Bachelor's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

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The rate of convergence to a fixed point in the complex plane

Bachelor's thesis in Mathematical Sciences
Supervisor: Berit Stensønes and John Erik Fornæss
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Preface

For this thesis to be best understood, it is desirable for the reader to have prior knowledge within analysis and complex analysis. I would also like to note that most of the information in this text was acquired from my supervisors directly, which is also the reason as to why there are such few sources.

Abstract

In this thesis we study the function $f(z) := z\left(1+z^k\right)$, where $k \in \mathbb{N}$, and its behaviour at different points in $\mathbb{C}$ when we iterate it n-fold on itself, $f^n(z)$. Our main focus is on its rate of convergence to the fixed point 0, which we show to behave as $\left(-\frac{1}{n \cdot k}\right)^{\frac{1}{k}}$, given it meets the right criteria.

Sammendrag

I denne oppgaven studerer vi funksjonen $f(z) := z\left(1+z^k\right)$, der $k \in \mathbb{N}$, og dens atferd ved forskjellige punkter i $\mathbb{C}$ når vi itererer den n ganger på seg selv, $f^n(z)$. Vårt hovedfokus er på dens konvergenshastighet til det fikserte punktet 0, som vi viser at oppfører seg som $\left(-\frac{1}{n \cdot k}\right)^{\frac{1}{k}}$, gitt at den oppfyller de rette kriteriene.

Chapter 1

The rate of convergence to a fixed point in the complex plane

Introduction

In this thesis we will be looking at a function $f : \mathbb{C} \to \mathbb{C}$, when we iterate it n-times on itself.

$$f(f(...f(z)...)) = f^n(z) \tag{1.1}$$

And what type of behaviour it has at different points, but mostly when, and how, it converges to a fixed point. To do this we will need to introduce some definitions, which we obtain from [1].

Definition 1.0.1 (Normal family)**.** Let $S \subseteq \mathbb{C}$, and Ω be a collection of holomorphic maps $f : S \to \mathbb{C}$. This is called a normal family, if every sequence of maps from Ω contains either a subsequence which converges uniformly on compact subsets of S, or it diverges to infinity.

Definition 1.0.2 (Normal)**.** Let $S \in \mathbb{C}$, and $f : S \to \mathbb{C}$ a non-constant holomorphic mapping, where $f^n : S \to S$ is its n-fold iterate. If there exists some neighborhood U of a fixed point $z_0 \in S$, such that the sequence of iterates $\{f^n\}$ restricted to U forms a normal family, then we say that z_0 is a normal point.

We can split $\mathbb{C}$ into two sets, the Fatou set, $\mathcal{F}$: the set of all the normal points. And the Julia set, $\mathcal{J} = \mathbb{C} \setminus \mathcal{F}$. In this thesis we will not describe a way to find the elements in $\mathcal{J}$, but we will look at some of them.

The function f, that we will work with is defined as follows, where $k \in \mathbb{N}$.

$$f(z) = z + z^{k+1} = z(1 + z^k) \tag{1.2}$$

In the first chapter we will be looking at the specific case when $k = 1$, and in the following chapter we will look at the general case where k is unspecified. The main focus of this thesis is to prove the following theorem:

Theorem. For all points $z \in \mathbb{C}$, such that $f^n(z) \to 0$ and $f^n(z) \neq 0$, with $m = 0, 1, \cdots, k-1$ we have that.

$$\lim_{n\to\infty} \frac{f^n(z)}{\frac{1}{n}^{\frac{1}{k}}} \to e^{i\left(\frac{\pi}{k}+\frac{2m\pi}{k}\right)} \left(\frac{1}{k}\right)^{\frac{1}{k}}$$

Which will tell us about the behaviour of f^n when it converges to the fixed point 0. We will prove this theorem first for the specific case $k = 1$, and then use the same approach again in the general case where k is unspecified.

1.1 Specific case

Here we will consider the specific case where k=1 in the function f from (1.2).

$$f(z) = z(1+z) \tag{1.3}$$

We want to locate some points where $f^n(z)$ either diverges to ∞, or converges to 0 as $n \to \infty$. If these points also have the properties from 1.0.2 we can conclude that they belong in the Fatou set $\mathcal{F}$. However if we find points, which differ in behaviour from the points around them, and does not have the properties from 1.0.2, we can say they are in the Julia set $\mathcal{J}$. To start with, we will consider the case where $|z| > 2$. Observer that the triangle inequality implies the following.

$$\begin{aligned} |1+z| &\geq |z| - 1 > 2 - 1 = 1 \\ \Rightarrow |z||1+z| &> |z| \end{aligned}$$

Therefore we can say that it grows, at least up to some point, but it is not enough to say it diverges. For us to conclude that it diverges, it needs to grow at a more rapid pace, $|z| > 3$ should do the trick. And again from the triangle inequality we get the following.

$$\begin{aligned} |1+z| &\geq |z| - 1 > 3 - 1 = 2 \\ \Rightarrow |z||1+z| &> 2|z| \end{aligned}$$

Thus $f^n(z) \to \infty$ as $n \to \infty$ when $|z| > 3$. From this we can define some elements in the Fatou set. If $3 < |z|$ this obviously also means that there are points in a neighborhood of z that also has an absolute value greater than three, so f^n on all points in this neighborhood diverges to infinity. From which we can conclude that, $3 < |z| \Rightarrow z \in \mathcal{F}$, because this means z is a normal point.
Now lets find some points where it approaches 0 instead. It is obvious that $z \in (-1, 0) \Rightarrow f(z) \in (-1, 0)$, and also that $f^n(z) \to 0$ as $n \to \infty$ in that case. We will consider a area in $\mathbb{C}$ where $|z| < \frac{1}{2}$, with $\pi - \frac{\pi}{8} < arg(z) < \pi + \frac{\pi}{8}$, in which i propose $f^n(z) \to 0$ as $n \to \infty$.

