TOTALLY ACYCLIC APPROXIMATIONS

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Abstract. Let $Q \to R$ be a surjective homomorphism of Noetherian rings such that Q is Gorenstein and R as a Q-bimodule admits a finite resolution by modues which are projective on both sides. We define an adjoint pair of functors between the homotopy category of totally acyclic R-complexes and that of Q-complexes. This adjoint pair is analogous to the classical adjoint pair of functors between the module categories of R and Q. As a consequence, we obtain a precise notion of approximations of totally acyclic R-complexes by totally acyclic Q-complexes.

Introduction

This paper is the result of a desire to approximate in a meaningful way totally acyclic complexes over a Noetherian ring R by simpler totally acyclic complexes, possibly even periodic ones. By definition, (nontrivial) totally acyclic complexes are (necessarily unbounded) exact complexes of finitely generated projective modules, whose dual complex is also exact. Nontrivial ones exist abundantly over any Gorenstein ring, and their behavior can be quite varied and unexpected (see, for example, [JoŞe].) Approximating such complexes by simpler ones was the motivation for our main theorem below. Our interest in this project was also motivated by recent work in [Kr] and [Ne2] on approximations and adjoints in homotopy categories.

The homotopy category $\mathbf{K}_{\text{tac}}(R)$ of totally acyclic R-complexes is a thick triangulated subcategory of the homotopy category of R-complexes. The main result of the paper is the establishment of an adjoint pair of functors which are triangle versions of the classical adjoint pair of functors between the module categories through change of rings.

Theorem A. *Assume that $\varphi : Q \to R$ is a surjective homomorphism of Noetherian rings such that Q is Gorenstein, and such that R as a Q-bimodule admits a finite resolution by modules which are projective as both left and right Q-modules. Then there exists an adjoint pair of triangle functors*

$$\mathbf{K}_{\text{tac}}(Q) \underset{T_\varphi}{\overset{S_\varphi}{\rightleftarrows}} \mathbf{K}_{\text{tac}}(R)$$

As is the case in the classical setting, the functor S_φ is simply the base change functor $R \otimes_Q -$. However, as nontrivial totally acyclic R-complexes are never totally acyclic Q-complexes when $R \neq Q$, obtaining the right adjoint T_φ requires a modification of the forgetful functor. Its construction is given in Section 2, where we prove in detail that it is a triangle functor. It is well known that adjoints of triangle

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functors are themselves triangulated (see, for example, [Ne1, Lemma 5.3.6]), hence since S_φ is triangulated, it follows immediately that so is T_φ. However, we find it instructive to give the direct arguments for T_φ. We remark also that the conditions of Q being Gorenstein and the finiteness of the projective dimension of R over Q are essential to the existence of S_φ and T_φ. Moreover, the existence of a finite bimodule resolution of R by two-sided projective Q-modules is key to the adjunction. When Q is a commutative local ring, then this condition amounts to saying that R has finite projective dimension as a Q-module. We prove theorem A in Section 3 through explicit computations of the unit and counit natural transformations.

In Section 4 we discuss the approximations we sought. Our main result here is the following.

Theorem B. *The isomorphism closure in $\mathbf{K}_{\mathrm{tac}}(R)$ of the image of S_φ is functorially finite in $\mathbf{K}_{\mathrm{tac}}(R)$. In other words, both left and right approximations exist in $\mathbf{K}_{\mathrm{tac}}(R)$ by objects in the isomorphism closure of the image of S_φ.*

That right approximations exist in $\mathbf{K}_{\mathrm{tac}}(R)$ by objects in the isomorphism closure of the image of S_φ is an immediate consequence of Theorem A. To show that left approximations exist, we use the existence of right approximations and the duality properties inherent in $\mathbf{K}_{\mathrm{tac}}(R)$. We also give several examples illustrating aspects of Theorem B.

After an earlier version of this paper appeared on the arXiv, the paper [OpPsSt] was published, with some similar results. The paper studies ring homomorphisms of Noetherian algebras, and the first part of its Theorem I, where an adjoint pair on the level of singularity categories is established, is analogous to our Theorem A.

1. Preliminaries

Let R be an associative ring with unity. Unless otherwise stated, we assume all modules to be left modules. By an *R-complex* C we mean a sequence of left R-module homomorphisms

$$C : \cdots \to C_{n+1} \xrightarrow{\partial^C_{n+1}} C_n \xrightarrow{\partial^C_n} C_{n-1} \to \cdots$$

graded homologically, so that n is the *homological degree* of C_n. If C is such a complex, then $\mathrm{Hom}_R(C, R)$ is a complex of right R-modules via the canonical right action on each $\mathrm{Hom}_R(C_n, R)$. We often regard R-modules as complexes concentrated in homological degree 0.

Given a complex C, we denote by ΣC the *shift* of C, where $(\Sigma C)_n = C_{n-1}$; one has the natural map $\sigma : C \to \Sigma C$, where for $x \in C_n$, the element $\sigma_n(x)$ is the same element x, but now has degree $n+1$ in ΣC. This map σ has the obvious inverse $\sigma^{-1} : \Sigma C \to C$. the differential of ΣC is taken to be $\partial^{\Sigma C}_n = -\sigma_{n-2}\partial^C_{n-1}\sigma^{-1}_n$. For a morphism of complexes $f : C \to D$ we have the induced morphism $\Sigma f : \Sigma C \to \Sigma D$. We also have the inverse shift Σ^{-1}, where $(\Sigma^{-1}C)_n = C_{n+1}$ and $\partial^{\Sigma^{-1}}_n = -(\Sigma^{-1}\sigma)^{-1}\partial^C_{n+1}\Sigma^{-1}\sigma$.

From this point on, unless stated to the contrary, we assume that R is a Noetherian ring (on both sides).

Definition. Recall from [AvMa] that an R-complex C of finitely generated projective modules is called *totally acyclic* if

$$\mathrm{H}(C) = 0 = \mathrm{H}(\mathrm{Hom}_R(C, R)).$$

Note that if P is a finitely generated projective R-module and C is a totally acyclic R-complex, then $\mathrm{H}(\mathrm{Hom}_R(C,P)) = 0$.

Recall that the homotopy category of R-complexes is a triangulated category with shift functor Σ. We denote by $\mathbf{K}_{\mathrm{tac}}(R)$ the subcategory of the homotopy category of R-complexes consisting of the totally acyclic R-complexes; the objects in $\mathbf{K}_{\mathrm{tac}}(R)$ are the totally acyclic R-complexes, and the morphisms are homotopy equivalence classes of morphisms of R-complexes. This is a thick triangulated subcategory of the homotopy category. For a morphism $f : C \to C'$ of R-complexes we write $[f]$ for its homotopy equivalence class. Thus for two morphisms $f, g : C \to C'$ of R-complexes, one has $f \sim g$ if and only if $[f] = [g]$.

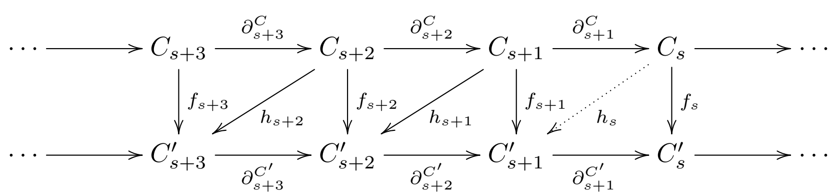
The following facts we use often in the rest of the paper. Given an R-complex C, we write from now on C^* for the R-complex $\mathrm{Hom}_R(C,R)$.

1.1. Suppose that s is an integer, $C \in \mathbf{K}_{\mathrm{tac}}(R)$, C' is a complex of finitely generated projective R-modules and $f : C \to C'$ is a morphism of R-complexes. If there exist maps $h_n : C_n \to C'_{n+1}$ satisfying

$$f_{n+1} = h_n \partial^C_{n+1} + \partial^{C'}_{n+2} h_{n+1}$$

for all $n > s$, then the h_n can be extended to a homotopy showing that $f \sim 0$.

Proof. We need to complete the diagram



By induction it suffices to show there exists a map $h_s : C_s \to C'_{s+1}$ such that $f_{s+1} = h_s \partial^C_{s+1} + \partial^{C'}_{s+2} h_{s+1}$.

Note that since C is a totally acyclic complex and C'_{s+1} is finitely generated projective, the complex $\mathrm{Hom}_R(C, C'_{s+1})$ is exact. We have

$$\begin{aligned}
\partial^{\mathrm{Hom}_R(C,C'_{s+1})}_{s+1}(f_{s+1} - \partial^{C'}_{s+2} h_{s+1}) &= \left(f_{s+1} - \partial^{C'}_{s+2} h_{s+1}\right) \partial^C_{s+2} \\
&= f_{s+1}\partial^C_{s+2} - \partial^{C'}_{s+2}\left(f_{s+2} - \partial^{C'}_{s+3} h_{s+2}\right) \\
&= \left(f_{s+1}\partial^C_{s+2} - \partial^{C'}_{s+2} f_{s+2}\right) + \partial^{C'}_{s+2}\partial^{C'}_{s+3} h_{s+2} \\
&= 0
\end{aligned}$$

Thus $f_{s+1} - \partial^{C'}_{s+2} h_{s+1} \in \operatorname{Ker} \partial^{\mathrm{Hom}_R(C,C'_{s+1})}_{s+1} = \operatorname{Im} \partial^{\mathrm{Hom}_R(C,C'_{s+1})}_{s}$. Therefore there exists a map $h_s : C_s \to C'_{s+1}$ such that $\partial^{\mathrm{Hom}_R(C,C'_{s+1})}_{s}(h_s) = f_{s+1} - \partial^{C'}_{s+2} h_{s+1}$, in other words $f_{s+1} = h_s \partial^C_{s+1} + \partial^{C'}_{s+2} h_{s+1}$. □

1.2. Suppose that s is an integer, $C \in \mathbf{K}_{\mathrm{tac}}(R)$ and C' is a complex of finitely generated projective R-modules. If there exist maps $f_n : C_n \to C'_n$ for $n \geq s$ satisfying $f_{n-1}\partial^C_n = \partial^{C'}_n f_n$ for all $n > s$, then the f_n can be extended to a morphism of R-complexes $f : C \to C'$. Any two such extensions are homotopic.

Proof. To prove the first statement, we need to complete the diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & C_{s+2} & \xrightarrow{\partial^C_{s+2}} & C_{s+1} & \xrightarrow{\partial^C_{s+1}} & C_s & \xrightarrow{\partial^C_s} & C_{s-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{s+2}} & & \downarrow{\scriptstyle f_{s+1}} & & \downarrow{\scriptstyle f_s} & & \vdots{\scriptstyle f_{s-1}} & & \\
\cdots & \longrightarrow & C'_{s+2} & \xrightarrow[\partial^{C'}_{s+2}]{} & C'_{s+1} & \xrightarrow[\partial^{C'}_{s+1}]{} & C'_s & \xrightarrow[\partial^{C'}_s]{} & C'_{s-1} & \longrightarrow & \cdots
\end{array}$$

and for this it suffices by induction to show that there exists a map $f_{s-1} : C_{s-1} \to C'_{s-1}$ such that $f_{s-1}\partial^C_s = \partial^{C'}_s f_s$.

