $x \in D$ whenever t is in a fixed open neighbourhood U_k of $[t_k, t_{k+1}]$. With this partition we define maps $\theta_k : D \times [0,1] \to U_0$ by

$$\theta_k(x,t) = \begin{cases} id & t < t_k \\ T(x,t,t_k) & t \in [t_k, t_{k+1}] \\ T(x,t_{k+1},t_k) & t > t_{k+1}. \end{cases}$$

The three partial definitions glue together to a continuous map. To make it smooth, we modify both θ_k and θ_{k+1} in the neighbourhood $U_k \cap U_{k+1}$ of t_{k+1}, giving the smooth fragmentation

$$\theta(x,t) = \theta_M(x,t) \cdot \theta_{M-1}(x,t) \cdot ... \cdot \theta_0(x,t). \tag{3.10}$$

Combining this with the definition of θ gives

$$g(x,t) = \theta_M(x,t) \cdot \theta_{M-1}(x,t) \cdot ... \cdot \theta_0(x,t)(g_0(x)).$$

We now inductively apply τ'_n to get our map. Explicitly, if $(x, L) \in g_0^* \Lambda^{i\mathbb{R}^n}(\mathbb{C}^n)$, i.e. $g_0(x) = \rho(L)$, we take

$$\Theta(x, L, t) = (x, t, \tau'_n(\theta_M(x,t), \tau'_n(\theta_{M-1}(x,t), ...\tau'_n(\theta_0(x,t), L)...).$$

At t=0, all the θ_k's are id, so by Lemma 3.25 each application of τ'_n stabilizes by h^n. This means that taking $N = M \cdot n$ is sufficient. □

The following lemma identifies the fiber of ρ. It appears as Lemma 2.15 in [TGNL], and we cite it here without proof.

Lemma 3.27. *If $L \in \Lambda_0(E)$, then the map*

$$\begin{aligned} Q &\longrightarrow \rho^{-1}(H) \\ q &\longmapsto L \cdot q \end{aligned}$$

is a homotopy equivalence.

To get a genuine fibration, the idea is now to get rid of the h^N factor above by passing to a colimit where this action is the invertible.

Definition 3.28. We let $\Lambda^V_\infty(E)$ and $\mathcal{Q}_\infty$ denote the following colimts.

$$\Lambda^V_\infty(E) = \operatorname{colim}\left(\Lambda^V(E) \xrightarrow{h\cdot} \Lambda^V(E) \xrightarrow{h\cdot} \dots\right)$$

$$\mathcal{Q}_\infty = \operatorname{colim}\left(\mathcal{Q} \xrightarrow{h\cdot} \mathcal{Q} \xrightarrow{h\cdot} \dots\right)$$

Note that since ρ is invariant with respect to both actions of $\mathcal{Q}$, it descends to the colimit giving a map $\rho\colon \Lambda^V_\infty(E) \to \Lambda_0(E)$. Note also that the left and right actions of $\mathcal{Q}$ on $\Lambda^V(E)$ commute, as does the actions on $\mathcal{Q}$. Therefore we get right actions of $\mathcal{Q}$ on both the colimits. With these definitions we can turn the "homotopy lifting up to stabilization" statement of Proposition 3.26 into a genuine Serre fibration.

Corollary 3.29. *The map $\rho\colon \Lambda_\infty(E) \to \Lambda_0(E)$ is a Serre fibration with fiber homotopy equivalent to $\mathcal{Q}_\infty$.*

Proof. Assume we are given the following commutative diagram.

$$\begin{tikzcd}
D^n \arrow[r, "h_0"] \arrow[d, "i_0"'] & \Lambda^V_\infty(E) \arrow[d, "\rho"] \\
D^n \times [0,1] \arrow[r, "g"'] & \Lambda_0(E)
\end{tikzcd} \tag{3.11}$$

Since D^n is compact, h_0 factors through some finite step of the colimit. As we have remarked, reduction is invariant with respect to the maps of the colimit, so we get the following diagram.

$$\begin{tikzcd}
D^n \arrow[r, "\tilde{h}_0"] \arrow[d, "i_0"'] & \Lambda^V(E) \arrow[r] \arrow[dr, "\rho"] & \Lambda^V_\infty(E) \arrow[d, "\rho"] \\
D^n \times [0,1] \arrow[rr, "g"'] & & \Lambda_0(E)
\end{tikzcd} \tag{3.12}$$

We see that $\tilde{h}_0$ is a lift of g_0, so by definition it is a section of $g_0^*\Lambda^V_\infty(E)$. By Proposition 3.26 we have a map $\Theta\colon g_0^*\Lambda^V(E) \to g^*\Lambda^V(E)$, so we consider the following diagram.

$$\begin{tikzcd}
D^n \arrow[rrr, "\tilde{h}_0"] \arrow[dd, "i_0"'] & & & \Lambda^V(E) \arrow[r] \arrow[ddr, "\rho"] & \Lambda^V_\infty(E) \arrow[dd, "\rho"] \\
& g_0^*\Lambda^V(E) \times [0,1] \arrow[r, "\Theta"] & g^*\Lambda^V(E) \arrow[ur] & & \\
D^n \times [0,1] \arrow[ur, "\tilde{h}_0 \times id"] \arrow[rrrr, "g"'] & & & & \Lambda_0(E)
\end{tikzcd} \tag{3.13}$$

This diagram does not quite commute, since the two parallel arrows $D^n \to \Lambda^V(E)$ differ by a factor h^N coming from Θ. This difference can be killed off by including at different steps of the colimit $\Lambda^V_\infty(E)$, so the diagram displays the required lift. It is clear from Lemma 3.27 that the fiber of $\rho : \Lambda^V_\infty(E) \to \Lambda_0(E)$ must be homotopy equivalent to Q_∞. □

We are now ready to apply the classifying space construction to get rid of the fiber Q_∞. The two main properties we will need is the fact that Θ from Proposition 3.26 is Q-equivariant, and the fact that the functor $|B(-, Q)|$ is well behaved with respect to certain categorical constructions. We also remark that the monoid Q is locally a finite dimensional manifold, and hence a locally finite CW-complex. All the right Q-spaces are manifolds or colimits of manifolds, and hence CW-complexes. In other words, the hypothesis of Lemma 2.39 is satisfied, so all $|B(F, Q)|$'s appearing in this section are CW-complexes.

Corollary 3.30. *The map $|B(\Lambda^V_\infty(E), Q)| \to \Lambda_0(E)$ induced by symplectic reduction is a Serre fibration with contractible fibers, and hence a homotopy equivalence.*

Proof. For the fibration statement one can repeat the proof for Corollary 3.29 with appropriate modifications; starting with a diagram similar to (3.11), we use Lemma 2.41 to factor h_0 through some $|B(\Lambda_V(E), Q)|$, so we get a diagram similar to 3.12. Since Θ from Proposition 3.26 is Q-equivariant, we can apply Lemma 2.42 thrice, to get the following diagram.

$$\begin{tikzcd} g_0^*|B(\Lambda^V(E), Q)| \times [0,1] \arrow[r, "\Theta'"] \arrow[d, phantom, "\cong" rotate=-90] & g^*|B(\Lambda^V(E), Q)| \arrow[d, phantom, "\cong" rotate=-90] \\ |B(g_0^*\Lambda^V(E)) \times [0,1], Q)| \arrow[r, "|B(\Theta, Q)|"] & |B(g^*\Lambda^V(E)), Q)| \end{tikzcd}$$

One can check that Θ' satisfies appropriate conditions to give a diagram similar to (3.13).

By the invariance of ρ, the inclusion of any fiber $\rho^{-1}(L) \hookrightarrow F$ is an equivariant map, and it is not hard to see that passing this to $|B(-, Q)|$ gives an isomorphism $|B(\rho^{-1}(L), Q)| \cong (\rho')^{-1}(L)$. Thus by Lemma 3.27 we know the fiber of ρ' is homotopy equivalent to $|B(Q_\infty, Q)|$. By Lemma 2.41 we have that,

$$|B(Q_\infty, Q)| = \operatorname{colim}\big(|B(Q, Q)| \longrightarrow |B(Q, Q)| \longrightarrow ...\big).$$

Each $|B(Q, Q)|$ is contractible by Lemma 2.34, so the colimit is also contractible. □

As with the stable Gauss map, we pass to the limit $\mathbb{C}^\infty$.

Definition 3.31. Paralleling Definition 3.10, we let

$$\Lambda^{i\mathbb{R}^\infty}_\infty(\mathbb{C}^\infty) = \operatorname{colim}\big(\Lambda^{i\mathbb{R}}_\infty(\mathbb{C}) \xrightarrow{\mathbb{R}\oplus -} \Lambda^{i\mathbb{R}^2}_\infty(\mathbb{C}^2) \xrightarrow{\mathbb{R}\oplus -} ...\big)$$

Note that when we act with $\mathbb{R} \oplus -$ we do not use the permutation σ of definition 3.17. This means that the left actions $\mathbb{R} \oplus -$ and $h \cdot -$ commute.

Each square

$$\begin{tikzcd}
\Lambda_\infty^{i\mathbb{R}^n}(\mathbb{C}^n) \arrow[r, "\mathbb{R}\oplus -"] \arrow[d, "\rho"'] & \Lambda_\infty^{i\mathbb{R}^{n+1}}(\mathbb{C}^{n+1}) \arrow[d, "\rho"] \\
\Lambda_0(\mathbb{C}^n) \arrow[r, "\mathbb{R}\oplus -"'] & \Lambda_0(\mathbb{C}^{n+1})
\end{tikzcd}$$

commutes, so ρ descends to a map $\Lambda_\infty^{i\mathbb{R}^\infty}(\mathbb{C}^\infty) \to \Lambda_0(\mathbb{C}^\infty)$ which we will also denote by ρ. It is a purely formal result that this map is also a Serre-fibration with fiber $\mathcal{Q}_\infty$. The maps $\mathbb{R} \oplus -$ are also $\mathcal{Q}$-equivariant, so we get a natural $\mathcal{Q}$-action on $\Lambda_\infty^{i\mathbb{R}^\infty}(\mathbb{C}^\infty)$. By Lemma 2.41, applying $|B(-, \mathcal{Q})|$ commutes with equivariant colimits, so Corollary 3.30 also carries over. We summarize these results in the following corollary.

Corollary 3.32. *The map*

$$\rho\colon \Lambda_\infty^{i\mathbb{R}^\infty}(\mathbb{C}^\infty) \longrightarrow \Lambda_0(\mathbb{C}^\infty)$$

is a $\mathcal{Q}$ invariant Serre-fibration with fiber $\mathcal{Q}_\infty$. The induced map

$$|B(\Lambda_\infty^{i\mathbb{R}^\infty}(\mathbb{C}^\infty), \mathcal{Q})| \longrightarrow \Lambda_0(\mathbb{C}^\infty)$$

is a homotopy equivalence.

The next results are concerned with computing the homotopy type of the spaces involved. The first two will be important in the sequel, while the latter shows how the theorems so far are related to Bott periodicity.

Lemma 3.33. *For all $n, m \in \mathbb{N}$, the space $\Lambda_n^{i\mathbb{R}^m}(\mathbb{C}^m)$ is $(m-1)$-connected, and so the colimit $\Lambda_n^{i\mathbb{R}^\infty}(\mathbb{C}^\infty)$ is contractible.*

Proof. The space $\Lambda_0^{i\mathbb{R}^{m+n}}(\mathbb{C}^{m+n})$ is isomorphic to the affine space of quadratic forms on $\mathbb{R}^{n+m}$. The space $\Lambda_n^{i\mathbb{R}^m}(\mathbb{C}^m)$ is a submanifold which is isomorphic to the set of quadratic forms whose graphs are transverse to $\mathbb{C}^m \oplus \mathbb{R}^n$. We claim that the compliment of this set is contained in a submanifold of codimension $m+1$. To see this, note that $\mathbb{C}^m + \mathbb{R}^n + \mathrm{graph}(\mathrm{d}q)$ is the image of a $(3m+2n) \times (2m+2n)$ matrix. The dimension of this image is equal to the rank of the matrix, and we have transversaility if and only if this rank is $2m+2n$. If the rank is not maximal, the determinant of all the $\binom{m+2(m+n)}{2(m+n)}$ maximal minors must vanish. It is well known that $m+1 \le \binom{m+k}{k}$, so the non transversal set is contained in a set cut out by at least $m+1$ smooth functions.

