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Trykket av NTNU Grafisk senter

To my grandfather, Christen Øygard

Abstract

We define geometric Hodge filtered complex cobordism groups $MU^n(p)(X)$ for complex manifolds X. Refining the Pontryagin–Thom construction, we give a natural isomorphism $MU^n(p)(X) \simeq MU^n_{\mathcal{D}}(p)(X)$, where $MU^n_{\mathcal{D}}(p)(X)$ are the Hodge filtered complex cobordism groups defined in [30]. We establish a pushforward map $g_*\colon MU^n(p)(X) \to MU^{n+2d}(p+d)(Y)$ for each proper holomorphic map $g\colon X \to Y$. Using g_*, we get for algebraic manifolds X a map $\Omega^n_{alg}(X) \to MU^{2n}(n)(X)$, where $\Omega^n_{alg}(X)$ denotes the algebraic cobordism group of X. This induces an Abel–Jacobi map. Using a cycle model for Deligne-cohomology similar to that of [21], we describe the Thom-morphism $MU^n(p)(X) \to H^n_{\mathcal{D}}(X;\mathbb{Z}(p))$, and verify that our Abel–Jacobi map refines that of Griffiths.

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Knut Bjarte Haus
Trondheim, May 2022

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Chapter 1

Introduction

Let X be a compact Riemann surface of genus g, and let $\omega \in H^0(X;\Omega^1) = \Omega^1(X)$, where Ω^1 denotes the sheaf of holomorphic 1-forms. Given a pair of points $p, q \in X$ we can integrate ω over a path γ connecting p and q, to obtain a number

$$\Phi(p,q)(\omega) = \int_\gamma \omega \in \mathbb{C}.$$

Thus $\Phi(p,q)$ defines an element of the dual space $H^0(X;\Omega^1)'$. Clearly $\Phi(\gamma)$ depends on the choice of the path γ. We can understand this dependence: If $\gamma_1,\ \gamma_2$ are two paths, then the difference between integrating ω along γ_1 and γ_2 is

$$\int_{\gamma_2} \omega - \int_{\gamma_1} \omega = \int_{\gamma_2-\gamma_1} \omega.$$

The functional $\int_{\gamma_2-\gamma_1} \in H^0(X;\Omega^1)^*$ depends only on the homology class of the cycle $\gamma_2 - \gamma_1$. Hence we define the lattice

$$\Lambda = \left\{ \int_\gamma \ :\ \gamma \in H_1(X;\mathbb{Z}) \right\} \subset H^0(X;\Omega^1)'.$$

The quotient

$$H^0(X;\Omega^1)'/\Lambda =: J(X)$$

is a torus of complex dimension g. The pair of points p, q defines an element of $J(X)$.

This was generalized by Griffiths in the following way. Let now X be a Kähler manifold of dimension n. Hodge theory implies

$$F^{n-k+1}H^{2n-2k+1}(X;\mathbb{C}) = \frac{F^{n-k+1}\mathcal{A}^{2n-2k+1}(X;\mathbb{C})_{\mathrm{cl}}}{dF^{n-k+1}\mathcal{A}^{2n-2k}(X;\mathbb{C})}$$

where $\mathcal{A}^*(X;\mathbb{C})$ is the $\mathbb{C}$-valued smooth de Rham complex of X, the subscript cl denotes the closed forms, and F^p is the Hodge filtration. Consider now a formal sum $Z = \sum_i a_i Z_i$ of analytic subvarieties $Z_i \subset X$ of codimension p. The primary topological invariant of Z is its homology class $[Z]$. Suppose $[Z] = 0$, say $Z = \partial\Gamma$. Then integration over Γ defines a homomorphism

$$\int_\Gamma \colon F^{n-p+1}H^{2n-2p+1}(X;\mathbb{C}) \to \mathbb{C}.$$

For this, we are using that for $\eta \in F^{n-p+1}\mathcal{A}^{2n-2p}(X;\mathbb{C})$ we have

$$\int_\Gamma d\eta = \int_Z \eta = 0$$

where the first equality follows from Stokes' theorem and the second equality holds, since each Z_i is analytic of dimension $n-p$ and η has no component of degree $(n-p, n-p)$. The map $\int_\Gamma \in \left(F^{n-p+1}H^{2n-2p+1}(X;\mathbb{C})\right)^*$ depends on Γ. However, just as in the case of curves, the equivalence class

$$\left[\int_\Gamma\right] \in \frac{\left(F^{n-p+1}H^{2n-2p+1}(X;\mathbb{C})\right)'}{H^{2n-2p+1}(X;\mathbb{Z})} =: J^{2p-1}(X)$$

depends only on Z. The torus $J^{2p-1}(X)$ is called Griffiths' intermediate Jacobian. One important property of the Deligne cohomology groups $H^{2p}_{\mathcal{D}}(X;\mathbb{Z}(p))$ is that when X is a compact Kähler manifold, it sits in a short exact sequence

$$0 \to J^{2p-1} \to H^{2p}_{\mathcal{D}}(X;\mathbb{Z}(p)) \to \mathrm{Hdg}^{p,p}(X) \to 0$$

where $\mathrm{Hdg}^{p,p}(X) = H^{2p}(X;\mathbb{Z}) \cap H^{p,p}(X;\mathbb{C})$ are the integral Hodge classes on X. The cycle map which takes an analytic subvariety to its fundamental class,

$$Z \mapsto [Z] \in H^{2p}(X;\mathbb{Z})$$

takes values in the Hodge classes. This cycle map factors through $H^{2p}_{\mathcal{D}}(X;\mathbb{Z}(p))$. This explains how Griffiths' intermediate Jacobian J^{2p-1} is the target of secondary invariants of algebraic cycles which are homologous to 0. For the Griffiths Abel–Jacobi map is just the morphism induced on the kernel of the cycle map. This shows that Deligne cohomology not only carries the topological information of singular cohomology but also the more subtle geometric information of the secondary invariant given by the Abel–Jacobi map.

One aim of this thesis is to expand on ideas from [46] and tell a story similar to the above for proper holomorphic maps $f\colon Z \to X$ which are nullbordant, meaning that $f_*[1_Z] = 0$ in $MU^{2p}(X)$. We now consider the Hodge filtered complex cobordism groups as defined in [30]. If X is compact Kähler, these too sit in a short exact sequence

$$0 \to J^{2p-1}_{MU}(X) \to MU^{2p}_{\mathcal{D}}(p)(X) \to \mathrm{Hdg}^{p,p}_{MU}(X) \to 0. \tag{1.1}$$

The groups $MU^n_{\mathcal{D}}(p)(X)$ are by definition represented by $MU_{\mathcal{D}}(p)$, defined as the homotopy pullback

$$\begin{tikzcd}
MU_{\mathcal{D}}(p) \arrow[r] \arrow[d] & MU \arrow[d, "\phi^p"] \\
H(\Omega^{*\geqslant p}(\mathcal{V}_*)) \arrow[r] & H(\Omega^*(\mathcal{V}_*))
\end{tikzcd}$$

in the category of presheaves of spectra on the site of complex manifolds. Here MU is the constant presheaf $X \mapsto MU$, $\mathcal{V}_* = MU_* \otimes \mathbb{C}$, and for a presheaf of chain complexes A, HA denotes the presheaf which assigns to X the Eilenberg–MacLane spectrum associated to the chain complex $A(X)$. Finally, ϕ^p is defined as the composition $MU \to H\mathcal{V}_* \to H(\Omega^*(\mathcal{V}_*))$ where $MU \to H\mathcal{V}_*$

is any map of spectra inducing on π_n the multiplication by $(2\pi i)^{n+p}$ map and $H\mathcal{V}_* \to H(\Omega^*(\mathcal{V}_*))$ is induced by the inclusion of constant functions into the holomorphic de Rham complex $\mathcal{V}_* \to \Omega^*(\mathcal{V}_*)$.

In order to better understand the theory $MU_{\mathcal{D}}$ we consider geometrically defined groups $MU^n(p)(X)$, denoted without the subscript $\mathcal{D}$. Let us now briefly describe the construction of $MU^n(p)(X)$. It is inspired by the geometric differential complex cobordism groups of [8] and Karoubi's multiplicative K-theory, [34]. The multiplicative K-theory groups $MK(X)$ are generated by triples (E, ∇, h) where $E \to X$ is a complex vector bundle with connection ∇, and h is a sequence of forms such that $ch_{2p}(\nabla) + dh_{2p} \in F^p\mathcal{A}^{2p}(X)$ where $ch_{2p}(\nabla)$ is the $2p$-th Chern–Weil Chern character form of ∇. For Hodge filtered complex cobordism, we essentially replace vector bundles with connection by the differential cobordism cycles of [8].

Consider the genus $\phi \colon MU_* \to \mathcal{V}_*$ given by multiplication by $(2\pi i)^n$ in degree $2n$. By Thom's theorem, MU_n is the bordism group of n-dimensional almost complex manifolds. Hirzebruch showed that if R is an integral domain over $\mathbb{Q}$, then any genus $\phi \colon MU_* \to R$ is of the form

$$\phi(Z) = \int_Z (K^\phi(TZ))^{-1}$$

for a multiplicative sequence K^ϕ, which yield an R-valued characteristic class of complex vector bundles. For us $R = \mathcal{V}_* = MU_* \otimes \mathbb{C}$, and we consider the characteristic class $K^p = (2\pi i)^p \cdot K^\phi$. If ∇ is a connection on a complex vector bundle E, Chern–Weil theory gives a form $K^p(\nabla)$ representing $K^p(E)$. Given a form ω on Z and a proper oriented map $f \colon Z \to X$, we consider the pushforward current $f_*\omega$, which acts on compactly supported forms on X by $\sigma \mapsto \int_Z \omega \wedge f^*\sigma$.

Now we can describe the group of Hodge filtered cycles $ZMU(p)(X)$. We use triples (f, ∇, h) where f is a complex oriented map $f \colon Z \to X$, ∇ is a connection on the complex stable normal bundle of f and h is a current on X so that $f_*K^p(\nabla) + dh$ is a smooth form in $F^p\mathcal{A}^n(X; \mathcal{V}_*)$. Here the convention we use is that

$$F^p\mathcal{A}^n(X; \mathcal{V}_*) = \bigoplus_j F^{p+j}\mathcal{A}^{n+2j}(X; \mathcal{V}_{2j}).$$

De Rham proved that all closed currents are cohomologous to forms, so it is always possible to find an h such that $f_*K^p(\nabla) + dh$ is smooth. We grade $ZMU(p)(X)$ by the codimension of f, so for $(f, \nabla, h) \in ZMU^n(p)(X)$ we have $\dim X - \dim Z = n$.

The first main result of the thesis is to prove:

Theorem 1.1. *For every complex manifolds X and integer p, there are Hodge filtered complex cobordism groups $MU^n(p)(X)$ given as the equivalence classes of Hodge filtered geometric cycles modulo a suitable bordism relation. Moreover, there is an isomorphism of Hodge filtered cohomology groups*

$$MU^n_{\mathcal{D}}(p)(X) \cong MU^n(p)(X)$$

which respects pullbacks.

Our proof of the existence of the isomorphism proceeds in two steps. First we define another spectrum of simplicial presheaves $MU_{\mathrm{hs}}(p)$ as a homotopy pullback similar to that defining $MU_{\mathcal{D}}(p)$, with tailor made models for all three presheaves. The constant presheaf MU is replaced by the spectrum of simplicial presheaves with n-th simplicial presheaf given by $X \mapsto \mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU_n)$, where $QMU_n = \mathrm{colim}_k\, \Omega^k MU_{n+k}$, and the superscript sm indicates the subspace of maps which in the appropriate sense are smooth. Then we define a map of spectra

$$\phi_{\mathrm{sm}} \colon \mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU_n) \to \mathcal{A}^n_\Sigma(\mathcal{V}_*)(X \times \Delta^\bullet).$$

Here $\mathcal{A}^*_\Sigma(\mathcal{V}_*)$ is a presheaf of simplicial spectra over $\mathbf{Man}_{\mathbb{C}}$ which is weakly equivalent to the Eilenberg–Maclane spectrum $H\mathcal{V}_*$. We also define a spectrum $F^p\mathcal{A}^*_\Sigma(\mathcal{V}_*)$ which is weakly equivalent to $H(F^p\mathcal{A}^*(\mathcal{V}_*))$, and comes with a natural objectwise inclusion $F^p\mathcal{A}^*_\Sigma(\mathcal{V}_*)) \to \mathcal{A}^*_\Sigma(\mathcal{V}_*)$, corresponding to the objectwise inclusion $\Omega^{*\geqslant p} \to \Omega^*$. Our definition of the spectrum $\mathcal{A}^*_\Sigma(\mathcal{V}_*)$ follows essentially from the work of Hopkins–Singer [31], in honor of which we denote the resulting theory with subscript hs. A key ingredient for the construction of the map ϕ_{sm} is the existence of natural Thom forms for Hermitian vector bundles with unitary connection. We use the Mathai–Quillen Thom forms of [42]. We show that there is a weak equivalence of presheaves of spectra $MU_{\mathcal{D}} \simeq MU_{\mathrm{hs}}$. We are then able to directly define a natural map

$$\kappa \colon MU^n_{\mathrm{hs}}(p)(X) \to MU^n(p)(X).$$

For the construction of κ we need all the deatils of the internal mechanics of the Pontryagin–Thom construction. For the proof that κ is an isomorphism, we furthermore depend on knowing the particular isomorphism $\rho \colon MU^*_h(X) \to MU^*(X)$ in play, to see that κ is compatible with it. Here $MU^*(X)$ denotes Quillen's geometric model for $MU^*(X)$ from [47], and $MU^*_h(X)$ are the homotopy theoretic cohomology groups of X represented by the complex cobordism spectrum MU in the stable homotopy category. Therefore we find it natural to provide along the way a concrete description of ρ, and a proof that it is an isomorphism. Though the map ρ by no means is new, and probably is what Quillen had in mind in [47], we could find no sources treating it in full detail as a map of cohomology theories. We hope that our exposition of ρ can contribute to make this version of the Pontryagin–Thom construction more accessible for non-experts. Finally, we note that it should be possible to adapt the construction of κ to give a more concrete geometric description of the isomorphism between the smooth cobordism of [8] and the differential cobordism of [31]. Though the existence of such an isomorphism follows from [9], a concrete description might for some purposes be useful.

The second main achievement of the thesis is the construction of pushforwards. This is also inspired by [8]. We are able to prove:

Theorem 1.2. *Let $g \colon X \to Y$ be a proper holomorphic map between complex manifolds. Then there is a natural pushforward homomorphism*

$$g_* \colon MU^n(p)(X) \to MU^{n+2d}(p+d)(Y)$$

where d denotes the complex codimension $d = \dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} X$.

In [8] the pushforward exist only for proper submersions with smooth MU-orientations. Let us explain why our pushforward exist for all proper holomorphic maps. We first define a notion of $MU_{\mathcal{D}}$-orientations, which again is inspired by [8] and [34]. Then we use ideas from [34] to obtain canonical $MU_{\mathcal{D}}$-orientations of holomorphic maps. Concretely, [34, Theorem 6.7] says that choosing a Bott connection D induces a map $K^0_{\mathrm{hol}}(X) \to MK^0(X)$ by $[E] \mapsto [E, D, 0]$. Essentially applying the corresponding result for $MU_{\mathcal{D}}$-orientations to the virtual holomorphic normal bundle of a holomorphic map g, grants g an $MU_{\mathcal{D}}$ orientation. The pushforward map of [8] could have been extended to arbitrary proper smooth maps with smooth MU-orientations, essentially by the same formulas, had the authors considered the corresponding currential differential extension of MU. The currential differential extension of MU would however not be isomorphic with the smooth differential extension, simply because the space of closed currents $\mathcal{D}^n(X)_{cl}$ is larger than the space of closed forms $\mathcal{A}^n(X)_{cl}$. This is developed for differential K-theory in [19], where particularly the exact sequences [19, (2.20)] and [19, (2.29)] makes it clear that the smooth and currential differential K-theory groups are different. In the Hodge filtered context, the currential and smooth theories are however canonically isomorphic. This is because instead of $\mathcal{A}^n(X)_{cl}$ and $\mathcal{D}^n(X)_{cl}$, we use $H^n(X; F^p\mathcal{A}^*)$ and $H^n(X; F^p\mathcal{D}^*)$, and instead of $\mathcal{A}^n(X)/\mathrm{Im}\,(d)$ and $\mathcal{D}^n(X)/\mathrm{Im}\,(d)$, we use $H^n\left(X; \frac{\mathcal{A}^*}{F^p}\right)$ and $H^n\left(X; \frac{\mathcal{D}^*}{F^p}\right)$. Essentially because the Dolbeault-Grothendieck lemma holds both for currents and forms, in both cases the canonical map from the first to the second group is an isomorphism.

We return to the short exact sequence (1.1). Our pushforward is compatible with the pushforward along proper complex oriented maps for the cohomology theory MU^*. Hence if $f\colon Z \to X$ is a nullbordant proper holomorphic map, then $f_*1 \in MU^{2p}(p)(X)$ lands in $J^{2p-1}_{MU}(X)$. Using the geometric description of the pushforward map we are able to give formulas for $AJ(f)$. They are however not easily evaluated as they feature certain Chern–Simons transgression forms mediating between an arbitrary connection on N_f, and a connection related to Bott connections on TZ and f^*TX. The Abel–Jacobi map is constructed in Section 8.

The Thom morphism $MU^n(X) \to H^n(X; \mathbb{Z})$ induces a Hodge filtered Thom morphism $MU^n_{\mathcal{D}}(p)(X) \to H^n_{\mathcal{D}}(X; \mathbb{Z}(p))$. We give a cycle description of this map in Section 9. For this, we use a cycle model for Deligne cohomology inspired by that of Gillet and Soulé in [21], but differing in that we for simplicity use currents of integration instead of integral currents in the sense of Federer [16].

Before we construct our geometric theory, we discuss an axiomatic definition of the notion of Hodge filtered cohomology theories in Chapter 3. This definition is inspired by the axiomatic definition of differential cohomology theories of [9]. In [9] it is shown that under mild assumptions these axioms suffice to characterize differential cohomology up to isomorphism. The hope is that our axioms too suffice to characterize Hodge filtered cohomology up to isomorphism. Such a uniqueness result would have saved us of the considerable effort of Chapters 5 and 6.

The difficulty of a straight forward translation of the proof from [9] is essentially the well known phenomenon that holomorphic maps are much more rigid than smooth maps. Using Oka-theory, we did however obtain partial results for Stein manifolds. We report on this in Section 3.5. The necessary assumption is that the underlying cohomology theory for which we would like to prove uniqueness of Hodge filtered extensions can be represented by spaces which can be approximated by manifolds which are Oka–Stein, meaning they are both Oka and Stein. Our only example of such a theory is complex K-theory. We note that it is an open problem in complex analysis whether the homotopy-types representable by Oka–Stein manifolds are the same as those representable by smooth manifolds. In light of the successful construction of the isomorphism in Theorem 1.1, which does not make use of the complex manifold structure of the Grassmannians and the tautological bundles, a new attempt at proving uniqueness seems due.

Finally a brief comment on Section 2. There we begin with both notation and recollections as well as the proof of some results which are relevant later in the thesis but would take the focus off the main argument if they were discussed where they are needed. Most of the subsections are independent from one another. Instead of collecting them in a long appendix at the end we decided to put them at the beginning. Hence one may first skip over Section 2 and get back to it when needed.

Chapter 2

Conventions, notation, recollections, and lemmas

In this chapter we explain notation and recollect material that will be used throughout the thesis. We also prove some results that would unduly interrupt the flow of the text if given in the main body. It is not advised to read this chapter in its entirety, but rather to read its sections only when they are needed for the main story.

2.1 Transversality

Let X be a smooth manifold and $Z \subset X$ a properly embedded submanifold of codimension d. For each $p \in S$ there is a neighborhood $U_p \subset X$ and a smooth map $f_p \colon U_p \to \mathbb{R}^d$ having 0 as a regular value and satisfying $Z \cap U_p = f_p^{-1}(0)$, see [38, Proposition 5.16]. We say that f_p is a local defining function for Z.

We say that a smooth map $g \colon Y \to X$ is transverse to Z, denoted $g \pitchfork Z$, if whenever $g(y) = z \in Z$, we have $D_y g(T_y Y) + T_z Z = T_z X$.

Proposition 2.1. *Let $Z \subset X$ be a submanifold of codimension d. If $g \colon Y \to X$ is transverse to Z, then $g^{-1}(Z)$ is a submanifold of Y of codimension d.*

Proof. Let y be an arbitrary point in $g^{-1}(Z)$, say $g(y) = p$. Being a submanifold is a local property, since Hausdorfness and second countability is inherited by any subset. Therefore it is enough to give a local defining function for $g^{-1}(Z)$ around y. Let U be a neighborhood of p in X, and let $f \colon U \to \mathbb{R}^d$ be a local defining function for Z around p. Then on the open set $V := g^{-1}(U)$ consider the function

$$F := f \circ g|_V \colon V \to \mathbb{R}^d.$$

It suffices to show that 0 is a regular value of F. Observe that $D_p f(T_p Z) = 0$. Since 0 is a regular value of f, and $T_p Z + D_y g(T_y Y) = T_p X$, we must have

$$D_p f(D_y g(T_y Y)) = T_0 \mathbb{R}^d.$$

Then 0 is a regular value of F by the chain rule

$$D_y(f \circ g) = (D_p f) \circ D_y g. \qquad \blacksquare$$

More generally, we say that for a map $f \colon Z \to X$, $g \colon Y \to X$ is transverse to f, written $g \pitchfork f$, if whenever $g(y) = x = f(z)$ we have

$$D_y g(T_y Y) + D_z f(T_z Z) = T_x X.$$

We define the codimension of a map by $\operatorname{codim} f := \dim X - \dim Z$.

Proposition 2.2. *If $g \pitchfork f$, there is a cartesian diagram of manifolds:*

$$\begin{tikzcd} Z' \arrow[r, "g'"] \arrow[d, "f'"'] & Z \arrow[d, "f"] \\ Y \arrow[r, "g"'] & X \end{tikzcd}$$

If f is proper, then f' is proper. We have $\operatorname{codim} f' = \operatorname{codim} f$.

Proof. Consider the two maps $f \times g \colon Z \times Y \to X \times X$ and $\Delta \colon X \to X \times X$ given by $\Delta(x) = (x, x)$. Then Δ is an embedding, and $f \pitchfork g \implies (f \times g) \pitchfork \Delta$. Hence by Proposition 2.1, we get a cartesian diagram of manifolds

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "(g',f')"'] & X \arrow[d, "\Delta"] \\ Z \times Y \arrow[r, "f \times g"'] & X \times X. \end{tikzcd}$$

Then the diagram

$$\begin{tikzcd} Z' \arrow[r, "g'"] \arrow[d, "f'"'] & Z \arrow[d, "f"] \\ Y \arrow[r, "g"'] & X \end{tikzcd}$$

is also cartesian. Suppose f is proper, and let $K \subset Y$ be a compact set. Then $f^{-1}(g(K))$ is compact. We have

$$\begin{aligned} f'^{-1}(K) &= \{(z, y) \in Z \times K \ : \ f(z) = g(y)\} \\ &\subset K \times f^{-1}(g(K)). \end{aligned}$$

Since f' is continuous, $f'^{-1}(K)$ is a closed set. Since it is a subset of a compact set, it is compact. For the last statement we need only note that by Proposition 2.1 the following equalities hold:

$$\dim Z + \dim Y - \dim Z' = \operatorname{codim}(g', f') = \operatorname{codim} \Delta = \dim X$$

Definition 2.3. *We define*

$$\operatorname{Map}^{\pitchfork}(Y, X; f) := \{g \in \mathcal{C}^\infty(Y, X) \ : \ g \pitchfork f\}.$$

When f is the inclusion of a submanifold $f \colon S \hookrightarrow X$, we may write

$$\operatorname{Map}^{\pitchfork}(Y, X; S) = \operatorname{Map}^{\pitchfork}(U, X; S).$$

We recall the following important theorem from [26, p. 35].

Theorem 2.4 (Thom's transversality theorem)**.** *If $Z \subset X$ is a proper submanifold, then the space $\operatorname{Map}^{\pitchfork}(Y, X; Z)$ is open and dense in $\mathcal{C}^\infty(Y, X)$ in the strong topology.*

Remark 2.5. The point is that transversality is a generic property.

2.2 Normal bundles

Given an embedding $\iota : Z \to X$, there is the normal bundle

$$N\iota := \iota^* TX/TZ.$$

Suppose $g\colon Y \to X$ is smooth and transverse to ι. Then $Z' := g^{-1}(\iota(Z))$ is a submanifold of Y. We denote the inclusion $Z' \hookrightarrow Y$ by ι', and the restriction $g|_{\iota'(Z')}$ by g'. We will need the following proposition:

Proposition 2.6. *The differential Dg induces an isomorphism*

$$N\iota' \simeq g'^* N\iota.$$

Proof. Consider the commutative diagram:

$$\begin{tikzcd} Z' \arrow[r, "g'"] \arrow[d, "\iota'"'] & Z \arrow[d, "\iota"] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

If γ is a curve in Z', then $g \circ \iota' \circ \gamma$ is a curve in $\iota(Z)$. Hence $Dg(D\iota'(TZ')) \subset D\iota(TZ)$, so that Dg induces a map

$$\overline{Dg}\colon N\iota' \to g'^* N\iota.$$

It follows from Proposition 2.1 that the two bundles have the same dimension, so it is enough to show that $\overline{Dg}$ is surjective. Consider $[v_x] \in N\iota$, for $v_x \in T_x X$, $x = \iota(z) = g(y)$. Using transversality, we can write

$$v_x = D_y g(v_y) + D\iota(v_z)$$

for $v_z \in T_z Z$, $v_y \in T_y Y$. Then $[v_x] = \overline{Dg}([v_y])$. ∎

We shall also need the following basic fact:

Proposition 2.7. *Let $\pi\colon E \to X$ be a vector bundle, and let $i_0\colon X \to E$ denote the 0-section. Then $Ni_0 \simeq E$.*

Proof. The proposition will follow once we establish an isomorphism

$$i_0^* TE \simeq TX \oplus E.$$

Since π is a submersion, we get a short exact sequence

$$0 \longrightarrow VE \longrightarrow TE \xrightarrow{D\pi} \pi^* TX \longrightarrow 0$$

of vector bundles on E. Here VE is just the kernel of $D\pi$, and is called the vertical bundle of E. It suffices to show that $i_0^*(VE) \simeq E$. We define a bundle map $\psi\colon E \to VE$ covering i_0 by

$$\psi(v_x) = \left.\frac{\partial}{\partial t}\right|_{t=0} (t \cdot v_x).$$

Since $\pi(t \cdot v_x) = x$ for each $t \in \mathbb{R}$, it is clear that $\psi(v_x) \in VE$, so we have a commutative square:

$$\begin{tikzcd} E \arrow[r, "\psi"] \arrow[d, "\pi"'] & VE \arrow[d] \\ X \arrow[r, "i_0"'] & E \end{tikzcd}$$

Furthermore, ψ is clearly fiberwise linear, and fiberwise injective. Then ψ is a fiberwise isomorphism for dimensional reasons. Thus ψ induces an isomorphism $E \simeq i_0^* VE$, which proves the proposition. ■

2.3 Forms and Currents

2.3.1 Forms

We will denote the de Rham complex of a real manifold X with real coefficients by $\mathcal{A}^*_{\mathbb{R}}(X)$. The de Rham complex with complex coefficients will be denoted by $\mathcal{A}^*(X)$. Thus the relationship between these groups is

$$\mathcal{A}^k(X) = \mathcal{A}^k_{\mathbb{R}}(X) \otimes_{\mathbb{R}} \mathbb{C}.$$

We follow de Rham [12], and consider $\mathcal{A}^*(X)$ with the C^∞ compact-open topology, which is defined by the family of seminorms obtained as follows: Let $U \subset X$ be a coordinate patch. With respect to these coordinates, write a differential form ω as

$$\omega = \sum \omega_I dx_I.$$

For each compact set $L \subset U$ and $m, k \in \mathbb{N}$, we have a seminorm defined by

$$\rho^L_{m,k}(\omega) = \sup_{x \in L} \max_{|I|=k, |\alpha| \leqslant m} (D^\alpha \omega_I(x)).$$

Here $\alpha = (\alpha_1, \dots, \alpha_{\dim X}) \in \mathbb{N}^{\dim X}$, $|\alpha| := \sum \alpha_i$ and

$$D^\alpha \omega_I = \frac{\partial^{|\alpha|} \omega_I}{\partial x_1^{\alpha_1} \cdots \partial x_{\dim X}^{\alpha_{\dim X}}}.$$

The support of a differential form is the smallest closed set outside of which it vanish:

$$\operatorname{supp}(\omega) = \bigcap_{U \text{ open } : \ \omega|_U = 0} (X \backslash U)$$

We denote the space of compactly supported forms on X by

$$\mathcal{A}^*_c(X) := \{\omega \in \mathcal{A}^*(X) \ : \ \operatorname{supp}(\omega) \text{ is compact}\}.$$

We give $\mathcal{A}^*_c(X)$ the subspace topology from $\mathcal{A}^*(X)$.

Remark 2.8. We here follow the convention from topology. For example Bott and Tu [5], Lee [38], and Madsen–Tornehaven all use subscript c to denote compact support. Also [23] use our convention. It seems the convention in analysis is to write $\mathscr{D}^*$ instead of $\mathcal{A}_c^*$, and $\mathscr{E}^*$ instead of $\mathcal{A}^*$. De Rham [12], directly inspired by Schwartz, followed the analysis convention. Also Demailly [11] and Hörmander [28] use the analysis convention.

When X is a complex manifold we have the Hodge decomposition of forms

$$\mathcal{A}^n = \bigoplus_{p+q=n} \mathcal{A}^{p,q}(X),$$

where $\mathcal{A}^{p,q}(X)$ are those $n = p+q$ forms ω so that for any $v_1, \ldots, v_n \in TX$ and complex number z we have

$$\omega(zv_1, \ldots, zv_n) = z^p \overline{z}^q \omega(v_1, \ldots, v_n).$$

This is the same as requiring that with respect to local holomorphic coordinates z_i, if we write

$$\omega = \sum_{I,J} \omega_{IJ} dz_I \wedge \overline{dz}_J$$

then $\omega_{IJ} = 0$ unless $|I| = p$ and $|J| = q$. With respect to this bigrading, we decompose the exterior differential d as $d = \partial + \overline{\partial}$, where ∂ has bi-degree $(1,0)$, and $\overline{\partial}$ has bi-degree $(0,1)$.

2.3.2 Currents

In this section, we assume for simplicity throughout that all manifolds are oriented. Note that in [12], the theory of currents is developed also for non-oriented manifolds.

We define the space of current on X, denoted $\mathcal{D}^*(X)$, as the topological dual of the space of compactly supported smooth forms $\mathcal{A}_c^*(X)$. Unless otherwise stated, we are considering $\mathcal{D}^*(X)$ with the weak* topology, which is induced by the seminorms

$$\rho_\sigma(T) = |T(\sigma)|$$

for $\sigma \in \mathcal{A}_c^*(X)$. Thus $\lim_{t\to 0} T_t = T$ if and only if $\lim_{t\to 0} T_t(\sigma) = T(\sigma)$ for every $\sigma \in \mathcal{A}_c^*(X)$. This corresponds to the topology used for distributions in [28]. Some theorems, such as the Schwartz kernel theorem, are however better stated using the strong topology, see [50, p. 198].

Remark 2.9. Following up Remark 2.8, our notation $\mathcal{D}^*(X)$ is that of Griffiths and Harris [23]. The authors denoting $\mathcal{A}_c^*(X)$ by $\mathscr{D}^*(X)$, such as de Rham and Hörmander, denotes $\mathcal{D}^*(X)$ by $\mathscr{D}'^*(X)$. Curiously, in [50] where only open subsets of euclidean spaces, U, are considered, $\mathcal{A}_c^0(U)$ is denoted by $\mathscr{C}_c^\infty(U)$, yet its dual by $\mathcal{D}'(U)$.

There is a map $\mathcal{A}^*(X) \to \mathcal{D}^*(X)$ denoted $\omega \mapsto T_\omega$, defined by

$$T_\omega(\sigma) = \int_X \omega \wedge \sigma.$$

The integral converges since σ has compact support. It is clear that this map is $\mathbb{C}$-linear and continuous. Now we show:

Proposition 2.10. *The map $\omega \mapsto T_\omega$ is injective.*

Proof. Suppose $\omega \in \mathcal{A}^*(X)$ is non-zero. Pick a Riemannian metric on X, let vol_X be the associated Riemannian volume form, and let $*$ be the Hodge-star. Then we have

$$\omega \wedge *\omega = f \cdot vol_X$$

for $f\colon X \to [0, \infty)$ a smooth function. Let ρ be a compactly supported bump function with $\operatorname{supp}\rho \cap \operatorname{supp}\omega \neq \emptyset$. Then $T_\omega(\rho \cdot (*\omega)) > 0$, so $T_\omega \neq 0$. ■

Hence we view currents as generalized forms, in the same way that distributions are generalized functions. We call a current smooth if it is of the form T_ω for some ω. Later on, we will not always take care to distinguish between the smooth form ω and the smooth current T_ω. The space of smooth currents on X is a dense subspace of the space of all currents on X. This follows for example from [12, Theorem 12].

We grade $\mathcal{D}^*(X)$ such that the inclusion $\mathcal{A}^*(X) \to \mathcal{D}^*(X)$ preserve grading: $\mathcal{D}^k(X)$ are those currents which vanishes on $\mathcal{A}^i_c(X)$ for $i \neq \dim_{\mathbb{R}} X - k$. There is a multiplication

$$\wedge\colon \mathcal{D}^n(X) \otimes \mathcal{A}^k(X) \to \mathcal{D}^{n+k}(X)$$

defined as follows. For $T \in \mathcal{D}^*(X)$, $\omega \in \mathcal{A}^*(X)$ and $\sigma \in \mathcal{A}^*_c(X)$ we put

$$(T \wedge \omega)(\sigma) := T(\omega \wedge \sigma).$$

If $\mathcal{V}_*$ is a graded algebra, not only a graded vector space, we get an induced multiplication

$$\wedge\colon \mathcal{D}^*(X; \mathcal{V}_*) \otimes \mathcal{A}^*(X; \mathcal{V}_*) \to \mathcal{D}^*(X; \mathcal{V}_*).$$

We define the support of T as the largest closed set such that $T(\sigma) = 0$ whenever the interior of $\operatorname{supp}(\sigma) \cap \operatorname{supp}(T)$ is empty, i.e.,

$$\operatorname{supp}(T) := \bigcap_{U \text{ open } : \ T|_U = 0} (X \backslash U)$$

Let $f\colon W \to X$ be a smooth map, let $T \in \mathcal{D}^*(W)$ and let $\sigma \in \mathcal{A}^*_c(X)$. The expression

$$f_* T(\sigma) = T(f^* \sigma) \tag{2.1}$$

is valid whenever $\operatorname{supp}(T) \cap \operatorname{supp}(f^*\sigma)$ is compact. In particular, f_*T defines by (2.1) a linear map $\mathcal{A}^*_c(Y) \to \mathbb{C}$ if $f|_{\operatorname{supp}(T)}$ is proper. Hence we define the space

$$\mathcal{D}^*_{fvc}(W) = \{T \in \mathcal{D}^*(W) \ : \ f|_{\operatorname{supp}(T)} \text{ is proper}\},$$

and conclude that we have a well defined pushforward map

$$f_*\colon \mathcal{D}^*_{fvc}(W) \to \mathcal{D}^{*+\operatorname{codim} f}(X). \tag{2.2}$$

In particular, if $f\colon W \to X$ is proper, then (2.1) defines a map

$$f_*\colon \mathcal{D}^*(W) \to \mathcal{D}^{*+\operatorname{codim} f}(X). \tag{2.3}$$

We denote for a complex vector space V the space of currents with coefficients in V by $\mathcal{D}^*(X;V) = \mathcal{D}^*(X) \otimes_{\mathbb{C}} V$. We define the pushforward of currents with coefficients

$$f_*\colon \mathcal{D}^*(W;V) \to \mathcal{D}^{*+d}(X;V)$$

by tensoring (2.3) with the identity map of V. We have the projection formula:

Proposition 2.11. *Let $f\colon W \to X$ be a smooth map, let $\omega \in \mathcal{A}^*(X)$, and let $T \in \mathcal{D}^*(W)$ be such that $f|_{\operatorname{supp} T}$ is proper. Then*

$$f_*(T \wedge f^*\omega) = (f_*T) \wedge \omega.$$

Proof. For $\sigma \in \mathcal{A}^*_c(X)$ we have

$$\begin{aligned} f_*(T \wedge f^*\omega)(\sigma) &= (T \wedge f^*\omega)(f^*\sigma) \\ &= T(f^*\omega \wedge f^*\sigma) \\ &= f_*T(\omega \wedge \sigma) \\ &= ((f_*T) \wedge \omega)(\sigma) \end{aligned}$$

which proves the formula. ∎

It is natural to consider the boundary operation on currents b, defined by

$$b(T)(\sigma) = T(d\sigma).$$

Let ω be a k-form, and σ a compactly supported form. Then

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma$$

and using Stokes' theorem we get

$$\begin{aligned} T_{d\omega}(\sigma) &= \int_X d\omega \wedge \sigma \\ &= \int_X (-1)^{k+1} \omega \wedge d\sigma + \int_X d(\omega \wedge \sigma) \\ &= (-1)^{k+1} bT_\omega(\sigma) + \int_{\partial X} \omega \wedge \sigma. \end{aligned}$$

Define the operator

$$\mathbf{w}\colon \mathcal{D}^*(X) \to \mathcal{D}^*(X) \tag{2.4}$$

by defining it for each k on $\mathcal{D}^k(X)$ by $\mathbf{w} = (-1)^k$. Let $\iota_{\partial X}$ be the inclusion $\partial X \hookrightarrow X$ of the boundary with its induced orientation, and consider the integration current of ∂X defined by $\delta_{\partial X} = (\iota_{\partial X})_* 1$. From the above computation, we conclude that for smooth currents T_ω,

$$T_{d\omega} = \mathbf{w} b T_\omega + \delta_{\partial X} \wedge T_\omega.$$

We would like to define dT so that $dT_\omega = T_{d\omega}$. Hence we are forced to attempt

$$dT = \mathbf{w} b T + \delta_{\partial X} \wedge T. \tag{2.5}$$

However $\delta_{\partial X} \wedge T$ is not well defined for arbitrary $T \in \mathcal{D}^*(X)$, and so we will in general consider the exterior differential of currents only on manifolds without boundary. We note that using the techniques and language of Section 2.6 below, we can define the product $\delta_{\partial X} \wedge T$ whenever $WF(T) \cap N(\partial X) = \emptyset$. That is, we can define $T \wedge \delta_{\partial X}$ if the singularities of T meet with the boundary in a controlled way. In any case, (2.5) provides a useful expression for dT_ω on a manifold with boundary. If $\partial X = \emptyset$ we define

$$d \colon \mathcal{D}^*(X) \to \mathcal{D}^{*+1}(X)$$

by

$$dT = \mathbf{w} b T. \tag{2.6}$$

Lemma 2.12. *When f_*T is defined, we have the relation $bf_*T = f_*bT$.*

Proof. For $\sigma \in \mathcal{A}_c^*(X)$, we have

$$bf_*T(\sigma) = f_*T(d\sigma) = T(df^*\sigma) = bT(f^*\sigma) = f_*bT(\sigma).$$ ■

Proposition 2.13. *Let X be a manifold, W a manifold with boundary, $f \colon W \to X$ a smooth map of codimension d, and $T \in \mathcal{D}^*_{fvc}(W)$ a current for which $\delta_{\partial W} \wedge T$ is well defined. Then*

$$df_*T = (-1)^d \left(f_* \mathbf{w} b T + f_*(\delta_{\partial W} \wedge T) \right),$$

where $\delta_{\partial W} = (i_{\partial W})_* 1$, for $i_{\partial W}$ the inclusion of the boundary with the induced orientation, $i_{\partial W} \colon \partial W \to W$. In particular if $\partial W = \emptyset$, then for any current $T \in \mathcal{D}^*(W)$ with $f|_{\operatorname{supp} T}$ proper, we have

$$df_*T = (-1)^d f_* dT.$$

Proof. Since the degree of T and f_*T differ by d, we have

$$\begin{aligned} \mathbf{w} b f_* T &= \mathbf{w} f_* b T \\ &= (-1)^d f_* \mathbf{w} b T. \end{aligned}$$

This proves the second claim. The first claim then follows from (2.5). ■

Remark 2.14. To remedy that f_* does not commute with d, let us again consider the identity $bf_* = f_*b$. It means that the equality

$$\int_W \omega \wedge df^*\sigma = f_*(\omega)(d\sigma)$$

holds for all $\sigma \in \mathcal{A}_c^*(X)$ and $\omega \in \mathcal{A}^*(W)$; the same is true if we demand that ω is compactly supported, and are agnostic regarding the support of σ. Hence we should consider the current T'_ω which acts on $\sigma' \in \mathcal{A}_c^*(W)$ by $T'_\omega(\sigma') = \int_W \sigma' \wedge \omega$. If ω is homogeneous of degree k, then the number $T'_\omega(\sigma')$ depends only on the homogeneous component of σ' of degree $\dim W - k$. Therefore $T'_\omega = (-1)^{k(\dim W - k)}T_\omega$. We let $\mathbf{w}'$ be the operator defined on homogeneous $T \in \mathcal{D}^k(W)$ by $\mathbf{w}'T = (-1)^{k + k\dim W}$. The above discussion culminates in the equality

$$\mathbf{w}' f_* \mathbf{w}' \mathbf{w} b = \mathbf{w} b \mathbf{w}' f_* \mathbf{w}'. \tag{2.7}$$

In any case, (2.7) is readily verified directly using $b\mathbf{w}' = (-1)^{1+\dim W}\mathbf{w}'b$, together with $f_*\mathbf{w} = (-1)^d \mathbf{w} f_*$ and $(-1)^{d+\dim W} = (-1)^{\dim X}$, where d is the codimension of f.

Remark 2.15. If we were working with non-oriented manifolds, we would have to specify an orientation of f, i.e. a local matching of the possible local orientations of X and Y, in order to define f_*. This is developed in [12]. It amounts to giving an orientation of the stable normal bundle of f. Since X and Y are oriented, the stable normal bundle of f has a canonical orientation, and so $f\colon X \to Y$ has a canonical orientation.

When X is a complex manifold, the space of currents is bi-graded: $\mathcal{D}^{p,q}(X)$ are those currents which vanishes on compactly supported (p', q')-forms unless $p' + p = \dim_{\mathbb{C}} X = q' + q$. If $\partial X = \emptyset$, we extend ∂ to $\mathcal{D}^*$ by

$$\partial T = \mathbf{w} b_\partial T$$

where $b_\partial T(\sigma) = T(\partial\sigma)$ and $\mathbf{w}$ is the operator of (2.4). Similarly $\overline{\partial}$ is defined on $\mathcal{D}^*(X)$ by $\overline{\partial} = \mathbf{w} b_{\overline{\partial}}$. Then the inclusion $\mathcal{A}^{*,*}(X) \to \mathcal{D}^{*,*}(X)$, $\omega \mapsto T_\omega$ is a map of double complexes. If $f\colon X \to Y$ is holomorphic of complex codimension d, then f^* respects the bigrading, and it follows that f_* has bidegree (d, d). Furthermore, since the real codimension of f is even, the proof of (2.13) shows

$$\partial f_* T = f_* \partial T, \quad \overline{\partial} f_* T = f_* \overline{\partial} T.$$

2.3.3 Dolbeault–Grothendieck and de Rham quasi-isomorphisms

We note that $\mathcal{D}^k$ is a sheaf on X. I.e. if $U \subset V$, then we can view compactly supported functions on U as compactly supported functions on V by extending by 0, and so apply currents defined on V to them. If $\{U_i\}$ is a collection of open subsets of X, and for each i we have $T_i \in \mathcal{D}^k(U_i)$ such that for each pair i, j, $T_i|_{U_i \cap U_j} = T_j|_{U_i \cap U_j}$, then we can combine the currents T_i thus: Let $U = \cup_i U_i$, and let $\{\lambda_i\}$ be a partition of unity on U subordinate to a locally finite refinement of $\{U_i\}$. Then $T = \sum \lambda_i \cdot T_i$ is the unique current $T \in \mathcal{D}^k(U)$, such that for each i we get $T|_{U_i} = T_i$.

We denote the holomorphic de Rham complex by $\Omega^*(X)$. Thus

$$\Omega^n(X) \subset \mathcal{A}^{n,0}(X)$$

are those $(n,0)$-forms that have holomorphic coefficient functions. Equivalently, Ω^n is the kernel of $\overline{\partial} : \mathcal{A}^{n,0} \to \mathcal{A}^{n,1}$.

Theorem 2.16. *The natural map of complexes of sheaves on* $\mathbf{Man}_{\mathbb{C}}$

$$\Omega^p \to \mathcal{A}^{p,*},$$

and the natural maps of complexes of sheaves on X

$$\mathcal{A}^* \to \mathcal{D}^*, \quad \text{and} \quad \mathcal{A}^{p,*} \to \mathcal{D}^{p,*}$$

are quasi-isomorphisms.

Proof. The first statement is known as the $\overline{\partial}$- Poincare lemma, or as the Dolbeault-Grothendieck lemma. For a proof, we refer to [23, p.25]. The second and third assertion are proven at [23, p.382–385]. The second statement was essentially first proven by de Rham, see [12, Theorem 14]. We note that both the first and third statements follow from the version of the Dolbeault-Grothendieck lemma of [11, p.28]. ∎

2.4 Grading-conventions and the Hodge filtration

Let $\mathcal{V}_*$ and $\mathcal{W}_*$ be degree-wise finite dimensional evenly graded complex and real vector spaces respectively. We define

$$\mathcal{A}^n(X;\mathcal{V}_*) = \bigoplus_j \mathcal{A}^{n+j}(X;\mathcal{V}_j) \tag{2.8}$$

$$\mathcal{D}^n(X;\mathcal{V}_*) = \bigoplus_j \mathcal{D}^{n+j}(X;\mathcal{V}_j) \tag{2.9}$$

where $A^{n+j}(X;\mathcal{V}_j) = \mathcal{A}^{n+j}(X) \otimes_{\mathbb{C}} \mathcal{V}_j$, and similarly for the other groups. We stress that we will never mean by $\mathcal{A}^n(X;\mathcal{V}_*)$ the space $\mathcal{A}^n(X) \otimes \mathcal{V}_*$, unless $\mathcal{V}_* = \mathcal{V}_0$ of course. We note that with our convention, we get

$$H^n(X;\mathcal{V}_*) = \bigoplus_j H^{n+j}(X;\mathcal{V}_j).$$

Let X be a complex manifold. We define the Hodge filtration by

$$F^p\mathcal{D}^n(X) = \bigoplus_{i \geqslant p} \mathcal{D}^{i,n-i}(X)$$

$$F^p\mathcal{A}^n(X) = \bigoplus_{i\geqslant p} \mathcal{A}^{i,n-i}(X).$$

We always mean the Hodge filtration when we write $F^p\mathcal{A}^n$ or $F^p\mathcal{D}^n$, unless explicitly saying otherwise. We have the subcomplex

$$\Omega^{\geqslant p}(X) \subset F^p\mathcal{A}^*(X)$$

which in degree k is $\Omega^k(X)$ if $k \geqslant p$, and 0 if $k < p$. We further define

$$\begin{aligned}
\Omega^{\geqslant p}(X;\mathcal{V}_*) &= \bigoplus_j \Omega^{\geqslant p+j}(X;\mathcal{V}_{2j}) \\
F^p\mathcal{A}^n(X;\mathcal{V}_*) &= \bigoplus_j F^{p+j}\mathcal{A}^{n+2j}(X;\mathcal{V}_{2j}) \qquad (2.10)\\
\frac{\mathcal{A}^n}{F^p}(X;\mathcal{V}_*) &= \bigoplus_j \frac{\mathcal{A}^{n+2j}(X;\mathcal{V}_{2j})}{F^{p+j}\mathcal{A}^{n+2j}(X;\mathcal{V}_{2j})} \\
F^p\mathcal{D}^n(X;\mathcal{V}_*) &= \bigoplus_j F^{p+j}\mathcal{D}^{n+2j}(X;\mathcal{V}_{2j}) \\
\frac{\mathcal{D}^n}{F^p}(X;\mathcal{V}_*) &= \bigoplus_j \frac{\mathcal{D}^{n+2j}(X;\mathcal{V}_{2j})}{F^{p+j}\mathcal{D}^{n+2j}(X;\mathcal{V}_{2j})}.
\end{aligned}$$

2.5 Cohomology of filtered forms and currents

Let $\mathcal{V}_*$ be an evenly graded complex vector-space. Let $\frac{\mathcal{A}^n}{F^p}(\mathcal{V}_*)$ and $F^p\mathcal{A}^n(\mathcal{V}_*)$ be the sheaves

$$U \mapsto \frac{\mathcal{A}^n}{F^p}(U;\mathcal{V}_*), \quad U \mapsto F^p\mathcal{A}^n(U;\mathcal{V}_*),$$

respectively, where the right hand sides are defined by (2.10).

Lemma 2.17. *The sheaves $F^p\mathcal{A}^n(\mathcal{V}_*)$ and $\frac{\mathcal{A}^n}{F^p}(\mathcal{V}_*)$ are acyclic.*

Proof. Both $F^p\mathcal{A}^n(\mathcal{V}_*)$ and $\frac{\mathcal{A}^n}{F^p}(\mathcal{V}_*)$ are fine, in the sense that they are sheaves of modules over $\mathcal{A}^0(\mathbb{C})$, which admits a partition of unity. Fine sheaves are well known to be acyclic. See for example [51, 4.36]. ∎

We will now describe the hypercohomology of the complexes of sheaves

$$\begin{aligned}
\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*) &:= \left(\cdots \xrightarrow{d} \tfrac{\mathcal{A}^0}{F^p}(\mathcal{V}_*) \xrightarrow{d} \tfrac{\mathcal{A}^1}{F^p}(\mathcal{V}_*) \xrightarrow{d} \cdots \right), \qquad (2.11)\\
F^p\mathcal{A}^*(\mathcal{V}_*) &:= \left(\cdots \xrightarrow{d} F^p\mathcal{A}^0(\mathcal{V}_*) \xrightarrow{d} F^p\mathcal{A}^1(\mathcal{V}_*) \xrightarrow{d} \cdots \right)
\end{aligned}$$

on a complex manifold X. We define:

$$\begin{aligned}
\widetilde{F}^p\mathcal{A}^n(X;\mathcal{V}_*) &:= F^p\mathcal{A}^n(X;\mathcal{V}_*) + d\mathcal{A}^{n-1}(X;\mathcal{V}_*), \qquad (2.12)\\
d^{-1}(F^p\mathcal{A}^{n+1}(X;\mathcal{V}_*))^n &:= \{\omega \in \mathcal{A}^n(X;\mathcal{V}_*) \ : \ d\omega \in F^p\mathcal{A}^{n+1}(X;\mathcal{V}_*)\}\\
F^p\mathcal{A}^n(X;\mathcal{V}_*)_{cl} &:= \{\omega \in F^p\mathcal{A}^n(X;\mathcal{V}_*) \ : \ d\omega = 0\}.
\end{aligned}$$

Proposition 2.18. *We have*

$$H^n\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \simeq \frac{d^{-1}(F^p\mathcal{A}^{n+1}(X;\mathcal{V}_*))^n}{\widetilde{F}^p\mathcal{A}^n(X;\mathcal{V}_*)}$$

and

$$H^n\left(X; F^p\mathcal{A}^*(\mathcal{V}_*)\right) \simeq \frac{F^p\mathcal{A}^n(X;\mathcal{V}_*)_{cl}}{dF^p\mathcal{A}^{n-1}(X;\mathcal{V}_*)}$$

where the left hand sides are hyper-cohomology groups.

Proof. The previous lemma implies, for example by [51, Proposition 8.8], that the hyper-cohomology group $H^n\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right)$ can be computed as

$$H^n\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \simeq \frac{\ker\left(d\colon \frac{\mathcal{A}^n(X;\mathcal{V}_*)}{F^p\mathcal{A}^n(X;\mathcal{V}_*)} \to \frac{\mathcal{A}^{n+1}(X;\mathcal{V}_*)}{F^p\mathcal{A}^{n+1}(X;\mathcal{V}_*)}\right)}{\mathrm{Im}\left(d\colon \frac{\mathcal{A}^{n-1}(X;\mathcal{V}_*)}{F^p\mathcal{A}^{n-1}(X;\mathcal{V}_*)} \to \frac{\mathcal{A}^n(X;\mathcal{V}_*)}{F^p\mathcal{A}^n(X;\mathcal{V}_*)}\right)}.$$

The kernel are the classes of forms modulo F^p which map to $F^p\mathcal{A}^{n+1}(X;\mathcal{V}_*)$ under d. The image are the classes modulo F^p of exact forms. This establish the first isomorphism of the proposition. Similarly we can write

$$H^n\left(X; F^p\mathcal{A}^*(\mathcal{V}_*)\right) \simeq \frac{\ker\left(d\colon F^p\mathcal{A}^n(X;\mathcal{V}_*) \to F^p\mathcal{A}^{n+1}(X;\mathcal{V}_*)\right)}{\mathrm{Im}\left(d\colon F^p\mathcal{A}^{n-1}(X;\mathcal{V}_*) \to F^p\mathcal{A}^n(X;\mathcal{V}_*)\right)},$$

which implies the second isomorphism. ∎

Consider now the short exact sequence of complexes of sheaves

$$0 \longrightarrow F^p\mathcal{A}^*(\mathcal{V}_*) \xrightarrow{\ i\ } \mathcal{A}^*(\mathcal{V}_*) \xrightarrow{\ \pi\ } \tfrac{\mathcal{A}^*}{F^p}(\mathcal{V}_*) \longrightarrow 0.$$

For each complex manifold X, there is an associated long exact sequence of cohomology groups

$$H^n(X;F^p\mathcal{A}^*(\mathcal{V}_*)) \xrightarrow{\ i_*\ } H^n(X;\mathcal{V}_*) \xrightarrow{\ \pi_*\ } H^n\left(X; \tfrac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \xrightarrow{\ d\ }$$

$$H^{n+1}(X;F^p\mathcal{A}^*(\mathcal{V}_*)) \xrightarrow{\ i_*\ } H^{n+1}(X;\mathcal{V}_*) \xrightarrow{\ \pi_*\ } H^{n+1}\left(X; \tfrac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \xrightarrow{\ d\ } \cdots.$$

Proposition 2.19. *The connecting homomorphism d is, using the descriptions of the groups given by Proposition 2.18, induced by the exterior differential*

$$d\colon d^{-1}(F^p\mathcal{A}^{n+1}(X;\mathcal{V}_*))^n \to F^p\mathcal{A}^{n+1}(X;\mathcal{V}_*).$$

Proof. In the proof, we will for the sake of clarity denote the connecting homomorphism d by δ. Since our complexes of sheaves are acyclic, we can describe δ concretely in terms of cycles as follows: Let $[\alpha] \in H^n\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right)$. We can write $\alpha = \pi(\beta)$ for some $\beta \in \mathcal{A}^n(X; \mathcal{V}_*)$. Since

$$\pi(d\beta) = d\alpha = 0 \in \frac{\mathcal{A}^{n+1}}{F^p}(X; \mathcal{V}_*)$$

we have $d\beta = i_*(\gamma)$ for some $\gamma \in F^p\mathcal{A}^{n+1}(X; \mathcal{V}_*)$. Then the connecting homomorphism δ is by definition

$$\delta[\alpha] = [\gamma].$$

From this description of δ we see that $\delta[\alpha]$ is represented by $d\alpha$. This proves the proposition. For the readers convenience, we remark that the class of γ is independent of the choice of β for the following reason: If $\pi(\beta) = \pi(\beta')$, then $\beta - \beta' \in F^p\mathcal{A}^n(X; \mathcal{V}_*)$. Hence $d\beta' - d\beta \in dF^p\mathcal{A}^{n+1}(X; \mathcal{V}_*)$, which is to say that $d\beta$ and $d\beta'$ represent the same class in $H^{n+1}(X; F^p\mathcal{A}^*(\mathcal{V}_*))$. ■

We define complexes of sheaves on X, $\frac{\mathcal{D}^*}{F^p}(\mathcal{V}_*)$ and $F^p\mathcal{D}^*(\mathcal{V}_*)$, by replacing $\mathcal{A}$ with $\mathcal{D}$ in (2.11). These too are complexes of modules of the sheaf $\mathcal{A}^0$. Hence replacing $\mathcal{A}$ by $\mathcal{D}$ in (2.12) and the proof of Proposition 2.18, we get

$$H^n\left(X; \frac{\mathcal{D}^*}{F^p}(\mathcal{V}_*)\right) \simeq \frac{d^{-1}(F^p\mathcal{D}^n(X; \mathcal{V}_*))^{n-1}}{\widetilde{F}^p\mathcal{D}^n(X; \mathcal{V}_*)}$$

and

$$H^n\left(X; F^p\mathcal{D}^*(\mathcal{V}_*)\right) \simeq \frac{F^p\mathcal{D}^n(X; \mathcal{V}_*)_{cl}}{dF^p\mathcal{D}^{n-1}(X; \mathcal{V}_*)}.$$

Finally, we have the following important result:

Theorem 2.20. *The maps of complexes of sheaves on X*

$$F^p\mathcal{A}^*(\mathcal{V}_*) \to F^p\mathcal{D}^*(\mathcal{V}_*)$$
$$\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*) \to \frac{\mathcal{D}^*}{F^p}(\mathcal{V}_*)$$

are quasi-isomorphisms.

Proof. Both statements are trivial consequences of Theorem 2.16, which states that the map of complexes of sheaves on X

$$\mathcal{A}^{p,*} \to \mathcal{D}^{p,*}$$

is a quasi isomorphism. ■

Remark 2.21. It is of course not true that $\mathcal{A}^n$ and $\mathcal{D}^n$ are quasi-isomorphic. Nor are the complexes $\mathcal{A}^{\geqslant n}$ and $\mathcal{D}^{\geqslant n}$ quasi-isomorphic. This is the basic reason why the currential K-theory of [19] differ from differential K-theory. [1] Theorem 2.20 will however imply that currential Hodge filtered cohomology theories are canonically isomorphic with smooth Hodge filtered cohomology theories. This is the fundamental reason why Hodge filtered cohomology theories have good pushforward maps, not only along submersions, but along arbitrary proper maps.

