William Tell

A Conditional Bound for the Riemann zeta-function in the Critical Line

Master's thesis in Mathematical Sciences
Supervisor: Kristian Seip
Co-supervisor: Andrés Chirre
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Abstract

This paper presents explicit constants for the upper bound for the logarithm of the Riemann zeta-function in the critical line, while assuming the Riemann hypothesis. This requires two considerations. First, we need an upper bound for the logarithm of the Riemann zeta-function when we are close to the critical line. We follow section 13.2 "Estimates for the zeta function" in "Multiplicative Number Theory I. Classical Theory" by Montgomery and Vaughan (2006) to obtain this. Second, we must implement a lower bound for the logarithmic derivative of the Riemann zeta-function, which is given in a result by Carneiro, Chirre and Milinovich (2019). We integrate this bound over an interval with length approaching zero. We optimize the argument by determining what interval length gives the best result.

Sammendrag

Denne oppgaven bestemmer eksplisitte konstanter for den øvre begrensningen av logaritmen til Riemann zeta-funksjonen på den kritiske linja, under antakelsen av Riemann-hypotesen. Dette krever to betraktninger. Vi finner først en øvre begrensning for logaritmen til Riemann zeta-funksjonen når vi er nær den kritiske linja. Dette gjør vi ved å følge seksjon 13.2 "Estimates for the zeta function" i "Multiplicative Number Theory I. Classical Theory" av Montgomery og Vaughan (2006). Deretter benytter vi oss av et resultat fra Carneiro, Chirre og Milinovich (2019) som gir en nedre begrensning for den logaritmiske deriverte av Riemann zeta-funksjonen. Vi integrerer denne over et intervall med lengde som går mot null. Vi optimaliserer argumentet ved å avgjøre hvilken lengde på intervallet som gir det beste resultatet.

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Contents

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function, where $s = \sigma + it$. It is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

which is absolutely convergent for $\sigma > 1$. From integration by parts one can get the integral representation formula, as Eq.(1.24) in [2]

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} (u - [u])u^{s-1}\, du,$$

which holds for $\sigma > 0$. From this half-plane of convergence, the function can be continued analytically to the whole complex plane via its functional equation, except at the simple pole $s = 1$. The function is non-zero for $\sigma \geq 1$. For $\sigma \leq 0$, the trivial zeros are situated at the negative even integers. This leaves only the non-trivial zeros that lie in the critical strip, i.e. where $0 < \sigma < 1$. The Riemann hypothesis, hereby referred to as RH, states that the non-trivial zeros of $\zeta(s)$ all lie on the line $\sigma = \frac{1}{2}$. A classical result of Littlewood [6] states that, under the assumption of RH, there is a constant $C > 0$ such that

$$\log|\zeta(\tfrac{1}{2} + it)| \leq C\,\frac{\log t}{\log\log t}, \tag{1.1}$$

for $t > 0$ sufficiently large. For simplicity, we let $\log_2(t)$ and $\log_3(t)$ denote $\log\log t$ and $\log\log\log t$, respectively. The order of magnitude has not been improved over the last ninety years, and the endeavors have since been concentrated in optimizing the value of C. While improving his own work [8] and building on the works of Ramachandra and Sankaranarayanan [7], Soundararajan together with Chandee [3] established in 2009 that

$$\log|\zeta(\tfrac{1}{2} + it)| \leq \frac{\log 2}{2}\,\frac{\log t}{\log_2 t} + O\left(\frac{\log t \log_3 t}{(\log_2 t)^2}\right). \tag{1.2}$$

This is the best up-to-date result for C in the upper bound of the Riemann zeta-function in the critical line. Going further, Carneiro and Chandee [4] improved the second order term on the right-hand side in (1.2), proving that

$$\log|\zeta(\tfrac{1}{2} + it)| \leq \frac{\log 2}{2}\,\frac{\log t}{\log_2 t} + O\left(\frac{\log t}{(\log_2 t)^2}\right). \tag{1.3}$$

The proofs in [4] and [3] rely on the use of the Hadamard product, the Guinand-Weil explicit formula and the construction of a certain extremal minorant for the function

$$\log\left(\frac{4+x^2}{(\sigma-\frac{1}{2})^2+x^2}\right), \quad \text{where } \tfrac{1}{2}\leq\sigma\leq 1 \text{ and } x\in\mathbb{R}.$$

The purpose of this paper is to establish a version of (1.3) with explicit constants. We have reached the following result.

Theorem 1. *Assume RH. For $t\geq 10^{1295}$, we have*

(1.4)

$$\log|\zeta(\tfrac{1}{2}+it)|\leq\frac{\log 2}{2}\frac{\log t}{\log_2 t}+2\frac{\log t}{(\log_2 t)^2}+0.696500\frac{\log t}{(\log_2 t)^3}+375.5451\frac{\log t}{(\log_2 t)^4}.$$

We will follow in the footsteps of Carneiro, Chirre and Milinovich and their proof of (1.3) in [1]. The idea is given in two steps, where the first step will be to obtain an explicit bound for $\log|\zeta(\sigma+it)|$, when $\sigma=\delta:=\frac{1}{2}+\frac{\log_3 t}{\log_2 t}$. One can acquire this by following the proof of Corollary 13.16 in [2], where the starting point will be a formula due to Selberg. The second step is lower bounds for

$$\Re\frac{\zeta'}{\zeta}(\sigma+it), \quad \text{when } \frac{1}{2}<\sigma\leq\delta.$$

It can be obtained, by following the proof of Theorem 2 (i) from [1] and computing the error terms explicitly. This proof depends on the use of the Guinand-Weil explicit formula and the construction of an extremal minorant for the Poisson kernel

$$m_\beta=\frac{\beta}{\beta^2+x^2}.$$

This extremal function is explicitly given, and it is easy to work with. Then we have to integrate over the lower bound from $\frac{1}{2}$ to δ. Note that it is possible to optimize this argument. By choosing

$$\delta_\lambda:=\frac{1}{2}+\lambda\frac{\log_3 t}{\log_2 t}$$

and optimizing over $\lambda>0$, will give the best possible result.