Now for $k \le m-1$, let $a \in \pi_k(\Lambda_n^{i\mathbb{R}^m}(\mathbb{C}^m))$. By Whitney approximation, we can pick a smooth representative $a = [f]$. Since $\Lambda_0^{i\mathbb{R}^n+m}$ is affine, f is smoothly contractible in this space, i.e. it extends to a smooth map $\tilde{f}\colon D^{k+1} \to \Lambda_0^{i\mathbb{R}^n+m}$. Since $k+1 \le m < m+1$, standard transversality theory implies that this map can be modified relative to the boundary to avoid the set of codimension $m+1$ from before, giving an extension $f'\colon D^n \to \Lambda_n^{i\mathbb{R}^m}(\mathbb{C}^m)$, showing that $a = 0$. $\square$

This gives the following immediate consequence

Corollary 3.34. $\Lambda^{i\mathbb{R}^\infty}_\infty(\mathbb{C}^\infty)$ *is homotopy equivalent to* Z *as right* Q *spaces, where* Q *acts on* $\mathbb{Z}$ *by* $n \cdot q = n + \dim(q)$.

Proof. Since the maps defining the colimts commute, we have an isomorphism

$$\Lambda^{i\mathbb{R}^\infty}_\infty(\mathbb{C}^\infty) \cong \Big(\coprod_{m\in\mathbb{N}} \Lambda^{i\mathbb{R}^m}(\mathbb{C}^m) \times \mathbb{N}\Big)/\sim.$$

Where the equivalence relation is generated by $(L, i) \sim (\mathbb{R} \oplus L, i)$ and $(L, i) \sim (h \cdot L, i + 1)$. Consider the maps

$$\begin{aligned} \Lambda^{i\mathbb{R}^m}(\mathbb{C}^m) \times \mathbb{N} &\longrightarrow \mathbb{Z} \\ (L, i) &\longmapsto \dim(L) - m - 2i. \end{aligned} \tag{3.14}$$

These descend to the above quotient since $\dim(\mathbb{R} \oplus L) = \dim(L) + 1$ and $\dim(h \cdot L) = \dim(L) + 2$. Moreover the map is Q equivariant since $\dim(L \cdot q) = \dim(L) + \dim(q) = \dim(L) \cdot q$. An explicit computation shows that the preimage of any $a \in \mathbb{Z}$ is homeomorphic to $\Lambda^{i\mathbb{R}^\infty}_n(\mathbb{C}^\infty)$ for some $n \in \mathbb{N}$, which is contractible by Lemma 3.33. □

We also state the following lemma which is not strictly necessary, but which allows relating the result to Bott-periodicity.

Lemma 3.35. *There is a homotopy equivalence* $Q_\infty \to \mathbb{Z} \times \mathbb{Z} \times BO$.

Proof. We give a somewhat informal proof of this lemma. As in the proof of 3.34, we identify the colimit as

$$Q_\infty = \big(Q \times N\big)/\sim$$

where $(q, i) \sim (h \cdot q, i + 1)$. Consider the maps

$$\begin{aligned} Q \times N &\longrightarrow \mathbb{Z} \\ (q, i) &\longmapsto \dim(q) - 2i \end{aligned}$$

and

$$\begin{aligned} Q \times N &\longrightarrow \mathbb{Z} \\ (q, i) &\longmapsto \mathrm{ind}(q) - i. \end{aligned}$$

Both of these respect the above equivalence relation since $\dim(h) = 2$ and $\mathrm{ind}(h) = 1$, so they descend to maps $Q_\infty \to \mathbb{Z}$. We also get a map

$$E^- : Q_\infty \longrightarrow BO$$

by mapping any $q \in Q$ to its negative eigenspace (of its symmetric matrix representative). Formally this is an element of the Grassmanian $\mathbf{Gr}\big(\mathrm{ind}(q). \dim(q)\big)$, but we include it into the colimit BO. Now the usual stabilizations defining BO from $\mathbf{Gr}(n, m)$ are not compatible with the action of h, but this can be resolved by a change of basis. To show that this map is a fibration, we use the fact that the action $O(n) \to \mathbf{Gr}(k, n)$ is a fibration. We then associate any quadratic form to its symmetric matrix, and let $O(n)$ act on Q_n by conjugation. This satisfies

$$E^-(SAS^T) = S(E^-(A)),$$

so each $Q_n \to \coprod_k \mathbf{Gr}(k, n)$ is a fibration. This induces a fibration $Q_\infty \to BO$ in the colimit. We claim that the fiber of this map over any point is homotopy equivalent to the fiber of $E^- : Q_n^k \to \mathbf{Gr}(k, n)$, where Q_n^k is the space of nondegenerate quadratic forms on $\mathbb{R}^n$ with index k for some $n, k \in \mathbb{N}$. This should be believable since the two $\mathbb{Z}$ factors keep track of the stable index and dimension, its just an issue of choosing i and representative in BO compatibly.

To see that this fiber is contractible, note that it consists of a contractible choice of compliment E^+, as well as a contractible choice of eigenvalues. To be more precise, fix a single compliment V. Then the space of compliments is in bijection with the space of linear transformations $V \to E^-$, which is contractible. Just as in the preceding paragraph, the map E^+ is a fibration, so we can lift a contraction to get a retraction of our fiber onto a subspace of matrices whose eigenvector decompositions respect $E^+ \oplus E^-$. The space of possible eigenvalues in our fiber is $(-\infty, 0)^k \times (0, \infty)^{n-k}$, which is also contractible. If we contract this to $(-1, ..., -1, 1, ..., 1)$, we get a contraction of our fiber to the unique quadratic form given by the matrix $\mathrm{diag}(-1, ..., -1, 1, ..., 1)$. ☐

A corollary of the last few results is that the fibration sequence

$$Q_\infty \longrightarrow \Lambda_\infty^{i\mathbb{R}^\infty}(\mathbb{C}^\infty) \longrightarrow U/O$$

from Corollary 3.29 is actually the fibration

$$BO \times \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow U/O.$$

The following map of fiber sequences then shows $\Omega(U/O) \simeq BO \times \mathbb{Z}$, which is one of the eight homotopy equivalences of real Bott periodicity.

$$\begin{tikzcd}
\Omega(U/O) \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & U/O \arrow[d] \\
BO \times \mathbb{Z} \times \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & U/O
\end{tikzcd}$$

In [18], the arguments of this section are modified to actually prove all ten homotopy equivalences of real and complex Bott periodicity.

We make one last refinement of the results, namely switching $\mathbb{Z}$ for $\mathbb{N}$. This will be important when we define twisted generating functions in the next subsection.

Lemma 3.36. *The map $\mathbb{Z} \xrightarrow{+2} \mathbb{Z}$ induces a map $|B(\mathbb{Z}, Q)| \to |B(\mathbb{Z}, Q)|$ homotopic to the identity. Furthermore the inclusion $\mathbb{N} \to \mathbb{Z}$ induces a homotopy equivalence $|B(\mathbb{N}, Q)| \to |B(\mathbb{Z}, Q)|$.*

Proof. Consider the diagram

$$\begin{tikzcd}
\Lambda^{i\mathbb{R}^k}(\mathbb{C}^k) \arrow[r, "h\cdot"] \arrow[d, "h\cdot"] & \Lambda^{i\mathbb{R}^k}(\mathbb{C}^k) \arrow[r, "h\cdot"] \arrow[d, "h\cdot"] & \cdots \\
\Lambda^{i\mathbb{R}^k}(\mathbb{C}^k) \arrow[r, "h\cdot"] & \Lambda^{i\mathbb{R}^k}(\mathbb{C}^k) \arrow[r, "h\cdot"] & \cdots .
\end{tikzcd}$$

Since this commutes, we get an induced map in the colimit $\Lambda_\infty^{i\mathbb{R}^k}(\mathbb{C}^k)$. As we remarked in definition 3.10, the left action of h is compatible with passing to the limit $\mathbb{C}^\infty$, so we get a map of $\Lambda_\infty^{i\mathbb{R}^\infty}(\mathbb{C}^\infty)$. In the notation of the proof of Corollary 3.34, this map is equivalent to $(L, i) \mapsto (h \cdot L, i)$, which in view of equation (3.14) covers $\mathbb{Z} \xrightarrow{+2} \mathbb{Z}$. We now apply $|B(-, Q)|$ to get the following diagram where the vertical arrows are homotopy equivalences.

$$\begin{tikzcd}
{|B(\mathbb{Z}, Q)|} \arrow[rr, "+2"] & & {|B(\mathbb{Z}, Q)|} \\
{|B(\Lambda_\infty^{i\mathbb{R}^\infty}(\mathbb{C}^\infty), Q)|} \arrow[u, "\dim"] \arrow[rr, "h\cdot"] \arrow[d, "\rho"] & & {|B(\Lambda_\infty^{i\mathbb{R}^\infty}(\mathbb{C}^\infty), Q)|} \arrow[u, "\dim"'] \arrow[d, "\rho"] \\
\Lambda_0(\mathbb{C}^\infty) \arrow[rr, "id"] & & \Lambda_0(\mathbb{C}^\infty)
\end{tikzcd}$$

By inserting homotopy inverses in this digram, we get the first part of the statement.

Let D be any compact CW-complex. We use the bijection of 2.60 to define a map $[D, |B(\mathbb{Z}, Q)| \to [D, |B(\mathbb{N}, Q)|]$. In view of the preceding calculation, we can stabilize any Q-twisted map to $\mathbb{Z}$ by $+2$ without changing the homotopy class. Since D is compact, we can refine within the homotopy class to a finite cover, so by stabilizing a finite number of times we get a Q-twisted map to $\mathbb{N}$. It should be clear that this map is inverse to the map $[D, |B(\mathbb{N}, Q)|] \to [D, |B(\mathbb{Z}, Q)|]$ induced by the inclusion. In particular, taking $D = S^n$ we get a weak equivalence, which by Whitehead must be a homotopy equivalence. $\square$

Remark 3.37. We have included this alternative proof of Lemma 2.22 of [TGNL] since we were not quite able to verify the proof given there. In [TGNL] the following diagram is displayed, and is claimed to be a map of (quasi-)fibrations

$$\begin{tikzcd}
{|B(\mathbb{N}, Q)|} \arrow[r] \arrow[d] & {|BQ|} \arrow[r] \arrow[d, "id"] & {|B\mathbb{N}|} \arrow[d] \\
{|B(\mathbb{Z}, Q)|} \arrow[r] & {|BQ|} \arrow[r] & {|B\mathbb{Z}|.}
\end{tikzcd}$$

In view of Example 2.50, the rightmost map is a homotopy equivalence. If the rows were quasi-fibrations we could pass to the long exact sequence in homotopy groups, and use the five lemma to conclude that $|B(\mathbb{N}, Q)| \to |B(\mathbb{Z}, Q)|$ is a weak equivalence.

The best result we were able to find concerning this quasi-fibration is Proposition 2.45, which requires the extra assumption that any $n \in \mathbb{N}$ gives a homotopy equivalence

$$|B(\mathbb{N}, \mathcal{Q})| \xrightarrow{+n} |B(\mathbb{N}, \mathcal{Q})|.$$

This is not argued in [TGNL], and we were not able to come up with a direct proof of this.