Let $arg(z) = \pi - \alpha < \pi$ for $\alpha \in \left(0, \frac{\pi}{8}\right)$. This implies that $arg(1+z) = \beta > 0$, where $\beta < \alpha$. This is because $|z| < \frac{1}{2}$ indicates that, $|Re(z)| < \frac{1}{2}$, from which we get that $|Re\,(1+z)\,| > \frac{1}{2}$. However

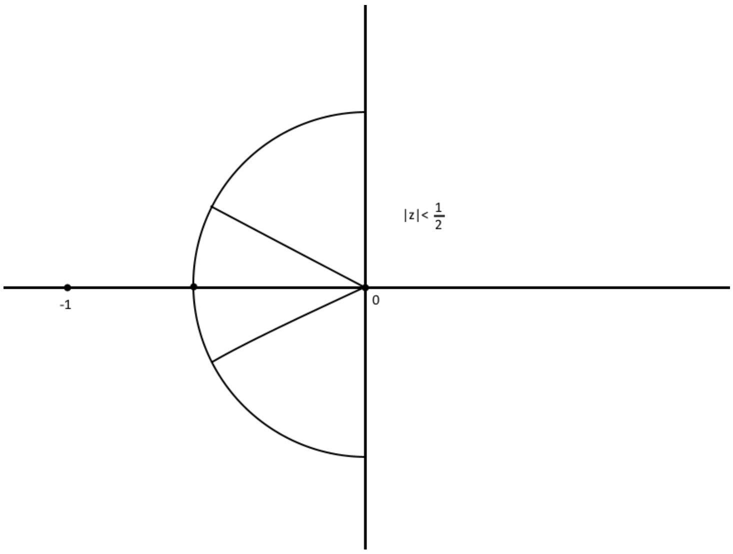

Figure 1.1: Sketch of the area we are looking at in $\mathbb{C}$.

$Im(z) = Im\,(1+z)$, so we get that the line is stretched further out along the real axis, but since the length along the imaginary axis is the same, we get that β is a smaller angle than α. This can also be seen in figure 1.2. Now we will use this to find the argument of $f(z)$.

$$z(1+z) = |z|e^{i(\pi-\alpha)}|1+z|e^{i\beta} = |z||1+z|e^{i(\pi-(\alpha-\beta))}$$
$$arg(z(1+z)) = \pi - (\alpha - \beta) \tag{1.4}$$
$$\Rightarrow \pi - arg(f(z)) < \pi - arg(z) \tag{1.5}$$

And since $\beta < \alpha$ we also have $\alpha - \beta > 0$, thus $arg(f(z))$ will never go over π, no matter how many times we iterate it, but it could converge to π.

We get the same result for $arg(z) = \pi + \alpha > \pi \Rightarrow arg(1+z) = 2\pi - \beta < 2\pi$, where $\beta < \alpha$, as seen in figure 1.2.

$$z(1+z) = |z|e^{i(\pi+\alpha)}|1+z|e^{i(2\pi-\beta)} = |z||1+z|e^{i(\pi+\alpha-\beta)}$$
$$arg(z(1+z)) = \pi + \alpha - \beta \tag{1.6}$$
$$\Rightarrow \pi - arg(z) < \pi - arg(f(z)) \tag{1.7}$$

and again because $\beta < \alpha$ we also have $\alpha - \beta > 0$, thus $arg(f(z))$ will never go under π, also here this will be the case no matter how many times we iterate, but it could converge to π.

We will now look at $|1+z|$ when we have $\pi - \frac{\pi}{8} < arg(z) < \pi + \frac{\pi}{8}$ and $|z| < \frac{1}{2}$. We want $|1+z|$ to

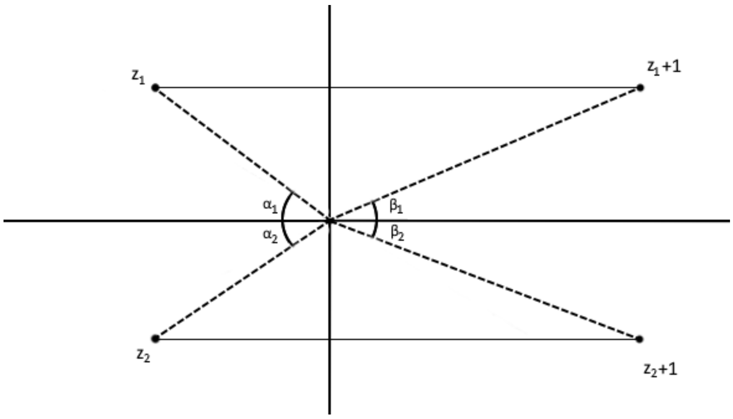

Figure 1.2: Sketch of four points z_1, z_2, z_1+1 and z_2+1, and the angels $\alpha_1, \alpha_2, \beta_1$ and β_2 associated with each of them respectively.

be less than one in order for us to can conclude that, z, gets smaller by every iteration of $f(z)$.

$$\begin{aligned} |1+z|^2 &= Im(z)^2 + (Re(z)+1)^2 && (1.8) \\ &= (|z|\sin\theta)^2 + (|z|\cos\theta+1)^2 \\ &= |z|^2\sin^2\theta + |z|^2\cos^2\theta + 2|z|\cos\theta + 1 \\ &= |z|^2 + 2|z|\cos\theta + 1 && (1.9) \end{aligned}$$

Where $\theta = arg(z)$. Then since $-1 = \cos\pi \leq \cos\theta \leq \cos\left(\pi \pm \frac{\pi}{8}\right) < -0.9$ and $|z| < \frac{1}{2}$, we can find an upper bound of $|1+z|^2$ by using this on (1.9).

$$\begin{aligned} |z|^2 + 2|z|\cos\theta + 1 &\leq |z|^2 - 1.8|z| + 1 \\ |z|^2 - 1.8|z| &\leq 0 \\ |z|^2 - 1.8|z| + 1 &\leq 1 && (1.10) \end{aligned}$$

where it is equal to one if $z = 0$, but $f(0) = 0$ so this can be ignored in this instance, if $z \neq 0$ then $|1+z|^2 < \frac{1}{4} - 1.8\frac{1}{2} + 1 = 0.35$. By using what we found in (1.10), we are able to determine the upper bound of $|1+z|$, and thus that $f(z)$ gets smaller for every iteration.

$$\begin{aligned} |1+z|^2 < 0.35 &\Rightarrow |1+z| < \sqrt{0.35} < 0.6 \\ &\Rightarrow |f(z)| = |z| \cdot |1+z| < |z|\frac{3}{5} && (1.11) \end{aligned}$$

If we consider both that $f(z)$ shrinks at a good pace, and the fact that the argument of f approaches π when n $\rightarrow\infty$, so it will continue to shrink this way as it stays in the same domain as it started in. We can thus conclude that $f^n(z) \rightarrow 0$ as $n \rightarrow \infty$ when $\pi - \frac{\pi}{8} < arg(z) < \pi + \frac{\pi}{8}$ and $|z| < \frac{1}{2}$.

Since we can take a neighborhood around a point in this domain and get the same result when we use f^n on it, with the exception of $z = 0$, we can conclude that if $\pi - \frac{\pi}{8} < arg(z) < \pi + \frac{\pi}{8}$ and $0 < |z| < \frac{1}{2}$ then we have that $z \in \mathcal{F}$, since that means it is a normal point .