Since C'_{s-1} is a finitely generated projective R-module and $C \in \mathbf{K}_{\text{tac}}(R)$, the complex $\operatorname{Hom}_R(C, C'_{s-1})$ is exact. We have

$$\begin{aligned}
\partial^{\operatorname{Hom}_R(C,C'_{s-1})}_s(\partial^{C'}_s f_s) &= \partial^{C'}_s f_s \partial^C_{s+1} \\
&= \partial^{C'}_s \partial^{C'}_{s+1} f_{s+1} \\
&= 0
\end{aligned}$$

Therefore $\partial^{C'}_s f_s \in \operatorname{Ker} \partial^{\operatorname{Hom}_R(C,C'_{s-1})}_s = \operatorname{Im} \partial^{\operatorname{Hom}_R(C,C'_{s-1})}_{s-1}$, and so there exists a map $f_{s-1} : C_{s-1} \to C'_{s-1}$ such that $f_{s-1}\partial^C_s = \partial^{\operatorname{Hom}_R(C,C'_{s-1})}_{s-1}(f_{s-1}) = \partial^{C'}_s f_s$.

The second statement follows immediately from 1.1: if $g : C \to C'$ is another morphism of R-complexes such that $g_n = f_n$ for all $n \geq s$, then $f - g$ is eventually zero, in particular, eventually nullhomotopic. Therefore 1.1 says that $f - g \sim 0$, in other words, $f \sim g$. □

We now recall an important definition from [AvMa].

Definition. A *complete resolution* of a finitely generated R-module M is a diagram

$$U \xrightarrow{\rho} P \xrightarrow{\pi} M$$

such that $U \in \mathbf{K}_{\text{tac}}(R)$, P is a projective resolution of M, ρ is a morphism of R-complexes, and ρ_n is bijective for all $n \gg 0$. We will often abuse terminology and call U a complete resolution of M.

The following is [AvMa, 5.3], and is key to defining our functor T. We include a different proof, using 1.1 and 1.2.

1.3. If $U \xrightarrow{\rho} P \xrightarrow{\pi} M$ and $U' \xrightarrow{\rho'} P' \xrightarrow{\pi'} M'$ are complete resolutions of finitely generated R-modules M and M', and $\mu : M \to M'$ is an R-module homomorphism, then there exists a unique up to homotopy morphism of R-complexes $\overline{\mu}$ making the right-hand square of the diagram

$$\begin{array}{ccccc}
U & \xrightarrow{\rho} & P & \xrightarrow{\pi} & M \\
\vdots{\scriptstyle \widehat{\mu}} & & \vdots{\scriptstyle \overline{\mu}} & & \downarrow{\scriptstyle \mu} \\
U' & \xrightarrow{\rho'} & P' & \xrightarrow{\pi'} & M'
\end{array}$$

commute, and for each choice of $\overline{\mu}$ there exists a unique up to homotopy morphism $\widehat{\mu}$ making the left-hand square commute up to homotopy. If two such $\overline{\mu}$ are homotopic, then so are the respective $\widehat{\mu}$. If $\mu = \operatorname{Id}_M$, then $\overline{\mu}$ and $\widehat{\mu}$ are homotopy equivalences.

Proof. The statement regarding $\overline{\mu}$ is classical, so we first concern ourselves with the first statement regarding $\widehat{\mu}$.

Suppose that the morphism $\overline{\mu}$ has been chosen. Since ρ_n and ρ'_n are bijective for all $n \gg 0$, there obviously exist maps $\widehat{\mu}_n : U_n \to U'_n$ such that $\widehat{\mu}_{n-1}\partial^U_n = \partial^{U'}_n \widehat{\mu}_n$ for all $n \gg 0$. Thus 1.2 says we obtain a unique up to homotopy morphism of R-complexes $\widehat{\mu} : U \to U'$. Now since $\overline{\mu}_n \rho_n = \rho'_n \widehat{\mu}_n$ for all $n \gg 0$, 1.1 says that $\overline{\mu}\rho \sim \rho'\widehat{\mu}$, as desired.

Suppose that $\overline{\mu}_1 : P \to P'$ and $\overline{\mu}_2 : P \to P'$ are two morphisms such that $\overline{\mu}_1 \sim \overline{\mu}_2$. Since ρ_n and $\rho'_n$are bijective for all $n \gg 0$, we see that $\widehat{\mu}_1$ and $\widehat{\mu}_2$ are eventually homotopic. Therefore 1.1 shows that $\widehat{\mu}_1 \sim \widehat{\mu}_2$.

If $\mu = \mathrm{Id}_M$, then one can also define morphisms $\overline{\mu}' : P' \to P$ and $\widehat{\mu}' : U' \to U$. Uniqueness up to homotopy then shows that $\overline{\mu}'\overline{\mu} \sim \mathrm{Id}_F$, $\overline{\mu\mu}' \sim \mathrm{Id}_{F'}$, $\widehat{\mu}'\widehat{\mu} \sim \mathrm{Id}_U$, and $\widehat{\mu}\widehat{\mu}' \sim \mathrm{Id}_{U'}$. □

The final statement of 1.3 says that complete resolutions are uniquely defined up to homotopy equivalence.

One needs to know that complete resolutions exist. The following result follows from [AvMa, Constructions 3.6 and 3.7]. Recall that a Noetherian (on both sides) ring is called *Gorenstein* if R has finite injective dimension as a module over itself on both sides.

1.4. Suppose that R is Gorenstein, and let M be a finitely generated R-module. Then for each projective resolution P of M there exists a complete resolution $U \xrightarrow{\rho} P \xrightarrow{\pi} M$ such that $\rho_n = \mathrm{Id}_{P_n}$ for all $n \gg 0$; one may in addition choose U such that ρ_n is surjective for all n.

1.5. Suppose that Q and R are associative rings with unity, and $\varphi : Q \to R$ is a surjective ring homomorphism. Let D be a complex of finitely generated projective left Q-modules. Then

$$\mathrm{Hom}_Q(D, Q) \otimes_Q R \text{ and } \mathrm{Hom}_R(R \otimes_Q D, R)$$

are isomorphic as complexes of right R-modules. Similarly, if D is a complex of finitely generated projective right Q-modules, then

$$R \otimes_Q \mathrm{Hom}_Q(D, Q) \text{ and } \mathrm{Hom}_R(D \otimes_Q R, R)$$

are isomorphic as complexes of left R-modules.

Proof. We only proof the first isomorphism, as the second is similar. By Hom-tensor adjunction for complexes, and after making the canonical identification of R-R bimodules $\mathrm{Hom}_R(R, R)$ with R, we immediately get that $\mathrm{Hom}_R(R \otimes_Q C, R)$ and $\mathrm{Hom}_Q(C, R)$ are isomorphic. Therefore it suffices to prove that $\mathrm{Hom}_Q(C, Q) \otimes_Q R$ and $\mathrm{Hom}_Q(C, R)$ are isomorphic.

Define maps $\alpha_n : \mathrm{Hom}_Q(C_{-n}, Q) \otimes_Q R \to \mathrm{Hom}_Q(C_{-n}, R)$ by $\alpha_n(f \otimes 1_R)(x) = \varphi(f(x))$, and $\beta_n : \mathrm{Hom}_Q(C_{-n}, R) \to \mathrm{Hom}_Q(C_{-n}, Q) \otimes_Q R$ by $\beta_n(g) = g' \otimes_Q 1_R$ where $\varphi g' = g$. Since φ is surjective and C_{-n} is projective, we know such a map g' exists. Furthermore, it is easy to see that both α_n and β_n are homomorphisms of right R-modules. We have $\alpha_n(\beta_n(g))(x) = \alpha_n(g' \otimes 1_R)(x) = \varphi(g'(x)) = g(x)$, and so $\alpha_n\beta_n = \mathrm{Id}_{\mathrm{Hom}_Q(C_{-n},R)}$. Also, $\beta_n(\alpha_n(f \otimes 1_R)) = g' \otimes 1_R$ where $\varphi(g'(x)) = \varphi(f(x))$. This implies that $g' \otimes 1_R = f \otimes 1_R$, and so $\beta_n\alpha_n = \mathrm{Id}_{\mathrm{Hom}_Q(C_{-n},Q)\otimes_Q R}$. Thus α_n and β_n are inverses of one another. Finally one just needs to check that α_n and β_n commute with the differentials.

We have $\alpha_{n-1}(\operatorname{Hom}_Q(\partial^C_{-n+1}, Q)\otimes 1_R)(f\otimes 1_R)(x) = \alpha_{n-1}(f\partial^C_{-n+1}\otimes 1_R)(x) = \varphi(f\partial^C_{-n+1}(x))$. On the other hand, $\operatorname{Hom}_Q(\partial^C_{-n+1}, R)\alpha_n(f\otimes 1_R)(x) = \alpha_n(f\otimes 1_R)\partial^C_{-n+1}(x) = \varphi(f\partial^C_{-n+1}(x))$. Therefore we see that $\alpha_{n-1}(\operatorname{Hom}_Q(\partial^C_{-n+1}, Q)\otimes 1_R) = \operatorname{Hom}_Q(\partial^C_{-n+1}, R)\alpha_n$, and so the α_n form a morphism of complexes. Next we have $(\operatorname{Hom}_Q(\partial^C_{-n+1}, Q)\otimes 1_R)\beta_n(g) = (\operatorname{Hom}_Q(\partial^C_{-n+1}, Q)\otimes 1_R)(g'\otimes 1_R) = g'\partial^C_{-n+1}\otimes 1_R$. On the other hand, $\beta_{n-1}\operatorname{Hom}_Q(\partial^C_{-n+1}, R)(g) = \beta_{n-1}(g\partial^C_{-n+1}) = (g\partial^C_{-n+1})'\otimes 1_R$. Now g' is defined such that $\varphi g' = g$, and $(g\partial^C_{-n+1})'$ is defined such that $\varphi(g\partial^C_{-n+1})' = g\partial^C_{-n+1}$. Thus $\varphi g'\partial^C_{-n+1} = \varphi(g\partial^C_{-n+1})'$, and it follows that $g'\partial^C_{-n+1}\otimes 1_R = (g\partial^C_{-n+1})'\otimes 1_R$. This shows that the β_n also form a morphism of complexes. □

2. The 'forgetful' triangle functor

From now on we assume that Q is a Gorenstein ring. We further assume that $\varphi : Q \to R$ a surjective ring homomorphism such that R has a finite projective resolution over Q.