3.6 Twisted generating functions

We are now ready to define twisted generating functions, and to state the main theorem of this section. We will give an outline of the proof, focusing on the parts concerned with the linear level, but we leave the details to [TGNL].

Definition 3.38. A twisted generating function on a manifold M consists of the following data:

- A directed open cover $(M_i)_{i \in I}$ of M.
- For each $i \in I$, a generating function (n_i, U_i, f_i) on M_i
- For each $i < j$, a map $q_{ij} : M_{ij} \to \mathcal{Q}_{n_j - n_i}$.

Note that this implicitly requires $i < j \implies n_i \leq n_j$. We use the notation $n_{ij} = n_j - n_i$. We will require a twisting condition and a cocycle condition. To make the twisting well defined, we require that for all $i < j$, $U_j \cap (M_{ij} \times \mathbb{R}^{n_j}) \subset (U_i \cap (M_{ij} \times \mathbb{R}^{n_i})) \times \mathbb{R}^{n_{ij}}$. Then the twisting condition is that for all $i < j$,

$$f_j(x, v, u) = f_i(x, v) \oplus q_{ij}(x)(u) \qquad \text{for } (x, v, u) \in U_j \cap (M_{ij} \times \mathbb{R}^{n_j}) \subset M_{ij} \times \mathbb{R}^{n_i} \times \mathbb{R}^{n_{ij}}.$$

The cocycle condition is that for all $i < j < k$,

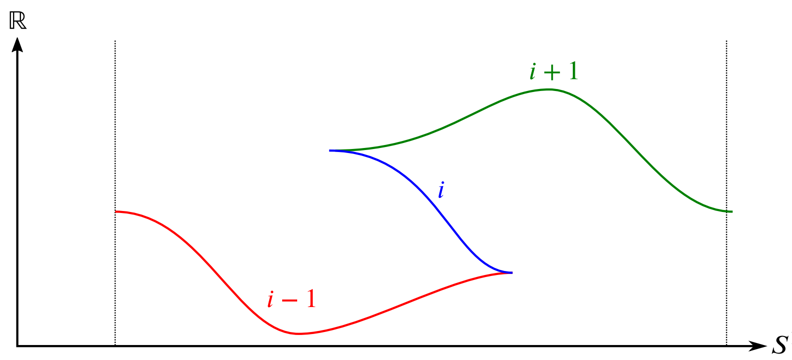
$$q_{ik}(x) = q_{ij}(x) \oplus q_{jk}(x) \qquad\qquad \text{for } x \in M_{ijk}.$$

Each of the generating functions (n_i, U_i, f_i) defines a singular manifold Σ_{f_i} and a Legendrian immersion $i_{f_i} \times f_i : \Sigma_{f_i} \to J_1(M)$. Since each nondegenerate quadratic form $q_{ij}(x)$ has a unique critical point at 0, the relation $f_j = f_i \oplus q_{ij}$ implies that

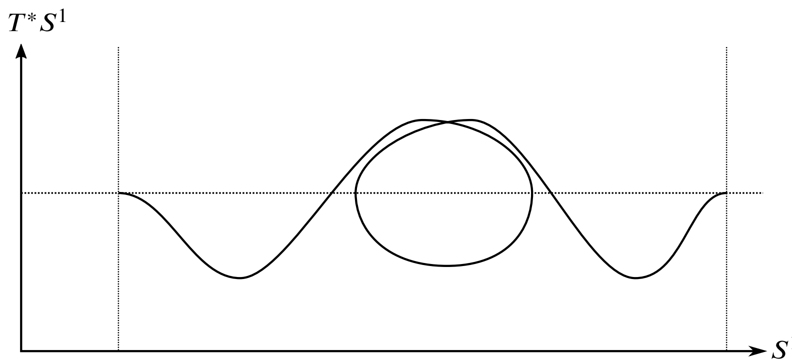
$$\Sigma_{f_j} \cap (M_{ij} \times \mathbb{R}^{n_j}) = \big(\Sigma_{f_i} \cap (M_{ij} \times \mathbb{R}^{n_i})\big) \times \{0\}.$$

This means that we can take the union over all these spaces to get a manifold Σ_f. Since the critical value of q_{ij} is zero, we have moreover that $f_i = f_j$ on the above sets. This, together with $f_j = f_i \oplus q_{ij}$ implies that we can glue together all the $i_{f_i} \times f_i$'s to a Legendrian immersion $i_f : \Sigma_f \to J^1(M)$. As before we say that a given Legnedrian immersion $\varphi \times z : L \to J^1(M)$ admits the twisted generating function $(M_i, n_i, U_i, f_i, q_{ij})$ if there exists a diffeomorphism $\psi : L \to \Sigma_f$ giving the factorization $\varphi \times z = (i_f \times f) \circ \psi$. Before we state the main result, we indicate how allowing twisting with a cocycle of quadratic forms can remove obstructions to having a generating function.

Example 3.39. Consider the following front diagram and corresponding Lagrangian over $M = S^1$.



(a) The front projection of a Legendrian in $J^1(S^1)$ with an attempt to coherently assign indices.



(b) The corresponding Lagrangian.

Figure 3: A Legendrian immersion that does not admit a generating funciton.

Imagine we wanted to find a generating function for this Legendrian. The index of the fiberwise critical points would have to be constant along the smooth branches of the front, and at each cusp, we would expect a birth/death event like in figure 2a. As we can see from the failed attempt to assign indices to such a function in a coherent way in figure 3a, it seems impossible to come up with a generating function for this Legendrian. Indeed, looking at the Lagrangian 3b, we can see that the "loop" represents a nontrivial Gauss map; when we go once around S^1, the tangent space also spins around one time.

The problem of coherent indices can be solved in this instance by allowing a twisted generating function. By locally adding a quadratic form of index 2 to the left hand side of this picture, we could at least have some hope of finding a generating function. The next theorem will formalize this intuition.

Note that the above example does not constitute a counterexample to the nearby Lagrangian conjecture since the Lagrangian projection is not embedded. This also highlights an interesting feature of the conjecture; while it is easy to construct Legendrians with nontrivial invariants that guarantee they cannot be isotopic to zero, there is no way to guarantee that the Lagrangian projection is embedded. In particular there is no Whitney trick for Lagrangians, since the dimension must always be half of the ambient dimension.

Proposition 3.40. *Let L and M be closed, smooth, n-dimensional manifolds. A Legendrian immersion $\varphi \times z : L \to J^1(M)$ admits a twisted generating function if and only if the stable Gauss map factors through $\pi : L \to M$ up to homotopy. In other words, if there exists $h : M \to \Lambda_0(\mathbb{C}^\infty)$ such that $h \circ \pi$ is homotopic to g_φ.*

Outline of proof. Assume first that $\varphi \times z$ admits a twisted generating function. The open cover M_i, the integers n_i and the quadratic forms define precisely a Q-twisted map to $\mathbb{N}$ over M, or equivalently (see Proposition 2.57) a simplicial map $MV(M_\bullet) \to B(\mathbb{N}, Q)$. Denote $L_i = \pi^{-1}(M_i)$, $\varphi_i = \varphi|_{L_i}$ and $E_i = E|_{L_i}$. Then the generating function $f_i : U_i \to \mathbb{R}$ and the diffeomorphism $\psi|_{L_i}$ define a lift ϕ_i as in Example 3.12.

$$\begin{tikzcd} & \Lambda^V_{n_i}(E_i) \arrow[d, "\rho"] \\ L_i \arrow[ur, "\phi_i"] \arrow[r, "G_{\varphi_i}"'] & \Lambda_0(E_i) \end{tikzcd}$$

The relations $f_j = f_i \oplus q_{ij}$ imply that on the tangent space level, we have $\phi_j = \phi_i \oplus \mathrm{graph}(\mathrm{d}q_{ij}) = \phi_i \cdot q_{ij}$ on the double intersection L_{ij}. After trivializing the bundle E and passing to the colimits, the sections ϕ_i turn into maps $\phi' : L_i \to \Lambda^{i\mathbb{R}^\infty}_\infty(\mathbb{C}^\infty)$. We define new maps $q'_{ij} : L_{ij} \to Q$ by $x \mapsto q_{ij}(\pi(x))$ These give a cocycle on $L_\bullet$, and satisfy the twisting condition $\phi'_j = \phi'_i \cdot q'_{ij}$. By construction each ϕ_i lifts the stable Gauss map, i.e. $\rho \circ \phi'_j = g_{\varphi_i}$. Together, the ϕ'_i's and q'_{ij}'s give precisely the data of a Q-twisted map to $\Lambda^{i\mathbb{R}^\infty}_\infty(\mathbb{C}^\infty)$, or equivalently a simplicial map $(\phi'_\bullet, q_{\bullet\bullet}) : MV(L_\bullet) \to B(\Lambda^{i\mathbb{R}^\infty}_\infty(\mathbb{C}^\infty), Q)$, which again lifts g_φ over symplectic reduction. Considering the explicit form of the homotopy equivalence $\dim : \Lambda^{i\mathbb{R}^\infty}_\infty(\mathbb{C}^\infty) \to \mathbb{Z}$ given in equation (3.14), it should be apparent that $\dim \circ \phi'_i = n_i$. All of the above information is summarized in the following commutative diagram of simplicial spaces, where arrows labeled "$\sim$" become homotopy equivalences after passing to geometric realizations, and where M, L and $\Lambda_0(\mathbb{C}^\infty)$ are seen as constant simplicial spaces.

$$\begin{tikzcd} M & MV(M_\bullet) \arrow[l, "\sim"'] \arrow[r, "n_\bullet"] & B(\mathbb{N}, Q) \arrow[r, "\sim"] & B(\mathbb{Z}, Q) \\ L \arrow[u, "\pi"] \arrow[drrr, "g_\varphi"'] & MV(L_\bullet) \arrow[l, "\sim"'] \arrow[rr, "\phi_\bullet"] & & B(\Lambda^{i\mathbb{R}^\infty}_\infty(\mathbb{C}^\infty), Q) \arrow[u, "\sim"] \arrow[d, "\sim"', "\rho"] \\ & & & \Lambda_0(\mathbb{C}^\infty) \end{tikzcd}$$

Passing to geometric realizations and picking homotopy inverses where necessary gives the required map $h : M \to \Lambda_0(\mathbb{C}^\infty)$.