2.6 Computations with wave front sets

One of the drawbacks of currents is that they cannot, in general, be pulled back along a smooth map $g\colon Y \to X$. Of course, the smooth currents can be pulled back, but $g^*\colon \mathcal{A}^*(X) \to \mathcal{A}^*(Y)$ does in general not extend continuously to a map $\mathcal{D}^*(X) \to \mathcal{D}^*(Y)$. That is; we can find a smooth map $f\colon X \to Y$ and a current $T \in \mathcal{D}^k(Y)$ so that if we let ω_t be forms on Y with $\lim_{t\to 0}\omega_t = T$, the limit $\lim_{t\to 0} f^*\omega_t$ depends on the particular family ω_t, not just T. To us the most important fact is that when $f\colon Z \to X$ is proper and transverse to g, then there is a good pullback g^* on the image of $f_*\colon \mathcal{A}^*(Z) \to \mathcal{D}^*(X)$. In order to define g^*T we need the current $T \in \mathcal{D}^*(X)$ to look smooth from the point of view of $\mathrm{Im}\,(g)$. Hence de Rham defines the singular locus of T, $\operatorname{sing\,supp} T \subset X$, as the set of points x such that $T|_U$ is not smooth for any open U containing x. Clearly, then g^*T is defined whenever $\mathrm{Im}\, g \cap \operatorname{sing\,supp} T = \emptyset$. This is not good enough for our purposes, and we therefore now introduce more sophisticated analytic tools, from microlocal analysis, which refine the singular locus. We will not provide all the definitions, since our subject is not microlocal analysis, and building up the whole context would take us to far astray. We will rather just state what we need and refer to precise locations of [28, Chapter 8].

Let for $x \in X$ the 0 element of $T^*_x X$ be denoted by 0_x, and define $0_X = \cup_{x\in X}\{0_x\}$. For a distribution $T \in \mathcal{D}^0(X)$, the wave front set, $WF(T)$, is defined in [28, Definition 8.1.2], see also the comments preceding example 8.2.5. [28, p.265]. It is a conic set (in the sense that it is closed under scalar multiplication)

$$WF(T) \subset T^*X \backslash 0_X.$$

If we view currents as distributional sections of the bundle $\bigwedge^*(T^*X)$, then the comments preceding [28, example 8.2.5, p.265] also defines the wave front set of a current. Let us expand that. For $U \subset X$ a coordinate patch, $\mathcal{D}^*(U)$ is a finite sum of copies of $\mathcal{D}^0(U)$. To be concrete, we can write a current T locally as $T = \sum_I T_I dx_I$ for distributions $T_I \in \mathcal{D}^0(X)$ and we then define its wave front set by

$$WF(T) = \bigcup_I WF(T_I).$$

The wave front set is a measure of the singularities of a current, which takes into account also the “directions” of the singularities. For π_X the projection onto X, we have

$$\pi_X(WF(T)) = \operatorname{sing\,supp}(T).$$

[1] We mention K-theory in particular only because currential K-theory is the only “currential differential cohomology theory” we could find in the literature.

In particular, if $T = T_\omega$ is a smooth current, then $WF(T) = \emptyset$. Given a closed cone $\Gamma \subset T^*X \backslash 0_X$, we define

$$\mathcal{D}^*_\Gamma(X) = \{T \in \mathcal{D}^*(X) \ : \ WF(T) \subset \Gamma\}.$$

There is a natural topology for the space $\mathcal{D}^*_\Gamma(X)$, see [28, Definition 8.2.2].

The topology of $\mathcal{D}^*_\Gamma(X)$ is finer than the weak* topology, so that the inclusion $\mathcal{D}^*_\Gamma(X) \to \mathcal{D}^*(X)$ is continuous. As an immediate consequence of [28, Theorem 8.2.3], we have:

Theorem 2.22. *$\mathcal{A}^*(X)$ is dense in $\mathcal{D}^*_\Gamma(X)$.* ∎

Let $f : Z \to Y$ be a smooth map. We define the conormal bundle of f by

$$N^*(f) = \{(z, v) \in Z \times_X T^*X \ : \ v(D(T_z Z)) = 0\}.$$

Then the normal set of f, N_f is defined as the image under the projection $N^*(f) \to T^*X$ of $N^*(f)$.

We now state the key result.

Theorem 2.23. *There is a unique way to define $f^*T \in \mathcal{D}^*(Z)$ for those $T \in \mathcal{D}^*(X)$ which satisfy $WF(T) \cap N_f = \emptyset$ such that the following is true:*

- *If $T = T_\omega$ is smooth, then $f^*T_\omega = T_{f^*\omega}$.*
- *If $\Gamma \subset T^*X$ is a closed cone in $T^*X \backslash 0_{sec}(X)$ such that $\Gamma \cap N(f) = \emptyset$, then f^* defines a continuous map*

$$f^* : \mathcal{D}^*_\Gamma(X) \to \mathcal{D}^*_{f^*\Gamma}(Z).$$

In particular, whenever $WF(T) \cap N(f) = \emptyset$ we have $WF(f^*T) \subset f^*WF(T)$.

Proof. It suffices to consider the case when $X \subset \mathbb{R}^n$ is open, since we can patch locally defined pullbacks together using a partition of unity. The case when T is a distribution is [28, Theorem 8.2.4], and all we need to do is to translate this result into our context. We can write $T = \sum_I T_I dx_I$ for distributions T_I. If

$$WF(T) \cap N_f = \emptyset,$$

then $WF(T_I) \cap N_f = \emptyset$, so that we can define

$$f^*T = \sum_I f^*(T_I) f^*(dx_I).$$

That this is the unique way to define f^*T such that the stated properties hold now follows at once from the case of distributions. ∎

We now establish the version of the Schwartz kernel theorem which we will need. Essentially, we will perform the “obvious” verification Hörmander indicates after [29, Theorem 18.1.34’], only we omit the half-orientation bundles, since we assume our manifolds are oriented. For $\omega_Z \in \mathcal{A}^*(Z)$ and $\omega_X \in \mathcal{A}^*(X)$ we mean by $\omega_Z \otimes \omega_X \in \mathcal{A}^*(Z \times X)$ the form which at (z, x) is the multilinear map deifned by

$$\omega_Z \otimes \omega_X(v_z^1, \ldots, v_z^{k_1}, v_x^1, \ldots, c_x^{k_2}) = \omega_Z(v_z^1, \ldots, v_z^{k_1}) \cdot \omega_X(v_x^1, \ldots, v_x^{k_2}).$$

Clearly for π_Z and π_X the projections onto Z and X respectively, we have

$$\omega_Z \otimes \omega_X = \pi_Z^* \omega_Z \wedge \pi_X^* \Omega_X.$$

If $T_Z \in \mathcal{D}^*(Z)$ and $T_X \in \mathcal{D}^*(X)$, we can also define $T_Z \otimes T_X$ by

$$(T_Z \otimes T_X)(\sigma_Z \otimes \sigma_X) = T_Z(\sigma_Z) \cdot T_X(\sigma_X). \tag{2.13}$$

Alternatively, since $\pi_Z^* T^* Z \cap \pi_X^* T^* X = 0_{Z \times X}$, we can apply Theorem 2.23 to define the tensor product by

$$(T_Z \otimes T_X)(\sigma_Z \otimes \sigma_X) = \pi_Z^* T_Z \wedge \pi_X^* T_X.$$

Theorem 2.24. *Given a kernel $K \in \mathcal{D}^*(Z \times X)$, the associated linear transformation*

$$\mathscr{K} : \mathcal{A}_c^*(Z) \to \mathcal{D}^*(X)$$

given by

$$\mathscr{K}(\sigma)(\tau) = K(\sigma \otimes \tau)$$

is continuous. This defines a linear bijection between $\mathcal{D}^*(Z \times X)$ and continuous linear maps $\mathcal{A}_c^*(Z) \to \mathcal{D}^*(X)$.

Proof. That $\mathscr{K}$ is continuous is obvious. We can assume that Z and X are open subsets of Euclidean spaces, since we can patch things together using partitions of unity. Using the Euclidean coordinates

$$z_1, \ldots z_{\dim Z}, x_1, \ldots, x_{\dim X}$$

we can write $K \in \mathcal{D}^*(Z \times X)$ as a finite sum over pairs of increasing multi-indices

$$K = \sum_{I = (I_Z, I_X)} u_I dz_{I_Z} \otimes dx_{I_X}$$

for distributions $u_I \in \mathcal{D}^0(Z \times X)$. Identifying the forms $dz_i \in T^* Z$ with $\pi_Z^* dz_i \in T^*(Z \times X)$, and similarly for the dx_i, we can replace $\otimes$ by $\wedge$. Then $\mathscr{K}$ is entirely characterized by

$$(\mathscr{K}(f dz_J))(f' dx_{J'}) = \sum_I u_I \left(f \otimes f' dz_{I_Z} \wedge x_{I_X} \wedge dz_J \wedge dx_{J'} \right).$$

Write $dx = dx_1 \wedge \cdots \wedge dx_{\dim X}$, and similarly for dz. Let I_Z' and I_X' be the unique multi-indices such, such that

$$dz_{I_Z} \wedge dx_{I_X} \wedge dz_{I_Z'} \wedge dx_{I_X'} = \pm dz \wedge dx.$$

The natural pairing of $\mathcal{D}^*(X)$ with $\mathcal{A}^*_c(X)$ restrict to a pairing of $\mathcal{D}^0(X) \cdot dx_{I_X}$ with $\mathcal{A}^0_c(X) \cdot dx_{I'_X}$ which induces an isomorphism

$$\begin{aligned} \mathcal{D}^0(X) \cdot dx_{I_X} &\simeq \left(\mathcal{A}^0_c(X) \cdot dx_{I'_X}\right)' \\ u \cdot dx_{I_X} &\mapsto (f \cdot dx_{I'_X} \mapsto u(f \cdot dx_{I_X} \wedge dx_{I'_X})). \end{aligned}$$

Let $\mathscr{K}_I$ be the map $\mathcal{A}^0_c(Z) \to \mathcal{D}^0(X)$ with Schwartz kernel $u_I \cdot dz$, i.e.

$$\mathscr{K}_I(f) = (\sigma_X \mapsto u_I(f dz \otimes \sigma_X)) .$$

The correspondence $u_I \mapsto \mathscr{K}_I$ is a bijection between elements of $\mathcal{D}^0(Z \times X)$ and continuous linear maps $\mathcal{A}^0_c(Z) \to \mathcal{D}^0(X)$ by [28, Theorem 5.2.1.]. [2] Then

$$\begin{tikzcd} \mathcal{A}^0_c(Z) \arrow[r, "\mathscr{K}_I"] \arrow[d, "\cdot dz_{I'_Z}"'] & \mathcal{D}^0(X) \arrow[d, "\cdot dx_{I_X}"] \\ \mathcal{A}^0_c(Z) \cdot dx_{I'_Z} \arrow[r, "\mathscr{K}"'] & \mathcal{D}^0(X) \cdot dx_{I_X} \end{tikzcd}$$

commutes for every pair of multi-indices $I = (I_Z, I_X)$. Summing over all pairs, the theorem is proven. ■

Remark 2.25. Not all maps $\mathcal{A}^*_c(Z) \to \mathcal{D}^*(X)$ extend to $\mathcal{A}^*(Z)$. However, all maps $\mathcal{A}^*(Z) \to \mathcal{D}^*(X)$ can be restricted to $\mathcal{A}^*_c(Z)$, and so correspond to a Schwartz kernel.

Given a map $f\colon Z \to X$, let $\iota_f\colon Z \to Z \times X$ be the graph map $\iota_f(z) = (z, f(z))$, and let $\iota'_f\colon Z \to X \times Z$ be the transposed graph map, given by $\iota'_f(z) = (f(z), z))$.

Lemma 2.26. *Let $f\colon Z \to X$ be smooth. Then the continuous map*

$$f^*\colon \mathcal{A}^*_c(X) \to \mathcal{A}^*(Z) \to \mathcal{D}^*(Z)$$

has Schwartz kernel $(\iota'_f)_*1$. If f furthermore is proper, the map $f_*\colon \mathcal{A}^*_c(Z) \to \mathcal{D}^*(X)$ has Schwartz kernel $(\iota_f)_*1$.

Proof. Let $\omega \in \mathcal{A}^*_c(X)$, and think of $f^*\omega$ as a current. For $\sigma \in \mathcal{A}^*_c(Z)$ we get:

$$\begin{aligned} (f^*(\omega))(\sigma) &= \int_Z f^*\omega \wedge \sigma \\ &= \int_Z (\iota'_f)^*(\omega \otimes \sigma) \\ &= ((\iota'_f)_*1)(\omega \otimes \sigma). \end{aligned}$$

[2] See also [50, Theorem 51.7], where the topologies are chosen such that the bijection is an isomorphism of topological vector spaces.

Similarly

$$\begin{aligned}(f_*\sigma)(\omega) &= \int_Z \sigma \wedge f^*\omega \\ &= \int_Z (\iota_f)^*(\sigma \otimes \omega) \\ &= ((\iota_f)_* 1)(\sigma \otimes \omega).\end{aligned}$$

■

Theorem 2.27. *Let $\mathscr{K} : \mathcal{A}_c^*(Z) \to \mathcal{D}^*(X)$ be the map with Schwartz kernel $K \in \mathcal{D}^*(Z \times X)$, as in the Theorem (2.24). We have*

$$WF(\mathscr{K}(\sigma)) \subset \{v_x \in T_x^*X \;:\; (0_z, v_x) \in WF(K) \textit{ for some } z \in \operatorname{supp}\sigma\}.$$

Proof. We can assume that Z and X are open subset of Euclidean spaces. Then this is essentially [28, Theorem 8.2.12.], which is the case of distributions. For $\sigma \in \mathcal{A}_c^*(Z)$, write $\sigma = \sum_J f_J dx_J$. In the notation of the proof of Theorem 2.24, if $v \in WF(\mathscr{K}\sigma)$, then $v \in WF(\mathscr{K}_I f_{J_I})$ for some I. Then [28, Theorem 8.2.12.] implies that for some $z \in \operatorname{supp} f_{J_I}$, we have $(0_z, v) \in WF(u_I)$. Realizing that then $z \in \operatorname{supp}\sigma$, and $(0_z, v) \in WF(K)$, the theorem is proven. ■

Theorem 2.28. *Let $K \in \mathcal{D}^*(Z \times X)$. There is a unique way to define $\mathscr{K}(T)$ for all $T \in \mathcal{D}_c^*(Z)$ satisfying $WF(T) \cap WF'(K)_Z = \emptyset$ where*

$$WF'(K)_Z = \{\xi_z, \in T^*Z \;:\; (-\xi_z, 0_x) \in WF(K) \textit{ for some } x \in X\}$$

so that for each compact $L \subset Z$, and closed conic $\Gamma \subset T^*(Z \times X)$ with $\Gamma \cap WF(K) = \emptyset$, the map $\mathcal{D}_c^*(L) \cap \mathcal{D}_\Gamma^*(Z) \to \mathcal{D}^*(X)$ defined by $\mathscr{K}$ is continuous. We have

$$WF(\mathscr{K}T) \subset WF(K)_X \cup WF'(K) \circ WF(T)$$

where

$$WF'(K) = \{(\xi_x, \eta_z) \in T^*X \times T^*Z \;:\; (\xi_x, -\eta_z) \in WF(K)\}$$

is considered as a relation mapping sets in $T^*Z\backslash 0_Z$ to sets in $T^*X\backslash 0_X$.

Proof. Using the adaptation techniques of the proof of Theorem (2.24), this follows from [28, Theorem 8.2.13]. ■

As an immediate consequence, we get:

Theorem 2.29. *If $K \in \mathcal{D}^*(Z \times X)$ is a smooth current, then for each $T \in \mathcal{D}_c^*(Z)$, $\mathscr{K}T$ is defined and smooth.*

Theorem 2.30. *Let $f : Z \to X$ be a smooth and proper map. For $\omega \in \mathcal{A}^*(Z)$ we have*

$$WF(f_*\omega) \subset N_f.$$

Proof. If f is an embedding, then this follows from [28, example 8.2.5.]. Hence $WF((\iota_f)_*1) = N_{\iota_f}$. Then the theorem follows from Theorem 2.27 together with Lemma 2.26. ■

Theorem 2.31. *Let g be a proper smooth map of codimension d, and let f be smooth with $f \pitchfork g$. We consider the cartesian diagram*

$$\begin{tikzcd} Z \times_X Y \arrow[r, "g'"] \arrow[d, "f'"'] & Z \arrow[d, "f"] \\ Y \arrow[r, "g"'] & X. \end{tikzcd}$$

Then we have an equality of continuous $\mathbb{C}$-linear maps

$$g^* \circ f_* = f'_* \circ g'^* \colon \mathcal{A}^*(Z) \to \mathcal{D}^{*+d}(Y).$$

Proof. It suffices to prove this equality on $\mathcal{A}_c^*(Z)$. We start by proving the special case that if f and g are embeddings, then

$$g^* f_*(1) = f'_* g'^* 1. \tag{2.14}$$

This is essentially [28, example 8.2.8.], but we must adapt the proof to our context. This is a local question; if it holds in a neighborhood of each point $z \in Z$, then we use a partition of unity to show that it holds everywhere. Locally in X we choose coordinates $(x_1, \ldots, x_{\dim X})$ so that Z is the 0-locus of the first $n' = \dim X - \dim Z$ coordinates, and Y is the 0-locus of the last $n''' := \dim X - \dim Y$ coordinates. We put $n'' = \dim Z + \dim Y - \dim X$. Write

$$\begin{aligned} x' &= (x_1, \ldots x_{n'}), \\ x'' &= (x_{n'+1}, \ldots x_{\dim Y}) \\ x''' &= (x_{\dim Y+1}, \ldots x_{\dim X}) \text{ and} \\ x &= (x', x'', x'''). \end{aligned}$$

Then $z = (x'', x''')$ are coordinates for Z, (x', x'') are coordinates for Y, and x'' are coordinates for $Z \times_X Y$. We also use the notation $dx' = dx_1 \wedge \cdots \wedge dx_{n'}$, and similarly for x'' and x'''. We analyze $f_* 1$. For $\sigma \in \mathcal{A}_c^*(X)$, we can write $\sigma = \sum_I \sigma_I dx_I$. Then

$$f_* 1(\sigma) = f_* 1(\sigma_{(n'+1, \ldots, \dim X)} dx'' \wedge dx'''),$$

so $f_* 1 = T \cdot dx'$ for some $T \in \mathcal{D}^0(X)$. Let $h(x')$ be a compactly supported real valued smooth function of x', satisfying $\int h(x')dx' = 1$. Put $h_t(x) = h(x'/t)t^{-n'}$. Then we have $\int h_t(x)dx' = 1$. Note $\operatorname{supp} h_t \subset t \operatorname{supp} h$, so that $\operatorname{supp}(\lim_{t \to 0} h_t) = Z$. We claim $T dx' = \lim_{t \to 0} h_t dx'$. To prove this, let $\sigma = \sigma_0 dx'' \wedge dx''' \in \mathcal{A}_c^{n''+n'''}(X)$. Put $S_t(x'', x''') = \sup_{x' \in \operatorname{supp} h_t} \sigma_0(x', x'', x''')$. Then $\lim_{t \to 0} S_t(x'', x''') = \sigma_0(0, x'', x''')$. We get

$$\int h_t(x)\sigma_0(x)dx' \leqslant \int h_t(x) S_t(x'', x''')dx' = S_t(x'', x''') \to \sigma_0(0, x'', x''').$$

Using instead infimum, we similarly get $\lim_{t\to 0}\int h_t(x)\sigma_0(x)dx' \geqslant S_t(x'',x''')$. Since $\sigma_0(0,x'',x''') = (g^*(\sigma))(x'',x''')$, we have

$$(h_t dx')(\sigma) = \int_Z \left(\int h_t(x)\sigma_0(x)dx'\right) dx'' \wedge dx''' \to \int_Z g^*\sigma dx'' \wedge dx''',$$

proving $f_* 1 = \lim_{t\to 0} h_t dx'$. Then $g^* f_* 1$ can be computed as the limit $\lim_{t\to 0} g^* h_t dx'$. Since $(g^* h_t)(x',x'') = h_t(x',x'',0)$, the techniques used above yields

$$\lim_{t\to 0} g^* h_t dx' = f'_* 1.$$

This proves equation (2.14). Using lemma (2.26), the Schwarts kernel of f_* is

$$K_1 := (\iota_f)_* 1 \in \mathcal{D}^*(Z \times X),$$

and the Schwarts kernel of g^* is

$$K_2 := (\iota'_g)_* 1.$$

We will compute the kernel K associated to

$$\mathscr{K} := \mathscr{K}_2 \circ \mathscr{K}_1 = g^* \circ f_*$$

The technique for computing K from K_1 and K_2 is described prior to [28, Theorem 8.2.14]. We start by showing that condition [28, (8.2.14), p.269], which we will refer to as condition (8.2.14), holds. The condition says that if $(\eta_x) \in T^*X$ is such that

$$(0_z, -\eta_x) \in WF(K_1) \subset T^*(Z \times X) = T^*Z \times T^*X$$

for some $z \in Z$, then there is no $y \in Y$ for which $(0_y, -\eta_x) \in WF(K_2)$. We have $WF(K_1) = N_{\iota_f}$ and $WF(K_2) = N_{\iota'_g}$. So, given such a covector η_x, there must be $z' \in Z$ and $y' \in Y$ with $g(y') = f(z') = x$. Then, that $(0_y, \eta_{g(y')})$ be normal to ι'_g means that $g^*_{y'}\eta_{g(y')} = 0$, and similarly for f. Hence η_x vanishes on $Dg(T_{z'}Z + D_f(T_{y'}Y) = T_xX$, where we use $g \pitchfork f$, and so $\eta_x = 0_x$. But $(0_y, 0_x) \notin WF(K_2)$, since the 0-section is excluded from the wave front set by definition. This establish the condition, and hence that $\mathscr{K}_2 \circ \mathscr{K}_1 = g^* \circ f_* \colon \mathcal{A}^*_c(Z) \to \mathcal{D}^*_c(Y)$ is a continuous map.

Hörmander then tells us how to compute K from K_1 and K_2: Consider first $K_1 \otimes K_2 \in \mathcal{D}^*(Z \times X \times X \times Y)$, which for us is $(\iota_f \times \iota'_g)_* 1$. Since the condition (8.2.14) holds, $K_1 \otimes K_2$ can be pulled back along $id_Z \times \Delta \times id_Y$, where $\Delta\colon X \to X \times X$ is the diagonal. Then K is the projection to $\mathcal{D}^*(Z \times Y)$ of this current. To compute the pullback, we use the following cartesian diagram

$$\begin{tikzcd}
Z \times_X Y \arrow[r, "g' \times f'"] \arrow[d, "j"'] & Z \times Y \arrow[d, "\iota_f \times \iota'_g"] \\
Z \times X \times Y \arrow[r, "id_Z \times \Delta \times id_Y"'] & Z \times X \times X \times Y
\end{tikzcd}$$

where $j(z,y) = (z, f(z), y) = (z, g(y), y)$, and $g' \times f'$ is the inclusion. We use the labels g' and f' since these maps are so named in the theorem we are in the middle of proving. By equation (2.14), we have

$$(id_Z \times \Delta \times id_Y)^* \left((\iota_f \times \iota'_g)_* 1\right) = j_* 1,$$

where we note that transversality of $id_Z \times \Delta \times id_Y$ and $\iota_f \times \iota'_g$ is exactly condition (8.2.14). Let $\pi_{Z\times Y} \colon Z \times X \times Y \to Z \times Y$ be the projection. Clearly $\pi_{Z\times Y} \circ j = g' \times f'$. Hence we have

$$K = (\pi_{Z\times Y})_* j_* 1 = (g' \times f')_* 1.$$

It remains to see that $(g' \times f')_* 1$ also is the Schwartz kernel of $f'_* \circ g'^*$. So let $\sigma_Z \in \mathcal{A}^*_c Z$ and $\sigma_Y \in \mathcal{A}^*_c(Y)$. Then

$$\begin{aligned} ((g' \times f')_* 1)(\sigma_Z \otimes \sigma_Y) &= \int_{Z\times_X Y} g'^* \sigma_Z \wedge f'^* \sigma_Y \\ &= (f'_* g'^*(\sigma_Z))(\sigma_Y) \end{aligned}$$

which concludes the proof. ■

Let $p \colon W \to X$ be a submersion. We define $\mathcal{A}^*_{pvc}(W)$ as the space of forms ω such that $p|_{\operatorname{supp}\omega}$ is proper. As another application of theorem (2.30) we get:

Proposition 2.32. *For $\omega \in \mathcal{A}^*_{pvc}(W)$, the current $p_*\omega$ is smooth.*

Proof. We have $WF(p_*\omega) \subset N_p \backslash 0_X$ by Theorem (2.30). Since p is a submersion, $N_p = \emptyset$. I.e. since $Dp(T_w W) = T_{p(w)} X$, the only covector at $p(w)$ which pulls back to 0_w is $0_{p(w)}$. Since the wave front set of a current projects to its singular locus, $WF(T) = \emptyset \implies$ sing supp$(T) = \emptyset$, which implies that T is smooth. ■

Remark 2.33. This is a local statement in X, and locally in X this is a form of Fubini's theorem. See [11, I-2.15] for a direct proof, along these lines.

Definition 2.34. *Let $p \colon W \to X$ be an oriented submersion of codimension d between closed manifolds. We define integration over the fiber*

$$\int_{W/X} : \mathcal{A}^*_{pcv}(W) \to \mathcal{A}^{*+d}(X)$$

by

$$T_{\int_{W/X} \omega} = \mathbf{w}' p_* T_\omega \mathbf{w}',$$

where $\mathbf{w}'$ is the sign operator acting on $\mathcal{D}^k(X)$ and $\mathcal{A}^k(X)$ by $(-1)^{k + k \dim_{\mathbb{R}} X}$. More generally, we define for an evenly graded vectorspace $\mathcal{V}_*$,

$$\int_{W/X} := \left(\sum_j \int_{W/X} \otimes \mathrm{id}_{\mathcal{V}_{2j}} \right) : \mathcal{A}^*_{pcv}(W; \mathcal{V}_*) \to \mathcal{A}^{*+d}(X; \mathcal{V}_*).$$

From (2.7), (2.5) and Proposition 2.11, we get:

Proposition 2.35. *Let $p\colon W \to X$ be a submersion. We assume $\partial X = \emptyset$. Integration over the fiber satisfies*

$$d\int_{W/X} \omega = \int_{W/X} d\omega - \int_{\partial W/X} \omega$$
$$\int_{W/X} p^*(\sigma) \wedge \omega = \sigma \wedge \int_{W/X} \omega$$

Remark 2.36. When $E \to X$ is a vector bundle, there is also a direct description of $\int_{E/X}$ at [5, p. 61]. That their integration map agrees with ours follows from [5, Proposition 6.15 (b)].

2.7 Thom spaces and the Thom isomorphism

Let $\pi\colon E \to X$ be a vector bundle of dimension n. If X is compact, we define the Thom space, denoted $\mathrm{Th}(E)$ or X^E, as the one-point compactification of E. In general we define it as the colimit over compacta $K \subset X$;

$$\mathrm{Th}(E) = X^E = \operatorname*{colim}_{K \subset X} K^E.$$

To be more explicit, the underlying set of X^E is just $E \sqcup \{\infty\}$. We say that $K \subset E$ is vertically compact, *vc* for short, if K is closed, and furthermore $K \cap \pi^{-1}(x)$ is compact for each $x \in X$. To topologize X^E, we declare:

- open subsets of E are also open in X^E,
- $X^E \backslash K$ is open whenever K is vertically compact.

We choose this approach to have the explicit inclusion $E \subset \mathrm{Th}(E)$ available. See [43, p.205] for another approach. Note that if $E \to X$ is the vector bundle of dimension 0, then ∞ is an isolated point. We consider forming Thom spaces as a functor

$$\mathrm{Th}\colon \mathit{Bun} \to \mathbf{Top}_*$$

from the category of smooth vector bundles and smooth, fiberwise linear maps, to that of pointed topological spaces and pointed maps. We will however keep in mind all the extra structure on $\mathrm{Th}(E)\backslash\{\infty\}$ endowed by the canonical inclusion $E \subset \mathrm{Th}(E)$.

Remark 2.37. Continuous fiberwise linear maps are not necessarily bundle maps, which are furthermore required to be fiberwise linear isomorphisms.

It is clear that whenever $F\colon E_1 \to E_2$ is continuous and fiberwise linear and $K \subset E_2$ is *vc*, then $F^{-1}(K)$ is *vc* too, so F extends to a map of Thom spaces, $\mathrm{Th}(F)\colon \mathrm{Th}(E_1) \to \mathrm{Th}(E_2)$.

We denote by $\underline{\mathbb{R}}^k_X$, respectively $\underline{\mathbb{C}}^k_X$, the k-dimensional trivial real, respectively complex, vector bundle over X. We define the k-sphere as the Thom space

$$S^k := \mathrm{Th}(\underline{\mathbb{R}}^k_{pt}).$$

We will always mean this exact space when writing S^k, unless otherwise specified.

Recall that for pointed topological spaces (X, x_0) and (Y, y_0), we define the wedge and smash product respectively by

$$X \vee Y = \{x_0\} \times Y \cup X \times \{y_0\} \subset X \times Y \quad \text{and} \quad X \wedge Y = \frac{X \times Y}{X \vee Y}.$$

One basic property of Thom spaces we will use is this:

Proposition 2.38. *Let $E_1 \to X_1$ and $E_2 \to X_2$ be two vector bundles. Then $E_1 \times E_2 \to X_1 \times X_2$ is a vector bundle, and we have a canonical pointed homeomorphism*

$$(X_1 \times X_2)^{(E_1 \times E_2)} \simeq X_1^{E_1} \wedge X_2^{E_2}$$

which is a diffeomorphism away from basepoints.

Proof. Removing the basepoint $\{\infty\}$ from the left hand side renders the manifold $E_1 \times E_2$. Removing the basepoint $\left[X_1^{E_1} \times \{\infty\} \cup \{\infty\} \times X_2^{E_2}\right]$ from the right hand side too renders the manifold $E_1 \times E_2$, so we have a canonical basepoint preserving bijection between the two spaces, which is a diffeomorphism away from the basepoints. If X_1 and X_2 are compact, we are done since both sides are the one-point compactification of $E_1 \times E_2$ in this case. Taking the colimit over compact subsets of X_1 and X_2, we see that our canonical bijection is a colimit of homeomorphisms, and we are done. ∎

Recall that for a pointed space Y, we define the reduced suspension $\Sigma^k Y = S^k \wedge Y$.

Proposition 2.39. *We have a canonical pointed homeomorphism*

$$X^{E \oplus \underline{\mathbb{R}}^k_X} \simeq \Sigma^k X^E$$

which is a diffeomorphism away from basepoints.

Proof. Apply proposition (2.38) with

$$(E_1 \to X_1) = (\underline{\mathbb{R}}^k_{pt} \to \mathrm{pt}) \quad \text{and} \quad (E_2 \to X_2) = (E \to X).$$ ∎

Further specialization yields:

Proposition 2.40. *We have $X^{\underline{\mathbb{R}}^n_X} \simeq \Sigma^n X_+$.*

Proof. Take $E = 0$ in Proposition 2.39. ∎

Let now E be a fixed real oriented vector bundle of dimension n. We denote by $\mathcal{D}^*_{vc}(E)$ the currents on E with vertically compact support, defined by

$$\mathcal{D}^*_{vc}(E) := \{T \in \mathcal{D}^*(E) : \exists\ K \subset E \text{ which is } vc \text{ s.t. } T|_{E\setminus K} = 0\}.$$

In the notation of 2.2, we have $\mathcal{D}^*_{vc}(E) = \mathcal{D}^*_{\pi vc}(E)$, so that we have a map

$$\pi_* \colon \mathcal{D}^*_{vc}(E) \to \mathcal{D}^{*-n}(X).$$

Theorem 2.41. *The pushforward map induces an isomorphism*

$$\pi_* \colon H^k(\mathcal{D}^*_{vc}(E)) \simeq H^{k-n}(X).$$

Proof. See [5, Theorem 6.17]. ∎

We define $\mathcal{A}^*_{vc}(E)$ to be the smooth forms on E with vertically compact support.

Proposition 2.42. *The inclusion $\mathcal{A}^*_{vc}(E) \to \mathcal{D}^*_{vc}(E)$ induces an isomorphism on cohomology.*

Proof. We have previously discussed de Rham's result that each current is cohomologous to a smooth form; for $T \in \mathcal{D}^k_{vc}(E)$ we can therefore find $\alpha \in \mathcal{D}^{k-1}(E)$ with $T + d\alpha \in \mathcal{A}^k(E)$. Since smooth forms are dense in currents, we can of course assume that $T + d\alpha$ is close to T and hence also vertically compactly supported. This loose argument can be made rigorous using [12, Theorem 12]. This proves surjectivity of the induced map on cohomology. Injectivity follows directly from injectivity of the induced map on cohomology of the map of complexes $\mathcal{A}^*(X) \to \mathcal{D}^*(X)$. ∎

We note that there are canonical isomorphisms

$$H^n(\mathcal{A}^*_{vc}(E)) \simeq H^n(E, E\setminus \mathrm{Im}\,(0_E)) \simeq \widetilde{H}^n(X^E), \tag{2.15}$$

for 0_E the 0-section. Hence $\mathcal{A}^*_{vc}(E)$ and $\mathcal{D}^*_{vc}(E)$ compute the reduced cohomology of the Thom space X^E. We define

$$\mathcal{D}^*(\mathrm{Th}(E)) = \mathcal{D}^*_{vc}(E).$$

We shall also need a slightly larger complex; that of rapidly decreasing currents. For the definition, we require that E is endowed with a Euclidean norm, which we denote by $\xi \mapsto |\xi|$. We need some notation. Let first

$$D_1(E) = \{\xi \in E \ : \ |\xi| \leqslant 1\}$$

be the unit disc bundle in E. Next, let $\mathcal{D}^*_1(E)$ denote currents with support in the unit disc bundle,

$$\mathcal{D}^*_1(E) = \{T \in \mathcal{D}^*_{vc}(E) \ : \ \mathrm{supp}(T) \subset D_1(E)\}.$$

There is a diffeomorphism $\psi\colon E \to D_1(E)$ defined by

$$\psi(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}.$$

We define

$$\mathcal{D}^*_{rd}(E) = \psi^*(\mathcal{D}^*_1(E)) \subset \mathcal{D}^*(E).$$

There is a pushforward map

$$\pi_*\colon \mathcal{D}^*_{rd}(E) \to \mathcal{D}^{*-n}(X)$$

because $\pi = \pi \circ \psi$. We also define

$$\mathcal{A}^*_{rd}(E) = \mathcal{A}^*(E) \cap \mathcal{D}^*_{rd}(E) \tag{2.16}$$

which comes with an integration over the fiber map

$$\int_{E/X} \colon \mathcal{A}^*_{rd}(E) \to \mathcal{A}^{*-n}(X)$$

where $\int_{E/X}$ is defined by the same formula as in Definition 2.34.

Remark 2.43. The space of forms $\mathcal{A}^*_{rd}(E)$ is the same as the space of rapidly decreasing forms considered by Mathai and Quillen in [42].

Definition 2.44. *If E has a Euclidean norm, we define the space of currents on* $\mathrm{Th}(E)$ *by*

$$\mathcal{D}^*(X^E) := \mathcal{D}^*_{rd}(E).$$

We also define

$$\mathcal{A}^*(X^E) = \mathcal{A}^*_{rd}(E).$$

If $\mathcal{V}_*$ is a graded vector space, we apply the conventions of (2.8) *to define* $\mathcal{D}^*(X^E;\mathcal{V}_*)$ *and* $\mathcal{A}^*(X^E;\mathcal{V}_*)$.

Thus we have overloaded the notation $\mathcal{D}^*(\mathrm{Th}(E))$. We will always mean $\mathcal{D}^*(\mathrm{Th}(E)) = \mathcal{D}_{vc}(E)$ unless we specify the metric. Usually we will state which complex we mean. We note that the canonical inclusion $\mathcal{D}^*_{vc}(E) \to \mathcal{D}^*_{rd}(E)$ is a quasi-isomorphism.

The Thom class $\tau_E \in \widetilde{H}^n(X^E)$ is defined as the class corresponding to $1 \in H^0(X)$ under the Thom isomorphism $\widetilde{H}^n(X^E) \simeq H^0(X)$. It is characterized by $\pi_*\tau_E = 1$. Let $\iota_E\colon X \to E$ denote the 0-section. Since $\pi \circ \iota_E = id_X$ we have $\pi_*\left(\iota_E\right)_* 1 = 1$, proving that the current $\left(\iota_E\right)_* 1$ represent the Thom class of E. We record this for future reference.

Proposition 2.45. *Let $\iota_E\colon X \to E$ denote the* 0*-section of the oriented n-plane bundle $E \to X$. The current*

$$\left(\iota_E\right)_* 1 \in \mathcal{D}^n(X^E)$$

represents the Thom class $\tau_E \in \widetilde{H}^n(X^E)$.

2.8 Mathai–Quillen Thom forms

We recall the construction of the Mathai–Quillen Thom forms, originaly defined in [42]. We rely on [2, Section 1.6]. Let $E \to X$ be an oriented Euclidean vector bundle of dimension $2m$[3] with compatible connection ∇. Let $\Lambda E \to X$ denote the exterior bundle of E. There is an induced Euclidean metric, and compatible connection on ΛE. The Berezin integral $T\colon \mathcal{A}^*(X; \Lambda E) \to \mathcal{A}^*(X)$ is defined as follows. For $\sigma \in \mathcal{A}^*(X; \Lambda^j E)$, we let $T(\sigma) = 0$ unless $j = 2m$. There is a canonical global non-vanishing section of the bundle $\Lambda^{2m} E \to X$, locally expressed as $x_1 \wedge \cdots \wedge x_{2m}$, where $x_1, \ldots, x_{2m}$ forms a local oriented orthonormal frame for E. We let

$$T(\omega \otimes x_1 \wedge \cdots \wedge x_{2m}) = \omega.$$

Extrapolating linearly, T is defined on arbitrary elements of $\mathcal{A}^*(X; \Lambda E)$. Let $\mathbf{x} \in \mathcal{A}^0(E; \Lambda^1 \pi^* E)$ denote the tautological section:

$$\begin{tikzcd}
E \arrow[r, "\mathbf{x}"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & \pi^* E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
& E \arrow[r, "\pi"] & X
\end{tikzcd}$$

Let $\exp\colon \Lambda E \to \Lambda^E$ be defined by the usual power series. We define the Mathai–Quillen Thom form by

$$U = U(E, \nabla, |-|) = \frac{1}{(2\pi)^m} T\left(\exp\left(-\frac{|\mathbf{x}|^2}{2} - i\nabla_{\mathbf{x}} - \pi^* F\right)\right)$$

where F is the curvature of ∇, and $\nabla_{\mathbf{x}}$ is the covariant derivative of $\mathbf{x}$ with respect to the induced connection on $\Lambda \pi^* E$. These forms are shown to be Thom forms in [2, Proposition 1.52]. Inspecting the definition of U, we conclude:

Proposition 2.46. *The Thom form U depends only on the orientation, Euclidean norm, $|-|$ and compatible connection ∇ on E. Furthermore, U depends on this data in a natural way, in the sense that if $f\colon Y \to X$ is a smooth map, we have*

$$\overline{f}^*(U(E, \nabla, |-|)) = U(f^*E, f^*\nabla, f^*|-|) \in \mathcal{A}^{2m}_{vc}(f^*E),$$

where $\overline{f}\colon f^*E \to E$ is the canonical bundle-map. ■

Next we relate this to the Thom-current of Proposition 2.45. We remark first that since E is of even dimension, for a Thom form U, we get $\pi_* T_U = T_{\int_{E/X} U}$. We use for $t \in \mathbb{R}_{>0}$ the notation

$$\Omega_t = t^2 |\mathbf{x}|^2/2 + it\nabla_{\mathbf{x}} + \pi^* F \in \mathcal{A}^*(E; \Lambda \pi^* E),$$

[3] In [2] this is formulated so that it works also for odd dimensional bundles.

Then we let $U_t = \frac{1}{(2\pi)^m} \cdot T(\exp(-\Omega_t))$. Notice that U_t is the Thom form obtained by scaling the metric by t. We have the transgression formula, [2, Proposition 1.53]

$$\frac{dU_t}{dt} = -i \cdot dT(\mathbf{x}\exp(-\Omega_t)). \tag{2.17}$$

We define

$$\alpha = \int_1^\infty iT(\mathbf{x}\exp(-\Omega_t))dt.$$

Note that α too depends naturally on the metric and connection. We also note that α is a well defined, and rapidly decreasing form on E, though it need not be smooth. We apply (2.17) to get

$$d\alpha = U_1 - \lim_{t\to\infty} U_t.$$

Lemma 2.47. *We have* $\lim_{t\to\infty} U_t = (\iota_E)_*1$, *for* ι_E *the* 0*-section of* E.

Proof. Denote for $t \in \mathbb{R}$ the multiplication by t map by $m_t \colon E \to E$. Then $\langle m_t v, m_t w\rangle = t^2\langle v, w\rangle$, so that $m_t^* U = U_t$; rescaling the metric amounts to the same thing as rescaling the vectors. Hence we prove the lemma by proving more generally that for any Thom form U, we have

$$\lim_{t\to\infty} m_t^* U = (\iota_E)_* 1.$$

In local coordinates, we can write U as $\tau \otimes 1$ for $\tau \in \mathcal{A}^{2m}_{rd}(\mathbb{R}^{2m})$ such that $\int_{\mathbb{R}^{2m}} \tau = 1 \in \mathbb{R}$ and 1 the constant function $1 \in \mathcal{A}^0(X)$. In this vein, $(\iota_E)_*1 = \delta_0 \otimes 1$ for $\delta_0 \in \mathcal{D}^0(\mathbb{R}^{2m})$ the Dirac delta function, $\delta_0(f) = f(0)$. Thus it suffices to show

$$\lim_{t\to\infty} m_t^* \tau = \delta_0.$$

Using the substitution $x^{new} = x^{old} \cdot t$ we have for compactly supported $f \colon \mathbb{R}^{2m} \to \mathbb{R}$

$$\int_{\mathbb{R}^{2m}} m_t^*(\tau(x)) \cdot f(x) = \int_{\mathbb{R}^{2m}} \tau(x) \cdot f\left(x/t\right).$$

Using the dominated convergence theorem with $T(\tau) \cdot \sup_{x\in\mathbb{R}^{2m}} f(x)$ we get

$$\lim_{t\to\infty} \int_{\mathbb{R}^{2m}} m_t^*(\tau(x)) \cdot f(x) = \int_{\mathbb{R}^{2m}} \lim_{t\to\infty} \tau(x) \cdot f\left(x/t\right) = f(0) \cdot \int_{\mathbb{R}^{2m}} \tau = f(0). \quad \blacksquare$$

We summarize our work compactly for convenient future reference:

Proposition 2.48. *Let E be an oriented vector bundle of dimension $2m$, endowed with a Euclidean metric and compatible connection. Then there is a current $\alpha \in \mathcal{D}^{2m-1}_{rd}(E)$ satisfying*

$$d\alpha = U - (\iota_E)_*1 \in \mathcal{D}^{2m}_{rd}(E),$$

where ι_E denotes the 0*-section of E. Furthermore, α depends naturally on E, the metric and the connection.*

2.9 Simplicial presheaves

Let $\mathbf{Man}_{\mathbb{C}}$ denote the category of complex manifolds and holomorphic maps. The Grothendieck topology defined by open coverings turns $\mathbf{Man}_{\mathbb{C}}$ into an essentially small site with enough points. We denote by $\mathbf{sPre} = \mathbf{sPre}$ the category of simplicial presheaves on $\mathbf{Man}_{\mathbb{C}}$, i.e., contravariant functors from $\mathbf{Man}_{\mathbb{C}}$ to the category $\mathbf{sS}$ of simplicial sets. We will often refer to objects in $\mathbf{sPre}$ just as *spaces*. Recall that sending an object X of $\mathbf{Man}_{\mathbb{C}}$ to the presheaf of sets it represents defines a fully faithful embedding of $\mathbf{Man}_{\mathbb{C}}$ into the category of presheaves of sets on $\mathbf{Man}_{\mathbb{C}}$. Since any presheaf of sets defines an object in $\mathbf{sPre}$ of simplicial dimension zero, we can embed $\mathbf{Man}_{\mathbb{C}}$ into $\mathbf{sPre}$. On the other hand, every simplicial set K defines a simplicial presheaf by sending every object to K. By abuse of notation, we denote this simplicial presheaf by K as well.

We will consider $\mathbf{sPre}$ with the local projective model structure (see e.g. [13], [14]). We will not discuss the specific properties of this model structure, but just recall that a map $\mathcal{F} \to \mathcal{G}$ in $\mathbf{sPre}$ is called a (local) weak equivalence if the induced map of stalks $\mathcal{F}_x \to \mathcal{G}_x$ is a weak equivalence in $\mathbf{sS}$ for every point x in $\mathbf{Man}_{\mathbb{C}}$. Furthermore, a map $\mathcal{F} \to \mathcal{G}$ is called an objectwise fibration if $\mathcal{F}(X) \to \mathcal{G}(X)$ is a fibration in $\mathbf{sS}$ for every $X \in \mathbf{Man}_{\mathbb{C}}$. A map is a local projective fibration if it is an objectwise fibration and satisfies descent for all hypercovers in $\mathbf{Man}_{\mathbb{C}}$ (see [14, Corollary 7.1]). We denote the corresponding homotopy category of $\mathbf{sPre}$ by $\mathrm{ho}\mathbf{sPre}$.

Let Δ^n be the standard topological n-simplex

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | 0 \le t_j \le 1, \sum t_j = 1\}. \tag{2.18}$$

For a topological space Z, we will consider the simplicial presheaf $\mathrm{Sing}\, Z$ on $\mathbf{Man}_{\mathbb{C}}$ defined by

$$X \mapsto \mathrm{Sing}\, Z(X) = \mathrm{Sing}\,(Z, X)$$

whose n-simplices are continuous maps

$$f \colon X \times \Delta^n \to Z.$$

For any CW-complex Z, the simplicial presheaf $\mathrm{Sing}\, Z$ is objectwise fibrant and satisfies descent for hypercovers in $\mathbf{Man}_{\mathbb{C}}$ by [15, Theorem 1.3] (see also [30, Lemma 2.3]). By [14, Corollary 7.1], this implies that $\mathrm{Sing}\, Z$ is a fibrant object in the local projective model structure on $\mathbf{sPre}$. For a topological spectrum E, we denote by $\mathrm{Sing}\, E$ the spectrum of presheaves on $\mathbf{Man}_{\mathbb{C}}$ whose nth presheaf is $\mathrm{Sing}\,(E_n)$.

For a simplicial set K, let $|K|$ be its geometric realization in the category of CW-complexes. By [30, Proposition 2.4], the natural map

$$K \to \mathrm{Sing}\, |K| =: \mathrm{Sing}\, K$$

is a weak equivalence of simplicial presheaves. Hence we can use the assignment $K \mapsto \mathrm{Sing}\, |K|$ as a natural fibrant replacement in $\mathbf{sPre}$ for simplicial presheaves coming from simplicial sets.

2.10 Eilenberg–MacLane spaces

Let $\mathcal{C}^*$ be a cochain complex of presheaves of abelian groups on $\mathbf{Man}_{\mathbb{C}}$. For any integer n, we denote by $\mathcal{C}^*[n]$ the cochain complex given in degree q by $\mathcal{C}^q[n] := \mathcal{C}^{q+n}$. The differential on $\mathcal{C}^*[n]$ is the one of $\mathcal{C}^*$ multiplied by $(-1)^n$. The hypercohomology $H^*(X;\mathcal{C}^*)$ of $X \in \mathbf{Man}_{\mathbb{C}}$ with coefficients in $\mathcal{C}^*$ is the graded group of morphisms $\mathrm{Hom}(\mathbb{Z}_X, a\mathcal{C}^*)$ in the derived category of cochain complexes of sheaves, where $a\mathcal{C}^*$ denotes the complex of associated sheaves of $\mathcal{C}^*$. We will denote by $K(\mathcal{C}^*, n)$ the Eilenberg–MacLane space, i.e., simplicial presheaf, associated to the connective cover of $\mathcal{C}^*[-n]$ by the Dold–Kan correspondence. For every integer n, this defines a functor $K(-,n)$ from the category of presheaves of complexes on $\mathbf{Man}_{\mathbb{C}}$ to $\mathbf{sPre}$. The following result is a version of Verdier's hypercovering theorem due to Ken Brown.

Proposition 2.49. ([7, Theorem 2]) *Let $\mathcal{C}^*$ be a cochain complex of presheaves of abelian groups on $\mathbf{Man}_{\mathbb{C}}$. Then for every integer n and $X \in \mathbf{Man}_{\mathbb{C}}$, there is a canonical isomorphism*

$$H^n(X;\mathcal{C}^*) \cong \mathrm{Hom}_{\mathbf{hosPre}}(X, K(\mathcal{C}^*, n)).$$

2.11 Concrete model for Eilenberg–MacLane space of forms

Let Δ^k be standard topological k-simplex, as in (2.18). Then $\Delta^\bullet$ is a simplicial space, with the face-maps $\partial_i \colon \Delta^k \to \Delta^{k+1}$ given by $\partial_i(t_0, \dots, t_k) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_k)$, i.e., inserting a 0 in the ith position. The degeneracy maps $s_i \colon \Delta^k \to \Delta^{k-1}$ are the maps adding the ith and $(i+1)$th element, i.e., $s_i(t_0, \dots, t_k) = (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_k)$. Then also $X \times \Delta^\bullet$ is a simplicial space, and we get a simplicial presheaf $X \mapsto \mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)$, which on X has face maps $\partial_i^* := (\mathrm{id}_X \times \partial_i)^*$. In this section, we will prove:

Proposition 2.50. *The simplicial presheaf $X \mapsto \mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)$ on $\mathbf{Man}$ is weakly equivalent to $K(\mathcal{A}^*(\mathcal{V}_*), n)$.*

We recall that the Dold–Kan correspondence is an equivalence of categories between simplicial abelian groups and connective chain-complexes under which weak equivalences correspond to quasi-isomorphisms. It associates to a simplicial presheaf $F_\bullet$ its the normalized chain complex, $N(F_\bullet)$. There is also the Moore complex of a simplicial presheaf, $M(F_\bullet)$, which has nth presheaf F_n, and differential $\sum(-1)^i \partial_i$, where ∂_i are the face maps of $F_\bullet$. There is a natural map $N(F_\bullet) \to M(F_\bullet)$ which is a quasi-isomorphism by [22, Theorem III.2.1]. We will therefore establish Proposition 2.50 by showing the following lemma.

Lemma 2.51. *Integration over the fiber $\int_{X \times \Delta^l / X}$ induces a chain homotopy* *equivalence*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial^*"] & \mathcal{A}^n(X\times\Delta^k;\mathcal{V}_*)_{\mathrm{cl}} \arrow[r, "\partial^*"] \arrow[d, "\int_{X\times\Delta^k/X}"'] & \cdots \arrow[r, "\partial^*"] & \mathcal{A}^n(X\times\Delta^1;\mathcal{V}_*)_{\mathrm{cl}} \arrow[r, "\partial^*"] \arrow[d, "\int_{X\times\Delta^1/X}"'] & \mathcal{A}^n(X;\mathcal{V}_*)_{\mathrm{cl}} \arrow[d, "\mathrm{id}"'] \\
\cdots \arrow[r, "d"] & \mathcal{A}^{n-k}(X;\mathcal{V}_*) \arrow[r, "d"] & \cdots \arrow[r, "d"] & \mathcal{A}^{n-1}(X;\mathcal{V}_*) \arrow[r, "d"] & \mathcal{A}^n(X;\mathcal{V}_*)_{\mathrm{cl}}
\end{tikzcd}$$

Remark 2.52. This is [31, Corollary D.14]. We provide the proof, because we need the technique in the next section.

Proof. Let $v_i \in \Delta^k$ be the point with $t_i = 1$. Let $p_k\colon \Delta^k\backslash v_0 \to \Delta^{k-1}$ be radial projection onto the 0th face,

$$p_k(t_0,\dots,t_k) = \left(\frac{t_1}{1-t_0},\dots,\frac{t_k}{1-t_0}\right).$$

Then we have the identities

$$\begin{aligned} p_k \circ \partial_0 &= \mathrm{id}, \\ p_k \circ \partial_i &= \partial_{i-1} \circ p_{k-1} \quad \text{for } i > 0. \end{aligned} \tag{2.19}$$

Let $g\colon [0,1] \to \mathbb{R}$ be smooth, and vanishing 0 in a neighborhood of 0 and equaling 1 in a neighborhood of 1. We note that $\partial_0^*(g) = 0$ and $\partial_1^*(g) = 1$, and furthermore both ∂_0^* and ∂_1^* takes all derivatives of g to 0. We define

$$\begin{aligned} h_{k-1}\colon \mathcal{A}^n(X\times\Delta^{k-1};\mathcal{V}_*) &\to \mathcal{A}^n(X\times\Delta^k;\mathcal{V}_*) \\ \omega &\mapsto g(1-t_0)\cdot(\mathrm{id}\times p_k)^*\omega. \end{aligned}$$

We will now show that h is a contraction of the complex $\mathcal{A}^n(X\times\Delta^\bullet;\mathcal{V}_*)$, with

$$\partial^* = \sum_{i=0}^{k}(-1)^i\partial_i^*\colon \mathcal{A}^n(X\times\Delta^k;\mathcal{V}_*) \to \mathcal{A}^n(X\times\Delta^{k-1};\mathcal{V}_*)$$

as differential. For h_0, this means $\partial^* h_0 = \mathrm{id}$. This follows from (2.19), which implies $\partial_0^* h_0 = \partial_0^*(g(1-t_0)\cdot(\mathrm{id}\times p_1)^*) = g(1)\cdot \mathrm{id}^* \times (p_k\circ\partial_0)^* = \mathrm{id}$ using $g(1) = 1$. Since $g(0) = 0$, we similarly have the second of the following equalities:

$$\partial_0^* h_0 = id \quad \text{and} \quad \partial_1^* h_0 = 0. \tag{2.20}$$

More generally, for any k we use (2.19) to compute:

$$\begin{aligned} \partial^* h_k &= \sum_{i=0}^{k+1}(-1)^i\partial_i^* g(1-t_0)\cdot(\mathrm{id}\times p_{k+1})^* \\ &= \partial_0^*(g(1-t_0)\cdot(\mathrm{id}\times p_{k+1})^*) + \sum_{i=1}^{k+1}(-1)^i\partial_i^*(g(1-t_0)\cdot(\mathrm{id}\times p_{k+1})^*) \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{id} + \sum_{i=1}^{k+1} (-1)^i g(1-t_0)(\mathrm{id}^* \times \partial^* p_{k+1}^*) \\
&= \mathrm{id} + \sum_{i=1}^{k+1} (-1)^i g(1-t_0)(\mathrm{id} \times p_k)^* \partial_{i-1}^* \\
&= \mathrm{id} - g(1-t_0)(\mathrm{id} \times p_k)^* \sum_{i=0}^{k} (-1)^i \partial_i^* \\
&= \mathrm{id} - h_{k-1}\partial^*
\end{aligned}
$$

Using the contraction h_k we can define the homotopy inverse of $\int_{X\times\Delta^\bullet/X}$ by $\tau_0 = \mathrm{id}\colon \mathcal{A}^n(X;\mathcal{V}_*)_{cl} \to \mathcal{A}^n(X;\mathcal{V}_*)_{cl}$, and for $k>0$,

$$
\begin{aligned}
\tau_k\colon \mathcal{A}^{n-k}(X;\mathcal{V}_*) &\to \mathcal{A}^n(X\times\Delta^k;\mathcal{V}_*)_{cl} \\
\tau_k &= d\circ h_{k-1}\circ d\circ\cdots\circ d\circ h_1\circ d\circ h_0.
\end{aligned}
\tag{2.21}
$$

Since $\tau_k(\omega)$ is exact, it is closed. Next we show that $\tau_\bullet$ is a chain map. Using (2.20) we have $\partial^*\tau_1 = \partial^* dh_0 = d\partial^* h_0 = d = \tau_0\circ d$. Inductively we get

$$
\begin{aligned}
\partial^*\tau_{k+1} &= \partial^* dh_k\tau_k \\
&= d\partial^* h_k\tau_k \\
&= d(\mathrm{id} - h_{k-1}\partial^*)\tau_k \\
&= dh_{k-1}\partial^*\tau_k \\
&= dh_{k-1}\tau_{k-1}d \\
&= \tau_k d
\end{aligned}
$$

where we also use that $d\tau_k = ddh_{k-1}\tau_{k-1} = 0$. We now show $\int_{X\times\Delta^k/X}\tau_k(\omega) = \omega$ for all $\omega\in\mathcal{A}^{n-k}(X;\mathcal{V}_*)$. For $k=1$ we have

$$
\begin{aligned}
\int_{X\times\Delta^1/X}\tau_1\omega = \int_{X\times\Delta^1/X} dh_0\omega &= \int_{X\times\Delta^1/X}\frac{\partial}{\partial t_0} g(1-t_0)dt_0\wedge(\mathrm{id}\times p_1)^*\omega \\
&= -(g(1-1)-g(1-0))\cdot\omega = \omega
\end{aligned}
\tag{2.22}
$$

where we use that $\int_{X\times\Delta^1/X} g(1-t_0)d(\mathrm{id}\times p_1)^*\omega = 0$, since there is no dt_0-factor. Similarly for the induction step:

$$
\begin{aligned}
\int_{X\times\Delta^{k+1}/X}\tau_{k+1}(\omega) &= \int_{X\times\Delta^{k+1}/X} dh_k\tau_k(\omega) \\
&= \int_{X\times\Delta^k/X}\left(\int_0^1 \frac{\partial}{\partial t_0} g(1-t_0)dt_0\right)(\mathrm{id}\times p_{k+1})^*\tau_k(\omega) \\
&= \int_{X\times\Delta^k/X}\tau_k(\omega) = \omega.
\end{aligned}
$$

Instead of showing that $\tau_k \int_{\Delta^k}$ is homotopic to the identity, we show that the two complexes $\mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)^n_{cl}$ and $\mathcal{A}^{n-\bullet}(X; \mathcal{V}_*)$ are quasi-isomorphic. We consider the following double complex:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
\cdots \arrow[r] & \mathcal{A}^{n-k}(X \times \Delta^k; \mathcal{V}_*) \arrow[r] \arrow[d] & \mathcal{A}^{n-k}(X \times \Delta^{k-1}; \mathcal{V}_*) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{A}^{n-k}(X; \mathcal{V}_*) \arrow[d] \\
 & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
\cdots \arrow[r] & \mathcal{A}^{n-1}(X \times \Delta^k; \mathcal{V}_*) \arrow[r] \arrow[d] & \mathcal{A}^{n-1}(X \times \Delta^{k-1}; \mathcal{V}_*) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{A}^{n-1}(X; \mathcal{V}_*) \arrow[d] \\
\cdots \arrow[r] & \mathcal{A}^{n}(X \times \Delta^k; \mathcal{V}_*)_{\mathrm{cl}} \arrow[r] & \mathcal{A}^{n}(X \times \Delta^{k-1}; \mathcal{V}_*)_{\mathrm{cl}} \arrow[r] & \cdots \arrow[r] & \mathcal{A}^{n}(X; \mathcal{V}_*)_{\mathrm{cl}}
\end{tikzcd}$$

Here the horizontal arrows are $\partial^* = \sum_i (-1)^i \partial_i^*$, and the vertical arrows are the exterior differential d. We consider the two spectral sequences of this double complex. Using the homotopy operators h, we see that computing horizontal cohomology first, the second page has only one non-zero row, namely the cohomology of the complex $\mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}$. Computing vertical cohomology first, we observe that all columns have the cohomology of the complex

$$\mathcal{A}^{n-k}(X; \mathcal{V}_*) \to \cdots \to \mathcal{A}^{n-1}(X; \mathcal{V}_*) \to \mathcal{A}^n(X; \mathcal{V}_*)_{cl}.$$

In fact $\partial_i \colon X \times \Delta^{k-1} \to X \times \Delta^k$ is a homotopy equivalence for each i. Furthermore, all the ∂_i are homotopic, so their induced maps on cohomology coincide. Hence

$$\partial^* = \sum_{i=0}^{j} (-1)^i \partial_i^* \colon H^{n-j}(X \times \Delta^j; \mathcal{V}_*) \to H^{n-j}(X \times \Delta^{j-1}; \mathcal{V}_*)$$

is an isomorphism if j is even, and 0 if j is odd. In particular the rightmost column is hit by the 0-map $\partial_0^* - \partial_1^*$, while all groups outside of the rightmost column are either the source or the target of an isomorphism, and so disappears on page 2. Since the spectral sequences converge to the same complex, we deduce that the cohomology of the complexes $\mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)_{cl}$ and $\mathcal{A}^{n-k}(X; \mathcal{V}_*) \to \cdots \to \mathcal{A}^n(X; \mathcal{V}_*)_{cl}$ are isomorphic. Since $\int_{\Delta^k} \circ \tau_k = \mathrm{id}$, we conclude that $\int_{\Delta^k}$ is a quasi-isomorphism. ■

2.12 A concrete Eilenberg-MacLane space for filtered forms

We continue using the notation of the previous section. Let $\pi \colon X \times \Delta^j \to X$ denote the projection. We can write any form on $\omega \in \mathcal{A}^*(X \times \Delta^k; \mathcal{V}_*)$ as a sum of forms of the form

$$\omega = \pi^*\theta \wedge (f(x,t)dt_I) \tag{2.23}$$

for $\theta \in \mathcal{A}^*(X;\mathcal{V}_*)$ and dt_I some product of the forms $dt_1,\dots,dt_k$ (leaving out dt_0 since $\sum_{i=0}^k dt_i = 0$) and $f\colon X\times\Delta^k \to \mathbb{C}$ a smooth function. We define $F^p\mathcal{A}^n(X\times\Delta^k)$ as the linear span of forms $\omega \in \mathcal{A}^n(X\times\Delta^k;\mathcal{V}_*)$, as in (2.23), with $\theta \in F^p\mathcal{A}^*(X;\mathcal{V}_*)$.