We will now introduce an invertible function ψ, to help us find more locations where $f^n(z)$ will converge to 0. We use ψ to construct a new function, g, that will give us equations that are easier to work with after n-iterations. We will use g to find f, by first mapping z to $g(z)$ using ψ, ψ^{-1} and $f(z)$, and then mapping it back to our function f also by using ψ and ψ^{-1}.

$$g = \psi^{-1} \circ f \circ \psi \tag{1.12}$$
$$g^n = \psi^{-1} \circ f \circ \psi \circ \psi^{-1} \circ f \circ \psi^{-1} \circ \cdots \circ \psi^{-1} \circ f \circ \psi$$
$$g^n = \psi^{-1} \circ f^n \circ \psi \tag{1.13}$$

If we let $\psi(z) = \frac{1}{z}$ where $z \neq 0$, then the inverse has to be $\psi^{-1} = \psi$. By inserting this into (1.12) we get a clearer definition of g as a function of z, and are able to search for ways to define g^n.

$$\begin{aligned} g(z) &= \psi^{-1} \circ f \circ \psi(z) \\ &= \psi^{-1} \circ f\left(\frac{1}{z}\right) \\ &= \psi^{-1} \circ \left(\frac{1}{z} + \frac{1}{z^2}\right) \\ &= \frac{1}{\frac{1}{z} + \frac{1}{z^2}} \\ g(z) &= \frac{z}{1 + \frac{1}{z}} \end{aligned} \tag{1.14}$$

This reminds us of the geometric series. We know that for a $|\omega| < 1$ where $\omega \in \mathbb{C}$, the geometric series will converge to something similar to the result in (1.14), as shown under.

$$\begin{aligned} S_N &= \sum_{j=0}^{N} (\omega)^j \\ &= 1 + \omega + \omega^2 + \cdots + \omega^N \\ \omega S_N &= \omega + \omega^2 + \cdots + \omega^N + \omega^{N+1} \\ &= S_N - 1 + \omega^{N+1} \\ \Rightarrow S_N\,(1-\omega) &= 1 - \omega^{N+1} \\ S_N &= \frac{1-\omega^{N+1}}{1-\omega} \end{aligned} \tag{1.15}$$
$$N \to \infty \Rightarrow \omega^{N+1} \to 0$$
$$\sum_{j=0}^{\infty} (\omega)^j = \frac{1}{1-\omega} \tag{1.16}$$

Thus if we have $\left|\frac{1}{z}\right| < 1$ then (1.16) gives us that we can rewrite equation (1.14) to the following, by replacing ω with $-\frac{1}{z}$. Underneath we refer to the j-th iteration of g on z as z_j, so $g^j(z) = z_j$

for $j = 1, \cdots, n$.

$$\begin{aligned} g(z) &= z\sum_{j=0}^{\infty}\left(-\frac{1}{z}\right)^{j} \\ &= z\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right) \\ &= z-1+\frac{1}{z}-\frac{1}{z^2}+\cdots \\ &= z-1+\mathcal{O}\left(\frac{1}{z}\right) \\ &= z_1 \\ g^2(z) &= g(z_1) \\ \Rightarrow g^2(z) &= z_1-1+\frac{1}{z_1}-\frac{1}{z_1^2}+\cdots \\ &= z-2+\frac{1}{z_1}+\frac{1}{z}-\frac{1}{z_1^2}-\frac{1}{z^2}+\cdots \\ &= z-2+\mathcal{O}\left(\frac{1}{z}\right) \\ &= z_2 \end{aligned} \tag{1.17}$$

However this only works as long as $|z_j|$ constantly is larger than one, if we ever get a $|z_j| \leq 1$ this approach will not work anymore. In the cases where $|z_j|$ always is larger than one, for every j, we start noticing a pattern, such that we can define $g^n(z)$ like the following.

$$g^n = z_n = z - n + \mathcal{O}\left(\frac{1}{z}\right) \tag{1.18}$$

Because the z part doesn't grow or shrink, it stays the same, and $\mathcal{O}\left(\frac{1}{z}\right)$ will see little change in value after a while. They will end up being small in comparison to $-n$ when n gets large, and this will be the dominating part, which we will consider to find an approximation of f^n. We will now use what we found in (1.18) and the definition of g^n from (1.13) to find f^n. Here we use that $0 \neq |\xi| < 1$ such that $\left|\frac{1}{\xi}\right| > 1$, and also assume we meet the criteria we set to find g^n in (1.18).

$$\begin{aligned} g^n &= \psi^{-1} \circ f^n \circ \psi \\ \Rightarrow f^n(\xi) &= \psi \circ g^n \circ \psi^{-1}(\xi) \\ &= \psi \circ g^n\left(\frac{1}{\xi}\right) \\ &= \psi\left(\frac{1}{\xi} - n + \mathcal{O}(\xi)\right) \end{aligned} \tag{1.19}$$

$$f^n(\xi) = \frac{1}{\frac{1}{\xi} - n + \mathcal{O}(\xi)} \approx -\frac{1}{n} \tag{1.20}$$

Thus $f^n(\xi) \to 0$ when $n \to \infty$. Also since we can use f^n on neighborhood around a point ξ which satisfies the criteria we set, and get the same result, this means that it is a normal point. Thus if $0 \neq |\xi| < 1$ and $f^n(\xi) \to 0$ then we have that $\xi \in \mathcal{F}$.

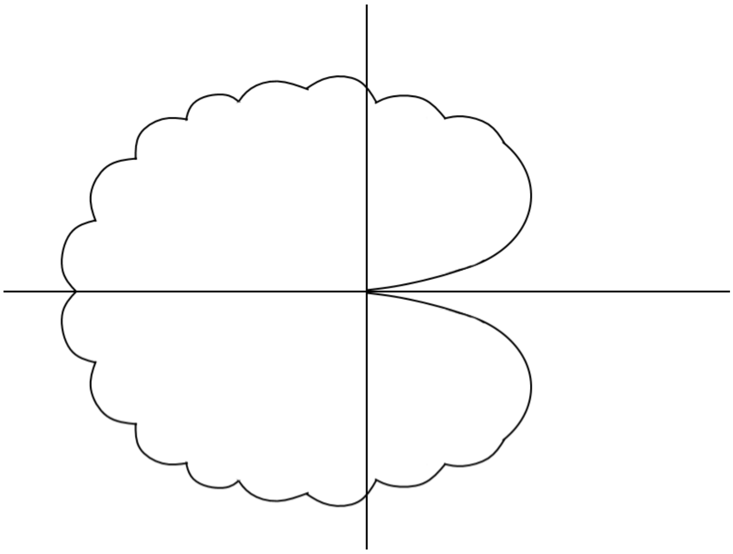

Figure 1.3: Sketch of the domain for k=1, inside the figure it converges to zero, and outside it diverges to infinity, both of these areas belong in the Fatou set. On the boundary it has a more chaotic behaviour and belongs to the Julia set.