For the opposite direction, we assume that $h : M \to \Lambda_0(\mathbb{C}^\infty)$ factors the stable Gauss map up to homotopy. We compose h with a homotopy inverse of $\rho : |B(\Lambda_\infty^{i\mathbb{R}^\infty}(\mathbb{C}^\infty), \mathcal{Q})| \to \Lambda_0(\mathbb{C}^\infty)$. Since M is compact, and by using Lemma 3.28 we can represent this map at a finite step of the colimits, giving a map $M \to |B(\Lambda^{i\mathbb{R}^m}(\mathbb{C}^m), \mathcal{Q})|$. We assume here that m is sufficiently large that g_φ also factors through some $g_1' : M \to \Lambda_0(\mathbb{C}^m)$. By Proposition 2.57 this gives a directed open cover $(M_i)_{i\in I}$ of M, and a $\mathcal{Q}$-twisted map to $\Lambda_\infty^{i\mathbb{R}^m}(\mathbb{C}^m)$ over this cover. By Lemma 2.64 we may assume this is given by smooth maps $\phi_i : M_i \to \Lambda_{n_i}^{i\mathbb{R}^m}(\mathbb{C}^m)$ and $q_{ij} : M_{ij} \to \mathcal{Q}_{n_{ij}}$, and by Lemma 2.63 we may assume M_i is finite and totally ordered. We now pull this data back along π by defining $L_i = \pi^{-1}(M_i)$, $\phi_i' = \phi_i \circ \pi|_{L_i}$ and $q_{ij}' = q_{ij} \circ \pi|_{L_{ij}}$. The way we have constructed these maps, θ_i' are not quite lifts of g', but since everything commutes up to homotopy, the maps $\rho \circ \theta_i'$ glue together into a map $g_0' : L \to \Lambda_0(\mathbb{C}^m)$ which is homotopic (at least after potentially increasing m) to g_1'. To remedy this, we use the smooth homotopy lifting property from Proposition 3.26. Letting g' be a homotopy from g_0' to g_1', we get a smooth $\mathcal{Q}$-equivariant map

$$\Theta : (g_0')^* \Lambda^{i\mathbb{R}^m}(\mathbb{C}^m) \times [0,1] \longrightarrow (g')^* \Lambda^{i\mathbb{R}^m}(\mathbb{C}^m).$$

We can view each ϕ_i' as a local section of $(g_0')^* \Lambda_0^{i\mathbb{R}^m}(\mathbb{C}^m)$, so composing each such section with Θ_1, we get new local sections

$$\phi_i'' = \Theta(\phi_i', 1) : L_i \longrightarrow (g_1')^* \Lambda^{i\mathbb{R}^m}(\mathbb{C}^m).$$

By definition these maps are local lifts of g_1. By the equivariance of Θ, the twisting condition $\phi_j'' = \phi_i'' \cdot q_{ij}'$ is still satisfied. It now remains to undo the trivialization; we need to turn the maps θ_i'' into local sections of $\Lambda^V(E)$ lifting the Gauss section $G_\varphi : L \to \Lambda_0(E)$. We have been intentionally vague about the exact procedure of trivializations, so we will not go into the details of this. Suffice it to say that this can be done at the cost of adding m extra auxiliary variables.

The goal is now to turn each θ_i'' into a genuine function. We note that a generating function (k, U, f) for a Legendrian immersion $\varphi : L \to J^1(M)$ is equivalent (through the association $f \mapsto j^1 f$) to an embedded Legendrian submanifold of $J^1(M \times \mathbb{R}^k)$ which is graphical over $U \subset M \times \mathbb{R}^k$, and which meets $J^1(M) \times \mathbb{R}^k$ transversally along an embedding of L. The idea for acheiving this is to use the Whitney embedding theorem to fix a map $\psi' : L \to \mathbb{R}^k$ for some $k \in \mathbb{N}$ such that $\psi = \pi \times \psi' : L \to M \times \mathbb{R}^k$ is an embedding. Then $\theta = \varphi \times \psi' \times z$ is an isotropic embedding of L in the contact manifold $J^1(M \times \mathbb{R}^k)$. Now the Weinstein neighborhood theorem for isotropic submanifolds of contact manifolds states that given a distribution of Legendrian planes tangent to an isotropic embedding, we can integrate to a Legendrian embedding. The goal is therefore to modify each of the lifts $\phi_i'' : L_i \to \Lambda_{k+n_i''}^V(E)$ to be tangent to the embedding $\varphi \times \psi' \times 0 \times z : L_i \to J^1(M \times \mathbb{R}^k \times \mathbb{R}^{n_i''})$, while keeping the reduction constant, and the twisting condition satisfied. The inductive procedure for doing this is detailed in [TGNL], and is based on the method found in [11]. For us it suffices to say that this is possible at the cost of further stabilization, and that the induction is possible since we took M_i to be a totally ordered finite cover. $\square$

To conclude this section we state the following immediate corollary which will be the starting point for the following section.

Corollary 3.41. *Any nearby Lagrangian $\varphi\colon L \to T^*M$ admits a twisted generating function.*

Proof. By definition of exact Lagrangian, the embedding $\varphi \times \varphi^*\lambda\colon L \to J^1(M)$ is Legendrian. By [1] the map $\pi\colon L \to M$ is a homotopy equivalence, so if we pick a homotopy inverse $r\colon M \to L$ and define $h = g_\varphi \circ r$, the hypothesis of Proposition 3.40 is satisfied. $\square$

4 Morse Homology

Using the Morse homology of a generating function to extract geometric information about the Lagrangian is a fundamental technique in symplectic topology. It has for instance been used to put topological bounds on Lagrangian intersections as in [7]. In the case of nearby Lagrangians we are especially interested in the Morse theory of each f_x. Passing to this homology can be seen as the passage from a generating function for L to a microlocal sheaf with singular support L [26]. Techniques from microlocal sheaf theory have shown great promise for proving theorems about symplectic topology [13]. In [TGNL], many of the homological computations of section 3 are carried out in the context of derived sheaves, which is natural considering the aforementioned connection. We will opt for a different yet formally equivalent argument, using the perhaps more well known language of spectral sequences and local systems.

Throughout this section we fix closed n-manifolds L and M, and a Legendrian embedding $\varphi \times z \to J^1 L$ covering a Lagrangian embedding $\varphi : L \to T^* M$. From Corollary 3.41, we know that $\varphi \times z$ admits a twisted generating function. There are several obstructions to doing Morse theory on such functions. First of all we only constructed *local* generating functions defined on open subsets $U_i \subset M_i \times R^{k_i}$. Extending each f_i to a function which is well behaved at infinity is done by the so called doubling trick. A second problem is the twisting. To do global computations we would like to "untwist" the functions $f_i : M_i \times R^{k_i}$ to form a single function $F : M \times \mathbb{R}^k \to \mathbb{R}$. This might not be possible directly, but after passing to the difference functions δf_i and stabilizing by quadratic forms, we will be able to untwist. The structure of the difference function means that we can still extract homological information about $(f_i)_x$ from F_x. The setup of all this will mostly be deferred to [TGNL], but we will give a brief summary in Subsection 4.2. In Subsection 4.3, we will use the Morse–Bott theory to show that

$$H^*(\{F \leq -4s_1\}, \{F \leq \infty\}; \mathbb{Z}) = H_{*+d}(M; \mathbb{Z}).$$

We will also set up a fibration $(\{F \leq -4s_1\}, \{F \leq \infty\}) \to M$ and use the Serre spectral sequence of this fibration to draw the desired conclusions. Some of our computations are sensitive to both compactness and actions of $\pi_1(M)$, so we will need to be a bit careful. In particular we will use the Galois theory of covering spaces $p : \tilde{M} \to M$ to come up with suitable finite covers. This theory is introduced in subsection 4.1. The finiteness of covers requires that we use coefficients in $\mathbb{Z}/P$ rather than $\mathbb{Z}$. Lemma 3.26 of [TGNL] will allow us to carry our results back to $\mathbb{Z}$ coefficients.

4.1 Covers and local systems

When $E \xrightarrow{p} B$ is a Serre fibration of CW-complexes with a simply connected base, the fibration condition and the connectedness of B imply that all the fibers $p^{-1}(x)$ are homotopy equivalent. To see this, consider a path γ connecting x to y, and apply the homotopy lifting property to the following diagram.

$$\begin{tikzcd} p^{-1}(x) \arrow[rr, hook] \arrow[d, "i_0"'] & & E \arrow[d, "p"] \\ p^{-1}(x) \times I \arrow[r, "pr_2"'] \arrow[urr, dotted, "G^\gamma"] & I \arrow[r, "\gamma"'] & B \end{tikzcd}$$

The commutativity of the diagram implies that restricting G^γ to $p^{-1}(x) \times \{1\}$ gives a map $G_1^\gamma : p^{-1}(x) \to p^{-1}(y)$. The homotopy lifting property also implies that homotopic maps give homotopic lifts. If we consider the reversed path γ^{-1}, both the composed paths $\gamma^{-1} \circ \gamma$ and $\gamma \circ \gamma^{-1}$ are homotopic to constant paths, so we can see that $G_1^{\gamma^{-1}}$ is a homotopy inverse of G_1^γ. Moreover the simply connectedness of B implies that the induced isomorphism $(G_1^\gamma)_* : H_*(p^{-1}(x); R) \to H_*(p^{-1}(y); R)$ is independent of choice of path. This information allows proving the existence of a homological Serre spectral sequence

$$E^2_{p,q} = H_q(B; H_q(F; R)) \Rightarrow H_{p+q}(E; R)$$

where F denotes an arbitrary fiber. If we however drop the simply connected assumption and only assume that B is connected, we get something slightly more complicated. The induced isomorphisms $(G_1^\gamma)_*$ are still independent of the homotopy class of the path γ, but when B is not simply connected there might exist several such homotopy classes. The preceding discussion essentially shows that the homology of all the fibers with coefficients in the ring R defines a *local coefficient system* of R-modules.

Definition 4.1. A local coefficient system $\mathcal{L}$ of R-modules on a space X consists of:

- For every point $x \in X$, an R-module $\mathcal{L}_x$.
- For any path $\gamma : [0, 1] \to X$, a module homomorphism $\gamma_* : \mathcal{L}_{\gamma(0)} \to \mathcal{L}_{\gamma(1)}$.

The morphisms are subject to the conditions that the constant path at x induces $id_{\mathcal{L}_x}$, and that $(\gamma \cdot \gamma')_* = \gamma_* \circ \gamma'_*$. This data is equivalent to a functor $\mathcal{L} : \Pi_1(X) \to R\textbf{-mod}$, where $\Pi_1(X)$ denotes the fundamental groupoid of X.

For path connected spaces X we work with the monodromy representation of a local system. This is given by a single R-module L and for every point $x \in X$ a representation $\pi_1(X, x) \to \mathrm{Aut}(L)$ together with compatible transfer morphisms for any path γ.

It is possible to define the homology of a space X with coefficients in a local coefficient system. If one denotes the local system associated to a fibration by $\mathcal{H}_*(F; R)$ there is a spectral sequence

$$E^2_{p,q} = H_q(B; \mathcal{H}_q(F; R)) \Rightarrow H_{p+q}(E; R).$$

Remark 4.2. It is also possible to get a Serre spectral sequence in relative homology. If $F \to E \to B$ and $F' \to E' \to B$ are fibrations, and we have a cofibration $E' \to E$ covering the identity on B, the relative homology $\mathcal{H}_*(F, F'; R)$ defines a local system on B, and we have a spectral sequence

$$E^2_{p,q} = H_q(B; \mathcal{H}_q(F, F'; R)) \Rightarrow H_{p+q}(E, E'; R).$$

This statement appears as exercise 5.6 in [20].

In our approach we will try to avoid working with homology of local systems. The only fact we will need is that if the monodromy $\pi_1(X, x) \to \mathrm{Aut}(L)$ is trivial, then the associated homology theory coincides with ordinary homology with coefficients in L. To turn any local system into a trivial one, we will use the following relationship between covers and $\pi_1(X, x)$ representations.

Proposition 4.3. *Let X be path connected and locally simply connected. Fix a basepoint $x \in X$ The functor Fib_x sending a covering space $p\colon \tilde{X} \to X$ to the fiber $p^{-1}(\{x\})$ induces an equivalence of categories between the category of covers and the category of left $\pi_1(X, x)$ sets. Connected covers correspond to sets with a transitive $\pi_1(X, x)$ action.*

Proof. See [25, Theorem 2.3.4]. □

The following corollary extracts the part of the above that will be relevant to us.

