Getting the valuation formulas right when it comes to annuities.

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Abstract

Design/methodology/approach: Inspired by recent observations in the literature concerning cash flows, discount rates and values in DCF methods, we mathematically derive DCF valuation formulas for annuities.

Purpose: The purpose of this paper is to establish the flow-to-equity method, the free cash flow (FCF) method, the adjusted present value method, and the relationships between these methods when the FCF appears as an annuity. More specifically, we depart from the two most widely used evaluation settings. The first setting is that of Modigliani and Miller from 1958 and 1963 who based their analysis on a stationary FCF. The second setting is that of Miles and Ezzell from 1980 and 1985 who worked with an FCF that represents an autoregressive possess of first order.

Findings: The following relationships are established: (a) the correct discount rate of the tax shield when the free cash flow takes the form of a first-order autoregressive annuity, (b) the direct valuation of the tax shield from the free cash flow for a first-order autoregressive annuity, (c) the correct translation from the required return on unlevered equity to the levered equity, when the free cash flow is a stationary annuity, and (d) direct calculation of the unlevered and levered firm values, and the value of the tax shield for a stationary annuity.

Originality/value: Until now the complete set of formulas for the valuation of stochastic annuities by different DCF methods has not been established in the literature. These formulas are developed here. These formulas are important for practitioners and academics when it comes to the valuation of cash flows of finite lifetime.

1 Introduction

Discounted cash flow (DCF) methods constitute one of the most widely used approaches to evaluate firms and investment projects in practice and academia (Mukhlynina & Nyborg, 2016). This is confirmed by contemporary textbooks in corporate finance (e.g., Copeland et al. 2014, chapters 13 & 14; Berk & DeMarzo, 2019, chapters 18 & 19; Brealey et al., 2020, chapters 17-19) and firm valuation (e.g., Damodaran, 2006, chapters 5-6; Koller et al., 2010, chapter 6; Kruschwitz & Löffler, 2020) as well as dozens of research articles (references will appear during this paper). The most prominent DCF methods are the flow-to-equity method, the free cash flow (FCF) method, the adjusted present value (APV) method (accredited to Myers, 1974) and the capital cash flow (CCF) method (e.g., McConnell & Sandberg, 1975; Nantell & Carlson, 1975; Ruback, 2002). All these methods are required to be consistent in the sense that they need to give the same firm value when they are applied to the same practical case with the same set of assumptions and input data. This consistency is ensured by formulas that translate between the discount rates that are used in these methods. To the best of our knowledge, Modigliani & Miller are the first to have developed such translation formulas in their seminal papers from 1958 and 1963. They show the translation between the required return on unlevered equity and the discount rate in the CCF method (their formula 11.c), the translation between the required return on unlevered and levered equity (their formula 12c), the translation between the required return on unlevered equity and the discount rate in the FCF method (inherent in their formula 31.c) and an adjusted present value formulation (their formula 3).

At this point, it is advisable to keep in mind that Modigliani & Miller (1963) aimed to explain the relevance of capital structure for the value of the firm. Hence, their paper presents a milestone in capital structure theory which was later supplemented by the trade-off theory (Kraus & Litzenberger, 1973), pecking order theory (Myers & Majluf, 1984), market timing theory (Baker

& Wurgler (2002) and the capital structure substitution theory (Timmer, 2012). However, the formulas that they have applied for the derivation of their well-known propositions have also entered normative models for valuing firms and investment projects. This can raise some issues because the approach of Modigliani & Miller (1958, 1963), and many subsequent researchers such as Farrar & Selwyn (1967), Nantell & Carlson (1975), Haris & Pringle (1985) and others, assume that the FCF appears as a constant perpetuity. However, in practice many investment projects either do not possess an infinite life, or firm value analysts tend to split the life of a firm into different valuation stages (such as an explicit planning period, a period with growth, a period without growth; see, for example, Mukhlynina & Nyborg, 2016, p. 20). Many researchers have therefore questioned to what extent the framework of Modigliani & Miller (1963) and their successors is applicable for cases when the FCF is not a constant perpetuity. Myers (1974), Beranek (1975), Arditty & Levi (1977), Ben-Horim (1979), Miles & Ezzel (1980), Tham & Vélez-Pareja (2005, 2019), Brusov et al. (2011, 2014) and Becker (2020, 2021) have investigated cases where cash flows have a limited lifetime. Nevertheless, until today no complete and plausible set of valuation equations has been developed for an FCF that comes as a stochastic annuity. By "complete", we mean to formulate both the flow-to-equity method, FCF method, CCF method, APV method and all the necessary translation formulas between the required returns or discount rates that are applied in these methods. By "plausible", we mean that these methods and translation formulas are not only mathematically connected, but that the underlying assumptions are compatible with each other and plausible with respect to the absence of arbitrage and the additivity of net present values; see, for example, the dispute by Fernandez (2004 and 2005), Fieten et al. (2005) and Cooper & Nyborg (2006). Furthermore, assumptions concerning the underlying stochasticity in the cash flows should not contradict other assumptions. For example, assuming that both the required return on unlevered equity and the required return for the tax shield are constant throughout time is not necessarily a plausible assumption for cash flows with a finite lifetime (Becker, 2021), although this assumption would allow for establishing a mathematically complete set of valuation equations as in Brusov et al. (2011). Furthermore, many studies do not clearly differentiate between the two mutually exclusive settings of Modigliani & Miller (1963) and Miles & Ezzell (1980, 1985).

The aim of this paper is to close these gaps. This paper therefore departs from Becker (2021), who has accentuated the differences in the valuation frameworks of Modigliani & Miller (1963) and Miles & Ezzell (1980 and 1985). For these two settings, he has illustrated the timely behavior of the discount rates when the FCF is an annuity. However, a complete set of valuation formulas and a rigorous mathematical derivation have not been tried in his contribution. Therefore, his results have limited usefulness for practical valuation cases. We will establish the complete and plausible set of formulas for consistent firm valuation of annuities. In addition, we will give an overview of the already known formulas. This makes a comparison of the traditional valuation formulas and the newly developed formulas easier, and the valuation practitioner will be able to pick the correct formula for her valuation case.