Now let's look at another way to write our function $f^n(\xi)$, this time by using the fact that $x = e^{log(x)}$. We do this to find an approximation to the convergence of our function, where we also show that it can converge to zero without necessarily having a starting value $|\xi| < 1$, as long at it shrinks to it after some iterations.

$$\begin{aligned} f\left(\xi\right) &= \xi\left(1+\xi\right) \\ \xi_j &= f^j\left(\xi\right) \\ f^n\left(\xi\right) &= \xi\left(1+\xi\right)\left(1+\xi_1\right)\cdots\left(1+\xi_{n-1}\right) \\ &= \xi e^{log(1+\xi)} e^{log(1+\xi_1)} \cdots e^{log(1+\xi_{n-1})} \\ f^n(\xi) &= \xi e^{\Sigma_{j=0}^{n-1} log\left(1+\xi_j\right)} \end{aligned} \tag{1.21}$$

We want to make this easier to work with, which can be done by replacing the log part in (1.21). The derivatives of $log(1+x)$ are as shown under, and we will use them to substitute the $log(1+\xi_j)$

terms of our equation, by implementing the Taylor series of it.

$$
\begin{aligned}
h(x) &= log\,(1+x) \\
h'(x) &= \frac{1}{1+x} \\
h''(x) &= -\frac{1}{(1+x)^2} \\
h'''(x) &= \frac{2}{(1+x)^3} \\
h^{(k)} &= (-1)^{k-1}\frac{1\cdot 2\cdot 3\cdots(k-1)}{(1+x)^k} \\
h^{(k)} &= (-1)^{k-1}\frac{(k-1)!}{(1+x)^k} \qquad (1.22)
\end{aligned}
$$

Now that we have the derivatives of $log(1+x)$, we can use this to calculate the Taylor series at $a=0$, also known as the Maclaurin series.

$$
\begin{aligned}
h(x) &= \sum_{k=1}^{\infty}\frac{h^k(0)}{k!}x^k \\
h(x) &= \sum_{k=1}^{\infty}(-1)^{k-1}\frac{(k-1)!}{k!}x^k \\
h(x) &= \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k} \qquad (1.23)
\end{aligned}
$$

We are now able to replace the $log(1+\xi_j)$ term in (1.21) with the Taylor series and thus get the following result, where we assume $|\xi_j|<1$ for all $j=t,t+1,\cdots,n$, where $j=t$ is the iteration where $f^j(\xi)$ first had a absolute value less than one.

$$
\begin{aligned}
log\,(1+\xi_j) &= \xi_j - \frac{\xi_j^2}{2} + \frac{\xi_j^3}{3} - \cdots \\
log\,(1+\xi_j) &= \xi_j + \mathcal{O}\left(\xi_j^2\right) \qquad (1.24) \\
\Rightarrow f^n(\xi) &= \xi e^{\Sigma_{j=0}^{t-1} log\,(1+\xi_j)} e^{\Sigma_{j=t}^{n-1}\xi_j + \mathcal{O}\left(\xi_j^2\right)} \\
f^n(\xi) &= \xi_t e^{\Sigma_{j=t}^{n-1}\xi_j + \mathcal{O}\left(\xi_j^2\right)} \qquad (1.25)
\end{aligned}
$$

This is the function form we will work with, and estimate a value of. We start off by looking at a part of the function. By using the result we got in (1.20) we are able to give an approximation of $e^{\Sigma_{j=j_0}^{n-1}\xi_j}$, where j_0 is a sufficiently large number.

$$
\begin{aligned}
f^n(\xi) &= \xi_t e^{\Sigma_{j=t}^{n-1}\xi_j + \mathcal{O}\left(\xi_j^2\right)} \\
f^n(\xi) &= \xi_t e^{\Sigma_{j=t}^{j_0}\xi_j} e^{\Sigma_{j=j_0}^{n-1}\xi_j} e^{\Sigma_{j=t}^{n-1}\mathcal{O}\left(\xi_j^2\right)} \qquad (1.26)
\end{aligned}
$$

$$
e^{\Sigma_{j=j_0}^{n-1}\xi_j} \approx e^{\Sigma_{j=j_0}^{n-1}\frac{1}{-j}} \qquad (1.27)
$$

We want to give an estimate of $\sum_{j=j_0}^{n-1} \frac{1}{-j}$. I propose that we can do this by estimating the sum of $\frac{1}{m}$, which we will do by comparing it to the integral of a function $\frac{1}{x}$.

$$\int_{m_0}^{M} \frac{1}{x} dx = log(M) - log(m_0) \tag{1.28}$$

$$\sum_{m=m_0}^{M} \frac{1}{m} \approx log(M) - log(m_0) \tag{1.29}$$

We also want to look at the error value this approximation gives us, compared to the exact value.

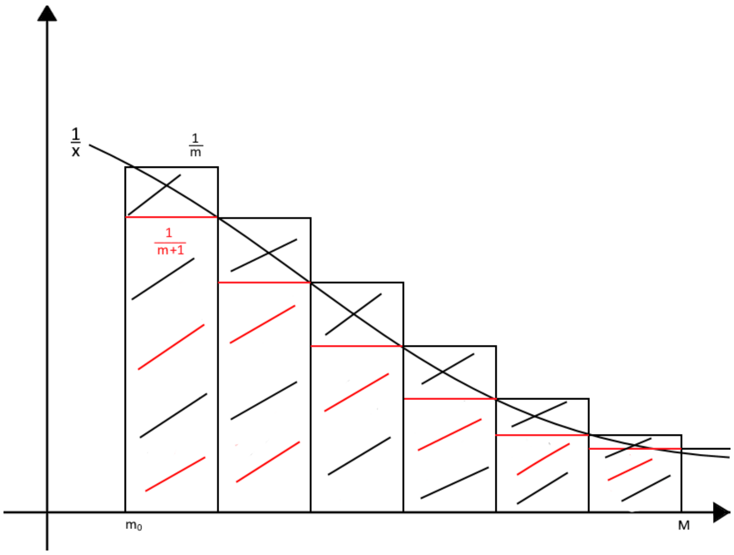

Figure 1.4: Sketch of the sums of $\frac{1}{m}$ (in black), and $\frac{1}{m+1}$ (in red), compared to the area under $\frac{1}{x}$.