Corollary 4.4. *Let X be path connected and locally simply connected. For any subgroup $H \subset \pi_1(X, x)$, there exists a connected cover $p\colon X_H \to X$ such that $p^{-1}(x) \cong \pi_1(X, x_0)/H$ (the set of left cosets of H). Moreover, for a choice of basepoint $\tilde{x}_0 \in p^{-1}(x_0)$ we have $p_*(\pi_1(\tilde{X}, x_0) = H$.*

Proof. The action $g_1 \cdot (gH) = (g_1 g)H$ is clearly transitive, so by 4.3 we get a connected cover $p\colon X_H \to X$. The statement about $p_*(\pi_1(\tilde{X}, x_0))$ can be seen from the long exact sequence of the fibration p, whose bottom terms are

$$0 \to \pi_1(X_H, \tilde{x}_0) \xrightarrow{p_*} \pi_1(X, x) \to \pi_1(X, x)/H \to 0. \tag{4.1}$$

□

Definition 4.5. Let $f : X \to Y$ be a continuous map, and let $\mathcal{L}$ be a local system on Y. Define a local system $f^*\mathcal{L}$ on X by $(f^*\mathcal{L})_x = \mathcal{L}_{f(x)}$ for any $x \in X$. The pushforward along any path $\gamma : I \to X$ is given by the pushforward in $\mathcal{L}$ along the composition $f \circ \gamma : I \to Y$. If the spaces are connected and $\phi : \pi_1(Y, y) \to \mathrm{Aut}(L)$ is a monodromy representation of $\mathcal{L}$, it should be clear that the monodromy representation of the pullback $f^*\mathcal{L}$ is given by the composition

$$\pi_1(X, x) \xrightarrow{f_*} \pi_1(Y, y) \xrightarrow{\phi} \mathrm{Aut}(L) \tag{4.2}$$

for any $x \in f^{-1}(y)$.

We call a local system finite if each $\mathcal{L}_*$ is a finite module.

Corollary 4.6. *If $\mathcal{L}$ is a finite local system on a path connected locally simply connected space X, then there exists a finite cover $p : \tilde{X} \to X$ such that $p^*\mathcal{L}$ is trivial.*

Proof. Let $\mathcal{L}$ have the monodromy representation $\phi : \pi_1(X, x) \to \mathrm{Aut}(L)$. Since L is a finite module, $\mathrm{Aut}(L)$ is a finite group. Now $\ker(\phi)$ is a subgroup of $\pi_1(X, x)$, and the first isomorphism theorem gives $\pi(X, x)/\ker(\phi) \cong \mathrm{Aut}(L)$. Applying Corollary 4.4 to this situation we get a finite cover $p : \tilde{X} \to X$. Since $p_*(\pi_1(\tilde{X}, \tilde{x})) = \ker(\phi)$, the composed mondromy representation (4.2) is trivial. □

In the special case of local systems of the form $\mathcal{H}_*(F)$ for a fibration $F \to E \to B$, we can further characterize the pullback along a map.

Lemma 4.7. *Consider the following pullback of fibrations of CW-complexes where B and B' are connected.*

$$\begin{tikzcd}
F \arrow[d] & F \arrow[d] \\
f^*E \arrow[r] \arrow[d, "p'"] & E \arrow[d, "p"] \\
B' \arrow[r, "f"] & B
\end{tikzcd}$$

Then for any ring R, the local system associated to the fibration $f^*E \to B'$ is isomorphic to the pullback $f^*\mathcal{H}_*(F; R)$.

Proof. Since the spaces are connected we work with monodromy representations. The homotopy lifting property of f^*E is defined in terms of the universal property of the pullback, so looking back at the construction of $\mathcal{H}_*(F)$ we get the following commutative diagram.

$$\begin{tikzcd}
(p')^{-1}(x) \arrow[r, hook] \arrow[d, "i_0"'] & f^*E \arrow[r] \arrow[d] & E \arrow[d] \\
(p')^{-1}(x) \times I \arrow[r, "\gamma \circ pr_2"'] \arrow[ur, dotted] \arrow[urr, dotted] & B' \arrow[r, "f"'] & B
\end{tikzcd}$$

The left dotted arrow computes the action of $\gamma : I \to B'$ on $H_*(p^{-1}(x); R)$, while the right arrow computes the action of $f \circ \gamma$ on $H_*(p^{-1}(f(x)))$. The commutativity shows that the isomorphism $(p')^{-1}(x) \to p^{-1}(f(x))$ induces an isomorphism of the required local systems. □

4.2 The doubling trick

In this subsection, we explain the passage from a local twisted generating function of L to a globalized difference function F. Smooth functions have good extension properties, so we can always extend any $f : U \to \mathbb{R}$ to a smooth function $\tilde{f} : M \times \mathbb{R}^k \to \mathbb{R}$. The problem is that this operation tends to create a lot of new critical points, so $\tilde{f}$ might generate a Legendrian that is larger than the one we started with. The doubling trick is a trick to control these extra critical points, while also guaranteeing good behaviour at infinity. Specifically it turns a local generating function for $\varphi \times z$ into a global generating function which is *linear at infinity* for the s^0-double $\varphi \times (z \pm s^0) : L \coprod L \to J^1(M)$ for a sufficiently small constant s^0. This is carried out in Lemma 3.11 of [TGNL]. To recover L from its double. we wish to pull these two copies apart so that we can restrict our attention to a single copy. This is one of the places where the hypothesis that φ is an embedding is crucial; if any $\varphi \times (z \pm s)$ has a double point, then the projection to T^*M must be a double point of φ. Such a homotopy can be extended to an ambient compactly supported contact isotopy [10]. In [7] a homotopy lifting property for generating functions is shown. It can be modified slightly to show that having a generating function linear at infinity is invariant under compactly supported contact isotopy. When M is compact we can pick a s_1 such that $\varphi \times (z + s_1)$ is contained in $T^*M \times (0, \infty)$, while $\varphi \times (z - s_1)$ is contained in $T^*M \times (-\infty, 0)$. The whole construction so far is outlined in Figure 4. Explicitly, the doubled function is

$$f^t(x, v, w) = \tilde{f}(x, v) + w + \left(\frac{1}{4} + t\right) \alpha(x, v)(D(w) - w),$$

where $\tilde{f}$ is an arbitrary extension of f, $\alpha(x, v)$ is a smooth bump function which is 1 near Σ_f and 0 near $\Sigma_{\tilde{f}} - \Sigma_f$, and $D(w)$ is a smooth interpolation of w and $w^3 - 3w$ as illustrated in (4c). This function is linear at infinity since it is of the form

$$f(x, v, w) = w + g(x, v) + \varepsilon(x, v, w)$$

where $\mathrm{supp}(\varepsilon) \to M$ is proper [TGNL, Definition 3.1]

Extra care is needed to make all of this work for twisted generating functions. To even define a twisted generating function linear at infinity, one needs to modify the action of Q on functions by defining $\oplus_b$ in such a way that $g \oplus_b q$ is linear at infinity when g is. There is also some work needed to show that there is a homotopy lifting property for twisted functions under contact isotopy (see [TGNL, Theorem 3.9]).

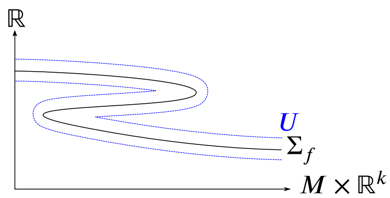
We now describe the untwisting. For any function $g : M \times \mathbb{R}^k$, we define the difference function

$$\delta g(x, v_1, v_1', ..., v_k, v_k') = g(x, v_1, ..., v_k) - g(x, v_1', ..., v_k').$$

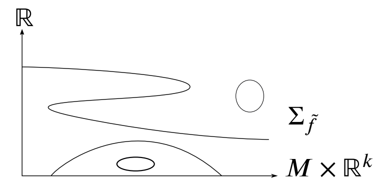
After further modifying $\oplus$, one can show that by applying the above construction to each g_i in a twisted generating function, we can find quadratic forms Q_i such that the functions $\delta g_i \oplus_b^\delta Q_i$ glue together to a single function $G : M \times \mathbb{R}^k \to \mathbb{R}$ [TGNL, Lemma 3.19]. Now the specific structure of the difference function allows a very explicit computation in Morse homology, essentially relating the homology computed by g_x and δg_x. Specifically, [TGNL, Lemma 3.20] states that if the critical points of g are contained in $(-2c, -c) \cup (c, 2c)$ for some $c > 0$ (which is the case for the $3s^1$-double with $c = 2s_1$), then there exists a chain complex C such that

$$H_*(\{\delta g_x < 2c\}, \{\delta g_x < -\infty\}) \simeq H_*(\{g_x < c\}, \{g_x < -\infty\}) \otimes^{\mathbf{L}} C. \tag{4.3}$$

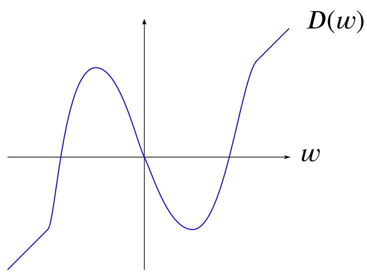
Note that $\otimes^{\mathbf{L}}$ denotes the total left derived functor of the tensor product (see [27, section 10.6]). Lemma 3.26 of [TGNL] will allow us to get rid of this C later, and it also allows us to work with field coefficients $\mathbb{Z}/p$, since $H_*(X, A; \mathbb{Z}/p) \simeq H_*(X, A; \mathbb{Z}) \otimes^{\mathbf{L}} \mathbb{Z}/p[0]$.



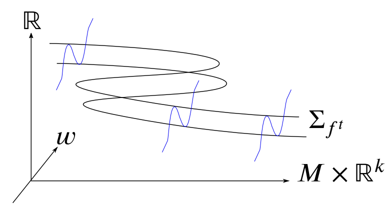
(a) Alocal generating function.



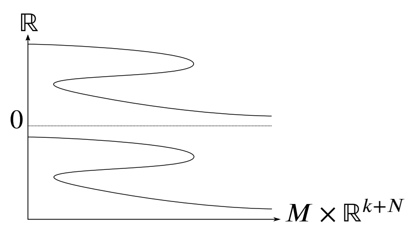
(b) An arbitrary extension with more critical points.



(c) The auxilliary function $D(w)$



(d) The doubled generating function f^t



(e) After the isotopy pulling the copies of L apart

Figure 4: The doubling trick illustrated.

4.3 Proof of the main result

We are now ready to state and prove the following, which is a precise formulation of Theorem 1.3.

Theorem 4.8. *Let L and M be closed smooth n-manifolds, and let $\varphi \times z : L \to J^1(M)$ be a Legendrian embedding covering a Lagrangian embedding φ. Then for sufficiently large $s > 0$, the s-double $\varphi \times (z \pm s)$ admits a twisted generating function $(b, M_i, n_i, f_i^s, q_{ij})$ such that for all i and all $x \in M_i$*

$$H_*\left(\left\{(f_i^s)_x \leq -4s_1\right\}, \left\{(f_i^s)_x \leq -\infty\right\}; \mathbb{Z}\right) = \mathbb{Z}[d_i].$$

Proof. **Step 1: Doubling and identifying critical points**

By Corollary 3.41, $\varphi \times z$ admits a local twisted generating function. Theorem 3.13 of [TGNL] states that we can apply the doubling trick to each f_i in a compatible way to get a twisted generating function linear at infinity for the s_0-double $\varphi \times (z \pm s_0)$. Now let $s_1 > \max_{x \in M}(|z(x)|)$. As remarked before, the hypothesis that φ is an embedding implies that

$$(L \coprod L) \times [0, 3s_1 - s_0] \xrightarrow{\varphi \times (z \pm (s_0 + t))} J^1(M)$$

is a homotopy of Legendrian embeddings. By the isotropic isotopy extension theorem [10, Theorem 2.41], this extends to a compactly supported ambient contact isotopy. Then by the homotopy lifting property for twisted functions linear at infinity [TGNL, Lemma 3.7], we get a twisted generating function $(b, M_i, n_i, f_i^{3s_1}, q_{ij})$ linear at infinity for the $3s_1$-double. Now consider one of the difference functions $\delta f_i^{3s_1}(x, v, v') = f_i^{3s_1}(x, v) - f_i^{3s_1}(x, v')$. Critical points (x, v) of this function satisfy the system

$$\partial_x f_i^{3s_1}(x, v) = \partial_x f_i^{3s_1}(x, v) \tag{4.4}$$

$$\partial_v f_i^{3s_1}(x, v) = 0 \tag{4.5}$$

$$-\partial_v f_i^{3s_1}(x, v') = 0. \tag{4.6}$$

Equations (4.5) and (4.6) imply that both (x, v) and (x, v') are fiber-wise critical points of $f_i^{3s_1}$, i.e., points in $\Sigma_{f_i^{3s_1}}$. This critical set is diffeomorphic to $L_i \coprod L_i$ through some $\psi_i : L_i \to \Sigma_{f_i^{3s_1}}$ satisfying $\varphi \times (z \pm 3s_1) = i_{f_i^{3s_1}} \circ \psi_i$. Equation (4.4) implies that

$$i_{f_i^{3s_1}}(x, v) = i_{f_i^{3s_1}}(x, v'),$$

so we must have $\varphi(\psi_i^{-1}(x, v)) = \varphi(\psi_i^{-1}(x, v'))$. Since $\varphi : L_i \coprod L_i \to T^*M$ is an embedding on each copy of L_i, we know that $\psi_i^{-1}(x, v)$ and $\psi_i^{-1}(x, v)$ are actually the same point p in L_i, but potentially living in different components of the disjoint union.