The main objective of this section is to define the 'forgetful' functor

$$T = T_\varphi : \mathbf{K}_{\mathrm{tac}}(R) \to \mathbf{K}_{\mathrm{tac}}(Q)$$

and prove that it is a triangle functor. The definition of T is as follows.

Definition. Let $C \in \mathbf{K}_{\mathrm{tac}}(R)$. Then $TC \in \mathbf{K}_{\mathrm{tac}}(Q)$ is a complete resolution of $\operatorname{Im}\partial^C_0$ over Q (which exists by 1.4 and is uniquely defined by 1.3.) Given a morphism $[f] : C \to C'$ in $\mathbf{K}_{\mathrm{tac}}(R)$, we have the Q-module homomorphism $\mu : \operatorname{Im}\partial^C_0 \to \operatorname{Im}\partial^{C'}_0$ induced by the morphism $f : C \to C'$ of R-complexes. Then $T[f] : TC \to TC'$ is the homotopy equivalence class $[\widehat{\mu}]$ of the comparison map $\widehat{\mu} : TC \to TC'$ between complete resolutions (which is uniquely determined by μ, by 1.3.)

Proposition 2.1. *$T : \mathbf{K}_{\mathrm{tac}}(R) \to \mathbf{K}_{\mathrm{tac}}(Q)$ is an additive functor.*

Proof. That T is actually a functor boils down to checking that T is well-defined on morphisms in $\mathbf{K}_{\mathrm{tac}}(R)$, so we prove this first. Let $[f] : C \to C'$ be the zero morphism in $\mathbf{K}_{\mathrm{tac}}(R)$, and $\mu : \operatorname{Im}\partial^C_0 \to \operatorname{Im}\partial^{C'}_0$ be the Q-module homomorphism induced by f. Then there exists a homotopy h satisfying $f_n = h_{n-1}\partial^C_n + \partial^{C'}_{n+1}h_n$ for all $n \in \mathbb{Z}$. Let ϵ denote R-module homomorphism $C'_0 \to \operatorname{Im}\partial^{C'}_0$ induced by $\partial^{C'}_0$ and $\overline{h}_{-1}$ denote the restriction $h_{-1}|_{\operatorname{Im}\partial^C_0}$. We claim that $\mu = \epsilon\overline{h}_{-1}$. Indeed, for $\partial^C_0(x) \in \operatorname{Im}\partial^C_0$, $x \in C_0$, we have $\mu(\partial^C_0(x)) = \epsilon f_0(x) = \epsilon(h_{-1}\partial^C_0 + \partial^{C'}_1 h_0)(x) = \epsilon\overline{h}_{-1}(\partial^C_0(x))$, since $\epsilon\partial^{C'}_1 = 0$.

Now let P and P' be projective resolutions of $\operatorname{Im}\partial^C_0$ and $\operatorname{Im}\partial^{C'}_0$ over Q, respectively, and K a finite projective resolution of C'_0 over Q (which exists since we are assuming R has as finite projective resolution as a Q-module.) Then any chain map

$\overline{\mu} : P \to P'$ lifting μ is homotopic to a lifting of the composition

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_1 & \xrightarrow{\partial_1^P} & P_0 & \longrightarrow & \operatorname{Im} \partial_0^C & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \downarrow{\scriptstyle \overline{h}_{-1}} & & \\
\cdots & \longrightarrow & K_1 & \xrightarrow{\partial_1^K} & K_0 & \longrightarrow & C'_0 & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \downarrow{\scriptstyle \epsilon} & & \\
\cdots & \longrightarrow & P'_1 & \xrightarrow{\partial_1^{P'}} & P'_0 & \longrightarrow & \operatorname{Im} \partial_0^{C'} & \longrightarrow & 0
\end{array}$$

which is eventually zero since K is a finite complex. This shows that $\widehat{\mu} \sim 0$, in other words, $T[f] = 0$.

It follows now that T is well-defined on objects as well, meaning that if C and C' are homotopically equivalent in $\mathbf{K}_{\text{tac}}(R)$, then TC and TC' are homotopically equivalent in $\mathbf{K}_{\text{tac}}(Q)$.

It is easy to see that T preserves compositions of morphisms, and takes the identity morphism to the identity morphism. Finally, one can easily check that for morphisms $[f], [g] : C \to C'$, we have $T([f] + [g]) = T[f] + T[g]$. $\square$

We want to show that T is moreover a triangle functor. For this we need some preparation.

For an arbitrary associative ring A, consider a short exact sequence of finitely generated left A-modules $0 \to X \to Y \xrightarrow{\pi} Z \to 0$. Let P and P' be projective resolutions of Y and Z, respectively. Let $\operatorname{cone}(\phi)$ denote the mapping cone of the morphism $\phi : P \to P'$ lifting the surjection π:

$$\operatorname{cone}(\phi) : \cdots \to P_1 \oplus P'_2 \xrightarrow{\begin{pmatrix} -\partial_1^P & 0 \\ \phi_1 & \partial_2^{P'} \end{pmatrix}} P_0 \oplus P'_1 \xrightarrow{\begin{pmatrix} \phi_0 & \partial_1^{P'} \end{pmatrix}} P'_0.$$

Lemma 2.2. *In the notation of the discussion above, given projective resolutions P of Y and P' of Z by finitely generated projective A-modules, the truncated mapping cone*

$$\operatorname{con}(\phi) : \cdots \to P_2 \oplus P'_3 \xrightarrow{\begin{pmatrix} -\partial_2^P & 0 \\ \phi_2 & \partial_3^{P'} \end{pmatrix}} P_1 \oplus P'_2 \xrightarrow{\begin{pmatrix} -\partial_1^P & 0 \\ \phi_1 & \partial_2^{P'} \end{pmatrix}} \ker \begin{pmatrix} \phi_0 & \partial_1^{P'} \end{pmatrix}$$

is (after shift) a projective resolution of X by finitely generated projective A-modules.

Proof. From the short exact sequence of complexes $0 \to P' \to \operatorname{cone}(\phi) \to \Sigma P \to 0$, we get the long exact sequence of homology

$$\cdots \to \mathrm{H}_1(P') \to \mathrm{H}_1(\operatorname{cone}(\phi)) \to \mathrm{H}_0(P) \xrightarrow{\pi} \mathrm{H}_0(P') \to \mathrm{H}_0(\operatorname{cone}(\phi)) \to 0.$$

Since π is surjective we have $\mathrm{H}_0(\operatorname{cone}(\phi)) = 0$, and so the map $\begin{pmatrix} \phi_0 & \partial_1^{P'} \end{pmatrix}$ is surjective. It follows that $\ker \begin{pmatrix} \phi_0 & \partial_1^{P'} \end{pmatrix}$ is projective, and thus $\operatorname{con}(\phi)$ is a complex of finitely generated projective modules. This long exact sequence of homology therefore reduces to the short exact sequence

$$0 \to \mathrm{H}_1(\operatorname{cone}(\phi)) \to Y \xrightarrow{\pi} Z \to 0,$$

and so $X \cong \mathrm{H}_1(\operatorname{cone}(\phi))$. Since $\mathrm{H}_i(\operatorname{cone}(\phi)) = \mathrm{H}_i(\operatorname{con}(\phi))$ for all $i \geq 1$, the result follows, that is $\Sigma^{-1} \operatorname{con}(\phi)$ is a projective resolution of X. $\square$

Let $C \in \mathbf{K}_{\mathrm{tac}}(R)$. Applying the lemma to the short exact sequence of Q-modules $0 \to \operatorname{Im} \partial_0^C \to C_{-1} \to \operatorname{Im} \partial_{-1}^C \to 0$ we obtain the following proposition. Note that since R has finite projective dimension over Q, so does any projective R-module.

Proposition 2.3. *Let $C \in \mathbf{K}_{\mathrm{tac}}(R)$, K be a finite projective resolution of C_{-1} over Q, and P a projective resolution of $\operatorname{Im} \partial_{-1}^C$ over Q. Consider the morphism of complexes $\phi : K \to P$ lifting the surjection $C_{-1} \to \operatorname{Im} \partial_{-1}^C$. Then $\Sigma^{-1} \operatorname{con}(\phi)$ is a projective resolution of $\operatorname{Im} \partial_0^C$ over Q. If $c = \operatorname{pd}_Q R < \infty$ then $(\Sigma^{-1} \operatorname{con} \phi)_{\geq c+1} = \Sigma^{-1}(P_{\geq c+2})$.*

Theorem 2.4. *$T : \mathbf{K}_{\mathrm{tac}}(R) \to \mathbf{K}_{\mathrm{tac}}(Q)$ is a triangle functor.*

Proof. We first show that T commutes with shifts, that is, there is a natural isomorphism between $T\Sigma$ and ΣT. Let $C \in \mathbf{K}_{\mathrm{tac}}(R)$. From Proposition 2.3 we see that there exists a projective resolution of $\operatorname{Im} \partial_0^C$ over Q which agrees with one of $\operatorname{Im} \partial_{-1}^C$, up to shift, beginning at degree $\operatorname{pd}_Q R + 1$. Specifically, there exists a projective resolution P' of $\operatorname{Im} \partial_{-1}^C$ over Q, and a projective resolution P of $\operatorname{Im} \partial_0^C$ over Q such that $P_{\geq c+1} = \Sigma^{-1}(P'_{\geq c+2})$, for $c = \operatorname{pd}_Q R$. By 1.4 there exist complete resolutions U and U' of $\operatorname{Im} \partial_0^C$ and $\operatorname{Im} \partial_{-1}^C$, respectively, and an integer n such that $U_{\geq n} = \Sigma^{-1}(U'_{\geq n+1})$. From 1.2 it follows that $U \simeq \Sigma^{-1} U'$. Since $\operatorname{Im} \partial_0^{\Sigma C} = \operatorname{Im} \partial_{-1}^C$, U' can be identified with $T\Sigma C$. Thus we have that $TC \simeq \Sigma^{-1} T\Sigma C$, in other words $\Sigma TC \simeq T\Sigma C$. It is clear that this isomorphism respects morphisms in $\mathbf{K}_{\mathrm{tac}}(R)$, that is, if $[f] : C \to C'$ is a morphism in $\mathbf{K}_{\mathrm{tac}}(R)$, then the following diagram commutes.