We observe that $\int_{X\times\Delta^k/X}$, and each ∂_i preserve the filtration. Furthermore, the contraction h restricts to a contraction of $F^p\mathcal{A}^*(X\times\Delta^\bullet)$. That is, inspecting the definition of h_k we observe that

$$h_k(F^p\mathcal{A}^*(X\times\Delta^k;\mathcal{V}_*) \subset F^p\mathcal{A}^*(X\times\Delta^{k+1};\mathcal{V}_*),$$

and so the equality $\partial^* h_k + h_{k+1}\partial^* = \mathrm{id}$ must still hold in $F^p\mathcal{A}^*(X\times\Delta^k;\mathcal{V}_*)$. It follows that τ_k preserve the filtration. As before we have $\int_{X\times\Delta^k/X}\tau_k = \mathrm{id}$ on $\mathcal{A}^{n-k}(X;\mathcal{V}_*)$. For the spectral sequence argument of the previous section, we need to know that each ∂_i define the same isomorphism $H^n(F^p\mathcal{A}^*(X\times\Delta^k);\mathcal{V}_*) \to H^n(F^p\mathcal{A}^*(X\times\Delta^{k-1};\mathcal{V}_*))$. For this we prove:

Lemma 2.53. *The projection $\pi\colon X\times\Delta^k \to X$ induces an isomorphism*

$$\pi^*\colon H^n(F^p\mathcal{A}^*(X;\mathcal{V}_*)) \to H^n(F^p\mathcal{A}^*(X\times\Delta^k;\mathcal{V}_*)).$$

Proof. As Karoubi points out in [34, Theorem 4.10], this is a version of the Poincaré lemma. Let us expand on that. Let $S_p\colon \mathcal{A}^p(\Delta^k;\mathcal{V}_*) \to \mathcal{A}^{p-1}(\Delta^k;\mathcal{V}_*)$ be maps satisfying $dS_p - S_{p+1}d = \mathrm{id}$ for $p>0$, and $S_1 d = \mathrm{id} - ev$, where ev maps a function $f\in\mathcal{A}^0(\Delta^k;\mathcal{V}_*)$ to the constant function at $f(\mathrm{pt})$ for $\mathrm{pt}\in\Delta^k$ a point of our choosing. See for example [41, Theorem 3.15]. We also set $S_0 = 0$. Set $S = \sum_i S_i\colon \mathcal{A}^*(\Delta^k;\mathcal{V}_*) \to \mathcal{A}^{*-1}(\Delta^k;\mathcal{V}_*)$. Then define

$$K\colon \mathcal{A}^*(X\times\Delta^k;\mathcal{V}_*) \to \mathcal{A}^{*-1}(X\times\Delta^k;\mathcal{V}_*)$$

as the linear map with $K\omega = (-1)^q(\mathrm{id}\otimes S)\omega$ when ω is a form as in (2.23) with θ homogeneous of degree q. We observe that K preserves the filtration. Let $\omega\in\mathcal{A}^*(X\times\Delta^k;\mathcal{V}_*)$ be of the form (2.23) with θ homogeneous of degree q, and $|I| = l > 0$. We have

$$Kd\omega = (\mathrm{id}\otimes S)d\omega = ((-1)^{q+1}d\otimes S_l - \mathrm{id}\otimes S_{l+1}d)\omega$$

and

$$dK\omega = d(\mathrm{id}\otimes S_l)\omega = ((-1)^q d\otimes S_l + \mathrm{id}\otimes dS_l)\omega.$$

Therefore

$$(dK+Kd)\omega = \mathrm{id}\otimes(dS_l - S_{l+1}d)\omega = \omega.$$

If $|I| = 0$, we get

$$Kd\omega = ((-1)^{q+1}d\otimes S_0 - id\otimes(id-ev))\omega = (\mathrm{id}\otimes(id-ev))\omega,$$

so that $\omega = Kd\omega + (\mathrm{id} \otimes ev)\omega$. Suppose now that $\omega = \sum_I \omega_I = f_I \cdot \pi^*\theta_I \wedge dt_I$ is closed. Let J be the empty multi-index. We get

$$\omega = dK\left(\sum_{I,\ |I|>0} \omega_I\right) + \pi^*\theta_J \cdot f_J(x, pt).$$

Since $f(x, pt)$ is independent of t, we can absorb it into θ; we conclude that ω is cohomologous to a form of the form $\pi^*\theta$. This proves the lemma. ■

Let $\pi_k\colon X \times \Delta^k \to X$ be the projection. Then we have $\pi_{k-1} = \pi_k \circ \partial_i$ for each face map ∂_i. The induced equation on pullbacks, $\partial_i^* \circ \pi_k^* = \pi_{k-1}^*$, implies that each ∂_i induces the same isomorphism $\partial_i^*\colon H^n(X \times \Delta^k; \mathcal{V}_*) \to H^n(X \times \Delta^{k-1}; \mathcal{V}_*)$. Now the spectral sequence argument of 2.11 applies and shows:

Proposition 2.54. *Integration over the fiber $\int_{X\times\Delta^k/X}$ induces a chain homotopy equivalence*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial^*"] & F^p\mathcal{A}^n(X \times \Delta^k; \mathcal{V}_*)_{\mathrm{cl}} \arrow[r, "\partial^*"] \arrow[d] & \cdots \arrow[r, "\partial^*"] & F^p\mathcal{A}^n(X \times \Delta^1; \mathcal{V}_*)_{\mathrm{cl}} \arrow[r, "\partial^*"] \arrow[d] & F^p\mathcal{A}^n(X; \mathcal{V}_*)_{\mathrm{cl}} \arrow[d] \\
\cdots \arrow[r, "d"] & F^p\mathcal{A}^{n-k}(X; \mathcal{V}_*) \arrow[r, "d"] & \cdots \arrow[r, "d"] & F^p\mathcal{A}^{n-1}(X; \mathcal{V}_*) \arrow[r, "d"] & F^p\mathcal{A}^n(X; \mathcal{V}_*)_{\mathrm{cl}}
\end{tikzcd}$$

with inverse τ, given by (2.21)

Therefore we can use $F^p\mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)_{cl}$ as a model for $K(F^p\mathcal{A}^*(X; \mathcal{V}_*), n)$.

2.13 Concrete Eilenberg-MacLane spectrum of presheaves for forms

Let $A_\bullet$ be a pointed simplicial set. The simplicial loop space $\Omega^{\mathrm{simp}} A_\bullet$ is the simplicial set with set of k-simplices given by the set of maps of simplicial sets $h\colon \Delta^k_\bullet \times \Delta^1_\bullet \to A_\bullet$ such that $h(x, 0) = h(x, 1) = a_0$, where a_0 denotes the basepoint of $A_\bullet$. As described in [31, pages 379+381], for $A_\bullet = \mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}$, a k-simplex of the simplicial loop space can be described as a sequence

$$\omega_0, \ldots, \omega_k \in \mathcal{A}^n(X \times \Delta^{k+1}; \mathcal{V}_*)_{\mathrm{cl}}$$

satisfying the conditions

$$\begin{aligned} \partial_i^*\omega_i &= \partial_i^*\omega_{i-1} \\ \partial_0^*\omega_0 &= 0 = \partial_{k+1}^*\omega_k \end{aligned} \tag{2.24}$$

We will now construct natural simplicial weak equivalences

$$s_\bullet^{\mathcal{A}}\colon \mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}} \to \Omega^{\mathrm{simp}}\mathcal{A}^{n+1}(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}.$$

Let $g\colon \Delta^1 \to \mathbb{R}$ be a smooth map which is 0 on a neighborhood of 0 and 1 on a neighborhood of 1. Then we use the map τ_0 from Section 2.11

$$\tau_0\omega = d(g(1-t_0)\cdot p_0^*\omega),$$

where ω is a form on $X \times \Delta^k$, and $p_0\colon X \times \Delta^1 \times \Delta^k \to X \times \Delta^k$ is the projection. From the fundamental theorem of calculus it follows that τ_0 satisfies

$$\int_{X\times\Delta^1\times\Delta^k/X\times\Delta^k} \tau_0\omega = \omega.$$

Since g is constant on a neighborhood of the endpoints, we have $\tau_0\omega$ restricting to 0 at the endpoints. The resulting simplicial map

$$\mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}} \to \mathcal{A}^{n+1}(X \times \Delta^1 \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}$$

is a weak equivalence, since the underlying map of chain complexes is a quasi-isomorphism.

Now we restrict $\tau_0\omega$ to the $(k+1)$-simplices in the standard triangulation of $\Delta^1 \times \Delta^k$ to get a sequence of forms $\omega_0, \dots, \omega_k$ in $\mathcal{A}^{n+1}(X \times \Delta^{k+1}; \mathcal{V}_*)_{\mathrm{cl}}$ which satisfy the identities (2.24). As pointed out in [31, page 381], this defines a natural weak equivalence of simplicial sets

$$s_\bullet^{\mathcal{A}}\colon \mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}} \to \Omega^{\mathrm{simp}}\mathcal{A}^{n+1}(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}.$$

Since $\mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}$ is a simplicial abelian group and hence a Kan complex for each n, the sequence $n \mapsto \mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}$ is an Ω-spectrum of simplicial sets for every X. Moreover, since each simplicial presheaf $\mathcal{A}^n(- \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}$ satisfies descent with respect to hypercovers, we obtain a fibrant object in the local projective model structure of presheaves of sequential spectra on $\mathbf{Man}_{\mathbb{C}}$. This provides a proof of the result:

Proposition 2.55. *The sequence $\mathcal{A}^*_\Sigma(\mathcal{V}_*)\colon n \mapsto \mathcal{A}^n(- \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}$ defines an Ω-spectrum in the category of sequential spectra of presheaves on $\mathbf{Man}_{\mathbb{C}}$.*

Furthermore, we note that the maps $s_\bullet^{\mathcal{A}}$ restrict to natural weak equivalences

$$s_\bullet^{F^p\mathcal{A}}\colon F^p\mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}} \to \Omega^{\mathrm{simp}}F^p\mathcal{A}^{n+1}(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}$$

as the filtration on forms on X is independent of the simplicial identities which define the map. Since each simplicial presheaf $F^p\mathcal{A}^n(- \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}$ satisfies descent with respect to hypercovers, this proves:

Proposition 2.56. *For every integer p, the sequence*

$$F^p\mathcal{A}^*_\Sigma(\mathcal{V}_*)\colon n \mapsto F^p\mathcal{A}^n(- \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}$$

defines an Ω-spectrum in the category of presheaves of sequential spectra on $\mathbf{Man}_{\mathbb{C}}$. ■

2.14 Fundamental cocycles

Let E be a rationally even spectrum, i.e., a spectrum such that $E_* \otimes_{\mathbb{Z}} \mathbb{Q}$ is concentrated in even degrees. As in [30, §4.1], let $\iota\colon E \to E \wedge H\mathbb{C}$ be a map of spectra[4] which induces for every n the map

$$E_{2n} \to E_{2n} \otimes \mathbb{C}$$

defined by multiplication by $(2\pi i)^n$. For given integer p, we consider the map

$$E \xrightarrow{(2\pi i)^p \iota} E \wedge H\mathbb{C}.$$

Let $E \wedge H\mathbb{C} \to H(E_* \otimes \mathbb{C})$ be a map of spectra which induces the canonical isomorphism $\pi_{2*}(E \wedge H\mathbb{C}) \cong E_{2*} \otimes \mathbb{C}$ where $H(E_{2*} \otimes \mathbb{C})$ denotes the Eilenberg–MacLane spectrum corresponding to $E_{2*} \otimes \mathbb{C}$. Let ϕ_p^E be the composition of the above maps

$$\phi_p^E\colon E \to E \wedge H\mathbb{C} \to H(E_* \otimes \mathbb{C}). \tag{2.25}$$

We will often write ϕ^E for ϕ_p^E.

2.15 A lemma on invertible power series

Given a ring R we let $R[[x_1, \ldots, x_n]]$ denote the formal power series in the variables $x_1, \ldots, x_n$ with coefficients in R. We have a canonical isomorphism

$$R[[x_1, \ldots, x_n]] \simeq R[[x_1, \ldots, x_{n-1}]][[x_n]].$$

Lemma 2.57. *Let R be a commutative ring. A formal power series*

$$f \in R[[x_1, \ldots, x_n]]$$

is invertible if and only if the constant term is invertible.

Proof. The case $n = 0$ is trivial. We proceed by induction. For the induction step, we consider

$$f(x_n) = f_0 + f_1 \cdot x_n + \cdots \in R[[x_1, \ldots, x_{n-1}]][[x_n]].$$

The constant term of f is the constant term of f_0, which is therefore invertible by the induction hypothesis. We now solve the equation

$$f(x_n) \cdot (g_0 + g_1 \cdot x_n + g_2 \cdot x_n^2 + \ldots) = 1$$

[4] We omit to choose a model for the category of spectra. If in doubt one could choose symmetric spectra in order to make it possible to construct mutliplicative structures.

by inductively defining suitable g_i by comparing coefficients. We put $g_0 = (f_0)^{-1}$. Then the equation $g_0 \cdot f_1 + g_1 \cdot f_0 = 0$ force $g_1 = -g_0 \cdot f_1 \cdot g_0$. More generally, having defined g_i we define g_{i+1} from the requirement

$$\sum_{j=0}^{i+1} f_j \cdot g_{i-j} = 0;$$

this gives

$$g_{i+1} \cdot f_0 = -\sum_{j=0}^{i} f_j \cdot g_{i-j}$$

which has a unique solution for g_{i+1}, since f_0 is invertible. ∎

Chapter 3

Hodge filtered cohomology theories

We establish an axiomatic framework for Hodge filtered cohomology theories, similar to that used for differential cohomology in [8]. The hope is that these axioms will characterize Hodge filtered cohomology theories uniquely up to isomorphism, as in differential cohomology in [9]. We have not succeeded in proving this. Thus we must work to establish isomorphisms between cycle models of Hodge filtered cohomology groups, and the abstractly defined groups of [30] in each case separately.

We first give our axiomatic definition of Hodge filtered cohomology theories. Then we show that Deligne cohomology is a Hodge filtered cohomology theory. We then generalize to see that the Hodge filtered cohomology theories of [30] also are Hodge filtered cohomology theories in our sense of the words. Then we show that there is a natural way to associate a Hodge filtered cohomology theory with a differential cohomology theory. This construction goes some way towards giving a general recipe for translating constructions for differential cohomology theories into constructions in Hodge filtered cohomology theories.

3.1 Axioms for Hodge filtered cohomology theories

Let $\mathcal{V}_*$ be an evenly graded complex vector space, let h^* be a rationally even cohomology theory,[1] in the sense that $h^*(pt) \otimes \mathbb{Q}$ is an evenly graded vector space, and let for $p \in \mathbb{Z}$

$$c(p) \colon h^* \to H^*(-;\mathcal{V}_*)$$

be a map of cohomology theories. Here $H^*(X;\mathcal{V}_*)$ is computed using the grading-convention of (2.8). Then a Hodge filtered cohomology theory over $(h^*, c(p))$ is a functor $h^*_{\mathcal{D}}(p) \colon \mathbf{Man}_{\mathbb{C}} \to \mathrm{Ab}^{\mathbb{Z}}$ together with natural transformations

- $a \colon H^{n-1}\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \to h^n_{\mathcal{D}}(p)(X)$
- $I \colon h^n_{\mathcal{D}}(p)(X) \to h^n(X)$
- $R \colon h^n_{\mathcal{D}}(p)(X) \to H^n\left(X; F^p\mathcal{A}^*(\mathcal{V}_*)\right)$

[1] It might be interesting to weaken this assumption. Concretely, it would be interesting to allow the sub"functor" of $MU^n(X)$, defined on complex manifolds X, consisting of cobordism classes which are representable by holomorphic maps. We use quotation marks since this may not be a functor, as the failure of a general transversality theorem for holomorphic maps make it difficult to construct pullbacks. For projective manifolds, the image of algebraic cobordism lives over this sub"functor".

such that:

- The following diagram commutes:

$$\begin{tikzcd} h^n_{\mathcal{D}}(p)(X) \arrow[r, "I"] \arrow[d, "R"'] & h^n(X) \arrow[d, "c(p)"] \\ H^n(X; F^p\mathcal{A}^*(\mathcal{V}_*)) \arrow[r, "\mathrm{inc}_*"] & H^n(X; \mathcal{V}_*) \end{tikzcd}$$

  Here inc_* is the map induced by the map of complexes of sheaves

$$F^p\mathcal{A}^*(\mathcal{V}_*) \xrightarrow{\mathrm{inc}} \mathcal{A}^*(\mathcal{V}_*).$$

- $R \circ a = d$, where d is explained by Proposition (2.19).
- The sequence

$$\cdots \longrightarrow h^{n-1}(X) \xrightarrow{\overline{c}(p)} H^{n-1}\left(X; \tfrac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \xrightarrow{a} h^n_{\mathcal{D}}(p)(X) \xrightarrow{I}$$

$$h^n(X) \longrightarrow H^n\left(X; \tfrac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \longrightarrow \cdots$$

  is exact, where $\overline{c}(p)$ is the composition

$$h^n(X) \xrightarrow{c(p)} H^n(X; \mathcal{V}_*) \longrightarrow H^n\left(X; \mathcal{A}^* F^p(\mathcal{V}_*)\right).$$

  We note that exactness of the above sequence is equivalent to exactness of

$$h^{n-1}(X) \xrightarrow{c(p)} H^{n-1}\left(X; \tfrac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \xrightarrow{a}$$

$$h^n_{\mathcal{D}}(p)(X) \xrightarrow{I} F^p h^n(X) \longrightarrow 0$$

  where we write

$$F^p h^n(X) := c(p)^{-1}(F^p H^n(X; \mathcal{V}_*)).$$

  Equivalently, $F^p h^*(X)$ is the pullback

$$\begin{tikzcd} F^p h^*(X) \arrow[r] \arrow[d] & h^*(X) \arrow[d] \\ F^p H^*(X; \mathcal{V}_*) \arrow[r] & H^*(X; \mathcal{V}_*). \end{tikzcd}$$

So far the theories $h^*_{\mathcal{D}}(p)$ for different values of p have not interacted with each other. Suppose now that $\mathcal{V}_*$ is a graded algebra, that h^* is a multiplicative cohomology theory, and that $c(p)$ is a map of multiplicative cohomology theories. We say that $h^*_{\mathcal{D}}(*)$ is a multiplicative Hodge filtered cohomology theory if it comes with an exterior product map

$$\mu\colon h^n_{\mathcal{D}}(p)(X) \otimes h^{n'}_{\mathcal{D}}(p')(X') \to h^{n+n'}_{\mathcal{D}}(p+p')(X \times X'),$$

satisfying the following conditions. Denote by $\cdot$ the product on

$$h^*_{\mathcal{D}}(*)(X) = \bigoplus_{n,p} h^n_{\mathcal{D}}(p)(X)$$

induced by

$$h^{n_1}_{\mathcal{D}}(p_1)(X) \otimes h^{n_2}_{\mathcal{D}}(p_2)(X) \xrightarrow{\Delta^* \circ \mu} h^{n_1+n_2}(p_1+p_2)(X)$$

for $\Delta\colon X \to X \times X$ the diagonal map $\Delta(x) = (x, x)$. We require that the structure maps R and I are multiplicative, and that a satisfies for each $x \in h^*(*)(X)$ and $\omega \in H^k\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V})\right)$

$$a(\omega) \cdot x = a(\omega \wedge R(x)).$$

3.2 Deligne cohomology is Hodge filtered

The Deligne complex is defined as

$$\mathbb{Z}_{\mathcal{D}}(p) := \left(\mathbb{Z} \xrightarrow{(2\pi i)^p} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1} \longrightarrow 0 \right).$$

The Deligne cohomology groups are the (hyper-)cohomology groups of this complex,

$$H^n_{\mathcal{D}}(X; \mathbb{Z}(p)) := H^n(X; \mathbb{Z}_{\mathcal{D}}(p)).$$

Let R be the following map of complexes of sheaves:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "(2\pi i)^p"] & \Omega^0 \arrow[r, "d"] & \Omega^1 \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Omega^{p-1} \arrow[r] \arrow[d, "d"] & 0 & \\
 & & & 0 \arrow[r] & \Omega^p \arrow[r, "d"] & \Omega^{p+1} \arrow[r, "d"] & \cdots .
\end{tikzcd}$$

Let $I\colon \mathbb{Z}_{\mathcal{D}}(p) \to \mathbb{Z}$ be $(-1)^p$ times the natural projection. We consider the following diagram, where both rows are short exact sequences of complexes of sheaves:

$$\begin{tikzcd}
0 \arrow[r] & \frac{\Omega^*}{\Omega^{\geqslant p}}[1] \arrow[r, "a"] \arrow[dr, "d"'] & \mathbb{Z}_{\mathcal{D}}(p) \arrow[r, "I"] \arrow[d, "R"] & \mathbb{Z} \arrow[r] \arrow[d, "(2\pi i)^p"] & 0 \\
 & & \Omega^{\geqslant p} \arrow[r, "inc"] & \Omega^* &
\end{tikzcd} \tag{3.1}$$

Here a and *inc* denote the natural inclusions, and the [1] indicates shifting the complex 1 to the right. I.e. for a complex C, we put $(C[1])^n = C^{n-1}$. The triangle in diagram (3.1) is easily seen to be commutative, and we will now show that the square is homotopy commutative. Let σ be the operator $\Omega^* \to \Omega^*$ defined on Ω^j as multiplication with $(-1)^{p-j+1}$. We define a chain homotopy

$$H\colon \mathbb{Z}_{\mathcal{D}}(p) \to \Omega^*$$

for the square by

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "(2\pi i)^p"] & \Omega^0 \arrow[r, "d"] \arrow[dl, "\sigma"'] & \Omega^1 \arrow[r, "d"] \arrow[dl, "\sigma"'] & \cdots \arrow[r] & \Omega^{p-2} \arrow[r, "d"] \arrow[dl, "-id"'] & \Omega^{p-1} \arrow[r] \arrow[dl, "id"'] & 0 \\
\Omega^0 \arrow[r, "d"] & \Omega^1 \arrow[r] & \cdots \arrow[r, "d"] & \Omega^{p-2} \arrow[r, "d"] & \Omega^{p-1} \arrow[r, "d"] & \Omega^p \arrow[r, "d"] & \cdots .
\end{tikzcd}$$

Then

$$d \circ H + H \circ d = inc \circ R - (2\pi i)^p \circ I. \tag{3.2}$$

Diagram (3.1) establishes Deligne cohomology as a Hodge filtered cohomology theory over $(H^*(-;\mathbb{Z}), (2\pi i)^p)$. Indeed a, I and R induce the structure maps, which we denote by the same label:

$$I\colon H^n_{\mathcal{D}}(X;\mathbb{Z}(p)) \to H^n(X;\mathbb{Z})$$

$$a\colon H^{n-1}\left(X; \frac{\mathcal{A}^*}{F^p\mathcal{A}^*}\right) \to H^n_{\mathcal{D}}(X;\mathbb{Z}(p))$$

$$R\colon H^n_{\mathcal{D}}(X;\mathbb{Z}(p)) \to H^n(X;F^p\mathcal{A}^*)$$

For the latter two, we are using Theorem 2.20. The top short exact sequence of complexes of sheaves in (3.1) induces the following long exact sequence:

$$\begin{tikzcd}
h^{n-1}(X) \arrow[r, "(2\pi i)^p"] & H^{n-1}\left(X; \frac{\mathcal{A}^*}{F^p\mathcal{A}^*}\right) \arrow[r, "a"] & H^n_{\mathcal{D}}(X;\mathbb{Z}(p)) \arrow[r, "I"] & {} \\
H^n(X;\mathbb{Z}) \arrow[r] & H^n\left(X; \frac{\mathcal{A}^*}{F^p\mathcal{A}^*}\right) \arrow[r] & \cdots
\end{tikzcd}$$

Furthermore we have $R \circ a = d$, and from (3.2) it follows that the following square commutes:

$$\begin{tikzcd}
H^n_{\mathcal{D}}(X;\mathbb{Z}(p)) \arrow[r, "I"] \arrow[d, "R"'] & H^n(X;\mathbb{Z}) \arrow[d, "(2\pi i)^p"] \\
H^n(X;F^p\mathcal{A}^*) \arrow[r] & H^n(X;\mathbb{C})
\end{tikzcd}$$

We conclude:

Theorem 3.1. *Deligne cohomology is a Hodge filtered cohomology theory.*

3.3 Hopkins–Quick theories are Hodge filtered

Let E be a rationally even spectrum. We will now show that the theories $E_{\mathcal{D}}$ defined in [30, Definition 4.2] are Hodge filtered theories in the above sense. We briefly recall their construction.

Let $\mathcal{V}_*$ denote the evenly graded $\mathbb{C}$-algebra $\mathcal{V}_* = E_* \otimes \mathbb{C}$. Let $\phi_p^E \colon E \to H\mathcal{V}_*$ be a fundamental cocycle as in (2.25), where $H\mathcal{V}_*$ is the Eilenberg–MacLane spectrum of $\mathcal{V}_*$. Considering $\mathcal{V}_*$ as a constant presheaf, there is a map of presheaves $\mathcal{V}_* \to \mathcal{A}^0(\mathcal{V}_*)$ obtained as the inclusion of the constant functions. Applying H, we view ϕ_p^E as a map $E \to H(\mathcal{A}^*(\mathcal{V}_*))$. Then $E_{\mathcal{D}}(p)$ is defined as the homotopy pullback

$$\begin{tikzcd}
E_{\mathcal{D}} \arrow[r] \arrow[d] & E \arrow[d, "\phi_p^E"] \\
H(F^p\mathcal{A}^*(\mathcal{V}_*)) \arrow[r] & H(\mathcal{A}^*(\mathcal{V}_*))
\end{tikzcd}$$

in the category of spectra of simplicial presheaves on $\mathbf{Man}_{\mathbb{C}}$, which we denote $\mathrm{Sp}(\mathbf{sPre})$. The groups $E^n_{\mathcal{D}}(p)(X)$ are defined by

$$E^n_{\mathcal{D}}(p)(X) = \mathrm{Hom}_{\mathrm{hoSp}(\mathbf{sPre})}(X, \Sigma^n E_{\mathcal{D}}(p)).$$

The map of spectra ϕ_p^E induces a map of cohomology theories

$$c(p) = (\phi_p^E)_* \colon E^*(X) \to H^*(X; \mathcal{V}_*).$$

We claim that $E_{\mathcal{D}}$ is a Hodge filtered cohomology theory over $(h, c(p)) = (E, \phi_p^E)$. We first provide the requisite data: The natural transformations

$$I \colon E^n_{\mathcal{D}}(p)(X) \to E^n(X) \quad \text{and} \quad R \colon E^n_{\mathcal{D}}(p)(X) \to H^n(X; F^p\mathcal{A}^*(\mathcal{V}_*))$$

are induced by the canonical maps $E_{\mathcal{D}}(p) \to E$ and $E_{\mathcal{D}}(p) \to H(F^p\mathcal{A}^*(\mathcal{V}_*))$ respectively. It is clear that $inc_* \circ R = \phi^E(p)_* \circ I$.

To define

$$a \colon H^{n-1}\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \to E^n_{\mathcal{D}}(p)(X)$$

we consider the following pushout diagram of complexes of presheaves

$$\begin{tikzcd}
F^p\mathcal{A}^*(\mathcal{V}_*) \arrow[r] \arrow[d] & \mathcal{A}^*(\mathcal{V}_*)) \arrow[d] \\
0 \arrow[r] & \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)
\end{tikzcd}$$

where the 0 in the bottom left hand corner denotes the 0-complex. Since the Eilenberg-MacLane functor functor H is a Quillen equivalence between stable model categories by [49], it preserves homotopy pushout diagrams. Hence in the following diagram in the category of presheaves of spectra

$$\begin{tikzcd} E_{\mathcal{D}}(p) \arrow[r] \arrow[d] & E \arrow[d] \\ H(F^p\mathcal{A}^*(\mathcal{V}_*)) \arrow[r] \arrow[d] & H(\mathcal{A}^*(\mathcal{V}_*)) \arrow[d] \\ 0 \arrow[r] & H\left(\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \end{tikzcd} \tag{3.3}$$

the bottom square is a pushout square where the bottom left hand corner denotes the zero object in the category of spectra. Since the category of presheaves of spectra is stable, this implies that the outer square is homotopy cartesian as well. Taking the homotopy cofiber of the right hand vertical map, we can extend the outer square in diagram (3.3) to the following homotopy-commutative diagram:

$$\begin{tikzcd} E_{\mathcal{D}}(p) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & H\left(\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \arrow[r] & \Sigma E_{\mathcal{D}}(p) \end{tikzcd}$$

We then let a be the map induced by $H\left(\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \to \Sigma E_{\mathcal{D}}(p)$. Equivalently a is the connecting homomorphism in the long exact sequence

$$\cdots \to E^{n-1}(X) \to H^{n-1}\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \to E^n_{\mathcal{D}}(p)(X) \to E^n(X) \to \cdots$$

It remains to show that $R \circ a = d$. This follows from the diagram (3.3) as follows. We can view Diagram (3.3) as a map of homotopy-cartesian squares from the outer to the lower square. This induces a map of the corresponding long exact sequences, and so $R \circ a = d$. Concretely, we form the cofiber of the bottom right hand vertical map and get a map $H\left(\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \to \Sigma H\left(F^p\mathcal{A}^*(\mathcal{V}_*)\right)$, which induces the connecting homomorphism

$$d\colon H^{n-1}\left(X; \frac{\mathcal{A}^*}{F^p}(Vh_*)\right) \to H^n(X; F^p\mathcal{A}^*(\mathcal{V}_*)).$$

Since the composition $E \to H\left(\frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \to \Sigma H\left(F^p\mathcal{A}^*(\mathcal{V}_*)\right)$ is nullhomotopic, we get an induced map $\Sigma E_{\mathcal{D}}(p) \to \Sigma H(F^p\mathcal{A}^*(\mathcal{V}_*))$; this map is well known to be $\Sigma\left(E_{\mathcal{D}}(p) \to H(F^p\mathcal{A}^*(\mathcal{V}_*))\right)$. Hence $R \circ a = d$, as required.

We summarize this discussion in the following theorem.

Theorem 3.2. *Let E be a rationally even spectrum E and ϕ^E be a fundamental cocycle as in* (2.25). *Then $E_{\mathcal{D}}$ is a Hodge filtered cohomology theory over (E^*, ϕ^E_*).*

3.4 From differential to Hodge filtered cohomology

We will here associate a Hodge filtered cohomology theory to a differential cohomology theory, in the sense of Bunke–Schick. First we recall their axioms from [9]:

Let h be a cohomology theory, let N be a $\mathbb{Z}$-graded real vector space, and let $c\colon h^*(X) \to H^*(X;N)$ be a natural transformation. The data of a smooth extension of the pair (h,c) is a functor $X \mapsto \check{h}^*(X)$, from the category of smooth manifolds to $\mathbb{Z}$-graded groups together with natural transformations

- $R\colon \check{h}^*(X) \to \mathcal{A}^*_{\mathbb{R}}(X,N)_{cl}$
- $I\colon \check{h}^*(X) \to h^*(X)$
- $a\colon \mathcal{A}^{*-1}_{\mathbb{R}}(X;N)/\mathrm{Im}\,(d) \to \check{h}^*(X)$.

We require the following axioms to hold:

- The following diagram commutes:

$$\begin{tikzcd} \check{h}^*(X) \arrow[r, "I"] \arrow[d, "R"] & h(X) \arrow[d, "c"] \\ \mathcal{A}_{\mathbb{R}}(X;N)_{cl} \arrow[r, "Rham"] & H^*(X;N) \end{tikzcd}$$

  where *Rham* denotes taking de Rham cohomology, followed by the de Rham isomorphism.

- $R \circ a = d$.
- The sequence

$$h^{*-1}(X) \xrightarrow{c} \mathcal{A}^{*-1}_{\mathbb{R}}(X;N)/\mathrm{Im}\,(d) \xrightarrow{a} \check{h}^*(X) \xrightarrow{I} h^*(X) \longrightarrow 0$$

  is exact.

To prepare for our main construction, we complexify the differential cohomology theory $\check{h}^*(X)$. Let $N_{\mathbb{C}} := N \otimes_{\mathbb{R}} \mathbb{C}$, and let

$$c_{\mathbb{C}}\colon h^n(X) \to H^n(X;N_{\mathbb{C}})$$

denote c composed with the inclusion-map of coefficients. Let $\check{h}^n_{\mathbb{C}}(X)$ be the group

$$\check{h}^n_{\mathbb{C}}(X) := \check{h}^n(X) \bigoplus_{\mathcal{A}^{n-1}(X;N)/\mathrm{Im}\,(d)} \mathcal{A}^{n-1}(X;N_{\mathbb{C}}),$$

i.e., we represent a class $\check{x}_{\mathbb{C}} \in \check{h}^n_{\mathbb{C}}(X)$ as a pair $(\check{x},\phi)$ with $\check{x} \in \check{h}^n(X)$ and $\phi \in \mathcal{A}^{n-1}(X;N_{\mathbb{C}})$, and the pairs $(\check{x}+a(\omega),\phi)$ and $(\check{x},\phi+\omega)$ represent the same class in $\check{h}^n_{\mathbb{C}}(X)$ for each

$$\omega \in \mathcal{A}^{n-1}_{\mathbb{R}}(X;N) \subset \mathcal{A}^{n-1}(X;N_{\mathbb{C}}).$$

The complexified structure-maps are defined in the obvious way:

- $R_{\mathbb{C}} \colon \check{h}^*_{\mathbb{C}}(X) \to \mathcal{A}^*(X, N_{\mathbb{C}})_{cl}$ $\qquad R_{\mathbb{C}}(\check{x}, \phi) = R(\check{x}) + d\phi$
- $I_{\mathbb{C}} \colon \check{h}^*_{\mathbb{C}}(X) \to h^*(X)$ $\qquad I_{\mathbb{C}}(\check{x}, \phi) = I(\check{x})$
- $a_{\mathbb{C}} \colon \mathcal{A}^{*-1}(X; N_{\mathbb{C}})/\mathrm{Im}\,(d) \to \check{h}^*_{\mathbb{C}}(X)$ $\qquad a_{\mathbb{C}}(\omega) = (0, \omega)$

These maps satisfy the following properties:

- The diagram
$$\begin{tikzcd} \check{h}^n_{\mathbb{C}} \arrow[r, "I_{\mathbb{C}}"] \arrow[d, "R_{\mathbb{C}}"'] & h^n(X) \arrow[d, "c_{\mathbb{C}}"] \\ \mathcal{A}^n(X; N_{\mathbb{C}}) \arrow[r] & H^n(X; N_{\mathbb{C}}) \end{tikzcd}$$
commutes.
- $R_{\mathbb{C}} \circ a_{\mathbb{C}} = d$.
- The sequence
$$h^{*-1}(X) \xrightarrow{c_{\mathbb{C}}} \mathcal{A}^{*-1}(X; N_{\mathbb{C}})/\mathrm{Im}\,(d) \xrightarrow{a_{\mathbb{C}}} \check{h}^*_{\mathbb{C}}(X) \xrightarrow{I_{\mathbb{C}}} h^*(X) \longrightarrow 0$$
is exact.

All of these properties are easy to check. We call a quadruple $(h^*_{\mathbb{C}}, R_{\mathbb{C}}, a_{\mathbb{C}}, I_{\mathbb{C}})$ satisfying the above properties a complex differential cohomology theory, or a complex differential extension of (h, c).

Remark 3.3. The proof of the uniqueness result of [9] applies without change to complex differential cohomology theories.

Suppose now that X is a complex manifold, and let $(\check{h}^*, R, a, I)$ be a complex differential cohomology theory, over (h, c), and suppose the graded complex vector space N is evenly graded. By composing with the forgetful functor $\mathbf{Man}_{\mathbb{C}} \to \mathbf{Man}$, we can consider $\check{h}^*(X)$. We define

$$F^p\check{h}^*(X) = R^{-1}(F^p\mathcal{A}^*(X; N)) \subset \check{h}^*(X)$$

to be the group of classes with curvature in $F^p\mathcal{A}^*(X; N)$. We now define

$$\check{h}^n(p)(X) := F^p\check{h}^n(X)/a(\widetilde{F}^p\mathcal{A}^{n-1}(X; N)).$$

Since $I \circ a = 0$, $I \colon \check{h}^n(X) \to h^n(X)$ induces a map $\check{h}^n(p)(X) \to h^n(X)$ which we still denote by I. Using Proposition 2.18, we get a map

$$a \colon H^{n-1}\left(X; \frac{\mathcal{A}^*}{F^p\mathcal{A}^*}(N)\right) \to \check{h}^n(p)(X)$$

defined as the cokernel morphism in the following commutative diagram with short exact rows

$$\begin{tikzcd}
\widetilde{F}^p\mathcal{A}^{n-1}(X;N) \arrow[r] \arrow[d] & d^{-1}\left(F^p\mathcal{A}^n(X;N)\right)^{n-1} \arrow[r] \arrow[d] & H^{n-1}\left(X;\frac{\mathcal{A}^*}{F^p\mathcal{A}^*}(N)\right) \arrow[d] \\
a\left(\widetilde{F}^p\mathcal{A}^{n-1}(X;N)\right) \arrow[r] & F^p\check{h}^n(X) \arrow[r] & \check{h}^n(p)(X)
\end{tikzcd}$$

where all vertical arrows are induced by a.

Since $R \circ a = d$, the map $R : \check{h}^n(X) \to \mathcal{A}^n(X;N)_{cl}$ induces

$$R\colon \check{h}^n(p)(X) \to \frac{F^p\mathcal{A}^n(X;N)_{cl}}{dF^p\mathcal{A}^{n-1}(X;N)} \simeq H^n(X;F^p\mathcal{A}^*(N)),$$

where the isomorphism comes from Proposition 2.18.

Theorem 3.4. *Let $(\check{h}^*, R, a, I)$ be a complex differential extension of (h, c). Then, with the structure maps defined above, $(\check{h}^n(p), R, a, I)$ is a Hodge filtered cohomology theory over (h, c).*

Proof. Having defined all the data, it remains to check that they satisfy the required properties. The equality $R \circ a = d$ is clear. It is also clear that the diagram

$$\begin{tikzcd}
\check{h}^n(p)(X) \arrow[r, "I"] \arrow[d, "R"'] & h^n(X) \arrow[d, "c"] \\
H^n(X;F^p\mathcal{A}^*(N)) \arrow[r] & H^n(X;N)
\end{tikzcd}$$

commutes, since the corresponding diagram for $\check{h}^n(X)$ commutes. It remains to show exactness of the sequence

$$\cdots \longrightarrow H^{n-1}\left(X;\tfrac{\mathcal{A}^*}{F^p}(N)\right) \xrightarrow{a} \check{h}^n(p)(X) \xrightarrow{I} h^n(X) \xrightarrow{\bar{c}(p)} H^n\left(X;\tfrac{\mathcal{A}^*}{F^p}(N)\right).$$

We start with exactness at $\check{h}^n(p)(X)$. $I \circ a = 0$ is clear. Suppose $I([\check{x}]) = 0$ for $\check{x} \in F^p\check{h}^n(X)$. Then $\check{x} = a(\omega)$ for some $\omega \in \mathcal{A}^{n-1}(X;N)$. Since $R \circ a = d$, and $R(\check{x}) \in F^p\mathcal{A}^n(X;N)$, we have $\omega \in d^{-1}(F^p\mathcal{A}^n(X;N))$. Letting $[\omega]$ denote the class represented by ω in $H^{n-1}\left(X;\frac{\mathcal{A}^*}{F^p}(N)\right)$, we have $a([\omega]) = [\check{x}] \in \check{h}^n(p)(X)$, proving exactness at $\check{h}^n(p)(X)$. Next we show exactness at $h^n(X)$. To see that $\bar{c}(p) \circ I = 0$, note that $c \circ I(\check{x}) = [R(\check{x})] \in H^n(X;N)$. This class vanishes in $H^n\left(X;\frac{\mathcal{A}^*}{F^p}(N)\right)$, since $R(\check{x}) \in H^n(X;F^p\mathcal{A}^*(N))$ and the sequence

$$H^n(X;F^p\mathcal{A}^*(N)) \to H^n(X;N) \to H^n\left(X;\frac{\mathcal{A}^*}{F^p}(N)\right)$$

is exact. Now we suppose that $\overline{c}(p)(x) = 0$. Then $c(x)$ is in the image of

$$H^n(X; F^p\mathcal{A}^*(N)) \to H^n(X; N).$$

Hence we can find a form $\omega \in F^p\mathcal{A}^n(X;N)$ representing $c(x)$. We can find $\check{x} \in \check{h}^n(X)$ with $R(\check{x}) = \omega$ and $I(\check{x}) = x$. Then $\check{x} \in F^p\check{h}^n(X)$ represents a class in $\check{h}^n(p)(X)$ such that $I([\check{x}]) = x$. Finally we need to show exactness at $H^n\left(X; \frac{\mathcal{A}^*}{F^p}(N)\right)$. This follows easily from exactness of the sequence

$$h^{n-1}(X) \to \mathcal{A}^{n-1}(X;N)/\mathrm{Im}\,(d) \to \check{h}^n(X). \qquad \blacksquare$$

Remark 3.5. Note that if we have a geometric description of $\check{h}^n(X)$, the construction of $\check{h}^n(p)(X)$ can be done geometrically, in the following sense:

- Consider only the geometric $\check{h}^*(X)$ cycles with curvature in $F^p\mathcal{A}^*(X;N)$.
- In addition to the equivalence relation you were going to impose in order to create $\check{h}^*(X)$, also impose the relation $a(\widetilde{F^p}\mathcal{A}^*(X;N)) \sim 0$.

In retrospect, the construction of $MU^n(p)(X)$ in Section 4 follows this recipe. The present section came about as a meditation on the adaptation of smooth cobordism [9] to a Hodge filtered cohomology theory, in Section 4.

3.5 Thoughts on uniqueness

We have attempted to prove that the axioms of Section 3.1 are strong enough to characterize a unique Hodge filtered cohomology theory. Our most serious attempt was inspired by the successful proof in [9] of the corresponding theorem for differential cohomology theories. We give a brief summary of their construction: Let E be an Ω-spectrum, so that the nth space E_n represent E^n in the sense that $E^n(X) = [X, E_n]$ for any CW-complex X. Then we can approximate E_n by a sequence of manifolds, E_{ni}, by letting E_{ni} be a manifold with the homotopy type of the ith skeleton of E_n. Next, if X is a manifold, then up to homotopy, any map $f\colon X \to E_n$ factor through some finite skeleton. Since smooth maps are dense in continuous maps, we can assume f to be a smooth map $f\colon X \to E_{ni}$ for i large enough. The proof of the uniqueness theorem of [9] proceeds by constructing compatible universal classes $x_{ni} \in \hat{E}^n(E_{ni})$, where $\hat{E}^n$ is a differential extension. Then it is shown that any class $\widehat{x} \in \hat{E}^n(X)$ can be written in the form $f^*\hat{x}_{ni} + a(\omega)$. If $\hat{E}_1^n$ is a second differential extension, we use the same manifolds E_{ni}, and construct compatible universal classes $\hat{x}'_{ni} \in \hat{E}_1(E_{ni})$. In order construct a map $\hat{E}^n \to \hat{E}_1^n$ it suffices to put $\hat{x}_{ni} \mapsto \hat{x}'_{ni}$. Then naturality and compatability with the structure maps uniquely extends this to a map of differential cohomology theories. Under mild assumptions, this is shown to be an isomorphism.

In the Hodge filtered setting, it is not true that any homotopy class of maps between complex manifolds can be represented by a holomorphic map. The exception is that if S is Stein, and X is Oka, then the inclusion $\mathcal{O}(S, X) \to C(S, X)$ is a weak homotopy equivalence, see for example [36] or [17]. Indeed already in [24, 0.7.B.], Gromov suggests using Oka manifolds to encode topological information in holomorphic terms. So let $E^n(p)$ and $E_1^n(p)$ be two Hodge filtered extensions of $(E, c(p))$. Suppose we can approximate E_n using Oka manifolds E_{ni}, and find compatible classes $\hat{x}_{ni} \in E^n(p)(E_{ni})$ and $\hat{x}'_{ni} \in E_1^n(p)(E_{ni})$. Then it is true that the assignment $\hat{x}_{ni} \mapsto \hat{x}'_{ni}$ defines a unique map $E^n(p) \to E_1^n(p)$, but only after we restrict to the Stein site. It turns out to be an unsolved problem in complex analysis, weather any finite CW-complex has the homotopy type of an Oka manifold. This was first asked in [24, 0.7.B". Problem]. No negative results are known. Furthermore, for the construction of compatible classes $\hat{x}_{ni} \in E^n(p)(E_{ni})$, we assumed that E_{ni} was Stein in addition to being Oka. It is not known weather any Oka manifold is weakly equivalent, in the sense of [37], to a manifold which is both Oka and Stein. It might be possible to eliminate the assumption that E_{ni} is Stein.

The only example of a cohomology theory we could find which is representable by Oka-Stein manifolds is K-theory. Hence we are able to prove that the axioms of Section 3.1 suffice to characterize Hodge filtered extensions of K-theory on the Stein site. If it turns out that the homotopy types of the Oka-Stein manifolds are exactly the homotopy types of smooth manifolds, the corresponding result holds for Hodge filtered extensions of any rationally even cohomology theory.

We made no progress on the problem of globalizing from Stein manifolds to arbitrary complex manifolds.

Let us add one final rather vague remark. It seems likely that the Hopkins–Quick theories in some sense are the maximally complicated Hodge filtered extensions, while the Hodge filtered extensions associated to differential cohomology theories are minimally complicated. If we accept this heuristic, it is interesting to note that in retrospect, our geometric Hodge filtered complex cobordism $MU^n(p)$ of Section 4 is the Hodge filtered cohomology theory associated to differential bordism, as defined in [8]. We prove in Theorem 6.33 that $MU^n(p)$ is also isomorphic to Hopkins–Quick Hodge-filtered complex bordism. Maybe this is some heuristic evidence that our axioms suffice to characterize Hodge filtered extensions.

Chapter 4

Geometric Hodge filtered cobordism

The purpose of this chapter is to give a geometric description of Hodge filtered complex cobordism as a Hodge filtered extension of complex cobordism for complex manifolds. Most of the chapter could have been written in the style of [34, §5], with F^p being a (more or less) arbitrary filtration of the de Rham complex $\mathcal{A}^*$. We will focus, however, on F^p being the Hodge filtration.

The chapter is organised as follows. We will first recall the cycle description of $MU^*(X)$, for X a manifold. We then discuss complex genera, their extensions to maps of cohomology theories and Chern–Weil theory. Then we are ready to define geometric Hodge filtered cobordism, and show some basic properties. We end the chapter by constructing a "currential" geometric Hodge filtered cobordism group. This currential group is convenient for the construction of the pushforward in Chapter 7.

4.1 Cycle description of complex cobordism

Let X be a smooth manifold. We give a geometric description of $MU^n(X)$, similar to that in [9]. This sort of description of $MU^n(X)$ was first given in [47]

4.1.1 Complex cobordism cycles

We denote by $\underline{\mathbb{R}}^k_X$ ($\underline{\mathbb{C}}^l_X$) the trivial real (complex) vector bundle of rank k (l) on X. In the case $k = 1$ and $l = 1$, we just write $\underline{\mathbb{R}}_X$ and $\underline{\mathbb{C}}_X$. When the base-space is clear, or irrelevant, we omit the subscript. When no confusion can arise, we identify $\underline{\mathbb{R}}^k \oplus \underline{\mathbb{C}}^l$ with $\underline{\mathbb{R}}^{k+2l}$ using the isomorphism $\underline{\mathbb{R}}^{k+2l} \to \underline{\mathbb{R}}^k \oplus \underline{\mathbb{C}}^l$ given by

$$(x_1, \cdots, x_{k+2l}) \mapsto ((x_1, \cdots x_k),\ (x_{k+1} + ix_{k+2}, \cdots, x_{k+2l-1} + ix_{k+2l}))$$

where i denotes the imaginary unit. Let $f : Z \to X$ be a smooth map. A complex orientation of f is represented by a pair (N, Φ) where $N \to Z$ is a complex vector bundle, and Φ is a short exact sequence of real vector bundles

$$\Phi = \left(0 \longrightarrow TZ \xrightarrow{(Df,\varphi_1)} f^*TX \oplus \underline{\mathbb{R}}^k_Z \xrightarrow{\varphi_2} N \longrightarrow 0 \right).$$

Another representative, denoted by primes, (N', Φ'), is isomorphic to (N', Φ') if $\varphi_1 = \varphi_1'$, and there is an isomorphism $\psi : N \to N'$ such that $\varphi_2' = \psi \circ \varphi_2$. For homotopies of bundle maps φ_1^t and φ_2^t, such that

$$\Phi_t = \left(0 \longrightarrow TZ \xrightarrow{(Df,\varphi_1^t)} f^*TX \oplus \underline{\mathbb{R}}^k_Z \xrightarrow{\varphi_2^t} N \longrightarrow 0 \right).$$

is short exact for each t, we say that (Φ_1, N) and (Φ_0, N) are homotopic. Furthermore we say that $(N \oplus \underline{\mathbb{C}}^l_Z, \Phi(l))$ defined by

$$\Phi(l) = \left(0 \longrightarrow TZ \xrightarrow{(Df,\varphi_1)} f^*TX \oplus \underline{\mathbb{R}}^k_Z \oplus \underline{\mathbb{C}}^l_Z \xrightarrow{\varphi_2 \oplus id_{\underline{\mathbb{C}}^l_Z}} N \oplus \underline{\mathbb{C}}^l_Z \longrightarrow 0 \right).$$

is stably equivalent with (N, Φ). A complex orientation of f is then an equivalence class of pairs (N, Φ), defined modulo isomorphism, homotopy and stable equivalence. We will ease notation and refer to f as a complex oriented map. If we need a representative (N, Φ) of the complex orientation of f, we will write $N_f := N$ and $\Phi_f := \Phi$. An isomorphism of complex oriented maps $f_1 \colon Z_1 \to X$, $f_2 \colon Z_2 \to X$, is a diffeomorphism $\psi \colon Z_1 \to Z_2$ such that $f_2 \circ \psi = f_1$ and $(\psi^* N_{f_2}, \psi^*(\Phi_{f_2}))$ represent the same complex orientation of f_1 as (N_{f_1}, Φ_{f_1}).