This implies that $f_i^{3s_1}(x,v) = z(p) \pm 3s_1 = f_i^{3s_1}(x,v')$. The possible critical values of $\delta f_i^{3s_1}$ are then

$$\begin{aligned}
&z(p) + 3s_1 - (z(p) + 3s_1) = 0\\
&z(p) - 3s_1 - (z(p) - 3s_1) = 0\\
&z(p) + 3s_1 - (z(p) - 3s_1) = 6s_1\\
&z(p) - 3s_1 - (z(p) + 3s_1) = -6s_1.
\end{aligned}$$

For any point $p \in L_i$ we denote p^+ and p^- for the corresponding points in each copy of $L_i \coprod L_i$, with $z(p^\pm) = z \pm 3s_1$. We write the diffeomorphism ψ as $\psi(p^\pm) = (\pi(p), v(p^\pm))$. (The π that appears here must be the $\pi : L \to M$ from before as one can easily check.) It should now be apparent that the critical points of $\delta f_i^{3s_1}$ with critical value $\leq -4s_1$ form an embedded copy of L_i. Concretely, an embedding is given by

$$\begin{aligned}
L_i &\to M \times \mathbb{R}^{k_i} \times \mathbb{R}^{k_i}\\
p &\mapsto (\pi(p), v_i(p^-), v_i(p^+)).
\end{aligned}$$

We now apply [TGNL, Lemma 3.19] to pick quadratic forms $Q_i : \mathbb{R}^{k_i} \to \mathbb{R}$ and glue together the functions $\delta f_i^{3s1} \oplus_b^\delta Q_i \to \mathbb{R}$ to a single function F, which is δ-linear at infinity [TGNL, Definition3.15]. Since the critical sets and values are unaffected by adding quadratic forms (up to adding 0 in the last k_i-coordinates), we can glue together the above embeddings to a single embedding

$$\begin{aligned}
L &\to M \times \mathbb{R}^{k} \times \mathbb{R}^{k}\\
p &\mapsto (\pi(p), v(p^-), v(p^+)),
\end{aligned} \tag{4.7}$$

rendering L as a critical submanifold of F.

Step 2: Fibration

Throughout this step, we let $a = -4s_1$ or $-\infty$. We wish to show that $\{F \leq a\} \to M$ is a fibration. We will begin by showing that the map is a submersion. The Ehresmann fibration theorem [6, Theorem 8.5.10] tells us that any *proper* surjective submersion is a locally trivial fibration. The specific structure of the function F at infinity will allow us to adapt the proof of the Ehresmann theorem found in [6] to our situation, even though the projection is definitely not proper.

For this computation we write the domain of F as $(x, v, v') \in M \times \mathbb{R}^k \times \mathbb{R}^k$, and denote the projection to M by pr_M. On the full domain, pr_M is trivially a submersion, so the same is true on the interior of the codimension 0 submanifold $\{F \leq a\}$. On the boundary $\{F = a\}$ we need to be more careful. The tangent space is $\ker(\mathrm{d}F)$, which, since a is a regular value, has codimension 1. The restriction of pr_M is a submersion if and only if this tangent space is transversal to the fibers $\mathbb{R}^{2k}$ of pr_M. The only way this transversality can fail is if $T_p\mathbb{R}^{2k} \subset \ker(\mathrm{d}_pF)$ at some point $p = (x, v_1, v_2) \in \{F = a\}$. This again is equivalent to p being a fiberwise critical point of F.

Since F locally looks like $\delta f_i^{3s_1} \oplus_b^\delta Q_i$, we know $p = (x, (v_1', 0), (v_2', 0))$ where $p' = (x, v_1', v_2')$ is a fiberwise critical point of $\delta f_i^{3s_1}$, and moreover that $F(p) = \delta f_i^{3s_i}(p')$. The situation now mirrors the computation of the critical values of F in Step 1, except that we have slightly less control. In particular, fiberwise critical points need only satisfy equations (4.5) and (4.6). This means (x, v_1) and (x, v_2) are fiberwise critical points of $f_i^{3s_1}$, not necessarily embedding in the same cotangent fibre. Still, by definition of s_1,

$$f_i^{3s_1}(x, v_1), f_i^{3s_1}(x, v_1) \in (-4s_1, -2s_1) \cup (2s_1, 4s_1),$$

which implies

$$\delta f_i^{3s_1}(x, v, v') = f_i^{3s_1}(x, v_1) - f_i^{3s_1}(x, v_1) \in (-8s_1, -4s_1) \cup (-2s_1, 2s_1) \cup (4s_1, 8s_1).$$

This shows that $\{F = a\}$ cannot contain any fiberwise critical points, and so we have our submersion.

The key to adapting the Ehresmann theorem to our situation is using that F is δ-linear at infinity [TGNL, Definition 3.15, Lemma 3.19]. We therefore now write the domain of F as $(x, w, w', v, v') \in M \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k$, and assume F has the form

$$F(x, w, w', v, v') = w - w' + G(x, v, v') + \varepsilon(x, w, v) - \varepsilon'(x, w', v'),$$

where $\mathrm{supp}(\varepsilon) \to M$ and $\mathrm{supp}(\varepsilon') \to M$ are proper. We may take some ball $B \subset \mathbb{R}^{2+2k}$ such that both supports are contained in $M \times B$, and consider the diffeomorphism

$$\begin{aligned} M \times (\mathbb{R} \times [0, -\infty) \times \mathbb{R}^{2k} - B') &\longrightarrow \{F \leq a\} - M \times B \\ (x, t, w', v, v') &\longmapsto (x, w' + a + G(x, v, v') - t, w', v, v'). \end{aligned} \tag{4.8}$$

This shows that outside the compact subset $M \times B$ the sublevel set is a trivial fiber bundle. On the compact set $M \times B$, the arguments of the Ehresmann theorem are valid. We will provide some more details on how this proof is adapted, but we encourage the reader to skip ahead if they are comfortable.

Let $f : M \to N$ be a proper surjective submersion. The proof of the Ehresmann theorem found in [6] can be summed up as follows.

1. Cover each fiber $f^{-1}(p_0)$ by charts U_p on which f looks like a projection. Since f is proper, we may pass to a finite subcover. Intersecting the images of U_p, we may fix a single chart U around p_0 such that $f : U_p \to U$ is a projection in these coordinates for all p. This reduces the situation to $N = \mathbb{R}^n$.

2. Lift the vector fields ∂x_i on $\mathbb{R}^n$ to each coordinate chart U_p. Glue these lifts together to vector fields X_i on M using a partition of unity. The ∂x_i have globally defined flows at all times, and since f is proper we can show that the same holds for X_i.

3. In $\mathbb{R}^n$, any point $(t_1, t_2, ..., t_n)$ can be flowed to 0 by first flowing along $-\partial x_i$ for time t_1, then along $-\partial x_2$ for time t_2 and so on. In M, restricting the flows along $-X_i$ to $f^{-1}(t_1, t_2, ..., t_n)$, and flowing for the same times as before, gives a diffeomorphism $f^{-1}(t_1, t_2, ..., t_n) \to f^{-1}(0)$. All these glue together to a diffeomorphism $M \to \mathbb{R}^n \times f^{-1}(0)$.

Note that while [6] assumes there is no boundary, this can easily be fixed when we assume $f : \partial M \to N$ is a submersion; just lift the ∂x_i's to vector fields on ∂M, and then smoothly extend to a lifts on M tangent to the boundary.

The two places where compactness is essential in this proof is to find a finite cover of each fiber by submersion charts, and to guarantee global existence of flows. The first can be remedied in our case by first choosing a finite subcover of $p_0 \times B \cap \{F \leq a\}$, and then using the chart from (4.8). For existence of flows of X_i, [6, Lemma 7.3.6] tells us that if the flow from a point q is not defined for all $t \in \mathbb{R}$, then the flowline must leave every compact set in finite time. This means the flowline from q must exit $M \times B$ a last time, and since the fibration is trivial outside $M \times B$, all flows exist here, giving a contradiction. Hence the flow of each X_i is defined for all $t \in \mathbb{R}$, and the rest of the above proof goes through.

Step 3: Morse–Bott theory

We show that L is a Morse–Bott critical submanifold for F. For any of the generating functions $f = f_i^{3s_1} : M \times \mathbb{R}^k \to \mathbb{R}$, the Hessian of the difference function δf is

$$H_{\delta f}(x, v, v') = \begin{pmatrix} \partial_x \partial_x f(x, v) - \partial_x \partial_x f(x, v') & \partial_x \partial_v f(x, v) & -\partial_x \partial_v f(x, v') \\ \partial_x \partial_v f(x, v) & \partial_v \partial_v f(x, v) & 0 \\ -\partial_x \partial_v f(x, v') & 0 & \partial_v \partial_v f(x, v') \end{pmatrix}.$$

At the embedded critical submanifold L_i, the tangent bundle splits as

$$T(M \times \mathbb{R}^{2k}) = TL_i \oplus NL_i.$$

The hessian is a bilinear form on $T(M_i \times \mathbb{R}^{2k})$. It is a standard part of the theory that $H_{\delta f}$ vanishes on TL_i [3], so it induces a bilinear form $h_{\delta f}$ on $NL_i \cong T(M_i \times \mathbb{R}^{2k})/TL_i$. The critical submanifold is said to be Morse–Bott if $h_{\delta f}$ is nonsingular.

Some linear algebra shows that the induced form $h_{\delta f}$ must have the same rank as $H_{\delta f}$ since we are quotienting out a subspace of the kernel of $H_{\delta f}$. In our case the rank of both TL_i and TM_i is n, so the rank of NL is $2k$. Since f is a generating function, 0 is a regular value of the function $(x, v) \mapsto \partial_v f(x, v) \subset (\mathbb{R}^k)^*$. As we argued in part 1, any critical point (x, v, v') of δf has $\partial_v f(x, v) = \partial_v f(x, v') = 0$, so these are regular points. The derivatives must therefore have rank k, and they have the form

$$d_{(x,v)}(\partial_v f) = \begin{pmatrix} \partial_x \partial_v f(x, v) & \partial_v \partial_v f(x, v) \end{pmatrix}$$

which we recognize in the bottom $2k$ rows of the Hessian, and whose transpose we recognize in the last $2k$ columns of H_f. We may perform row operations in the middle k-rows and column operations in the final k-columns of $H_{\delta f}$ to get $2k$ pivots. We then have

$$2k \leq \operatorname{rank}(H_{\delta f}) = \operatorname{rank}(h_{\delta f}) \leq \operatorname{rank}(NL_i) = 2k.$$

Hence $h_{\delta f}$ must be nonsingular, and so L_i is Morse–Bott. If Q is a nonsingular quadratic form, it is not hard to see that $L_i \times \{0\}$ becomes a Morse–Bott critical submanifold of $\delta f \oplus Q$. Since F has this form near the critical submanifold L, it must be Morse–Bott.