2. Estimates for the zeta-function via the Selberg moment formula

As already stated, we will follow the proof of Corollary 4 in [1]. We start off with the following integral

$$\int_{1/2}^{\delta} \Re\frac{\zeta'}{\zeta}(\sigma+it)\,d\sigma = \log|\zeta(\delta+it)| - \log|\zeta(\tfrac{1}{2}+it)|$$

which we can rearrange as follows

$$\log|\zeta(\tfrac{1}{2}+it)| = -\int_{1/2}^{\delta} \Re\frac{\zeta'}{\zeta}(\sigma+it)\,d\sigma + \log|\zeta(\delta+it)|. \tag{2.1}$$

Our choice of δ will be $\delta_\lambda = \frac{1}{2} + \lambda\frac{\log_3 t}{\log_2 t}$. So the problem is to find a lower bound for $\Re\frac{\zeta'}{\zeta}(\sigma+it)$ and an upper bound for $\log|\zeta(\delta+it)|$. This section will deal with the latter of the two. We will use the proof of the following result, Corollary 13.16 from [2], that while assuming RH, we have

$$\log|\zeta(\sigma+it)| \le \log\frac{1}{1-\sigma} + O\left(\frac{(\log t)^{2-2\sigma}}{(1-\sigma)\log_2 t}\right) \tag{2.2}$$

uniformly for $1/2+1/\log_2 t \le \sigma \le 1-1/\log_2 t$ and $t \ge 3$. We will retrace the steps in the proof, and get an estimate for $\log|\zeta(\sigma+it)|$ when $\sigma=\delta$. This corollary is proved by Corollary 13.15, which again is proved by Theorem 13.13 in [2].

We will begin with identifying a few important components which we will need later. The Riemann ξ-function, which is defined by

$$\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(s/2)\pi^{-s/2}, \tag{2.3}$$

is an entire function of order 1. It would be remiss not to mention, that it also satisfies the functional equation $\xi(s)=\xi(1-s)$ for all s. It carries significant information since the zeros of $\xi(s)$ are exactly the non-trivial zeros of $\zeta(s)$. We can write ξ as an Hadamard product over these zeros, as in Theorem 10.12 in [2]

$$\xi(s) = \frac{1}{2}e^{Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}.$$

By taking the logarithmic derivative of the product and applying the functional equation, one can find that B is given by

$$B = -\sum_{\rho}\Re\frac{1}{\rho}$$

If we now also take the logarithmic derivative of the right hand side in (2.3), we can establish the following relevant formula. First note that $1/2\, s\,\Gamma(s/2) = \Gamma(s/2+1)$, then

$$\frac{\zeta'}{\zeta}(s) + \frac{1}{s-1} + \frac{1}{2}\frac{\Gamma'}{\Gamma}(s/2+1) - \frac{1}{2}\log\pi = B + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{2.4}$$

We also introduce here the von Mangoldt function. It is denoted by $\Lambda(n)$ and defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and some integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then let $\psi(z)$ denote the function which sums over the von-Mangoldt function, i.e.

$$\psi(z) = \sum_{n \leq z} \Lambda(n).$$

Then, for $\sigma > 1$ we have the following relation

$$|\log\zeta(s)| \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}\, \log n}. \tag{2.5}$$

We start by proving Theorem 13.13 from [2], which asserts

$$\left|\frac{\zeta'}{\zeta}(s)\right| \leq \sum_{n \leq (\log t)^2} \frac{\Lambda(n)}{n^{\sigma}} + O\left((\log t)^{2-2\sigma}\right) \tag{2.6}$$

uniformly for $1/2 + 1/\log_2 t \leq \sigma \leq 3/2$. It will be necessary to find an explicit constant in front of the error term, because this will eventually lead to a contribution in the third-order term in our result. Eq.(13.35) in [2], which has been referred to as the Selberg moment formula, states that for $x \geq 2$ and $y \geq 2$

$$\begin{aligned} \sum_{n \leq xy} w(n)\frac{\Lambda(n)}{n^s} &= -\frac{\zeta'}{\zeta}(s) + \frac{(xy)^{1-s} - x^{1-s}}{(1-s)^2 \log y} \\ &\quad - \sum_{\rho} \frac{(xy)^{\rho-s} - x^{\rho-s}}{(\rho-s)^2 \log y} - \sum_{k=1}^{\infty} \frac{(xy)^{-2k-s} - x^{-2k-s}}{(2k-s)^2 \log y}, \end{aligned}$$

where $\rho = \beta + i\gamma$ are the non-trivial zeros of the Riemann zeta-function and $w(u)$ is defined as

$$w(u) = \begin{cases} 1 & \text{if } 1 \leq u \leq x, \\ 1 - \frac{\log(u/x)}{\log y} & \text{if } x \leq u \leq xy, \\ 0 & \text{if } u \geq xy. \end{cases}$$

From this, we see that $|w(n)| \le 1$ for $n \le xy$. After rearranging, we get

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n\le xy} w(n)\frac{\Lambda(n)}{n^s} + \sum_{\rho} \frac{x^{\rho-s}-(xy)^{\rho-s}}{(\rho-s)^2\log y} + \frac{(xy)^{1-s}-x^{1-s}}{(1-s)^2\log y} + \sum_{k=1}^{\infty} \frac{x^{-2k-s}-(xy)^{-2k-s}}{(2k-s)^2\log y}. \tag{2.7}$$

In [2], x and y are chosen to be

$$y = \exp\Big(\frac{1}{\sigma-1/2}\Big), \quad x = (\log t)^2/y$$

This is where we assume RH, and estimate the sum over the non-trivial zeros.

$$\Big|\sum_{\rho} \frac{x^{\rho-s}-(xy)^{\rho-s}}{(\rho-s)^2\log y}\Big| \le \sum_{\rho} \frac{x^{\frac12-\sigma}(1+y^{\frac12-\sigma})}{|s-\rho|^2\log y} = \sum_{\gamma} \frac{x^{\frac12-\sigma}(1+e^{-1})(\sigma-\frac12)}{(\sigma-\frac12)^2+(t-\gamma)^2}. \tag{2.8}$$