Definition 4.1. *A complex cobordism cycle, or cycle for short, of degree n on X is an isomorphism class of proper complex oriented maps*

$$f \colon Z \to X$$

of codimension $\operatorname{codim} f = \dim X - \dim Z = -n$. *We let $ZMU^n(X)$ be the monoid of cycles over X of degree n. The monoid structure is induced by disjoint union, with the empty cycle as identity. We will denote the cobordism cycle represented by f also by f.*

4.1.2 Exterior product

Let $f_i \colon Z_i \to X_i$ be proper, complex oriented maps for $i = 1, 2$. Then the product-map $f_1 \times f_2 \colon Z_1 \times Z_2 \to X_1 \times X_2$ is proper and has a natural complex orientation, which we now describe. Let $\pi_i \colon Z_1 \times Z_2 \to Z_i$ denote the projection for $i = 1, 2$. We have canonical isomorphisms

$$T(Z_1 \times Z_2) \simeq \pi_1^* TZ_1 \oplus \pi_2^* TZ_2$$

and

$$(f_1 \times f_2)^* T(X_1 \times X_2) \simeq \pi_1^* f_1^* TX_1 \oplus \pi_2^* f_2^* TX_2.$$

Hence we can define the product complex orientation by $(\Phi_{f_1 \times f_2}, N_{f_1 \times f_2})$, where $N_{f_1 \times f_2} = \pi_1^* N_{f_1} \oplus \pi_2^* N_{f_2}$, and $\Phi_{f_1 \times f_2}$ is the term-wise direct sum

$$\Phi_{f_1 \times f_2} := \pi_1^* \Phi_{f_1} \oplus \pi_2^* \Phi_{f_2}.$$

The exterior product is associative on cycles, because there is an obvious isomorphism of complex oriented maps

$$f_1 \times (f_2 \times f_3) \simeq (f_1 \times f_2) \times f_3.$$

4.1.3 Pullback of cycles

Let $f\colon Z \to X$ be a cycle and let $g\colon Y \to X$ be transverse to f. Then proposition (2.2) allows us to form the following pullback diagram, where f' is proper whenever f is, and satisfies $\operatorname{codim} f' = \operatorname{codim} f$.

$$\begin{tikzcd} Z' \arrow[r, "g'"] \arrow[d, "f'"'] & Z \arrow[d, "f"] \\ Y \arrow[r, "g"] & X \end{tikzcd} \tag{4.1}$$

We now define the induced complex orientation on f'. Represent the complex orientation of f by

$$\Phi_f = \left(0 \longrightarrow TZ \xrightarrow{(Df,\varphi_1)} f^*TX \oplus \underline{\mathbb{R}}^k_{\mathbb{R}} \xrightarrow{\varphi_2} N_f \longrightarrow 0 \right).$$

Consider the following commutative diagram of vector bundles on Z'. To ease notation, we omit writing all the pullbacks; in our opinion $Df\colon TZ \to TX$ is clearer than $g'^*(Df)\colon g'^*TZ \to g'^*f^*TX$.

$$\begin{tikzcd} TZ' \arrow[r, "Dg'"] \arrow[d, "(Df',\varphi_1')"'] & TZ \arrow[r] \arrow[d, "(Df,\varphi_1)"] & \frac{TZ}{Dg'(TZ')} \arrow[d] \\ TY \oplus \underline{\mathbb{R}}^k \arrow[r, "Dg\oplus id_{\underline{\mathbb{R}}^k}"] \arrow[d, "\varphi_2'"'] & TX \oplus \underline{\mathbb{R}}^k \arrow[r] \arrow[d, "\varphi_2"] & \frac{TX}{Dg(TY)} \\ N_f \arrow[r, "\mathrm{id}"] & N_f & \end{tikzcd} \tag{4.2}$$

Here $\varphi_1' = \varphi_1 \circ Dg'$, and $\varphi_2' = \varphi_2 \circ (Dg \oplus id_{\underline{\mathbb{R}}^k_Z})$. The middle column and the first two rows are short exact sequences of vector bundles. The map

$$\frac{g'^*TZ}{Dg'(TZ')} \to \frac{g'^*f^*TX}{f'^*Dg(TY)}$$

induced by the differential Df is onto since $g \pitchfork f$; then it is an isomorphism since the two bundles have the same dimension. It follows from the snake lemma that φ_2' is onto. Since $(f', g')\colon Z' \to Y \times Z$ is an embedding, it follows that (Df', Dg') is injective. Then $(Df', (Df, \varphi_1) \circ Dg')$ is injective. Since $Df \circ Dg' = Dg \circ Df'$, we conclude that (Df', φ_1') is injective. Finally it is clear that $\varphi_2' \circ (Df', \varphi_1') = 0$. Put $N' := g'^*N_f$. For dimensional reasons we can conclude that the sequence

$$\Phi' = \left(0 \longrightarrow TZ' \xrightarrow{(Df',\varphi_1')} f'^*TY \oplus \underline{\mathbb{R}}^k_{Z'} \xrightarrow{\varphi_2'} N' \longrightarrow 0 \right).$$

is short exact. We give f' the complex orientation represented by (N', Φ'). It is clear that the stable normal complex structure represented by (N', Φ') depends only on the stable normal complex structure represented by (N, Φ). Hence we have defined a pullback map

$$g^* \colon ZMU^n_g(X) \to ZMU^n(Y)$$

where

$$ZMU^n_g(X) = \{f \in ZMU^n(X) \ : \ g \pitchfork f\}.$$

for $g_1 \colon X_1 \to X_2$ and $g_2 \colon X_2 \to X_3$, we put

$$\begin{aligned} ZMU^n_{g_1 g_2}(X_3) &= \{f \in ZMU^n(X_3) \ : \ g_2 \pitchfork f \text{ and } g_2 \circ g_1 \pitchfork f\} \\ &= ZMU^n_{g_2}(X_3) \cap ZMU^n_{g_2 \circ g_1}(X_3). \end{aligned}$$

Then g_2^* restricts to a map

$$g_2^* \colon ZMU^n_{g_1 g_2}(X_3) \to ZMU^n_{g_1}(X_2),$$

and we have

$$g_1^* \circ g_2^* = (g_2 \circ g_1)^* \colon ZMU^n_{g_1 g_2}(X_3) \to ZMU^n(X_1).$$

This observation essentially follows from the following cartesian diagram:

$$\begin{tikzcd} Z'' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & Z \arrow[d, "f"] \\ X_1 \arrow[r, "g_1"'] & X_2 \arrow[r, "g_2"'] & X_3 \end{tikzcd}$$

We note that since transversality is a generic property, $ZMU^n_{g_1 g_2}(X_3)$ is "almost all" of $ZMU^n(X_3)$. For maps $g_1 \colon Y_1 \to X$ and $g_2 \colon Y_2 \to X$, we also write

$$ZMU^n_{g_1, g_2}(X) = ZMU^n_{g_1}(X) \cap ZMU^n_{g_2}(X). \tag{4.3}$$

4.1.4 Composition of complex oriented maps

Let $g_i \colon X_i \to X_{i+1}$ be a complex oriented map for $i = 1, 2$. We now describe the composed complex orientation on $g_2 \circ g_1$. We put $N_{g_2 \circ g_1} = N_{g_1} \oplus g_1^* N_{g_2}$. Choose a splitting $s \colon g_2^* TX_3 \oplus \underline{\mathbb{R}}^{k_2}_{X_2} \to TX_2$. We then get the following commutative diagram, where the first row is Φ_{g_1}, and the second row is obtained from $g_1^* \Phi_{g_2}$ by adding $id_{\underline{\mathbb{R}}^{k_1}}$ to the first map.

$$\begin{tikzcd} TX_1 \arrow[r] \arrow[d] \arrow[dr] & g_1^* TX_2 \oplus \underline{\mathbb{R}}^{k_1} \arrow[r] & N_1 \\ g_1^* TX_2 \oplus \underline{\mathbb{R}}^{k_1} \arrow[r] & g_1^* g_2^* TX_3 \oplus \underline{\mathbb{R}}^{k_2} \oplus \underline{\mathbb{R}}^{k_1} \arrow[r] \arrow[u, "(g_1^* s) \oplus id_{\underline{\mathbb{R}}^{k_1}}"'] & g_1^* N_2 \end{tikzcd}$$

We claim that the induced sequence

$$\Phi_{g_2 \circ g_1} := \left(0 \longrightarrow TX_1 \longrightarrow g_1^* g_2^* TX_3 \oplus \underline{\mathbb{R}}^{k_2} \oplus \underline{\mathbb{R}}^{k_1} \xrightarrow{\varphi} N_1 \oplus g_1^* N_2 \longrightarrow 0 \right)$$

is short exact. Injectivity and surjectivity is clear. It also clear that the composition is 0. Then exactness follows for dimensional reasons. Any two choices for the splitting, s, s' yield homotopic short exact sequences. Indeed $t \cdot s + (1-t) \cdot s'$ is a homotopy between splittings s and s', which induces the required homotopies.

4.1.5 Basic examples of cycles and pullback

Consider $id_X \colon X \to X$. It has a natural complex orientation represented by

$$0 \to TX \to TX \to 0 \to 0.$$

We denote id_X with this complex orientation by $1_X \in ZMU^n(X)$. For $\pi \colon X \to pt$, we have $1_X = \pi^*(1_{pt})$. There is another natural complex orientation of id_X represented by

$$0 \longrightarrow TX \xrightarrow{(id,0)^t} TX \oplus \underline{\mathbb{R}}^2_{\mathbb{R}} \xrightarrow{(0,-1,i)} \underline{\mathbb{C}} \longrightarrow 0$$

where we use matrix notation and view elements of direct sums as column-vectors, i.e., $(0,-1,i)(v_x, a_1, a_2)^t = -a_1 + i \cdot a_2$. We denote id_X with this complex orientation by $-1_X \in ZMU^n(X)$. We have the following equalities in $ZMU^0(X)$

$$-1_X = \pi^*(-1_{\mathrm{pt}}) = (-1_{\mathrm{pt}}) \times 1_X.$$

We note that $-1_{pt} \times -1_{pt} = 1_{pt}$. This follows from the fact that two automorphisms of $\mathbb{R}^n$ are homotopic if and only if their determinants have the same sign. This also distinguish -1_{pt} from 1_{pt} as cycles.

Next, let

$$a \colon \mathbb{R} \to \mathbb{R} \tag{4.4}$$

be the map $a(t) = t^2 - t$. With respect to the canonical coordinates, the matrix of $D_t a$ is $2t-1$. Viewing $T\mathbb{R}$ as a trivial bundle, we give a the complex orientation represented by

$$0 \longrightarrow T\mathbb{R} \xrightarrow{(2t-1,1,0)^t} a^*T\mathbb{R} \oplus \underline{\mathbb{R}}^2 \xrightarrow{(1,1-2t,i)} \underline{\mathbb{C}} \longrightarrow 0\ .$$

Since 0 is a regular value of a, a is transverse to the map $\iota \colon pt \to \mathbb{R}$ where $pt = \{*\}$ is the one-point manifold, and $\iota(*) = 0$. Let us examine $a' = \iota^* a$. As a map, it is of course the inclusion $\{0,1\} \hookrightarrow \mathbb{R}$. The complex orientation is given by

$$0 \longrightarrow T\{0,1\} \xrightarrow{(Da', \varphi_1')^t} a'^*T\{0\} \oplus \underline{\mathbb{R}}^2 \xrightarrow{(1-2t,i)} \underline{\mathbb{C}} \longrightarrow 0$$

which amounts to the isomorphism $\underline{\mathbb{R}}^2 \to \underline{\mathbb{C}}$ given by $(x_1, x_2) \mapsto (x_1 + ix_2)$ over 0, and the isomorphism $(x_1, x_2) \mapsto (-x_1, x_2)$ over 1. Hence we have

$$\iota^* a = 1_{pt} + (-1_{pt}) \in ZMU^0(pt).$$

This justifies the notation -1; as we can see, $1_{pt} + (-1_{\mathrm{pt}})$ is a boundary in a sense we now make precise.

4.1.6 Bordism data

Definition 4.2. *Let $i_t\colon X \to \mathbb{R} \times X$ be the map $i_t(x) = (t, x)$. A bordism datum over X is a cycle*

$$b \in ZMU^n_{i_0,i_1}(\mathbb{R} \times X),$$

as defined in (4.3)*. More concretely, a bordism datum over X is a proper complex oriented map $b = (a_b, f_b)\colon W \to \mathbb{R} \times X$ such that* 0 *and* 1 *are regular values for a. We then define the boundary cycle*

$$\partial b := i_1^*(b) + (-1_{pt}) \times i_0^*(b)$$

where $i_t^*(b)$ is considered with the induced complex orientation. We define

$$BMU^n(X) \subset ZMU^n(X)$$

as the submonoid generated by the elements ∂b as b range over bordism data over X.

Remark 4.3. Given a bordism datum over X, $b = (a_b, f_b)$, we can consider $b' = (a \times id_X) \circ (a_b, f_b)$, where a is the map (4.4). Then b' is also a bordism datum over X. We have $i_1^* b' = 0$, and

$$i_0^* b' = (-1_{pt}) \times i_1^*(b) + i_0^*(b) = (-1_{pt}) \times \partial b$$

It follows that $BMU^n(X)$ also can be described as the submonoid generated by $i_0^* b$ for $b = (a_b, f_b)$ a bordism datum with a_b bounded.

Remark 4.4. Let us say a word about why we reverse the orientation in $\partial b = i_1^* b + (-1_{pt}) \times i_0^* b$. We compare the example $a\colon \mathbb{R} \to \mathbb{R}$, $a(t) = t^2 - t$ in (4.4) to the example $1_{\mathbb{R}} \in ZMU^0_{\pitchfork i_0,i_1}(\mathbb{R})$. We have $i_0^* 1_{\mathbb{R}} = i_1^* 1_{\mathbb{R}} = 1_{pt}$, for example because $1_{\mathbb{R}} = \pi^*(1_{\mathrm{pt}})$ and $\pi \circ i_0 = \pi \circ i_1$, where $\pi\colon \mathbb{R} \to pt$ is the only map. We should view a and $1_{\mathbb{R}}$ as essentially the same bordism datum over pt; it is just that a is bent and turned. So we should describe the induced complex orientation on $i_t^* b$ in a way that is invariant under "bending" or "turning around". The correct concept is the induced orientation: If W is an oriented manifold with boundary, let $\omega \in \mathcal{A}^*(W)$ be an orientation form. Then we orient ∂W by the form $o \lrcorner \omega$, for o an outward-pointing vector field $o \in TW|_{\partial W}$. Write $W_t = a_b^{-1}(t)$, and let NW_t denote the normal bundle of W_t in W. In the definition of $i_t^* b$, we use the isomorphism $NW_t \simeq a_b^*(N\{t\}) \simeq \mathbb{R}$ where the first isomorphism induced by the differential Da_b, and the second by the coordinate vector-field on $\mathbb{R}$. For the purposes of getting the correct complex orientation of the boundary, we should instead use the isomorphism of an outward pointing vector field. On $1 \in [0, 1]$, the coordinate vector-field of $\mathbb{R}$ is outward pointing, while on 0 it is not.

4.1.7 Complex cobordism groups of X

By taking disjoint unions of bordism data over X, the relation $x \sim y$ for $x, y \in ZMU^n(X)$ such that $x + (-1_{pt}) \times y \in BMU^n(X)$ is a congruence, in the sense that if $x_1 \sim y_1$ and $x_2 \sim y_2$, then $x_1 + x_2 \sim y_1 + y_2$. Hence the following definition is valid:

Definition 4.5. *The geometric n-th cobordism group of X is defined by*

$$MU^n(X) = \frac{ZMU^n(X)}{BMU^n(X)}.$$

Proposition 4.6. *For each $(f\colon Z \to X) \in ZMU^n(X)$, we have*

$$f + (-1_{pt} \times f) \in BMU^n(X).$$

In particular, $MU^n(X)$ is a group, with inverses given by $-[f] = [(-1_{pt}) \times f]$.

Proof. Let $\pi_X : \mathbb{R} \times X \to X$ be the projection, and let b the the bordism datum $b = \pi_X^* f$. The underlying map of b is

$$b = \mathrm{id} \times f \colon \mathbb{R} \times Z \to \mathbb{R} \times X.$$

Then $\partial b = f + (-1_{pt} \times f)$. ■

From now on we write $-f := -1_{pt} \times f$, even though $f - f \neq 0 \in ZMU^n(X)$. Let us show that the cobordism relation is stronger than the homotopy relation:

Proposition 4.7. *Let $f_0 \colon Z \to X$ be a proper, complex oriented map, and let f_1 be a proper map which is homotopic to f_0. Then there is a complex orientation of f_1 such that $f_0 - f_1 \in BMU^n(X)$.*

We first prove a lemma:

Lemma 4.8. *In the context of Proposition* (4.7), *there exist a smooth map*

$$H \colon \mathbb{R} \times Z \to X$$

such that $H \circ i_t = f_t$ for $i_t \colon Z \to \mathbb{R} \times Z$ given by $i_t(z) = (t, z)$, $t = 0, 1$. Furthermore, we can make H such that there is $\epsilon > 0$ so that for $t < \epsilon$ we have $H(t, z) = H(0, z)$ and $H(1 - t, z) = H(1, z)$.

Proof. Let $H' \colon I \times X \to Y$ be a homotopy between f_0 and f_1. Define $H'' \colon \mathbb{R} \times X \to Y$ by

$$H''(t, x) = \begin{cases} H'(0, x) & t \leqslant \frac{1}{3} \\ H'(\frac{3t}{2}, x) & \frac{1}{3} \leqslant t \leqslant \frac{2}{3} \\ H'(1, x) & t \geqslant \frac{2}{3} \end{cases}$$

Then H'' is continuous everywhere and smooth outside of $[1/3, 2/3]$. We get H as in the statement of the lemma by an application of [26, Theorem 2.5, p.48]. The point is that we need not perturb H'' on $(-\infty, 1/4] \cup [3/4, \infty)$ since H'' is already smooth on a neighborhood of that set. ■

Proof of Proposition 4.7. There is a smooth homotopy $H\colon \mathbb{R} \times Z \to X$ between f_0 and f_1 by Lemma 4.8. We can choose a connection for H^*TX. Then parallel transport along the curves $t \mapsto (t, z)$ gives an isomorphism between f_0^*TX and f_1^*TX. This allows us to use the complex orientation of f_0 to define one for f_1. The space of connections is connected, as we will show in Proposition 4.15, and a path of connections induces a homotopy of isomorphisms between f_0^*TX and f_1^*TX. Therefore the obtained complex orientation of f_1 is independent of the choice of connection. To obtain the requisite bordism datum, we play this game with the map

$$\begin{aligned} b\colon \mathbb{R} \times Z &\to \mathbb{R} \times X \\ b(t, z) &= (t, H(t, z)). \end{aligned}$$

We observe that b is homotopic to the map $\mathrm{id}_R \times f_0$, which is complex oriented using the orientation of id_R representing $1_{\mathbb{R}}$. For a concrete homotopy, take $s \mapsto \big((t, z) \mapsto (t, H(st, z)\big)$. Then the complex orientation of $\mathrm{id}_{\mathbb{R}} \times f_0$ endows b with a complex orientation. Then b is a bordism datum, and we have $\partial b = f_1 - f_0$. ■

We now establish that MU^n is a contravariant functor. Let $g\colon Y \to X$ be a smooth map. We define the pullback by

$$\begin{aligned} g^*\colon MU^n(X) &\to MU^n(Y) \\ g^*[f] &= [g^* f] \end{aligned} \tag{4.5}$$

where g^*f is defined in (4.1) when $g \pitchfork f$.

Lemma 4.9. *The pullback* (4.5) *is well defined and makes MU^n a contravariant functor.*

Proof. First we note that by Proposition 4.7 and Thom's transversality theorem, here recalled in Theorem 2.4, we can represent $[f]$ by a cycle with underlying map f such that $f \pitchfork g$. We must show that (4.5) is independent of the choice of such a cycle f. Let $b = (a_b, f_b)\colon W \to \mathbb{R} \times X$ be a bordism datum over X. We can assume that f_b is transverse to g by Thom's transversality theorem. Indeed, on a neighborhood of $a_b^{-1}(\{0,1\})$, g is transverse to f_b, so we need only perturb f_b away from $a^{-1}(\{0, 1\})$, so that (a_b, f_b) remains a cobordism between f_0 and f_1 through the perturbation. We still have a complex orientation, also after the perturbation, by the technique in the proof of Proposition 4.7. Then we form the following pullback square

$$\begin{array}{ccc} W' & \xrightarrow{(a_b', f_b')} & \mathbb{R} \times Y \\ \downarrow & & \downarrow{\scriptstyle id_{\mathbb{R}} \times g} \\ W & \xrightarrow{(a_b, f_b)} & \mathbb{R} \times X \end{array}$$

and give $b' = (a'_b, f'_b)$ the induced complex orientation. For $t = 0, 1$ we now consider the following diagram where all squares are pullbacks:

$$
\begin{tikzcd}
 & & W_t \arrow[d] \arrow[drr] & & \\
W'_t \arrow[urr] \arrow[d] \arrow[drr] & & t \times X \arrow[drr] & & W \arrow[d, "{(a_b, f_b)}"] \\
t \times Y \arrow[urr] \arrow[drr] & & W' \arrow[urr] \arrow[d, "{(a'_b, f'_b)}"] & & \mathbb{R} \times X \\
 & & \mathbb{R} \times Y \arrow[urr, "id_{\mathbb{R}} \times g"'] & &
\end{tikzcd}
$$

This diagram proves $g^*\partial b = \partial(id_{\mathbb{R}} \times g)^* b$, so (4.5) is well defined. ■

4.1.8 Ring structure

We now make $MU^*(X) = \bigoplus_n MU^n(X)$ into a graded ring. We first note that the exterior product of cycles induces a map

$$\times \colon MU^n(X) \times MU^m(Y) \to MU^{n+m}(X \times Y).$$

Indeed, if we have a bordism datum $b = (a_b, f_b) \colon W \to \mathbb{R} \times X$, and a cycle $f \colon Z \to Y$, then $b \times f \colon W \times Z \to \mathbb{R} \times X \times Y$ is a bordism datum, and clearly $\partial(b \times f) = (\partial b) \times f$.

Let $\Delta \colon X \to X \times X$ be the diagonal, $\Delta(x) = (x, x)$. We define the product on

$$MU^*(X) := \bigoplus_n MU^n(X)$$

by

$$[f_1] \cdot [f_2] := \Delta^*([f_1] \times [f_2]).$$

We note that any map $g \colon X \to Y$ makes $MU^*(X)$ into a $MU^*(Y)$ module. In particular, for each manifold X, we see that $MU^*(X)$ is a $MU^* := MU^*(\mathrm{pt})$-module. We define the graded ring MU_* to have the same underlying ring structure as MU^*, but with the grading $MU_n = MU^{-n}$.

4.2 Genera

In this section, we recall some material from [27]. We recommend also [43, §19]. Recall that a (complex) genus is a homomorphism of rings $\phi \colon MU_* \to R$. If R is an integral domain over $\mathbb{Q}$, there is a nice classification of genera $MU_* \to R$, which we now discuss. Let $E \to X$ be a complex vector bundle. We will use the well known splitting principle. For a proof we refer to [27, Section 4.4].

Theorem 4.10. *There exist a map $\pi : Y \to X$ such that*

$$\pi^*\colon H^*(X;\mathbb{Z}) \to H^*(Y;\mathbb{Z})$$

is a monomorphism, and

$$\pi^* E \simeq L_1 \oplus \cdots \oplus L_n$$

for line bundles L_i on Y.

Using the splitting principle we can often pretend that the total Chern class of E factorize as

$$c(E) := 1+c_1(E)+c_2(E)+\cdots+c_n(E) = (1+x_1(E))(1+x_2(E))\cdots(1+x_n(E)).$$

Namely, we can compute with $\pi^*(c(E)) = c(\pi^*E)$, which does factorize as above in $H^*(Y;\mathbb{Z})$, with $x_i(E) = c_i(L)$. Since π^* is a monomorphism, most true statements about $c(\pi^*E)$ are also true about $c(E)$. Note that $c_i(\pi^*E) = \pi^*c_i(E)$ are symmetric polynomials in the classes $x_i(E)$.

Let $Q \in R[x]$ be a power series starting with 1. We can define characteristic classes of E in $H^*(X;R)$ by

$$K^Q(E) := Q(1+x_1)\cdots Q(1+x_n).$$

To make sense of this, we note that $K^Q(\pi^*E)$ belong to the image of π^* since it is symmetric in the x_i. Hence there really is a class $K^Q(E) \in H^*(X;R)$ so that $\pi^*K^Q(E) = Q(1+x_1)\cdots Q(1+x_n) \in H^*(Y;R)$, and this is what is meant. Let K^Q_k denote the homogeneous part of K^Q of degree k. Then $K^Q_k(E)$ is a polynomial of homogeneous degree k in the Chern classes $c_i(E)$. As such K^Q_k depends only on $c_i(E)$ for $2i \leqslant k$. The sequence $K^Q_2(c_1),\ K^Q_4(c_1,c_2),\cdots$ is called a multiplicative sequence, because the map

$$K := \left(\sum_{i=0}^{\infty} q_i z^i \mapsto \sum_{n=0}^{\infty} K_n(q_1,\cdots,q_n) z^n\right)$$

is an endomorphism of the multiplicative monoid $\left(R[x_1,x_2,\cdots][[z]]\right)^0$, where we give x_i degree i and z degree -1. That is, for powerseries a and b of total degree 0, we have $K(ab) = K(a)K(b)$. Using multiplicative sequences, Hirzebruch constructed complex genera as follows. Let $Z \to pt$ be a complex oriented map of codimension n, so that Z defines a class in MU_n. The stable complex structure on NZ gives us a total Chern class which we factorize as above

$$c(NZ) = (1+x_1)\cdots(1+x_n).$$

Now we define

$$\phi^Q(Z) := K^Q(NZ)[Z] = K^Q_n(NZ)[Z]$$

where $[Z]$ is the fundamental class of Z. It follows from Theorem 4.12 below that ϕ^Q in fact is a genus, at least when R is a $\mathbb{C}$-algebra. To complete the picture, we draw the following conclusion from [27, Section 1.8]:

Theorem 4.11. *If R is an integral domain over $\mathbb{Q}$, then for any genus $\phi\colon MU_* \to R$, there is a power series $Q \in R[[x]]$ such that $\phi = \phi^Q$.* ■

4.3 Extension of genera to maps $MU \to HR$.

We use the notation of Section 4.2.

Theorem 4.12. *Let R be a $\mathbb{R}$ algebra and let $\phi\colon MU_* \to R$ be a genus. There is an extension of ϕ to a map of multiplicative cohomology theories*

$$\phi\colon MU^*(X) \to H^*(X; R),$$

defined by,

$$\phi([f]) = f_* K^Q(N_f)$$

where Q is the power-series of Theorem 4.11 giving $\phi = \phi^Q$.

Remark 4.13. The assumption that R is a $\mathbb{R}$-algebra is unnecessarily strong. We make it only to simplify the discussion, as we have defined the pushforward along proper, oriented maps in terms of currents. Note also that we strictly speaking only have defined f_* in the case that X and Z are oriented manifolds, since we soon will assume that X is a complex manifold. The below proof is however valid for arbitrary X and arbitrary $\mathbb{Q}$-algebras R.

Proof. To show that $\phi\colon MU^n(X) \to H^*(X; R)$ is well defined, consider a bordism datum $b = (a_b, f_b)\colon W \to \mathbb{R} \times X$ over X. We write $W_{[0,1]} = a_b^{-1}([0,1])$, and allow ourselves to write f_b also for the restriction $f_b|_{W_{[0,1]}}$. Consider then for $t = 0, 1$ the following commutative diagram, where the square is cartesian:

$$\begin{tikzcd}
Z_t \arrow[r, "g'_t"] \arrow[d, "f_t"'] & W_{[0,1]} \arrow[d, "{(a_b,f_b)}"'] \arrow[dd, bend left=60, "f_b"] \\
\{t\} \times X \arrow[r, "g_t"'] \arrow[dr, "\mathrm{id}_X"'] & {[0,1] \times X} \arrow[d] \\
 & X
\end{tikzcd}$$

We must show that

$$(f_1)_* K^Q(N_{f_1}) - (f'_0)_* K^Q(N_{f_1}) = 0 \in H^*(X; R). \tag{4.6}$$

Let $\omega \in \mathcal{A}^*(W; \mathcal{V}_*)$ be a closed form representing $K^Q(N_b)$. Then $g_t'^*\omega$ is a closed form representing $K^Q(N_{f_t})$. Since ω is closed, we get by Proposition 2.13

$$\begin{aligned}
d(f_b)_*(\omega) &= (f_b)_*(\delta_{W_{[0,1]}} \wedge \omega) \\
&= (f_b)_* \left((g'_1)_* g_1'^*\omega - (g'_0)_* g_0'^*\omega\right) \\
&= (f_1)_* g_1'^*\omega - (f_0)_* g_0'^*\omega
\end{aligned}$$

which proves (4.6).

To show naturality, let $g\colon Y \to X$ be transverse to the cycle $f\colon Z \to X$. We recall that $g^*(f, N_f)$ is defined as $(f', N_{f'})$ as in the following diagram with both squares pullback.

$$\begin{tikzcd}
N_{f'} \arrow[r] \arrow[d] & N_f \arrow[d] \\
Z' \arrow[r, "g'"] \arrow[d, "f'"'] & Z \arrow[d, "f"] \\
Y \arrow[r, "g"'] & X
\end{tikzcd}$$

Then $K^Q(N_{f'}) = (g')^* K^Q(N_f)$ by naturality of characteristic classes. It then follows from Theorem 2.31 that

$$f'_* K^Q(N_{f'}) = g^* f_* K^Q(N_f),$$

which means that ϕ is a natural transformation. It remains to see that ϕ is multiplicative. Let $f_i\colon Z_i \to X_i \in ZMU^{n_i}(X_i)$ for $i = 0, 1$. We have

$$N_{f_1 \times f_2} = \pi^*_{Z_1} N_{f_1} \oplus \pi^*_{Z_2} N_{f_2},$$

where π_{Z_i} is the projection $Z_1 \times Z_2 \to Z_i$, for $i = 1, 2$. Sine K^Q is multiplicative, we get the following equalities

$$\begin{aligned}
K^Q(N_{f_1 \times f_2}) &= \pi^*_{Z_1} K^Q(N_{f_1}) \wedge \pi^*_{Z_2} K^Q(N_{f_2}) \\
&= K^Q(N_{f_1}) \otimes K^Q(N_{f_2})
\end{aligned}$$

in $H^0(Z_1; R) \otimes H^0(Z_2; R)$. Recall the which by the Künneth theorem is isomorphic to $H^0(Z_1 \times Z_2; R)$. Under this isomorphism and the corresponding isomorphism for $X_1 \times X_2$, the map $(f_1 \times f_2)_*$ corresponds to $(f_1)_* \otimes (f_2)_*$. This implies

$$(f_1 \times f_2)_* K^Q(N_{f_1 \times f_2}) = (f_1)_* K^Q(N_{f_1}) \otimes (f_2)_* K^Q(N_{f_2})$$

which establishes multiplicativity and so finishes the proof. ∎

4.4 Chern–Weil theory

Chern–Weil theory provides the means to express characteristic classes in geometric terms. We will define the relevant notions, and give a brief overview of the theory. For a better written account of Chern–Weil theory, we refer the reader to the appendix of [43].

Let $\pi : E \to X$ be a smooth complex vector bundle over a manifold X. Let $\mathcal{A}^i(X; E)$ denote the space of E valued i-forms on X. It can be constructed by first defining $\mathcal{A}^0(X; E)$ to be the space of smooth sections of π, and then define

$$\mathcal{A}^i(X; E) = \mathcal{A}^i(X) \otimes_{\mathcal{A}^0(X)} \mathcal{A}^0(X; E).$$

We say that a linear map

$$\nabla\colon \mathcal{A}^0(X;E) \to \mathcal{A}^1(X;E)$$

is a connection if it satisfies the Leibniz rule

$$\nabla(f \otimes s) = df \otimes s + f \cdot \nabla s. \tag{4.7}$$

Note that the data of a connection equivalently can be encoded by a map of sheaves on X, $\mathcal{A}^0(E) \to \mathcal{A}^1(E)$, where $\mathcal{A}^*(E)$ is the sheaf $U \mapsto \mathcal{A}^*(U, E|_U)$. If E is trivial and $S = (s_1, \cdots, s_n)$ is a frame of E, then ∇ is determined by the matrix of 1-forms θ defined by

$$\nabla(s_i) = \sum \theta_{ij} s_j.$$

We say that θ is the connection matrix of ∇ with respect to the frame S. There are no restrictions on the forms θ_{ij}, so we see that the trivial bundle admits plenty of connections.

Proposition 4.14. *All vector bundles $E \to X$ admits connections.*

Proof. Take a locally finite cover of X, $\{U_i\}$, such that for each i, $E|_{U_i}$ is trivial. Let $\{\lambda_i\}$ be a partition of unity subordinate to $\{U_i\}$. Pick a connection ∇_i for each $E|_{U_i}$. Then it is easily verified that

$$\sum_i \lambda_i \nabla_i$$

is a connection on E. ■

In the above proof, we take affine combinations of connections. Here is an explanation of why that is possible:

Proposition 4.15. *The space of connections on E is affine over $\mathcal{A}^1(X; \mathrm{End}(E))$.*

Proof. Let ∇ and ∇' be two connections on E, and let $a \in \mathcal{A}^0(X; \mathrm{End} E)$. The action of $\mathcal{A}^0(X; \mathrm{End} E)$ on the space of connections is by:

$$(\nabla + a)(s) = \nabla(s) + a(s).$$

This is easily seen to define a faithful action. We have:

$$\begin{aligned}
(\nabla - \nabla')(fs) &= \nabla(fs) - \nabla'(fs) \\
&= df \otimes s + f\nabla(s) - df \otimes s - f\nabla' s \\
&= f \cdot (\nabla - \nabla')(s)
\end{aligned}$$

This shows that $\nabla - \nabla'$ is $\mathcal{A}^0(X)$ linear, and so defines a section of $\mathcal{A}^0(X; \mathrm{End} E)$. This proves that the action is also transitive. ■

Remark 4.16. Proposition 4.15 says that not only can we always find a connection; the space of connections is contractible.

Proposition 4.17. *Let $S_1 \oplus S_2 \simeq E \to X$ be a vector bundle with connection ∇. There is a natural induced connection ∇_i on S_i.*

Proof. The induced connection is the composition

$$\mathcal{A}^0(X;S) \xrightarrow{\nabla|_{\mathcal{A}^0(X;S)}} \mathcal{A}^1(X;E) \xrightarrow{P^{S^2}} \mathcal{A}^1(X;S)$$

where P^{S_2} is induced by the projection $E \to S_1$ along S_2. Concretely, let $s_1, \cdots, s_k$ be a local frame for S_1, and let $s_{k+1}, \cdots, s_l$ be a local frame for S_2. Then we can write

$$\nabla(s_i) = \sum_{j=1}^{l} \theta_{ij} s_j$$

for a matrix of 1-forms $\theta = (\theta_{ij})$. We define ∇_1 by

$$\nabla_1(s_i) = \sum_{j=1}^{k} \theta_{ij} s_j,$$

the only difference being that we sum only to k, and that $\nabla_1(s_i)$ only is defined for $i \leqslant k$. That ∇_1 is a connection is clear. ■

Proposition 4.18. *A connection ∇ on $E \to X$ extends uniquely to a linear map*

$$\nabla : \mathcal{A}^*(X;E) \to \mathcal{A}^{*+1}(X;E),$$

which we still denote ∇, satisfying the Leibniz rule

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla(s) \tag{4.8}$$

for $\omega \in \mathcal{A}^k(X)$ and $s \in \mathcal{A}^0(X;E)$.

Proof. Any element of $\mathcal{A}^*(X;E)$ is a sum of elements of the form $\omega \otimes s$. Hence requiring linearity, (4.8) defines the extensions of ∇. ■

For $f \in \mathcal{A}^0(X)$ and s a local section of E, we have

$$\begin{aligned}\nabla^2(fs) &= \nabla(df \otimes s + f \cdot \nabla(s)) \\ &= d^2 f \otimes s - df \otimes \nabla(s) + df \cdot \nabla(s) + f \cdot \nabla^2(s) \\ &= f \cdot \nabla^2(s),\end{aligned}$$

so $\nabla^2|_{\mathcal{A}^0(X;E)}$ is $\mathcal{A}^0(X)$-linear. Hence we may define the curvature, K^∇, to be the unique element in $\mathcal{A}^2(X;\mathrm{End}(E))$ such that for each $s \in \mathcal{A}^0(X;E)$ we have

$$\nabla^2(s) = K^\nabla \cdot s.$$

Here $\cdot$ denotes multiplication by the form, and application of the endomorphism.

Let $M_n(\mathbb{C})$ be the algebra of $n\times n$-matrices over $\mathbb{C}$. By an invariant polynomial we mean a polynomial mapping

$$P : M_n(\mathbb{C}) \to \mathbb{C}$$

such that for each $Q \in Gl(\mathbb{C}^n)$ we have

$$P(A) = P(QAQ^{-1}).$$

If P is an invariant polynomial, then the expression $P(K^\nabla)$ can be defined as follows: Pick a frame $S = (s_i)$ of E over an open set $U \subset X$. We can write

$$K^\nabla \cdot s_i = \sum_j \Omega_{ij} \otimes s_j,$$

where $\Omega_{ij} \in \mathcal{A}^2(X)$ are 2-forms on X. Since $\mathcal{A}^{2*}(X)$ is a commutative algebra over $\mathbb{C}$, it makes sense to define:

$$P(K^\nabla)|_U = P(\Omega).$$

If we change the frame to (s'_i), then Ω gets changed to $\Omega' = Q\Omega Q^{-1}$. Since P is an invariant polynomial, the local forms $P(K^\nabla)|_U$ glue to a well defined form on X, independent of the chosen local frames of E.

Letting I^* denote the ring of invariant polynomials, the assignment $P \mapsto P(K^\nabla)$ defines a homomorphism of graded rings

$$I^* \to \mathcal{A}^*(X)$$

depending on the connection ∇. This is the Chern–Weil homomorphism. We will write

$$P(\nabla) := P(K^\nabla).$$

It is easily verified that

$$P(f^*\nabla) = f^*P(\nabla).$$

Lemma 4.19. *For each invariant polynomial P, $P(\nabla) \in \mathcal{A}^*(X)$ is a closed form.*

Proof. We refer to [43, Fundamental Lemma, Appendix C]. ■

Let ∇_0 and ∇_1 be two connections on the vector bundle $E \to X$. Let P be an invariant polynomial and let γ be a piece-wise smooth path from ∇_0 to ∇_1 in the space of connections on E. Then we can view γ as a connection, ∇_γ on the vector bundle π^*E for $\pi\colon I \times X \to X$ the projection. We consider the form

$$P^\gamma_{CS} = \pi_* P(\nabla_\gamma) \in \mathcal{A}^{*-1}(X).$$

Lemma 4.20. *The form P^γ_{CS} satisfies the equation*

$$P(\nabla_1) = P(\nabla_0) + dP^\gamma_{CS}.$$

Proof. Let $i_t : X \to I \times X$ be defined by $i_t(x) = (t, x)$. Then ∇_γ is a connection on $\pi^* E$ such that $i_t^*(\nabla_\gamma) = \nabla_t$ for $t = 0, 1$. It follows that $i_t^* P(\nabla_\gamma) = P(\nabla_t)$ for $t = 0, 1$. By Proposition 2.13, using $\delta_{\partial(I\times X)} = (i_1)_* 1 - (i_0)_* 1$, we have

$$d\pi_* P(\nabla_\gamma) = i_1^* P(\nabla_\gamma) - i_0^* P(\nabla_\gamma),$$

which proves the lemma. ∎

We will make use the form $P_{CS}^\gamma(\nabla_0, \nabla_1)$ later on. For now, it serves to prove:

Theorem 4.21. *Let P be a homogeneous invariant polynomial, of degree k, and let $E \to X$ be a complex vector bundle. Then there is a well defined cohomology class $[P(\nabla)] \in H^{2k}(X; \mathbb{C})$ depending only on the vector bundle E.*

Proof. Given a connection ∇, Lemma 4.19 shows that $P(\nabla)$ defines a cohomology class. Since we can always find a path between two connections, it follows from Lemma 4.20 that the cohomology class of $P(\nabla)$ is independent of the choice of connection ∇. ∎

Proposition 4.22. *The class*

$$\widetilde{P}_{CS}(\nabla_0, \nabla_1) := [P_{CS}^\gamma] \in \mathcal{A}^{*-1}(X)/\mathrm{Im}\,(d)$$

is independent of the choice of path γ.

Proof. We follow the proof of [48, Proposition 1.6] to prove that $\widetilde{P}_{CS}(\nabla, \nabla')$ is well defined. It suffices to show that P_{CS}^γ is exact whenever γ is a closed path of connections. It is clear that P_{CS}^γ is closed in this case. It is enough to show that the integral of P_{CS}^γ over any closed smooth integral homology cycle is 0. So let Z be such a cycle. We have

$$\begin{aligned} P_{CS}^\gamma(Z) &= \int_{Z\times S^1} P(\nabla_\gamma) \\ &= P(\pi^* E)(Z \times S^1) \\ &= \pi^* P(E)(Z \times S^1) \\ &= P(E)(\pi_*(Z \times S^1)) = 0. \end{aligned}$$

The last equality is using the observation that $\pi_*(Z \times S^1)$ is a boundary, namely $\partial \Pi_*(Z \times D^2)$, where Π is the projection $\Pi\colon X \times D^2 \to X$. ∎

Definition 4.23. *We will call the form*

$$\widetilde{P}_{CS}(\nabla_0, \nabla_1) = [P_{CS}^\gamma] = \pi_* \left[P(\nabla_\gamma)\right] \in \mathcal{A}^{*-1}(X)/\mathrm{Im}\, d$$

the Chern–Simons transgression form.

Thus by means of the theory of connections we are able to define characteristic classes of complex vector bundles. For example, the Chern character of E arise in this way by the formula

$$ch(\nabla) = \mathrm{tr}\ \exp\left(\frac{iK^{\nabla}}{2\pi}\right),$$

and the total Chern class by the formula

$$c(\nabla) = \det\left(\frac{iK^{\nabla}}{2\pi} + I\right). \tag{4.9}$$

Taking homogeneous parts, we get forms representing the Chern classes. One can also first extract the concrete invariant polynomial for c_k thus: For $0 \leqslant k \leqslant n$, let c_k be polynomial mappings on $M_n(\mathbb{C})$ defined by

$$\det\left(\frac{iAt}{2\pi} + I\right) = \sum_k c_k(A)t^k; \tag{4.10}$$

then $c_k(\nabla)$ represents the k-th Chern class of ∇. Any characteristic class

$$\psi \in H^*(BU(n);\mathbb{C})$$

is a polynomial of the Chern classes, $\psi = p(c_1, \cdots, c_n)$, by [43, Theorem 14.5]. We define

$$\psi(\nabla) = p(c_1(\nabla), \cdots, c_n(\nabla)).$$

We summarize:

Theorem 4.24. *Let ∇ be a connection on the complex vector bundle $E \to X$, and let $\psi(E) \in H^*(BU;\mathbb{C})$ be a characteristic class. Then $\psi(E)$ has a canonical representative Chern–Weil form $\psi(\nabla)$.*

Remark 4.25. We have only obtained a preferred Chern–Weil form for each characteristic class. We remark that it is a classical theorem that the Chern–Weil homomorphism $I^* \to \mathcal{A}^*(X)$ is injective, so the preferred representative Chern–Weil form is the unique Chern–Weil form representing that class.

We remark furthermore that the Chern–Weil forms are unique in the following strong sense: If $\psi(\nabla) \in \mathcal{A}^*(X)$ is a natural assignment of forms to connections, then there exist an invariant polynomial P such that $\psi(\nabla) = P(\nabla)$. This is among the main theorems of [18].

4.5 Geometric Hodge filtered cobordism cycles

We let $\phi\colon MU_* \to \mathcal{V}_* = MU_* \otimes \mathbb{C}$ be the genus which in degree k multiplies by $(2\pi i)^k$. Let

$$K = 1 + K_2(x_1) + K_4(x_1, x_2) + \cdots$$

be the multiplicative sequence corresponding to ϕ by Theorem 4.11. By Theorem 4.12, we extend ϕ to a map of multiplicative cohomology theories by sending the class of a cycle $(f\colon Z \to X) \in ZMU^n(X)$ to

$$\phi([f]) = f_* K\left(N_f\right). \tag{4.11}$$

We consider for $p \in \mathbb{Z}$ a twist of the multiplicative sequence K

$$K^p = (2\pi i)^p \cdot K.$$

Let ∇ be a connection on N_f. Then by Theorem 4.24, there is a well defined form

$$K^p(\nabla) = 1 + K_2^p(c_1(\nabla) + K_4^p(c_1(\nabla), c_2(\nabla)), \dots \in \mathcal{A}^0(Z; \mathcal{V}_*)$$

representing $K^p(N_f)$. We view Z with the orientation induced from the sequence Φ_f; TZ is the only bundle in that sequence without a canonical orientation. Then the current $f_* K^p(\nabla_f)$ is well defined, and it represents the class $f_* K^p(N_f)$.

Definition 4.26. *A geometric cycle over X is a quadruple $\widetilde{f} = (f, \Phi, N, \nabla)$ where (N, Φ) is a complex orientation of $f\colon Z \to X$ and ∇ a connection on N. We will omit Φ from the notation, and refer to it by Φ_f if we need it. We say that $\widetilde{f}$ and $\widetilde{f'}$ are isomorphic if there is a commutative diagram*

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "f"'] & & Z' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

where g is a diffeomorphism such that $g^*(N_{f'}, \nabla_{f'}) \simeq (N_f, \nabla_f)$, in the sense that $g^*(\Phi_{f'}, N_{f'})$ represent the same complex orientation as (Φ_f, N_f), and under the induced isomorphism $\Psi\colon g^* N_{f'} \oplus \underline{\mathbb{C}}_Z^{l_1} \to N_f \oplus \underline{\mathbb{C}}_Z^{l_2}$ the connection $\Psi^* \nabla_f$ is gauge equivalent to $\nabla_{f'}$. Let

$$\widetilde{ZMU}^n(X)$$

be the abelian group generated by isomorphism classes of geometric cycles over X of codimension n with the relations $\widetilde{f_1} + \widetilde{f_2} = \widetilde{f_1} \sqcup \widetilde{f_2}$.

For a geometric cycle $\widetilde{f} \in \widetilde{ZMU}^n(X)$ we define

$$\phi^p(\widetilde{f}) = f_* K^p(\nabla_f) \in \mathcal{D}^n(X; \mathcal{V}_*).$$

By deRhams theorem, recalled as part of Theorem 2.16, we can find a current $h \in \mathcal{D}^{n-1}(X; \mathcal{V}_*)$ such that

$$\phi^p(\widetilde{f}) + dh = f_* K^p(\nabla_f) + dh \text{ is a form.}$$

Definition 4.27. *Let X be a complex manifold. We define the group of Hodge filtered cycles of degree (n,p) as the subgroup*

$$ZMU^n(p)(X) \subset \widetilde{ZMU}^n(X) \times \mathcal{D}^{n-1}(X;\mathcal{V}_*)$$

consisting of pairs $\gamma = (\widetilde{f}, h)$ satisfying

$$\phi^p(\widetilde{f}) + dh \in F^p\mathcal{A}^n(X;\mathcal{V}_*).$$

Remark 4.28. To ease notation, we will often write ϕ instead of ϕ^p.

We now define the structure maps on the level of cycles:

$$\begin{aligned}
&R : ZMU^n(p)(X) \to F^p\mathcal{A}^n(X;\mathcal{V}_*)_{cl}, && R(\widetilde{f}, h, \omega) = \phi^p(\widetilde{f}) + dh \\
&a : d^{-1}\left(F^p\mathcal{A}^n(X;\mathcal{V}_*)\right)^{n-1} \to ZMU^n(p)(X), && a(h) = (0, h, dh) \qquad (4.12)\\
&I : ZMU^n(p)(X) \to ZMU^n(X), && I(\widetilde{f}, h, \omega) = f
\end{aligned}$$

Remark 4.29. We may sometimes write a Hodge filtered cobordism cycle as $\gamma = (\widetilde{f}, \omega, h)$, meaning that $(\widetilde{f}, h) \in ZMU^n(p)(X)$, and $R(\widetilde{f}, h) = \omega$.

4.6 Hodge filtered cobordism relation

We will now introduce the cobordism relation. Let $g\colon Y \to X$ be a smooth map, and let $\widetilde{ZMU}^n_g(X) \subset \widetilde{ZMU}(X)$ denote the set of geometric cycles $\widetilde{f}$ with $g \pitchfork f$. For $\widetilde{f} = (f, N_f, \nabla_f) \in \widetilde{ZMU}^n_g(X)$, recall that the pullback cobordism cycle $g^*(f, N_f)$ is defined by $g^*(f, N_f) = (f', g'^*N)$ in the notation of diagram (4.1), see there for the definition of the short exact sequence $\Phi_{f'}$. We define the pullback of geometric cycles by

$$\begin{aligned} g^* \colon \widetilde{ZMU}^n_g(X) &\to \widetilde{ZMU}^n(Y) \\ g^*\widetilde{f} &= (f', g'^*N_f, g'^*\nabla_f) \in \widetilde{ZMU}. \end{aligned} \tag{4.13}$$

Given smooth maps $g_i\colon Y_i \to X$ for $i = 0, 1$, we furthermore define

$$\widetilde{ZMU}^n_{g_1,g_2}(X) = \widetilde{ZMU}^n_{g_1}(X) \cap \widetilde{ZMU}^n_{g_2}(X).$$

Definition 4.30. *Let for $t = 0, 1$ the map $i_t\colon X \to \mathbb{R}\times X$ be given by $i_t(x) = (t, x)$. The group of geometric bordism data over X is the group*

$$\widetilde{ZMU}^n_{i_0,i_1}(\mathbb{R}\times X).$$

That is, 0 and 1 are regular values of a_b. Then $W_t = a_b^{-1}(t)$ is a closed manifold for $t = 0, 1$. We define

$$\partial\widetilde{b} := i_1^*\widetilde{b} - i_0^*(\widetilde{b}) \in \widetilde{ZMU}^n(X)$$

and letting $W_{[0,1]} = a_b^{-1}([0,1])$, *we define:*

$$\psi^p(\widetilde{b}) = \mathbf{w}\left(f_b|_{W_{[0,1]}}\right)_*(K^p(\nabla_b))$$

where $\mathbf{w}$ *is the operator of* (2.4), *acting on* $\mathcal{D}^k(X)$ *by* $(-1)^k$.

Remark 4.31. We will often write ψ instead of ψ^p to ease the notation.

Proposition 4.32. *For* $\widetilde{b}$ *a geometric bordism datum over* X, *we have*

$$\phi^p(\partial\widetilde{b}) + d\psi^p(\widetilde{b}) = 0$$

Proof. Let $\sigma \in \mathcal{A}_c^*(X;\mathcal{V}_*)$. Recall the boundary operator on currents, $bT(\sigma) = T(d\sigma)$, from Section 2.3.2. We use the notation of the following pullback square, where we view j_t with the pullback orientation.

$$\begin{tikzcd}
W_t \arrow[r, "j_t"] \arrow[d, "f_t"'] & W \arrow[d, "(a_b, f_b)"'] \arrow[dd, bend left, "f_b"] \\
\{t\} \times X \arrow[r, "i_t"'] \arrow[dr, "\mathrm{id}"'] & \mathbb{R} \times X \arrow[d] \\
 & X
\end{tikzcd}$$

Since $K^p(\nabla_b)$ is closed, and of even degree, we have $d(K^p(\nabla_b)\wedge\sigma) = K(\nabla_b)\wedge d\sigma$, and so by Stokes theorem, $bK^p(\nabla_b) = \delta_{\partial W} \wedge K^p(\nabla_b)$. We observe that $\delta_{\partial W_{[0,1]}} = (j_1)_*1 - (j_0)_*1$, since the pullback orientation coincides with the boundary orientation at 1, but not at 0. See Remark 4.4. Using Lemma 2.12 and Proposition 2.11 we have

$$\begin{aligned}
b(f_b|_{[0,1]})_*K^p(\nabla_b) &= (f_b|_{[0,1]})_* bK^p(\nabla_b) \\
&= (f_b|_{W_{[0,1]}})_*(\delta_{\partial W_{[0,1]}} \wedge K^p(\nabla_b)) \\
&= (f_b|_{W_{[0,1]}})_*((j_1)_*j_1^*K(\nabla_b) - (j_0)_*j_0^*K^p(\nabla_b)) \\
&= (f_1)_*K^p(\nabla_{f_1}) - (f_0)_*K^p(\nabla_{f_0}) \\
&= \phi^p(\widetilde{f}_1) - \phi^p(\widetilde{f}_0).
\end{aligned}$$

Since $d\mathbf{w} = -\mathbf{w}d = -b$, this finishes the proof. ∎

4.7 Definition of geometric Hodge filtered cobordism groups

Leaning on Proposition 4.32, we consider $(\partial\widetilde{b}, \psi^p(\widetilde{b}))$ as a Hodge filtered cycle of degree $(\operatorname{codim} b, p)$. We call such cycles nullbordant. We recall from (2.12), the definition:

$$\widetilde{F}^p\mathcal{A}^{n-1}(X;\mathcal{V}_*) := F^p\mathcal{A}^{n-1}(X;\mathcal{V}_*) + d\mathcal{A}^{n-2}(X;\mathcal{V}_*)$$

Let $BMU^n_{geo}(X) \subset ZMU^n(p)(X)$ denote the subgroup of nullbordant cycles. Then we define the group of Hodge filtered cobordism relations by

$$BMU^n(p)(X) = BMU^n_{geo}(X) + a\left(\widetilde{F}^p \mathcal{A}^{n-1}(X; \mathcal{V}_*)\right). \tag{4.14}$$

Definition 4.33. *The geometric Hodge filtered cobordism groups of X are defined as the quotient*

$$MU^n(p)(X) := \frac{ZMU^n(p)(X)}{BMU^n(p)(X)}.$$

We denote the Hodge filtered cobordism class of $\gamma = (\widetilde{f}, h) \in ZMU^n(p)(X)$ by $[\gamma] = [\widetilde{f}, h]$.

Lemma 4.34. *Let $\widetilde{f}_0 = (f, N, \nabla_0)$, and $\widetilde{f}_1 = (f, N, \nabla_1) \in \widetilde{ZMU}^n(X)$ be two geometric cycles over X, with the same underlying complex oriented map. There is a geometric bordism $\widetilde{b}$ with $\partial\widetilde{b} = \widetilde{f}_1 - \widetilde{f}_0$ and*

$$\psi(\widetilde{b}) = \mathbf{w} f_* \widetilde{K}_{CS}(\nabla_0, \nabla_1).$$

Proof. Let $b = 1_{\mathbb{R}} \times f \colon \mathbb{R} \times Z \to \mathbb{R} \times X$, and let $\pi_Z \colon \mathbb{R} \times Z \to Z$ denote the projection. With the product complex orientation, using $N_{1_{\mathbb{R}}} = 0$, we have $N_b = N_{1_{\mathbb{R}} \times f} = \pi_Z^* N_f$. On $\pi_Z^* N_f$ we consider the connection

$$\nabla_b := t \cdot \pi_Z^* \nabla_0 + (1-t) \cdot \pi_Z^* \nabla_1$$

where t is the $\mathbb{R}$ coordinate. With this connection, we promote b to a geometric bordism $\widetilde{b} = (b, N_b, \nabla_b)$. Then $\partial\widetilde{b} = \widetilde{f}_1 - \widetilde{f}_0$. Using Definition 4.23 and $f \circ \pi_Z = \pi_X \circ b$, the lemma follows from

$$\begin{aligned} f_* \widetilde{K}_{CS}(\nabla_0, \nabla_1) &= f_* \circ (\pi_Z|_{[0,1]\times Z})_* (K(\nabla_b)) \\ &= \left((\pi_X \circ b)|_{[0,1]\times Z}\right)_* K(\nabla_b) \end{aligned}$$

■

Remark 4.35. In the notation of Lemma 4.34, if $\widetilde{K}_{CS}(\nabla_0, \nabla_1) = 0$ we see that for $(\widetilde{f}, h) \in ZMU^n(p)(X)$ we have $(\widetilde{f}_1, h) - (\widetilde{f}_0, h) \in BMU^n(p)(X)$. Therefore the Hodge filtered cobordism class of $(\widetilde{f}, h)$ depends only on ∇_f modulo the equivalence relation that $\nabla \sim \nabla'$ whenever $\widetilde{K}_{CS}(\nabla, \nabla') = 0$. This equivalence relation makes for a sort of differential K-theory group in the style of [48], with $K \colon K^0(X) \to H^0(X; MU_* \otimes \mathbb{C})$ replacing $ch \colon K^0(X) \to H^0(X; K_* \otimes \mathbb{C})$. Since K is not additive, this "differential" K-theory group does not satisfy the axioms of differential cohomology. A similar group features in [8] as the group of smooth MU-orientations of a map. It would be interesting to understand the relationship between these groups and differential K-theory. We have not investigated this.

We will show that Hodge filtered cobordism is a Hodge filtered cohomology theory over (MU^*, ϕ^p), in the sense of Section 3.1. The structure maps are induced by the maps (4.12), i.e.,

$$\begin{aligned} &R: MU^n(p)(X) \to H^n(X; F^p\mathcal{A}^*(\mathcal{V}_*)), && R[\widetilde{f}, \omega, h] = [\omega] \\ &a: H^{n-1}\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \to MU^n(p)(X), && a(h) = [0, dh, h] \\ &I: ZMU^n(p)(X) \to MU^n(X), && I[\widetilde{f}, h, \omega] = [f] \end{aligned} \tag{4.15}$$

We are here using the descriptions of the occurring sheaf cohomology groups from Proposition 2.18.

Proposition 4.36. *The structure maps* (4.15) *are well defined.*

Proof. We first show that I and R vanish on $BMU^n(p)(X)$, as defined in (4.14): For $\gamma = (\partial\widetilde{b}, \psi(\widetilde{b})) \in BMU^n_{geo}(X)$, we have

$$I(\gamma) = \partial b \in BMU^n(X), \quad \text{and} \quad R(\gamma) = 0,$$

where the second equality is Lemma 4.32, and b is the bordism datum underlying $\widetilde{b}$. We have $I \circ a = 0$, so in particular

$$I(a(h)) = 0, \quad h \in \widetilde{F}^p\mathcal{A}^{n-1}(X; \mathcal{V}_*),$$

which finishes the proof that I is well defined. We have

$$R \circ a\left(\widetilde{F}^p\mathcal{A}^{n-1}(X; \mathcal{V}_*)\right) = d\left(F^p\mathcal{A}^{n-1}(X; \mathcal{V}_*)\right)$$

which is the group of relations for $H^n(X; F^p\mathcal{A}^*(\mathcal{V}_*))$, so R to is well defined. That a is well defined is clear from Proposition 2.18 and the definition of $BMU^n(p)(X)$. ∎

It is clear that $R \circ a = d$, and by construction we have

$$[R(\gamma)] = \phi^p(I(\gamma)) \quad \text{in } H^n(X; \mathcal{V}_*),$$

so the diagram

$$\begin{tikzcd} MU^n(p)(X) \arrow[r, "I"] \arrow[d, "R"'] & MU^n(X) \arrow[d, "\phi^p"] \\ H^n(X; F^p\mathcal{A}^*(\mathcal{V}_*)) \arrow[r] & H^n(X; \mathcal{V}_*) \end{tikzcd}$$

commutes. It remains to establish contravariant functoriality and the long exact sequence, and we will have proved:

Theorem 4.37. *Geometric Hodge filtered complex cobordism $MU^n(p)(X)$ with the structure maps R, I and a from* (4.15) *form a Hodge filtered refinement of* (MU^*, ϕ^p).

4.8 A fundamental long exact sequence

Let $\overline{\phi}$ denote the composition of ϕ with the reduction modulo F^p map:

$$\overline{\phi} = \left(MU^n(X) \xrightarrow{\phi} H^n(X;\mathcal{V}_*) \to H^n\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right)\right).$$

Theorem 4.38. *There is a long exact sequence:*

$$\begin{array}{c} \cdots \longrightarrow H^{n-1}\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \xrightarrow{a} MU^n(p)(X) \xrightarrow{I} \\ MU^n(X) \xrightarrow{\overline{\phi}} H^n\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \xrightarrow{a} MU^{n+1}(p)(X) \longrightarrow \cdots \end{array}$$

Proof. We start with exactness at $MU^n(p)(X)$. First observe

$$I(a([h])) = I[0, dh, h] = 0.$$

The converse requires more work. We work at the cycle level, so let $\gamma = (\widetilde{f}, h, \omega) \in ZMU^n(p)(X)$ and suppose $I(\gamma) = 0$. That means $f = \partial b$ for some bordism datum b. We may extend the geometric structure of $\widetilde{f}$ over b and obtain a geometric bordism datum $\widetilde{b}$ such that $\partial\widetilde{b} = \widetilde{f}$. We have

$$(\widetilde{f}, \omega, h) - (\partial\widetilde{b}, 0, \psi(\widetilde{b})) = (0, \omega, h') = a(h').$$

The last equality follows from the observation that since $(0, \omega, h') \in ZMU^n(p)(X)$ is a Hodge filtered cycle, we must have

$$dh' = \omega \in F^p\mathcal{A}^n(X;\mathcal{V}_*).$$

Hence $\gamma \in BMU^n(p)(X)$. Next we show exactness at $MU^n(X)$. That $\overline{\phi} \circ I = 0$ follows from the following commutative diagram, where the row is the obvious exact sequence:

$$\begin{tikzcd} MU^n(p)(X) \arrow[r, "I"] \arrow[d, "R"'] & MU^n(X) \arrow[d, "\phi"'] \arrow[dr, "\overline{\phi}"] & \\ H^n(X; F^p\mathcal{A}^*(\mathcal{V}_*)) \arrow[r, "inc_*"] & H^n(X;\mathcal{V}_*) \arrow[r] & H^n\left(X; \frac{\mathcal{A}}{F^p}(\mathcal{V}_*)\right) \end{tikzcd}$$

Conversely, suppose $\overline{\phi}([f]) = 0$. That means we can find $\omega \in F^p\mathcal{A}^n(X;\mathcal{V})$ such that

$$\phi([f]) = inc_*([\omega]).$$

Let ∇_f be a connection on N_f, so that we get a geometric cycle $\widetilde{f}$ with $I(\widetilde{f}) = f$. Then $\phi(\widetilde{f})$ is a current representing $\phi([f])$; hence $\phi(\widetilde{f})$ and ω are cohomologous. That is to say, there is a current $h \in \mathcal{D}^{n-1}(X;\mathcal{V}_*)$ such that $\phi(\widetilde{f}) + dh = \omega$. Then $\gamma := (\widetilde{f}, \omega, h)$ is a Hodge filtered cycle with $I(\gamma) = f$.