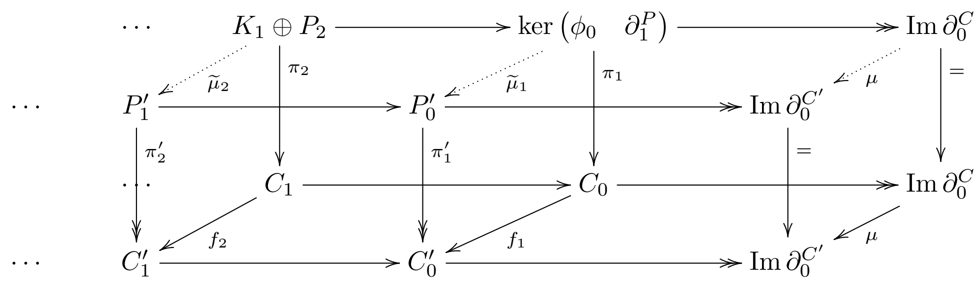
$$\begin{array}{ccc} T\Sigma C & \xrightarrow{T\Sigma[f]} & T\Sigma C' \\ \downarrow \simeq & & \downarrow \simeq \\ \Sigma TC & \xrightarrow{\Sigma T[f]} & \Sigma TC' \end{array}$$

Next we show that T takes distinguished triangles to distinguished triangles. Any distinguished triangle in $\mathbf{K}_{\mathrm{tac}}(R)$ is isomorphic as a triangle to one of the form $C \xrightarrow{[f]} C' \to \operatorname{cone}([f]) \to \Sigma C$. It suffices to complete the diagram.

$$\begin{array}{ccccccc} TC & \xrightarrow{T[f]} & TC' & \longrightarrow & T\operatorname{cone}([f]) & \longrightarrow & \Sigma(TC) \\ \| & & \| & & \vdots & & \| \\ TC & \xrightarrow{T[f]} & TC' & \longrightarrow & \operatorname{cone}(T[f]) & \longrightarrow & \Sigma(TC) \end{array}$$

with an isomorphism $T\operatorname{cone}([f]) \to \operatorname{cone}(T[f])$ so that the second two squares commute.

Choose projective resolutions P of $\operatorname{Im} \partial_{-1}^C$ (assumed concentrated in degree 0) and P' of $\operatorname{Im} \partial_0^{C'}$ over Q. We may construct a chain map $\pi' : P' \to C'_{\geq 0}$ which is surjective in each degree (see, for example [Ve, 1.3.4]). Let K be a projective resolution of C_{-1} over Q, and $\phi : K \to P$ a chain map lifting the surjection $C_{-1} \to \operatorname{Im} \partial_{-1}^C$. Without loss of generality we may assume that ϕ_0 is surjective, and then by adding projective summands to P_0 and P_1 if necessary, that in fact ϕ_0 is an isomorphism. We have by Proposition 2.3 that $\Sigma^{-1} \operatorname{con}(\phi)$ is a projective resolution $\operatorname{Im} \partial_0^C$ over Q. By the standard argument for lifting maps to chain maps (see, for example, [Ve, Proposition 2.2.4], we can complete the diagram



with maps $\widetilde{\mu}_i$ and π_i, $i \geq 1$, such that all squares commute.

We have the commutative square

$$\begin{array}{ccccc} K_0 & \xrightarrow{\epsilon'} & C_{-1} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \phi_0} & & \downarrow & & \\ P_0 & \xrightarrow{\epsilon} & \operatorname{Im} \partial^C_{-1} & \longrightarrow & 0 \end{array}$$

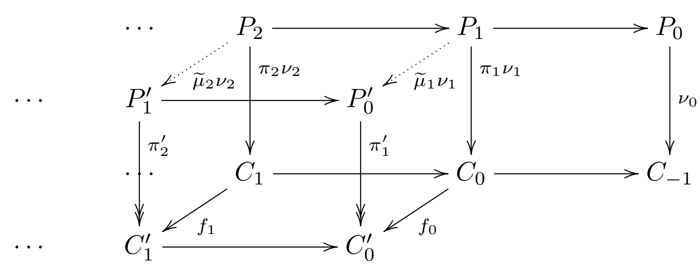
and since ϕ_0 is an isomorphism we have $\epsilon = \partial^C_{-1}\epsilon'\phi_0^{-1}$. Now we establish a chain map

$$\begin{array}{ccccccccc} P_2 & \xrightarrow{\partial^P_2} & P_1 & \xrightarrow{\partial^P_1} & P_0 & \xrightarrow{\epsilon} & \operatorname{Im} \partial^C_{-1} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \nu_2} & & \downarrow{\scriptstyle \nu_1} & & \downarrow{\scriptstyle \nu_0} & & \downarrow{\scriptstyle =} & & \\ K_1 \oplus P_2 & \longrightarrow & \ker \begin{pmatrix} \phi_0 & \partial^P_1 \end{pmatrix} & \longrightarrow & C_{-1} & \longrightarrow & \operatorname{Im} \partial^C_{-1} & \longrightarrow & 0 \end{array}$$

where $\nu_n : P_n \to K_{n-1} \oplus P_n$ is the natural injection for $n \geq 2$, ν_1 is given by

$$x \mapsto \begin{pmatrix} -\phi_0^{-1}\partial^P_1(x) \\ x \end{pmatrix}$$

and $\nu_0 = \epsilon'\phi_0^{-1}$. Combining diagrams we now have a commutative diagram



Setting $\xi_i = (-1)^{i+1}\Sigma^{-1}\sigma\widetilde{\mu}_{i+1}\nu_{i+1}$ and $\zeta_i = (-1)^{i+1}\Sigma^{-1}\sigma\pi_{i+1}\nu_{i+1}$ we achieve a commutative diagram of morphism of complexes

$$\begin{array}{ccc} \Sigma^{-1}P & \xrightarrow{\xi} & P' \\ \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \pi'} \\ C_{\geq -1} & \xrightarrow{f} & C'_{\geq 0} \end{array}$$

This diagram gives rise to a commutative diagram of short exact sequences of morphisms of Q-complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P' & \longrightarrow & \operatorname{cone}(\xi) & \longrightarrow & P & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \left(\begin{smallmatrix} \Sigma\zeta & 0 \\ 0 & \pi' \end{smallmatrix}\right)} & & \big\downarrow{\scriptstyle \Sigma\zeta} & & \\
0 & \longrightarrow & C'_{\geq 0} & \longrightarrow & \operatorname{cone}(f)_{\geq 0} & \longrightarrow & \Sigma(C_{\geq -1}) & \longrightarrow & 0
\end{array}$$

The resulting commutative diagram of long exact sequences of homology shows that the morphism of Q-complexes $\operatorname{cone}(\xi) \to \operatorname{cone}(f)_{\geq 0}$ is a quasiisomorphism. Thus $\operatorname{cone}(\xi)$ is a projective resolution of $\operatorname{Im} \partial_0^{\operatorname{cone}(f)}$. It follows from 1.4 that $T \operatorname{cone}(f)$ may be chosen such that $\operatorname{cone}(\xi)_{\geq n} = T \operatorname{cone}(f)_{\geq n}$ for large enough n.

On the other hand, 1.4 shows that TC and TC' can be chosen to eventually agree with $\Sigma^{-1} \operatorname{con}(\phi)$ and P', and the comparison map $\overline{\mu}$ lifting $\mu : \operatorname{Im} \partial_0^C \to \operatorname{Im} \partial_0^{C'}$ can be taken to be $\overline{\mu}_n = (-1)^{n+1} \Sigma^{-1} \sigma \widetilde{\mu}_{n+1}$. Since ν is the identity map in all large degree, we see that actually $\xi_n = \overline{\mu}_n$ for all large n. Thus we have that $T \operatorname{cone}(f)_{\geq n} = \operatorname{cone}(Tf)_{\geq n}$ for some n, and this completes the diagram as desired. □

3. Adjunction

We continue to assume that $\varphi : Q \to R$ is a surjective ring homomorphism with Q Gorenstein. Moreover, we now assume that R as a Q-bimodule admits a finite resolution by modules which are projective as both left and right Q-modules. The goal of this section is to compare $\mathbf{K}_{\mathrm{tac}}(R)$ with $\mathbf{K}_{\mathrm{tac}}(Q)$ by means of an adjoint pair of triangle functors.

The descension functor $S = S_\varphi : \mathbf{K}_{\mathrm{tac}}(Q) \to \mathbf{K}_{\mathrm{tac}}(R)$ is easy; it is defined by

$$SC = R \otimes_Q C \text{ and } S[f] = [R \otimes_Q f]$$

for C an object and $[f]$ a morphism in $\mathbf{K}_{\mathrm{tac}}(Q)$. This is a triangle functor due in part to 1.5. Indeed, if C is acyclic complex of projective Q-modules, then $R \otimes_Q C$ is an acyclic complex of projective R-modules (since R has a finite resolution by projective right Q-modules), and $\operatorname{Hom}_Q(C, Q)$ being acyclic implies $\operatorname{Hom}_Q(C, Q) \otimes_Q R$ is acyclic. Hence $\operatorname{Hom}_R(R \otimes_Q C, R)$ is acyclic by 1.5. It is easy to see that S takes homotopic morphisms of complexes to homotopic morphisms of complexes, commutes with shifts and takes distinguished triangles to distinguished triangles.

The ascension functor $T = T_\varphi : \mathbf{K}_{\mathrm{tac}}(R) \to \mathbf{K}_{\mathrm{tac}}(Q)$ is the functor defined in Section 2. Our main result for this section is the following.

Theorem 3.1. *The triangle functors S and T form an adjoint pair, that is, they satisfy the following property: for all $C \in \mathbf{K}_{\mathrm{tac}}(R)$ and $D \in \mathbf{K}_{\mathrm{tac}}(Q)$ there exist a bijection*

$$\operatorname{Hom}_{\mathbf{K}_{\mathrm{tac}}(Q)}(D, TC) \to \operatorname{Hom}_{\mathbf{K}_{\mathrm{tac}}(R)}(SD, C)$$

which is natural in each variable.

In the discussion below, we fix a finite Q-bimodule resolution K of R, of length c, by modules that are projective Q-modules on the left and on the right; we assume, without loss of generality, that $K_0 = Q$. Before engaging the proof, we observe that for $D \in \mathbf{K}_{\mathrm{tac}}(Q)$ one has

$$TSD \simeq K \otimes_Q D$$

Indeed, since K is a finite bimodule resolution of R by one-sided projective Q-modules, one has that $K \otimes_Q D_{\geq 0}$ is a complex of projective Q-modules. It is exact (except in degree 0) by the finiteness of K. Thus we see that $K \otimes_Q D_{\geq 0}$ is a projective resolution of $\operatorname{Im} \partial_0^{SD} \cong R \otimes_Q \operatorname{Im} \partial_0^D$ over Q. The assertion is now clear.