From figure 1.4 we observe that one way we can find the error is as follows, by comparing the sum of $\frac{1}{m}$, the sum of $\frac{1}{m+1}$ and the integral of $\frac{1}{x}$. Here $\epsilon \in \mathbb{R}$ denotes the error.

$$\sum_{m=m_0}^{M} \frac{1}{m} > log(M) - log(m_0)$$

$$\sum_{m=m_0+1}^{M+1} \frac{1}{m} = \sum_{m=m_0}^{M} \frac{1}{m+1} < log(M) - log(m_0)$$

$$\Rightarrow \sum_{m=m_0}^{M} \frac{1}{m} - \sum_{m=m_0+1}^{M+1} \frac{1}{m} = \frac{1}{m_0} - \frac{1}{M}$$

$$\epsilon \leq \frac{1}{m_0} - \frac{1}{M} \tag{1.30}$$

$$\Rightarrow \sum_{m=m_0}^{M} \frac{1}{m} = log(M) - log(m_0) + \epsilon \tag{1.31}$$

We insert this into (1.26), and also note that the sum of ξ_j for $j = t, \cdots, j_0$ will not provide a big value to the equation, as the imaginary part will only obtain a rotation since it is the exponent of Euler's number. So the real part is the only component providing any size to the value, and since $|\xi_j| < 1$ for $j = t, \cdots, j_0$ this will not become a large number. Also we have found earlier in the text that if $|\xi| > 3$ then $f^n(\xi)$ would diverge to infinity, so we have that $|\xi_t| < 3$. Therefore $\xi_t e^{\sum_{j=t}^{j_0} \xi_j}$ will not be of significant size. Also we have that ϵ is a small number, so we get the following approximation.

$$\begin{aligned} f^n(\xi) &= \xi_t e^{\sum_{j=t}^{j_0} \xi_j} e^{-log(n-1)+log(j_0)-\epsilon} e^{\sum_{j=t}^{n-1} \mathcal{O}\left(\xi_j^2\right)} \\ &= \xi_t e^{\sum_{j=t}^{j_0} \xi_j} e^{-\epsilon} e^{\sum_{j=t}^{n-1} \mathcal{O}\left(\xi_j^2\right)} \cdot \frac{j_0}{n-1} \\ e^{-\epsilon} &\approx 1 \end{aligned} \tag{1.32}$$

We also need to look at $e^{\sum_{j=t}^{n-1} \mathcal{O}\left(\xi_j^2\right)}$ to give our result. Remember that $\mathcal{O}\left(\xi_j^2\right)$ represents all the extra terms in the Maclaurin series of $log(1+\xi_j)$. This will be the key to giving an approximation of its value combined with the result from (1.20). From the Maclaurin series we notice that we will have a small value, and when j gets large enough we can switch ξ_j to $\frac{1}{-j}$. Which gives us the idea of comparing our result with $\sum_{j=1}^{\infty} \frac{1}{j^2}$, which we know the value of, and thus can give an approximation of the value of $\left| e^{\sum_{j=t}^{n-1} \mathcal{O}\left(\xi_j^2\right)} \right|$.

$$\begin{aligned} \mathcal{O}\left(\xi_j^2\right) &= -\frac{\xi_j^2}{2} + \frac{\xi_j^3}{3} - \frac{\xi_j^4}{4} + \cdots \\ &\approx -\frac{1}{2j^2} + \frac{1}{-3j^3} - \frac{1}{4j^4} + \cdots \\ &\approx -\frac{1}{j^2} \\ \left| \sum_{j=t}^{n-1} \mathcal{O}\left(\xi_j^2\right) \right| &\approx \sum_{j=1}^{\infty} \frac{1}{j^2} \\ &= \frac{\pi^2}{6} \\ \left| e^{\sum_{j=t}^{n-1} \mathcal{O}\left(\xi_j^2\right)} \right| &\approx e^{\frac{\pi^2}{6}} \approx 5.2 \end{aligned} \tag{1.33}$$

Finally by combining all the terms we have looked at we will have an approximation of $f^n(z)$, and when $n \to \infty$ it will be really big compared to our other terms, which are not of significant value, and thus be the main factor in our result, combined with some sort of angle ϕ which it comes in from.

$$
\begin{aligned}
f^n(\xi) &= \xi_t e^{\Sigma_{j=t}^{j_0}\xi_j} e^{-\epsilon} e^{\Sigma_{j=t}^{n-1}\mathcal{O}\left(\xi_j^2\right)} \cdot \frac{j_0}{n-1} \\
&\approx \xi_t e^{\Sigma_{j=t}^{j_0}\xi_j} \cdot 1 \cdot 5.2 \cdot \frac{j_0}{n-1} \\
f^n(\xi) &\approx e^{i\phi}\frac{1}{n} \qquad (1.34)
\end{aligned}
$$

However since $0 \neq |\xi_t| < 1$ we are able to find the angle ϕ, as we could put ξ_t into (1.20) and compare the value with (1.34) when $n \to \infty$. This would give $e^{i\phi}\frac{1}{n} = -\frac{1}{n} \Rightarrow \phi = \pi$. This means that as long as a value converges to zero, then we can use (1.20) to calculate its rate of convergence.

We now introduce a theorem that will help us describe the behaviour of the rate of convergence for $f^n(z)$ to 0.

Theorem 1.1.1. *For all points $z \in \mathbb{C}$, such that $f^n(z) \to 0$ and $f^n(z) \neq 0$, we have that.*

$$\lim_{n\to\infty} \frac{f^n(z)}{\frac{1}{n}} \to -1$$

Proof. Since we have that $f^n(z) \to 0$ and $f^n(z) \neq 0$ we can use what we found in (1.20), and thus we have.

$$f^n(z) = \frac{1}{\left(\frac{1}{z} - n + \mathcal{O}(z)\right)}$$

So now we are able to to calculate $\frac{f^n(z)}{\frac{1}{n}}$, by using this definition of $f^n(z)$.

$$
\begin{aligned}
\frac{f^n(z)}{\frac{1}{n}} &= \frac{\frac{1}{\left(\frac{1}{z}-n+\mathcal{O}(z)\right)}}{\frac{1}{n}} \\
&= \frac{\left(\frac{1}{z} - n + \mathcal{O}(z)\right)^{-1}}{n^{-1}} \\
&= \left(\frac{\frac{1}{z} - n + \mathcal{O}(z)}{n}\right)^{-1} \\
&= \left(\frac{1}{n \cdot z} - 1 + \frac{\mathcal{O}(z)}{n}\right)^{-1} \qquad (1.35)
\end{aligned}
$$

Now we let $n \to \infty$ and then we obtain the desired result.

$$
\begin{aligned}
\lim_{n\to\infty} \frac{f^n(z)}{\frac{1}{n}} &= \lim_{n\to\infty} \left(\frac{1}{n \cdot z} - 1 + \frac{\mathcal{O}(z)}{n}\right)^{-1} \\
\lim_{n\to\infty} \frac{f^n(z)}{\frac{1}{n}} &\to (-1)^{-1} = -1 \qquad (1.36)
\end{aligned}
$$

□

Therefore we have from theorem 1.1.1 that our function in (1.20) behaves as $-\frac{1}{n}$ if n is large. We will adjust this theorem for the general case later.

$$\begin{aligned} f^n(z) &= \frac{1}{\left(\frac{1}{z} - n + \mathcal{O}(z)\right)} \\ f^n(z) &\sim -\frac{1}{n} \end{aligned} \tag{1.37}$$

Also since we can use f^n on neighborhood around a point z which satisfies theorem 1.1.1 and get the same result, this means that z is a normal point. Therefore if $f^n(z) \to 0$ and $f^n(z) \neq 0$ then we have that $z \in \mathcal{F}$.