This paper is structured as follows. In section 2, we will give an overview of the assumptions and models that are considered in this paper, and which form the basis for the selection of the relevant literature that is presented in section 3. In the third section, we will also highlight inconsistencies in some of the existing approaches that we attempt to correct in this paper. The fourth section gives a summary on the observations made by Becker (2021) concerning the evolution of cash flows, discount rates and values of the firm, debt, equity and tax shields when the FCF is an annuity. Sections 5 and 6 are devoted to the derivation of the new valuation formulas. Section 7 shows that our formulas collapse to the well-known formulas for perpetuities. Finally, section 8 gives a tabular summary of all formulas and concludes the paper.

2 Assumptions and Discounted Cash Flow Methods

Beside the immense amount of research that conceptually deals with capital structure policy (some key references are given in the previous section), there exist numerous contributions on DCF methods and their discount rates. Before we can present the strand of literature relevant to this article in the subsequent section, we need to explain the common assumptions underlying this literature and our analysis. In most of the contributions, we cannot find complete compilations of assumptions on which the analyses are based, although some more or less extensive attempts have been made (Miles & Ezzell, 1980, pp. 722-723; Copeland et al. 2014, p. 525; Becker, 2020, pp. 468-469; Becker 2021). In many analyses, assumptions are not explicitly formulated but are tacitly applied in valuation models. The assumptions underlying the literature in the next chapter and our subsequent analysis are the following:

(1) **Existence of pricing operator/No arbitrage:** We assume that there exists a pricing operator or mechanism (stochastic discount factors, vector of state prices, risk-neutral probabilities, betas from the capital asset pricing model, or similar) that facilitates the assignment of values (prices) in point of time t to stochastic cash flows (payoffs) in $t+1$. For this pricing operator to make sense in financial markets, we require the principle of no arbitrage (Barbi, 2012, p. 254, pp. 473-475, Copeland et al. 2014, p. 525; Tham & Vélez-Pareja, 2019; Kruschwitz & Löffler, 2020, pp. 28-31).

(2) **Deterministic and time-invariance of pricing operator:** The pricing operator is deterministic and remains unchanged throughout time. By this we mean that a stochastic cash flow with a specific probability distribution has the same one-period discount rate, no matter when the cash flow appears in time (See also Miles & Ezzel, 1980, p. 722;

Barbi, 2012, p. 254, footnote 2 on the constancy of the required return on unlevered equity).

(2a) **Deterministic and time-invariant risk-free rate**: Assumption 2 implies that the risk-free rate $r_{\mathrm{f},t}$ is non-stochastic (Sick, 1990, p. 1434) and constant across time: $r_{\mathrm{f},t} = r_{\mathrm{f}}$ (Modigliani & Miller, 1963, p. 436).

(3) **Discrete points in time:** All cash flows appear at discrete and equidistant points in time $t = 1, \dots, T$.

(4) **Specification of annuity:** The free cash flow (FCF) is assumed to be a stochastic annuity: More precisely, we assume that $\{FCF_t\}$ with $t = 1, \dots, T$ (where T may tend to infinity in case of a perpetuity) is a stochastic process with constant unconditional expectation $\mathbb{E}[FCF_t] = A$ for all $t = 1, \dots, T$. A denotes this constant. A is furthermore positive to ensure a positive value of the firm. Figures 1 and 3 illustrate two examples of such a stochastic annuity (the first four points in time) by means of a scenario tree.

Since this paper deals with stochastic finite life annuities, we will also require that there is no continuing value beyond the lifetime $t = T$.

(5) **FCF independence of leverage:** The FCF is unaffected by changes in the capital structure (Copeland et al. 2014, p. 525; Berk & DeMarzo, 2020, p. 556; Kruschwitz & Löffler, 2020, p. 72).

(6) **Types of financing:** The firm/project to be valued issues only two types of claims, namely pure debt and pure equity (Reilly & Wecker, 1973, p. 123; Barbi, 2012, p. 253, Copeland et al. 2014, p. 525).

(7) **Constant leverage:** We assume a constant equity-to-firm-value ratio q throughout the lifetime of the FCF (Boudreaux & Long, 1979, p. 9; Harris & Pringle, p. 240; 1985,

Miles & Ezzell, 1980, p. 722; Miles & Ezzell, 1985, p. 1486; Taggart, 1991, p. 14; Barbi, 2012, p. 255).

(8) **Frictionless levering/delevering:** There do not exist any transaction/information costs or restraints for levering or delevering the firm (Modigliani & Miller, 1963, pp. 440 and 442).

(9) **Corporate taxation only**: We apply only corporate taxation; there are no wealth taxes or personal taxation (Sick, 1990, p. 1434; Fieten et al., 2005, p. 185; Copeland et al. 2014, p. 525).

(10) **Deterministic, time-invariant, and independent corporate tax rate**: The corporate tax rate is assumed to be deterministic (non-stochastic) (Farrar & Selwyn, 1967, p. 445; Sick, 1990, p. 1434), time-invariant (Boudreaux & Long, 1979), and does not depend on the size of the earnings before interest and taxes (EBIT) which is related to the FCF by $FCF = EBIT \cdot (1 - \tau) + C$ where τ is the tax rate and C denotes the difference between operating cash flow and operating income (this assumption is discussed in Modigliani & Miller, 1963, p. 435, footnote 5 and p. 438, footnote 9.)

(11) **Tax symmetry on gains and losses**: There is either no negative income before taxes, or in the case of negative income before taxes, there is a tax transfer to the firm (reverse taxation) (Sick, 1990, p. 1434; Fieten et al., 2005, p. 185; see Barbi, 2012, p. 253 footnote 1 for an alternative formulation). Appendix 2 shows that the effective tax rates cannot be the same in the unlevered and levered firm if this assumption does not hold.