By taking real parts in (2.4), we get

$$\begin{aligned}\sum_{\gamma} \frac{(\sigma-\frac12)}{(\sigma-\frac12)^2+(t-\gamma)^2} &= \Re\frac{\zeta'}{\zeta}(s) + \frac12\Re\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}+1\Big) - \frac12\log\pi + \frac{\sigma-1}{(\sigma-1)^2+t^2} \\ &\le \Re\frac{\zeta'}{\zeta}(s) + \frac12\Re\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}+1\Big), \end{aligned} \tag{2.9}$$

for t sufficiently large. Here we use an explicit bound by Chandee [5] which states that $\Re(\Gamma'/\Gamma)(s) \le \log|s|$, for $\sigma \ge \frac14$. Thus

$$\begin{aligned}\frac12\Re\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}+1\Big) \le \frac12\log\Big|\frac{s}{2}+1\Big| &= \frac14\log\Big(\Big(\frac{\sigma}{2}+1\Big)^2+\Big(\frac{t}{2}\Big)^2\Big) \\ &= \frac12\log\Big(\frac{t}{2}\Big) + \frac14\log\Big(\Big(\frac{2}{t}\Big(\frac{\sigma}{2}+1\Big)\Big)^2+1\Big) \\ &\le \frac12\log t - \frac12\log 2 + \frac{1}{t^2}\Big(\frac{\sigma}{2}+1\Big)^2 \le \frac12\log t,\end{aligned}$$

for t sufficiently large. Combining what we have shown above, we get

$$\begin{aligned}\Big|\sum_{\rho} \frac{x^{\rho-s}-(xy)^{\rho-s}}{(\rho-s)^2\log y}\Big| &\le (1+e^{-1})x^{\frac12-\sigma}\Re\frac{\zeta'}{\zeta}(s) + \frac{(1+e^{-1})}{2}x^{\frac12-\sigma}\log t \\ &= (e+1)(\log t)^{1-2\sigma}\Re\frac{\zeta'}{\zeta}(s) + \frac{(e+1)}{2}(\log t)^{2-2\sigma}.\end{aligned} \tag{2.10}$$

In the last line we used that $x^{1/2-\sigma} = (\log t)^{1-2\sigma}y^{\sigma-1/2} = (\log t)^{1-2\sigma}e$. We also have to estimate the two last terms in (2.7). These are substantially smaller in magnitude so we allow ourselves to be more crude.

$$\Big|\frac{(xy)^{1-s}-x^{1-s}}{(1-s)^2\log y}\Big| \le \frac{(\sigma-\frac12)(xy)^{1-\sigma}(1+y^{\sigma-1})}{t^2} \le (1+e^{1/2})\frac{(\log t)^{2-2\sigma}}{t^2}.$$

The sum is a bit more intricate.

$$\begin{aligned}
\Big|\sum_{k=1}^{\infty}\frac{x^{-2k-s}-(xy)^{-2k-s}}{(2k-s)^2\log y}\Big| &\le \frac{x^{-2-\sigma}+(xy)^{-2-\sigma}}{\log y}\sum_{k=1}^{\infty}\frac{1}{|2k-s|^2}\\
&\le \frac{(\sigma-\frac{1}{2})(y^{2+\sigma}+1)}{(xy)^{2+\sigma}}\int_1^{\infty}\frac{dx}{(2x-\sigma)^2+t^2}\\
&\le \frac{\left(e\cdot\exp\left(\frac{5\log_2 t}{2\lambda\log_3 t}\right)+1\right)}{(\log_2 t)^{2\lambda}\,(\log t)^5\,t^2}\int_1^{\infty}\frac{dx}{((2x-\sigma)/t)^2+1}\\
&\le \frac{\pi\left(e\cdot\exp\left(\frac{5\log_2 t}{2\lambda\log_3 t}\right)+1\right)}{4\,(\log_2 t)^{2\lambda}\,(\log t)^5\,t}.
\end{aligned}$$

We can now insert these bounds into (2.7), and it can now be written as

$$\begin{aligned}
\Big|\frac{\zeta'}{\zeta}(s)\Big| \le -\sum_{n\le xy} w(n)\frac{\Lambda(n)}{n^s} + |\theta|\,(e+1)(\log t)^{1-2\sigma}\Big|\Re\frac{\zeta'}{\zeta}(s)\Big| + \frac{(e+1)}{2}(\log t)^{2-2\sigma}\\
+(1+e^{1/2})\frac{(\log t)^{2-2\sigma}}{t^2} + \frac{\pi\left(e\cdot\exp\left(\frac{5\log_2 t}{2\lambda\log_3 t}\right)+1\right)}{4\,(\log_2 t)^{2\lambda}\,(\log t)^5\,t}, \qquad (2.11)
\end{aligned}$$

where θ is a complex number such that $|\theta|\le 1$, according to [2]. As long as we ensure that

$$(e+1)(\log t)^{1-2\sigma} \le (e+1)(\log t_0)^{1-2\sigma} < 1,$$

we can write

$$\begin{aligned}
\left(1-\frac{e+1}{(\log t)^{2\sigma-1}}\right)\cdot\Big|\frac{\zeta'}{\zeta}(s)\Big| \le \Big|\sum_{n\le xy} w(n)\frac{\Lambda(n)}{n^s}\Big| + \frac{(e+1)}{2}(\log t)^{2-2\sigma}\\
+(1+e^{1/2})\frac{(\log t)^{2-2\sigma}}{t^2} + \frac{\pi\left(e\cdot\exp\left(\frac{5\log_2 t}{2\lambda\log_3 t}\right)+1\right)}{4\,(\log_2 t)^{2\lambda}\,(\log t)^5\,t}. \qquad (2.12)
\end{aligned}$$

We now turn to the sum in the line above. For $\sigma\ge 1/2$, we have

$$\Big|\sum_{n\le xy} w(n)\frac{\Lambda(n)}{n^s}\Big| \le \sum_{n\le xy}\frac{\Lambda(n)}{n^{1/2}}.$$

When integrating this sum by parts, it turns out to be useful to apply the following bound, also from Chandee [5], which says that for $T\ge 2$

$$\sum_{n\le T}\Lambda(n) \le (1.006)T.$$

Then

$$\sum_{n\le xy}\frac{\Lambda(n)}{n^{1/2}} = \psi(xy)\cdot(xy)^{-1/2} - \psi(2^-)\cdot 2^{-1/2} - \int_2^{xy}\psi(t)(t^{-1/2})'\,dt$$
$$\le (1.006)(xy)^{1/2} + \frac{1.006}{2}\int_2^{xy} t^{-1/2}\,dt$$
$$= 2.012\log t - 1.006\cdot 2^{1/2}.$$