We will now examine two points which i propose are in $\mathcal{J}$, i and $-i$. Lets begin with looking at i. Remember that if $z = a + ib$, then $arg(z) = tan^{-1}\left(\frac{b}{a}\right)$, we use this when we find the angles.

$$\begin{aligned} f(i) &= i(1+i) \\ &= e^{i\frac{\pi}{2}}\left(\sqrt{2}e^{i\frac{\pi}{4}}\right) \\ &= \sqrt{2}e^{i\frac{3\pi}{4}} \\ f\left(\sqrt{2}e^{i\frac{3\pi}{4}}\right) &= \sqrt{2}e^{i\frac{3\pi}{4}}\left(1+\sqrt{2}e^{i\frac{3\pi}{4}}\right) \\ &= \sqrt{2}e^{i\frac{3\pi}{4}}(1-1+i) \\ &= \sqrt{2}e^{i\left(\frac{3\pi}{4}+\frac{\pi}{2}\right)} \\ &= \sqrt{2}e^{i\frac{5\pi}{4}} = -1-i \\ f\left(\sqrt{2}e^{i\frac{5\pi}{4}}\right) &= \sqrt{2}e^{i\frac{5\pi}{4}}\left(1+\sqrt{2}e^{i\frac{5\pi}{4}}\right) \\ &= \sqrt{2}e^{i\frac{5\pi}{4}}(1-1-i) \\ &= \sqrt{2}e^{i\left(\frac{5\pi}{4}-\frac{\pi}{2}\right)} \\ &= \sqrt{2}e^{i\frac{3\pi}{4}} = -1+i \end{aligned}$$

Now lets look at $-i$.

$$\begin{aligned} f(-i) &= -i(1-i) \\ &= e^{i\frac{-\pi}{2}}\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right) \\ &= \sqrt{2}e^{i\frac{-3\pi}{4}} \\ &= \sqrt{2}e^{i\frac{5\pi}{4}} \\ f\left(\sqrt{2}e^{i\frac{5\pi}{4}}\right) &= \sqrt{2}e^{i\frac{3\pi}{4}} \end{aligned}$$

So we see that both i and $-i$ end up in a loop between the two points, $-1+i$ and $-1-i$, when we iterate the function f on them. However if we were to look at the neighborhoods around these points, that is including $-1+i$ and $-1-i$, we would notice that f^n would get varying results. Such as it diverging to infinity, or it converging to zero, nevertheless it would not converge uniformly to a limit or to infinity, therefore they are not normal points, meaning that $i, -i, -1+i, -1-i \in \mathcal{J}$. From [1] we also have that for any point z where $f^j(z)$ obtains a value in the Julia set, then z also belongs in the Julia set. More precisely if $z \in \mathcal{J}$ and $f^p(z) = f^q(z')$ for some $p \geq 0$ and $q \geq 0$, then $z' \in \mathcal{J}$. The last thing we do in this chapter is establishing that 0 and -1 are in the Julia set, which follows the same reasoning. 0 is a fixed point, and using f^n on a neighborhood around it will give us varying results. Like diverging to infinity or converging to zero, so it is not a normal point, meaning that $0 \in \mathcal{J}$. Also since $f(-1) = 0$ we have that $-1 \in \mathcal{J}$.

1.2 General case

We will now consider the general case of the function f, as defined in (1.2) with $k \in \mathbb{N}$.

$$f(z) = z\left(1 + z^k\right)$$

Also here we want to locate some points where $f^n(z)$ either diverges to ∞, or converges to 0 as $n \to \infty$. And we will find out if the points are normal and belong in the Fatou set $\mathcal{F}$, or if they belong in the Julia set $\mathcal{J}$. Let's start with considering $|z| > 2^{\frac{1}{k}}$, then we find that this implies the following results, by using the triangle inequality.

$$\begin{aligned} |1+z^k| &\geq |z^k| - 1 > 2 - 1 = 1 \\ \Rightarrow |z||1+z^k| &> |z| \end{aligned} \tag{1.38}$$

Therefore we can say that it grows, at least up to some point, but it is not enough for us to say that it diverges to infinity. For us to conclude with that, we need it to grow at a more rapid pace, $|z| > 3^{\frac{1}{k}}$ should fulfill this requirement. And again by using the triangle inequality we get the following.

$$\begin{aligned} |1+z^k| &\geq |z^k| - 1 > 3 - 1 = 2 \\ \Rightarrow |z||1+z^k| &> 2|z| \end{aligned} \tag{1.39}$$

Thus $f^n(z) \to \infty$ as $n \to \infty$, when $|z| > 3^{\frac{1}{k}}$. When $|z| > 3$ this obviously also means that a neighborhood around z also has an absolute value less than three. So by using f^n on the neighborhood it would uniformly diverge to infinity,and therefore z is a normal point and we have that $3 < |z| \Rightarrow z \in \mathcal{F}$.
Now lets find out what happens when it approaches 0, the first step is to locate where $z^k = -s$ for $0 \leq s \in \mathbb{R}$.

$$z^k = -s \tag{1.40}$$

$$z^k = (-1) \cdot s$$

$$z^k = e^{i(\pi + 2m\pi)} \cdot s \tag{1.41}$$

$$\Rightarrow z = e^{i\left(\frac{\pi}{k} + \frac{2m\pi}{k}\right)} \cdot s^{\frac{1}{k}}$$

$$z = e^{i\left(\frac{\pi}{k} + \frac{2m\pi}{k}\right)} \cdot r \tag{1.42}$$

Where $m = 0, 1, \cdots, k-1$, and $r = s^{\frac{1}{k}}$.