Now we show exactness at $H^n\left(X; \frac{\mathcal{A}}{F^p}(\mathcal{V}_*)\right)$. Let $f\colon Z \to X$ be a bordism cycle on X. We will show $a(\overline{\phi}([f])) = 0 \in MU^{n+1}(p)(X)$. Lift f to a geometric cycle $\widetilde{f} \in \widetilde{ZMU}^n(X)$; then we can write

$$a(\overline{\phi}([f]) = [0, 0, \phi(\widetilde{f})].$$

We may build from $\widetilde{f}$ a geometric bordism datum $\widetilde{b}$ with underlying map

$$Z \xrightarrow{(\frac{1}{2}, f)} \mathbb{R} \times X$$

where $\frac{1}{2}$ is the constant map at $\frac{1}{2}$. Clearly $\partial\widetilde{b} = 0$. More interesting is the observation that $\psi(\widetilde{b}) = \mathbf{w}\phi(\widetilde{f})$. Hence

$$(\phi(\partial\widetilde{b}), 0, \psi(\widetilde{b})) = (0, 0, \mathbf{w}\phi(\widetilde{f})) \in BMU^n_{geo}(X)$$

and we conclude that $a(\overline{\phi}([f])) = 0$.

Conversely, suppose that $h \in (d^{-1}F^p\mathcal{A}^n(X; \mathcal{V}_*))^{n-1}$ is such that $a(h) = (0, dh, h)$ represent 0. Then we must have

$$[dh] = R(a([h])) = R(0) = 0 \in H^n(X; F^p\mathcal{A}^*(\mathcal{V})),$$

so $dh = dh'$ for some $h' \in F^p\mathcal{A}^{n-1}(X; \mathcal{V})$. For such h', $[0, dh', h'] = 0$, so $[a(h)] = [0, 0, h - h'] = 0$. Hence there must be a geometric bordism datum $\widetilde{b}$ with underlying map $(a_b, f_b)\colon W \to \mathbb{R} \times X$ such that

$$(0, 0, h - h') = (\partial\widetilde{b}, 0, \psi(\widetilde{b})).$$

Since $\partial\widetilde{b} = 0$, we have that

$$f := f_b|_{a_b^{-1}([0,1])} \in ZMU^n(X)$$

is a bordism cycle. By definition of ψ, we have $\psi(\widetilde{b}) = \pm\phi(\widetilde{f})$ where $\widetilde{f}$ is the obvious geometric cycle over f. We have shown

$$[h] = [h - h'] = [\psi(\widetilde{b})] = \pm\overline{\phi}([f]) \in H^n\left(X; \frac{\mathcal{A}}{F^p}(\mathcal{V}_*)\right),$$

which finishes the proof. ■

This proves the first part of Theorem 1.1 in the introduction.

4.9 Pullback in geometric Hodge filtered cobordism

We now establish the contravariant functoriality of $MU^n(p)(X)$. Let $g\colon Y \to X$ be a holomorphic map. We define $ZMU^n_g(p)(X)$ to be the subset of $ZMU^n(p)(X)$ consisting of such $\gamma = (\widetilde{f}, h)$ that

- $WF(h) \cap N(g) = \emptyset$, and
- $g \pitchfork f$.

For $\gamma \in ZMU^n_g(p)(X)$, we can define $g^*\gamma \in ZMU^n(p)(Y)$ by

$$g^*(\gamma) = (g^*\widetilde{f}, g^*h). \tag{4.16}$$

This use Theorem 2.23, and Proposition 2.2. Here $g^*\widetilde{f}$ is defined in (4.13). The aim of this section is to show:

Theorem 4.39. *The above pullback construction induces a map*

$$g^* \colon MU^n(p)(X) \to MU^n(p)(Y)$$

for any holomorphic map $g : Y \to X$, making $MU^n(p)$ a contravariant functor of complex manifolds.

The proof proceeds in three steps:

Proposition 4.40. *Given a holomorphic map $g \colon Y \to X$, and a Hodge filtered cycle $\gamma \in ZMU^n(p)(X)$, there exist $b \in BMU^n(p)(X)$ such that*

$$\gamma + b \in ZMU^n(p)(X)_g.$$

Proposition 4.41. *If*

$$\gamma \in ZMU^n_g(p)(X) \cap BMU^n(p)(X),$$

then $g^*\gamma \in BMU^n(p)(Y)$.

For holomorphic maps $g_1 \colon X_1 \to X_2$ and $g_2 \colon X_2 \to X_3$, define

$$ZMU^n_{g_1 g_2}(p)(X_3) = ZMU^n_{g_2}(p)(X_3) \cap ZMU^n_{g_2 \circ g_1}(p)(X_3).$$

Proposition 4.42. *We have $g_2^*\left(ZMU^n_{g_1 g_2}(p)(X_3)\right) \subset ZMU^n_{g_1}(p)(X_2)$, and*

$$g_1^* \circ g_2^* = (g_2 \circ g_1)^* \colon ZMU^n_{g_1 g_2}(p)(X_3) \to ZMU^n(p)(X_1).$$

It is clear that these three propositions together prove Theorem 4.39.

Proof of Proposition 4.42. Let $\gamma = (\widetilde{f}, h) \in ZMU^n_{g_1 g_2}(p)(X)$. The equality is evident $g_1^* g_2^* h = (g_1 \circ g_2)^* h$. See the discussion of naturality of pullback of cycles in Section 4.1.3. We need only add that pullback of connections also is natural. ■

Proof of 4.40. By Thom's transversality theorem (2.4) we may choose f_1 homotopic to f so that $f_1 \pitchfork g$. Recall from (4.7) that there is a complex orientation of f_1 and a cobordism between f_1 and f of the form $b = \mathbb{R} \times Z \to \mathbb{R} \times X$ with $b(t, z) = (t, H(t, z))$ where H is a homotopy between f_1 and f, in the sense that $H(1,z) = f_1(z)$ and $H(0,z) = f(z)$. To give b a geometric structure, let first ∇ be any connection on N_b. We can assume the homotopy H is constant over $[0,\epsilon]$. Then we can identify $\pi_Z^* N_f|_{[0,\epsilon)\times Z}$ with $N_b|_{[0,\epsilon)\times Z}$. Let $\{\lambda_1, \lambda_2\}$ be a partition of unity subordinate to $\{[0,\epsilon), (\epsilon/2, 1]\}$. Then $\nabla_b := \lambda_1 \cdot \pi_Z^* \nabla_f + \lambda_2 \cdot \nabla$ is a connection on N_b which restricts to ∇_f on $\{0\} \times Z$, and (b, N_b, ∇_b) is a geometric bordism datum. We give f_1 the geometric structure $i_1^*(\widetilde{b})$, for $i_1 \colon X \to \mathbb{R} \times X$ the inclusion $i_1(x) = (1, x)$. By design we have $\partial \widetilde{b} = \widetilde{f_1} - \widetilde{f}$. Hence

$$(\widetilde{f_1}, \omega, h + \psi(\widetilde{b})) - (\widetilde{f}, \omega, h) \in BMU^n_{geo}(X),$$

and $g^* \widetilde{f_1}$ is well defined. We turn to $g^* h$, which is well defined if

$$WF(h) \cap N(g) = \emptyset,$$

as discussed at Theorem 2.23. From Theorem 2.30 we have that $\phi(\widetilde{f_1}) = (f_1)_* K(\nabla_{f_1})$ has wavefront set contained in $N(f_1)$. Since $g \pitchfork f_1$, we get $WF(\phi(f_1)) \cap N(g) = \emptyset$ so that $g^* \phi(f_1)$ is well defined. Furthermore, applying Theorem 2.31, we have

$$g^* \phi(\widetilde{f_1}) = g^* (f_1)_* K(\nabla_{f_1}) = f'_* g'^* K(\nabla_{f_1}) = f'_* K(\nabla_{f'}) = \phi(g^* \widetilde{f_1}),$$

with the notation of (4.1). Since $\phi(\widetilde{f}) + dh$ is smooth, we must have $WF(\phi(\widetilde{f})) = WF(dh)$. Since d is a differential operator we have the containment $WF(dh) \subset WF(h)$, [28, 8.1.11], but equality need not hold. That is, h may have additional singularities which disappear when we apply d, but d cannot create new singularities. We are done once we have shown that upon replacing h with a cohomologous current if necessary, we can assume $WF(h) = WF(dh)$, since $a(d\beta) \in BMU^n(p)(X)$ by definition. Thus we have reduced to proving the following lemma. ∎

Lemma 4.43. *Let $\alpha \in \mathcal{D}^n(X; \mathcal{V}_*)$. There exist a current $\beta \in \mathcal{D}^{n-1}(X; \mathcal{V}_*)$ so that $WF(d\alpha) = WF(\alpha + d\beta)$.*

Proof. This is [8, Lemma 4.11]. For the readers convenience, we recount their proof. Choose a Riemannian metric on X. Let d^* be formally adjoint to d. Then we consider the Laplacian $\Delta = d^* d + d d^*$, which is an elliptic differential operator

$$\Delta \colon \mathcal{D}^*(X) \to \mathcal{D}^*(X).$$

Using [29, Theorem 18.1.24] we can find a parametrix $\mathscr{P} \colon \mathcal{D}^*(X) \to \mathcal{D}^*(X)$, properly supported in the sense that both projections from the support of the Schwartz kernel of $\mathscr{P}$, which we denote P, $\mathcal{D}^*(X \times X) \supset \operatorname{supp} P \to X$ are proper maps, such that both $\Delta \circ \mathscr{P} - id$ and $\mathscr{P} \circ \Delta - id$ are smoothing operators. We put $G = d^* \mathscr{P}$. This pseudo-differential operator satisfies

$$dG + Gd = 1 + S$$

for a smoothing operator S. Let $\beta = G\alpha$. Then we get

$$\begin{aligned}\alpha - d\beta &= \alpha - dG\alpha \\ &= \alpha - (1 - Gd + S)\alpha \\ &= -Gd\alpha + S\alpha.\end{aligned}$$

Since G is a pseudo-differential operator we have $WF(Gd\alpha) \subset WF(d\alpha)$. This can for example be seen by taking $\Gamma = T^*X\backslash 0$ in [29, Proposition 18.1.26.]. This finishes the proof. ■

Proof of Proposition 4.41. Let $\gamma = (\widetilde{f}, h) \in ZMU^n(p)(X)_g \cap BMU^n(p)(X)$. We can then write $\gamma = (\partial\widetilde{b}, \phi(\widetilde{b})) + a(\omega)$ where $\widetilde{b}$ is a geometric cobordism with underlying map $b : W \to \mathbb{R} \times X$ and $\omega \in \widetilde{F}^p\mathcal{A}^{n-1}(X; \mathcal{V}_*)$. We must show that $g^*\gamma \in BMU^n(p)(Y)$. We start by noting

$$a(g^*\omega) = g^*a(\omega)$$

$$g^*\omega \in \widetilde{F}^p\mathcal{A}^{n-1}(Y; \mathcal{V}_*)$$

which implies

$$g^*(a(\omega)) \in BMU^n(p)(Y).$$

Now we turn to the cobordism relation. As in establishing the pullback for MU^n at 4.5, we can perturb f_b slightly so as to ensure $f_b \pitchfork g$. As in the proof of Proposition 4.40, we can extend the geometric structure over the perturbing homotopy and restrict to the other end, so that we still have a geometric bordism datum $\widetilde{b}$ with $\partial\widetilde{b} = \widetilde{f}$. As in establishing 4.5, we consider the pullback geometric bordism datum $g^*\widetilde{b}$, which is formally defined as the geometric cycle $(id_{\mathbb{R}} \times g)^*\widetilde{b}$. It has underlying map $(a_b', f_b') \colon W' \to Y$, fitting into the following diagram where all squares are pullback squares of manifolds for $t \in \{0, 1\}$.

$$\begin{tikzcd}
 & & W_t \arrow[d] \arrow[drr] & & \\
W_t' \arrow[urr] \arrow[d] \arrow[drr] & & t \times X \arrow[drr] & & W \arrow[d, "{(a_b, f_b)}"] \\
t \times Y \arrow[urr] \arrow[drr] & & W' \arrow[urr, "G"'] \arrow[d, "{(a_b', f_b')}"] & & \mathbb{R} \times X \\
 & & \mathbb{R} \times Y \arrow[urr, "id_{\mathbb{R}} \times g"'] & &
\end{tikzcd}$$

It is then clear that $\partial g^*\widetilde{b} = g^*\partial\widetilde{b}$. We get from Theorem 2.31 that:

$$g^*\psi(\widetilde{b}) = \mathbf{w}g^*(f_b|_{a_b^{-1}([0,1])})_*K(\nabla_b) = \mathbf{w}\left(f_b'|_{a_b'^{-1}([0,1])}\right)_* G^*K(\nabla_b) = \psi(g^*\widetilde{b})$$

Hence we have

$$g^*(\partial\widetilde{b}, \psi(\widetilde{b})) = (\partial g^*\widetilde{b}, \psi(g^*\widetilde{b})) \in BMU^n(p)(Y).$$

This finishes the proof. ■

This finishes the proof of Theorem 4.37.

4.10 Exterior products

We will now define exterior products:

$$\times \colon MU^{n_1}(p_1)(X_1) \times MU^{n_2}(p_2)(X_2) \to MU^{n_1+n_2}(p_1+p_2)(X_1 \times X_2). \quad (4.17)$$

Let for $i = 1, 2$

$$\gamma_i = (\widetilde{f}, h_i) \in ZMU^{n_i}(p_i)(X_i)$$

be Hodge filtered cycles, and let π_i be the projection $Z_1 \times Z_2 \to Z_i$. We define the exterior product of geometric cycles by

$$\widetilde{f}_1 \times \widetilde{f}_2 = (f_1 \times f_2,\ N_1 \times N_2,\ \nabla_1 \times \nabla_2),$$

where we abbreviate N_{f_1} by N_1, and so on. Here $N_1 \times N_2 = \pi_1^* N_1 \oplus \pi_2^* N_2$, and $\nabla_1 \times \nabla_2 = \pi_1^*\nabla_1 \oplus \pi_2^*\nabla_2$, and the short exact sequence $\Phi_{f_1 \times f_2}$ is defined in Section 4.1.2, and as usual suppressed from the notation. Recall from (2.13) the tensor product

$$\otimes : \mathcal{D}^{n_1}(X_1) \times \mathcal{D}^{n_2}(X_2) \to \mathcal{D}^{n_1+n_2}(X_1 \times X_2),$$

satisfying $T_1 \otimes T_2 = \pi_1^* T_1 \wedge \pi_2^* T_2$. Since K is multiplicative, we have

$$K^{p_1+p_2}(\nabla_1 \times \nabla_2) = K^{p_1}(\nabla_1) \otimes K^{p_2}(\nabla_2).$$

We now go back to supressing the p in $K^p = (2\pi i)^p \cdot K$ and ϕ^p from the notation. Since $(f_1 \times f_2)_*(T_1 \otimes T_2) = (f_1)_* T_1 \otimes (f_2)_* T_2$ we get

$$\phi(\widetilde{f}_1 \times \widetilde{f}_2) = \phi(\widetilde{f}_1) \otimes \phi(\widetilde{f}_2). \quad (4.18)$$

We want $R(\gamma_1 \times \gamma_2) = R(\gamma_1) \otimes R(\gamma_2)$. We compute:

$$\begin{aligned}
R(\gamma_1) \otimes R(\gamma_2) &- \phi(\widetilde{f}_1) \otimes \phi(\widetilde{f}_2) \\
&= (\phi(\widetilde{f}_1) + dh_1) \otimes (\phi(\widetilde{f}_2) + dh_2) - \phi(\widetilde{f}_1) \otimes \phi(\widetilde{f}_2) \\
&= dh_1 \otimes \phi(\widetilde{f}_2) + \phi(\widetilde{f}_1) \otimes dh_2 + dh_1 \otimes dh_2 \\
&= d\left(h_1 \otimes \phi(\widetilde{f}_2) + (-1)^{n_1}\phi(\widetilde{f}_1) \otimes h_2 + h_1 \otimes dh_2\right) \\
&= d\left(h_1 \otimes R(\gamma_2) + (-1)^{n_1}\phi(\widetilde{f}_1) \otimes h_2\right)
\end{aligned}$$

Therefore we define the exterior product of Hodge filtered cycles by

$$\gamma_1 \times \gamma_2 = \left(\widetilde{f}_1 \times \widetilde{f}_2, \quad h_1 \otimes R(\gamma_2) + (-1)^{n_1}\phi(\widetilde{f}_1) \otimes h_2\right). \quad (4.19)$$

Remark 4.44. In the above computation, we chose $h_1 \otimes dh_2$ as a form with exterior derivative $dh_1 \otimes dh_2$. Had we chosen instead $(-1)^{n_1} dh_1 \otimes h_2$, we would have been led to define

$$\gamma_1 \times \gamma_2 = \left(\widetilde{f}_1 \times \widetilde{f}_2, \quad h_1 \otimes \phi(\widetilde{f}_2) + (-1)^{n_1} R(\gamma_1) \otimes h_2 \right).$$

When we quotient out $a(\widetilde{F}^{p_1+p_2}\mathcal{A}^{n_1+n_2}(X_1 \times X_2))$, this choice becomes immaterial, since

$$h_1 \otimes dh_2 - (-1)^{n_1} dh_1 \otimes h_2 = d\left((-1)^{n_1+1} h_1 \otimes h_2\right).$$

Proposition 4.45. *Let $s\colon X_1 \times X_2 \to X_2 \times X_1$ be the swap map $s(x_1, x_2) = (x_2, x_1)$. The exterior product* (4.19) *satisfies*

$$\begin{aligned} (\gamma_1 + \gamma_1') \times \gamma_2 &= \gamma_1 \times \gamma_2 + \gamma_1' \times \gamma_2, \\ \gamma_1 \times \gamma_2 &= (-1)^{n_1} s^* \left(\gamma_2 \times \gamma_1 + a(d\left((-1)^{n_1+1} h_1 \otimes h_2\right))\right) \end{aligned}$$

Proof. The isomorphism of geometric cycles underlying the first equation is obvious. Then the first equation follows since the expression $h_1 \otimes \phi(\widetilde{f}_2) + (-1)^{n_1} R(\gamma_1) \otimes h_2$ is linear in h_1. The second equality follows from Remark 4.44. ∎

We now show that the cobordism class of $\gamma_1 \times \gamma_2$ depends only on the cobordism class of γ_i. Because of the symmetry, it suffices to show that if γ_2 represent 0, i.e. $\gamma_2 \in BMU^{n_2}(p_2)(X_2)$, then

$$\gamma_1 \times \gamma_2 \in BMU^{n_1+n_2}(p_1 + p_2)(X_1 \times X_2).$$

We can write

$$\gamma_2 = (\partial\widetilde{b}, \psi(\widetilde{b}) + h) = (\partial\widetilde{b}, \psi(\widetilde{b})) + a(h)$$

for $\widetilde{b}$ a geometric bordism datum over X_2, with underlying map (a_b, f_b), and $h \in \widetilde{F}^{p_2}\mathcal{A}^{n_2}(X_2; \mathcal{V}_*)$. We note first that since $\widetilde{f}_1 \times 0 = 0$ we get

$$\begin{aligned} \gamma_1 \times a(h) &= (0, \quad h_1 \otimes dh + \phi(\widetilde{f}_1) \otimes h) \\ &= a\left(R(\gamma_1) \otimes h + d\left((-1)^{n_1+1} h_1 \otimes h\right)\right). \end{aligned} \tag{4.20}$$

Since

$$R(\gamma_1) \otimes h + d\left((-1)^{n_1+1} h_1 \otimes h\right) \in \widetilde{F}^{p_1+p_2}\mathcal{A}^{n_1+n_2}(X_1 \times X_2; \mathcal{V}_*)$$

we conclude that $\gamma_1 \times a(h)$ represent 0. We now suppose $\gamma_2 = (\partial\widetilde{b}, \psi(\widetilde{b}))$. Then $R(\gamma_2) = 0$, and we get

$$\begin{aligned} \gamma_1 \times \gamma_2 &= \gamma_1 \times (\partial\widetilde{b},\ \psi(\widetilde{b})) \\ &= (\widetilde{f}_1 \times \partial\widetilde{b}, (-1)^{n_1} \phi(\widetilde{f}_1) \otimes \psi(\widetilde{b})) \\ &= (\partial(\widetilde{f}_1 \times \widetilde{b}), \psi(\widetilde{f}_1 \times \widetilde{b})) \end{aligned}$$

where we interpret $\widetilde{f}_1 \times \widetilde{b}$ as a geometric bordism datum on $X_1 \times X_2$, and the sign is absorbed by the sign operator $\mathbf{w}$ in the definition of ψ. We have now established the exterior product, (4.17).

4.11 Ring structure

Using (4.17) we can now turn

$$MU^*(*)(X) := \bigoplus_{n,p} MU^n(p)(X)$$

into a ring with a product

$$MU^n(p)(X) \times MU^m(q)(X) \to MU^{n+m}(p+q)(X) \tag{4.21}$$

defined by

$$[\gamma_1] \cdot [\gamma_2] = \Delta^*([\gamma_1 \times \gamma_2])$$

where $\Delta \colon X \to X \times X$ is the diagonal. We can now improve Theorem 4.37, and show:

Theorem 4.46. *Geometric Hodge filtered cobordism is a multiplicative Hodge filtered extension of (MU^*, ϕ^p) in the sense of Section 3.1.*

Proof. It is clear that I is multiplicative from the description of the multiplication on $MU^*(X)$ in Section 4.1.8. That R is multiplicative follows from the computations prior to (4.19). It remains to establish the equation

$$a([h]) \cdot \gamma = a([h \wedge R(\gamma)])$$

for $h \in H^{n_1-1}\left(X; \frac{\mathcal{A}^*}{F^{p_1}}(\mathcal{V}_*)\right)$ and $\gamma \in MU^{n_2}(p_2)(X)$. From (4.19) we get

$$a([h]) \cdot [\widetilde{f}, h_2] = [0, h \wedge R(\widetilde{f}, h_2)]$$

which proves the theorem. ∎

4.12 Currential Hodge filtered complex cobordism

We now define a currential version of geometric Hodge filtered complex cobordism $MU^n_\delta(p)(X)$, and establish a canonical isomorphism

$$MU^n(p)(X) \to MU^n_\delta(p)(X).$$

As in [19], the currential theory serves to clarify the pushforward.

We define the group of currential Hodge filtered cycles

$$ZMU^n_\delta(p)(X)$$

to be the subgroup of $ZMU^n(X) \times \mathcal{D}^{n-1}(X; \mathcal{V}_*)$ consisting of pairs $(\widetilde{f}, h)$ such that

$$\phi(\widetilde{f}) + dh \in F^p\mathcal{D}^n(X; \mathcal{V}_*).$$

We will also sometimes write a currential Hodge filtered cycle as a triple $(\widetilde{f}, T, h)$, meaning that $(\widetilde{f}, h)$ is a Hodge filtered cycle, and $T = \phi(\widetilde{f}) + dh$.

We define structure maps for $ZMU^n_\delta(p)(X)$ by essentially the same formulae as we used for $ZMU^n(p)(X)$ in (4.12):

$$\begin{aligned} &R\colon ZMU^n_\delta(p)(X) \to F^p\mathcal{D}^n(X;\mathcal{V}_*) && R(\widetilde{f},T,h)=T \\ &a\colon d^{-1}\left(F^p\mathcal{D}^*(X;\mathcal{V}_*)\right)^{n-1} \to ZMU^n_\delta(p)(X) && a(h)=(0,dh,h) \\ &I\colon ZMU^n_\delta(p)(X) \to ZMU^n(X) && I(\widetilde{f},T,h)=f \end{aligned} \tag{4.22}$$

We have a canonical inclusion $ZMU^n(p)(X) \to ZMU^n_\delta(p)(X)$ given by

$$(\widetilde{f},\omega,h) \mapsto (\widetilde{f},T_\omega,h)$$

where T_ω is the current associated to ω by $T_\omega = \left(\sigma \mapsto \int_X \omega\wedge\sigma\right)$. This inclusion is compatible with the structure maps of (4.12) in the sense that

$$\begin{tikzcd} ZMU^n(p)(X) \arrow[r, "R"] \arrow[d] & F^p\mathcal{A}^n(X;\mathcal{V}_*) \arrow[d] \\ ZMU^n_\delta(p)(X) \arrow[r, "R"] & F^p\mathcal{D}^n(X;\mathcal{V}_*), \end{tikzcd}$$

and the corresponding diagrams for a and I, all commute. We therefore view $ZMU^n(p)(X)$ as a submonoid of $ZMU^n_\delta(p)(X)$, and say that $\gamma \in ZMU^n_\delta(p)(X)$ is smooth if it belongs to $ZMU^n(p)(X)$.

We define the monoid of currential cobordism relations by

$$BMU^n_\delta(p)(X) = BMU^n_{geo}(X) + a\left(\widetilde{F}^p\mathcal{D}^*(X;\mathcal{V}_*)\right).$$

Then we define the currential Hodge filtered cobordism groups by

$$MU^n_\delta(p)(X) := ZMU^n_\delta(p)(X)/BMU^n_\delta(p)(X).$$

Comparing $BMU^n(p)(X)$ and $BMU^n_\delta(p)(X)$, it is clear that the inclusion

$$ZMU^n(p)(X) \hookrightarrow ZMU^n_\delta(p)(X)$$

induces a map

$$MU^n(p)(X) \to MU^n_\delta(p)(X).$$

It is also clear that the structure maps of (4.22) descend to give structure maps for $MU^n_\delta(p)(X)$:

$$\begin{aligned} &R\colon MU^n_\delta(p)(X) \to H^n\left(X;F^p\mathcal{D}^*(\mathcal{V}_*)\right) && [\widetilde{f},T,h]\mapsto [T], \\ &a\colon H^{n-1}\left(X;\frac{\mathcal{D}^*}{F^p}(\mathcal{V}_*)\right) \to MU^n_\delta(p)(X) && [h]\mapsto [0,dh,h] \\ &I\colon MU^n_\delta(p)(X) \to MU^n(X) && [\widetilde{f},h,T]\mapsto [f] \end{aligned}$$

Let $\overline{\phi}$ denote the composition of ϕ with the reduction modulo F^p map:

$$MU^n(X) \xrightarrow{\phi} H^n(X;\mathcal{V}_*) \longrightarrow H^n\left(X;\tfrac{\mathcal{D}^*}{F^p}(\mathcal{V}_*)\right).$$

Proposition 4.47. *Let X be a complex manifold. There is a long exact sequence*

$$\begin{tikzcd} \cdots \arrow[r] & H^{n-1}\left(X; \frac{\mathcal{D}^*}{F^p}(\mathcal{V}_*)\right) \arrow[r, "a"] & MU^n_\delta(p)(X) \arrow[r, "I"] & {} \\ MU^n(X) \arrow[r, "\overline{\phi}"] & H^n\left(X; \frac{\mathcal{D}^*}{F^p}(\mathcal{V}_*)\right) \arrow[r, "a"] & MU^{n+1}_\delta(p)(X) \arrow[r] & \cdots \end{tikzcd} \tag{4.23}$$

Proof. The exactness follows in the same way as for the long exact sequence of Theorem 4.38. ■

Theorem 4.48. *The natural map $MU^n(p)(X) \to MU^n_\delta(p)(X)$ is an isomorphism.*

Proof. Consider the diagram

$$\begin{tikzcd} \cdots \arrow[r] & H^{n-1}\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \arrow[r, "a"] \arrow[d] & MU^n(p)(X) \arrow[r, "I"] \arrow[d] & MU^n(X) \arrow[r, "\overline{\phi}"] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^{n-1}\left(X; \frac{\mathcal{D}^*}{F^p}(\mathcal{V}_*)\right) \arrow[r, "a"] & MU^n_\delta(p)(X) \arrow[r, "I"] & MU^n(X) \arrow[r, "\overline{\phi}"] & \cdots \end{tikzcd}$$

where the rows are the long exact sequences of Theorem 4.38 and Proposition 4.47. The left and right vertical arrows are isomorphisms by (2.20). The diagram commutes since the map $MU^n(p)(X) \to MU^n_\delta(p)(X)$ is compatible with the structure maps a and I. The theorem now follows from the five-lemma. ■

Remark 4.49. As we discussed in the introduction, this Theorem is an important difference between Hodge filtered cohomology and differential cohomology. Differential cohomology is similar to Hodge filtered cohomology, but using the filtration $\mathcal{A}^{\geqslant p}$ instead of the Hodge filtration. The groups $\mathcal{A}^n(X;\mathcal{V}_*)_{cl}$ are used in the axiomatization since $H^n(X;\mathcal{A}^{\geqslant p}(\mathcal{V}_*)$ only has interesting cohomology groups for $n = p$, when it is $\mathcal{A}^p(X)_{cl}$, and similarly for $\mathcal{A}^n(X;\mathcal{V}_*)/\mathrm{Im}\,(d)$. Therefore, replacing $\mathcal{A}^{\geqslant p}$ in the construction of a differential cohomology theory by $\mathcal{D}^{\geqslant p}$, has the effect of replacing $\mathcal{A}^p(X)_{cl}$ by $\mathcal{D}^p(X)_{cl}$. The latter is a strictly larger group. The pushforward will in general land in the currential group, since the pushforward of a form in general is a current.

As a concrete example of this phenomenon, we posit the currential differential K-theory groups [19, 2.28]. The difference of the differential extensions of K-theory using forms and currents, respectively, becomes apparent in the equations [19, (2.20)] and [19, (2.29)]. In [19, 4.1] Freed–Lott construct the pushforward along a proper embedding. It goes from the smooth to the currential theory. Their pushforward does preserve smoothness along submersions.

Remark 4.50. We remark also that the pushforward of [8] easily could be extended to a general pushforward for a corresponding currential theory. Theorem 4.48 will allow us to get a pushforward $MU^n(p)(X) \to MU^{n+2d}(p+d)(Y)$ along every proper holomorphic map $X \to Y$ of complex codimension d, even though on cycles it will be of the form $ZMU^n(p)(X) \to ZMU^{n+2d}_\delta(p+d)(Y)$.

Chapter 5

Pontryagin–Thom

In this chapter we discuss the Pontryagin–Thom construction, as pertaining to the cohomology theory MU. We give the proof of [47, Proposition 1.2], eluded to there. That is, we construct an isomorphism between the cycle model for $MU^n(X)$ given in Section 4.1 and the cohomology groups of the spectrum MU. At the heart of our exposition is the Pontryagin–Thom construction,

$$\rho\colon \mathrm{Map}_*^{\pitchfork}(\Sigma^k X_+, MU(m,l)) \to ZMU^n(X)$$

where $k+n=2m$. The domain of ρ, to be defined in 5.8, is a certain dense subset of the mapping space $\mathrm{Map}(\Sigma^k X_+, MU(m,l))$. One reason why we give the full construction of ρ, is that there we will discuss a minor modification of it which we call the geometric Pontryagin–Thom construction:

$$\rho_\nabla\colon \mathrm{Map}_*^{\pitchfork}(\Sigma^k X_+, MU(m,l)) \to \widetilde{ZMU}(X)$$

This geometric Pontryagin–Thom construction is the crucial geometric component in the map we will construct in a chapter6.

5.1 Canonical structures on the tautological bundles

Let $\mathrm{Gr}_m(\mathbb{C}^{m+l})$ denote the Grassmannian manifold of m-dimensional vector subspaces of $\mathbb{C}^{m+l}$. Let

$$\gamma_{m,l} = \{(v,V) \in \mathbb{C}^{m+l} \times \mathrm{Gr}_m(\mathbb{C}^{m+l}) \ : \ v \in V\}$$

denote the tautological bundle over $\mathrm{Gr}_m(\mathbb{C}^{m+l})$. We note that $\gamma_{m,l}$ in a natural way is a subbundle of the trivial bundle $\underline{\mathbb{C}}^{m+l}_{\mathrm{Gr}_m(\mathbb{C}^{m+l})}$. We agree to write $\underline{\mathbb{C}}^{m+l}$ instead of $\underline{\mathbb{C}}^{m+l}_{\mathrm{Gr}_m(\mathbb{C}^{m+l})}$ when no confusion can arise. The bundle $\underline{\mathbb{C}}^{m+l}$ is Hermitian, with the standard Hermitian inner product of $\mathbb{C}^{m+l}$. Also, $\underline{\mathbb{C}}^{m+l}$ has a natural connection, namely the exterior derivative d. Thus $\gamma_{m,l}$ inherits two important geometric features; a Hermitian metric $H_{m,l}$ and a compatible connection. Let $\gamma_{m,l}^\perp$ denote the orthogonal complement to $\gamma_{m,l}$ in $\underline{\mathbb{C}}^{m+l}$.

Definition 5.1. *Let $\nabla_{m,l}$ be the connection on $\gamma_{m,l}$ induced from the trivial connection d by the direct sum decomposition*

$$\underline{\mathbb{C}}^{m+l}_{\mathrm{Gr}_m(\mathbb{C}^{m+l})} \simeq \gamma_{m,l} \oplus \gamma_{m.l}^\perp$$

as in Proposition 4.17. Concretely, let $s_1, \cdots, s_{m+l}$ be an orthonormal frame for $\underline{\mathbb{C}}^{m+l}$ with $s_1, \cdots s_m$ sections of $\gamma_{m,l}$. We can write

$$ds_i = \sum_{j=1}^{m+l} \theta_{ij} \otimes s_j$$

for a Hermitian matrix of 1*-forms* $\theta = (\theta_{ij})$*. We define* $\nabla_{m,l}$ *by*

$$\nabla_{m,l} s_i = \sum_{j=1}^{m} \theta_{ij} \otimes s_j.$$

Remark 5.2. These connections are the same as those used by Narashimhan Ramanan in [44]. They are there shown to be universal among unitary connections, in the sense that if ∇ is a unitary connection on a complex vector bundle E of dimension m, then there is a bundle map $F\colon E \to \gamma_{m,l}$ for l large enough, so that ∇ and $F^*\nabla_{m,l}$ are gauge equivalent.

We consider $\mathbb{C}^k$ as a subspace of $\mathbb{C}^{k+1}$ by the inclusion

$$\mathbb{C}^k \to \mathbb{C}^{k+1}, \quad (z_1, \cdots, z_k) \mapsto (z_1, \cdots, z_k, 0).$$

This consideration induces the right of the following commutative diagrams:

$$\begin{tikzcd} \underline{\mathbb{C}} \oplus \gamma_{m,l} \arrow[r, "\overline{j_{m,l}}"] \arrow[d] & \gamma_{m+1,l} \arrow[d] \\ \mathrm{Gr}_m(\mathbb{C}^{m+l}) \arrow[r, "j_{m,l}"'] & \mathrm{Gr}_{m+1}(\mathbb{C}^{m+1+l}) \end{tikzcd} \qquad \begin{tikzcd} \gamma_{m,l} \arrow[r, "\overline{i_{m,l}}"] \arrow[d] & \gamma_{m,l+1} \arrow[d] \\ \mathrm{Gr}_m(\mathbb{C}^{m+l}) \arrow[r, "i_{m,l}"'] & \mathrm{Gr}_m(\mathbb{C}^{m+l+1}) \end{tikzcd} \tag{5.1}$$

For the left diagram, $\overline{j_{m,l}}$ is defined by

$$(z, (v, V)) \mapsto (v + z \cdot e_{m+l+1},\ V \oplus \mathbb{C} \cdot e_{m+l+1})$$

where $z \in \mathbb{C}$, $v \in V \in \mathrm{Gr}_m(\mathbb{C}^{m+l})$ and $e_{m+l+1} = (0, \cdots, 0, 1) \in \mathbb{C}^{m+l+1}$. We define $j_{m,l}$ by $V \mapsto V \oplus \mathbb{C} \cdot e_{m+l+1}$. The diagrams (5.1) are cartesian, and $\overline{i_{m,l}}$ and $\overline{j_{m,l}}$ are bundle maps, i.e. continuous fiberwise linear isomorphisms.

Proposition 5.3. *The connections* $\nabla_{m,l}$ *are compatible in the sense that*

$$\overline{i_{m,l}}^* \nabla_{m,l+1} = \nabla_{m,l}, \quad \text{and} \quad \overline{j_{m,l}}^* \nabla_{m+1,l} = d \oplus \nabla_{m,l}.$$

Here d *denotes the exterior derivative, thought of as a connection on the trivial bundle.*

Proof. There is a map $\mathrm{Gr}_m(\mathbb{C}^{m+l}) \to \mathrm{Gr}_l(\mathbb{C}^{m+l})$ given by $V \mapsto V^\perp$. We denote by $\perp$ the bundle map $\gamma_{m,l}^\perp \to \gamma_{l,m}$ given by $(v, V) \mapsto (v, V^\perp)$. This is a diffeomorphism. The bundle map

$$\overline{i_{m,l}} \oplus (\perp^{-1} \circ \overline{j_{l,m}} \circ (\mathrm{id}_{\underline{\mathbb{C}}} \oplus \perp))\colon \gamma_{m,l} \oplus (\underline{\mathbb{C}} \oplus \gamma_{m,l}^\perp) \to \gamma_{m,l+1} \oplus \gamma_{m,l+1}^\perp = \underline{\mathbb{C}}^{m+l+1}_{\mathrm{Gr}_m(\mathbb{C}^{m+l+1})}$$

equals the canonical map

$$\gamma_{m,l} \oplus \underline{\mathbb{C}} \oplus \gamma_{m,l}^\perp = \underline{\mathbb{C}}^{m+l+1}_{\mathrm{Gr}_m(\mathbb{C}^{m+l})} \to \underline{\mathbb{C}}^{m+l+1}_{\mathrm{Gr}_m(\mathbb{C}^{m+l+1})}$$

covering $i_{m,l}$. This proves both claims: For the first claim, we must observe that the connection ∇_1 induced on E_1 from a connection ∇ on $E_1 \oplus E_2$, is also induced on E_1 from $E_1 \oplus E_2 \oplus \underline{\mathbb{C}}$ with the connection $\nabla \oplus d$. For the second claim, we must observe that $\nabla \oplus d$ induces on $E_1 \oplus \underline{\mathbb{C}}$ the connection $\nabla_1 \oplus d$. ∎

Contemplating the above proof, we observe that the metrics too are compatible. We record this for future reference:

Proposition 5.4. *The Hermitian metrics $H_{m,l}$ on $\gamma_{m,l}$ are compatible in the sense that $\overline{j_{m,l}}$ and $\overline{i_{m,l}}$ are metric preserving bundle maps. Here we consider $\underline{\mathbb{C}} \oplus \gamma_{m,l}$ with the product metric of the standard metric on $\underline{\mathbb{C}}$, and $H_{m,l}$.*

5.2 The spectra MU and QMU

We define

$$MU(m,l) = \mathrm{Gr}_m(\mathbb{C}^{m+l})^{\gamma_{m,l}} = \mathrm{Th}(\gamma_{m,l}) = \gamma_{m,l} \sqcup \{\infty\}$$

as the Thom space of the tautological bundle. See Section (2.7) for our definition of the Thom space. There are natural maps

$$q_{m,l} \colon MU(m,l) \to MU(m,l+1) \tag{5.2}$$

defined using the maps of (5.1) as $q_{m,l} = \mathrm{Th}(\overline{i_{m,l}})$. Taking the colimit over l we obtain

$$MU(m) = \operatorname*{colim}_l MU(m,l).$$

Similarly, using Proposition 2.39 we get maps

$$s_{m,l} \colon S^2 \wedge MU(m,l) \to MU(m+1,l) \tag{5.3}$$

by $s_{m,l} = \mathrm{Th}(\overline{j_{m,l}})$. Taking the colimit over l we get maps

$$s_m \colon S^2 \wedge MU(m) \to MU(m+1) \tag{5.4}$$

which form the structure maps of the spectrum MU. To make it a sequential spectrum, let $MU_{2m} = MU(m)$ and $MU_{2m+1} = \Sigma MU(m)$, and use the structure maps

$$S^1 \wedge MU_{2m} = MU_{2m+1}, \qquad S^1 \wedge MU_{2m+1} = S^2 \wedge MU_{2m} \to MU_{2m+2}.$$

Then the homotopy theoretic complex cobordism groups of X are defined by

$$MU_h^n(X) = [\Sigma^\infty X, \Sigma^n MU]$$

where $\Sigma^\infty X$ is the suspension spectrum of X, and $[-,-]$ denotes the set of maps in the stable homotopy category.

We will now give an alternative description of the maps in the stable homotopy category on the level of spaces for the case that X is a manifold. Let QMU be the spectrum whose nth space is defined as the colimit

$$QMU_n = \operatorname*{colim}_k \Omega^k MU_{n+k}$$

where Ω^k denotes the kth iterated loop space. The colimit is along the maps

$$\Omega^{2k} MU(n+k) \to \Omega^{2k+2} MU(n+k+1), \qquad f \mapsto s_{n+k} \circ (\mathrm{Id}_{S^2} \wedge f),$$

and the structure maps of QMU are induced by those of MU. Since QMU is an Ω-spectrum, it is a fibrant object in the Bousfield–Friedlander model structure on sequential spectra. Hence the adjunction between Σ^∞ and Ω^∞ induces a natural isomorphism

$$MU_h^n(X) \cong [X, QMU_n]$$

where $[-,-]$ now denotes the set of homotopy classes of maps of spaces.

5.3 The map A.

Maps $X \to QMU_n$ may arise in the following manner: Consider a continuous pointed map $g\colon \Sigma^k X_+ \to MU(m,l)$. It induces a map $\Sigma^k X_+ \to MU_{n+k}$ by composing with the canonical map $MU(m,l) \to MU_{n+k}$ with $n+k = 2m$. This map corresponds to a unique pointed map $X_+ \to \Omega^k MU_{n+k}$ under the adjunction between Σ^k and Ω^k. Composing with the canonical map

$$\Omega^k MU_{n+k} \to \operatorname*{colim}_{k'} \Omega^{k'} MU_{n+k'} = QMU_n$$

we get a map $X_+ \to QMU_n$ corresponding to g. Using that forgetting the basepoint induces an isomorphism between the set of pointed continuous maps $X_+ \to QMU_n$ and the set of unpointed continuous maps $X \to QMU_n$ we get a map

$$A(g)\colon X \to QMU_n.$$

This defines a map

$$A\colon \mathrm{Map}_*(\Sigma^k X_+, MU(m,l)) \to \mathrm{Map}(X, QMU_n),\ g \mapsto A(g). \tag{5.5}$$

Since X is finite dimensional the Freudenthal suspension theorem [35, Corollary 3.2.3], implies:

Proposition 5.5. $A\colon \mathrm{Map}_*(\Sigma^k X_+, MU(m,l)) \to \mathrm{Map}(X, QMU_n)$ *induce a bijection on connected components for* m, l *large enough.*

Hence for many purposes it will therefore suffice to study the subspace of maps $g \in \mathrm{Map}(X, QMU_n)$ such that $g = A(g')$ for some $g' \in \mathrm{Map}_*(\Sigma^k X_+, MU(m,l))$. For our later purposes, we need to understand to what extent the choice of such g' is unique. We start with the following lemma:

Lemma 5.6. *Let*

$$\begin{aligned} g\colon \Sigma^k X_+ &\to MU(m,l), \\ g'\colon \Sigma^{k+2} X_+ &\to MU(m+1,l) \quad \textit{and} \\ g''\colon \Sigma^k X_+ &\to MU(m,l+1) \end{aligned}$$

be such that

$$A(g) = A(g') = A(g'')\colon X \to QMU_n.$$

Then we have: $g' = s_{m,l} \circ \Sigma^2 g$ *and* $g'' = q_{m,l} \circ g$

Proof. The map $A\colon \mathrm{Map}_*(\Sigma^k X_+, MU(m,l)) \to \mathrm{Map}(X, QMU_n)$ associates to g the adjoint of g composed with the canonical map

$$\Omega^k MU(m,l) \to QMU_n = \operatorname*{colim}_{k',l'} \Omega^{k'} MU_{n+k',l'}.$$

The colimit is computed along the maps $f \mapsto q_{m,l} \circ f$ and $f \mapsto s_{m,l} \circ (\mathrm{Id}_{S^2} \wedge f)$. Both $s_{m,l}$ and $q_{m,l}$ are injective. Hence the assumption $A(g) = A(g') = A(g'')$ implies $g' = s_{m,l} \circ \Sigma^2 g$ and $g'' = q_{m,l} \circ g$. ∎

This implies the following useful result:

Proposition 5.7. *Let* $\mathrm{Map}^A(X, QMU_n)$ *denote the set of maps* $g\colon X \to QMU_n$ *such that* $g = A(g')$ *for some* $g' \in \mathrm{Map}(\Sigma^k X_+, MU(m,l))$*, let* S *be a set, and let for another set* B $\mathrm{Fun}(B,S)$ *denote the set of functions* $B \to S$*. Then the assignments*

$$\mathrm{Fun}\left(\mathrm{Map}^A(X, QMU_n), S\right) \to \mathrm{Fun}\left(\mathrm{Map}_*(\Sigma^k X_+, MU(m,l)), S\right)$$
$$f \mapsto (g \mapsto f(A(g)))$$

defines a bijection between the set of functions $\mathrm{Map}^A(X, QMU_n) \to S$*, and systems of functions*

$$f_{m,l}\colon \mathrm{Map}(\Sigma^k X_+, MU(m,l)) \to S$$

which are compatible in the sense that

$$f_{m+1,l}(s_{m,l} \circ \Sigma^2 g) = f_{m,l}(g) = f_{m,l+1}(q_{m,l} \circ g).$$

5.4 Definition of ρ and ρ_∇

Recall that

$$\Sigma^k X_+ = \mathrm{Th}(\underline{\mathbb{R}}^k_X) = \mathbb{R}^k \times X \sqcup \{\infty\}$$

and

$$MU(m,l) = \mathrm{Th}(\gamma_{m,l}) = \gamma_{m,l} \sqcup \{\infty\}.$$

are smooth manifolds away from the basepoints, ∞. Let

$$\iota_{m,l}\colon \mathrm{Gr}_m(\mathbb{C}^{m+l}) \to \gamma_{m,l} \tag{5.6}$$

denote the 0-section, whereby we consider $\mathrm{Gr}_m(\mathbb{C}^{m+l})$ as a submanifold of $\gamma_{m,l}$.

Definition 5.8. *We let*

$$\mathrm{Map}_*^{\pitchfork}\left(\Sigma^k X_+, MU(m,l)\right) \subset \mathrm{Map}_*\left(\Sigma^k X_+, MU(m,l)\right)$$

denote the space of pointed maps $g\colon \Sigma^k X_+ \to MU(m,l)$ *such that* g *restricted to* $g^{-1}(\gamma^{m,l})$ *is smooth and transverse to* $\mathrm{Gr}_m(\mathbb{C}^{m,l})$*. We also define*

$$\mathrm{Map}^{\pitchfork}(X, QMU_n) = \{g \;:\; g = A(g_\pitchfork) \text{ for some } g_\pitchfork \in \mathrm{Map}_*^{\pitchfork}(\Sigma^k X_+, MU(m,l))\}.$$

Proposition 5.9. *For every $g \in \mathrm{Map}(X, QMU_n)$ there is a map*

$$g_{\pitchfork} \in \mathrm{Map}^{\pitchfork}(X, QMU_n)$$

such that g is homotopic to $g_{\pitchfork}$.

Proof. It follows from Thom's transversality theorem that

$$\mathrm{Map}_*^{\pitchfork}\left(\Sigma^k X_+, MU(m,l)\right)$$

is dense in the space of all continuous pointed maps $\mathrm{Map}_*\left(\Sigma^k X_+, MU(m,l)\right)$. In particular these transverse maps are present in every path component, so for each $g' \colon \Sigma^k X_+ \to MU(m,l)$ there is a map $g_{\pitchfork} \in \mathrm{Map}_*^{\pitchfork}\left(\Sigma^k X_+, MU(m,l)\right)$ so that g' is homotopic to $g_{\pitchfork}$. Now the proposition follows from Proposition 5.5. ∎

For $g \in \mathrm{Map}_*^{\pitchfork}\left(\Sigma^k X_+, MU(m,l)\right)$ we define

$$Z_g := g^{-1}\left(\mathrm{Gr}_m\left(\mathbb{C}^{m+l}\right)\right).$$

Since g is pointed we have an inclusion

$$i \colon Z_g \hookrightarrow g^{-1}(\gamma_{m,l}) \hookrightarrow \mathbb{R}^k \times X.$$

Let $f_g \colon Z_g \to X$ be the map $\pi \circ i$, for π the projection $\mathbb{R}^k \times X \to X$. Note that the codimension of f_g is $k - 2m = -n$. We give f_g a complex orientation as follows: Let

$$N_{f_g} = (g|_{Z_g})^* \gamma_{m,l},$$

and use Propositions 2.6 and 2.7 to obtain the following short exact sequence

$$\Phi_{f_g} = \left(0 \longrightarrow TZ_g \xrightarrow{Di} i^*T(\mathbb{R}^k \times X) \longrightarrow N_{f_g} \longrightarrow 0 \right).$$

Identifying $i^*T\mathbb{R}^k$ with $\underline{\mathbb{R}}^k_{Z_g}$, this gives f_g a complex orientation. The Pontryagin–Thom construction is induced by the following maps:

$$\rho_{k,m,l} \colon \mathrm{Map}_*^{\pitchfork}\left(\Sigma^k X_+, MU(m,l)\right) \to ZMU^n(X)$$
$$\rho_{k,m,l}(g) = (f_g, N_g).$$

We also define

$$\nabla_g = (g|_{Z_g})^* \nabla_{m,l},$$

and

$$\rho_{\nabla,k,m,l}(g) = (f_g, N_g, \nabla_g).$$

The maps $\rho_{\nabla,k,m,l}$ form the basis for the geometric Pontryagin–Thom construction. These definitions requires the following verification:

Proposition 5.10. *f_g as defined above is proper.*

Proof. Let $K \subset X$ be a compact set. Then $f_g^{-1}(K)$ fits into the following cartesian diagram of spaces:

$$\begin{tikzcd}
f_g^{-1}(K) \arrow[r] \arrow[d, "i"'] & Z_g \arrow[r] \arrow[d] & \mathrm{Gr}_m(\mathbb{C}^{m+l}) \arrow[d, "\iota'_{m,l}"] \\
\Sigma^k K_+ \arrow[r] & \Sigma^k X_+ \arrow[r] & MU(m,l).
\end{tikzcd}$$

Here $\iota'_{m,l}$ is the composition of $\iota_{m,l} \colon \mathrm{Gr}_m(\mathbb{C}^{m,l}) \to \gamma_{m,l}$ with the canonical inclusion $\gamma_{m,l} \to \mathrm{Th}(\gamma_{m,l})$. Since $\iota'_{m,l}$ is proper, i is proper. Since $\Sigma^k K_+$ is compact, it follows that $f_g^{-1}(K)$ is compact. ■

Recall from (5.2) and (5.3) the maps

$$s_{m,l} \colon S^2 \wedge MU(m,l) \to MU(m+1,l) \quad \text{and} \quad q_{m,l} \colon MU(m,l) \to MU(m,l+1)$$

Proposition 5.11. *Let $g_{\pitchfork} \in \mathrm{Map}_*^{\pitchfork}(\Sigma^k X_+, MU(m,l))$. Then*

$$\begin{aligned}
\rho_{k,m,l}(g_{\pitchfork}) &= \rho_{k,m,l+1}(q_{m,l} \circ g_{\pitchfork}) \\
\rho_{\nabla,k,m,l}(g_{\pitchfork}) &= \rho_{\nabla,k,m,l+1}(q_{m,l} \circ g_{\pitchfork})
\end{aligned}$$

Furthermore, we have $s_{m,l} \circ \Sigma^2 g_{\pitchfork} \in \mathrm{Map}_*^{\pitchfork}(\Sigma^{k+2} X_+, MU(m+1,l))$, and

$$\begin{aligned}
\rho_{k,m,l}(g_{\pitchfork}) &= \rho_{k+2,m+1,l}(s_{m,l} \circ \Sigma^2 g_{\pitchfork}) \\
\rho_{\nabla,k,m,l}(g_{\pitchfork}) &= \rho_{\nabla,k+2,m+1,l}(s_{m,l} \circ \Sigma^2 g_{\pitchfork}).
\end{aligned}$$

Proof. We use the maps of (5.1) and (5.6). Set $U = {g_{\pitchfork}}^{-1}(\gamma_{m,l})$. The first claim follows from the following commutative diagram, where all squares are cartesian.

$$\begin{tikzcd}
& Z_{q_{m,l} \circ g_{\pitchfork}} \arrow[d] \arrow[dr] & \\
Z_{g_{\pitchfork}} \arrow[ur] \arrow[d] \arrow[dr] & \mathrm{Gr}_m(\mathbb{C}^{m+l+1}) \arrow[dr, "\iota_{m,l+1}"] & U \arrow[d] \arrow[dd, bend left] \\
\mathrm{Gr}_m(\mathbb{C}^{m+l}) \arrow[ur] \arrow[dr, "\iota_{m,l}"'] & U \arrow[ur, "\mathrm{id}"] \arrow[d, "g_{\pitchfork}"'] \arrow[dr] & \gamma_{m,l+1} \\
& \gamma_{m,l} \arrow[ur, "\overline{i_{m,l}}"'] & X
\end{tikzcd}$$

For the second property, we first describe $s_{m,l} \circ \Sigma^2 g_{\pitchfork}$. Recall from Section 2.7 that $\Sigma^2 \Sigma^k X_+ = \mathrm{Th}(\underline{\mathbb{C}}_X \oplus \underline{\mathbb{R}}_X^k)$, and $S^2 \wedge MU(m,l) = \mathrm{Th}(\underline{\mathbb{C}} \oplus \gamma_{m,l})$. The map $s_{m,l} \circ \Sigma^2 g_{\pitchfork}$ is characterized by

$$\left(s_{m,l} \circ \Sigma^2 g_{\pitchfork}\right)|_{\mathbb{C} \times \mathbb{R}^k \times X} = \overline{j_{n,k}} \circ (\mathrm{id}_{\mathbb{C}} \times (g_{\pitchfork}|_{\mathbb{R}^k \times X})).$$

It is clear that $\mathrm{id}_{\mathbb{C}} \times (g_{\pitchfork}|_{\mathbb{R}^k \times X})$ is transverse to $\{0\} \times \mathrm{Gr}_m(\mathbb{C}^{m+l})$, with inverse image $\{0\} \times Z_{g_{\pitchfork}}$. We consider the following commutative diagram, where all squares are cartesian squares of manifolds.

$$\begin{tikzcd}
 & & Z_{s_{m,l} \circ \Sigma^2 g_{\pitchfork}} \arrow[d] \arrow[drr] & & \\
\{0\} \times Z_{g_{\pitchfork}} \arrow[urr] \arrow[d] \arrow[drr] & & \mathrm{Gr}_{m+1}(\mathbb{C}^{m+1+l}) \arrow[drr] & & \mathbb{C} \times U \arrow[d] \arrow[dd, bend left] \\
\mathrm{Gr}_m(\mathbb{C}^{m+l}) \arrow[urr] \arrow[dr, "0 \oplus \iota_{m,l}"'] & & \mathbb{C} \times U \arrow[urr, "\mathrm{id}"] \arrow[d, "\mathrm{id}_{\mathbb{C}} \times g_{\pitchfork}"'] \arrow[drr] & & \gamma_{m+1,l} \\
 & & \underline{\mathbb{C}} \oplus \gamma_{m,l} \arrow[urr, "\overline{j_{n,k}}"'] & & X
\end{tikzcd}$$

It is now clear that the map $\{0\} \times Z_{g_{\pitchfork}} \to Z_{s_{m,l} \circ \Sigma^2 g_{\pitchfork}}$ is an isomorphism of complex oriented maps over X, proving the proposition. ∎

Based on Propositions 5.11 and 5.7, we can now define the Pontryagin–Thom map

$$\begin{aligned} \rho\colon \mathrm{Map}^{\pitchfork}(X, QMU_n) &\to ZMU^n(X) \\ \rho(g) &= \rho_{k,m,l}(g_{\pitchfork}) \end{aligned} \tag{5.7}$$

for any $g_{\pitchfork} \in \mathrm{Map}_*^{\pitchfork}\left(\Sigma^k X_+, MU(m,l)\right)$ with $A(g_{\pitchfork}) = g$. We also get a geometric version of ρ:

Definition 5.12. *We define the geometric Pontryagin–Thom map*

$$\begin{aligned} \rho_{\nabla}\colon \mathrm{Map}^{\pitchfork}(X, QMU_n) &\to \widetilde{ZMU}^n(X) \\ \rho_{\nabla}(g) &= \rho_{\nabla,k,m,l}(g_{\pitchfork}) \end{aligned}$$

for any $g_{\pitchfork} \in \mathrm{Map}_*^{\pitchfork}\left(\Sigma^k X_+, MU(m,l)\right)$ *with* $A(g_{\pitchfork}) = g$.