We now wish to compute $H_*(\{F \leq -4s_1\}, \{F \leq -\infty\})$ by adapting a standard Morse theoretic argument (see for instance [22, Theorem 3.14]) to the situation at hand. Since F is δ-linear at infinity, we fix a pseudo-gradient vector field X which agrees with ∂_w outside a compact set. Let W_L^- denote the unstable manifold of L. As in [22] we use the negative gradient flow and a tubular neighbourhood to retract $\{F \leq 4s_1\}$ onto $\{F \leq -\infty\} \cup W_L^-$. Since L is Morse–Bott, it has a well defined negative eigenbundle $\nu_L^- \subset NL$ of rank $d = \mathrm{ind}(L)$. We denote the disc and sphere bundle of ν_L^- by $D(\nu_L^-)$ and $S(\nu_L^-)$ respectively. Note that any flowline in W_L^- reaches $\{F \leq -\infty\}$ in finite time, since any such line must enter the subspace where $X = \partial_w$ in finite time. The Morse–Bott lemma then gives a parametrization $D(\nu_L^-) \xrightarrow{\cong} W_L^- \cap \{F \geq -\infty\}$, such that $D(\nu_L^-) \cap \{F \leq -\infty\} = S(\nu_L^-)$ up to identifications. If ν_L^- is orientable, we get the following isomorphisms of integral homology groups.

$$H_*(\{F \leq -4s_1\}, \{F \leq -\infty\}) \cong H_*(D(\nu_L^-), S(\nu_L^-)) \cong H_{*+d}(L) \cong H_{*+d}(M).$$

The first isomorphism follows from excision. The second isomorphism is the Thom isomorphism for an orientable rank d vector bundle, and the final isomorphism follows from the homotopy equivalence $\pi\colon\ M \to L$. If ν_L^- is not orientable, L has a double cover such that the pullback of ν_L^- to this cover is orientable. The passage to such a cover is a process we will need several times, so we formulate it as a lemma, the proof of which we postpone.

Lemma 4.9. *Any finite connected cover of either L or M induces a finite connected cover of the other such that the following square commutes*

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\tilde{\pi}} & \tilde{M} \\ \big\downarrow & & \big\downarrow p \\ L & \xrightarrow{\pi} & M. \end{array}$$

Moreover, if we set

$$\begin{aligned} p^*F\colon\ \tilde{M} \times \mathbb{R}^k &\to \mathbb{R} \\ (x, v) &\mapsto F(p(x), v) \end{aligned}$$

then $\mathrm{crit}_{\leq -4s_1}(p^*F)$ is an embedded copy of $\tilde{L}$ and a Morse–Bott critical submanifold. Moreover, for any regular value, the projection $\{p^*F \leq a\} \to \tilde{M}$ is the pullback of the fibration $\{F \leq a\} \to M$, so by Lemma 4.7 there is an isomorphism of local systems

$$p^*H_*(\{F_x \leq -4s_1\}, \{F_x \leq -\infty\}; R\} \cong H_*(\{p^*F_x \leq -4s_1\}, \{p^*F_x \leq -\infty\}; R\})$$

for any ring R.

Step 4: Spectral sequence

For this step, we assume that M is orientable. If it is not, we make the necessary replacements as in Lemma 4.9 so that M is replaced with an orientable cover. For any prime P, the pair of fibrations

$$(\{F \leq -4s_1\}, \{F \leq -\infty\}) \to M$$

defines a local system in homology with $\mathbb{Z}/P$ coefficients which we denote G_* for convenience. Since F is δ-linear at infinity, the critical points of F_x are contained in a compact set, so the homology is finitely generated. Since Z/P is a finite ring, this means that G_* is a finite local system. If this local system is not trivial, we use corollaries 4.6 and 4.9 and make the necessary replacements so that the local system becomes constant. Assume now that the smallest nonzero degree of G_* is a, while the largest nonzero degree is b. Since we have only passed to finite connected covers, M is still a closed, connected and orientable n-manifold, which means that M has Poincare duality. The E^2 page of the relative homology Serre spectral sequence is shown below.

$$
\begin{array}{c|ccccccc}
\vdots & & & & & & & \\
b+1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
b & G_b & H_1(M;G_b) & \cdots & H_{n-1}(M;G_b) & G_b & 0 & \cdots \\
\vdots & & & & & & & \\
d & G_d & H_1(M;G_d) & \cdots & H_{n-1}(M;G_d) & G_d & 0 & \cdots \\
\vdots & & & & & & & \\
a & G_a & H_1(M;G_a) & \cdots & H_{n-1}(M;G_a) & G_a & 0 & \cdots \\
a-1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
\vdots & & & & & & & \\
\hline
 & 0 & 1 & \cdots & n-1 & n & n+1 &
\end{array}
$$

Note that since this converges to $H_{*+d}(M;\mathbb{Z}/P)$, the only total degrees that can have surviving nonzero elements are d through $n+d$. The differentials go up and to the left, so the highlighted G_a placed at $E^2_{0,a}$ will survive. Its total degree is a, so this implies $a \geq d$. Likewise, the highlighted G_b placed in $E^2_{n,b}$ will survive, but has total degree $n+b$. This implies $b \leq d$. All this means that G_* is concentrated in degree d, and so the spectral sequence collapses at E^2. By considering total degree d, we get that $G_d \cong H_0(M;\mathbb{Z}/P) = \mathbb{Z}/P$. If we combine this with [TGNL, Lemma 3.20], we have that for all primes P, all $i \in I$ and all $x \in M_i$, there exists a bounded chain complex C

such that

$$\begin{aligned}
&H_*(\{(f_i^{3s_1})_x \le 0\}, \{(f_i^{3s_1})_x \le -\infty\}; \mathbb{Z}) \otimes^{\mathbf{L}} C \otimes^{\mathbf{L}} \mathbb{Z}/P \\
\simeq &H_*(\{(\delta f_i^{3s_1})_x \le -4s_1\}, \{(\delta f_i^{3s_1})_x \le -\infty\}; \mathbb{Z}/P) \\
\simeq &H_*(\{F_x \le -4s_1\}, \{F_x \le -\infty\}; \mathbb{Z}/P)[r_i] = G_*[r_i] = \mathbb{Z}/P[d + r_i].
\end{aligned}$$

Note that the shift by r_i appears since adding the quadratic form

$$F_x = (\delta f_i^{3s_1})_x \oplus_b^\delta Q_i$$

has the effect of suspending the sublevel set by $\mathrm{ind}(Q_i) = r_i$. We are also using that

$$H_*(X, A; Z) \otimes^{\mathbf{L}} \mathbb{Z}/P \simeq H_*(X, A; \mathbb{Z}/P)$$

by definition. Setting $d_i = d + r_i$ and applying [TGNL, Lemma 3.26] to this situation gives

$$H_*(\{(f_i^{3s_1})_x \le 0\}, \{(f_i^{3s_1})_x \le -\infty\}; \mathbb{Z}) = \mathbb{Z}[d_i]$$

which is the desired result for $s = 3s_1$. It is easily seen that the computation remains valid for any $s \ge 3s_1$.

Step 5: Proof of Lemma 4.9

Let $p: \tilde{M} \to M$ be a cover. We define

$$\begin{aligned}
p^*F: \tilde{M} \times \mathbb{R}^k &\to \mathbb{R} \\
(x, v) &\mapsto F(p(x), v)
\end{aligned}$$

Since p is a local diffeomorphism, it should be clear that (x, v) is a critical point of p^*F if and only if $(p(x), v)$ is a critical point of F. This means that we have the following pullback square.

$$\begin{tikzcd}
\mathrm{crit}_{\le -4s_1}(p^*F) \arrow[r, "p\times id"] \arrow[d, "pr_{\tilde{M}}"] & \mathrm{crit}_{\le -4s_1}(F) \arrow[d, "pr_M"] \\
\tilde{M} \arrow[r, "p"] & M
\end{tikzcd}$$

The pullback of a cover is a cover, so $p \times id$ is a covering map with the same discrete fiber as p. Considering the explicit form of the embedding in (4.7) it is clear that the following commutes.

$$\begin{tikzcd}
L \arrow[r, "\cong"] \arrow[dr, "\pi"'] & \mathrm{crit}_{\le -4s_1}(F) \arrow[d, "pr_M"] \\
& M
\end{tikzcd}$$

In particular, pr_M must be a homotopy equivalence. This means that the pullback $pr_{\tilde{M}}$ is also a homotopy equivalence. In summary, the critical points of p^*F form a cover $\tilde{L} \to L$ which is homotopy equivalent to $\tilde{M}$. Note that by picking a homotopy inverse of π we could just as well have started with a cover of L. $\square$

Remark 4.10. The spectral sequence in Step 4 of this proof bears some resemblance to the one appearing in [17]. which is given by

$$E_2^{*,*} = H^*(M; HF_*(L, T_x^*)) \Rightarrow HF_*(L, L).$$

Here, T_x^* denotes the cotangent fiber over $x \in M$, and $HF_*(-,-)$ denotes Floer intersection homology. Note that when M is not simply connected, $HF_*(L, T_x^*)$ defines a local system on M. In the proof of the Conley–Zehnder theorem found in [21], a generating function appears as a finite dimensional approximation of the action functional on the loop space, allowing one to replace fixpoint Floer homology with Morse homology of the generating function. Therefore it is a natural question to ask what the relationship between $HF_*(L, T_x^*)$ and the local system G_* appearing in our proof is. Maybe there is a concrete way to view the Morse homology of $(\{F_x \leq 0\}, \{F_x \leq -\infty\})$ as a finite dimensional approximation to this Floer homology. In that case, the above proof could be replaced with a direct appeal to the computation in [17], rather than using the homotopy equivalence $\pi : L \to L$.

4.4 From homology to geometry

What can the homological result 4.8 tell us about the geometry of L? In sufficiently high dimension, the H-cobordism theorem holds, and this certainly gives one procedure to turn homological results in to geometric ones. This procedure will be explained in more detail below, and will show that we can homotope each $(f_i)_x$ to have the simple form $D \oplus Q$, where D is the function from 4c, and Q is a quadratic form. If one could perform this homotopy for all f_x simultaneously, we could actually untwist. Specifically, assume that we have functions

$$F_i : [0,1] \times U_i \times \mathbb{R}^{k_i} \to \mathbb{R}$$

such that:

1. $F_i \oplus q_{ij} = F_j$ on U_{ij}.
2. $F_i\big|_{t=0} = f_i$
3. $F_i\big|_{t=1} = D \oplus Q_i$ for some $Q_i : U_i \to Q$.

The twisting condition at $t = 1$ implies that $Q_i \oplus q_{ij} = Q_j$. Up to reordering, we could now define a new twisted generating function by $(U_i, f_i \oplus -Q_i, q_{ij} \oplus -q_{ij})$. Performing an argument similar to the proof of [TGNL, Lemma 3.19], it now seems we could untwist this function.

The obstruction to turning the fiberwise homotopies into a simultaneous homotopy comes from higher parametric Morse theory. At the present time the possibility of this is unclear. In section 4 of [TGNL] however, the machinery of algebraic K-theory of spaces is applied to the situation. The main result obtained from this is that the stable Gauss map vanishes on homotopy groups.