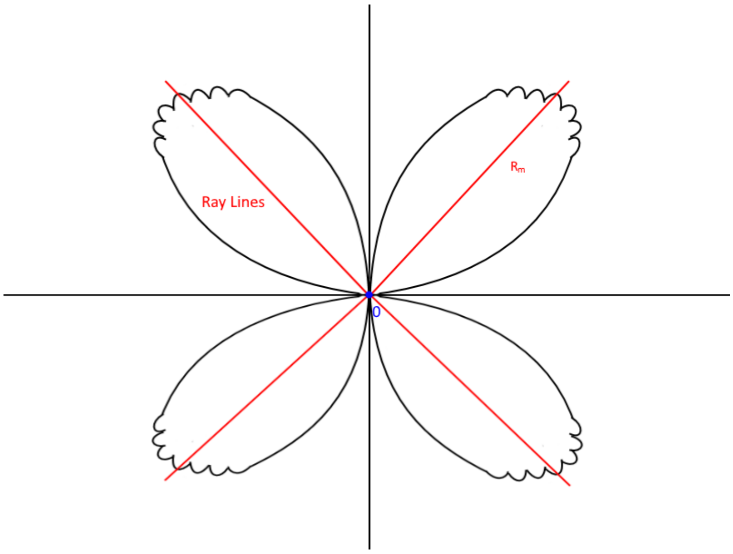

Figure 1.5: A sketch of what the domain could look like, here it would be for $k = 4$. Inside the "leaves" it converges to zero, outside it diverges to infinity, both of which are in the Fatou set. On the boundary it has a more chaotic behavior and belongs to the Julia set.

So we get k "Ray Lines" as seen in figure 1.5, this is what we are going to refert to them as. Lets call each ray line for R_m. If $0 < r < 1$ and $z \in R_m$ then we have the following.

$$z = re^{i\left(\frac{\pi}{k}+\frac{2m\pi}{k}\right)}$$

$$0 < \left(1 + z^k\right) \in \mathbb{R}$$

$$f(z) = z\left(1 + z^k\right) \in R_m$$

$$f(z) = re^{i\left(\frac{\pi}{k}+\frac{2m\pi}{k}\right)}\left(1 + r^k e^{i(\pi+2m\pi)}\right)$$

$$f(z) = e^{i\left(\frac{\pi}{k}+\frac{2m\pi}{k}\right)} r\left(1 - r^k\right) \tag{1.43}$$

$$r_j = \frac{f^j(z)}{e^{i\left(\frac{\pi}{k}+\frac{2m\pi}{k}\right)}} \tag{1.44}$$

$$f^n(z) = e^{i\left(\frac{\pi}{k}+\frac{2m\pi}{k}\right)} r\left(1 - r^k\right)\left(1 - r_1^k\right)\cdots\left(1 - r_{n-1}^k\right) \tag{1.45}$$

We will now show that $r_n \to 0$ as $n \to \infty$, first assume that we have a $0 < r_j < 1$.

$$
\begin{aligned}
r_j\left(1 - r_j^k\right) &= r_{j+1} && (1.46)\\
\left(1 - r_j^k\right) &< 1\\
\Rightarrow r_j\left(1 - r_j^k\right) &< r_j\\
r_{j+1} &< r_j && (1.47)
\end{aligned}
$$

So it will get smaller and smaller as n gets larger.
We will now use the invertible function $\psi(z)$ from the previous chapter, that is its own inverse $\psi^{-1}(z) = \psi(z)$, and we will also introduce another invertible function $\omega(z)$. We will use $\omega(z)$ on a new function, $f_k(z)$, which will help us find the rate of convergence of $f(z)$, this will become more apparent as we go on. Now lets define these functions.

$$
\begin{aligned}
\omega(z) &= z^k && (1.48)\\
\omega^{-1}(z) &= z^{\frac{1}{k}} && (1.49)\\
f_k(z) &= z\left(1+z\right)^k && (1.50)\\
\Rightarrow f(z) &= \omega^{-1} \circ f_k \circ \omega(z) && (1.51)\\
&= \omega^{-1} \circ f_k(z^k)\\
&= \omega^{-1}\left(z^k\left(1+z^k\right)^k\right)\\
&= z\left(1+z^k\right)\\
g_k &= \psi^{-1} \circ f_k \circ \psi && (1.52)\\
g_k^n &= \psi^{-1} \circ f_k \circ \psi(z) \circ \psi^{-1} \circ f_k \circ \psi \circ \cdots \circ \psi^{-1} \circ f_k \circ \psi\\
g_k^n &= \psi^{-1} \circ f_k^n \circ \psi && (1.53)
\end{aligned}
$$

So we use the same approach as in the last chapter, with the definition of ψ and f_k, which gives us a way to calculate what the function g_k will look like.

$$
\begin{aligned}
\psi(z) &= \frac{1}{z} && (1.54)\\
g_k(z) &= \psi^{-1} \circ f_k \circ \psi(z)\\
&= \psi^{-1} \circ f_k \left(\frac{1}{z}\right)\\
&= \psi^{-1} \left(\frac{1}{z}\left(1+\frac{1}{z}\right)^k\right)\\
&= \frac{1}{\frac{1}{z}\left(1+\frac{1}{z}\right)^k}\\
&= \frac{1}{\frac{1}{z}\left(1+k\frac{1}{z}+\mathcal{O}\left(\frac{1}{z^2}\right)\right)}\\
g_k(z) &= \frac{z}{1+k\frac{1}{z}+\mathcal{O}\left(\frac{1}{z^2}\right)} && (1.55)
\end{aligned}
$$

This looks like the geometric series, so if we have $|\frac{k}{z_j}+\mathcal{O}\left(\frac{1}{z_j^2}\right)|<1$ for every z_j, equation (1.16) gives us the following.

$$
\begin{aligned}
g_k(z) &= z\sum_{j=0}^{\infty}(-1)^j\left(k\frac{1}{z}+\mathcal{O}\left(\frac{1}{z^2}\right)\right)^j\\
&= z\left(1-k\frac{1}{z}+\mathcal{O}\left(\frac{1}{z^2}\right)\right)\\
&= z-k+\mathcal{O}\left(\frac{1}{z}\right)\\
&= z_1\\
\Rightarrow g_k^2 &= z_1-k+\mathcal{O}\left(\frac{1}{z_1}\right)\\
&= z-2\cdot k+\mathcal{O}\left(\frac{1}{z}\right)\\
&= z_2\\
\Rightarrow g_k^n &= z-n\cdot k+\mathcal{O}\left(\frac{1}{z}\right) && (1.56)
\end{aligned}
$$

From this we can now find f_k^n, by using what we found in (1.56) and the definition of g_k^n from

(1.53). We use that $z \neq 0$ and $|kz_j + \mathcal{O}\left(z_j^2\right)| < 1$.