5.5 Naturality of ρ_{∇}

We will now show that ρ and ρ_{∇} are somewhat natural in X. There is a minor problem in that for $g_{\pitchfork} \in \mathrm{Map}_*^{\pitchfork}(\Sigma^k X_+, MU(m,l))$, and a smooth map $F\colon Y \to X$, the cycle $F^*\rho(g_{\pitchfork})$ may not be well defined, and the composition $g_{\pitchfork} \circ \Sigma^k F_+$ may not belong to $\mathrm{Map}_*^{\pitchfork}(\Sigma^k Y_+, MU(m,l))$. We therefore define the space

$$\mathrm{Map}_*^{\pitchfork,F}\left(\Sigma^k X_+, MU(m,l)\right) \subset \mathrm{Map}_*^{\pitchfork}\left(\Sigma^k X_+, MU(m,l)\right) \tag{5.8}$$

to be the subset of maps g such that $g \circ \Sigma^k F_+ \in \mathrm{Map}_*^{\pitchfork}(\Sigma^k Y_+, MU(m,l))$. The maps in this space are generic; it follows immediately from Thom's transversality theorem that

$$(\Sigma^k F_+)^* \Big(\mathrm{Map}_*^{\pitchfork}(\Sigma^k X_+, MU(m,l))\Big) \bigcap \mathrm{Map}_*^{\pitchfork}(\Sigma^k Y_+, MU(m,l))$$

is a dense subset of $(\Sigma^k F_+)^* \mathrm{Map}_*(\Sigma^k X_+, MU(m,l))$.

Theorem 5.13. *Both ρ and ρ_∇ are natural in the following sense: For $F\colon Y \to X$ a smooth map, and $g \in \mathrm{Map}_*^{\pitchfork, F}\left(\Sigma^k X_+, MU(m,l)\right)$ we have:*

$$\rho_\nabla(g \circ \Sigma^k F_+) = F^* \rho_\nabla(g) \in \widetilde{ZMU}^n(Y)$$

$$\rho(g \circ \Sigma^k F_+) = F^* \rho(g) \in ZMU^n(Y)$$

Proof. It is well known that given smooth maps $f_1\colon M_1 \to M_2$ and $f_2\colon M_2 \to M_3$ such that $f_2 \pitchfork S \subset M_3$, we have $f_1 \pitchfork (f_2^{-1}(S))$ if and only if $(f_2 \circ f_1) \pitchfork S$. From this we see that F^* is defined on $\rho(g)$. Now we consider the commutative diagram

$$\begin{tikzcd}
Z_Y \arrow[r] \arrow[d, "i_Y"'] & Z_X \arrow[r, "g_\pitchfork|_Z"] \arrow[d, "i_X"'] & \mathrm{Gr}_m\left(\mathbb{C}^{m+l}\right) \arrow[d, "\iota_{m,l}"] \\
U_Y \arrow[r, "\mathrm{id}\times F"] \arrow[d, "\pi_Y"'] & U_X \arrow[r, "g_\pitchfork|U"] \arrow[d, "\pi_X"] & MU(m,l) \\
Y \arrow[r, "F"] & X &
\end{tikzcd}$$

with $Z_X = Z_g = (g)^{-1}(\mathrm{Gr}_m(\mathbb{C}^{m+l}))$ and $Z_Y = (\mathrm{id} \times F)^{-1} Z_X$. We use that F is transverse to g, and we write:

$$U_X = g^{-1}(\gamma_{m,l}) \subset \mathbb{R}^k \times X \subset \Sigma^k X_+$$
$$U_Y = (\mathrm{id} \times F)^{-1}(U_X) \subset \mathbb{R}^k \times Y \subset \Sigma^k Y_+$$

This proves the theorem. ∎

5.6 The isomorphism $MU^n(X) \simeq MU_h^n(X)$.

We now prove that ρ as defined in (5.7) induces a natural isomorphism $MU_h^n(X) \to MU^n(X)$. We continue the convention that $n + k = 2m$.

Proposition 5.14. *The map*

$$\rho\colon \mathrm{Map}^\pitchfork(X, QMU_n) \to ZMU^n(X)$$

is surjective.

Remark 5.15. We remark that ρ_∇ is not surjective. The image of ρ_∇ consist of all geometric cycles (f, N_f, ∇_f) for which ∇_f is unitary, in the sense that there exist a Hermitian metric on N_f with which ∇_f is compatible. That ρ_∇ only yields unitary connections is clear, since $\nabla_{m,l}$ is unitary with respect to the canonical Hermitian metric on $\gamma_{m,l}$. That we get all geometric cycles with unitary connection follows from [44], and the proof of Proposition 5.14. If we wanted ρ_∇ to be surjective, we would have to replace $\gamma_{m,l}$ with a bundle carrying a universal connection. It seems likely that the construction of [45] suffices, but we have not pursued this.

Proof of 5.14. Let $f\colon Z \to X$ be a proper complex oriented map of codimension $-n$. By Whitney's embedding theorem, we can find a proper embedding $Z \to \mathbb{R}^k$. Taking the product with f, we get a proper embedding

$$i\colon Z \to \mathbb{R}^k \times X.$$

We can choose k so that $k - n = 2m$ is even. Then the complex orientation of f endows the normal bundle Ni with a complex structure. If l is large enough, there exist a bundle map $\xi\colon Ni \to \gamma_{m,l}$, by the complex analogue of [43, Lemma 5.3].[1]. By the tubular neighborhood theorem [26, Theorem 5.2] we can extend i to an open embedding $\Xi\colon Ni \to \mathbb{R}^k \times X$. Then define $g_f\colon \Sigma^k X_+ \to MU(m,l)$ by:

$$g_f(x) = \begin{cases} \xi \circ \Xi^{-1}(x) & x \in \mathrm{Im}\,(\Xi) \\ \infty & x \notin \mathrm{Im}\,(\Xi) \end{cases}$$

Alternatively, g_f is the composition

$$g_f = \mathrm{Th}(\xi) \circ \mathit{col}$$

where $\mathit{col}\colon \Sigma^k X_+ \to \mathrm{Th}(Ni)$ is the Pontryagin–Thom collapse, equaling Ξ^{-1} on $\mathrm{Im}\,(\Xi)$, and collapsing everything outside of $\mathrm{Im}\,(\xi)$ to ∞. Then $(f_{g_f}, N_{g_f}) = (f, N_f)$ by construction. ∎

Proposition 5.16. *ρ induces a map $MU^n_h(X) = [X, QMU_n] \to MU^n(X)$.*

Proof. We must show that if $g_0,\ g_1 \in \mathrm{Map}^{\pitchfork}(X, QMU_n)$ are homotopic, then f_{g_0} and f_{g_1} are cobordant. By Definition 5.8, we can for $t = 0, 1$ find $g_t^{\pitchfork}\colon \Sigma^k X_+ \to MU(m,l)$ such that $A(g_t^{\pitchfork}) = g_t$. Furthermore, by Lemma 5.5 we can assume that $g_0^{\pitchfork}$ and $g_1^{\pitchfork}$ are homotopic as maps $\Sigma^k X_+ \to MU(m,l)$. By Lemma 4.8 we choose a smooth homotopy

$$G\colon \mathbb{R} \times \Sigma^k X_+ \to MU(m,l).$$

I.e. for $i_t\colon X \to \mathbb{R} \times X$ the map $i_t(x) = (t,x)$, we have $G \circ i_t = g_t^{\pitchfork}$ for $t = 0, 1$. Then evidently we have

$$G|_{\{0,1\}\times\Sigma^k X_+} \pitchfork \mathrm{Gr}_m(\mathbb{C}^{m+l}),$$

and by Thom's transversality theorem, we can perturb G slightly, so that G is transverse to $\mathrm{Gr}_m(\mathbb{C}^{m+l})$ everywhere. We can do this without changing $G|_{\{0,1\}\times\Sigma^k X_+}$ so that G remains a homotopy between $g_{0\pitchfork}$ and $g_{1\pitchfork}$. Now let

$$(b_G,\ N_b) := \rho(G) \in ZMU^{n-1}(\mathbb{R} \times X).$$

[1] The proof of [43, Lemma 5.3] works equally well in the complex case. Annoyingly the corresponding lemma for complex vector bundles is not stated. Rather they state only the theorem they need the lemma for proving, [43, Theorem 14.6], stating the proof is the same as in the real case.

This is a bordism datum over X, which is equivalent to saying

$$G \in \mathrm{Map}_*^{\pitchfork, i_0 \sqcup i_1}(\mathbb{R} \times \Sigma^k X_+, MU(m,l)).$$

Since ρ is natural, as we showed in Theorem 5.13, we finish the proof thus: For $t = 0, 1$ we have

$$i_t^* \rho(G) = \rho(G \circ i_t) = \rho(g_t^{\pitchfork}).$$ ■

Recall that the addition in $[\Sigma^k X_+, MU(m,l)]$ is induced by the binary operation on $\mathrm{Map}_*(\Sigma^k X_+, MU(m,l))$ given by

$$g_1 * g_2 = (g_1 \vee g_2) \circ \mathit{pinch}$$

for $\mathit{pinch}\colon \Sigma^k X_+ \to \Sigma^k X_+ \vee \Sigma^k X_+$ the pinch map which collapses

$$\mathbb{R}^{k-1} \times \{0\} \times X$$

onto

$$\infty \in \Sigma^k X_+ = \mathbb{R}^k \times X \sqcup \{\infty\}.$$

Here we view the open upper and lower half planes as disjoint copies of $\mathbb{R}^k$ via

$$\begin{aligned} \mathbb{R}^{k-1} \times (0,\infty) \times X &\to \mathbb{R}^k \times X \\ (v,t,x) &\mapsto \left(v, \frac{t^2-1}{t}, x\right) \end{aligned}$$

and similarly for the lower half plane.

Proposition 5.17. *The maps*

$$\begin{aligned} \rho\colon \mathrm{Map}^{\pitchfork}(X, QMU_n) &\to ZMU^n(X) \qquad \textit{and} \\ \rho_\nabla\colon \mathrm{Map}^{\pitchfork}(X, QMU_n) &\to \widetilde{ZMU}^n(X) \end{aligned}$$

are homomorphisms in the sense that for $g_1,\ g_2 \in \mathrm{Map}_*^{\pitchfork}(\Sigma^k X_+, MU(m,l))$ *we have*

$$\begin{aligned} \rho_\nabla(A((g_1 \vee g_2) \circ \mathit{pinch})) &= \rho_\nabla(A(g_1)) + \rho_\nabla(A(g_2)), \quad \textit{and} \\ \rho(A((g_1 \vee g_2) \circ \mathit{pinch})) &= \rho(A(g_1)) + \rho(A(g_2)). \end{aligned}$$

Proof. We note that there is an evident diffeomorphism

$$(g_1 * g_2)^{-1}(\gamma_{m,l}) \simeq g_1^{-1}(\gamma_{m,l}) \sqcup g_0^{-1}(\gamma_{m,l})$$

such that the following diagram commutes:

$$\begin{tikzcd} (g_1 * g_2)^{-1}(\gamma_{m,l}) \arrow[rr] \arrow[dr] & & g_1^{-1}(\gamma_{m,l}) \sqcup g_2^{-1}(\gamma_{m,l}) \arrow[dl] \\ & \gamma_{m,l} & \end{tikzcd}$$

The point is that the union is disjoint since $g_t^{-1}(\gamma_{m,l})$ are in different half-planes for $t=0$ and $t=1$. In particular Z_{g_1} and Z_{g_2} are disjoint, and the complex orientations of these versions of f_{g_1} and f_{g_2} as well as the connections ∇_{g_1}, ∇_{g_2}, are the same as the original ones. Hence we get

$$f_{(g_1 \vee g_2)\circ pinch} = f_{g_1} + f_{g_2}. \qquad \blacksquare$$

Theorem 5.18. *The Pontryagin–Thom construction ρ induces an isomorphism $MU_h^n(X) \to MU^n(X)$.*

Proof. We recap our work. There is an induced map by Proposition 5.16. The induced map is a homomorphism by Proposition 5.17, and it is surjective since ρ is surjective by Proposition 5.14. It remains to show injectivity. Let

$$g \in \mathrm{Map}^{\pitchfork}(X, QMU_n).$$

It suffices to show that if f_g represents 0, then g is nullhomotopic. If f_g represents 0, we can find a bordism datum over X with underlying map

$$b = (a_b, f_b)\colon W \to \mathbb{R} \times X$$

such that 0 and 1 are regular values of a_b and

$$f_b|_{a_b^{-1}(0)} = f_g, \qquad a_b^{-1}(1) = \emptyset. \tag{5.9}$$

In particular $b \in ZMU^n(\mathbb{R} \times X)$. By Proposition 5.14, we can for sufficiently large k' and l', and for $k' + n = 2m'$, find a map

$$G \in \mathrm{Map}^{\pitchfork}(\Sigma^{k'}(\mathbb{R} \times X), MU(m', l'))$$

such that $\rho(G) = b$. We view G as a homotopy of pointed maps by $G_t = G|_{\Sigma^{k'}\{t\}\times X_+}$. Then by (5.9) we have $G_0 = g$. Since $a_b^{-1}(1) = \emptyset$, G_1 does not meet $\mathrm{Gr}_{m'}(\mathbb{C}^{m'+l'})$. The space $MU(m', l')\backslash\mathrm{Gr}_{m'}(\mathbb{C}^{m'+l'})$ is contractible. The contraction is given by multiplication by $1/t$, where we interpret multiplication by $1/0$ as collapsing everything to the point ∞. Hence we conclude that g is nullhomotopic, finishing the proof. $\blacksquare$

5.7 Geometric fundamental currents

Given $g_{\pitchfork} \in \mathrm{Map}^{\pitchfork}(X, QMU_n)$, we consider the geometric cycle $\rho_\nabla(g_{\pitchfork}) \in \widetilde{ZMU}^n(X)$. In this Section we describe

$$\phi(\rho_\nabla(g_{\pitchfork})) = (f_{g_{\pitchfork}})_* K(\nabla_{g_{\pitchfork}})$$

in universal terms. We continue letting

$$\iota_{m,l}\colon \mathrm{Gr}_m(\mathbb{C}^{m+l}) \to \gamma_{m,l}$$

be the 0-section, and recall the connections $\nabla_{m,l}$ from Definition 5.1.

Definition 5.19. *We define the geometric fundamental currents by*

$$\phi_{\nabla_{m,l}} = (\iota_{m,l})_* K(\nabla_{m,l}) \in \mathcal{D}^{2m}_{vc}(\gamma_{m,l}; \mathcal{V}_*).$$

Recall from (5.1) the maps

$$\overline{j_{m,l}} \colon \underline{\mathbb{C}} \oplus \gamma_{m,l} \to \gamma_{m+1,l} \quad \text{and} \quad \overline{i_{m,l}} \colon \gamma_{m,l} \to \gamma_{m,l+1}.$$

Proposition 5.20. *The geometric fundamental currents are compatible in the sense that the following pullbacks are well defined, and verifies the following equalities:*

$$\begin{aligned} \overline{i_{m,l}}^* \phi_{\nabla_{m,l+1}} &= \phi_{\nabla_{m,l}} \\ \overline{j_{m,l}}^* \phi_{\nabla_{m+1,l}} &= \delta_0 \otimes \phi_{\nabla_{m,l+1}} \end{aligned}$$

Here $\delta_0 \in \mathcal{D}^0(\mathbb{C})$ *is the Dirac delta at* 0*, operating by* $\delta_0(f dx \wedge dy) = f(0)$*.*

Proof. Consider the following commutative diagrams, obtained by taking the 0-sections in the diagrams (5.1).

$$\begin{tikzcd} Gr_m(\mathbb{C}^{m+l}) \arrow[r, "j_{m,l}"] \arrow[d, "0\times\iota_{m,l}"'] & Gr_{m+1}(\mathbb{C}^{m+1+l}) \arrow[d, "\iota_{m+1,l}"] \\ \underline{\mathbb{C}} \oplus \gamma_{m,l} \arrow[r, "\overline{j_{m,l}}"'] & \gamma_{m+1,l} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Gr}_m(\mathbb{C}^{m+l}) \arrow[r, "i_{m,l}"] \arrow[d, "\iota_{m,l}"'] & \mathrm{Gr}_m(\mathbb{C}^{m+l+1}) \arrow[d, "\iota_{m,l+1}"] \\ \gamma_{m,l} \arrow[r, "\overline{i_{m,l}}"'] & \gamma_{m,l+1} \end{tikzcd}$$

Note that $\overline{j_{m,l}} \pitchfork \iota_{m+1,l}$. This is so because $\overline{j_{m,l}}$ is a fiberwise isomorphism, and so onto the normal bundle of $\iota_{m+1,l}$ where they meet. Similarly $\overline{i_{m,l}} \pitchfork \iota_{m,l+1}$. Using Proposition 5.3, the assertion follows from Theorem 2.31, upon observing

$$K(d \oplus \nabla_{m,l}) = K(\nabla_{m,l}) \in \mathcal{A}^0(\mathrm{Gr}_m(\mathbb{C}^{m+l}); \mathcal{V}_*). \qquad \blacksquare$$

Recall from Proposition 2.40 that we write

$$\Sigma^k X_+ = \mathrm{Th}(\mathbb{R}^k_X) = \mathbb{R}^k \times X \sqcup \{\infty\}.$$

Let $g \in \mathrm{Map}^{\pitchfork}(X, QMU_n)$, say $g = A(g_{\pitchfork})$ for $g_{\pitchfork} \in \mathrm{Map}^{\pitchfork}_*(\Sigma^k X_+, MU(m,l))$. Let

$$U = {g_{\pitchfork}}^{-1}(\gamma_{m,l}) \subset \mathbb{R}^k \times X.$$

Then $g_{\pitchfork}|_U$ is smooth and transverse to $\mathrm{Gr}_m(\mathbb{C}^{m+l})$. Using the cartesian square

$$\begin{tikzcd} Z \arrow[r, "g_{\pitchfork}|_Z"] \arrow[d, "i"'] & \mathrm{Gr}_m\left(\mathbb{C}^{m+l}\right) \arrow[d, "\iota_{m,l}"] \\ U \arrow[r, "g_{\pitchfork}|_U"'] & \gamma_{m,l} \end{tikzcd} \tag{5.10}$$

and Theorem 2.31 we get the following equalities in $\mathcal{D}^*(\mathbb{R}^k \times X; \mathcal{V}_*)$:

$$\begin{aligned}(g_{\pitchfork}|_U)^* \phi_{\nabla_{m,l}} &= (g_{\pitchfork}|_U)^* (\iota_{m,l})_* K(\nabla_{m,l}) \\ &= i_* \left(g_{\pitchfork}|_Z\right)^* K(\nabla_{m,l}) \\ &= i_* K(\nabla_{g_{\pitchfork}})\end{aligned}$$

By Proposition 5.10, $f_{g_{\pitchfork}} = \pi \circ i$ is a proper map, where $\pi\colon \mathbb{R}^k \times X \to X$ denotes the projection onto X. Hence we have $i_* K(\nabla_{g_{\pitchfork}}) \in \mathcal{D}^*_{vc}(\mathbb{R}^k \times X; \mathcal{V}_*)$. Applying the map

$$\pi_*\colon \mathcal{D}^*_{vc}(\mathbb{R}^k \times X; \mathcal{V}) \to \mathcal{D}^{*-k}(X; \mathcal{V}_*)$$

we obtain the following theorem.

Theorem 5.21. *With the above notation, we have*

$$\pi_*(g_{\pitchfork}|_U)^*(\phi_{\nabla_{m,l}}) = (f_{g_{\pitchfork}})_* K(\nabla_{g_{\pitchfork}}) \in \mathcal{D}^*(X; \mathcal{V}_*). \quad \blacksquare$$

Chapter 6

Comparison of Hodge filtered cobordism theories

In this chapter we construct an isomorphism between the geometric Hodge filtered cobordism theory of Section 4 and the homotopy-theoretical construction of Hopkins–Quick. In the definition of Hopkins–Quick Hodge filtered bordism, recalled below in Section 6.1, there features a choice of a fundamental cocycle, i.e., a map of simplicial presheaves of spectra $MU \to H\mathcal{V}_*$ inducing the natural transformation ϕ of (4.11). Composing with the canonical map $H\mathcal{V}_* \to H(\mathcal{A}^*(\mathcal{V}_*))$ we get a fundamental cocycle.

The guiding principle of our construction is that the geometric fundamental currents $\phi_{\nabla_{m,l}}$, being compatible by Proposition 5.20, should combine to form a map $\phi_\nabla \colon MU \to H(\mathcal{D}^*(\mathcal{V}_*))$. Since the map $\mathcal{A}^*(\mathcal{V}_*) \to \mathcal{D}^*(\mathcal{V}_*)$ on each manifold X is a weak equivalence of complexes of presheaves, we should be allowed to use $H(\mathcal{D}^*(\mathcal{V}_*))$ in place of $H(\mathcal{A}^*(\mathcal{V}_*))$ in the Hopkins–Quick construction, and induce a fundamental cocycle from the geometric fundamental currents $\phi_{\nabla_{m,l}}$.

There are two problems with a straightforward implementation of this plan. First, MU is interpreted as the constant presheaf $X \mapsto MU$, so $\phi_{\nabla_{m,l}} \in \mathcal{D}^{2m}(MU(m,l); \mathcal{V}_*)$ cannot combine to a map $MU \to \mathcal{D}^*(\mathcal{V}_*)$. To make our plan work, we must replace MU by a spectrum of simplicial presheaves $X \mapsto \mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU)$, which exploits the smooth structure on $\gamma_{m,l} \subset MU(m,l)$.

Secondly, $\mathcal{D}^*(\mathcal{V}_*)$ is not a presheaf on $\mathbf{Man}_\mathbb{C}$, since it does not have pullbacks along arbitrary holomorphic maps. To get around this obstruction, we define fundamental forms $\phi_{m,l} \in \mathcal{A}^{2m}(MU(m,l); \mathcal{V}_*)$, and comparison currents $\alpha_{m,l}$ satisfying $\phi_{\nabla_{m,l}} + d\alpha_{m,l} = \phi_{m,l}$.

Using the forms $\phi_{m,l}$, and the spectra defined in Section 2.13, we give one more model of Hodge filtered complex cobordism, MU_{hs}. We then first establish that $MU_{\mathrm{hs}}(p)$ and $MU_{\mathcal{D}}(p)$ represent the same object of $\mathrm{hoSp}(\mathbf{sPre}_*)$. Then we give a concrete description of the groups $MU^n_{\mathrm{hs}}(p)(X)$, amenable for application of the Pontryagin–Thom construction. Finally we define $\kappa \colon MU^n_{\mathrm{hs}}(p)(X) \to MU^n(p)(X)$ and show that it is an isomorphism.

Remark 6.1. Note that here we consider sequential topological spectra. This has the consequence that the comparison map κ we will construct between the two Hodge filtered cobordism theories only is a homomorphism of groups. However, we expect this map to respect the product structures in both theories once the additional structure is taken care of in the construction.

6.1 Hodge filtered cobordism of Hopkins–Quick

We briefly recall the construction of Hodge filtered cobordism groups in the stable homotopy category of presheaves of spectra of [30].

Let Z be a topological space and let $\operatorname{sing}(Z)$ denotes its singular simplicial set. Let $\operatorname{sing}(MU)$ denote the spectrum of simplicial sets whose n-th simplicial set is given by $\operatorname{sing}(MU_n)$. We consider $\operatorname{sing}(MU)$ as a spectrum of constant presheaves on $\mathbf{Man}_{\mathbb{C}}$. Let $H(\mathcal{V}_*)$ denote the spectrum of presheaves whose nth simplicial set is the simplicial Eilenberg–MacLane space $K(\mathcal{V}_*, n)$. We also consider $H(\mathcal{V}_*)$ as a spectrum of constant presheaves on $\mathbf{Man}_{\mathbb{C}}$. Let $\operatorname{sing}(MU) \to H(\mathcal{V}_*)$ be a map of spectra which induces the genus $\phi_* \colon MU_* \to \mathcal{V}_*$ on homotopy groups, as in Section 2.14. We consider ϕ also as a map of presheaves of spectra on $\mathbf{Man}_{\mathbb{C}}$. The canonical inclusion map of constant functions into the de Rham complex $\mathcal{V}_* \to A^*(X; \mathcal{V}_*)$ induces a map of presheaves of Eilenberg–MacLane spectra $H(\mathcal{V}_*) \to H(\mathcal{A}(\mathcal{V}_*))$ under the Dold–Kan correspondence. Composition then yields a map

$$\phi \colon \operatorname{sing}(MU) \to H(\mathcal{A}^*(\mathcal{V}_*))$$

which we also denote by ϕ. For a given integer p, let $\phi_p = (2\pi i)^p \cdot \phi$. Then $MU_{\mathcal{D}}(p)$ is defined to be the homotopy pullback of the diagram of presheaves of spectra

$$\begin{tikzcd} MU_{\mathcal{D}}(p) \arrow[r] \arrow[d] & \operatorname{sing}(MU) \arrow[d, "\phi_p"] \\ H(F^p\mathcal{A}^*(\mathcal{V})) \arrow[r] & H(\mathcal{A}^*(\mathcal{V}_*)). \end{tikzcd}$$

For a complex manifold X, the nth Hodge filtered cobordism group of X is defined as the group of homotopy classes of maps of presheaves of spectra

$$MU^n_{\mathcal{D}}(p)(X) = \operatorname{Hom}_{\mathrm{hoSp}(\mathbf{sPre}_*)}(\Sigma^\infty(X_+), \Sigma^n MU_{\mathcal{D}}(p)).$$

6.2 Mathai–Quillen Thom forms and the induced fundamental cocycle

The first step is to replace the abstract map ϕ with a concrete functorial assignment of forms to maps into MU. Recall from (2.44) the space of forms

$$\mathcal{A}^*(MU(m,l); \mathcal{V}_*) = \mathcal{A}^*_{rd}(\gamma_{m,l}; \mathcal{V}_*).$$

We consider $\gamma_{m,l}$ with the canonical Hermitian metric and connection, as explained in the beginning of Section 5. Let

$$U_{m,l} \in \mathcal{A}^{2m}(MU(m,l); \mathcal{V}_*)$$

denote the Mathai–Quillen Thom forms of Proposition 2.46 on $\gamma_{m,l}$ with respect to these structures. The bundle maps

$$\overline{i_{m,l}}\colon \gamma_{m,l} \to \gamma_{m,l+1} \quad \text{and} \quad \overline{j_{m,l}}\colon \underline{\mathbb{C}} \oplus \gamma_{m,l} \to \gamma_{m+1,l}$$

of diagram (5.1) are compatible with both the Hermitian metrics, and the connections, by Propositions 5.3 and 5.4. Hence the Mathai–Quillen Thom forms are compatible too. For the statement of the compatibility, let $U_{\mathbb{C}}$ be the Mathai–Quillen Thom form of $\underline{\mathbb{C}}_{pt}$ with the standard Hermitian metric, and the trivial connection d.

Proposition 6.2. *With the above notation, we have* $\overline{i_{m,l}}^* U_{m,l+1} = U_{m,l}$ *and* $\overline{j_{m,l}}^* U_{m+1,l} = U_{\mathbb{C}} \otimes U_{m,l}$. ■

Now we define forms on $MU(m,l)$ which will induce our fundamental cocycle. We set

$$\phi_{m,l} = U_{m,l} \wedge \pi_{m,l}^* K(\nabla_{m,l}) \tag{6.1}$$

where $\pi_{m,l}\colon \gamma_{m,l} \to \mathrm{Gr}_m(\mathbb{C}^{m+l})$ is the projection. Combining Propositions 6.2 and 5.3 we get, using $K(d) = 1 \in \mathcal{A}^0(X;\mathcal{V}_*)$, the following result:

Proposition 6.3. *With the above notation, we have* $\overline{i_{m,l}}^* \phi_{m,l+1} = \phi_{m,l}$ *and* $\overline{j_{m,l}}^* \phi_{m+1,l} = U_{\mathbb{C}} \otimes \phi_{m,l}$. ■

Let $MU \to QMU$ again be the fibrant replacement of MU with QMU being the Ω-spectrum with $QMU_n = \mathrm{colim}_k\, \Omega^k MU_{n+k}$. Let $\mathrm{Map}_*^{\mathrm{sm}}(\Sigma^k X_+, MU(m,l))$ denote the space of pointed maps $\Sigma X_+ \to MU(m,l)$ which are smooth on the preimage of $\gamma_{m,l}$.

Definition 6.4. *We define* $\mathrm{Map}^{\mathrm{sm}}(X, QMU_n)$ *as the set of maps* $g\colon X \to QMU_n$ *such that* $g = A(g_{\mathrm{sm}})$ *for some* $g_{\mathrm{sm}} \in \mathrm{Map}_*^{\mathrm{sm}}(\Sigma^k X_+, MU(m,l))$.

We define a map $\phi_{sm}^{m,l}\colon \mathrm{Map}_*^{\mathrm{sm}}(\Sigma^k X_+, MU(m,l)) \to \mathcal{A}^{2m}(\Sigma^k X_+;\mathcal{V}_*)_{\mathrm{cl}}$ by $g_{\mathrm{sm}} \mapsto g_{\mathrm{sm}}^* \phi_{m,l}$. For $n + k = 2m$, integrating over the suspension coordinates provides a form in $\mathcal{A}^n(X;\mathcal{V}_*)$.

Lemma 6.5. *We have a well defined map*

$$\phi_{\mathrm{sm}}^n\colon \mathrm{Map}^{\mathrm{sm}}(X, QMU_n) \to \mathcal{A}^n(X;\mathcal{V}_*)_{\mathrm{cl}},$$

defined by

$$g = A(g_{\mathrm{sm}}) \mapsto \int_{\Sigma^k X_+/X} g_{\mathrm{sm}}^* \phi_{m,l}$$

Proof. We will use Proposition 6.3 to show that the conditions of Proposition 5.7 are satisfied, so that $\phi_{\mathrm{sm}}^n(g)$ is independent of the choice of g_{sm} with $A(g_{\mathrm{sm}}) = g$. We first note that since $q_{m,l} = \mathrm{Th}(\overline{i_{m,l}})$, it is immediate from Proposition 6.3 that

$$\int_{\Sigma^k X_+/X} g_{\mathrm{sm}}^* \phi_{m,l} = \int_{\Sigma^k X_+/X} (q_{m,l} \circ g_{\mathrm{sm}})^* \phi_{m,l+1}.$$

Next observe that the map $s_{m,l} \circ \Sigma^2 g_{\mathrm{sm}}$ is characterized by restricting to $\overline{j_{m,l}} \circ (\mathrm{id}_{\mathbb{C}} \times g_{\mathrm{sm}}|_{\mathbb{R}^k \times X})$ on $\mathbb{C} \times (\mathbb{R}^k \times X)$. Proposition 6.3 implies

$$\begin{aligned}\left(\left(s_{m,l} \circ \Sigma^2 g_{\mathrm{sm}}\right)|_{\mathbb{C}\times\mathbb{R}^k\times X}\right)^* \phi_{m+1,l} &= (\mathrm{id}_{\mathbb{C}} \times g_{\mathrm{sm}}|_{\mathbb{R}^k \times X})^* \left(\overline{j_{m,l}}^* \phi_{m+1,l}\right) \\ &= U_{\mathbb{C}} \otimes g_{\mathrm{sm}}^* \phi_{m,l}.\end{aligned} \tag{6.2}$$

The projection $\pi_E \colon \underline{\mathbb{C}} \oplus E \to E$ is a Hermitian vector bundle. There is an integration along the fiber map

$$\int_{\Sigma^2(\mathrm{Th}(E)/\mathrm{Th}(E)} : \mathcal{A}^*_{rd}(\underline{\mathbb{C}} \oplus E; \mathcal{V}_*) \to \mathcal{A}^{*-2}_{rd}(E; \mathcal{V}_*).$$

By Proposition 2.35, we have for any $\omega \in \mathcal{A}^*_{rd}(E)$

$$\int_{\Sigma^2\mathrm{Th}(E)/\mathrm{Th}(E)} \pi_E^* \omega \wedge U_{\mathbb{C}} = \omega.$$

We note that $U_{\mathbb{C}}$ is of even degree, and therefore is in the center of the ring $\mathcal{A}^*(X) \subset \mathcal{A}^*(X;\mathcal{V}_*)$. Now (6.2) implies

$$\int_{\Sigma^2(\Sigma^k X_+)/\Sigma^k X_+} (s_{m,l} \circ \Sigma^2 g_{\mathrm{sm}})^* \phi_{m+1,l} = g^*_{\mathrm{sm}} \phi_{m,l} \quad \text{in } \mathcal{A}^{2m}(\Sigma^k X_+; \mathcal{V}_*),$$

and hence

$$\int_{\Sigma^{k+2} X_+/X} (s_{m,l} \circ \Sigma^2 g_{\mathrm{sm}})^* \phi_{m+1,l} = \int_{\Sigma^k X_+/X} g^*_{\mathrm{sm}} \phi_{m,l} \quad \text{in } \mathcal{A}^n(X; \mathcal{V}_*). \quad \blacksquare$$

Remark 6.6. In light of Lemma 6.1 we may view ϕ^n_{sm} as corresponding to an element of $\mathcal{A}^n(QMU_n; \mathcal{V}_*)$, as follows. While we have not discussed infinite dimensional manifolds, it is by the Yoneda lemma reasonable to define $\mathcal{A}^n(F; \mathcal{V}_*)$ as the space of natural transformations $F \to \mathcal{A}^n(\mathcal{V}_*)$ for $F \in \mathbf{Pre}(\mathbf{Man})$. This is for example the approach taken in [18], where in particular the deRham complex of the classifying stack of smooth vector bundles with connections is computed. Hence if we identify QMU_n with the simplicial presheaf $X \mapsto \mathrm{Map}^{\mathrm{sm}}(X, QMU_n)$, there is a "form" $\phi_n \in \mathcal{A}^n(QMU_n; \mathcal{V}_*)$ corresponding to ϕ^n_{sm}. In any case, the correct intuition is that the map ϕ^n_{sm} is defined by pullback of a form on QMU_n.

Since both the domain and the codomain of the map ϕ^n_{sm} are defined for every finite-dimensional manifold, we can replace X with $X \times \Delta^k$ for any k. Moreover, the argument of the proof of Lemma 6.5 applies to $X \times \Delta^k$ as well. Hence we draw the following conclusion:

Proposition 6.7. *The maps $\phi^{m,l}_{\mathrm{sm}}$ induce maps of simplicial sets*

$$\phi^n_{\mathrm{sm}} \colon \mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU_n) \to \mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)$$

which fit into commutative diagrams of the form

$$\begin{tikzcd} \mathrm{Map}^{\mathrm{sm}}_*(\Sigma(X \times \Delta^\bullet)_+, QMU_{n+1}) \arrow[r, "\phi^{n+1}_{\mathrm{sm}}"] \arrow[d] & \mathcal{A}^{n+1}(\Sigma(X \times \Delta^\bullet)_+; \mathcal{V}_*)_{\mathrm{cl}} \arrow[d] \\ \mathrm{Map}^{\mathrm{sm}}_*((X \times \Delta^\bullet)_+, QMU_n) \arrow[r, "\phi^n_{\mathrm{sm}}"] & \mathcal{A}^n((X \times \Delta^\bullet)_+; \mathcal{V}_*)_{\mathrm{cl}} \end{tikzcd}$$

where the right-hand vertical map is given by integration over the fiber. ■

From the first assertion of the proposition, we conclude that we have the following diagram of simplicial sets

$$\begin{tikzcd} & \mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU_n) \arrow[d, "\phi^n_{\mathrm{sm}}"] \\ F^p\mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}} \arrow[r] & \mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}. \end{tikzcd} \tag{6.3}$$

Definition 6.8. *Let X be a complex manifold and n, p be integers. Let $MU_{\mathrm{hs}}(p)_n(X)$ be the homotopy pullback of diagram (6.3). We set*

$$MU^n_{\mathrm{hs}}(p)(X) := \pi_0(MU_{\mathrm{hs}}(p)_n(X)).$$

We denote by ι^1_t the map

$$\iota^1_t \colon X \times \Delta^0 \times \Delta^1 \hookrightarrow X \times \Delta^1 \times \Delta^1$$

with image $X \times \{t\} \times \Delta^1$, and by ι^2_s the map

$$\iota^2_s \colon X \times \Delta^1 \times \Delta^0 \hookrightarrow X \times \Delta^1 \times \Delta^1$$

with image $X \times \Delta^1 \times \{s\}$. Since (6.3) is a diagram of Kan complexes, the set $\pi_0(MU_{\mathrm{hs}}(p)_n(X))$ has the following concrete description:

Theorem 6.9. *Let X be a complex manifold. An element in $MU^n_{\mathrm{hs}}(p)(X)$ is represented by a triple*

$$(g, \omega, h) \in \mathrm{Map}^{\mathrm{sm}}(X, QMU_n) \times F^p\mathcal{A}^n(X; \mathcal{V}_*)_{\mathrm{cl}} \times \mathcal{A}^n(X \times \Delta^1; \mathcal{V}_*)_{\mathrm{cl}},$$

such that $\iota_1^* h = \phi^n_{\mathrm{sm}}(g)$ and $\iota_0^* h = \omega$. Two such triples (g_0, ω_0, h_0) and (g_1, ω_1, h_1) are homotopic if there is a triple $(g_\bullet, \omega_\bullet, h_\bullet)$ in

$$\mathrm{Map}^{\mathrm{sm}}(X \times \Delta^1, QMU_n) \times F^p\mathcal{A}^n(X \times \Delta^1; \mathcal{V}_*)_{\mathrm{cl}} \times \mathcal{A}^n(X \times \Delta^1 \times \Delta^1; \mathcal{V}_*)_{\mathrm{cl}}$$

which satisfies $(\iota^2_1)^* h_\bullet = \phi^n_{\mathrm{sm}}(g_\bullet)$ and $(\iota^2_0)^* h_\bullet = \omega_\bullet$ in $\mathcal{A}^n(X \times \Delta^1; \mathcal{V}_*)$, and such that $\iota_i^*(g_\bullet, \omega_\bullet, h_\bullet) = (g_i, \omega_i, h_i)$ for $i = 0, 1$. The latter means, in particular, $(\iota^1_0)^* h_\bullet = h_0$ and $(\iota^1_1)^* h_\bullet = h_1$.

6.3 Comparison of homotopy models for $MU(p)$

Now we show that there is a natural isomorphism $MU^n_{\mathcal{D}}(p)(X) \cong MU^n_{\mathrm{hs}}(p)(X)$. In fact, we will show that there is a zig-zag of weak equivalence between the defining homotopy pullbacks.

First we observe that $\mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU)$ is a simplicial spectrum with structure maps defined as follows: Recall that, as described in [31, page 379], a k-simplex of the simplicial loop space $\Omega^{\mathrm{simp}} A_\bullet$ of a simplicial set $A_\bullet$ can be described as a sequence

$$a_0, \ldots, a_k \in A_{k+1}$$

satisfying the conditions

$$\begin{aligned} \partial_i^* a_i &= \partial_i^* a_{i-1} \\ \partial_0^* a_0 &= * = \partial_{k+1}^* a_k \end{aligned} \tag{6.4}$$

The homeomorphism $QMU_n \xrightarrow{\cong} \Omega QMU_{n+1}$ induces for each n a natural isomorphism $\mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU_n) \xrightarrow{\cong} \mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, \Omega QMU_{n+1})$. The adjunction between the suspension and loop space functors then induces a natural isomorphism $\mathrm{Map}_*^{\mathrm{sm}}(\Sigma(X \times \Delta^\bullet)_+, QMU_{n+1}) \xrightarrow{\cong} \mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU_{n+1})$. A pointed map $\Sigma(X \times \Delta^\bullet)_+ \to QMU_{n+1}$ corresponds to a map $X \times \Delta^\bullet \times \Delta^1 \to QMU_{n+1}$ which restricts to the constant map on both subspaces $X \times \Delta^\bullet \times \{0\}$ and $X \times \Delta^\bullet \times \{1\}$ with value the canonical basepoint of QMU_{n+1}. The restriction of any such map to the $(k+1)$-simplices in the standard triangulation of $\Delta^k \times \Delta^1$ leads to a sequence of maps

$$g_0, \ldots, g_k \colon X \times \Delta^{k+1} \to QMU_{n+1},$$

i.e., a sequence of $(k+1)$-simplices in $\mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU_{n+1})$ satisfying the relations corresponding to (6.4). This defines a natural map of simplicial sets

$$\mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU_n) \to \Omega^{\mathrm{simp}} \mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU_{n+1})$$

which provides the sequence $n \mapsto \mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU_n)$ with the structure of a sequential spectrum of simplicial sets.

Next we show that the maps ϕ^n_{sm} induce a natural map of simplicial spectra

$$\mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU) \xrightarrow{\phi_{\mathrm{sm}}} \mathcal{A}^*_\Sigma(X, \mathcal{V}_*).$$

Since the maps ϕ^n_{sm} are natural in $X \times \Delta^\bullet$, a sequence $g_0, \ldots, g_k$ of $(k+1)$-simplices in $\mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU_{n+1})$ satisfying the relations corresponding to (6.4) induces a sequence $\phi^{n+1}_{\mathrm{sm}}(g_0), \ldots, \phi^{n+1}_{\mathrm{sm}}(g_k)$ of $(k+1)$-simplices in $\mathcal{A}^{n+1}(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}$ satisfying similar relations. Thus ϕ^n_{sm} induces a natural map on the simplicial loop spaces as well. In fact, we get a commutative diagram of the form

$$\begin{tikzcd} \mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU_n) \arrow[r, "\phi^n_{\mathrm{sm}}"] \arrow[d] & \mathcal{A}^n(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}} \arrow[d] \\ \Omega^{\mathrm{simp}} \mathrm{Map}^{\mathrm{sm}}(X \times \Delta^\bullet, QMU_{n+1}) \arrow[r, "\phi^{n+1}_{\mathrm{sm}}"] & \Omega^{\mathrm{simp}} \mathcal{A}^{n+1}(X \times \Delta^\bullet; \mathcal{V}_*)_{\mathrm{cl}}, \end{tikzcd}$$

since both vertical maps arise from the restriction to the standard triangulation of $\Delta^k \times \Delta^1$.

Now we construct the comparison map. The map $MU \to QMU$ induces a map $\mathrm{Sing}\,(MU) \to \mathrm{Sing}\,(QMU)$. We consider the map $\mathrm{sing}(MU) \to \mathrm{Sing}\,(QMU)$ given by precomposing with the isomorphism $\mathrm{sing}(MU) \cong \mathrm{Sing}\,(MU)(pt)$. It follows from [30, Proposition 3.11] that this map induces an isomorphism on stalks and so is a weak equivalence. Let $\mathrm{Map}^{\mathrm{sm}}(- \times \Delta^\bullet, QMU) \to \mathrm{Sing}\,(QMU)$ be the map of presheaves of spectra induced by forgetting smoothness. This is an objectwise weak equivalence, since every continuous map is homotopic to a smooth map. We note also that this is a map between fibrant objects by [30, Lemma 3.12]. Let $H(\mathcal{A}^*(\mathcal{V}_*)) \to \mathcal{A}^*_\Sigma(\mathcal{V}_*)$ be the map of presheaves of spectra which for the nth spaces is given by the map τ defined in (2.21). Its homotopy inverse is induced by integrating over the fiber. Similarly, let $H(F^p\mathcal{A}^*(\mathcal{V}_*)) \to F^p\mathcal{A}^*_\Sigma(\mathcal{V}_*)$ be the map of presheaves of spectra induced by the restriction of τ. Again, this map has a homotopy inverse is induced by integrating over the fiber. In total, we have a diagram of presheaves of spectra on $\mathbf{Man}_\mathbb{C}$

$$\begin{tikzcd}
H(F^p\mathcal{A}^*(\mathcal{V}_*)) \arrow[r] \arrow[dd, bend left, "\simeq"] & H(\mathcal{A}^*(\mathcal{V}_*)) \arrow[dd, bend left, "\simeq"] & \mathrm{sing}(MU) \arrow[l, "\phi"'] \arrow[d, "\simeq"] \\
 & & \mathrm{Sing}\,(QMU) \\
F^p\mathcal{A}^*_\Sigma(\mathcal{V}_*) \arrow[r] \arrow[uu, bend left, "\simeq"] & \mathcal{A}^*_\Sigma(\mathcal{V}_*) \arrow[uu, bend left, "\simeq"] & \mathrm{Map}^{\mathrm{sm}}(- \times \Delta^\bullet, QMU). \arrow[l, "\phi_{\mathrm{sm}}"'] \arrow[u, "\simeq"']
\end{tikzcd} \tag{6.5}$$

We write $MU_{\mathrm{hs}}(p)$ for the homotopy pullback of the bottom row of (6.5). Recall that the homotopy pullback of the top row is $MU_\mathcal{D}(p)$ by definition.

Proposition 6.10. *The homotopy pullbacks $MU_\mathcal{D}(p)$ and $MU_{\mathrm{hs}}(p)$ are isomorphic in the homotopy category* $\mathrm{hoSp}(\mathbf{sPre}_*)$ *of presheaves of spectra on* $\mathbf{Man}_\mathbb{C}$.

Proof. The assertion is a formal consequence of the observation on homotopy pullbacks in model categories formulated in the following lemma. ■

Lemma 6.11. *Let $\mathcal{C}$ be a proper simplicial model category. We consider the diagram*

$$\begin{tikzcd}
C_1 \arrow[r] \arrow[dd, bend left, "\simeq"] & B_1 \arrow[dd, bend left, "\simeq"] & A_1 \arrow[l] \arrow[dr, "\simeq"] & \\
 & & & A_0 \\
C_2 \arrow[r] \arrow[uu, bend left, "\simeq"] & B_2 \arrow[uu, bend left, "\simeq"] & A_2 \arrow[l] \arrow[ur, "\simeq"'] &
\end{tikzcd} \tag{6.6}$$

in which all vertical maps are weak equivalences and in which the left-hand squares commute. We also assume that A_0 and A_2 are fibrant. Then the homotopy pullbacks $C_1 \times^h_{B_1} A_1$ and $C_2 \times^h_{B_2} A_2$ of the top and bottom row, respectively, are weakly equivalent.

Proof. First we take the homotopy pullback $A_1 \times^h_{A_0} A_2$ of the right-hand vertical maps. The two induced maps $A_1 \times^h_{A_0} A_2 \to A_1$ and $A_1 \times^h_{A_0} A_2 \to A_2$ are weak equivalences. Hence the induced maps on homotopy pullback

$$C_1 \times^h_{B_1} (A_1 \times^h_{A_0} A_2) \to C_1 \times^h_{B_1} A_1 \text{ and } C_2 \times^h_{B_2} (A_1 \times^h_{A_0} A_2) \to C_2 \times^h_{B_2} A_2$$

are weak equivalences. Thus it remains to observe that the homotopy pullbacks $C_1 \times^h_{B_1} (A_1 \times^h_{A_0} A_2)$ and $C_2 \times^h_{B_2} (A_1 \times^h_{A_0} A_2)$ are weakly equivalent. This follows from the following diagram

$$\begin{tikzcd}
C_1 \arrow[r] \arrow[d, bend left, "\simeq"] & B_1 \arrow[d, bend left, "\simeq"] & A_1 \times^h_{A_0} A_2 \arrow[l] \arrow[d, equal] \\
C_2 \arrow[r] \arrow[u, bend left, "\simeq"] & B_2 \arrow[u, bend left, "\simeq"] & A_1 \times^h_{A_0} A_2 \arrow[l]
\end{tikzcd}$$

in which the right-hand square commutes up to homotopy and the vertical maps are weak equivalences. ∎

As a consequence of Proposition 6.10 we get that both homotopy pullbacks represent isomorphic cohomology groups. This implies the following result:

Theorem 6.12. *Let X be a complex manifold and n, p be integers. Then there is a natural isomorphism*

$$MU^n_{\mathcal{D}}(p)(X) \xrightarrow{\cong} MU^n_{\mathrm{hs}}(p)(X).$$

Proof. By Proposition 6.10, it remains to relate the groups $MU^n_{\mathrm{hs}}(p)(X)$ of Definition 6.8 to $MU_{\mathrm{hs}}(p)$-cohomology. Each of the presheaves of spectra in the bottom row of diagram (6.5) satisfies levelwise hypercover descent and the structure maps are objectwise weak equivalences. Hence each of the presheaves of spectra in the bottom row is an Ω-spectrum. Hence the nth space $MU_{\mathrm{hs}}(p)_n$ of $MU_{\mathrm{hs}}(p)$ represents $MU_{\mathrm{hs}}(p)$-cohomology, i.e., there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{hoSp}(\mathbf{sPre}_*)}(\Sigma^\infty(X_+), \Sigma^n MU_{\mathrm{hs}}(p)) \cong \mathrm{Hom}_{\mathrm{ho}\mathbf{sPre}}(X, MU_{\mathrm{hs}}(p)_n).$$

Moreover, we can compute the simplicial presheaf $MU_{\mathcal{D}}(p)_n$ levelwise as the homotopy pullback of the nth spaces of the presheaves of spectra in the bottom row of (6.5). By [46, Proposition 2.7], we can compute this homotopy pullback objectwise in the sense that there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{ho}\mathbf{sPre}}(X, MU_{\mathrm{hs}}(p)_n) \cong \pi_0(MU_{\mathrm{hs}}(p)_n(X)).$$

Since we have $MU^n_{\mathrm{hs}}(p)(X) = \pi_0(MU_{\mathrm{hs}}(p)_n(X))$ by definition, this proves the assertion of the theorem. ∎

6.4 The map κ from homotopy to geometry

We are going to define a natural isomorphism of groups

$$\kappa\colon MU^n_{\mathrm{hs}}(p)(X) \to MU^n(p)(X)$$

for every n and p. In order to construct geometric cycles from the data of classes in $MU^n_{\mathrm{hs}}(p)(X)$ we need suitable geometric representatives. Recall from Definition 5.8 the space $\mathrm{Map}^{\pitchfork}(X, QMU_n)$.

Theorem 6.13. *Let X be a complex manifold and n, p be integers. For every element $\gamma \in MU^n_{\mathrm{hs}}(p)(X)$, there is a representative (g, ω, h) as in Theorem 6.9 such that $g \in \mathrm{Map}^{\pitchfork}(X, QMU_n)$.*

To prove the theorem we are going to use the following construction:

Lemma 6.14. *Let $g_\bullet\colon X \times \Delta^1 \to QMU_n$ be a homotopy between $g_0 = \iota_0^* g_\bullet$ and $g_1 = \iota_1^* g_\bullet$. Assume we have a triple (g_0, ω, h_0) which represents an element in $MU^n_{\mathrm{hs}}(p)(X)$. Then there is a form $h_1 \in \mathcal{A}^n(X \times \Delta^1; \mathcal{V}_*)$ such that the triple (g_1, ω, h_1) is homotopic to (g_0, ω, h_0).*

Proof. The pullback along the projection $\pi_X\colon X \times \Delta^1 \to X$ of $\phi_{\mathrm{sm}}(g_0)$ yields a closed $\pi_X^* \phi^n_{\mathrm{sm}}(g_0) \in \mathcal{A}^n(X \times \Delta^1; \mathcal{V}_*)_{\mathrm{cl}}$ which is constant on Δ^1. We set $h_1 := h_0 + \phi_{\mathrm{sm}}(g_\bullet) - \pi_X^* \phi_{\mathrm{sm}}(g_0)$. The restrictions along $\iota_t\colon X \times \Delta^0 \hookrightarrow X \times \Delta^1$ yield

$$\begin{aligned}
\iota_1^* h_1 &= \iota_1^* h_0 + \iota_1^* \phi_{\mathrm{sm}}(g_\bullet) - \iota_1^* \pi_X^* \phi_{\mathrm{sm}}(g_0) \\
&= \phi_{\mathrm{sm}}(g_0) + \phi_X^0(g_1) - \phi_{\mathrm{sm}}(g_0) \\
&= \phi_{\mathrm{sm}}(g_1)
\end{aligned}$$

and

$$\begin{aligned}
\iota_0^* h_1 &= \iota_0^* h_0 + \iota_0^* \phi_{\mathrm{sm}}(g_\bullet) - \iota_0^* \pi_X^* \phi_{\mathrm{sm}}(g_0) \\
&= \omega + \phi_{\mathrm{sm}}(g_0) - \phi_{\mathrm{sm}}(g_0) \\
&= \omega.
\end{aligned}$$

Hence the triple (g_1, ω, h_1) represents an element in $MU^n_{\mathrm{hs}}(p)(X)$.

Now we construct a homotopy between (g_0, ω, h_0) and (g_1, ω, h_1). The homotopies $g_\bullet$ and $\omega_\bullet := \pi_X^* \omega$ satisfy the requirements of Theorem 6.9. It remains to find a compatible homotopy $h_\bullet$. To construct $h_\bullet$ we consider the map

$$G_\bullet\colon X \times \Delta^1 \times \Delta^1 \to QMU_n$$

defined by $G_s(x,t) = g_{st}(x)$. We think of $G_\bullet$ as a homotopy between the maps $G_0\colon (x,t) \mapsto g_0(x)$ and $G_1\colon (x,t) \mapsto g_t(x)$. We set

$$h_\bullet := \pi^*_{X \times \Delta^1}(h_0 - \pi_X^* \phi_{\mathrm{sm}}(g_0)) + \phi_{\mathrm{sm}}(G_\bullet) \in \mathcal{A}^n(X \times \Delta^1 \times \Delta^1; \mathcal{V}_*)_{\mathrm{cl}}.$$

Next we compute the pullbacks along the various inclusions into the copies of Δ^1. The restriction to 1 on the right most factor in $X \times \Delta^1 \times \Delta^1$ yields

$$\begin{aligned}(\iota_1^2)^* h_\bullet &= (\iota_1^2)^* \pi^*_{X\times\Delta^1}(h_0 - \pi_X^* \phi_{\mathrm{sm}}(g_0)) + (\iota_1^2)^* \phi_{\mathrm{sm}}(G_\bullet)\\ &= h_0 - \pi_X^* \phi_{\mathrm{sm}}(g_0) + \phi_{\mathrm{sm}}(g_\bullet)\\ &= h_1.\end{aligned}$$

The restriction to 0 yields

$$\begin{aligned}(\iota_0^2)^* h_\bullet &= (\iota_0^2)^* \pi^*_{X\times\Delta^1}(h_0 - \pi_X^* \phi_{\mathrm{sm}}(g_0)) + (\iota_0^2)^* \phi_{\mathrm{sm}}(G_\bullet)\\ &= h_0 - \pi_X^* \phi_{\mathrm{sm}}(g_0) + \phi_{\mathrm{sm}}(g_{0\cdot\bullet})\\ &= h_0 - \pi_X^* \phi_{\mathrm{sm}}(g_0) + \pi_X^* \phi_{\mathrm{sm}}(g_0)\\ &= h_0.\end{aligned}$$

The restriction to 1 on the middle factor in $X \times \Delta^1 \times \Delta^1$ yields

$$\begin{aligned}(\iota_1^1)^* h_\bullet &= (\iota_1^1)^* \pi^*_{X\times\Delta^1}(h_0 - \pi_X^* \phi_{\mathrm{sm}}(g_0)) + (\iota_1^1)^* \phi_{\mathrm{sm}}(G_\bullet)\\ &= \iota_1^*(h_0 - \pi_X^* \phi_{\mathrm{sm}}(g_0)) + \phi_{\mathrm{sm}}(g_\bullet)\\ &= \phi_{\mathrm{sm}}(g_0) - \phi_{\mathrm{sm}}(g_0) + \phi_{\mathrm{sm}}(g_\bullet)\\ &= \phi_{\mathrm{sm}}(g_\bullet).\end{aligned}$$

The restriction to 0 yields

$$\begin{aligned}(\iota_0^1)^* h_\bullet &= (\iota_0^1)^* \pi^*_{X\times\Delta^1}(h_0 - \pi_X^* \phi_{\mathrm{sm}}(g_0)) + (\iota_0^1)^* \phi_{\mathrm{sm}}(G_\bullet)\\ &= \iota_0^*(h_0 - \pi_X^* \phi_{\mathrm{sm}}(g_0)) + \phi_{\mathrm{sm}}(g_{0\cdot\bullet})\\ &= \omega - \phi_{\mathrm{sm}}(g_0) + \phi_{\mathrm{sm}}(g_0)\\ &= \omega.\end{aligned}$$

Thus the triple $(g_\bullet, \pi_X^*\omega, h_\bullet)$ is a homotopy between (g_0, ω, h_0) and (g_1, ω, h_1). ■

Now we can prove Theorem 6.13:

Proof. Let $\gamma \in MU^n_{\mathcal{D}}(p)(X)$ and (g, ω, h) be a representative as in Theorem 6.9. By Proposition 5.9, there is a map $g_\pitchfork \in \mathrm{Map}^\pitchfork(X, QMU_n)$ which is homotopic to g. By Lemma 6.14, there exists an h_1 such that $(g_\pitchfork, \omega, h_1)$ represents an element in $MU^n_{\mathrm{hs}}(p)(X)$ and a homotopy between (g, ω, h) and $(g_\pitchfork, \omega, h_1)$. Hence the latter too represents the class γ in $MU^n_{\mathrm{hs}}(p)(X)$. ■

Definition 6.15. *We denote by* $MU^\pitchfork_{\mathrm{hs}}(p)_n(X)_0$ *the subset of triples* (g, ω, h) *in* $MU_{\mathrm{hs}}(p)_n(X)_0$ *such that* $g \in \mathrm{Map}^\pitchfork(X, QMU_n)$.

We recall that we write $\iota_{m,l} \colon \mathrm{Gr}_m\left(\mathbb{C}^{m+l}\right) \to \gamma_{m,l}$ for the 0-section, and that the fundamental currents, $\phi_{\nabla_{m,l}}$ of Definition 5.19, are defined by

$$\phi_{\nabla_{m,l}} = (\iota_{m,l})_* K(\nabla_{m,l}) \in \mathcal{D}^{2m}(MU(m,l); \mathcal{V}_*).$$

Let $\alpha'_{m,l}$ be the current of Proposition 2.48 satisfying $d\alpha'_{m,l} = U_{m,l} - (\iota_{m,l})_* 1$. We define $\alpha_{m,l} = \alpha'_{m,l} \wedge \pi^*_{m,l} K(\nabla_{m,l})$. Then we have

$$d\alpha_{m,l} = \phi_{m,l} - \phi_{\nabla_{m,l}}.$$

Remark 6.16. We show in Section 6.7 below that it is possible to circumvent Proposition 2.48. Essentially, we may simply work with the class modulo $\mathrm{Im}\,(d)$ of $\alpha_{m,l}$, which we show depends only on $\phi_{m,l}$ and $\phi_{\nabla_{m,l}}$.