Proof. In order to prove the theorem we define natural transformations

$$\eta : \mathrm{Id}_{\mathbf{K}_{\mathrm{tac}}(Q)} \to TS$$

and

$$\epsilon : ST \to \mathrm{Id}_{\mathbf{K}_{\mathrm{tac}}(R)}$$

— the unit and counit, respectively, of the adjunction — as follows. For $D \in \mathbf{K}_{\mathrm{tac}}(Q)$ define $\eta_D : D \to TSD$ to be the morphism of complexes embedding D_n into the first summand of $TSD_n = \bigoplus_{i=0}^{c} K_i \otimes_Q D_{n-i}$ via $x \mapsto 1 \otimes x$ for all n. And for $C \in \mathbf{K}_{\mathrm{tac}}(R)$ define $\epsilon_C : STC \to C$ to be the morphism of complexes induced by the comparison map $F \to C_{\geq 0}$, where F is a projective resolution of $\operatorname{Im} \partial_0^C$ over Q. It follows from 1.2 and 1.1 that η and ϵ are natural in their arguments.

We just need to show that

$$T\epsilon_C \circ \eta_{TC} \sim \mathrm{Id}_{TC} \quad \text{and} \quad \epsilon_{SD} \circ S\eta_D \sim \mathrm{Id}_{SD}$$

First we discuss the map $T\epsilon_C$. By definition we have the morphism of complexes

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & TC_1 & \xrightarrow{\partial_1^{TC}} & TC_0 & \xrightarrow{p} & \operatorname{Im} \partial_0^{TC} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_0} & & \downarrow{\scriptstyle \overline{\rho}} & & \\
\cdots & \longrightarrow & F_1 & \xrightarrow{\partial_1^{F}} & F_0 & \xrightarrow{p'} & \operatorname{Im} \partial_0^{C} & \longrightarrow & 0
\end{array}$$

where F is a projective resolution of $\operatorname{Im} \partial_0^C$ over Q; by 1.4 we can assume that $\rho_n = \mathrm{Id}_{TC_n}$ for all $n \gg 0$. Consider the induced map $\widehat{\rho} : R \otimes_Q \operatorname{Im} \partial_0^{TC} \to \operatorname{Im} \partial_0^C$ where $\widehat{\rho}(r \otimes a) = r\overline{\rho}(a)$. By lifting $\widehat{\rho}$ one summand of $(K \otimes TC)_n$ at a time, for $n = 0, 1, \dots,$ we may achieve a morphism of complexes

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & (K \otimes_Q TC)_1 & \xrightarrow{\partial_1^{K \otimes TC_{\geq 0}}} & K_0 \otimes_Q TC_0 & \xrightarrow{\tau \otimes p} & R \otimes_Q \operatorname{Im} \partial_0^{TC} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} & & \downarrow{\scriptstyle \widehat{\rho}} & & \\
\cdots & \longrightarrow & F_1 & \xrightarrow{\partial_1^{F}} & F_0 & \xrightarrow{p'} & \operatorname{Im} \partial_0^{C} & \longrightarrow & 0
\end{array}$$

written succinctly as $u : K \otimes_Q TC_{\geq 0} \to F$, the former complex being a Q-free resolution of $R \otimes_Q \operatorname{Im} \partial_0^{TC}$, such that $u_n(x \otimes a) = x\rho_n(a)$ for $x \otimes a \in K_0 \otimes_Q TC_n$. It follows that we may achieve the morphism of complexes $T\epsilon_C : TSTC \to TC$ satisfying $(T\epsilon_C)_n(x \otimes a) = xa$ for $x \otimes a \in K_0 \otimes_Q TC_n$, and all $n \gg 0$. We also have the natural embedding $\eta_{TC} : TC \to TSTC$ with $(\eta_{TC})_n(a) = 1 \otimes a \in K_0 \otimes_Q TC_n$ for all $a \in TC_n$. Thus we have shown that $(T\epsilon_C)_n \circ (\eta_{TC})_n = \mathrm{Id}_{TC_n}$ for all $n \gg 0$. It follows from 1.2 that $T\epsilon_C \circ \eta_{TC} \sim \mathrm{Id}_{TC}$.

The morphism $S\eta_D : SD \to STSD$ embeds SD_n into the first component of $STSD_n \simeq \bigoplus_{i=0}^{c} R \otimes_Q (K_i \otimes_Q D_{n-i})$ for all n. And the morphism $\epsilon_{SD} : STSD \to SD$ takes the first component of $STSD \simeq \bigoplus_{i=0}^{c} R \otimes_Q (K_i \otimes_Q D_{n-i})$ to SD_n for all $n \in \mathbb{Z}$. Thus we have $\epsilon_{SD} \circ S\eta_D \sim \mathrm{Id}_{SD}$. □

4. Approximations of totally acyclic complexes

Our main application of Theorem 3.1 is a resulting notion of approximation in the homotopy category of totally acyclic complexes. We now recall the notion of approximation we use, due to Auslander and Smalø [AuSm], and independently, Enochs [En]. Let $\mathcal{X}$ be a full subcategory of a category $\mathcal{C}$. Then a *right $\mathcal{X}$-approximation* of $C \in \mathcal{C}$ is a morphism $X \xrightarrow{\epsilon} C$, with $X \in \mathcal{X}$, such that for all objects $Y \in \mathcal{X}$, the sequence

$$\operatorname{Hom}_{\mathcal{C}}(Y,X) \xrightarrow{\operatorname{Hom}(Y,\epsilon)} \operatorname{Hom}_{\mathcal{C}}(Y,C) \to 0$$

is exact. Dually, one has the concept of left $\mathcal{X}$-approximations. Specifically, a morphism $C \xrightarrow{\mu} X$, with $X \in \mathcal{X}$, is called a *left $\mathcal{X}$-approximation* of $C \in \mathcal{C}$ if for all objects $Y \in \mathcal{X}$, the sequence

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{\operatorname{Hom}(\mu,Y)} \operatorname{Hom}_{\mathcal{C}}(C,Y) \to 0$$

is exact. The full subcategory $\mathcal{X}$ is called *functorially finite* in $\mathcal{C}$ if for every object $C \in \mathcal{C}$, there exists a right $\mathcal{X}$-approximation of C and a left $\mathcal{X}$-approximation of C.

We let $S\,\mathbf{K}_{\mathrm{tac}}(Q)$ denote the isomorphism closure in $\mathbf{K}_{\mathrm{tac}}(R)$ of $\{R \otimes_Q D \mid D \in \mathbf{K}_{\mathrm{tac}}(Q)\}$. Our main application of Theorem 3.1 is the following.

Theorem 4.1. *$S\,\mathbf{K}_{\mathrm{tac}}(Q)$ is functorially finite in $\mathbf{K}_{\mathrm{tac}}(R)$.*

Proof. That right $S\,\mathbf{K}_{\mathrm{tac}}(Q)$-approximations exist in $\mathbf{K}_{\mathrm{tac}}(R)$ follows immediately from Theorem 3.1; it is well known that every adjoint pair of functors gives rise to one-sided approximations (see for example [AuRe, Proposition 1.1]). It is instructive to repeat the argument in the current context: the morphism $[\epsilon_C] : STC \to C$ is a right approximation in $\mathbf{K}_{\mathrm{tac}}(R)$. Indeed, if $[f] : SD \to C$ is any morphism in $\mathbf{K}_{\mathrm{tac}}(R)$ with $D \in \mathbf{K}_{\mathrm{tac}}(Q)$, then from the natural transformation $\epsilon : ST \to \mathrm{Id}_{\mathbf{K}_{\mathrm{tac}}(R)}$ we have the equality $[\epsilon_C] \circ ST[f] = [f] \circ [\epsilon_{SD}]$. Composing on the right with $S[\eta_D]$ we obtain $[\epsilon_C] \circ ST[f] \circ S[\eta_D] = [f]$, and thus $ST[f] \circ S[\eta_D] : SD \to STC$ is the morphism we seek.

Now we show that every $C \in \mathbf{K}_{\mathrm{tac}}(R)$ has a left approximation. This can be done by simply dualizing a right approximation. For this we will use several times the isomorphisms from 1.5. For $C \in \mathbf{K}_{\mathrm{tac}}(R)$ we denote by C^* the dual complex $\operatorname{Hom}_R(C,R)$ of right R-modules, which we consider as a complex of left R-modules over the opposite ring R^{op}, so that $C^* \in \mathbf{K}_{\mathrm{tac}}(R^{\mathrm{op}})$. We do the same over Q, so that for $D \in \mathbf{K}_{\mathrm{tac}}(Q)$ we have $D^* = \operatorname{Hom}_Q(D,Q) \in \mathbf{K}_{\mathrm{tac}}(Q^{\mathrm{op}})$. For a morphism of complexes $f : C \to C'$, $C, C' \in \mathbf{K}_{\mathrm{tac}}(R)$, we denote by f^* the dual morphism $\operatorname{Hom}_R(f,R)$. We let S^{op} and T^{op} denote the corresponding adjoint pair of functors between $\mathbf{K}_{\mathrm{tac}}(R^{\mathrm{op}})$ and $\mathbf{K}_{\mathrm{tac}}(Q^{\mathrm{op}})$.

We have the right approximation $[\epsilon_{C^*}] : S^{\mathrm{op}}T^{\mathrm{op}}C^* \to C^*$ of C^*. The claim is that $[\epsilon^*_{C^*}] : C \cong C^{**} \to (S^{\mathrm{op}}T^{\mathrm{op}}C^*)^*$ is a left approximation of C. Note that the target of $\epsilon^*_{C^*}$ is in $S\,\mathbf{K}_{\mathrm{tac}}(Q)$ by 1.5, that is, by the second isomorphism of 1.5 we have $(S^{\mathrm{op}}T^{\mathrm{op}}C^*)^* \cong S((T^{\mathrm{op}}C^*)^*)$. Now let $E \in \mathbf{K}_{\mathrm{tac}}(Q)$ and $f : C \to SE$ be a morphism in $\mathbf{K}_{\mathrm{tac}}(R)$. Then we have the morphism $f^* : (SE)^* \to C^*$, with $(SE)^* \cong S^{\mathrm{op}}E^*$ in $S^{\mathrm{op}}\,\mathbf{K}_{\mathrm{tac}}(Q^{\mathrm{op}})$. Therefore from the right approximation we have that $f^* \sim \epsilon_{C^*}g$ for some morphism $g : (SE)^* \to S^{\mathrm{op}}T^{\mathrm{op}}C^*$. Dualizing back we have that $f \sim g^*\epsilon^*_{C^*}$, which is what we needed to show. □

Motivated by results along the lines of [Ne2, Proposition 1.4], we ask the following:

Question 4.2. When is $S\,\mathbf{K}_{\mathrm{tac}}(Q)$ a thick subcategory of $\mathbf{K}_{\mathrm{tac}}(R)$?

The following example from S. Lindokken shows that this is not always the case.