(12) **Specification of the flow to debt holders**: The flow to debt consists of interest payments and changes in the principal of debt only. There do not exist additional fees, discounts, etc. Any debt that is issued throughout the lifetime of the FCF will be paid back until T (lifetime of FCF). Note that in the perpetual case of Modigliani & Miller (1963) only

interest payments occur while the outstanding principal is kept constant eternally. In the perpetual case of Miles & Ezzell (1980), the principal amount of debt is adjusted, but it will never be paid down entirely.

(13) **Risk-free debt.** We assume that debt is risk free (Modigliani & Miller, 1958, p. 268; Modigliani & Miller, 1963, p. 436; Miles & Ezzel, 1980, footnote 1; Taggart, 1991, p. 9). Few researchers have applied a cost of debt that is different from the risk-free rate, but still treated debt deterministically (for example, Ruback, 2002; Cooper & Nyborg, 2008). However, the modeling of risky debt requires additional assumptions and complicates the computations. In the case of risky debt, the interest payments may belong to another risk class than the down payments. One can presume that interest payments are lost before down payments if the cash flow of the firm can only partially satisfy the debt holders. The total debt will then have a risk that is composed of both these risk classes, meaning that the required return on debt is a compound. Therefore, the interest tax shield which is tied to the interest payments cannot be linked to the total flow to the debt holders in a linear fashion as is often done in previous research. Throughout this paper it will therefore be convenient to assume debt as risk-free.

(14) **No costs of financial distress**: We assume the absence of costs in case of bankruptcy or financial distress (See Modigliani & Miller, 1958, footnote 18.; See Barbi, 2012, p. 253; Copeland et al. 2014, p. 525).

(15) **Outstanding debt equals market value of debt**: The value of debt DV_t equals the nominal (contractual) amount of debt DN_t:

$$DV_t = DN_t \text{ for all } t$$

This implies that the nominal (contractual) interest rate equals the risk-free rate. This assumption is hardly mentioned explicitly in the literature. However, it is essential for deriving the discount rate in the FCF approach (expression (1) below). Appendix 1 shows how the weighted average costs would look like if this assumption does not hold.

Additional aspects that have been discussed in the literature include personal taxation (Sick, 1990; Taggart, 1991), asymmetric taxation (Kruschwitz & Löffler, 2018) and costs or benefits of financial distress (Stiglitz, 1969; Kraus & Litzenberger, 1973; Berk & DeMarco, 2019, chapter 16; Brealey et al., 2020, chapter 18). These aspects are outside the scope of this paper.

To avoid any confusion about the construction of the valuation methods addressed in this analysis, we will quickly outline these models here. Note that other approaches exist, such as the business risk-adjusted FCF method or the economic value added (EVA) approach (for an overview, see Fernandez, 2007). These approaches are outside the scope of our analysis.

The notation applied in the models below is given as follows:

DV .... Value of debt

EV_{L} .... Value of levered equity

FV_{L} .... Value of levered firm

FV_{U} .... Value of unlevered firm (unlevered equity)

TSV .... Value of interest tax shield

ΔDV .... Change of debt because of down payments or issues of new debt

FCF .... Free cash flow

I .... Interest payments

TS .... Interest tax shield

q .... Equity-to-firm-value ratio

τ .... Corporate tax rate

r_{CCF} .... Required discount rate in CCF method

r_{EL} .... Required return on levered equity

r_{FCF} .... Required discount rate in the FCF method

r_{D} .... Required return on debt

r_{f} .... Risk free rate

r_{TS} .... Required discount rate for the interest tax shield

r_{U} .... Required return on unlevered equity or unlevered firm

All methods will be stated in recursive form for $t = 0, \ldots, T-1$. Accordingly, all parameters receive a time index because their amounts can change over time.

Flow-to-equity method: In this method the value of the equity in the levered firm is calculated directly by discounting the cash flow to the equity holders by means of the required return on levered equity r_{EL}. The flow to equity is the cash flow that remains after paying interest, issuing or paying down debt (see Berk & DeMarzo, 2020, pp. 690-691):

$$EV_{\text{L},t} = \frac{FCF_{t+1} - I_{t+1} \cdot (1-\tau) - \Delta DV_{t+1} + EV_{\text{L},t+1}}{1 + r_{\text{EL},t}}$$

The change in debt is determined as: $\Delta DV_{t+1} = DV_{t+1} - DV_t$. The value of the levered firm can be calculated by adding the debt value to the equity value: $FV_{\text{L},t} = EV_{\text{L},t} + DV_t$.

FCF method: In this method the value of the levered firm (i.e., the value of equity and debt together) is calculated by discounting the FCF by means of a corresponding discount rate $r_{\text{FCF}} =$

$q \cdot r_{\mathrm{EL}} + (1-q) \cdot (1-\tau) \cdot r_{\mathrm{D}}$ (often referred to as the after-tax weighted average costs of capital: see Harris & Pringle, 1985, p. 237; McConnell & Sandberg, 1975, p. 885):

$$FV_{\mathrm{L},t} = \frac{FCF_{t+1} + FV_{\mathrm{L},t+1}}{1 + q_t \cdot r_{\mathrm{EL},t} + (1-q_t) \cdot (1-\tau) \cdot r_{\mathrm{D},t}} \tag{1}$$

The FCF method evolves from the flow-to-equity method, and this is a well-known relationship (see Becker, 2020, p. 475).