If we now insert this into (2.12), the right hand side becomes

$$2.012\log t - 1.006\cdot 2^{1/2} + \frac{(e+1)}{2}(\log t)^{2-2\sigma}$$
$$+ (1+e^{1/2})\frac{(\log t)^{2-2\sigma}}{t^2} + \frac{\pi\left(e\cdot\exp\left(\frac{5\log_2 t}{2\lambda\log_3 t}\right)+1\right)}{4\,(\log_2 t)^{2\lambda}\,(\log t)^5\,t}$$
$$\le 2.012\log t + \frac{(e+1)}{2}(\log t)^{2-2\sigma} = \log t\Big(2.012 + \frac{(e+1)}{2}(\log t)^{1-2\sigma}\Big),$$

for t sufficiently large. This means that

$$\left|\Re\frac{\zeta'}{\zeta}(s)\right| \le \left|\frac{\zeta'}{\zeta}(s)\right| \le \log t\bigg(\Big(1-\frac{e+1}{(\log t)^{2\sigma-1}}\Big)^{-1}\Big(2.012+\frac{(e+1)}{2}(\log t)^{1-2\sigma}\Big)\bigg),$$

and we finally insert this into (2.11). Then

$$\left|\frac{\zeta'}{\zeta}(s)\right| \le \sum_{n\le(\log t)^2}\frac{\Lambda(n)}{n^\sigma} + (\log t)^{2-2\sigma}\left[\frac{(e+1)\Big(2.012+\frac{(e+1)}{2\,(\log_2 t)^{2\lambda}}\Big)}{\Big(1-\frac{e+1}{(\log_2 t)^{2\lambda}}\Big)}\right.$$
$$\left. + \frac{(e+1)}{2} + \frac{(1+e^{1/2})}{t^2} + \frac{\pi\left(e\cdot\exp\left(\frac{5\log_2 t}{2\lambda\log_3 t}\right)+1\right)}{4\,(\log t)^6\,t}\right], \tag{2.13}$$

which proves the theorem and gives us a candidate for the constant in front of the error term. Let $m(t,\lambda)$ denote the expression inside the brackets in the line above. In one of the succeeding corollaries, Corollary 13.15 in [2], it is shown that

$$|\log\zeta(s)| \le \sum_{n\le(\log t)^2}\frac{\Lambda(n)}{n^\sigma\log n} + O\Big(\frac{(\log t)^{2-2\sigma}}{\log_2 t}\Big), \tag{2.14}$$

where $\frac{1}{2}+\frac{1}{\log_2 t}\le\sigma\le\frac{3}{2}$. The proof originates from the line

$$|\log\zeta(\sigma+it)| \le |\log\zeta(\tfrac{3}{2}+it)| + \int_\sigma^{3/2}\left|\frac{\zeta'}{\zeta}(\sigma+it)\right|d\sigma,$$

which we can now work on explicitly. Using the bound in (2.13), we have

$$|\log \zeta(\sigma+it)| \le \sum_{n\le(\log t)^2} \frac{\Lambda(n)}{\log n}\left(n^{-\sigma}-n^{-3/2}\right) + m(t,\lambda)\frac{(\log t)^{2-2\sigma}}{\log_2 t}, \tag{2.15}$$

By the relation in (2.5), the right hand side is equal to

$$\sum_{n\le(\log t)^2} \frac{\Lambda(n)}{n^{\sigma}\log n} + \sum_{n>(\log t)^2} \frac{\Lambda(n)}{n^{3/2}\log n} + m(t,\sigma)\left(\frac{(\log t)^{2-2\sigma}}{\log_2 t}\right), \tag{2.16}$$

We will see that the second sum is of lesser order than that of the error term. Here we will introduce another helpful bound. Since we are assuming RH, we are able to apply the following explicit bound due to Schoenfeld, Eq.(6.5) in [9],

$$|\psi(x)-x| \le \frac{1}{8\pi}x^{1/2}(\log x)^2 \text{ for } x\ge 59. \tag{2.17}$$

Lemma 2. *For* $t>e^{e^2}$,

$$\sum_{n>(\log t)^2} \frac{\Lambda(n)}{n^{3/2}\log n} \le \frac{1}{\log t\log_2 t}\cdot\left(1+\frac{(\log_2 t)^2}{8\pi\log t}+\frac{5}{16\pi}\frac{\log_2 t}{\log t}\right). \tag{2.18}$$

Proof. We consider the sum over $z=(\log t)^2<n\le Y$, and let $Y\to\infty$. As before, we use integration by parts with respect to $\psi(x)$. We use the trick of adding and subtracting a factor of z in order to use the bound in (2.17).

$$\sum_{z<n\le Y}\frac{\Lambda(n)}{n^{3/2}\log n} = -\left(\frac{\psi(z)-z}{\log z\cdot z^{3/2}}\right)-\frac{z^{-1/2}}{\log z\cdot}$$

$$-\int_z^Y(\psi(t)-t)\Big(\big(\log t\cdot t^{3/2}\big)^{-1}\Big)'\,dt-\int_z^Y t\cdot\Big(\big(\log t\cdot t^{3/2}\big)^{-1}\Big)'\,dt$$

$$\le -\frac{\log z}{8\pi z}-\frac{z^{-1/2}}{\log z}+\frac{1}{8\pi}\int_z^Y\frac{1}{t^2}\Big(1+\frac{3}{2}\log t\Big)\,dt+\int_z^Y\frac{1}{(\log t)^2\,t^{3/2}}\Big(1+\frac{3}{2}\log t\Big)\,dt$$

$$=-\frac{\log z}{8\pi z}-\frac{1}{\log z\cdot z^{1/2}}+\frac{1}{8\pi}\left(\frac{5}{2\cdot z}+\frac{3\log z}{2z}\right)+\frac{3}{\log z\cdot z^{1/2}}-\int_z^Y\frac{2}{(\log t)^2\,t^{3/2}}\,dt$$

$$\le\frac{2}{\log z\cdot z^{1/2}}+\frac{1}{16\pi}\frac{\log z}{z}+\frac{5}{16\pi\cdot z}=\frac{1}{\log t\log_2 t}+\frac{\log_2 t}{8\pi(\log t)^2}+\frac{5}{16\pi(\log t)^2}.$$

□

The idea in Corollary 13.16 in [2] is to consider the finite sum when $n\le z$ and $z=(\log t)^2$ and integrate by parts with respect to ψ. Then by Eq. (13.45) in [2]