Example 4.3. ([Li]). Let $Q = k[[x,y,z]]/(x^2+yz)$ and $R = Q/(y,z) \cong k[[x]]/(x^2)$, where k is an algebraically closed field. In this case $\mathbf{K}_{\mathrm{tac}}(Q)$ has precisely one non-zero indecomposable element, namely the complex

$$D : \cdots \to Q^2 \xrightarrow{\left(\begin{smallmatrix} x & -y \\ z & x \end{smallmatrix}\right)} Q^2 \xrightarrow{\left(\begin{smallmatrix} x & y \\ -z & x \end{smallmatrix}\right)} Q^2 \xrightarrow{\left(\begin{smallmatrix} x & -y \\ z & x \end{smallmatrix}\right)} Q^2 \to \cdots$$

Thus the complex SD has the summand $\cdots \to R \xrightarrow{x} R \xrightarrow{x} R \to \cdots$, which is not in $S\,\mathbf{K}_{\mathrm{tac}}(Q)$.

We illustrate Theorem 4.1 with an example.

Example 4.4. Let $R = k[x,y]/(x^2,y^2)$, and C be the totally acyclic R-complex with $\operatorname{Im}\partial_0^C = Rxy \cong k$:

$$C : \cdots \to R^3 \xrightarrow{\left(\begin{smallmatrix} x & 0 & -y \\ 0 & y & x \end{smallmatrix}\right)} R^2 \xrightarrow{(\,x\ y\,)} R \xrightarrow{(\,xy\,)} R \xrightarrow{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)} R^2 \to \cdots$$

Then a free resolution of $\operatorname{Im}\partial_0^C$ over $Q = k[x,y]/(x^2)$ is given by

$$F : \cdots \to Q^2 \xrightarrow{\left(\begin{smallmatrix} x & -y \\ 0 & x \end{smallmatrix}\right)} Q^2 \xrightarrow{\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)} Q^2 \xrightarrow{\left(\begin{smallmatrix} x & -y \\ 0 & x \end{smallmatrix}\right)} Q^2 \xrightarrow{(\,x\ y\,)} Q \to 0$$

The right approximation $\epsilon_C : STC \to C$ takes the form

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & R^2 & \xrightarrow{\left(\begin{smallmatrix} x & -y \\ 0 & x \end{smallmatrix}\right)} & R^2 & \xrightarrow{\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)} & R^2 & \xrightarrow{\left(\begin{smallmatrix} x & -y \\ 0 & x \end{smallmatrix}\right)} & R^2 & \xrightarrow{\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)} & R^2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \mathrm{Id}_{R^2}} & & \downarrow{\scriptstyle (\,1\ 0\,)} & & \downarrow{\scriptstyle (\,y\ 0\,)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} y & 0 \\ 0 & 0 \end{smallmatrix}\right)} & & \\
\cdots & \longrightarrow & R^3 & \xrightarrow[\left(\begin{smallmatrix} x & 0 & -y \\ 0 & y & x \end{smallmatrix}\right)]{} & R^2 & \xrightarrow[(\,x\ y\,)]{} & R & \xrightarrow[(\,xy\,)]{} & R & \xrightarrow[\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)]{} & R^2 & \longrightarrow & \cdots
\end{array}$$

Since C is self-dual in this example, that is $C \cong \Sigma^{-1}(C^*)$, the left approximation $[\epsilon_C^*] : C \to (STC)^*$ takes the form

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & R^3 & \xrightarrow{\left(\begin{smallmatrix} x & 0 & -y \\ 0 & y & x \end{smallmatrix}\right)} & R^2 & \xrightarrow{(\,x\ y\,)} & R & \xrightarrow{(\,xy\,)} & R & \xrightarrow{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)} & R^2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \left(\begin{smallmatrix} y & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} y & 0 \\ 0 & 0 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \mathrm{Id}_{R^2}} & & \\
\cdots & \longrightarrow & R^2 & \xrightarrow[\left(\begin{smallmatrix} x & 0 \\ -y & x \end{smallmatrix}\right)]{} & R^2 & \xrightarrow[\left(\begin{smallmatrix} x & 0 \\ y & x \end{smallmatrix}\right)]{} & R^2 & \xrightarrow[\left(\begin{smallmatrix} x & 0 \\ -y & x \end{smallmatrix}\right)]{} & R^2 & \xrightarrow[\left(\begin{smallmatrix} x & 0 \\ y & x \end{smallmatrix}\right)]{} & R^2 & \longrightarrow & \cdots
\end{array}$$

Approximations may be trivial, in particular, when the projective dimension of $\operatorname{Im}\partial_0^C$ is finite over Q, as is the case in the next example.

Example 4.5. Let $R = k[x,y]/(x^2,y^2)$ and C the totally acyclic R-complex with $\operatorname{Im}\partial_0^C = Ry$:

$$C : \cdots \to R \xrightarrow{(\,y\,)} R \xrightarrow{(\,y\,)} R \xrightarrow{(\,y\,)} R \xrightarrow{(\,y\,)} R \to \cdots$$

Then for $Q = k[x,y]/(x^2)$, $\operatorname{pd}_Q \operatorname{Im}\partial_0^C < \infty$ and the approximation is $[\epsilon_C] : 0 \to C$.

Recall (from [AuSm], for example) that a morphism $X \xrightarrow{\epsilon} C$ is called *right minimal* if for every morphism $X \xrightarrow{f} X$ such that $\epsilon f = \epsilon$, we have that f is an isomorphism. We show that the right approximation $[\epsilon_C] : STC \to C$ may or may not be right minimal.

Proposition 4.6. *Suppose that $D \in \mathbf{K}_{\mathrm{tac}}(Q)$. Then $[\epsilon_{SD}] : STSD \to SD$ is not a minimal approximation.*

Proof. Let K be a bimodule Q-free resolution of R. As described in the proof of 3.1, $\epsilon_{SD} : STSD \to SD$ takes the first component of $STSD_n \simeq \bigoplus_{i=0}^{c} R \otimes_Q (K_i \otimes_Q D_{n-i})$ to SD_n for all $n \in \mathbb{Z}$. Thus taking as $[f] : STSD \to STSD$ the morphism sending $R \otimes_Q (K_0 \otimes_Q D_n)$ to itself and all other summands to zero, for each n, we have $[\epsilon_{SD}] \circ [f] = [\epsilon_{SD}]$ and $[f]$ is not an isomorphism in $\mathbf{K}_{\mathrm{tac}}(R)$. $\square$

Example 4.7. Let $R = k[x, y]/(x^2, y^2)$ and C the totally acyclic complex

$$C : \cdots \to R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \to \cdots$$

Then $M = \operatorname{Im} \partial_0^C = Rx$ and a free resolution of M over $Q = k[x, y]/(x^2)$ is given by

$$\cdots \to Q^2 \xrightarrow{\left(\begin{smallmatrix} x & -y^2 \\ 0 & x \end{smallmatrix}\right)} Q^2 \xrightarrow{\left(\begin{smallmatrix} x & y^2 \\ 0 & x \end{smallmatrix}\right)} Q^2 \xrightarrow{\left(\begin{smallmatrix} x & -y^2 \\ 0 & x \end{smallmatrix}\right)} Q^2 \xrightarrow{\left(\begin{smallmatrix} x & y^2 \end{smallmatrix}\right)} Q \to 0$$

Thus STC takes the form

$$\cdots \to R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)} R^2 \to \cdots$$

and $\epsilon_C : STC \to C$ is given by $(\epsilon_C)_n = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)$ for all n. This is not a minimal right approximation. Indeed, consider the morphism $f : STC \to STC$ given by $f_n = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then one has $\epsilon_C f = \epsilon_C$ and f is not a homotopy equivalence.

We next state a few results for later reference. They have to do with compositions of approximations, in two different senses.

Proposition 4.8. *Consider a sequence of finite local ring homomorphisms*

$$Q \xrightarrow{\varphi} R' \xrightarrow{\psi} R$$

such that Q and R' are Gorenstein, $\operatorname{pd}_Q R' < \infty$, and $\operatorname{pd}_{R'} R < \infty$. Then $S_{\psi\varphi}$ and $T_{\psi\varphi}$ are naturally isomorphic to $S_\psi S_\varphi$ and $T_\varphi T_\psi$, respectively.

Proof. This follows from the fact that the assertion is clear for the S functors, and from uniqueness of adjoints. $\square$

4.9. *Resolutions.* Upon computing the right approximation $[\epsilon_C] : STC \to C$, one may iterate this process. Indeed, complete $[\epsilon_C]$ to a triangle in $\mathbf{K}_{\mathrm{tac}}(R)$ and rotate it to obtain

$$\Sigma^{-1} \operatorname{cone}([\epsilon_C]) \to STC \to C \to .$$

Now compute a right approximation of $\Sigma^{-1} \operatorname{cone}([\epsilon_C])$, and repeat. One then obtains a sequence of maps in $\mathbf{K}_{\mathrm{tac}}(R)$:

$$\mathbf{B} : \cdots \to B_3 \to B_2 \to B_1 \to B_0 \to C$$

where B_0 is a right approximation of C, B_1 is a right approximation of $\Sigma^{-1} \operatorname{cone}(B_0 \to C)$, etc. Note that since composing two consecutive maps in an exact triangle is the zero map, one has that the same holds for the maps in $\mathbf{B}$.

5. Maximal Cohen-Macaulay modules

In this section we maintain the assumption that Q is a Gorenstein ring and that $\varphi : Q \to R$ is a surjective ring homomorphism such that R admits a finite Q-bimodule resolution by one-sided projective Q-modules.

Recall that (in the current context) a finitely generated Q-module M is called *maximal Cohen-Macaulay* (MCM for short) if $\operatorname{Ext}^n_Q(M, Q) = 0$ for all $n > 0$. It is well-known that the stable module category of maximal Cohen-Macaulay Q-modules $\underline{\mathrm{MCM}}(Q)$ is a triangulated category, and is equivalent as such to $\mathbf{K}_{\mathrm{tac}}(Q)$. Indeed, this was first shown by Buchweitz in [Bu], the functor being, in his notation, $\Omega_0 : \mathbf{K}_{\mathrm{tac}}(Q) \to \underline{\mathrm{MCM}}(Q)$, where $\Omega_0(C) = \operatorname{Im} \partial_0^C$ for $C \in \mathbf{K}_{\mathrm{tac}}(Q)$ and $\Omega_0([f]) = \mu$ the induced map $\mu : \operatorname{Im} \partial_0^C \to \operatorname{Im} \partial_0^{C'}$ for a morphism $[f] : C \to C'$.