APV method: In this method the value of the levered firm is determined as the value of the unlevered firm plus the value of the interest tax shield. The value of the unlevered firm is computed by discounting the FCF with the required return on unlevered equity r_{U}. The value of the tax shield is computed by discounting the interest tax shield with the corresponding discount rate r_{TS}.

$$FV_{\mathrm{L},t} = FV_{\mathrm{U},t} + TSV_t \qquad FV_{\mathrm{U},t} = \frac{FCF_{t+1} + FV_{\mathrm{U},t+1}}{1 + r_{\mathrm{U},t}} \quad TSV_t = \frac{TS_{t+1} + TSV_{t+1}}{1 + r_{\mathrm{TS},t}}$$

CCF method: In this method the cash flow to the capital holders is discounted with the corresponding weighted average costs of capital $r_{\mathrm{CCF}} = q \cdot r_{\mathrm{EL}} + (1-q) \cdot r_{\mathrm{D}}$. The flow to the capital holders consists of the flow to both the equity and debt holders after corporate taxation:

$$FV_{\mathrm{L},t} = \frac{FCF_{t+1} + I_{t+1} \cdot \tau + FV_{\mathrm{L},t+1}}{1 + q_t \cdot r_{\mathrm{EL},t} + (1-q_t) \cdot r_{\mathrm{D},t}} \tag{2}$$

This method also evolves directly from the flow to equity method together with the valuation of debt. (See appendix in Becker, 2021). From (1) and (2) we can immediately derive the relationship between r_{CCF} and r_{FCF} which is:

$$r_{\mathrm{CCF},t} = r_{\mathrm{FCF},t} + (1-q_t) \cdot \tau \cdot r_{\mathrm{D},t}$$

Having all preliminaries (assumptions and models) in place, we will now discuss the relevant literature that deals with cash flows of limited lifetime.

3 Literature Review on Valuation of Cash Flows with Limited Lifetime

In this section we will look at how the existing literature has addressed cash flows that have a finite length of life. In this and all subsequent sections we will abbreviate Modigliani & Miller (1963) with M&M. Miles & Ezzell (1980, 1985) will be referred to as M&E.

Arditti (1973, p. 1002) claims that "*If annual expected earning in year* t, $\overline{EBIT}_t$ *varies with t, or the firm has a finite life* […] *then the average cost of capital* […] *will, in general, be inappropriate*". However, in his analysis he presumes that the weighted average cost of capital (the discount rate r_{FCF} or r_{CCF} in the FCF or CCF method; these two methods coincide in his analysis because there are no taxes) does not change over the lifetime of a project. The same doubt is expressed by Reilly & Wecker (1973), who doubt the correctness of the formula of the weighted average costs of capital (the discount rate r_{CCF} in the CCF method). Like Arditti (1973), they require this discount rate to be constant throughout time despite the temporal variation of the cash flows. We will see that this doubt is not justified for an autoregressive annuity of first order. However, on a general basis it can be expected that both the required return on levered equity r_{EL} and the equity-value-to-firm-value ratio $\frac{EV_{\mathrm{L}}}{FV_{\mathrm{L}}}$ (respectively the debt-value-to-firm-value ratio) change over time. This means that we cannot thoughtlessly assume that r_{FCF} and r_{CCF} remain constant across time. Nevertheless, their construction (see formulas (1) and (2) above) cannot be questioned.

Myers (1974, p. 1) states that "*capital budgeting rules based on the weighted average cost of capital formulas proposed by* [M&M] *and other authors are not generally correct*." With this statement Myers (1974) refers to the following relationship that was originally stated by M&M:

$$r_{\mathrm{FCF}} = r_{\mathrm{U}} - r_{\mathrm{U}} \cdot \tau \cdot (1 - q)$$

Myers (1974, pp. 10 and 11) shows that the appropriate discount rate for an FCF with a lifetime of one period is the following:

$$r_{\mathrm{FCF}} = r_{\mathrm{U}} - \tau \cdot (1 - q) \cdot r_{\mathrm{f}} \cdot \frac{1 + r_{\mathrm{U}}}{1 + r_{\mathrm{f}}} \tag{3}$$

This is the same formula that later will be shown valid for perpetuities by Miles & Ezzell (1980). This allows for two immediate conjectures. First, assuming a stochastic cash flow according to M&M, the discount rate in the FCF approach will change with the remaining maturity. Second, assuming a stochastic cash flow according to M&E, the discount rate in the FCF approach will be constant. Third, the discount rate in the FCF approach will be the same for both types of cash flows when the remaining lifetime of the FCF is a single period.

Arditti & Levi (1977) study the validity of the weighted average costs of capital for both perpetual and finite life annuities. Their formula (26) represents the FCF method, where they assume a constant discount rate. As we will discuss later, this formula is correct only for cash flows that fulfill the stochastic properties, according to M&E. Their formula (12) which corresponds to the FCF method, and their formula (27), which corresponds to the CCF method, assume both constant interest payments and constant discount rates. As we will show later, both these conditions are inconsistent with cash flows in both the settings of M&M and M&E. Furthermore, Arditti & Levi (1977) generate an inconsistency in the CCF method. The CCF method, with a discount rate $r_{\mathrm{CCF}} = \frac{EV_{\mathrm{L}}}{FV_{\mathrm{L}}} \cdot r_{\mathrm{EL}} + \frac{DV}{FV_{\mathrm{L}}} \cdot r_{\mathrm{D}}$, requires that the tax shield is calculated as $TS = \tau \cdot r_{\mathrm{D}} \cdot DV_L = \tau \cdot r_{\mathrm{D}} \cdot \frac{DV}{FV_{\mathrm{L}}} \cdot FV_{\mathrm{L}}$. Arditti & Levi (1977) in their formula (15) and (27) apply $TS = \tau \cdot r_{\mathrm{D}} \cdot \frac{DV}{FV_{\mathrm{L}}} \cdot Inv$ instead, where Inv is the initial investment outlay (acquisition costs). The last inconsistency is addressed and resolved by Ben-Horim (1979). Moreover, for finite life cash flows Bourdreaux & Long (1979, pp. 8 and 9) point out that (a) the discount rates may not remain constant across time, (b) the capital structure may change across time, (c) outstanding debt will not necessarily be equal to the debt value in any given period of time, (d) the market value of the project can be decreasing throughout time, and (e) the amount of the interest tax shield changes throughout time. Despite the importance of these statements, a rigorous mathematical analysis of these effects is not undertaken in their study. Furthermore, they provide an example where they illustrate that the FCF method and the CCF method need to be consistent for any lifetime. In this example they apply constant discount rates r_{FCF} and r_{CCF} across time. We will see later that this will be correct for an M&E type of annuity, but it will not be valid for an M&M type of annuity.