(2.19)

$$\sum_{n\le z}\frac{\Lambda(n)}{n^{\sigma}\,\log n}=\int_2^z\frac{du}{u^{\sigma}\log u}+\frac{\psi(z)-z}{z^{\sigma}\log z}\quad+\frac{2^{1-\sigma}}{\log 2}+\int_2^z\frac{\psi(u)-u}{u^{\sigma+1}\log u}\Big(\sigma+\frac{1}{\log u}\Big)\,du.$$

The first integral, with the substitution $v = u^{1-\sigma}$ becomes $\mathrm{li}(z^{1-\sigma}) - \mathrm{li}(2^{1-\sigma})$, where

$$\mathrm{li}(x) = \int_2^x \frac{du}{\log u}$$

is the logarithmic integral-function. The first of these two terms is what will give the greatest contribution when σ is close to $\frac{1}{2}$. Here we can apply a bound by Rosser, given in [10], which states that

$$\mathrm{li}(x) < \frac{x}{\log x}\left(1 + \frac{3}{2\,\log x}\right),$$

for $x \geq e^8$. This restriction on x is very strict with respect to how large $t = t_0$ needs to be chosen for this to hold. In our case, we have

$$x = z^{1-\sigma} = (\log t)^{2-2\sigma} = \frac{\log t}{(\log_2 t)^{2\lambda}}.$$

We will investigate how far down we can push x, if we substitute the $\frac{3}{2}$ with a greater constant. This is described in the following lemma.

Lemma 3.

$$\mathrm{li}(x) < \frac{x}{\log x}\left(1 + \frac{1.785}{\log x}\right), \quad x > 1. \tag{2.20}$$

Proof. We only have to consider the interval $(1, e^8)$, otherwise we are covered by the bound by Rosser. Differentiating the difference and solving

$$\frac{d}{dx}\left(\frac{x}{\log x}\left(1 + \frac{1.785}{\log x}\right) - \mathrm{li}(x)\right) = 0$$

gives the minimum point for the difference at $x = 94.421679$ and the difference evaluated at this point is 0.004064132923. □

Now that we have a bound valid for $x > 1$, we can check when our x satisfies this. We let $t \geq t_0 := e^{e^{e^a}}$, for some $a > 0$, thus

$$x = \frac{\log t_0}{(\log_2 t_0)^{2\lambda}} = \frac{e^{e^a}}{e^{2a\lambda}} = e^{(e^a - 2a\lambda)} > e^{\log 1}.$$

Hence we get the following restriction for λ

$$\lambda < \frac{e^a}{2a}. \tag{2.21}$$

We use the bound in Lemma 3 for $\mathrm{li}(z^{1-\sigma})$.

$$\begin{aligned}\mathrm{li}(z^{1-\sigma}) &\le \frac{\log t}{(\log_2 t)^{2\lambda}}\cdot\frac{1}{\big(\log_2 t-2\lambda\log_3 t\big)}\left(1+\frac{1.785}{\big(\log_2 t-2\lambda\log_3 t\big)}\right)\\ &= \frac{\log t}{(\log_2 t)^3}\cdot\frac{1}{(\log_2 t)^{2(\lambda-1)}\big(1-\frac{2\lambda\log_3 t}{\log_2 t}\big)}\\ &+\frac{\log t}{(\log_2 t)^4}\cdot\frac{1.785}{(\log_2 t)^{2(\lambda-1)}\big(1-\frac{2\lambda\log_3 t}{\log_2 t}\big)^2}.\end{aligned}$$

$\mathrm{li}(2^{1-\sigma})$ is problematic when σ is close to 1 since the function $1/\log u$ has a singularity at $u=1$. Also, on the interval $(1,2^{1/2}]$, $\mathrm{li}(x)$ is negative and increasing, thus

$$-\,\mathrm{li}(2^{1-\sigma})\le\mathrm{li}(2^{1/2-\lambda a/\exp a}).$$

We apply the bound in (2.17) again, and get an explicit bound for the second term in (2.19)

$$\frac{\psi(z)-z}{z^\sigma\log z}\le\frac{1}{8\pi}z^{1/2-\sigma}\log z=\frac{1}{8\pi}(\log t)^{1-2\sigma}\,2\log_2 t=\frac{1}{4\pi(\log_2 t)^{2\lambda-1}}.$$

Similarly, for the fourth term in (2.19), we get

$$\begin{aligned}\int_2^z\frac{\psi(u)-u}{u^{\sigma+1}\log u}\left(\sigma+\frac{1}{\log u}\right)du &\le\frac{1}{8\pi}\left(\int_2^z\sigma\,u^{-\sigma-\frac12}\log u\,du+\int_2^z u^{-\sigma-\frac12}\,du\right)\\ &\le\frac{1}{8\pi}\left(\Big(\frac{\log_2 t}{\lambda\log_3 t}\Big)^2+\frac{1}{\lambda}\Big(1+\frac{\log 2}{2}\Big)\frac{\log_2 t}{\log_3 t}+\log 2\right).\end{aligned}$$

The third term is only bounded by a constant, since

$$\frac{2^{1-\sigma}}{\log 2}\le\frac{2^{1/2}}{\log 2}.$$

For bookkeeping purposes, we will summarize our findings.

$$\begin{aligned}\log|\zeta(\delta_\lambda+it)| &\le\frac{\log t}{(\log_2 t)^3}\cdot\frac{1}{(\log_2 t)^{2(\lambda-1)}}\cdot\left(\frac{1}{\big(1-\frac{2\lambda\log_3 t}{\log_2 t}\big)}+m(t,\lambda)\right)\\ &+\frac{\log t}{(\log_2 t)^4}\cdot\frac{1.785}{(\log_2 t)^{2(\lambda-1)}\big(1-\frac{2\lambda\log_3 t}{\log_2 t}\big)^2}\\ &+\frac{1}{8\pi}\left(\Big(\frac{\log_2 t}{\lambda\log_3 t}\Big)^2+\frac{1}{\lambda}\Big(1+\frac{\log 2}{2}\Big)\frac{\log_2 t}{\log_3 t}+\log 2\right)\\ &+\mathrm{li}(2^{1/2-a/\exp a})+\frac{2^{1/2}}{\log 2}+\frac{1}{4\pi(\log_2 t)^{2\lambda-1}}\\ &+\frac{1}{\log t\log_2 t}\cdot\left(1+\frac{(\log_2 t)^2}{8\pi\log t}+\frac{5}{16\pi}\frac{\log_2 t}{\log t}\right).\end{aligned}\tag{2.22}$$