Let $\alpha_{\mathbb{C}}$ be the current on $\mathbb{C}$ obtained by applying Proposition 2.48 to $\mathbb{C} \to \mathrm{pt}$ with the standard metric and connection. As a consequence of the naturality of the construction of α, by Proposition 2.48, the Propositions 5.3 and 5.4 imply:

Proposition 6.17. *The currents $\alpha_{m,l}$ are compatible in the sense that, using the maps of* (5.1)*, we have*

$$\overline{j_{m,l}}^* \alpha_{m+1,l} = \alpha_{\mathbb{C}} \otimes \alpha_{m,l}, \quad \textit{and} \quad \overline{i_{m,l}}^* \alpha_{m,l+1} = \alpha_{m,l}.$$

We use this observation to make the following definition:

Definition 6.18. *We define*

$$\begin{aligned} pb^\alpha \colon \mathrm{Map}^{\pitchfork}(X, QMU_n) &\to \mathcal{D}^n(X; \mathcal{V}_*) \\ pb^\alpha(g) &= (-1)^n \pi_*({g_\pitchfork}^*(\alpha_{m,l})) \end{aligned}$$

where $g_\pitchfork \colon \Sigma^k X_+ \to MU(m,l)$ is any map satisfying $A(g_\pitchfork) = g$, and $\pi \colon \mathbb{R}^k \times X \to X$ is the projection.

Remark 6.19. By Propositions 6.17 and 5.7, the choice of $g_\pitchfork$ does not matter. Hence pb^α is a well-defined map.

Remark 6.20. We should say something about the sign in Definition 6.18. One reason it is here is that it is needed in Lemma 6.24 below, because of the sign in Proposition 2.13. We can also get at this in a more principled manner: In defining ϕ^n_{sm} in Lemma 6.5, we used integration along the fiber. For pb^α we use pushforward of currents. We need pb^α and ϕ_{sm} to be compatible. Therefore we recall from Definition 2.34 that integration along the fiber and pushforward of currents are related by

$$T_{\int_{X\times\mathbb{R}^k/X}\omega} = \mathbf{w}' \left(\pi_X^{\Sigma X_+}\right)_* \mathbf{w}' T_\omega$$

where T_ω is the current $\sigma \mapsto \int_X \omega \wedge \sigma$, and $\mathbf{w}'$ is the sign operator acting on $T \in \mathcal{D}^k(X)$ by $\mathbf{w}'T = (-1)^{k+k\dim X}$. Hence we must consider the map $\omega \mapsto \mathbf{w}'T_\omega$. We observe that $\mathbf{w}'T_\omega =: T'_\omega$ is the current acting by

$$T'_\omega(\sigma) = \int_X \sigma \wedge \omega. \tag{6.7}$$

Until this point, we have always treated forms as currents by $\omega \mapsto T_\omega$. We are now forced to consider T' in dealing with ϕ_{sm}, but also T, in dealing with geometric Hodge filtered cobordism cycles.

While T' is natural for pushforwards along proper submersions, T is natural for pullbacks. That is, for $f\colon Y \to X$ and $\omega \in \mathcal{A}^n(X)$ we have $T_{f^*\omega} = f^*T_\omega$. The map T' on the other hand satisfies

$$f^*T'_\omega = (-1)^{nd}T'_{f^*\omega}, \tag{6.8}$$

for $d = \operatorname{codim} f$. We make one more observation: We have $T'_\omega = (-1)^{n+n\cdot\dim X}T_\omega$, and $T'_{d\omega} = (-1)^{n+1+(n+1)\cdot\dim X}T_{d\omega}$, which together with $dT_\omega = T_{d\omega}$ yields

$$dT'_\omega = (-1)^{1+\dim X}T'_{d\omega}.$$

That is, d commutes with T' on odd dimensional manifolds, and anticommutes with T' on even dimensional manifolds.

We remark furthermore that for $\phi_{m,l} \in \mathcal{A}^{2m}_{rd}(\gamma_{m,l};\mathcal{V}_*)$ we get $T_{\phi_{m,l}} = T'_{\phi_{m,l}}$. Hence we have

$$d\alpha_{m,l} = T_{\phi_{m,l}} - \phi_{\nabla_{m,l}} = T'_{\phi_{m,l}} - \phi_{\nabla_{m,l}}.$$

However for the map $\phi^n_{\mathrm{sm}}\colon \mathrm{Map}^{\mathrm{sm}}(X, QMU_n) \to \mathcal{A}^n(X;\mathcal{V}_*)$ of Lemma 6.5 we get

$$T'_{\phi^n_{\mathrm{sm}}(g)} = (-1)^n T_{\phi^n_{\mathrm{sm}}(g)}. \qquad \blacksquare$$

We now define the comparison map κ. For the following definition recall the definition of the group $ZMU^n(p)(X)$ of Hodge filtered cobordism cycles from definition 4.27.

Definition 6.21. *We define the map* $\kappa\colon MU^{\pitchfork}_{\mathrm{hs}}(p)_n(X)_0 \to ZMU^n(p)(X)$ *by*

$$(g,\omega,h) \mapsto \left(\rho_\nabla(g),\ T'_\omega,\ pb^\alpha(g) + \left(\pi^{X\times\Delta^1}_X\right)_* T'_h\right).$$

Remark 6.22. We remark that both ρ_∇ and pb^α are defined on $\mathrm{Map}^{\pitchfork}(X, QMU_n)$, not on $\mathrm{Map}^{\pitchfork}_*(\Sigma^k X_+, MU(m,l))$. In particular, they depend only on $g = A(g_\pitchfork)$, not $g_\pitchfork$.

Remark 6.23. It is annoying that we are forced to work with a sign here, though we hide it in $T'_\omega = (-1)^n T_\omega$. This is reminiscent of the sign showing up for Deligne cohomology, in the definition of I in diagram (3.1). We could have put the sign on ρ instead, but there seem to be no way to avoid the sign entirely. We accept our faith and move on.

We now show that κ actually takes values in $ZMU^n(p)(X)$. For the proof, we need some context. Let (g,ω,h) be a triple in $MU^{\pitchfork}_{\mathrm{hs}}(p)_n(X)_0$. Recalling Definition 5.12, we put

$$\widetilde{f}_g = (f_g, N_g, \nabla_g) = \rho_\nabla(g).$$

Let $g_{\pitchfork} \in \mathrm{Map}_*^{\pitchfork}\left(\Sigma^k X_+, MU(m,l)\right)$ be a map with $A(g_{\pitchfork}) = g$. Recall from the discussion leading to Theorem 5.21 that we have the commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "g_{\pitchfork}|_Z"] \arrow[dd, "i"'] & \mathrm{Gr}_m\left(\mathbb{C}^{m+l}\right) \arrow[d, "\iota_{m,l}"'] & \\
 & \gamma_{m,l} \arrow[d] & \\
U_{g_{\pitchfork}} \arrow[ur, "g_{\pitchfork}|_U"] \arrow[r] \arrow[d, "\pi"'] & MU(m,l) \arrow[r] & QMU_{2m} \\
X & &
\end{tikzcd} \tag{6.9}$$

where $Z = Z_{g_{\pitchfork}} = (g_{\pitchfork})^{-1}(\mathrm{Gr}_m(\mathbb{C}^{m+l}))$, $U_{g_{\pitchfork}} = (g_{\pitchfork})^{-1}(\gamma_{m,l}) \subset \mathbb{R}^k \times X \subset \Sigma^k X_+$ and π is the restriction to $U_{g_{\pitchfork}}$ of the projection $R^k \times X \to \mathbb{R}^k$. By definition of ρ_∇, we have $f_g = \pi \circ i$. With this context, we can prove:

Lemma 6.24. *For*

$$(g, \omega, h) \in MU_{\mathcal{D}}^{\pitchfork}(p)_n(X)_0,$$

we have $\kappa(g, \omega, h) \in ZMU^n(p)(X)$.

Proof. The equation $\pi_* {g_{\pitchfork}}^* \phi_{\nabla_{m,l}} = (f_g)_* K(\nabla_g)$ of Theorem 5.21, together with $d\pi_* = (-1)^{\operatorname{codim} \pi} \pi_* d$ from Proposition 2.13 and the equalities $\operatorname{codim} \pi = k$ and $k + n = 2m$ implies

$$\begin{aligned}
(f_g)_* K(\nabla_g) + d(-1)^n \pi_* {g_{\pitchfork}}^* \alpha_{m,l} &= \pi_* {g_{\pitchfork}}^* (\phi_{\nabla_{m,l}} + d\alpha_{m,l}) \\
&= \pi_* {g_{\pitchfork}}^* (T'_{\phi_{m,l}}) \\
&= T'_{\phi^n_{\mathrm{sm}}(g)}.
\end{aligned}$$

Now we look at the contribution of $d\left(\pi_X^{X \times \Delta^1}\right)_* h$. Since we can assume that the triple (g, ω, h) has the form described in Theorem 6.9, we have by Proposition 2.35

$$d\left(\int_{X \times \Delta^1/X} h\right) = (\iota_1^* h - \iota_0^* h) = \phi^n_{sm}(g) - \omega.$$

Applying T', and the equalities $T'_{d\eta} = -dT'_\eta$ and $T'_{\left(\int_{X \times \Delta^1/X} h\right)} = \left(\pi_X^{X \times \Delta^1}\right)_* T'_h$, both of which holds by Remark 6.20, we can now conclude that

$$(f_g)_* K(\nabla_g) + d(-1)^n \pi_* {g_{\pitchfork}}^* \alpha_{m,l} + d\left(\pi_X^{X \times \Delta^1}\right)_* T'_h = T'_\omega. \tag{6.10}$$

Since $T'_\omega = \mathbf{w}' T_\omega$ is a smooth current in $F^p \mathcal{D}^n(X; \mathcal{V}_*)$, this shows that the image of κ is indeed contained in $ZMU^n(p)(X)$. ■

We will now show that homotopic triples yield cobordant Hodge filtered cycles. We continue to work within the context of diagram (6.9).

Lemma 6.25. *Let (g_0, ω_0, h_0) and (g_1, ω_1, h_1) be triples in $MU^{\pitchfork}_{\mathrm{hs}}(p)_0(X)$ and assume there is a homotopy $(g_\bullet, \omega_\bullet, h_\bullet)$ between them. Then*

$$\kappa(g_1, \omega_1, h_1) - \kappa(g_0, \omega_0, h_0) \in BMU^n(p)(X)$$

where $BMU^n(p)(X)$ is the group of bordism data defined in (4.14).

Proof. Say $g_0 = A(g_0^{\pitchfork})$ and $g_1 = A(g_1^{\pitchfork})$ with $g_i^{\pitchfork} \in \mathrm{Map}_*^{\pitchfork}(\Sigma^k X_+, MU(m,l))$. Since there is a homotopy $g_\bullet$ between g_0 and g_1, there is by Lemma 5.5 also a pointed homotopy $g_\bullet^{\pitchfork}$ between $g_0^{\pitchfork}$ and $g_1^{\pitchfork}$, if m and l are large enough. Using the technique of Lemma 4.8 we can take $g_\bullet^{\pitchfork}$ to be a pointed homotopy $\Sigma^k X_+ \times \mathbb{R} \to MU(m,l)$ which is smooth on the preimage of $\gamma_{m,l}$, such that for some $\epsilon > 0$ we have

$$g_t^{\pitchfork}(v,x) = g_0^{\pitchfork}(v,x) \ \text{ if } t < \epsilon, \qquad \text{and} \qquad g_t^{\pitchfork}(v,x) = g_1^{\pitchfork}(v,x) \text{ if } t > 1-\epsilon. \quad (6.11)$$

Since $g_\bullet^{\pitchfork}$ is a pointed homotopy, we can consider it as a map $\Sigma\Sigma^k X_+ \to MU(m,l)$. By Thom's transversality theorem, since $g_0^{\pitchfork}$ and $g_1^{\pitchfork}$ are transverse to $\mathrm{Gr}_m(\mathbb{C}^{m+l})$, we can perturb $g_\bullet^{\pitchfork}$ slightly to achieve $g_\bullet^{\pitchfork} \in \mathrm{Map}_*^{\pitchfork}(\Sigma\Sigma^k X_+, MU(m,l))$. Put

$$g'_\bullet = A(g_\bullet^{\pitchfork}) \in \mathrm{Map}^{\pitchfork}(\mathbb{R} \times X, QMU_n).$$

Then we let

$$\widetilde{b} = (b, N_b, \nabla_b) := \rho_\nabla(g'_\bullet) \in \widetilde{ZMU}^n(\mathbb{R} \times X),$$

with $b = (a_b, f_b) \colon W \to \mathbb{R} \times X$. This is a geometric bordism datum over X. Let $i_t \colon X \to \mathbb{R} \times X$ be the map $i_t(x) = (t,x)$. Then we have for $t = 0, 1$

$$i_t^* \widetilde{b} = \rho_\nabla(g_t)$$

since $g'_\bullet \circ i_t = g_t$, and ρ_∇ is natural by Theorem 5.13. Recalling the definition of $\psi(\widetilde{b})$ from Definition 4.30, it follows from (6.11) that

$$(\partial \widetilde{b}, \psi(\widetilde{b})) = (i_1^* \widetilde{b} - i_0^* \widetilde{b},\ (-1)^{k+1} (f_b)_* T_{K(\nabla_b)}).$$

We remark that (6.11) also implies

$$\delta_{\partial X \times \Delta^1} \wedge (g_\bullet^{\pitchfork})^* \alpha_{m,l} = (i_1)_* (g_1^{\pitchfork})^* \alpha_{m,l} - (i_0)_* (g_0^{\pitchfork})^* \alpha_{m,l}$$

and in particular that Proposition 2.13 applies to $\left((g_\bullet^{\pitchfork})^* \alpha_{m,l}\right)|_{\mathbb{R}^k \times X \times \Delta^1}$. Appealing once more to Theorem 5.21, and using $n + 1 + k + 1 = 2m + 2$, we have so far shown:

$$\left(\rho_\nabla(g_1) - \rho_\nabla(g_0),\ (-1)^{n+1} \left(\pi_X^{\mathbb{R}^k \times X \times \Delta^1}\right)_* (g_\bullet^{\pitchfork})^* (\phi_{\nabla_{m,l}})\right) \in BMU^n(p)(X).$$

In order to show that

$$\kappa(g_1, \omega_1, h_1) - \kappa(g_0, \omega_0, h_0) \in BMU^n(p)(X)$$

it remains to show that

$$(-1)^n \pi_* \left((g_1^{\pitchfork})^* \alpha_{m,l} - (g_0^{\pitchfork})^* \alpha_{m,l}\right) \\ + \left(\pi_X^{X\times\Delta^1}\right)_* \left(T'_{h_1} - T'_{h_0} + (-1)^n \left(\pi_{X\times\Delta^1}^{\mathbb{R}^k\times X\times\Delta^1}\right)_* (g_\bullet^{\pitchfork})^* (\phi_{\nabla_{m,l}})\right)$$

belongs to $\widetilde{F}^p\mathcal{A}^{n-1}(X;\mathcal{V}_*)$, as defined in (2.12) by

$$\widetilde{F}^p\mathcal{A}^{n-1}(X;\mathcal{V}_*) := F^p\mathcal{A}^{n-1}(X;\mathcal{V}_*) + d\mathcal{A}^{n-2}(X;\mathcal{V}_*).$$

From Proposition 2.13 we get that modulo the image of d we have

$$(g_1^{\pitchfork})^* \alpha_{m,l} - (g_0^{\pitchfork})^* \alpha_{m,l} = \left(\pi_{\mathbb{R}^k\times X}^{\mathbb{R}^k\times X\times\Delta^1}\right)_* d(g_\bullet^{\pitchfork})^* \alpha_{m,l}$$

in $\mathcal{D}_{vc}^{n+k-1}(\mathbb{R}^k\times X;\mathcal{V}_*)$. Applying π_*, this implies that in $\mathcal{D}^{n-1}(X;\mathcal{V}_*)/\mathrm{Im}\, d$ we have

$$\begin{aligned} \pi_*(g_1^{\pitchfork})^* \alpha_{m,l} - \pi_*(g_0^{\pitchfork})^* \alpha_{m,l} + \left(\pi_X^{X\times\Delta^1}\right)_* \left(\pi_{X\times\Delta^1}^{\mathbb{R}^k\times X\times\Delta^1}\right)_* (g_\bullet^{\pitchfork})^* (\phi_{\nabla_{m,l}}) \\ = \left(\pi_X^{X\times\Delta^1}\right)_* \left(\pi_{X\times\Delta^1}^{\mathbb{R}^k\times X\times\Delta^1}\right)_* (g_\bullet^{\pitchfork})^* (d\alpha_{m,l} + \phi_{\nabla_{m,l}}) \\ = \left(\pi_X^{X\times\Delta^1}\right)_* \left(\pi_{X\times\Delta^1}^{\mathbb{R}^k\times X\times\Delta^1}\right)_* (g_\bullet^{\pitchfork})^* T'_{\phi_{m,l}}. \end{aligned}$$

By Remark 6.20 we have

$$\left(\pi_{X\times\Delta^1}^{\mathbb{R}^k\times X\times\Delta^1}\right)_* (g_\bullet^{\pitchfork})^* T'_{\phi_{m,l}} = T_{\phi_{\mathrm{sm}}^{n+1}(g_\bullet)} \text{ in } \mathcal{D}^{n+1}(X\times\Delta^1;\mathcal{V}_*).$$

Hence we have shown:

$$\begin{aligned} \pi_*((g_1^{\pitchfork})^* \alpha_{m,l} - (g_0^{\pitchfork})^* \alpha_{m,l}) + \left(\pi_X^{X\times\Delta^1}\right)_* \left(\left(\pi_{X\times\Delta^1}^{\mathbb{R}^k\times X\times\Delta^1}\right)_* (g_\bullet^{\pitchfork})^* (\phi_{\nabla_{m,l}})\right) \\ = \left(\pi_X^{X\times\Delta^1}\right)_* T'_{\phi_{\mathrm{sm}}^{n+1}(g'_\bullet)}. \end{aligned}$$

By Stokes' theorem, we have

$$\int_{\Delta^1\times\Delta^1} d + d\int_{\Delta^1\times\Delta^1} = \int_{\Delta^1\times 0} + \int_{1\times\Delta^1} - \int_{\Delta^1\times 1} - \int_{0\times\Delta^1}.$$

Applying this, or rather its analogue for the currential pushforward following from Proposition (2.13) to the closed current $T'_{h_\bullet}$ yields

$$d\left(\pi_X^{X\times\Delta^1\times\Delta^1}\right)_* T'_{h_\bullet} = -\left(\pi_X^{X\times\Delta^1}\right)_* \left(T'_{\phi_{\mathrm{sm}}^{n+1}(g_\bullet)} - T'_{\omega_\bullet} + T'_{h_1} - T'_{h_0}\right)$$

This finishes the proof, since

$$\left(\pi_X^{X\times\Delta^1}\right)_* T'_{\omega_\bullet} = \mathbf{w}' T_{\int_{X\times\Delta^1/X}\omega_\bullet} \in F^p\mathcal{D}^{n-1}(X;\mathcal{V}_*)$$

is a smooth current of filtration p. ■

We consider $MU_{\mathrm{hs}}^{\pitchfork}(p)_n(X)_0$ with the binary operation

$$(A(g), \omega, h) + (A(g'), \omega', h') = (A((g \vee g') \circ pinch, \omega + \omega', h + h'),$$

for maps $g,\ g' \in \mathrm{Map}_*^{\pitchfork}(\Sigma^k X_+, MU(m,l))$. This binary operation induces the addition on $MU_{\mathrm{hs}}^{\pitchfork}(p)^n(X)$. Recall from Proposition 5.17 that

$$\rho_\nabla \colon \mathrm{Map}_*^{\pitchfork}(X, QMU_n) \to \widetilde{ZMU}^n(X)$$

is a homomorphism, in the sense that for $g_1, g_2 \in \mathrm{Map}_*^{\pitchfork}(\Sigma^k X_+, MU(m,l))$ we have

$$\rho_\nabla(A((g_1 \vee g_2) \circ pinch)) = \rho_\nabla(A(g_1)) + \rho_\nabla(A(g_2)).$$

We conclude:

Lemma 6.26. *The map $\kappa \colon MU_{\mathrm{hs}}^{\pitchfork}(p)_n(X)_0 \to ZMU^n(p)(X)$ is a homomorphism in the sense that if for $i = 1,2$ we have $\gamma_i = (A(g_i^{\pitchfork}), \omega_i, h_i) \in MU_{\mathrm{hs}}^{\pitchfork}(p)_n(X)_0$, then*

$$\kappa(A((g_1^{\pitchfork} \vee g_2^{\pitchfork}) \circ pinch), \omega_1 + \omega_2, h_1 + h_2) = \kappa(\gamma_1) + \kappa(\gamma_2).$$

Let γ be an arbitrary element in $MU_{\mathrm{hs}}^n(p)(X)_0$. By Theorem 6.13, γ can be represented by a triple (g, ω, h) in $MU_{\mathrm{hs}}^{\pitchfork}(p)_n(X)_0$. If (g', ω', h') is another representative of γ in $MU_{\mathrm{hs}}^{\pitchfork}(p)_n(X)_0$, then there is a homotopy between (g, ω, h) and (g', ω', h'). By Lemma 6.25, the images of (g, ω, h) and (g', ω', h') under κ agree in $MU^n(p)(X)$. Hence there is an induced map of sets $\kappa \colon MU_{\mathrm{hs}}^n(p)(X) \to MU^n(p)(X)$, which it follows from Lemma 6.26 is a homomorphism. Thus we have proven the following key result:

Theorem 6.27. *For every complex manifold X and integers n, p, the map κ induces a homomorphism of cohomology groups*

$$\kappa \colon MU_{\mathrm{hs}}^n(p)(X) \to MU^n(p)(X).$$

6.5 The map κ respects pullbacks

Now we show that κ commutes with the pullback along holomorphic maps. Let $F \colon Y \to X$ be a holomorphic map. Recall from (5.8) that we define

$$\mathrm{Map}_*^{\pitchfork, F}\left(\Sigma^k X_+, MU(m,l)\right) \subset \mathrm{Map}_*^{\pitchfork}\left(\Sigma^k X_+, MU(m,l)\right)$$

to be the subset of maps g such that $g \circ \Sigma^k F \in \mathrm{Map}_*^{\pitchfork}(\Sigma^k Y_+, MU(m,l))$.

Definition 6.28. *We denote by $MU_{\mathrm{hs}}^{\pitchfork, F}(p)_n(X)_0$ the subset of triples (g, ω, h) in $MU_{\mathrm{hs}}^{\pitchfork}(p)_n(X)_0$ such that $g = A(g_{\pitchfork})$ for a map $g_{\pitchfork}$ in $\mathrm{Map}_*^{\pitchfork, F}\left(\Sigma^k X_+, MU(m,l)\right)$.*

Then the pullback for MU_{hs} is induced by the map

$$F^* \colon MU_{\mathrm{hs}}^{\pitchfork, F}(p)_n(X)_0 \to MU_{\mathrm{hs}}^{\pitchfork}(p)_n(Y)_0$$
$$F^*(g, h, \omega) = (g \circ \Sigma^k F_+, F^*h, F^*\omega).$$

Lemma 6.29. *For every element $\gamma \in MU^n_{\mathrm{hs}}(p)(X)$, there is a representative (g, ω, h) in $MU^{\pitchfork,F}_{\mathrm{hs}}(p)_n(X)_0$. Furthermore, the following diagram commutes:*

$$\begin{tikzcd}
MU^{\pitchfork,F}_{\mathrm{hs}}(p)_n(X)_0 \arrow[r, "F^*"] \arrow[d, "\kappa"'] & MU^{\pitchfork}_{\mathrm{hs}}(p)_n(Y)_0 \arrow[d, "\kappa"] \\
ZMU^n_F(p)(X) \arrow[r, "F^*"'] & ZMU^n(p)(Y).
\end{tikzcd}$$

Proof. For the first part, Lemma 6.14 reduce us to showing that the set of maps $\mathrm{Map}^{\pitchfork,F}_*\left(\Sigma^k X_+, MU(m,l)\right)$ contains maps from every homotopy class of pointed maps $\Sigma^k X_+ \to MU(m,l)$. This follows from the fact that transversality is a generic property.

For the second part, we use naturality of pullback of currents, together with Proposition 5.13, which tells us that F^* is defined on the image of the left hand vertical arrow, and that $F^*(\rho_\nabla(g)) = \rho_\nabla(g \circ \Sigma^k F_+)$. We note also that since F has even codimension, we have $F^* T'_\omega = T'_{F^*\omega}$, and similarly for h since also $F \times \mathrm{Id}_{\Delta^1}$ has even co-dimension. The lemma follows. ∎

Since homotopies of triples are sent to Hodge filtered cobordism relations by Lemma 6.25 and every element in $MU^n_{\mathrm{hs}}(p)(X)$ can be represented by a triple in the subset $MU^{\pitchfork,F}_{\mathrm{hs}}(p)_n(X)_0$ by Lemma 6.29, we have proven the following result:

Theorem 6.30. *Let $F\colon Y \to X$ be a holomorphic map and n, p, be integers. Then we have $F^* \circ \kappa = \kappa \circ F^*_{\mathrm{hs}}$ where F^* denotes the pullback in $MU(p)^n(-)$ and F^*_{hs} the pullback in $MU^n_{\mathrm{hs}}(p)(-)$.*

6.6 The map κ is an isomorphism

We will now show that κ respects the structure maps of a Hodge filtered cohomology theory. The respective long exact sequences of both theories will then imply that κ is an isomorphism.

In Section 3.3 we showed that $MU_{\mathcal{D}}$ is a Hodge filtered cohomology theory. We denote its structure maps with a subscript $\mathcal{D}$. Recall that the maps $I_{\mathcal{D}}$ and $R_{\mathcal{D}}$ are induced from the canonical maps $MU_{\mathcal{D}}(p) \to MU$, and $MU_{\mathcal{D}}(p) \to H(F^p\mathcal{A}^*(\mathcal{V}_*))$, respectively, and $a_{\mathcal{D}}$ is a certain connecting homomorphism. We similarly define structure maps R_{hs}, I_{hs} and a_{hs} for $MU_{\mathrm{hs}}(p)$. Then the arguments from 3.3 also show that MU_{hs} is a Hodge filtered cohomology theory over $\phi\colon MU^*(X) \to H^*(X;\mathcal{V}_*)$. From these descriptions of the structure maps we deduce:

Proposition 6.31. *The isomorphism $MU^n_{\mathrm{hs}}(p)(X) \cong MU^n_{\mathcal{D}}(p)(X)$ of Theorem 6.12 is an isomorphism of Hodge filtered cohomology theories.*

For MU_{hs} we can lift the structure maps from the level of maps in the homotopy category, to the level of 0-simplices of the simplicial mapping space using Theorem 6.9. On this level, the structure maps are given by

$$\begin{aligned} I_{\mathrm{hs}}\colon MU^n_{\mathrm{hs}}(p)(X)_0 &\to \mathrm{Map}^{\mathrm{sm}}(X, QMU_n) \\ (g,\omega,h) &\mapsto g \\ R_{\mathrm{hs}}\colon MU^n_{\mathrm{hs}}(p)(X)_0 &\to F^p\mathcal{A}^n(X;\mathcal{V}_*) \\ (g,\omega,h) &\mapsto \omega, \\ a_{\mathrm{hs}}\colon d^{-1}\left(F^p\mathcal{A}^n(X;\mathcal{V}_*)\right)^{n-1} &\to MU^n_{\mathrm{hs}}(p)(X)_0 \\ h &\mapsto (0, dh, \tau_0 h) \end{aligned}$$

where τ_0 is the map $\tau_0 = d \circ h_0 \colon \mathcal{A}^{n-1}(X;\mathcal{V}_*) \to \mathcal{A}^n(X \times \Delta^1; \mathcal{V}_*)_{cl}$ from (2.21). Note that by (2.20) we have $i_0^*\tau h = 0$ and $i_1^*\tau h = dh$, so that using Theorem 6.9, $a_{\mathrm{hs}}(h) = (0, dh, \tau_0 h)$ represent a class in $MU^n_{\mathrm{hs}}(p)(X)$. Recall from (5.7) the classical Pontryagin–Thom map

$$\rho\colon \mathrm{Map}_*^{\pitchfork}(X, QMU_n) \to ZMU^n(X).$$

We denote also the induced map $MU^n_h(X) \to MU^n(X)$ by ρ.

Lemma 6.32. *The map $\kappa\colon MU^n_{\mathrm{hs}}(p)(X) \to MU^n(p)(X)$ respects the structure maps in the following sense:*

$$\begin{aligned} \kappa \circ a_{\mathrm{hs}} &= (-1)^{n-1} a, \\ I \circ \kappa &= \rho \circ I_{\mathrm{hs}} \text{ and} \\ R \circ \kappa &= (-1)^n R_{\mathrm{hs}}. \end{aligned}$$

Proof. Let $[h] \in H^{n-1}\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right)$ be a class represented by a an element $h \in \mathcal{A}^{n-1}(X;\mathcal{V}_*)$, with $dh \in F^p\mathcal{A}^n(X;\mathcal{V}_*)$. We have $\left(\pi_X^{X\times\Delta^1}\right)_* T'_{\tau_0 h} = T'_h$ by (2.22). Since $T'_h = (-1)^{n-1}T_h$ we get by the definition of κ, Definition 6.21,

$$\kappa \circ a_{\mathrm{hs}}([h]) = a([(-1)^{n-1}h]).$$

Compatibility with I holds on the level of triples $(g,\omega,h) \in MU^{\pitchfork}_{\mathrm{hs}}(p)_n(X)_0$, since the underlying cobordism cycle of $\rho_\nabla(g)$ is $\rho(g)$. Finally, the equality

$$R \circ \kappa[(g,\omega,h)] = [T'_\omega]$$

was verified on the level of forms in the proof of Lemma 6.24. We have $[T'_\omega] = (-1)^n[T_\omega]$, so that $R \circ \kappa[(g,\omega,h)] = (-1)^n R_{\mathrm{hs}}(g,\omega,h)$. ∎

Theorem 6.33. *The map $\kappa\colon MU_{\mathrm{hs}}(p)^n(X) \to MU^n(p)(X)$ is an isomorphism.*

Proof. It follows from Lemma 6.32 that κ fits into a morphism of long exact sequences:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^{n-1}(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) & \xrightarrow{a_{\mathrm{hs}}} & MU^n_{\mathrm{hs}}(p)(X) & \xrightarrow{I_{\mathrm{hs}}} & MU^n_h(X) & \longrightarrow \cdots \\ & \Big\downarrow{\scriptstyle (-1)^{n-1}\cdot \mathrm{id}} & & \Big\downarrow{\scriptstyle \kappa} & & \Big\downarrow{\scriptstyle \rho} & \\ \cdots \longrightarrow & H^{n-1}(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)) & \xrightarrow{a} & MU^n(p)(X) & \xrightarrow{I} & MU^n(X) & \longrightarrow \cdots \end{array}$$

Since the outer vertical maps are isomorphisms for each n and p, the five-lemma implies that κ is an isomorphism. ■

Together with Theorem 6.12, this finishes the proof of Theorem 1.1.

6.7 Alternative approach to $\alpha_{m,l}$

We end this chapter by describing an approach to the comparison currents $\alpha_{m,l}$ which is less canonical and concrete than that used in the above, but simpler in the sense that it follows from abstract nonsense.

The forms $\phi_{m,l}$ of (6.1) and currents $\phi_{\nabla_{m,l}} = (\iota_{m,l})_* K(\nabla_{m,l})$ of Definition 5.19 are different objects. However $T_{\phi_{m,l}} \in \mathcal{D}^{2m}(MU(m,l);\mathcal{V}_*)$ represent the same class as $\phi_{\nabla_{m,l}}$ in $H^{2m}(MU(m,l);\mathcal{V}_*)$. For this, recall that the Thom isomorphism is induced by integration over the fiber. Thus two currents in $\mathcal{D}^*_{vc}(\gamma_{m,l};\mathcal{V}_*)$ are cohomologous if and only if their images under the pushforward along $\pi_{m,l}\colon \gamma_{m,l} \to \mathrm{Gr}_m(\mathbb{C}^{m+l})$ are cohomologous. We have

$$\int_{\gamma_{m,l}/\mathrm{Gr}_m(\mathbb{C}^{m,l})} U_{m,l} \wedge \pi^* K(\nabla_{m,l}) = K(\nabla_{m,l})$$

by the projection formula for integration along the fiber, recalled at Proposition 2.35. Similarly, we have

$$(\pi_{m,l})_*(\iota_* K(\nabla_{m,l})) = K(\nabla_{m,l})$$

since $\pi_{m,l} \circ \iota_{m,l} = \mathrm{id}_{\mathrm{Gr}_m(\mathbb{C}^{m+l})}$. Since both $\mathrm{Gr}_m(\mathbb{C}^{m+l})$ and $\gamma_{m,l}$ are of even real dimension, we have for each $\omega \in \mathcal{A}^*(MU(m,l);\mathcal{V}_*)$ the equality

$$(\pi_{m,l})_* T_\omega = T_{\int_{\gamma_{m,l}/\mathrm{Gr}_m(\mathbb{C}^{m,l})} \omega} \in \mathcal{D}^*(\mathrm{Gr}_m(\mathbb{C}^{m+l});\mathcal{V}_*).$$

This establish:

$$[T_{\phi_{m,l}}] = [\phi_{\nabla_{m,l}}] \in H^{2m}(MU(m,l);\mathcal{V}_*).$$

It follows that there are currents

$$\alpha_{m,l} \in \mathcal{D}^{2m-1}(MU(m,l);\mathcal{V}_*)$$

such that

$$d\alpha_{m,l} + \phi_{\nabla_{m,l}} = \phi_{m,l} \text{ in } \mathcal{D}^{2m}(MU(m,l);\mathcal{V}_*). \tag{6.12}$$

Since $WF(d\alpha_{m,l}) = WF(\phi_{\nabla_{m,l}}) \subset N(\iota_{m,l})$, we can by Lemma 4.43 furthermore assume

$$WF(\alpha) \subset N(\iota_{m,l}).$$

This is important because then $g^*\alpha_{m,l}$ is defined whenever $g \pitchfork \iota_{m,l}$, by Theorem 2.23. The choice of $\alpha_{m,l}$ is unique up to an exact current, since

$$H^{2m-1}(MU(m,l);\mathcal{V}_*) = 0.$$

Concretely, this implies that given another current $\alpha' \in \mathcal{D}^{2m-1}(MU(m,l);\mathcal{V}_*)$ satisfying $d\alpha' + \phi_{\nabla_{m,l}} = \phi_{m,l}$, we have

$$d(\alpha_{m,l} - \alpha') = 0.$$

The vanishing of $H^{2m-1}(MU(m,l);\mathcal{V}_*) = 0$ means that we can find a current β in $\mathcal{D}^{2m-2}(MU(m,l);\mathcal{V}_*)$ such that $d\beta = \alpha_{m,l} - \alpha'$. We will eventually only care about $\alpha_{m,l}$ modulo the image of d, and as such we see that $\alpha_{m,l}$ is well-defined.

From propositions 6.2 and 5.20 it follows that modulo exact currents, we have

$$\overline{i_{m,l}}^*\alpha_{m,l+1} = \alpha_{m,l} \quad \text{and} \quad \pi_*(\overline{j_{m,l}}^*\alpha_{m+1,l}) = \alpha_{m,l}.$$

where $\pi\colon \underline{\mathbb{C}} \oplus \gamma_{m,l} \to \gamma_{m,l}$ is the projection. Hence we may by Proposition 5.7 define

$$pb^\alpha\colon \mathrm{Map}^{\pitchfork}(X, QMU_n) \to \mathcal{D}^*(X;\mathcal{V}_*)/\mathrm{Im}\,(d)$$

by $pb^\alpha(g) = \pi_* {g_\pitchfork}^*\alpha_{m,l}$ for any $g_\pitchfork \in \mathrm{Map}_*^{\pitchfork}(\Sigma^k X_+, MU(m,l))$ with $A(g_\pitchfork) = g$, where $\pi\colon \mathbb{R}^k \times X \to X$ denotes the projection. Using this definition of pb^α in Definition 6.21, κ becomes a map

$$\kappa\colon MU_{\mathcal{D}}^{\pitchfork}(p)_n(X)_0 \to ZMU^n(p)(X)/a(\mathrm{Im}\,(d)).$$

Chapter 7

Pushforward for geometric Hodge filtered cobordism

In [30], Hopkins and Quick show that there are pushforward maps

$$g_*\colon MU^n_{\mathcal{D}}(p)(X) \to MU^{n+2d}_{\mathcal{D}}(p+d)(Y)$$

for g a projective morphism between smooth projective complex varieties. In fact, they show that there are pushforward maps for a logarithmically refined version of $MU_{\mathcal{D}}$ for smooth complex varieties. This theory coincides with $MU_{\mathcal{D}}$ for projective smooth complex varieties. In [30, Section 7], the existence of such pushforwards is a rather formal consequence of the projective bundle formula.

In this chapter, we give a geometric description of a pushforward, inspired by the pushforward map of [8],

$$g_*\colon MU^n(p)(X) \to MU^{n+2d}(p+d)(Y)$$

for all proper homolorphic maps between complex manifolds, where d isthe complex codimension of g. Hence our pushforward exists for a significantly larger class of maps than the one in [30].

This presumable makes the restriction of the theory $MU^{2*}(*)$ to algebraic manifolds into an oriented Borel–Moore cohomology theory in the sense of [39], though we do not check all the axioms.

We believe that our geometric pushforward coincides with that of [30] when both are defined. To prove this one must establish the projective bundle formula for $MU^n(p)$, and show that the isomorphism with $MU^n_{\mathcal{D}}(p)$ preserves this structure. We have not checked the details.

The chapter is structured as follows: We define the notion of an $MU_{\mathcal{D}}$-orientation of a holomorphic map g. We then define a pushforward along proper and $MU_{\mathcal{D}}$-oriented maps. Next we establish some basic properties of the pushforward. Finally we use ideas from [34] to show that any holomorphic g has a canonical $MU_{\mathcal{D}}$-orientation.

7.1 Hodge filtered MU-orientations

We recall from(4.34) that a geometric bordism cycle $\widetilde{f}$ can be thought of as a proper complex oriented map $g\colon Z \to X$ with a lift of N_f from $K^0(Z)$ to the group of pairs (E, ∇) modulo the relation $(E, \nabla) \sim (E, \nabla')$ whenever $\widetilde{K}_{CS}(\nabla, \nabla') = 0$, where $\widetilde{K}_{CS}(\nabla, \nabla')$ is the Chern–Simons transgression form of Proposition 4.22. When f is holomorphic we can involve the Hodge filtration.

Definition 7.1. *Let Z be a complex manifold. We define the set of Hodge filtered MU-orientations, $MU_{\mathcal{D}}$-orientations for short, of filtration p on Z, which we denote by $\check{K}^0(p)(Z)$, as follows: An element is represented by a triple (E, ∇, σ) where E is a complex vector bundle on Z, ∇ is a connection on E and $\sigma \in \mathcal{A}^{-1}(Z; \mathcal{V}_*)$ such that*

$$K^p(\nabla) + d\sigma \in F^p\mathcal{A}^0(Z; \mathcal{V}_*).$$

Then $\check{K}^0(p)(Z)$ is the set of equivalence classes under the equivalence relation generated by the following relations:

- $(E, \nabla, \sigma) \sim (E \oplus \underline{\mathbb{C}}_Z, \nabla \oplus d, \sigma)$.
- $(E, \nabla, \sigma) \sim (E', \nabla', \sigma')$ *if there is an isomorphism $f \colon E \to E'$ such that $f^*\sigma' = \sigma + \widetilde{K}_{CS}(\nabla, f^*\nabla') + \alpha$ for some*

$$\alpha \in \widetilde{F}^p\mathcal{A}^{-1}(Z; \mathcal{V}_*) = F^p\mathcal{A}^{-1}(Z : \mathcal{V}_*) + d\mathcal{A}^{-2}(Z; \mathcal{V}_*).$$

We will continue to suppress the p from the notation and write K instead of K^p. We define the map $K \colon \check{K}^0(p)(Z) \to H^0(Z; F^p\mathcal{A}^*(\mathcal{V}_*))$ by

$$K[E, \nabla, \sigma] = K(\nabla) + d\sigma.$$

We now define the group of all $MU_{\mathcal{D}}$-orientations on Z by

$$\check{K}^0(\bullet)(Z) = \bigoplus_p \check{K}^0(p)(Z).$$

Here an element of the sum $\bigoplus_p \check{K}^0(p)(Z)$ is a sequence of elements $[E_p, \nabla_p, \sigma_p] \in K^0(p)(Z)$, so that $E_p \neq 0$ for only finitely many p. To define the addition on $\check{K}^0(\bullet)(Z)$, we first define addition of generators by

$$\begin{gathered} +\colon \check{K}^0(p)(Z) \times \check{K}^0(p')(Z) \to \check{K}^0(p + p')(Z) \\ [E, \nabla, \sigma] + [E', \nabla', \sigma'] := [E \oplus E',\ \nabla \oplus \nabla',\ \sigma \wedge K(\nabla') + \sigma' \wedge K[E, \nabla, \sigma]]. \end{gathered} \tag{7.1}$$

This operation is well-defined. We show that the choice of representative for the first argument is immaterial. The proof for the second variable is similar. For the first relation, we compute

$$\begin{aligned} &(E \oplus \underline{\mathbb{C}}_Z, \nabla \oplus d, \sigma) + (E', \nabla', \sigma') \\ &\qquad = (E \oplus \underline{\mathbb{C}} \oplus E',\ \nabla \oplus d \oplus \nabla',\ \sigma \wedge K(\nabla') + \sigma' \wedge K(E, \nabla, \sigma)) \\ &\qquad \sim (E \oplus E',\ \nabla \oplus \nabla',\ \sigma \wedge K(\nabla') + \sigma' \wedge K(E, \nabla, \sigma)). \end{aligned}$$

For the second relation we use that for $\alpha \in F^p\mathcal{A}^{-1}(Z; \mathcal{V}_*)$, we have

$$\alpha \wedge K(E', \nabla', \sigma') \in F^{p+p'}\mathcal{A}^{-1}(Z; \mathcal{V}_*)$$

together with the fact that

$$\widetilde{K}_{CS}(\nabla \oplus \nabla', \nabla'' \oplus \nabla') = \widetilde{K}_{CS}(\nabla, \nabla'') \wedge K(\nabla').$$

It remains to see that $K([E, \nabla, \sigma] + [E', \nabla', \sigma']) \in F^{p+p'}\mathcal{A}^0(Z; \mathcal{V}_*)$. This follows from the following lemma:

Lemma 7.2. $K\left([E,\nabla,\sigma]+[E',\nabla',\sigma']\right)=K[E,\nabla,\sigma]\wedge K[E',\nabla',\sigma']$.

Proof. This is a straight forward computation:

$$\begin{aligned}K\left([E,\nabla,\sigma]+[E',\nabla',\sigma']\right)&=K(\nabla\oplus\nabla')+d\left(\sigma\wedge K(\nabla')+\sigma'\wedge K[E,\nabla,\sigma]\right)\\&=K(\nabla)\wedge K(\nabla')+d\sigma\wedge K(\nabla')+d\sigma'\wedge K[E,\nabla,\sigma]\\&=(K(\nabla)+d\sigma)\wedge K(\nabla')+d\sigma'\wedge K[E,\nabla,\sigma]\\&=K[E,\nabla,\sigma]\wedge K[E',\nabla',\sigma'].\end{aligned}$$

■

We also define $I\colon \check{K}^0(p)(Z)\to\widetilde{K}^0(Z)$ by $I[E,\nabla,\sigma]=[E]$.

Proposition 7.3. *The addition on $\check{K}^0(\bullet)(Z)$ is commutative.*

Proof. Since σ' is of odd degree, we have the equality modulo the image of d

$$\sigma'\wedge d\sigma=d\sigma'\wedge\sigma.$$

Also $K(\nabla)$, $K[E,\nabla,\sigma]$, $d\sigma$ and $d\sigma'$ are of even degrees and hence in the center of the ring $\mathcal{A}^*(X;\mathcal{V}_*)$. Hence in $\mathcal{A}^{-1}(Z;\mathcal{V}_*)/\mathrm{Im}\,(d)$ we get the equalities

$$\begin{aligned}\sigma\wedge K(\nabla')+\sigma'\wedge K[E,\nabla,\sigma]&=\sigma\wedge K(\nabla')+\sigma'\wedge(K(\nabla)+d\sigma)\\&=\sigma\wedge K(\nabla')+\sigma'\wedge K(\nabla)+d\sigma'\wedge\sigma)\\&=(K(\nabla')+d\sigma')\wedge\sigma+\sigma\wedge K(\nabla').\end{aligned}$$

This resolves the apparent asymmetry in the definition of $+$. The proposition follows. ■

Proposition 7.4. *The set $\check{K}^0(\bullet)$ together with the above operation is a group.*

Proof. The cycle $(\underline{\mathbb{C}}_Z,d,0)$ represents an additive identity element. We must show existence of inverses, so let (E,∇,σ) be a cycle. Then we can find a complex vector bundle E' such that $E\oplus E'\simeq\mathbb{C}_Z^N$. Give E' the connection ∇' induced from d by the direct sum decomposition $E\oplus E'=\underline{\mathbb{C}}_Z^N$. We can write

$$K(\nabla)+d\sigma=1+K_2(c_1(\nabla))+d\sigma_2+K_4(c_1(\nabla),c_2(\nabla))+d\sigma_4+\cdots\in\mathcal{A}^0(X;\mathcal{V}_*)$$

with $K_{2i}(c_1(\nabla),\cdots,c_i(\nabla))\in\mathcal{A}^{2i}(Z;\mathcal{V}_{2i})$, and $\sigma_{2i}\in\mathcal{A}^{2i-1}(Z;\mathcal{V}_{2i})$. It follows from the Milnor-Quillen theorem on MU that there is an isomorphism

$$\mathcal{V}_*\simeq\mathbb{C}[b_2,b_4,\cdots].$$

Hence we can view $K(\nabla)+d\sigma$ as a powerseries over the commutative ring $\mathcal{A}^{2*}(Z)$ in the variables b_{2i} for $i\leqslant\dim_{\mathbb{C}}Z$. Since the constant term is 1, $K(\nabla)+d\sigma$ is invertible by Lemma 2.57. We put

$$\sigma'=\left(\widetilde{K}_{CS}(\nabla\oplus\nabla',d)-\sigma\wedge K(\nabla')\right)\wedge(K(\nabla)+d\sigma)^{-1}.$$

We allow ourselves to suppress the isomorphism $E \oplus E' \simeq \underline{\mathbb{C}}_Z^N$ from the notation, and pretend that d is a connection on $E \oplus E'$. We get

$$\begin{aligned}(E, \nabla, \sigma) + (E', \nabla', \sigma') &= (E \oplus E',\ \nabla \oplus \nabla',\ \sigma' \wedge (K(\nabla) + d\sigma) + \sigma \wedge K(\nabla')) \\ &= (E \oplus E', \nabla \oplus \nabla',\ \widetilde{K}_{CS}(\nabla \oplus \nabla', d) - \sigma \wedge K(\nabla') + \sigma \wedge K(\nabla')) \\ &\sim (\underline{\mathbb{C}}_Z^N,\ d,\ 0),\end{aligned}$$

which finishes the proof. ■

Definition 7.5. *We define an $MU_{\mathcal{D}}$-orientation (of filtration p) of a holomorphic map $g\colon Z \to X$ to be a class*

$$\overline{N}_g = [N_g, \nabla_g, \sigma_g] \in \check{K}^0(p)(Z)$$

and a representative of the canonical complex orientation of g of the form (N_g, Φ_g). More precisely, a representative of a $MU_{\mathcal{D}}$-orientation is a tuple

$$\overline{N}_g = (N_g, \Phi_g, \nabla_g, \sigma_g),$$

where (N_g, Φ_g) represents the canonical complex orientation of g, ∇_g is a connection on N_g, and $\sigma_g \in \mathcal{A}^{-1}(X; \mathcal{V}_*)$ is a form such that

$$K(\overline{N}_g) := K(\nabla_g) + d\sigma \in F^p\mathcal{A}^0(X; \mathcal{V}_*).$$

Given a representative of an $MU_{\mathcal{D}}$-orientation of g, we write $\check{g} = (g, N_g, \nabla_g, \sigma_g)$, and $\widetilde{g} = (g, N_g, \nabla_g)$. We also write $K(\check{g}) = K(\overline{N}_g)$.

7.2 Pushforward along proper maps with Hodge filtered MU-orientation

Let $\overline{N}_g = (N_g, \nabla_g, \sigma)$ represent an $MU_{\mathcal{D}}$-orientation of filtration p of a proper holomorphic map

$$g\colon X \to Y$$

of complex codimension $d = \dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} X$. We now construct a map

$$\check{g}_*\colon MU^n(p')(X) \to MU^{n+2d}(p' + p + d)(Y).$$

We use the currential description of $MU^n(p)(X)$ from Section 4.12. We write $\widetilde{g} \circ \widetilde{f} = (g \circ f,\ N_f \oplus f^*N_g,\ \nabla_f \oplus f^*\nabla_g)$ for the composed geometric cycle.

Definition 7.6. *We define the pushforward on currential cycles thus:*

$$\begin{aligned}\check{g}_*\colon ZMU^n_\delta(p')(X) &\to ZMU^{n+2d}_\delta(p' + p + d)(Y) \\ \check{g}_*(\widetilde{f}, h) &= \left(\widetilde{g} \circ \widetilde{f},\quad g_*\left[K(\nabla_g) \wedge h + \sigma \wedge R(\widetilde{f}, h)\right]\right).\end{aligned}$$

Lemma 7.7. *We have $\check{g}_*(\widetilde{f}, h) \in ZMU^{n+2d}_\delta(p' + p + d)(Y)$.*

Proof. We must verify that $R(\check{g}_*(\widetilde{f},h))$ is in the correct filtration stage:

$$\begin{aligned}
R(\check{g}_*(\widetilde{f},h)) &= g_*f_*K(\nabla_f \oplus f^*\nabla_g) + dg_*\left(K(\nabla_g)\wedge h + \sigma \wedge R(\widetilde{f},h)\right)\\
&= g_*((f_*K(\nabla_f))\wedge K(\nabla_g)) + g_*\left(K(\nabla_g)\wedge dh + d\sigma\wedge R(\widetilde{f},h)\right)\\
&= g_*\left(K(\nabla_g)\wedge(f_*K(\nabla_f)+dh) + d\sigma \wedge R(\widetilde{f},h)\right)\\
&= g_*\left(K(\check{g})\wedge R(\widetilde{f},h)\right).
\end{aligned}$$

Since $K(\check{g}) \in F^p\mathcal{A}^0(X;\mathcal{V}_*)$, $R(\widetilde{f},h) \in F^{p'}\mathcal{D}^n(X)$, and g is holomorphic of codimension d it follows that

$$g_*\left[K(\widetilde{N}_g)\wedge R(\widetilde{f},h)\right] \in F^{p+p'+d}\mathcal{D}^{n+2d}(Y;\mathcal{V}_*)$$

as required. ∎

Theorem 7.8. *Let $g\colon X\to Y$ be a proper, holomorphic and $MU_{\mathcal{D}}$-oriented map. The class*

$$[\check{g}_*(\widetilde{f},h)] \in MU_\delta^{n+2d}(p'+p+d)(Y)$$

is independent of the representative of the orientation class $[\overline{N}_g] \in K^0(p)(X)$, and of the representative $(\widetilde{f},h) \in ZMU^n(p)$ of the Hodge filtered cobordism class $[\widetilde{f},h] \in MU^n(p')(X)$.

We split the proof into the following two lemmas:

Lemma 7.9. *Let $\check{g} = (g, N_g, \nabla, \sigma)$ and $\check{g}' = (g, N_g, \nabla', \sigma')$ be two representatives of the same $MU_{\mathcal{D}}$-orientation of $g : X \to Y$. Then for each $\gamma \in ZMU_\delta^n(p')(X)$ we have*

$$[\check{g}_*\gamma] = [\check{g}'_*\gamma].$$

Proof. We have from definition (7.1) that

$$\sigma' = \sigma + \widetilde{K}_{CS}(\nabla',\nabla) + a(\alpha)$$

for some $\alpha \in \widetilde{F}^0\mathcal{A}^{-1}(X;\mathcal{V}_*)$. The difference between $\widetilde{g}\circ\widetilde{f}$ and $\widetilde{g}'\circ\widetilde{f}$ is that the connection $\nabla_f \oplus f^*\nabla$ is replaced with $\nabla_f \oplus f^*\nabla'$. We have

$$\widetilde{K}_{CS}(\nabla_f\oplus f^*\nabla,\ \nabla_f\oplus f^*\nabla') = K(\nabla_f)\wedge f^*\widetilde{K}_{CS}(\nabla,\nabla').$$

Hence

$$\begin{aligned}
\sigma\wedge f_*K(\nabla_f) &+ f_*\widetilde{K}_{CS}(\nabla_f\oplus f^*\nabla,\ \nabla_f\oplus f^*\nabla')\\
&= (\sigma + \widetilde{K}_{CS}(\nabla,\nabla'))\wedge f_*K(\nabla_f)\\
&= \sigma'\wedge f_*K(\nabla_f).
\end{aligned}$$

Using this, together with

$$K(\nabla) + d\sigma = K(\nabla') + d\sigma',$$

Proposition 7.3 and Lemma 4.34 we finish the proof with this computation:

$$\begin{aligned}
[\check{g}_*(\widetilde{f},h)] &= \Big[\widetilde{f}\circ\widetilde{g},\ \ g_*\left((K(\nabla)+d\sigma)\wedge h+\sigma\wedge f_*K(\nabla_f)\right)\Big] \\
&= \Big[\widetilde{f}\circ\widetilde{g}',\ \ g_*\Big(K(\nabla)+d\sigma)\wedge h+\sigma\wedge f_*K(\nabla_f) \\
&\qquad\qquad + f_*\widetilde{K}_{CS}\big(\nabla_f\oplus f^*\nabla',\nabla_f\oplus f^*\nabla\big)\Big)\Big] \\
&= \Big[\widetilde{f}\circ\widetilde{g}',\ \ g_*\Big((K(\nabla')+d\sigma')\wedge h+\sigma'\wedge f_*K(\nabla_f)\Big)\Big] \\
&= \Big[\check{g}'_*(\widetilde{f},h)\Big].
\end{aligned}$$

■

Lemma 7.10. *Given a representative of an $MU_{\mathcal{D}}$-orientation of g, we have*

$$\check{g}_*\left(BMU^n_\delta(p')(X)\right)\subset BMU^{n+2d}_\delta(p'+p+d)(Y).$$

Proof. First let $\alpha\in\widetilde{F}^{p'}\mathcal{A}^{n-1}(X;\mathcal{V}_*)$. Then we have

$$\check{g}_*(a(\alpha))=(0,g_*((K(\nabla)+d\sigma)\wedge\alpha)).$$

Clearly $g_*((K(\nabla)+d\sigma)\wedge\alpha)\in\widetilde{F}^{p+p'+d}\mathcal{D}^{n+2d}(Y;\mathcal{V}_*)$, so $g_*a(\alpha)$ represents 0 as it should. It remains to show

$$\check{g}_*(BMU^n_{geo}(X))\subset BMU^{n+2d}_{geo}(Y).$$

This is proven in [8, Lemma 4.35]. We give a proof. Let $\widetilde{b}\in\widetilde{ZMU}^n(\mathbb{R}\times X)$ be a geometric bordism datum on X, as defined in Definition 4.30. Let $\widetilde{e}$ denote the geometric cycle $\mathrm{id}_{\mathbb{R}}\times g\colon\mathbb{R}\times X\to\mathbb{R}\times Y$ with the product geometric structure, where $\mathrm{id}_{\mathbb{R}}$ has the geometric structure $(\mathrm{id}_{\mathbb{R}},\underline{\mathbb{C}}_{\mathbb{R}},d)$. Then $\widetilde{e}\circ\widetilde{b}$ is a geometric bordism datum over Y. We recall that $R\big(\partial\widetilde{b},\psi(\widetilde{b})\big)=0$. Using that g is of even real codimension we get

$$\psi(\widetilde{e}\circ\widetilde{b})=g_*(K(\widetilde{N}_g)\wedge\psi(\widetilde{b})).$$

Hence we finishes the proof by:

$$\begin{aligned}
\check{g}_*(\partial\widetilde{b},\psi(\widetilde{b})) &= \Big(\widetilde{g}\circ\partial\widetilde{b},\ g_*\left[K(\widetilde{N}_g)\wedge\psi(\widetilde{b})\right]\Big) \\
&= \Big(\partial(\widetilde{e}\circ\widetilde{b}),\ \psi(\widetilde{e}\circ\widetilde{b})\Big).
\end{aligned}$$

■

Theorem 7.8 now follows from Lemmas 7.9 and 7.10.

7.3 Properties of the pushforward

Next we establish some properties for the pushforward of Section 7.2.