Miles & Ezzell (1980, p. 720) depart from their observation that "*a number of authors have argued that the textbook approach does not generally provide correct valuations of uneven finite cash flows.*" On the one hand, they show (p. 727) that the discount rate r_{FCF} in the FCF method corresponds to the WACC, i.e., $r_{\mathrm{FCF},t} = q_t \cdot r_{\mathrm{EL},t} + (1 - q_t) \cdot (1 - \tau) \cdot r_{\mathrm{f}}$ for any lifetime of the cash flow. They also show (p. 726) that the relationship between the discount rate r_{FCF} and the required return on unlevered equity is given by formula (3) and that this relationship holds for any arbitrary lifetime of the FCF. In their analysis, they do not explicitly state the type of stochasticity of the FCF. However, it becomes clear that their analysis is based on the implicit assumption that the FCF cannot be stationary. In Miles & Ezzel (1985), the same authors state the stochastic process of the underlying FCF more explicitly. Specifically, and in contrast to M&M, they assume an expectation revision process of the form: $\mathbb{E}_{\tau}\big[\widetilde{FCF}_t\big] = \mathbb{E}_{\tau-1}[FCF_t] \cdot \big(1 + \tilde{e}_{\tau,t}\big)$ with $\mathbb{E}\big[\tilde{e}_{\tau,t}\big] = 0$. Miles & Ezzell (1985, formula 19) also derive the following formula for determining the value of the levered firm and the value of the tax shield for a finite-life FCF (not necessarily an annuity):

$$FV_{\mathrm{L},0} = FV_{\mathrm{U},0} + TSV_0 = \sum_{t=1}^{T} \frac{\mathbb{E}\big[\widetilde{FCF}_t\big]}{(1 + r_{\mathrm{U}})^t} + \sum_{t=1}^{T} \frac{\mathbb{E}\big[\widetilde{TS}_t\big]}{(1 + r_{\mathrm{U}})^{t-1} \cdot (1 + r_{\mathrm{f}})}$$

It is important to note that this formula assumes a particular stochastic process that generates the FCF. Moreover, this process needs to assure that both $\mathbb{E}\left[\widetilde{FCF}_t\right]$ and $\mathbb{E}\left[\widetilde{FV}_{\mathrm{U},t}\right]$, each for itself, can be discounted with the required return on unlevered equity r_{U}. This is not generally the case, and it is particularly not the case in M&M where continuing values (like $FV_{\mathrm{U},t}$) are deterministic and will be discounted with r_{f}. It is essential to recognize that even if the FCF is an annuity, the tax shield is not an annuity. Hence, in this formula we require the calculation of the amount of the tax shield for each point in time. Later, in our analysis, we will therefore derive a formula for calculating the value of the tax shield directly from the FCF annuity.

Brusov et al. (2011) intend to develop a valuation formula for an M&M style constant finite life annuity. In their analysis (more precisely their formula 14) they neglect the fact that the continuing value of debt (like all the other values) is decreasing throughout time. This fact has already been pointed out by Boudreaux & Long (1979, p. 8). If debt decreases, then tax savings will also decrease. Hence, the annuity formula applied in their analysis (see Brusov et al., 2011, formula 14) cannot be valid. Brusov et al. (2011) furthermore presume that the unlevered return on equity as well as the discount rate in the FCF method remain constant throughout time. However, as we will show below, these rates can be subject to change throughout the lifetime of the FCF. If discount rates change throughout time, the traditional annuity formula (as in their formula 20) is inapplicable.

Mukhlynina & Nyborg (2016, p. 21) in their survey present an example that aims at showing the share of the terminal value in relation to the total value of the firm. They split the stream of cash flows into two parts. The first part is an annuity with the growth factor h and lifetime $T-1$, and the second part (the terminal value) consists of a perpetuity with growth factor g starting in point in time T. They assume that the discount factor r_{FCF} in the FCF method is constant across time.

These expressions are valid for the kind of stochasticity assumed by M&E. However, this approach will be incorrect for a stationary FCF like in M&M.

Tham & Vélez-Pareja (2019, formula 12) develop the following translation formula:

$$r_{\mathrm{EL},t} = r_{\mathrm{U},t} + \left(r_{\mathrm{U},t} - r_{\mathrm{D},t}\right) \cdot \frac{DV_t}{EV_{\mathrm{L},t}} - \left(r_{\mathrm{U},t} - r_{\mathrm{TS},t}\right) \cdot \frac{TSV_t}{EV_{\mathrm{L},t}} \tag{4}$$

This formula is more general in the sense that it does not depend on the structure or the lifetime of the cash flow. However, Tham & Vélez-Pareja (2019) do not answer the question how the different discount rates relate to each other, or how they should be chosen. Furthermore, the ratios $\frac{DV_t}{EV_{\mathrm{L},t}}$ and $\frac{TSV_t}{EV_{\mathrm{L},t}}$ need to be established by means of additional information. It is furthermore important to notice that this formula is applicable in the recursive computation (backward induction) of firm values as illustrated in Becker (2020). This means that we cannot assume that some or all parameters are constant throughout time. Anyway, for the perpetual case of M&M we would have the additional relationship that $r_{\mathrm{TS},t} = r_{\mathrm{f},t}$, $TSV_t = \tau \cdot DV_t$, and that all parameters are constant throughout time. We then receive M&M's relationship:

$$r_{\mathrm{EL}} = r_{\mathrm{U}} + (r_{\mathrm{U}} - r_{\mathrm{f}}) \cdot \frac{DV}{EV_{\mathrm{L}}} \cdot (1 - \tau)$$

Becker (2020, formula 46) establishes another special case, where cash flows have a finite lifetime and debt financing is prespecified. In such a case, formula (4) can be written as follows:

$$r_{\mathrm{EL},t} = r_{\mathrm{U},t} + \left(r_{\mathrm{U},t} - r_{\mathrm{f},t}\right) \frac{DV_t}{EV_{\mathrm{L},t}} \cdot (1 - \tau \cdot v_t)$$

Here, the parameter v_t describes the value of the interest payments relative to the value of debt. In addition, this formula can only be used in a backward induction scheme.