3. Estimates via bandlimited approximations

The lower bound we will use in this section is due to Carneiro, Chirre and Milinovich, given by Theorem 2 in [1], which states

$$
\begin{aligned}
\Re\frac{\zeta'}{\zeta}(\sigma+it) \geq (\log t)^{2-2\sigma}&\left(\frac{1}{1+(\log t)^{1-2\sigma}}+\frac{2\sigma-1}{\sigma(1-\sigma)}\right)\\
&+O\left((\sigma-\tfrac{1}{2})\left(\frac{\log_2 t}{\pi}\right)^4\right)+O\left(\min\{1,(\sigma-\tfrac{1}{2})\log_2 t\}\right)\\
&+O\left(\frac{(\sigma-\frac{1}{2})\left(\frac{\log_2 t}{\pi}\right)^2\log t}{1+t\log_2 t}\right).
\end{aligned}
\tag{3.1}
$$

When we integrate over the bound in (3.1) we will retrieve the constant $\log 2/2$ in front of the first-order term. The search for the constant in front of the second-order term becomes easier than anticipated due to the fact that it also follows directly from this integral. As seen in the previous section, the second term in (2.1) only contributes to the third-order- and lower-order terms. To evaluate the integral over the first term in the expression above, we make the substitution $u = \log t^{-2\sigma}$. Hence, we get

$$
\begin{aligned}
\int_{1/2}^{\delta}\frac{(\log t)^{2-2\sigma}}{1+(\log t)^{1-2\sigma}}\,d\sigma &= -\frac{(\log t)^2}{2\log_2 t}\int_{(\log t)^{-1}}^{(\log t)^{-2\delta}}\frac{d\,u}{1+(\log t)\,u}\\
&=\frac{(\log t)^2}{2\log_2 t}\left(\frac{\log(1+(\log t)\,u)}{\log t}\right)\Bigg|_{(\log t)^{-2\delta}}^{(\log t)^{-1}}\\
&=\frac{\log 2}{2}\frac{\log t}{\log_2 t}-\log\left(1+\frac{1}{(\log_2 t)^2}\right)\frac{\log t}{\log_2 t}.
\end{aligned}
\tag{3.2}
$$

In the last line we used that $(\log t)^{-2\,\log_3 t/\log_2 t} = (\log_2 t)^{-2}$, which is a relation that will appear in some form several times in the proceeding calculations.
When inspecting the second term in (2.2) it might be tempting to perform a partial fraction decomposition, although this will result in two terms involving $\mathrm{li}(x)$, which will be hard to compare to each other. A solution to this obstacle will be to make the substitution $2\sigma - 1 = \alpha$, which will prove fruitful. Thus

$$
\begin{aligned}
\int_{1/2}^{\delta}\frac{2\sigma-1}{\sigma(1-\sigma)}(\log t)^{2-2\sigma}\,d\sigma &= \frac{1}{2}\int_0^{2\delta-1}\frac{\alpha}{(\frac{1+\alpha}{2})(\frac{1-\alpha}{2})}(\log t)^{1-\alpha}\,d\alpha\\
&=2\,\log t\int_0^{2\delta-1}\frac{\alpha}{1-\alpha^2}\,(\log t)^{-\alpha}\,d\alpha.
\end{aligned}
$$

We remark that we can write $\frac{1}{1-\alpha^2}$ as a geometric series, then the integral becomes

$$2\,\log t \int_0^{2\delta-1} \alpha\,(1+\alpha^2+\alpha^4+\dots)\,(\log t)^{-\alpha}\,d\alpha.$$

One can find a general formula, e.g. for $n \geq 1$

$$\int_0^{2\delta-1} \alpha^{2n-1}(\log t)^{-\alpha}\,d\alpha = \frac{(2n-1)!}{(\log_2 t)^{2n}} - \sum_{k=0}^{2n-1} \frac{2^k}{k!}\frac{(\log_3 t)^k}{(\log_2 t)^{2(n+1)}}.$$

One method to obtain this is to integrate quite a few terms and recognizing the pattern appearing, and then prove the formula by induction on n. There are also literature, in which this could be found explicitly. This gives us some insight about the behaviour of each term. The only non-negative term comes from the evaluation at 0 in the final integral after $2n-1$ successive rounds of integration by parts. We can ignore terms where there is a $(\log_3 t)^k$ in the numerator, unless we have something to gain from a cancellation. We decide it is unnecessary to include too many terms, so instead we write $\frac{1}{1-\alpha^2} = 1 + \frac{\alpha^2}{1-\alpha^2}$ and get

$$\begin{aligned}\int_0^{2\delta_\lambda-1} \alpha\,(\log t)^{-\alpha}\,d\alpha &= -\frac{\alpha}{\log_2 t}\,(\log t)^{-\alpha}\Big|_0^{2\delta_\lambda-1} + \frac{1}{\log_2 t}\int_0^{2\delta_\lambda-1}(\log t)^{-\alpha}\,d\alpha \\ &= -\frac{(2\delta_\lambda-1)}{\log_2 t}\,(\log t)^{1-2\delta_\lambda} - \left(\frac{1}{(\log_2 t)^2}\,(\log t)^{-\alpha}\right)\Big|_0^{2\delta_\lambda-1} \\ &= -\frac{2\lambda\,\log_3 t}{(\log_2 t)^{2(\lambda+1)}} - \frac{1}{(\log_2 t)^{2(\lambda+1)}} + \frac{1}{(\log_2 t)^2}.\end{aligned}$$

Also

$$\begin{aligned}&\int_0^{2\delta_\lambda-1} \frac{\alpha^3}{1-\alpha^2}\,(\log t)^{-\alpha}\,d\alpha \leq \frac{1}{1-\left(\frac{2\lambda\log_3 t}{\log_2 t}\right)^2}\int_0^{2\delta_\lambda-1}\alpha^3(\log t)^{-\alpha}\,d\alpha \\ &= \frac{1}{1-\left(\frac{2\lambda\log_3 t}{\log_2 t}\right)^2}\left(-\frac{\big(8\,(\lambda\log_3 t)^3+12\,(\lambda\log_3 t)^2+12\lambda\log_3 t+6\big)}{(\log_2 t)^{4+2\lambda}} + \frac{6}{(\log_2 t)^4}\right) \\ &\leq \frac{1}{1-\left(\frac{2\lambda\log_3 t}{\log_2 t}\right)^2}\cdot\frac{6}{(\log_2 t)^4}.\end{aligned}$$