Theorem 7.11. *Let $g\colon X \to Y$ be a proper holomorphic map of complex codimension d with $MU_{\mathcal{D}}$-orientation represented by $\check{g} = (g, N_g, \nabla_g, \sigma_g)$ Then the following diagrams commute:*

$$\begin{tikzcd}
H^{n-1}\left(X; \frac{\mathcal{D}^*}{F^p}(\mathcal{V}_*)\right) \arrow[r, "a"] \arrow[d, "g_*(K(\check{g})\wedge -)"'] & MU^n_\delta(p)(X) \arrow[r, "I"] \arrow[d, "\check{g}_*"] & MU^n(X) \arrow[d, "g_*"] \\
H^{n-1+2d}\left(Y; \frac{\mathcal{D}^*}{F^{p+p'+d}}(\mathcal{V}_*)\right) \arrow[r, "a"] & MU^{n+2d}_\delta(p+p'+d)(Y) \arrow[r, "I"] & MU^{n+2d}(Y)
\end{tikzcd}$$

and

$$\begin{tikzcd}
MU^n_\delta(p)(X) \arrow[r, "R"] \arrow[d, "\check{g}_*"'] & H^n(X; F^p\mathcal{D}^*(X;\mathcal{V}_*)) \arrow[d, "g_*(K(\check{g})\wedge -)"] \\
MU^{n+2d}_\delta(p+p'+d)(Y) \arrow[r, "R"] & H^{n+2d}(X; F^{p+p'+d}\mathcal{D}^*(Y;\mathcal{V}_*)).
\end{tikzcd}$$

Proof. Let $[h] \in H^{n-1}\left(X; \frac{\mathcal{D}^*}{F^p}(\mathcal{V}_*)\right)$. Then

$$\begin{aligned} g_*(a[h]) &= g_*[0, h] \\ &= [0, g_* K(\overline{N}_g) \wedge h] \\ &= a\left(g_*\left(K(\overline{N}_g) \wedge h\right)\right) \end{aligned}$$

which proves that the first square commutes. Next we have

$$g_* \circ I[\widetilde{f}, h] = g_*[f, N_f] = [g \circ f, N_f \oplus f^* N_g] = I(g_*(\widetilde{f}, h)).$$

Finally, we have already seen in the proof of Lemma 7.7 that

$$R(g_*(\gamma)) = g_*\left(R(\gamma) \wedge K(\overline{N}_g)\right)$$

which shows that also the second square commutes. ■

7.4 The canonical Hodge filtered MU-orientation of holomorphic maps

It is convenient to follow Karoubi in working on a larger category than that of complex manifolds. We extend the Hodge filtration to the de Rham complex of product manifolds $S \times X$, where S is a real and X a complex manifold, by

$$F^p\mathcal{A}^*(S \times X) = \mathcal{A}^*(S)\widehat{\otimes}F^p\mathcal{A}^*(X).$$

Here $\widehat{\otimes}$ denotes the closure of the algebraic tensor product in the space $\mathcal{A}^*(S\times X)$ with its usual Fréchet topology. Using this definition of F^p, we extend the definition of $\check{K}^0(0)$, as defined in Section 7.1 to products $S \times X$ as follows.

We define the sheaf Ω^0 by

$$\Omega^0(S \times X) = \{f : S \times X \to \mathbb{C} : df \in F^1\mathcal{A}^1(S \times X)\}.$$

When $S = \mathrm{pt}$ we recover the definition of Ω^0 as the sheaf of holomorphic functions. Let $\pi : E \to S \times X$ be a complex vector bundle of dimension n with local trivializations $\{(U_i, \rho_i)\}$. Define the transition functions $g_{ij} : U_{ij} = U_i \cap U_j \to M_n(\mathbb{C})$ by

$$\rho_i \circ \rho_j^{-1} = (\mathrm{id}_{U_{ij}}, g_{ij}(x)) : U_{ij} \times \mathbb{C}^n \to U_{ij} \times \mathbb{C}^n.$$

We say that E is partially flat if the trivializations can be chosen so that g_{ij} belongs to $M_n(\Omega^0(U_{ij}))$, where M_n denotes the set of $n \times n$-matrices. This is equivalent to saying

$$dg_{ij} \in F^1\mathcal{A}^1(U_{ij}; M_n(\mathbb{C})) = M_n(F^1\mathcal{A}^1(U_{ij})).$$

If $S = pt$, a partially flat vector bundle over $S \times X$ is just a holomorphic vector bundle over X. Let E and E' be two partially flat vector bundle over X of dimensions n and n', respectively, with local trivializations $\{(U_i, \rho_i)\}$ and $\{(U_i, \rho_i')\}$ as above. Then a map $\alpha : E \to E'$ is partially flat if

$$\rho_i' \circ \alpha \circ \rho_i^{-1} \in M_{n' \times n}(\Omega^0(U_i)).$$

We then define $K^0_{\mathrm{PF}}(S \times X)$ as the group generated by partially flat vector bundles, modulo the relation $[E_2] = [E_1] + [E_3]$ whenever there is a short exact sequence of partially flat vector bundles

$$0 \to E_1 \to E_2 \to E_3 \to 0.$$

When $S = \mathrm{pt}$, we have $K^0_{\mathrm{PF}}(S \times X) = K^0_{hol}(X)$.

Let ∇ be a connection on E. Then with respect to the local coordinates (U_i, ρ_i), ∇ acts as $d + \theta^i$, where $\theta^i = (\theta^i_{jk})$ is a matrix of 1-forms. Recall that we have

$$\theta^i = g_{ji}^{-1} dg_{ji} + g_{ji}^{-1}\theta^j g_{ji}$$

and that conversely any such cocycle $\{\theta^i\}$ defines a connection. We say that ∇ is a Bott-connection if for each i, j, k we have

$$\theta^i_{jk} \in F^1\mathcal{A}^1(X).$$

Then the curvature of ∇, which in local coordinates is represented by the matrix

$$d\theta^i + \theta^i \wedge \theta^i,$$

clearly belongs to $F^1\mathcal{A}^2(S \times X; \mathrm{End}(E))$. In particular, for any invariant polynomial of total degree 0 with coefficients in a commutative graded $\mathbb{C}$-algebra $\mathcal{R}$, we have

$$P(\nabla) \in F^0\mathcal{A}^0(S \times X; \mathcal{R}).$$

Our main examples are

$$ch(\nabla) \in F^0\mathcal{A}^0(S \times X; K_* \otimes \mathbb{C})$$

and

$$K(\nabla) \in F^0\mathcal{A}^0(S \times X; MU_* \otimes \mathbb{C}).$$

Karoubi shows in [34, Theorem 6.7] that picking a Bott connection D, the assignment

$$E \mapsto [E, D, 0] \in K^0(0)(S \times X)$$

defines a map

$$K^0_{\mathrm{PF}}(S \times X) \to K^0(0)(S \times X).$$

That is, the class $[E, D, 0] \in K^0(0)(S \times X)$ is independent of the choice of Bott-connection D, and the relations of $K^0_{\mathrm{PF}}(S \times X)$ also hold in $K^0(0)(S \times X)$. We adapt his proof to show:

Proposition 7.12. *There is a well defined map*

$$\kappa : K^0_{\mathrm{PF}}(S \times X) \to \check{K}^0(0)(S \times X)$$

given by

$$\kappa[E] = [E, D, 0],$$

where D is any Bott connection on E. In particular, κ gives a map $K^0_{\mathrm{hol}}(X) \to \check{K}^0(0)(X)$.

Proof. We first show that there always exists a Bott connection. Let $\{(U_i, \rho_i)\}$ trivialize E as above, so that the transition functions g_{ij} have $dg_{ij} \in F^1$. Let (λ_i) be a partition of unity subordinate to $\{U_i\}$. Then put

$$\theta^i = \sum_k \lambda_k g_{ki}^{-1} dg_{ki}.$$

Clearly $\theta^i \in F^1$. Over U_{ij} we compute, using that since $g_{ki} = g_{kj}g_{ji}$ we have $dg_{ki} = dg_{kj} \cdot g_{ji} + g_{kj} \cdot dg_{ji}$,

$$\begin{aligned}
\theta^i &= \sum_k \lambda_k g_{ki}^{-1} dg_{ki} \\
&= \sum_k \lambda_k g_{ji}^{-1} g_{kj}^{-1} \left(dg_{kj} \cdot g_{ji} + g_{kj} dg_{ji}\right) \\
&= g_{ji}^{-1} \sum_k \lambda_k \left(g_{kj}^{-1} dg_{kj} + d\right) g_{ji} \\
&= g_{ji}^{-1} \theta^j g_{ji} + g_{ji}^{-1} dg_{ji}
\end{aligned}$$

which demonstrates that the θ^i are the connection matrices of a Bott connection. Let D and D' be two Bott connections on E. Let $\pi\colon I \times S \times X \to S \times X$ denote the projection. Then $D'' = t \cdot \pi^* D + (1 - t) \cdot \pi^* D'$ is a Bott-connection on $\pi^* E$.

Let $i_t\colon S \times X \to I \times S \times X$ be given by $i_t(x) = (t, x)$. We observe that both i_t^* and π_* are filtration preserving maps. In particular, $\pi_* K(D'') \in F^0\mathcal{A}^{-1}(X; \mathcal{V}_*)$, which together with $\widetilde{K}_{CS}(D, D') = \pi_* K(D'')$ proves that

$$[E, D, 0] = [E, D', 0] \in \check{K}^0(0)(S \times X).$$

Hence $\kappa([E])$ is well defined; it remains to show that κ respects the relations of K^0_{PF}. Let

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

be a short exact sequence of partially flat vector bundles, and let D_i be a Bott-connection on E_i. Choosing a smooth splitting, we can view $D_1 \oplus D_3$ as a connection on E_2. Of course the actual connection on E_2 thus obtained depends on the splitting. However, the class $[E_2, D_1 \oplus D_2, 0]$ is independent of the splitting by the second relation in Definition 7.1. In general, $D_1 \oplus D_3$ is not a Bott connection, but it does satisfy

$$K(D_1 \oplus D_3) \in F^0\mathcal{A}^0(S \times X; \mathcal{V}_*)$$

since K is multiplicative. Considering the connection $t \cdot D_2 + (1 - t) \cdot D_1 \oplus D_3$ on $\pi^* E_2$, we again find

$$\widetilde{K}_{CS}(D_2, D_1 \oplus D_3) \in F^0\mathcal{A}^{-1}(S \times X; \mathcal{V}_*).$$

This finishes the proof. ■

The idea for defining the canonical $MU_{\mathcal{D}}$ orientation for a holomorphic map $g\colon X \to Y$ is to apply this proposition to the virtual holomorphic normal bundle

$$N_g := [g^*TY] - [TX] \in K^0_{\mathrm{hol}}(X).$$

We do however need a representative with an explicit complex vector bundle and a short exact sequence Φ_g representing the complex orientation. Let $i\colon X \to \mathbb{C}^k$ be a smooth, proper embedding. We get a short exact sequence

$$\mathcal{E} := \left(0 \longrightarrow TX \xrightarrow{D(g,i)} g^*TY \oplus \underline{\mathbb{C}}^k_X \longrightarrow N_{(g,i)} \longrightarrow 0 \right).$$

Let D_X be a Bott connection for TX, and D_Y a Bott connection for g^*TY. Let ∇_g be any connection on $N_{(g,i)}$. Then we have

$$\kappa(N_g) = [N_{g,i}, \nabla_g, \widetilde{K}_{CS}(\mathcal{E})] \in \check{K}^0(0)(X).$$

This is the canonical $MU_{\mathcal{D}}$ orientation of g.

Theorem 7.13. *Let $g\colon X \to Y$ be a proper holomorphic map of codimension d. Then there is a pushforward*

$$g_*\colon MU^n_\delta(p)(X) \to MU^{n+2d}_\delta(p + d)(Y).$$

Proof. This follows from applying Theorem 7.8 with $\kappa(N_g) \in K^0(0)(X;\mathcal{V}_*)$ as the lift of $[g^*TY] - TX \in K^0(X)$. ■

Remark 7.14. If the map g is projective, we can use the Euler sequence to get a holomorphic representative of the stable normal bundle of g. However, as we have seen, this is not necessary for our theory.

Remark 7.15. Sometimes there can be other ways of getting a lift of the virtual normal bundle to $K^0(0)(X)$. In particular, we have the following construction: When X and Y are Hermitian manifolds, one can consider the virtual Hermitian normal bundle $\mathcal{N}_g = [g^*T_Y] - [T_X] \in \hat{K}^0(X;\mathcal{V}_*)$. Here $\hat{K}^0(X;\mathcal{V}_*)$ is a group of virtual Hermitian holomorphic vector bundles, defined similarly to the group of virtual Hermitian vector bundles of Gillet and Soulé in [20], but with the multiplicative sequence K replacing the Chern character. That is, the generators are triples $[E, h^E, \eta]$ where E is a holomorphic vector bundle with Hermitian metric h^E, and η is a real form in $\mathcal{A}^{-1,-1}(X;\mathcal{V}_*)/(\mathrm{Im}\,(\partial) + \mathrm{Im}\,(\overline{\partial}))$. We impose for each short exact sequence of Hermitian vector bundles

$$\mathcal{E} = \Big(0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0 \Big)$$

the relations

$$(E_1, h^{E_1}, \eta_1) + (E_3, h^{E_3}, \eta_3) \sim (E_2, h^{E_2}, \eta_2 + \widetilde{K}_{BC}(\mathcal{E}))$$

where $\widetilde{K}_{BC}(\mathcal{E}) \in \mathcal{A}^{-1,-1}(X;\mathcal{V}_*)/(\mathrm{Im}\,(\partial) + \mathrm{Im}\,(\overline{\partial}))$ is the Bott–Chern double transgression form. It satisfies

$$dd^c \widetilde{K}_{BC}(\mathcal{E}) = K(\nabla_1) \wedge K(\nabla_3) - K(\nabla_2)$$

where ∇_i is the Chern connection of (E_i, h^{E_i}), and $d^c = \frac{1}{2}(\partial - \overline{\partial})$. We consider the diagram

$$\begin{tikzcd} \hat{K}^0(X;\mathcal{V}_*) \arrow[r] \arrow[d, "d^c"'] & K^0_{\mathrm{hol}}(X;\mathcal{V}_*) \arrow[dl] \\ \check{K}^0(0)(X) & \end{tikzcd} \tag{7.2}$$

which we warn in general does not commute. Here the map d^c takes (E, h^E, η) to $(E, \nabla_{h^E}, d^c\eta)$ for ∇_{h^E} the Chern connection of (E, h^E). The map d^c is well defined because

$$\partial \widetilde{K}_{BC}(\mathcal{E}) = \widetilde{K}_{CS}(\mathcal{E}) \in \mathcal{A}^{-1}(X;\mathcal{V}_*)/\mathrm{Im}\,(d)$$

where we mean the Chern–Simons transgression form with respect to the Chern-connections. Then $\mathcal{N}_g \in \hat{K}^0(X;\mathcal{V}_*)$ is a lift of the virtual holomorphic normal bundle $[g^*TY] - [TX]$ in $K^0_{\mathrm{hol}}(X)$. Let us analyse the commutativity of (7.2). The question is whether $[E, \nabla, d^c\eta] = [E, \nabla, 0]$. I.e. whether $[a(d^c\eta)] = 0$. This is the case if $d^c\eta \in \widetilde{F}^0\mathcal{A}^{-1}(X;\mathcal{V}_*)$. Modulo $F^0\mathcal{A}^{-1}(X;\mathcal{V}_*)$, we have $d^c\eta = \frac{1}{2}\overline{\partial}\eta$, so we are left with asking if $\overline{\partial}\eta$ is exact. This is the case for manifolds satisfying the $\partial\overline{\partial}$-lemma, such as compact Kähler manifolds, but it is not generally true, essentially because the Fröhlicher spectral sequence can be arbitrarily non-degenerate.

Recall that compact Kähler manifolds are Hermitian manifolds for which the Chern connection on the holomorphic tangent bundle agrees with the Levi-Civita connection of the underlying Riemannian manifold. We think of the difference of the two lifts of N_g from $K^0(X)$ to $K^0(0)(X)$ as a measure for the failure of the real geometry of g to coincide with its complex geometry. It is then maybe not surprising that, in the situation of a map between Hermitian manifolds, there are two pushforward maps: one induced by the complex geometry of the holomorphic map, and one induced by the real geometry of the Hermitian normal bundle, and that the two agree for manifolds satisfying the $\partial\overline{\partial}$-Lemma such as for example Kähler manifolds.

7.5 Functoriality

Let $g_i \colon X_i \to X_{i+1}$ be a proper holomorphic map of complex codimension d_i, and $MU_{\mathcal{D}}$ orientation $\overline{N}_{g_i} \in K^0(p_i)(X_i)$ for $i = 0, 1$. We define the composed $MU_{\mathcal{D}}$ orientation of $g_2 \circ g_1$ by

$$\overline{N}_{g_2 \circ g_1} := \overline{N}_{g_1} + g_1^* \overline{N}_{g_2} \in \check{K}^0(p_1 + p_2)(X_1),$$

using the addition defined in equation (7.1).

Proposition 7.16. *Let $g_2 \circ g_1$ be endowed with the composed $MU_{\mathcal{D}}$-orientation. Then we have $(g_2)_* \circ (g_1)_* = (g_2 \circ g_1)_*$ as maps*

$$MU_\delta^n(p')(X_1) \to MU_\delta^{n+2d_1+2d_2}(p' + p_1 + p_2 + d_1 + d_2)(X_3).$$

Proof. Represent the $MU_{\mathcal{D}}$-orientations in the form $\check{g}_i = (g_i, N_i, \nabla_i, \sigma_i)$. As before we let $\widetilde{g}_i$ denote the underlying geometric cycle obtained by dropping σ_i. Then the data of the composed $MU_{\mathcal{D}}$-orientation is contained in $\check{g}_{12} = (g_{12}, N_{12}, \nabla_{12}, \sigma_{12})$ defined by

$$\check{g}_{12} := (g_2 \circ g_1,\ N_2 \oplus g_1^* N_1,\ \nabla_2 \oplus g_1^* \nabla_2,\ g_1^* K(\overline{N}_{g_2}) \wedge \sigma_1 + g_1^* \sigma_2 \wedge K(\nabla_{g_1})).$$

Observe that the underlying geometric cycle of $\check{g}_{12}$, which we denote by $\widetilde{g}_{12}$, is the composed geometric cycle

$$\widetilde{g}_{12} = \widetilde{g}_2 \circ \widetilde{g}_1.$$

Let $\gamma = (\widetilde{f}, h) \in ZMU_\delta^n(p')(X_1)$. Then the underlying geometric cycles of

$$(\check{g}_{12})_* \gamma = (\widetilde{g}_{12} \circ \widetilde{f},\ h_{12})$$

and

$$(\check{g}_2)_* \circ (\check{g}_1)_* \gamma = (\widetilde{g}_2 \circ \widetilde{g}_1 \circ \widetilde{f},\ h')$$

agree. Hence we must show that $h_{12} = h'$, at least modulo $\widetilde{F}^p\mathcal{D}^*(X_3; \mathcal{V}_*)$. We use the projection formula (2.11) to compute

$$
\begin{aligned}
h' &= (g_2)_* \Big[K(\overline{N}_{g_2}) \wedge (g_1)_* \big(K(\overline{N}_{g_1}) \wedge h + \sigma_1 \wedge f_* K(\nabla_f)\big) \\
&\qquad\qquad\qquad\qquad + \sigma_2 \wedge (g_1)_* f_* \left(K(\nabla_f) \wedge f^* K(\nabla_{g_1})\right)\Big] \\
&= (g_2 \circ g_1)_* \Big[g_1^* K(\overline{N}_{g_2}) \wedge K(\overline{N}_{g_2}) \wedge h + g_1^* K(\overline{N}_{g_2}) \wedge \sigma_1 \wedge f_* K(\nabla_f) \\
&\qquad\qquad\qquad\qquad + g_1^* \sigma_2 \wedge K(\nabla_{g_1}) \wedge f_* K(\nabla_f)\Big] \\
&= (g_2 \circ g_1)_* \left[K\left(\overline{N}_{g_2 \circ g_1}\right) \wedge h + \left(g_1^* K(\overline{N}_{g_2}) \wedge \sigma_1 + g_1^* \sigma_2 \wedge K(\nabla_{g_1})\right) \wedge f_* K(\nabla_f)\right] \\
&= (g_2 \circ g_1)_* \left[K\left(\overline{N}_{g_2 \circ g_1}\right) \wedge h + \sigma_{12} \wedge f_* K(\nabla_f)\right] \\
&= h_{12}.
\end{aligned}
$$

■

Since the map $K^0_{hol}(X) \to \check{K}^0(0)(X)$ is a homomorphism, it is clear that the canonical $MU_{\mathcal{D}}$-orientation of $g_2 \circ g_1$ is the $MU_{\mathcal{D}}$-orientation composed of the canonical $MU_{\mathcal{D}}$-orientations of g_2 and g_1.

Chapter 8

An Abel–Jacobi map for algebraic cobordism

8.1 From algebraic to Hodge filtered cobordism

In this section, we let X denote an algebraic complex manifold. The functor

$$MU^{2*}(*)(-)\colon X \mapsto \bigoplus_p MU^{2p}(p)(X(\mathbb{C}))$$

is an oriented Borel Moore cohomology theory on the category of quasi-projective smooth complex varieties in the sense of [39]. Here $X(\mathbb{C})$ denotes the space of complex points of X with the analytic topology. This is complex manifold and we will from now on just write X instead of $X(\mathbb{C})$. In [39] it is shown that $\Omega^*_{\mathrm{alg}}(X)$ is the universal oriented Borel Moore cohomology theory. Hence there is a canonical map $\Omega^*_{\mathrm{alg}}(X) \to MU^{2*}(*)(X)$. We recall how this canonical map is defined.

Let $\mathcal{M}^*(X)$ be the free abelian group generated by isomorphism classes of projective morphisms $Z \to X$ graded by codimension. Oriented Borel Moore cohomology theories have pushforward maps along projective morphisms. In particular, there is a map $\mathcal{M}(X) \to \Omega^*_{\mathrm{alg}}(X)$ given by $[f\colon Z \to X] \mapsto f_*[1_Z]$. This is a surjection, with kernel the double point relations of [40]. More generally, if E^* is any oriented Borel Moore cohomology theory, then the canonical map $\mathcal{M}^*(X) \to E^*(X)$ factors through $\Omega^*_{\mathrm{alg}}(X)$. That is, the map $\mathcal{M}^*(X) \to MU^{2*}(*)(X)$ given by $[f\colon Z \to X] \mapsto f_*[1_Z]$ factors through a map $\Omega^*_{\mathrm{alg}}(X) \to MU^{2*}(*)(X)$.

This map can be described using our geometric pushforward. For this we again use the currential geometric Hodge filtered cobordism group $MU^{2p}_{\delta}(p)(X)$. For an algebraic cycle $[f\colon Z \to X]$, the image in $MU^{2p}_{\delta}(p)(X)$ is

$$f_*[1_Z] = [f, N_f, \nabla_f, f_*\sigma_f] = [\widetilde{f}, f_*\sigma_f] \tag{8.1}$$

where $[N_f, \nabla_f, \sigma_f] = [f^*TX, f^*D_X, 0] - [TZ, D_Z, 0]$ in the group of $MU_{\mathcal{D}}$-orientations $\check{K}^0(0)(Z)$ for Bott connections D_X and D_Z. That is, ∇_f is any connection on N_f, and $\sigma_f = \widetilde{K}_{CS}(\nabla_f \oplus D_Z, D_X \oplus d)$.

We recall the long exact sequence from Theorem 4.38.

$$\cdots \longrightarrow H^{n-1}\left(X; \tfrac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \xrightarrow{\ a\ } MU^n(p)(X) \xrightarrow{\ I\ }$$

$$MU^n(X) \xrightarrow{\ \phi\ } H^n\left(X; \tfrac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \longrightarrow \cdots$$

For X compact Kähler we split of the fundamental short exact sequence. Let $\mathrm{Hdg}^{2p}_{MU}(X) = I\left(MU^{2p}(p)(X)\right)$. Moreover, we write

$$J^{2p-1}_{MU}(X) = H^{2p-1}\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) / \phi\left(MU^{2p-1}\right).$$

Then we get a short exact sequence

$$0 \longrightarrow J^{2p-1}_{MU}(X) \longrightarrow MU^{2p}(p)(X) \longrightarrow \mathrm{Hdg}^{2p}_{MU}(X) \longrightarrow 0\,. \tag{8.2}$$

Hence, if X is a projective algebraic complex manifold, there is an Abel–Jacobi map, the map induced on the kernel in this commutative diagram:

$$\begin{tikzcd}
\mathrm{Ker}\,(\chi)(X) \arrow[r] \arrow[d, "AJ"'] & \Omega^*_{\mathrm{alg}}(X) \arrow[d] \arrow[dr, "\chi"] & \\
J^{2p-1}_{MU}(X) \arrow[r] & MU^{2p}(p)(X) \arrow[r] & \mathrm{Hdg}^{2p}_{MU}(X).
\end{tikzcd} \tag{8.3}$$

We will now give two descriptions of AJ. First, following the proof of Theorem 4.38. Secondly, we follow Voisin's account of the classical Abel–Jacobi map in [51], which can be understood without knowledge of $MU^{2p}(p)$, just like Griffiths' Abel–Jacobi map predates Deligne cohomology.

8.2 The Abel–Jacobi map - first description

For an algebraic cycle $[f\colon Z \to X]$, recall that its image in $MU^{2p}(p)(X)$ is described in (8.1). Now suppose $\chi([f]) = 0$. Then there is a bordism datum $b\colon W \to \mathbb{R} \times X$ such that $\partial b = f$. We can extend the connection ∇_f to get a connection ∇_b on N_b, and obtain a geometric cobordism datum $\widetilde{b}$. Then

$$\gamma - (\partial\widetilde{b}, \psi(\widetilde{b})) = (0, f_*\sigma_f - \psi(\widetilde{b}))$$

so that $[\gamma] = a[\sigma_f - \psi(\widetilde{b})]$. The class of $f_*\sigma_f - \psi(\widetilde{b})$ in

$$H^{2p-1}\left(X; \frac{\mathcal{D}^*}{F^p}(\mathcal{V}_*)\right) \simeq \frac{H^{2p-1}(X;\mathcal{V}_*)}{F^pH^{2p-1}(X;\mathcal{V}_*)}$$

depends on the choice of $\widetilde{b}$. This ambiguity is explained precisely by (8.2), and we get a well defined class

$$AJ(f) = [f_*\sigma_f - \psi(\widetilde{b})] \in \frac{H^{2p-1}(X;\mathcal{V}_*)}{F^pH^{2p-1}(X;\mathcal{V}_*) + \phi(MU^{2p}(X))} = J^{2p-1}_{MU}(X). \tag{8.4}$$

8.3 The Abel–Jacobi map - second description

Now we give the second description of AJ. This requires setting up Proposition 8.3. Write $\mathcal{V}'_*$ for the $\mathbb{C}$-dual graded algebra with homogeneous components

$$\mathcal{V}'_j = (\mathcal{V}_{-j})' = \mathrm{Hom}_{\mathbb{C}}(V_{-j}, \mathbb{C}).$$

Then the canonical pairing

$$ev\colon \mathcal{V}'_* \otimes \mathcal{V}_* \to \mathbb{C}$$

has degree 0, if $\mathbb{C}$ is interpreted as a graded vector space concentrated in degree 0. Let $n = \dim_{\mathbb{C}} X$. There is a pairing

$$\begin{aligned} H^k(X;\mathcal{V}_*) \times H^{2n-k}(X;\mathcal{V}'_*) &\to \mathbb{C} \\ \langle [\omega],[\eta]\rangle &= ev\left(\int_X \omega \wedge \eta\right), \end{aligned} \tag{8.5}$$

where $\omega \wedge \eta$ is interpreted as a $\mathcal{V}_* \otimes \mathcal{V}'_*$-valued form.

Lemma 8.1. *The pairing* (8.5) *is perfect.*

Proof. This is essentially by Poincaré duality. Let us expand that. We note that $H^m(X;\mathcal{V}_*)$ is a finite dimensional vector-space for each m, since X is finite dimensional and each $\mathcal{V}_j$ is finite dimensional. Hence it suffices to show (8.5) to be non-degenerate. By the symmetry of the situation, it suffices to show that the restriction of the map $H^k(X;\mathcal{V}_*) \to \left(H^{2n-k}(X;\mathcal{V}'_*)\right)'$ to each $H^{k+2j}(X;\mathcal{V}_{2j})$ is injective. Poincaré duality states that the pairing

$$H^{k+2j}(X;\mathbb{C}) \otimes H^{2n-k-2j}(X;\mathbb{C}) \to \mathbb{C}$$

is perfect. Given $b_j \in \mathcal{V}_j$, let $\beta_j \in \mathcal{V}'_{-j}$ be such that $\beta_j(b_j) \neq 0$. Then the restriction of (8.5) to $H^{k+2j}(X;\mathbb{C}\cdot b_j) \otimes H^{2n-k-2j}(X;\mathbb{C}\cdot\beta_j)$ is perfect. Hence (8.5) is perfect. ∎

So we may identify $H^{2p-1}(X;\mathcal{V}_*)$ with $\left(H^{2n-2p+1}(X;\mathcal{V}'_*)\right)'$.

Lemma 8.2. *Under this identification,*

$$F^pH^{2p-1}(X;\mathcal{V}_*) := \bigoplus_j F^{p+j}H^{2p-1+2j}(X;\mathcal{V}_{2j})$$

is identified with

$$\left(F^{n-p+1}H^{2n-2p+1}(X;\mathcal{V}'_*)\right)^{\perp}.$$

Proof. To see this, note first that the restriction of the pairing to

$$F^pH^{2p-1}(X;\mathcal{V}_*) \otimes F^{n-p+1}H^{2n-2p+1}(X;\mathcal{V}'_*)$$

is 0 because, for each j, the component in

$$H^{2p-1+2j}(X;\mathcal{V}_{2j}) \otimes H^{2n-2p+1-2j}(X;\mathcal{V}'_{-2j})$$

lie in the $(n+1)$-th filtration stage. The lemma now follows for dimensional reasons from Hodge symmetry and Serre-duality. ∎

This immediately implies:

Proposition 8.3. *We have isomorphisms*

$$\frac{H^{2p-1}(X;\mathcal{V}_*)}{F^pH^{2p-1}(X;\mathcal{V}_*)} \simeq \left(F^{n-p+1}H^{2n-2p+1}(X;\mathcal{V}'_*)\right)'$$

and

$$J^{2p-1}_{MU}(X) \simeq \frac{\left(F^{n-p+1}H^{2n-2p+1}(X;\mathcal{V}'_*)\right)'}{\phi(MU^{2p-1}(X))}.$$

Here ϕ is the map $\phi\colon MU^{2p-1}(X) \to H^{2p-1}(X;\mathcal{V}_*)$ followed by the identification following from (8.5). To be explicit,

$$[f\colon Z \to X] \in MU^{2p-1}(X)$$

is mapped to the element $\phi(f)$ in $\left(H^{2n-2p+1}(X;\mathcal{V}'_*)\right)'$ defined by

$$\phi(f)([\omega]) = ev\left(\int_Z f^*\omega \wedge K(N_f)\right). \tag{8.6}$$

We note that since Z is closed it is clear from Stokes' theorem that this pairing is independent of the choice of representatives of $[\omega]$ and $K(N_f)$.

Let us now trace the element

$$AJ(f) = f_*\sigma_f - \psi(\widetilde{b}) \in J^{2p-1}_{MU}(X)$$

through the isomorphism

$$J^{2p-1}_{MU}(X) \simeq \frac{\left(F^{n-p+1}H^{2n-2p+1}(X;\mathcal{V}'_*)\right)'}{\phi(MU^{2p-1}(X))}.$$

We need some notation. Let $\widetilde{b} = (b, N_b, \nabla_b)$ where $b = (a_b, f_b)\colon W \to \mathbb{R}\times X$. Let $W_{[0,1]} = a_b^{-1}([0,1])$, and $w = f_b|_{W_{[0,1]}}$. Since the codimension of f_b is odd, $\psi(\widetilde{b}) = -w_*K(\nabla_b)$. Now let ω be any form in $F^{n-p+1}\mathcal{A}^{2n-2p+1}(X;\mathcal{V}'_*)$. We get:

$$\begin{aligned}\int_X (f_*\sigma_f - \psi(\widetilde{b}))\wedge\omega &= \int_X f_*\sigma_f\wedge\omega - \int_X \psi(\widetilde{b})\wedge\omega\\ &= \int_Z \sigma_f\wedge f^*\omega + \int_{W_{[0,1]}} K(\nabla_b)\wedge w^*\omega\\ &= \int_{W_{[0,1]}} (d\sigma'_f + K(\nabla_b))\wedge w^*\omega\end{aligned}$$

where σ'_f is any extension of σ_f to W. For example first extend σ_f to a collar neighborhood, and then extend to all of $W_{[0,1]}$ using a partition of unity; we are essentially reproving that $\mathcal{A}^*$ is a soft sheaf on X. We conclude:

Theorem 8.4. *Under the isomorphism of Proposition 8.3, $AJ(f)$ is represented by the functional*

$$\phi_F(w) = \left([\omega]\mapsto ev\int_{W_{[0,1]}} K(N_b)\wedge\omega\right) \in \left(F^{n-p+1}H^{2n-2p+1}(X;\mathcal{V}'_*)\right)'.$$

Remark 8.5. We remark that the formula for $\phi_F(w)$ does not define an element of $H^{2n-2p+1}(X;\mathcal{V}'_*)'$. This is because if we tried to make $\phi_F(w)$ act on $[\omega] \in H^{2n-2p+1}(X;\mathcal{V}'_*)$ by the same formula, the result would depend both on ω, and the form chosen to represent N_b. As an element of

$$\left(F^{n-p+1}H^{2n-2p+1}(X;\mathcal{V}'_*)\right)'$$

independence of these choices was clear in our story, since we already knew $AJ(f)$ to be well defined. It can also be established directly using Stokes' theorem, that $K(N_b)|_Z = K(N_f) \in H^{0,0}(Z;\mathcal{V}_*)$, and the fact that

$$F^{n-p+1}H^{2n-2p}(Z;\mathcal{V}_* \otimes \mathcal{V}'_*) = 0.$$

Chapter 9

Hodge filtered Thom morphism

In this section we analyse the map from Hodge filtered cobordism to Deligne cohomology which is induced by the Thom morphism $MU \to H\mathbb{Z}$. In order to define a map on the level of cycles we will first construct a cycle model for Deligne cohomology. Our construction is inspired by that of Gillet–Soulé in [21], but our construction is more elementary in that it avoids the use of geometric measure theory. We are also inspired by [25]. We strive to provide complete proofs. Then we study the induced map on intermediate Jacobians and describe the kernel on the level of geometric cycles.

9.1 Cycle model for Deligne cohomology

Let $\overline{C}^k$ denote the presheaf of smooth relative chains

$$U \mapsto \overline{C}^k(U) := C^{\mathrm{diff}}_{\dim X - k}(X, X \backslash \overline{U}; \mathbb{Z}(p)).$$

We recall that this group is defined as the quotient

$$C^{\mathrm{diff}}_{\dim X - k}(X, X \backslash \overline{U}; \mathbb{Z}(p)) = \frac{C^{\mathrm{diff}}_{\dim X - k}(X; \mathbb{Z}(p))}{C^{\mathrm{diff}}_{\dim X - k}(X \backslash \overline{U}; \mathbb{Z}(p))}.$$

The restriction maps of $\overline{C}^k$ are the induced by quotient out the further appropriate extra chains, i.e., for $V \subset U$, we have the natural map

$$\overline{C}^k(U) = C^{\mathrm{diff}}_{\dim X - k}(X, X \backslash \overline{U}; \mathbb{Z}(p)) \to C^{\mathrm{diff}}_{\dim X - k}(X, X \backslash \overline{V}; \mathbb{Z}(p)) = \overline{C}^k(V).$$

The presheaf $\overline{C}^k$ is very close to being a sheaf; it satisfies the sheaf condition for coverings of X, but not for general collections of open subsets of X. Let C^k be the sheafification of $\overline{C}^k$. The sheaf C^k is not quite fine, but it is homotopically fine, meaning that its endomorphism sheaf admits a homotopy partition of unity. We refer to [6, p. 172], from which we also recall the implication that $H^*(H^j(\overline{C}^*(U))) = 0$ for $j > 0$. Hence the hypercohomology spectral sequence degenerates on the E_2-page, past which only the row $H^0(C^*(U))$ survives. On stalks the sheaf C^k coincides with the presheaf $\overline{C}^k$. Let U be a small contractible open subset of X. By excision we have

$$H^k(\overline{C}^*(U)) = H_{\dim X - k}(\mathbb{R}^{\dim X},\ \mathbb{R}^{\dim X} \backslash D;\ \mathbb{Z}(p)),$$

for D the closed unit disc. Hence we get

$$H^j(\overline{C}^*(U)) = \begin{cases} 0 & j > 0 \\ \mathbb{Z}(p) & j = 0. \end{cases}$$

This proves the following result:

Lemma 9.1. *The complex C^* is an acyclic resolution of the constant sheaf $\mathbb{Z}(p)$, as sheaves on X.* ∎

From [6, p. 32] we also have the following key result:

Lemma 9.2. *The canonical map $\overline{C}^* \to C^*$ induces an isomorphism of cohomology groups on global sections $H^k(\overline{C}^*(X)) = H^k(C^*(X))$.* ∎

In other words, the sheaf cohomology $H^k(X; \mathbb{Z}(p))$ can be computed as the cohomology of the complex $\overline{C}^*(X)$. Recall the operator $\mathbf{w}\colon \mathcal{D}^* \to \mathcal{D}^*$ which on $\mathcal{D}^k$ acts by $(-1)^k$. Now we consider the map

$$T\colon \overline{C}^*(X) \to \mathcal{D}^*(X)$$

given by $c \mapsto \mathbf{w} \int_c$. This is a chain map. Indeed by Stokes' theorem, $\int_{\partial c} = b \int_c$, where b is the operator on currents given by $bT(\sigma) = T(d\sigma)$. Since $d = \mathbf{w}b$ by definition, see (2.6), we have $T(\partial c) = \mathbf{w}bT(c) = dT(c)$. Let $\mathcal{D}^*_{\mathbb{Z}}(X)$ be the image of T in $\mathcal{D}^*(X)$.

Proposition 9.3. *The cochain map $T\colon \overline{C}^*(X) \to \mathcal{D}^*_{\mathbb{Z}}(X)$ induces an isomorphism on cohomology.*

Proof. By Whitehead's triangulation theorem, we may pick a smooth triangulation of X, i.e., a set $S = \{f_i\colon \Delta^{k_i} \to X\}$ such that each f_i is a continuous embedding which extend to a smooth mapping of a neighborhood of $\Delta^k \subset \mathbb{R}^k$, and each $x \in X$ is in the interior of a unique cell $S_i = \operatorname{Im}(f_i)$. It is well known that the inclusion of cellular chains $C_*(S; \mathbb{Z}(p)) \to C_*(X; \mathbb{Z}(p))$ is a quasi-isomorphism. Hence it suffices to show that T restricts to a quasi-isomorphism on the cellular chains of S. Since each point $x \in X$ is contained in the interior of a unique cell of S, we can show that T is injective on cellular chains as follows. We can construct for each i a form $\omega_i \in \mathcal{A}^{k_i}(X)$ such that $\int_{\Delta^{k_i}} f_i^* \omega_i \neq 0$, and such that the only k_i-cell intersecting the support of ω_i is S_i. Suppose $T(c) = 0$ for $c = \sum a_i f_i$. Then $T(c)(\omega_i) = a_i T(f_i)(\omega_i)$ is a nonzero multiple of a_i, and we get $a_i = 0$.

To see that the map induced by T from cellular homology is injective, we first note that the inclusion of cellular chains into singular chains is a chain homotopy equivalence. This is because it is a quasi-isomorphism between complexes of projective modules. In particular, we may choose a retraction r onto the cellular chains. Let now c be a cellular cycle with $T(c) = dT(\alpha)$ for α an arbitrary integral chain $\alpha \in C^*(X)$. Then

$$\begin{gathered} T(c) = dT(\alpha) = T(\partial\alpha) \implies \\ T(c) = T(r(c)) = T(r(\partial\alpha)) = T(\partial r(\alpha)) \end{gathered}$$

and since T is injective on cellular chains, we get $c = \partial r(\alpha)$. Hence c represents 0 in cellular homology, so the map induced by T on cellular homology is injective.

It remains to see that T restricted to cellular chains is surjective on cohomology. So let $\sum_i T(a_i \cdot f_i)$ represent a class in $H^k(X; \mathcal{D}_{\mathbb{Z}})$. Then

$$0 = d \sum_i T(a_i \cdot f_i) = \sum_i T(a_i \cdot \partial f_i),$$

and since T is injective on cellular chains, we see that $\sum_i a_i f_i$ is a cellular cycle. This completes the proof. ∎

We are now ready to give our presentation of Deligne cohomology. Let

$$i_F \colon F^p \mathcal{A}^* \to \mathcal{D}^*$$

be the canonical inclusion of sheaves, and $i_c \colon \mathcal{D}^*_{\mathbb{Z}}(X) \to \mathcal{D}^*(X)$ the inclusion. We will show that the following cochain complex computes the Deligne cohomology of X:

$$C^*_{\mathcal{D}}(p)(X) = \operatorname{cone}\left(\mathcal{D}^*_{\mathbb{Z}}(X) \oplus F^p\mathcal{A}^*(X) \xrightarrow{i_c - i_F} \mathcal{D}^*(X) \right).$$

In degree k we have the group

$$C^k_{\mathcal{D}}(p)(X) = \mathcal{D}^k_{\mathbb{Z}}(X) \oplus F^p\mathcal{A}^k(X) \oplus \mathcal{D}^{k-1}(X).$$

The differential is defined by

$$d(T, \omega, h) = (dT,\ d\omega,\ i_c(T) - dh + i_F(\omega)).$$

Theorem 9.4. *The cochain complex $C^*_{\mathcal{D}}(p)(X)$ computes Deligne cohomology $H^*_{\mathcal{D}}(X; \mathbb{Z}(p))$.*

We will use multicomplexes, which are more flexible than bicomplexes. We recall from [3] that a multicomplex of abelian groups consist of the data of a bigraded abelian group, $E^{s,t}$, and differentials $d_r^{s,t} \colon E^{s,t} \to E^{s+r,t-r+1}$ such that

$$\sum_{i+j=k} d_j^{s+i,t-i+1} \circ d_i^{s,t} = 0 \colon E^{s,t} \to E^{s+k,t-k+2}.$$

One can consider multicomplexes of objects in any abelian category. We are considering here multicomplexes of sheaves.

Proof. We will construct a series of quasi-isomorphisms of complexes of sheaves

$$\mathbb{Z}_{\mathcal{D}}(p) \simeq C'^*_{\mathcal{D}}(p) \simeq \operatorname{Tot}(M)$$

and a quasi-isomorphism of complexes of abelian groups $\operatorname{Tot}(M)(X) \to C^*_{\mathcal{D}}(X)$. Consider the multicomplex of sheaves on X:

$$M^{s,t} = \begin{cases} C^t & s = 0 \\ \mathcal{D}^{s-1,t} & 0 < s < p \\ F^p\mathcal{A}^{s,t} \oplus \mathcal{D}^{s-1,t} & p \leqslant i \end{cases}$$

To define the differentials, let $\Pi^{s,k-s}\colon \mathcal{D}^k \to \mathcal{D}^{s,k-s}$ be the projection. For $s > 0$, there is only d_0 and d_1. The differentials of M are:

$$d_0^{s,t} = \begin{cases} d\colon C^t \to C^{t+1} & s = 0 \\ -\overline{\partial}\colon \mathcal{D}^{s-1,t} \to \mathcal{D}^{s-1,t+1} & 0 < s < p \\ (\overline{\partial}, i_F - \overline{\partial})\colon F^p\mathcal{A}^{s,t} \oplus \mathcal{D}^{s-1,t} \to F^p\mathcal{A}^{s,t+1} \oplus \mathcal{D}^{s-1,t+1} & s \geqslant p \end{cases}$$

$$d_1^{s,t} = \begin{cases} \Pi^{0,t} \circ i_c\colon C^t \to \mathcal{D}^{0,t} & s = 0 \\ -\partial\colon \mathcal{D}^{s-1,t} \to \mathcal{D}^{s,t} & 0 < s < p \\ (\partial, i_F - \partial)\colon F^p\mathcal{A}^{s,t} \oplus \mathcal{D}^{s-1,t} \to F^p\mathcal{A}^{s+1,t} \oplus \mathcal{D}^{s,t} & s \geqslant p. \end{cases}$$

$$d_r^{0,t} = \Pi^{r,t-r} \circ i_c\colon C^t \to \mathcal{D}^{r,t-r}.$$

We can describe the total complex of M as the complex of sheaves

$$\mathrm{Tot}^*(M) = \mathrm{cone}\left(C^* \oplus F^p\mathcal{A}^* \xrightarrow{i_F - i_c} \mathcal{D}^* \right).$$

There is therefore a natural map $\mathrm{Tot}^*(M(X)) \to C^*_{\mathcal{D}}(p)(X)$ given by

$$\mathrm{Tot}^*(M(X)) \ni (c,\ \omega,\ h) \mapsto (aT(c),\ \omega,\ h) \in C^*_{\mathcal{D}}(p)(X)$$

where we write aT for the map of the sheafification induced by T. This map of complexes induces an isomorphism on cohomology, since each of the maps

$$id\colon F^p\mathcal{A}^*(X) \to F^p\mathcal{A}^*(X), \quad T\colon \overline{C}^*(X) \to \mathcal{D}^*_{\mathbb{Z}}(X) \quad \text{and} \quad id\colon \mathcal{D}^*(X) \to \mathcal{D}^*(X)$$

are quasi-isomorphisms. We define yet another complex of sheaves $C'^*_{\mathcal{D}}(p) := \mathrm{cone}(inc - (2\pi i)^p)$, where inc and $(2\pi i)^p$ are the maps of complexes in diagram (3.1). Concretely we have

$$C'^*_{\mathcal{D}}(p) = \left(\mathbb{Z}(p) \longrightarrow \Omega^0 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-2} \xrightarrow{(0,d)} \Omega^p \oplus \Omega^{p-1} \xrightarrow{\delta_p} \cdots \right)$$

with $\delta_p(\omega, \tau) = (d\omega, \omega - d\tau)$, and δ_i defined similarly for $i > p$. There is a map $f\colon \mathbb{Z}_{\mathcal{D}}(p)(X) \to C'^*_{\mathcal{D}}(p)(X)$ given by

$$\begin{tikzcd} \mathbb{Z}(p) \arrow[r] \arrow[d, "\mathrm{id}"] & \Omega^0 \arrow[r, "d"] \arrow[d, "\mathrm{id}"] & \Omega^1 \arrow[r, "d"] \arrow[d, "\mathrm{id}"] & \cdots \arrow[r, "d"] & \Omega^{p-2} \arrow[r, "d"] \arrow[d, "\mathrm{id}"] & \Omega^{p-1} \arrow[r] \arrow[d, "\alpha"] & 0 \\ \mathbb{Z}(p) \arrow[r] & \Omega^0 \arrow[r, "d"] & \Omega^1 \arrow[r] & \cdots \arrow[r, "d"] & \Omega^{p-2} \arrow[r, "{(0,d)}"] & \Omega^p \oplus \Omega^{p-1} \arrow[r, "\delta_p"] & \cdots \end{tikzcd}$$

with $\alpha(\omega) = (d\omega, \omega)$. This is a quasi-isomorphism of complexes of sheaves; this is clear in degrees $< p$, and in degrees $> p$ it follows since $C'^*_{\mathcal{D}}(p)$ is exact there. In degree p we must show that f induces an isomorphism on cohomology of stalks. Let U be a polydisc. Then

$$H^p_{\mathcal{D}}(U, \mathbb{Z}(p)) = \frac{\Omega^{p-1}(U)}{\mathrm{Im}\, d},$$

and
$$H^p\left(U, C'^{*}_{\mathcal{D}}(p)\right) = \frac{\{(\omega,\tau)\in\Omega^p(U)\oplus\Omega^{p-1}(U)\ :\ d\tau=\omega\}}{\operatorname{Im}(0,d)\}}.$$

It is clear that the map induced by f, which can be described as $[\tau]\mapsto[d\tau,\tau]$ is an isomorphism. Hence f is a quasi isomorphism as claimed. Next there is a natural map $C'^{*}_{\mathcal{D}}(p)\to M$ given by

$$\begin{tikzcd}
\mathbb{Z}(p) \arrow[r] \arrow[d, "\epsilon"] & \Omega^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^{p-2} \arrow[r] \arrow[d] & \Omega^p\oplus\Omega^{p-1} \arrow[r] \arrow[d] & \cdots \\
C^0 \arrow[r] & \mathcal{D}^{0,0} \arrow[r] & \cdots \arrow[r] & \mathcal{D}^{p-2,0} \arrow[r] & \mathcal{A}^p\oplus\mathcal{D}^{p-1} \arrow[r] & \cdots
\end{tikzcd}$$

where ϵ is the quasi-isomorphism $\mathbb{Z}(p)\to C^*$. The column $M^{i,*}$ is a resolution of the sheaf $C'^{i}_{\mathcal{D}}(p)$ by Lemma 9.1 and Theorem 2.16. Hence the natural map $C'^{*}_{\mathcal{D}}(p)\to M$ is a quasi-isomorphism. This concludes the proof. ■

9.2 The Hodge filtered Thom morphism

Let P_0 be the map
$$\mathcal{D}^*(X;\mathcal{V}_*)\to\mathcal{D}^*(X;\mathbb{C})$$
induced by the map on coefficients $\mathcal{V}_*=MU_*\otimes\mathbb{C}\to\mathbb{C}$ of the additive formal group law over $\mathbb{C}$. Concretely, this is the projection onto the degree 0 component of $MU_*\otimes\mathbb{C}$, which is $\mathbb{C}\cdot[\mathrm{pt}]$, followed by $z\cdot[\mathrm{pt}]\mapsto z$. Then P_0 is a chain map, and it preserves the Hodge filtration. Now we are ready to define our Hodge filtered Thom morphism on the level of cycles:

$$\begin{aligned}\tau_{\mathbb{Z}}\colon ZMU^n(p)(X)&\to C^n_{\mathcal{D}}(p)(X),\\ \gamma=(\widetilde{f},\ h)&\mapsto (f_*1,\ P_0(R(\gamma)),\ P_0(h)).\end{aligned}$$

Remark 9.5. If we choose a smooth triangulation of Z, then summing up the top-cells we get a smooth singular cycle c_Z representing the fundamental class $[Z]\in H_{\dim Z}(Z;\mathbb{Z})$. We have $T(c_Z)=1\in\mathcal{D}^0(Z)$, and so

$$f_*1=f_*T(c_Z)=T(f_*c_Z)\in\mathcal{D}^*_{\mathbb{Z}}(X).$$

The advantage of using $\mathcal{D}_{\mathbb{Z}}$ is that no choice of triangulation is needed in order to get the current f_*1.

Lemma 9.6. *We have $P_0(\phi(\widetilde{f}))=f_*1$.*

Proof. We have $\phi(\widetilde{f})=f_*K(\nabla_f)$. Here $K(\nabla_f)$ is the Chern–Weil form of ∇_f associated to the multiplicative sequence

$$K=K_0+K_2(c_1)+K_4(c_1,c_2)+\cdots$$

as discussed in Section 4.2. Hence the assertion follows from $K_0 = 1$. The latter holds for any multiplicative sequence. Indeed the equation $K(ab) = K(a)K(b)$ forces it: For each power series a, we have

$$K(a) = K(a \cdot 1) = K(a) \cdot K_0,$$

which implies $K_0 = 1$. ■

Theorem 9.7. *$\tau_{\mathbb{Z}}$ induces a map $\widehat{\tau}_{\mathbb{Z}}\colon MU^n(p)(X) \to H^n_{\mathcal{D}}(X;\mathbb{Z}(p))$.*

Proof. It is clear that $\tau_{\mathbb{Z}}$ is a group homomorphism. We must prove that for

$$\gamma = (\widetilde{f}, h) \in ZMU^n(p)(X),$$

we have $d\tau_{\mathbb{Z}}(\gamma) = 0$, and that

$$\tau_{\mathbb{Z}}\left(BMU^n(p)(X)\right) \subset dC^{n-1}_{\mathcal{D}}(p)(X). \tag{9.1}$$

We start with the former. Since f_*1 is a closed current, and $R(\gamma)$ is a closed form, we get

$$\begin{aligned} d\tau_{\mathbb{Z}}(\widetilde{f}, h) &= d(f_*1, P_0(R(\gamma)), P_0(h)) \\ &= (df_*1, P_0(dR(\gamma)),\ P_0(dh) + f_*1 - P_0(R(\gamma))) \end{aligned}$$

Since P_0 is a homomorphism and $R(\gamma) = f_*K(\nabla_f) + dh$, we get $d\tau_{\mathbb{Z}}(\gamma) = 0$ from Lemma 9.6. Recall that $BMU^n(p)(X) = BMU^n_{geo} + a(\widetilde{F}^p\mathcal{A}^*(X;\mathcal{V}_*))$. Let $\widetilde{b}$ be a geometric bordism datum. Then

$$\begin{aligned} \widehat{\tau}_{\mathbb{Z}}(\partial\widetilde{b}, \psi(\widetilde{b})) &= (P_0\phi(\partial\widetilde{b}),\ 0,\ P_0\psi(\widetilde{b})) \\ &= (P_0 d\psi(\widetilde{b}), 0, P_0\psi(\widetilde{b})) \\ &= d(P_0\phi(\widetilde{b}), 0, 0). \end{aligned}$$

Next let $h \in \widetilde{F}^p\mathcal{A}^{n-1}(X;\mathcal{V}_*)$. Then $P_0(h) \in \widetilde{F}^p\mathcal{A}^{n-1}(X)$, so that

$$(0, P_0(h), 0) \in C^{n-1}_{\mathcal{D}}(p)(X).$$

We have

$$\begin{aligned} \tau_{\mathbb{Z}}(a(h)) &= \tau_{\mathbb{Z}}(0,\ h) \\ &= (0,\ P_0(dh),\ P_0(h)) \\ &= d(0,\ P_0(h),\ 0) \end{aligned}$$

which finishes the proof. ■

We observe that $\hat{\tau}_{\mathbb{Z}}$ is a map of Hodge filtered cohomology theories. That is, if we denote the structure maps of Deligne cohomology by $a_{\mathcal{D}}$, $I_{\mathcal{D}}$ and $R_{\mathcal{D}}$, we have

$$\begin{aligned} a_{\mathcal{D}} \circ P_0 &= \hat{\tau}_{\mathbb{Z}} \circ a, \\ \tau \circ I &= I \circ \hat{\tau}_{\mathbb{Z}} \\ P_0 \circ R &= R_{\mathcal{D}} \circ \hat{\tau}_{\mathbb{Z}}. \end{aligned}$$

It follows that we get a map of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & H^{n-1}\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \arrow[r] \arrow[d, "P_0"] & MU^n(p)(X) \arrow[r] \arrow[d, "\hat{\tau}_{\mathbb{Z}}"] & MU^n(X) \arrow[r] \arrow[d, "\tau"] & \cdots \\ \cdots \arrow[r] & H^{n-1}\left(X; \frac{\mathcal{A}^*}{F^p}(\mathbb{C})\right) \arrow[r] & H^n_{\mathcal{D}}(X; \mathbb{Z}(p)) \arrow[r] & H^n(X; \mathbb{Z}) \arrow[r] & \cdots \end{tikzcd} \tag{9.2}$$

9.3 Kernel of the Hodge filtered Thom-morphism on compact Kähler manifolds

We now do a brief investigation of the kernel of $\hat{\tau}_{\mathbb{Z}}$ when X is compact Kähler. We start by chopping of (9.2) into a map of fundamental short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & J^{2p-1}_{MU}(X) \arrow[r] \arrow[d, "\tau_J"] & MU^{2p}(p)(X) \arrow[r] \arrow[d, "\widehat{\tau_{\mathbb{Z}}}"] & \mathrm{Hdg}^{2p}_{MU}(X) \arrow[r] \arrow[d, "\tau"] & 0 \\ 0 \arrow[r] & J^{2p-1}(X) \arrow[r] & H^{2p}_{\mathcal{D}}(X; \mathbb{Z}(p)) \arrow[r] & \mathrm{Hdg}^{2p}(X) \arrow[r] & 0 \end{tikzcd}$$

Since P_0 is an epimorphism of vector-spaces, the map τ_J is surjective. The snake lemma then implies that we get a short exact sequence

$$0 \to \ker \tau_J \to \ker \widehat{\tau}_{\mathbb{Z}} \to \ker \tau \to 0.$$

Hence $\ker \widehat{\tau}_{\mathbb{Z}}$ contains information of the failure of the Thom morphism τ to be injective on Hodge classes, but also of the failure of τ_J to be injective. Let us now briefly investigate the kernel of τ_J. There is a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & MU^{2p-1}(X)_{mt} \arrow[r] \arrow[d, "\tau"] & H^{2p-1}\left(X; \frac{\mathcal{A}^*}{F^p}(\mathcal{V}_*)\right) \arrow[r] \arrow[d, "\tau_{\overline{J}}"] & J^{2p-1}_{MU}(X) \arrow[r] \arrow[d, "\tau_J"] & 0\ . \\ 0 \arrow[r] & H^{2p-1}(X; \mathbb{Z})_{mt} \arrow[r, "i"] & H^{2p-1}\left(X; \frac{\mathcal{A}^*}{F^p}(\mathbb{C})\right) \arrow[r] & J^{2p-1}(X) \arrow[r] & 0 \end{tikzcd}$$

where the subscript mt means modulo torsion. Since P_0 is onto, it easily follows that $\tau_{\overline{J}}$ is onto. Then the snake lemma places $\ker \tau_J$ in the exact sequence

$$0 \to \ker \tau \to \ker \tau_{\overline{J}} \to \ker \tau_J \to \operatorname{coker} \tau \to 0. \tag{9.3}$$

Here $\ker \tau_{\overline{J}}$ is trivial in the sense that

$$\ker \tau_{\overline{J}} = \bigoplus_{j>0} H^{2p-1+2j} \left(X; \frac{\mathcal{A}^*}{F^{p+j}}(\mathcal{V}_{2j}) \right).$$

The non-trivial content of the exact sequence (9.3) is the following construction: Let

$$x \in H^{2p-1}(X; \mathbb{Z})_{mt} \backslash im(\tau).$$

Then $i(x) = \tau_{\overline{J}}(y)$ for some y. Since y does not come from MU^{2p-1}, $[y] \neq 0$ in $J^{2p-1}_{MU}(X)$. Since $y \mapsto i(x)$ we observe that $[y] \in \ker \tau_J$. Furthermore $[y] \mapsto [x]$ under $\ker \tau_J \to \operatorname{coker} \tau$, so y does not come from $\ker \tau_{\overline{J}}$, i.e., y does not belong to the trivial kernel.

To conclude, we have seen that $\ker \hat{\tau}_{\mathbb{Z}}$ contains information of the failure of τ to be injective on Hodge classes, and of the failure of τ to be surjective on MU^{2p-1}. Being an extension, it may contain more information still.

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