To summarize, we can say that it is possible to determine the value of the levered firm (FV_{L}), the value of the unlevered firm (unlevered equity, FV_{U}), the value of levered equity (EV_{L}), the value of debt (DV) and the value of the interest tax shield (TSV) for annuities in the context of M&E.

However, we will contribute some missing details. First, we will derive a formula for the calculation of the required return on the tax shield $r_{\mathrm{TS},t}$ for the recursive calculation of the form:

$$TSV_{t-1} = \frac{TS_t}{(1+r_{\mathrm{f}})} + \frac{\mathbb{E}\big[\widetilde{TSV}_t\big]}{(1+r_{\mathrm{U}})} = \frac{TS_t + \mathbb{E}\big[\widetilde{TSV}_t\big]}{\big(1+r_{\mathrm{TS},t}\big)}$$

Second, we will show the direct calculation of the value of the tax shield based on the FCF.

With respect to M&M the literature has not yet provided a valid framework for valuing annuities. In this paper we will develop a framework that consists of the complete set of formulas for determining the values of unlevered and levered equity, debt and the tax shield, as well as all translation formulas for the required returns.

4 Behavior of Cash Flows, Values and Discount Rates for FCF Annuity

In this section, we will introduce the results of Becker (2021) regarding the behavior of the cash flows, values, and discount rates (required returns) when the FCF takes the form of a stochastic annuity. Concerning the evolution of the FCF, two cases are differentiated. The first case corresponds to Modigliani & Miller (1963), who assume the following:

(16a) The FCF is represented by a stationary process of the form:

$$\widetilde{FCF}_t = A \cdot (1+\tilde{e}) \quad \text{with} \quad \mathbb{E}[\tilde{e}] = 0$$

with A being a constant. Here the tilde "$\sim$" indicates a stochastic variable.

This implies that besides the unconditional expected FCF (see assumption 4), the conditional expected FCF is also constant (time-invariant):

$$\mathbb{E}\left[\widetilde{FCF}_t \mid FCF_{t-1}\right] = A$$

The second case corresponds to Miles & Ezzell (1980, 1985) who assume a weak autoregressive process (See Kruschwitz & Löffler, 2020, pp. 50-58). For annuities or perpetuities this process is the following (see also Barbi, 2012, p. 254 with a growth component):

(16b) The FCF can be described by a simple autoregressive process of the form:

$$\widetilde{FCF}_t = FCF_{t-1} \cdot (1 + \tilde{e}) \quad \text{with} \quad \mathbb{E}[\tilde{e}] = 0$$

where $\tilde{e}$ is a stochastic parameter that is stationary across time. This implies that the conditional expected FCF at point in time t equals the realized FCF in the previous point in time $(t-1)$: $\mathbb{E}\left[\widetilde{FCF}_t \mid FCF_{t-1}\right] = FCF_{t-1}$

Figures 1 to 4 are supposed to help in understanding the summary given below. In these figures $FtE_{\mathrm{L},n}$ (FtD_n, TS_n) refers to the flow to levered equity (flow to debt, tax shield) in node *n* of the tree. All numerical quantities in these figures are taken from Becker (2021). Figure 1 shows a scenario tree that represents the evolution of the cash flows according to the setting of M&M. For simplicity, the branching of the scenarios in this tree is assumed with equal probability. Furthermore, the tax rate, the equity-to-firm-value ratio, and the risk-free rate are $\tau = 30$ %, $q = 40$ % and $r_{\mathrm{f}} = 5$ % respectively. Figure 2 shows the corresponding evolution of all values and discount rates. All values and discount rates are computed by means of risk-neutral probabilities, with the up-scenario and down-scenario having a risk-neutral probability of 40 % and 60 % respectively. The detailed computations are shown in Becker (2021) and will not be repeated here. Figures 3 and 4 exemplify the evolution of the cash flows, values and discount rates according to M&E.