Summarizing, we have

$$\begin{aligned}\int_{1/2}^{\delta_\lambda} \frac{2\sigma-1}{\sigma(1-\sigma)}(\log t)^{2-2\sigma}d\sigma &= 2\,\log t\int_0^{2\delta-1}\frac{\alpha}{1-\alpha^2}\,(\log t)^{-\alpha}\,d\alpha \\ &\leq \frac{2\,\log t}{(\log_2 t)^2} + \left(\frac{12}{1-\left(\frac{2\lambda\log_3 t}{\log_2 t}\right)^2} - \frac{2}{(\log_2 t)^{2(\lambda-1)}}\right)\frac{\log t}{(\log_2 t)^4}. \qquad (3.3)\end{aligned}$$

We will require $\lambda > 1$ to ensure that $1 \big/ (\log_2 t)^{2(\lambda-1)}$ is a decreasing function. Let us now consider the error terms from the bound on $\Re\frac{\zeta'}{\zeta}$. A brief introduction to the notation used in [1] is in order. We have $\beta = \sigma - 1/2$, where $0 < \beta < 1/2$. Δ is a real parameter, later chosen to be such that $\pi\Delta = \log_2 t$. Also denote by m_Δ^- the extremal minorant which is obtained for the Poisson kernel. By Lemma 9 in [1], this is given by

$$m_\Delta^-(z) = \left(\frac{\beta}{\beta^2 + z^2}\right)\left(\frac{e^{2\pi\beta\Delta} + e^{-2\pi\beta\Delta} - 2\cos(2\pi\Delta z)}{(e^{\pi\beta\Delta} + e^{-\pi\beta\Delta})^2}\right).$$

The first error term comes from

$$\sum_{n\le x} \frac{\Lambda(n)}{n^{1/2}}\left(\frac{x^\beta}{n^\beta} - \frac{n^\beta}{x^\beta}\right) = \frac{2\beta x^{\frac{1}{2}} - 2^{\frac{1}{2}-\beta}x^\beta(\frac{1}{2}+\beta)^2 + 2^{\frac{1}{2}+\beta}x^{-\beta}(\frac{1}{2}-\beta)^2}{(\frac{1}{4}-\beta^2)} + O\left(\beta x^\beta (\log x)^4\right),$$

where $x = e^{2\pi\Delta}$. We have found an explicit bound for the sum in the following lemma.

Lemma 4.

$$\sum_{n\le x} \frac{\Lambda(n)}{n^{1/2}}\left(\frac{x^\beta}{n^\beta} - \frac{n^\beta}{x^\beta}\right) \le \frac{2\beta x^{\frac{1}{2}} - 2^{\frac{1}{2}-\beta}x^\beta(\frac{1}{2}+\beta)^2 + 2^{\frac{1}{2}+\beta}x^{-\beta}(\frac{1}{2}-\beta)^2}{(\frac{1}{4}-\beta^2)}$$
$$+ \frac{1}{24\pi}\cdot\big((\log x)^3 - (\log 2)^3\big)\cdot\big((\tfrac{1}{2}+\beta)x^\beta - (\tfrac{1}{2}-\beta)x^{-\beta}\big)$$

Proof. We start by writing

$$\sum_{n\le x} \Lambda(n)\cdot\left(n^{-(1/2+\beta)}x^\beta - n^{-(1/2-\beta)}x^{-\beta}\right)$$

Integration by parts gives

$$\psi(x)\left(x^{-(1/2+\beta)}x^\beta - x^{-(1/2-\beta)}x^{-\beta}\right) - \int_2^x \psi(t)\Big(t^{-(1/2+\beta)}x^\beta - t^{-(1/2-\beta)}x^{-\beta}\Big)'\,dt,$$

and $\psi(2^-) = 0$. The first term is 0, and to apply the bound by Schoenfeld [9] again, we write the integral as

$$\int_2^x (\psi(t) - t)\cdot\left(t^{-(1/2-\beta)}x^{-\beta} - t^{-(1/2+\beta)}x^\beta\right)'\,dt$$
$$+ \int_2^x t\cdot\left(t^{-(1/2-\beta)}x^{-\beta} - t^{-(1/2+\beta)}x^\beta\right)'\,dt$$

By (2.17), we have that this is less than or equal to

$$\frac{1}{8\pi}\int_2^x t^{1/2}(\log t)^2\cdot\Big((\tfrac12+\beta)t^{-(3/2+\beta)}x^{\beta}-(\tfrac12-\beta)t^{-(3/2-\beta)}x^{-\beta}\Big)\,dt$$
$$+\int_2^x t\cdot\Big((\tfrac12+\beta)t^{-(3/2+\beta)}x^{\beta}-(\tfrac12-\beta)t^{-(3/2-\beta)}x^{-\beta}\Big)\,dt$$

We simplify the two integrands, and get

$$\int_2^x \Big((\tfrac12+\beta)t^{-(1/2+\beta)}x^{\beta}-(\tfrac12-\beta)t^{-(1/2-\beta)}x^{-\beta}\Big)dt$$
$$+\frac{1}{8\pi}\int_2^x(\log t)^2\cdot\left(\frac{(\tfrac12+\beta)x^{\beta}}{t^{(1+\beta)}}-\frac{(\tfrac12-\beta)x^{-\beta}}{t^{(1-\beta)}}\right)\,dt$$

The first integral we compute explicitly, and for the second integral we use the mean value theorem for the function $r\to(\frac12+r)x^r$, as in Lemma B.4. in [1]. Then the two integrals are less than or equal to

$$(\tfrac12+\beta)x^{\beta}\left(\frac{x^{1/2-\beta}}{1/2-\beta}-\frac{2^{1/2-\beta}}{1/2-\beta}\right)-\left((\tfrac12-\beta)x^{-\beta}\left(\frac{x^{1/2+\beta}}{1/2+\beta}-\frac{2^{1/2+\beta}}{1/2+\beta}\right)\right)$$
$$+\frac{1}{8\pi}\int_2^x(\log t)^2\cdot\left(\frac{(\tfrac12+\beta)x^{\beta}}{t}-\frac{(\tfrac12-\beta)x^{-\beta}}{t}\right)\,dt$$