Also shown in [Bu, Lemma 4.2.2] is that a Q-module M is MCM if and only if its dual $M^* = \operatorname{Hom}_Q(M, Q)$ is MCM as a Q^{op}-module. Furthermore, a MCM Q-module M is *reflexive*, meaning that the natural biduality map $M \to M^{**}$ is an isomorphism. One recognizes that these properties are precisely those defining *modules of G-dimension zero* in [AuBr, Proposition 3.8], also known as *totally reflexive modules*, which is the terminology introduced in [AvMa, Section 2]. It turns out that the subcategory of the category of left R-modules consisting of the totally reflexive R-modules is a Frobenius category (see, for example, [DaEsHo, Proposition 2.2]). Just as in the case of MCM modules over a Gorenstein ring, the projective-injective objects in the subcategory are just the projective objects in the module category. Hence one may form the stable category of totally reflexive modules $\underline{\mathrm{TR}}(R)$, which is a triangulated category. It is well-known that same functor Ω_0 yielding the equivalence of triangulated categories $\mathbf{K}_{\mathrm{tac}}(Q) \to \underline{\mathrm{MCM}}(Q)$ from [Bu] also yields an equivalence of triangulated categories $\mathbf{K}_{\mathrm{tac}}(R) \to \underline{\mathrm{TR}}(R)$. Thus our functors $S = S_\varphi$ and $T = T_\varphi$ induce an adjoint pair of functors $\underline{S}$ and $\underline{T}$ which make the following diagram commute.

$$\begin{array}{ccc}
\mathbf{K}_{\mathrm{tac}}(Q) & \xrightarrow[\cong]{\Omega_0} & \underline{\mathrm{MCM}}(Q) \\
{\scriptstyle S}\big\downarrow\big\uparrow{\scriptstyle T} & & {\scriptstyle \underline{S}}\big\downarrow\big\uparrow{\scriptstyle \underline{T}} \\
\mathbf{K}_{\mathrm{tac}}(R) & \xrightarrow[\cong]{\Omega_0} & \underline{\mathrm{TR}}(R)
\end{array}$$

The functor $\underline{S}$ still simply the base-change functor, $\underline{S}N = R \otimes_Q N$ for $N \in \underline{\mathrm{MCM}}(Q)$. The functor $\underline{T}$ is more interesting. For $M \in \underline{\mathrm{TR}}(R)$, $\underline{T}(M)$ is the essential MCM approximation of M, according to [AuBu].

To put the functors S and T in further perspective, Let $\mathbf{D}^{\mathrm{b}}_{\mathrm{sg}}(Q)$ denote the *singularity category of* R, which is the verdier quotient of the bounded derived category $\mathbf{D}^{\mathrm{b}}(Q)$ by the thick subcategory of perfect complexes $\operatorname{perf}(Q)$. Similarly, we let $\mathbf{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$ denote the singularity category $\mathbf{D}^{\mathrm{b}}(R)/\operatorname{perf}(R)$ of R. The top right horizontal arrow is the equivalence of categories proved in [Bu]; the functor β is defined by hard truncation of a totally acyclic complex to the right of zero. It is proved in [BeJoOp] that the same functor is fully faithfully in general, that is, when the ring is not necessarily Gorenstein, which explains the bottom right arrow.

$$\begin{array}{ccccc}
\underline{\mathrm{MCM}}(Q) & \xleftarrow[\cong]{\Omega_0} & \mathbf{K}_{\mathrm{tac}}(Q) & \xrightarrow[\cong]{\beta} & \mathbf{D}^{\mathrm{b}}_{\mathrm{sg}}(Q) \\
\underline{S}\downarrow\uparrow\underline{T} & & S\downarrow\uparrow T & & \sigma\downarrow\uparrow\tau \\
\underline{\mathrm{TR}}(R) & \xleftarrow[\cong]{\Omega_0} & \mathbf{K}_{\mathrm{tac}}(R) & \xhookrightarrow{\beta} & \mathbf{D}^{\mathrm{b}}_{\mathrm{sg}}(R)
\end{array}$$

The descension functor σ is the derived base-change functor $R\otimes^{\mathbb{L}}_{Q}$, but now the ascension functor τ is the simply the forgetful functor; σ and τ form an adjoint pair.

6. Relative codimension one approximations.

In this section we assume that Q is a commutative local Gorenstein ring, and that $\varphi : Q \to R$ is a map of local rings. We study the approximations $[\epsilon_C] : STC \to C$ in the case of relative codimension one, and give applications to Betti numbers. Later, we further assume that Q is a hypersurface ring, and study the approximations in this case.

We start with a concrete description of the approximation $[\epsilon_C] : STC \to C$ in the case of relative codimension one. Recall the following construction from [Ei]

6.1. Let f be a non-zerodivisor contained in the maximal ideal $\mathfrak{n}$ of Q, $R = Q/(f)$, and $C \in \mathbf{K}_{\mathrm{tac}}(R)$. Choose a sequence of free Q-modules $\widetilde{C}_n$ and maps $\widetilde{\partial}^C_n$ between them such that C and $\widetilde{C} \otimes_Q R$ are isomorphic as R-complexes. One can then write for all n

$$\widetilde{\partial}^C_{n-1}\widetilde{\partial}^C_n = f \cdot \widetilde{t}_n$$

for maps $\widetilde{t}_n : \widetilde{C}_n \to \widetilde{C}_{n-2}$.

Letting $t_n = \widetilde{t}_n \otimes_Q R$, Eisenbud [Ei] shows that $t = \{t_n\}$ defines a morphism of complexes $t : C \to \Sigma^2 C$.

The following is from [St], see also [BeJo].

Theorem 6.2. *Let f be a non-zerodivisor contained in the maximal ideal of Q, $R = Q/(f)$, and $C \in \mathbf{K}_{\mathrm{tac}}(R)$. Set $M = \operatorname{Im} \partial^C_0$. Then the sequence*

$$\cdots \longrightarrow \widetilde{C}_3 \oplus \widetilde{C}_2 \xrightarrow{\begin{pmatrix} \widetilde{\partial}^C_3 & f \\ -\widetilde{t}_3 & -\widetilde{\partial}^C_2 \end{pmatrix}} \widetilde{C}_2 \oplus \widetilde{C}_1 \xrightarrow{\begin{pmatrix} \widetilde{\partial}^C_2 & f \\ -\widetilde{t}_2 & -\widetilde{\partial}^C_1 \end{pmatrix}} \widetilde{C}_1 \oplus \widetilde{C}_0 \xrightarrow{\begin{pmatrix} \widetilde{\partial}^C_1 & f \end{pmatrix}} \widetilde{C}_0$$

is a Q-free resolution of M.

Corollary 6.3. *Let f be a non-zerodivisor contained in the maximal ideal of Q, $R = Q/(f)$, and $C \in \mathbf{K}_{\mathrm{tac}}(R)$. Then STC can be taken to be $\Sigma^{-1}\operatorname{cone}(t)$, and the approximation map*

$$[\epsilon_C] : \Sigma^{-1}\operatorname{cone}(t) \to C$$

the natural projection.

Proof. We see from Thereom 6.2 that TC is given by

$$\cdots \widetilde{C}_{n+1} \oplus \widetilde{C}_n \xrightarrow{\begin{pmatrix} \widetilde{\partial}^C_{n+1} & f \\ -\widetilde{t}_{n+1} & -\widetilde{\partial}^C_n \end{pmatrix}} \widetilde{C}_n \oplus \widetilde{C}_{n-1} \xrightarrow{\begin{pmatrix} \widetilde{\partial}^C_n & f \\ -\widetilde{t}_n & -\widetilde{\partial}^C_{n-1} \end{pmatrix}} \widetilde{C}_{n-1} \oplus \widetilde{C}_{n-2} \cdots$$

and the approximation $[\epsilon_C] : STC \to C$ takes the form

$$\begin{array}{ccccc} \cdots C_{n+1} \oplus C_n & \xrightarrow{\begin{pmatrix} \partial^C_{n+1} & 0 \\ -t_{n+1} & -\partial^C_n \end{pmatrix}} & C_n \oplus C_{n-1} & \xrightarrow{\begin{pmatrix} \partial^C_{n} & 0 \\ -t_{n} & -\partial^C_{n-1} \end{pmatrix}} & C_{n-1} \oplus C_{n-2} \cdots \\ \downarrow & & \downarrow & & \downarrow \\ \cdots C_{n+1} & \xrightarrow{\partial^C_{n+1}} & C_n & \xrightarrow{\partial^C_n} & C_{n-1} \cdots \end{array}$$

where the vertical maps are the natural projections. Finally, one recognizes that the top complex is $\Sigma^{-1}\operatorname{cone}(t)$. □

Corollary 6.4. *Let f be a non-zerodivisor contained in the maximal ideal of Q, $R = Q/(f)$, and $C \in \mathbf{K}_{\mathrm{tac}}(R)$. If $[\epsilon_C] : STC \to C$ is the right approximation of C, then $\operatorname{cone}([\epsilon_C])$ is isomorphic to $\Sigma^2 C$ in $\mathbf{K}_{\mathrm{tac}}(R)$, and we have the distinguished triangle*

$$STC \xrightarrow{[\epsilon_C]} C \xrightarrow{[t]} \Sigma^2 C \to \Sigma STC$$

Proof. Let $t : C \to \Sigma^2 C$ be the morphism of complexes defined in 6.1, and consider the corresponding distinguished triangle

$$C \xrightarrow{[t]} \Sigma^2 C \to \operatorname{cone}([t]) \to \Sigma C$$

Rotating this triangle we have

$$\Sigma^{-1}\operatorname{cone}([t]) \to C \xrightarrow{[t]} \Sigma^2 C \to \operatorname{cone}([t])$$

which by 6.3 is isomorphic to the triangle

$$STC \xrightarrow{[\epsilon_C]} C \xrightarrow{[t]} \Sigma^2 C \to \Sigma STC$$

and this proves both claims of the corollary. □

In the current case of relative codimension one, we have a concrete description of the form of resolutions defined in 4.9. The following proposition follows from the previous result, and may be regarded as an analogue in $\mathbf{K}_{\mathrm{tac}}(R)$ of the fact that modules over a hypersurface ring have free resolutions which are eventually periodic.

Proposition 6.5. *Let f be a non-zerodivisor of Q, $R = Q/(f)$, and $C \in \mathbf{K}_{\mathrm{tac}}(R)$. Then the triangle resolution, as in 4.9, of C in $\mathbf{K}_{\mathrm{tac}}(R)$ with respect to $\mathbf{K}_{\mathrm{tac}}(Q)$ has the form*

$$\cdots \to \Sigma^2 STC \to \Sigma STC \to STC \to C.$$

A complex C of R-modules is *minimal* if $\partial^C_i(C_i) \subseteq \mathfrak{m} C_{i-1}$ for all i. A complex is called *contractible* if it is homotopically equivalent to the zero complex. We note that every totally acyclic complex C may be decomposed as $C' \oplus Z$ where C' is minimal and Z is contractible. In this event we define the ith *Betti number* of C to be the rank of the free R-module C'_i. Note that this is equal to $\operatorname{rank}_k(\mathrm{H}_i(C \otimes_R k))$.