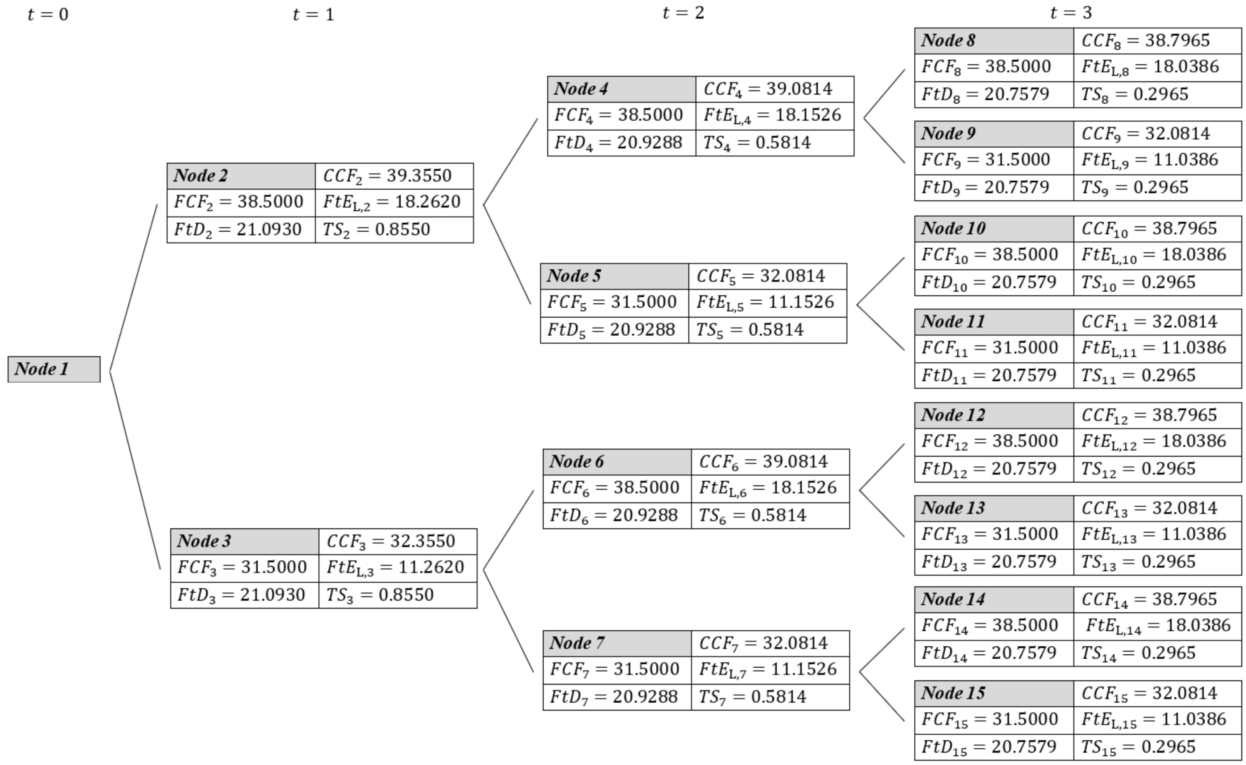

Figure 1: Modigliani-Miller Flows to Stakeholders

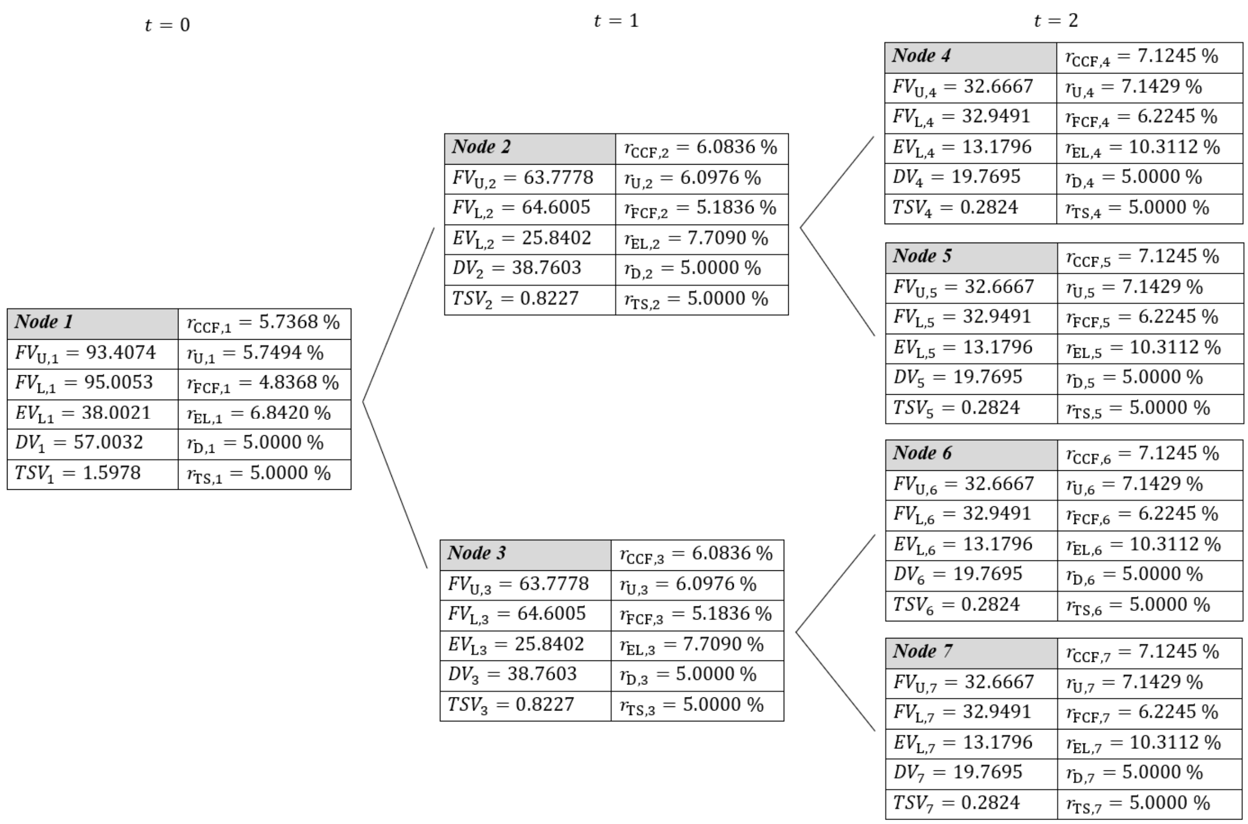

Figure 2: Modigliani-Miller Values and Discount Rates

$t = 0$ | $t = 1$ | $t = 2$ | $t = 3$

Node 1

Node 2	$CCF_2 = 39.3385$
$FCF_2 = 38.5000$	$FtE_{L,2} = 22.8611$
$FtD_2 = 16.4775$	$TS_2 = 0.8385$

Node 3	$CCF_3 = 32.3385$
$FCF_3 = 31.5000$	$FtE_{L,3} = 8.1850$
$FtD_3 = 24.1535$	$TS_3 = 0.8385$

Node 4	$CCF_4 = 42.9833$
$FCF_4 = 42.3500$	$FtE_{L,4} = 22.5749$
$FtD_4 = 20.4084$	$TS_4 = 0.6333$