$$\begin{aligned}
g_k &= \psi^{-1} \circ f_k \circ \psi \\
\Rightarrow f_k(z) &= \psi \circ g_k \circ \psi^{-1}(z) \quad (1.57)\\
&= \psi \circ g_k\left(\frac{1}{z}\right) \\
&= \psi\left(\frac{1}{z} - k + \mathcal{O}(z)\right) \\
f_k(z) &= \frac{1}{\frac{1}{z} - k + \mathcal{O}(z)} \quad (1.58)\\
g_k^n &= \psi^{-1} \circ f_k^n \circ \psi \\
\Rightarrow f_k^n(z) &= \psi \circ \psi \circ g_k^n \circ \psi^{-1}(z) \quad (1.59)\\
&= \psi \circ g_k^n\left(\frac{1}{z}\right) \\
&= \psi\left(\frac{1}{z} - n \cdot k + \mathcal{O}(z)\right) \\
f_k^n(z) &= \frac{1}{\frac{1}{z} - n \cdot k + \mathcal{O}(z)} \quad (1.60)
\end{aligned}$$

Finally we can use this result and what we found in (1.51) to calculate f^n. Also here we need $z \neq 0$ and $|kz_j^k + \mathcal{O}\left(z_j^{2k}\right)| < 1$.

$$\begin{aligned}
f &= \omega^{-1} \circ f_k \circ \omega \\
f^n &= \omega^{-1} \circ f_k \circ \omega \circ \omega^{-1} \circ f_k \circ \omega \circ \cdots \circ \omega^{-1} \circ f_k \circ \omega \\
f^n(z) &= \omega^{-1} \circ f_k^n \circ \omega(z) \quad (1.61)\\
&= \omega^{-1} \circ f_k\left(z^k\right) \\
&= \omega^{-1}\left(\frac{1}{\frac{1}{z^k} - n \cdot k + \mathcal{O}(z^k)}\right) \\
f^n(z) &= \frac{1}{\left(\frac{1}{z^k} - n \cdot k + \mathcal{O}(z^k)\right)^{\frac{1}{k}}} \quad (1.62)\\
& \quad (1.63)
\end{aligned}$$

Now we will introduce a theorem to help us describe the behaviour of the rate of convergence for $f^n(z)$ to 0.

Theorem 1.2.1. *For all points $z \in \mathbb{C}$, such that $f^n(z) \to 0$ and $f^n(z) \neq 0$, with $m = 0, 1, \cdots, k-1$ we have that.*

$$\lim_{n\to\infty} \frac{f^n(z)}{\frac{1}{n}^{\frac{1}{k}}} \to e^{i\left(\frac{\pi}{k} + \frac{2m\pi}{k}\right)} \left(\frac{1}{k}\right)^{\frac{1}{k}}$$

Proof. Since we have that $f^n(z) \to 0$ and $f^n(z) \neq 0$ we can use what we found in (1.62), and thus we have.

$$f^n(z) = \frac{1}{\left(\frac{1}{z^k} - n \cdot k + \mathcal{O}\left(z^k\right)\right)^{\frac{1}{k}}}$$

So now we are able to to calculate $\frac{f^n(z)}{\frac{1}{n}^{\frac{1}{k}}}$, by using this definition of $f^n(z)$.

$$\begin{aligned}
\frac{f^n(z)}{\frac{1}{n}^{\frac{1}{k}}} &= \frac{\frac{1}{\left(\frac{1}{z^k} - n\cdot k + \mathcal{O}\left(z^k\right)\right)^{\frac{1}{k}}}}{\frac{1}{n}^{\frac{1}{k}}} \\
&= \frac{\left(\frac{1}{z^k} - n \cdot k + \mathcal{O}\left(z^k\right)\right)^{-\frac{1}{k}}}{n^{-\frac{1}{k}}} \\
&= \left(\frac{\frac{1}{z^k} - n \cdot k + \mathcal{O}\left(z^k\right)}{n}\right)^{-\frac{1}{k}} \\
&= \left(\frac{1}{n \cdot z^k} - k + \frac{\mathcal{O}\left(z^k\right)}{n}\right)^{-\frac{1}{k}} \qquad (1.64)
\end{aligned}$$

Now we let $n \to \infty$ and then we obtain the desired result.

$$\begin{aligned}
\lim_{n\to\infty} \frac{f^n(z)}{\frac{1}{n}^{\frac{1}{k}}} &= \lim_{n\to\infty} \left(\frac{1}{n \cdot z^k} - k + \frac{\mathcal{O}\left(z^k\right)}{n}\right)^{-\frac{1}{k}} \\
\lim_{n\to\infty} \frac{f^n(z)}{\frac{1}{n}^{\frac{1}{k}}} &\to (-k)^{-\frac{1}{k}} = e^{i\left(\frac{\pi}{k} + \frac{2m\pi}{k}\right)} \left(\frac{1}{k}\right)^{\frac{1}{k}} \qquad (1.65)
\end{aligned}$$

□

Therefore we have from theorem 1.2.1 that our function in (1.62) behaves as $\left(-\frac{1}{n \cdot k}\right)^{\frac{1}{k}}$.

$$\begin{aligned}
f^n(z) &= \frac{1}{\left(\frac{1}{z^k} - n \cdot k + \mathcal{O}\left(z^k\right)\right)^{\frac{1}{k}}} \\
f^n(z) &\sim \left(-\frac{1}{n \cdot k}\right)^{\frac{1}{k}} = e^{i\left(\frac{\pi}{k} + \frac{2m\pi}{k}\right)} \left(\frac{1}{n \cdot k}\right)^{\frac{1}{k}} \qquad (1.66)
\end{aligned}$$

Also since we can use f^n on a neighborhood around a point z which satisfies theorem 1.2.1 and get the same result, this means that z is a normal point. Therefore if $f^n(z) \to 0$ and $f^n(z) \neq 0$ then we have that $z \in \mathcal{F}$.

The last thing we will look at, is a point in the Julia set, namely our fixed point 0. Since we get varying results when using f^n on a neighborhood around 0, such as it diverging to infinity or converging to zero, we can therefore say that it is not a normal point and thus $0 \in \mathcal{J}$. From this we can find many other points in the Julia set, such as the points shown below.

$$\begin{aligned} f(z) &= 0 \\ z(1+z^k) &= 0 \\ \Rightarrow z &= 0, e^{i(\frac{\pi}{k}+\frac{2m\pi}{k})} \end{aligned}$$

So for $m = 0, 1, \cdots, n-1$ we have that $e^{i(\frac{\pi}{k}+\frac{2m\pi}{k})} \in \mathcal{J}$. This we know from [1] where it is stated that if $z \in \mathcal{J}$ and $f^p(z) = f^q(z')$ for some $p \geq 0$ and $q \geq 0$, then $z' \in \mathcal{J}$.

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