For the maps t from 6.1, define $r_i = \operatorname{rank}_k(\operatorname{Im}(t_{i+2} \otimes_R k))$. The following is a consequence of 6.3. It compares the Betti numbers of a minimal totally acyclic R-complex C with those of its Q-approximation STC, and can be seen as an analogue in $\mathbf{K}_{\mathrm{tac}}(R)$ of comparisons of Betti numbers of modules.

Theorem 6.6. *Let C be a minimal totally acyclic R-complex. Set $c_i = \operatorname{rank} C_i$, and $b_i = \operatorname{rank}_k \mathrm{H}_i(STC \otimes_Q k)$.*

(1) *For all $n \in \mathbb{Z}$ we have*

$$b_n = c_n - r_{n-2} + c_{n-1} - r_{n-1}$$

(2) *Moreover, if for some n, both t_n and t_{n+1} are surjective, then*

$$b_n = c_n - c_{n-2}$$

(3) *and if for some n, both t_n and t_{n+1} are injective, then*

$$b_n = c_{n-1} - c_{n+1}$$

Proof. By 6.3, the complex $STC \otimes_Q k$ is isomorphic to

$$\cdots \to k^{c_{n+1}} \oplus k^{c_n} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ -u_{n+1} & 0 \end{smallmatrix}\right)} k^{c_n} \oplus k^{c_{n-1}} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ -u_n & 0 \end{smallmatrix}\right)} k^{c_{n-1}} \oplus k^{c_{n-2}} \to \cdots$$

where u_n represents $t_n \otimes_R k$. The results follow. □

The following is a triangulated analogue of the change of rings long exact sequence of cohomology for modules (see, for example, [CE, Chap. XVI, Sect. 5]).

Theorem 6.7. *Let f be a non-zerodivisor of Q, $R = Q/(f)$, and $C \in \mathbf{K}_{\mathrm{tac}}(R)$. If $[\epsilon_C] : STC \to C$ is the right approximation of C, then for $C' \in \mathbf{K}_{\mathrm{tac}}(R)$ we have the long exact sequences of cohomology*

$$\cdots \to \mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(Q)}(\Sigma^{n+1}TC, TC') \to \mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(R)}(\Sigma^{n+2}C, C') \to$$
$$\mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(R)}(\Sigma^{n}C, C') \to \mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(Q)}(\Sigma^{n}TC, TC') \to \cdots$$

Proof. From the distinguished triangle $STC \xrightarrow{[\epsilon_C]} C \xrightarrow{[t]} \Sigma^2 C \to$ we have the standard long exact sequence of cohomology

$$\cdots \to \mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(R)}(\Sigma^{n+1}STC, C') \to \mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(R)}(\Sigma^{n+2}C, C') \to$$
$$\mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(R)}(\Sigma^{n}C, C') \to \mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(R)}(\Sigma^{n}STC, C') \to \cdots$$

Now using Theorem 3.1, we have for all n the isomorphisms

$$\mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(R)}(\Sigma^{n+1}STC, C') \cong \mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(Q)}(\Sigma^{n+1}TC, TC')$$

and the long exact sequence becomes what was advertised. □

7. Approximations by periodic complexes and matrix factorizations

Recall that a commutative local ring Q is a *hypersurface ring* if Q is the quotient of a regular local ring by a principal ideal; hypersurface rings are Gorenstein. In this case, it follows from Eisenbud [Ei] that minimal totally acyclic complexes are always periodic of period at most two, and with constant Betti numbers. Thus when Q is a hypersurface (and, as always, that $\mathrm{pd}_Q R < \infty$ via $\varphi : Q \to R$), our approximations compare what are often aperiodic totally acyclic complexes with growing Betti numbers, to those of period two with constant Betti numbers. In fact, we will show that Theorem 4.1 allows us to approximate totally acyclic R-complexes by pairs of matrices.

Matrix factorizations. We first discuss the homotopy category of matrix factorizations. Matrix factorizatons were first defined by Eisenbud in [Ei] in his investigation of maximal Cohen-Macaulay modules. Let P be a commutative local ring and x an element in the maximal ideal of P. A *matrix factorization* (F, G, ϕ, ψ) of x is a diagram

$$F \xrightarrow{\phi} G \xrightarrow{\psi} F$$

where F and G are finitely generated free P-modules, and ϕ and ψ are homomorphisms satisfying $\psi\circ\phi = x1_F$ and $\phi\circ\psi = x1_G$. Since ϕ and ψ are maps between free modules, one is welcome to think of them as matrices (with respect to fixed bases of F and G) with entries in P. A morphism $\theta\colon (F_1, G_1, \phi_1, \psi_1) \to (F_2, G_2, \phi_2, \psi_2)$ between two matrix factorizations (of x) is a pair of homomorphisms $\alpha\colon F_1 \to F_2$ and $\beta\colon G_1 \to G_2$ such that the diagram

$$\begin{array}{ccccc} F_1 & \xrightarrow{\phi_1} & G_1 & \xrightarrow{\psi_1} & F_1 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ F_2 & \xrightarrow{\phi_2} & G_2 & \xrightarrow{\psi_2} & F_2 \end{array}$$

commutes. The category $\mathbf{MF}(P, x)$ of matrix factorizations of x and morphisms is additive, with the obvious notion of a zero object and direct sums.

The shift $\Sigma(F, G, \phi, \psi)$ of (F, G, ϕ, ψ) is the matrix factorization

$$G \xrightarrow{-\psi} F \xrightarrow{-\phi} G$$

of x. The *mapping cone* C_θ of the map θ above is the diagram

$$G_1 \oplus F_2 \xrightarrow{\left[\begin{smallmatrix} -\psi_1 & 0 \\ \beta & \phi_2 \end{smallmatrix}\right]} F_1 \oplus G_2 \xrightarrow{\left[\begin{smallmatrix} -\phi_1 & 0 \\ \alpha & \psi_2 \end{smallmatrix}\right]} G_1 \oplus F_2$$

which again is a matrix factorization of x. For a morphism θ of matrix factorizations as above, we have the natural short exact sequence

$$0 \to (P_2, Q_2, \phi_2, \psi_2) \xrightarrow{i_\theta} C_\theta \xrightarrow{\pi_\theta} \Sigma(P_1, Q_1, \phi_1, \psi_1) \to 0$$

of matrix factorizations in $\mathbf{MF}(P, x)$.

Two maps $\theta, \theta'\colon (F_1, G_1, \phi_1, \psi_1) \to (F_2, G_2, \phi_2, \psi_2)$ in $\mathbf{MF}(P, x)$, with the same source and target, are *homotopic* if there are diagonal maps in the diagram

$$\begin{array}{ccccc} F_1 & \xrightarrow{\phi_1} & G_1 & \xrightarrow{\psi_1} & F_1 \\ {\scriptstyle\alpha'}\downarrow\downarrow{\scriptstyle\alpha} & \overset{s}{\swarrow} & {\scriptstyle\beta'}\downarrow\downarrow{\scriptstyle\beta} & \overset{t}{\swarrow} & {\scriptstyle\alpha'}\downarrow\downarrow{\scriptstyle\alpha} \\ F_2 & \xrightarrow{\phi_2} & G_2 & \xrightarrow{\psi_2} & F_2 \end{array}$$

satisfying $\alpha - \alpha' = s\circ\phi_1 + \psi_2\circ t$ and $\beta - \beta' = t\circ\psi_1 + \phi_2\circ s$. This defines an equivalence relation on the abelian groups of morphisms in $\mathbf{MF}(P, x)$, and the equivalence class of the map θ is denoted by $[\theta]$. It is straightforward to show that homotopies are compatible with addition and composition of maps in $\mathbf{MF}(P, x)$. The *homotopy category* $\mathbf{HMF}(P, x)$ has the same objects as $\mathbf{MF}(P, x)$, but the morphism sets are homotopy equivalence classes of morphisms in $\mathbf{MF}(P, x)$. By the above, the morphism sets in $\mathbf{HMF}(P, x)$ are abelian groups, hence the homotopy category is also additive with the same zero object (which is now unique only up to homotopy) and the usual direct sums.

The homotopy category $\mathbf{HMF}(P, x)$ admits a natural structure of a triangulated category. The suspension defined above induces an additive automorphism $\Sigma\colon \mathbf{HMF}(P, x) \to \mathbf{HMF}(P, x)$, with Σ^2 the identity automorphism. Let Δ be the collection of all triangles in $\mathbf{HMF}(P, x)$ isomorphic to *standard triangles*, that is, triangles of the form

$$(F_1, G_1, \phi, \psi) \xrightarrow{[\theta]} (F_2, G_2, \phi_2, \psi_2) \xrightarrow{[i_\theta]} C_\theta \xrightarrow{[\pi_\theta]} \Sigma(F_1, G_1, \phi, \psi)$$

Then the triple $(\mathbf{HMF}(P, x), \Sigma, \Delta)$ is a triangulated category; the classical proof (cf. [HoJø, Theorem 6.7]) showing that the homotopy category of complexes over an additive category is triangulated carries over (see also [Or]).

Complete intersections. Now assume that $(P, \mathfrak{n})$ is a regular local ring and $x_1, \dots, x_c$ a regular sequence contained in $\mathfrak{n}^2$. Set $I = (x_1, \dots, x_c)$ and define $R = P/I$; this is a *complete intersection* of codimension c. Let $x \in I - \mathfrak{n} I$, and $Q = P/(x)$. Then Q is a hypersurface ring, and we have the natural projections $P \to Q \to R$.

Theorem 7.1. *We have the diagram of triangulated categories*

$$\begin{array}{ccccc} \mathbf{K}_{\mathrm{tac}}(Q) & \xrightarrow[\cong]{\beta} & \mathbf{D}^{\mathrm{b}}_{\mathrm{sg}}(Q) & \xrightarrow[\cong]{F} & \mathbf{HMF}(P, x) \\ S \big\downarrow \big\uparrow T & & \swarrow S' \quad \nearrow T' & & \\ \mathbf{K}_{\mathrm{tac}}(R) & & & & \end{array}$$

where S' and T' are an adjoint pair of functors induced from S and T.

Proof. The equivalence β is that of Buchweitz [Bu], and the equivalence F was noted by Buchweitz and proved by Orlov [Or]. □

Thus the functors S and T give approximations of totally acyclic R-complexes by matrix factorizations. We end with some questions for further study.

Questions. (1) How do the matrix factorization approximations in Theorem 7.1 compare with the higher matrix factorizations of Eisenbud and Peeva [EiPe], as one varies the generator x defining Q?

(2) Can one classify the objects of $\mathbf{K}_{\mathrm{tac}}(R)$ with finite data, in terms of the objects of $\mathbf{HMF}(P, x)$ when Q has finite Cohen-Macaulay type?

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