Up to stabilizing with an appropriate quadratic form, we may assume that each $f = (f_i^s)_x : \mathbb{R}^{n_i} \to \mathbb{R}$ has $n_i \geq 6$, and that no critical points have index $0, 1, n_i - 1$ or n_i. This means that using techniques from [22], we can smoothly modify f on a compact subset of $\{f \leq 0\}$ away from $f^{-1}(0)$ to achieve the following:

1. Make f self indexing.

2. Perform handle slides and pick orientations such that the critical points give a basis of $C_\bullet(f)$ in which all the Morse differentials ∂_n are diagonal with only 1 or 0 on the diagonal.

3. Cancel corresponding critical points from the bottom up leaving only the critical point of index d_i generating the homology.

Note that by flipping the function, everything we have done so far for $(\{f_i^s \leq 0\}, \{f_i^s \leq -\infty\})$ can be repeated for $(\{f_i^s \geq 0\}, \{f_i^s \geq \infty\})$. All this shows that we can modify $(f_i^s)_x$ smoothly through functions linear at infinity for which 0 is a regular value to a function with precisely two critical points, which must have indices d_i and $d_i + 1$ since $\mathbb{R}^{n_i}$ is contractible. We are now very close to showing that f_i^s is of *tube type*.

Definition 4.11. A function $f : \mathbb{R}\times\mathbb{R}^k \to \mathbb{R}$ is of tube type if there is a function linear at infinity $F : [0,1]\times\mathbb{R}\times\mathbb{R}^k \to \mathbb{R}$ and a quadratic form $Q : \mathbb{R}^k \to \mathbb{R}$ such that for all t, 0 is a regular value of the function $(x, w) \mapsto F(t, w, x)$, and such that the restrictions of F to 0 and 1 are f and $D \oplus_b Q$ respectively.

Definition 4.12. A twisted generating function of tube type is a twisted generating function linear at infinity such that all fiberwise functions $(f_i)_x$ are of tube type.

The final details needed to show the following can be found in [TGNL, Proposition 3.23].

Corollary 4.13. *For any sufficiently large s, let f_i^s be a twisted generating function linear at infinity for the s-double of a Legendrian embedding covering a Lagrangian embedding as in theorem 4.8. Then there exists some integer N such that $h^N \oplus f_i$ is of tube type.*

In section 4 of [TGNL], this corollary is linked to algebraic K-theory by Waldhausen's spaces of tubes. A space $\mathcal{T}$ of functions of tube type is defined. This space has compatible right and left actions of $\mathcal{Q}$ defined by the modified direct sum operation $- \oplus_b q$. Stabilization in $\mathcal{T}$ is defined by left acting with h as before, and the colimit over this is denoted $\mathcal{T}_\infty$. A stable $\mathcal{Q}$-equivariant dimension map $\mathcal{T}_\infty \to \mathbb{Z}$ is defined analogously to the proof of Lemma 3.34. Since $\mathbb{Z}$ is discrete, this map is a fibration. In [TGNL, p. 4.2] it is shown that for all $q \in \mathcal{Q}$, the action $\mathcal{T}_\infty \xrightarrow{\cdot q} \mathcal{T}_\infty$ is a homotopy equivalence, and so Lemma 2.37 gives a quasifibration of classifying spaces

$$|B(\mathcal{T}_\infty, \mathcal{Q})| \longrightarrow |B(\mathbb{Z}, \mathcal{Q})|. \tag{4.9}$$

By taking the level set at 0, we get a map from $\mathcal{T}$ to Waldhausen's space $\mathcal{W}$. The elements of $\mathcal{W}$ are hypersurfaces in some $\mathbb{R}\times\mathbb{R}^k$ isotopic to the space obtained by attaching a trivial handle to $\{0\}\times\mathbb{R}^k$. A linear subspace $V \subset \mathbb{R}^k$ determines such a hypersurface by attaching a tube along the unit circle in V, resulting in maps $r : \mathbf{Gr}(i, k) \to \mathcal{W}$. Two commuting stabilizations on $\mathcal{W}$ are defined in a way that is compatible with the stabilizations $\mathbf{Gr}(i, k) \to \mathbf{Gr}(i + 1, k)$ and $\mathbf{Gr}(i, k) \to \mathbf{Gr}(i, k + 1)$ under the map r. The resulting map on colimts is called the rigid tube map $r : BO \to \mathcal{W}_\infty$.

Böksted's theorem is a deep result of algebraic K-theory which states that r is a rational homotopy equivalence. In [TGNL, section 4.2] it is argued by considering the eight different homotopy groups of BO that Böksted's theorem actually implies r is injective on homotopy groups.

It is further argued that the fiber of $\mathcal{T}_\infty \to \mathbb{Z}$ (and hence of (4.9),) is homotopy equivalent to $\mathcal{W}_\infty \times \mathbb{Z}$, and that the fiber of $\mathcal{Q}_\infty \to \mathbb{Z}$ is homotopy equivalent to $BO \times \mathbb{Z}$. Moreover, it is shown that the map $\mathcal{Q}_\infty \to \mathcal{T}_\infty$ defined by right-acting on D determines the following map of fibration sequences.

$$\begin{tikzcd}
 & * \arrow[d, "\sim"] & \\
BO \times \mathbb{Z} \arrow[r] \arrow[d, "r\times id"] & {|B(\mathcal{Q}_\infty, \mathcal{Q})|} \arrow[r] \arrow[d] & {|B(\mathbb{Z}, \mathcal{Q})|} \arrow[d, "id"] \\
\mathcal{W}_\infty \times \mathbb{Z} \arrow[r] & {|B(\mathcal{T}_\infty, \mathcal{Q})|} \arrow[r] & {|B(\mathbb{Z}, \mathcal{Q})|}
\end{tikzcd}$$

This induces a map of long exact sequences of homotopy groups, so for any $k > 0$ we have the following diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \pi_k{|B(\mathbb{Z}, \mathcal{Q})|} \arrow[r] \arrow[d] & \pi_{k-1} BO \arrow[rr] \arrow[d, "r_*"] \arrow[dl, dashed] & & 0 \arrow[d] \\
\pi_k{|B(\mathcal{T}_\infty, \mathcal{Q})|} \arrow[r, "\alpha"] & \pi_k{|B(\mathbb{Z}, \mathcal{Q})|} \arrow[r, "\beta"] & \pi_{k-1}\mathcal{W}_\infty \arrow[r, "\gamma"] & \pi_{k-1}{|B(\mathcal{T}_\infty, \mathcal{Q})|} &
\end{tikzcd}$$

By commutativity, we have $\gamma \circ r_* = 0$, so by exactness r_* factors through β. Since β factors r_*, which is injective, β must also be injective. Exactness then implies that $\mathrm{im}(\alpha) = \ker(\beta) = 0$, so $\alpha = 0$. Since $|B(\mathbb{Z}, \mathcal{Q})|$ is connected the argument is trivial at $k = 0$, and we have shown that the map $|B(\mathcal{T}_\infty, \mathcal{Q})| \to |B(\mathbb{Z}, \mathcal{Q})|$ vanishes on homotopy group. Pretty much by definition of $\mathcal{T}$, Corollary 4.13 implies that we can lift the stable Gauss map of any nearby Lagrangian along this map up to homotopy. Thus the stable Gauss map must be trivial on homotopy groups.

Refrences

[TGNL] Mohammed Abouzaid, Sylvain Courte, Stephane Guillermou, and Thomas Kragh. "Twisted generating functions and the nearby Lagrangian conjecture". In: *arXiv preprint arXiv:2011.13178* (2020).

[1] Mohammed Abouzaid and Thomas Kragh. "Simple homotopy equivalence of nearby Lagrangians". In: *Acta Mathematica* 220.2 (2018), pp. 207–237.

[2] Nikolaos Adaloglou. "Uniqueness of Lagrangians in T^*RP2". In: *arXiv preprint arXiv:2201.09299* (2022).

[3] Augustin Banyaga and David E Hurtubise. "A proof of the Morse-Bott lemma". In: *Expositiones Mathematicae* 22.4 (2004), pp. 365–373.

[4] Marco Brunella. "On a theorem of Sikorav". In: *Enseign. Math.(2)* 37.1-2 (1991), pp. 83–87.

[5] Georgios Dimitroglou Rizell, Elizabeth Goodman, and Alexander Ivrii. "Lagrangian isotopy of tori in $S^2 \times S^2$ and $\mathbb{C}P^2$". In: *Geometric and Functional Analysis* 26.5 (2016), pp. 1297–1358.

[6] Bjørn Ian Dundas. *A short course in differential topology*. Cambridge University Press, 2018.

[7] Yasha Eliashberg and Misha Gromov. "Lagrangian intersection theory: finite-dimensional approach". In: *Translations of the American Mathematical Society-Series 2* 186 (1998), pp. 27–118.

[8] Zbigniew Fiedorowicz. "A counterexample to a group completion conjecture of JC Moore." In: *Algebraic & Geometric Topology* 2 (2002), pp. 31–33.

[9] Kenji Fukaya, Paul Seidel, and Ivan Smith. "The symplectic geometry of cotangent bundles from a categorical viewpoint". In: *Homological mirror symmetry*. Springer, 2008, pp. 1–26.

[10] Hansjörg Geiges. "Contact geometry". In: *Handbook of differential geometry*. Vol. 2. Elsevier, 2006, pp. 315–382.

[11] Emmanuel Giroux. "Formes génératrices d'immersions lagrangiennes dans un espace cotangent". In: *Géométrie Symplectique et Mécanique*. Springer, 1990, pp. 139–145.

[12] Stéphane Guillermou. "Quantization of conic Lagrangian submanifolds of cotangent bundles". In: *arXiv preprint arXiv:1212.5818* (2012).

[13] Stéphane Guillermou. "Sheaves and symplectic geometry of cotangent bundles". In: *arXiv preprint arXiv:1905.07341* (2019).

[14] Allen Hatcher. "A short exposition of the Madsen-Weiss theorem". In: *arXiv preprint arXiv:1103.5223* (2011).

[15] Allen Hatcher. *Algebraic Topology*. 1st ed. Cambridge University Press, 2001. ISBN: 978-0521795401.

[16] Richard Hind. "Lagrangian unknottedness in Stein surfaces". In: *Asian Journal of Mathematics* 16.1 (2012), pp. 1–36.

[17] Thomas Kragh. “Homotopy equivalence of nearby Lagrangians and the Serre spectral sequence”. In: *Mathematische Annalen* 368.3 (2017), pp. 945–970.

[18] François Latour. “Transversales lagrangiennes, périodicité de Bott et formes génératrices pour une immersion lagrangienne dans un cotangent”. In: *Annales scientifiques de l’École Normale Supérieure*. Vol. 24. 1. 1991, pp. 3–55.

[19] J Peter May. *Simplicial objects in algebraic topology*. Vol. 11. University of Chicago Press, 1992.

[20] John McCleary. *A user’s guide to spectral sequences*. 58. Cambridge University Press, 2001.

[21] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Vol. 27. Oxford University Press, 2017.

[22] John Milnor. “Lectures on the h-cobordism theorem”. In: *Lectures on the H-Cobordism Theorem*. Princeton university press, 2015.

[23] Daniel Quillen. “Higher algebraic K-theory: I”. In: *Higher K-theories*. Springer, 1973, pp. 85–147.

[24] Graeme Segal. “Categories and cohomology theories”. In: *Topology* 13.3 (1974), pp. 293–312.

[25] Tamás Szamuely. *Galois groups and fundamental groups*. Vol. 117. Cambridge University Press, 2009.

[26] Claude Viterbo. “An introduction to symplectic topology through sheaf theory”. In: *Preprint* (2011).

[27] Charles Weibel and MCR Butler. “An introduction to homological algebra”. In: *Bulletin of the London Mathematical Society* 28.132 (1996), pp. 322–323.

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