After simplifying the first line, and using that $\Big((\log t)^3/3\Big)'=(\log t)^2/t$ to compute the final integral, we have the result. □

We have that $x=e^{2\pi\Delta}=(\log t)^2$ and $\beta=\sigma-1/2$, so the integral over this error term becomes

$$\left(\frac{8(\log_2 t)^3-(\log 2)^3}{24\pi}\right)\int_{1/2}^{\delta_\lambda}\Big(\sigma(\log t)^{2\sigma-1}+(\sigma-1)(\log t)^{1-2\sigma}\Big)\,d\sigma$$
$$\le\frac{(\log_2 t)^4}{12\pi}+\frac{\lambda\,(\log_2 t)^3\log_3 t}{6\pi} \tag{3.4}$$

for t sufficiently large. This is the most problematic error term, since when we want to compare this to the fourth-order term in our result, the function $(\log_2 t)^8/\log t$ is decreasing when $t>e^{e^8}$, which is roughly 10^{1295}. The second error term in (3.1) comes from the Fourier transform of the minorant evaluated at 0, and by Eq.(3.4) in [1] this is

$$\widehat{m}_\Delta(0)=\pi\left(\frac{e^{\pi\beta\Delta}-e^{-\pi\beta\Delta}}{e^{\pi\beta\Delta}+e^{-\pi\beta\Delta}}\right)=\pi\tanh(\pi\beta\Delta)\le\pi\min\{1\,,\beta\Delta\}.$$

Thus

(3.5)
$$\int_{1/2}^{\delta_\lambda} \pi \, \min\{1, (\sigma - \tfrac{1}{2}) \log_2 t/\pi\} \, d\sigma \le \pi \, \min\{1, \lambda \log_3 t/\pi\} \int_{1/2}^{\delta_\lambda} d\sigma = \frac{\pi \, \lambda \log_3 t}{\log_2 t},$$

for $t > \frac{\pi}{\lambda} \cdot e^{e^e}$. The third error term in (3.1) comes from

$$\Big| m_\Delta^-(t - \tfrac{1}{2i}) + m_\Delta^-(t + \tfrac{1}{2i}) \Big|,$$

and by Eq.(3.8) and Eq.(3.21) in [1], this is

$$\Bigg| \frac{\beta \Delta^2 e^{2\pi\Delta 1/2}}{1 + \Delta \cdot |t + \frac{1}{2i}|} + \frac{\beta \Delta^2 e^{2\pi\Delta 1/2}}{1 + \Delta \cdot |t + \frac{1}{2i}|} \Bigg| = \frac{\beta \Delta^2 e^{\pi\Delta}}{|\Delta((t^2/4) + 1)^{1/2} + 1|} \le \frac{\beta \Delta^2 e^{\pi\Delta}}{\Delta t + 1}.$$

Hence

(3.6)
$$\int_{1/2}^{\delta_\lambda} \frac{(\sigma - \frac{1}{2})\big(\log_2 t/\pi\big)^2 \log t}{1 + t \log_2 t/\pi} \, d\sigma = \frac{\lambda^2}{2\pi} \cdot \frac{\log t (\log_3 t)^2}{(t \log_2 t + \pi)}.$$

4. Proof of Theorem 1

First we summarize the results from the two previous sections. By (2.1), (2.22), (3.2), (3.3), (3.4), (3.5) and (3.6), we have

$$
\begin{aligned}
&\log|\zeta(\tfrac{1}{2}+it)| = -\int_{1/2}^{\delta_\lambda} \Re\frac{\zeta'}{\zeta}(\sigma+it)\,d\sigma + \log|\zeta(\delta_\lambda+it)|.\\
&\le \frac{\log 2}{2}\frac{\log t}{\log_2 t} + \frac{2\log t}{(\log_2 t)^2} + \left(\frac{12}{1-\left(\frac{2\lambda\log_3 t}{\log_2 t}\right)^2} - \frac{2}{(\log_2 t)^{2(\lambda-1)}}\right)\frac{\log t}{(\log_2 t)^4}\\
&+ \frac{(\log_2 t)^4}{12\pi} + \frac{\lambda(\log_2 t)^3\log_3 t}{6\pi} + \frac{\pi\,\lambda\log_3 t}{\log_2 t} + \frac{\lambda^2}{2\pi}\cdot\frac{\log t(\log_3 t)^2}{(t\log_2 t+\pi)}\\
&+ \frac{\log t}{(\log_2 t)^3}\cdot\frac{1}{(\log_2 t)^{2(\lambda-1)}}\cdot\left(\frac{1}{\left(1-\frac{2\lambda\log_3 t}{\log_2 t}\right)} + m(t,\lambda)\right)\\
&+ \frac{\log t}{(\log_2 t)^4}\cdot\frac{1.785}{(\log_2 t)^{2(\lambda-1)}\left(1-\frac{2\lambda\log_3 t}{\log_2 t}\right)^2}\\
&+ \frac{1}{8\pi}\left(\left(\frac{\log_2 t}{\lambda\log_3 t}\right)^2 + \frac{1}{\lambda}\left(1+\frac{\log 2}{2}\right)\frac{\log_2 t}{\log_3 t} + \log 2\right)\\
&+ \mathrm{li}(2^{1/2-\lambda a/\exp a}) + \frac{2^{1/2}}{\log 2} + \frac{1}{4\pi(\log_2 t)^{2\lambda-1}}\\
&+ \frac{1}{\log t\log_2 t}\cdot\left(1+\frac{(\log_2 t)^2}{8\pi\log t} + \frac{5}{16\pi}\frac{\log_2 t}{\log t}\right),
\end{aligned}
$$

for $t > e^{e^{e^{3\log 2}}}$. We choose $\lambda = 1.7540$. Then

$$
\log|\zeta(\tfrac{1}{2}+it)| \le \frac{\log 2}{2}\frac{\log t}{\log_2 t} + 2\frac{\log t}{(\log_2 t)^2} + 0.696500\frac{\log t}{(\log_2 t)^3} + 375.5451\frac{\log t}{(\log_2 t)^4}. \tag{4.1}
$$

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