Node 5	$CCF_5 = 35.2833$
$FCF_5 = 34.6500$	$FtE_{L,5} = 10.5256$
$FtD_5 = 24.7577$	$TS_5 = 0.6333$

Node 6	$CCF_6 = 35.1681$
$FCF_6 = 34.6500$	$FtE_{L,6} = 18.4704$
$FtD_6 = 16.6978$	$TS_6 = 0.5181$

Node 7	$CCF_7 = 28.8681$
$FCF_7 = 28.3500$	$FtE_{L,7} = 8.6119$
$FtD_7 = 20.2563$	$TS_7 = 0.5181$

Node 8	$CCF_8 = 46.9438$
$FCF_8 = 46.5850$	$FtE_{L,8} = 21.8267$
$FtD_8 = 25.1171$	$TS_8 = 0.3588$

Node 9	$CCF_9 = 38.4738$
$FCF_9 = 38.1150$	$FtE_{L,9} = 13.3567$
$FtD_9 = 25.1171$	$TS_9 = 0.3588$

Node 10	$CCF_{10} = 38.4086$
$FCF_{10} = 38.1150$	$FtE_{L,10} = 17.8582$
$FtD_{10} = 20.5503$	$TS_{10} = 0.2936$

Node 11	$CCF_{11} = 31.4786$
$FCF_{11} = 31.1850$	$FtE_{L,11} = 10.9282$
$FtD_{11} = 20.5503$	$TS_{11} = 0.2936$

Node 12	$CCF_{12} = 38.4086$
$FCF_{12} = 38.1150$	$FtE_{L,12} = 17.8582$
$FtD_{12} = 20.5503$	$TS_{12} = 0.2936$

Node 13	$CCF_{13} = 31.4786$
$FCF_{13} = 31.1850$	$FtE_{L,13} = 10.9282$
$FtD_{13} = 20.5503$	$TS_{13} = 0.2936$

Node 14	$CCF_{14} = 31.4252$
$FCF_{14} = 31.1850$	$FtE_{L,14} = 14.6113$
$FtD_{14} = 16.8139$	$TS_{14} = 0.2402$

Node 15	$CCF_{15} = 25.7552$
$FCF_{15} = 25.5150$	$FtE_{L,15} = 8.9413$
$FtD_{15} = 16.8139$	$TS_{15} = 0.2402$

Figure 3: Miles-Ezzell Flows to Stakeholders

$t = 0$ | $t = 1$ | $t = 2$

Node 1	$r_{CCF,1} = 7.1245$ %
$FV_{U,1} = 91.6119$	$r_{U,1} = 7.1429$ %
$FV_{L,1} = 93.1682$	$r_{FCF,1} = 6.2245$ %
$EV_{L1} = 37.2673$	$r_{EL,1} = 10.3112$ %
$DV_1 = 55.9009$	$r_{D,1} = 5.0000$ %
$TSV_1 = 1.5563$	$r_{TS,1} = 6.0433$ %

Node 2	$r_{CCF,2} = 7.1245$ %
$FV_{U,2} = 69.4711$	$r_{U,2} = 7.1429$ %
$FV_{L,2} = 70.3642$	$r_{FCF,2} = 6.2245$ %
$EV_{L,2} = 28.1457$	$r_{EL,2} = 10.3112$ %
$DV_2 = 42.2185$	$r_{D,2} = 5.0000$ %
$TSV_2 = 0.8931$	$r_{TS,2} = 5.6957$ %

Node 3	$r_{CCF,3} = 7.1245$ %
$FV_{U,3} = 56.8400$	$r_{U,3} = 7.1429$ %
$FV_{L,3} = 57.5707$	$r_{FCF,3} = 6.2245$ %
$EV_{L3} = 23.0283$	$r_{EL,3} = 10.3112$ %
$DV_3 = 34.5424$	$r_{D,3} = 5.0000$ %
$TSV_3 = 0.7307$	$r_{TS,3} = 5.6957$ %

Node 4	$r_{CCF,4} = 7.1245$ %
$FV_{U,4} = 39.5267$	$r_{U,4} = 7.1429$ %
$FV_{L,4} = 39.8684$	$r_{FCF,4} = 6.2245$ %
$EV_{L,4} = 15.9474$	$r_{EL,4} = 10.3112$ %
$DV_4 = 23.9210$	$r_{D,4} = 5.0000$ %
$TSV_4 = 0.3417$	$r_{TS,4} = 5.0000$ %

Node 5	$r_{CCF,5} = 7.1245$ %
$FV_{U,5} = 32.3400$	$r_{U,5} = 7.1429$ %
$FV_{L,5} = 32.6196$	$r_{FCF,5} = 6.2245$ %
$EV_{L,5} = 13.0478$	$r_{EL,5} = 10.3112$ %
$DV_5 = 19.5718$	$r_{D,5} = 5.0000$ %
$TSV_5 = 0.2796$	$r_{TS,5} = 5.0000$ %

Node 6	$r_{CCF,6} = 7.1245$ %
$FV_{U,6} = 32.3400$	$r_{U,6} = 7.1429$ %
$FV_{L,6} = 32.6196$	$r_{FCF,6} = 6.2245$ %
$EV_{L,6} = 13.0478$	$r_{EL,6} = 10.3112$ %
$DV_6 = 19.5718$	$r_{D,6} = 5.0000$ %
$TSV_6 = 0.2796$	$r_{TS,6} = 5.0000$ %

Node 7	$r_{CCF,7} = 7.1245$ %
$FV_{U,7} = 26.4600$	$r_{U,7} = 7.1429$ %
$FV_{L,7} = 26.6888$	$r_{FCF,7} = 6.2245$ %
$EV_{L,7} = 10.6755$	$r_{EL,7} = 10.3112$ %
$DV_7 = 16.0133$	$r_{D,7} = 5.0000$ %
$TSV_7 = 0.2288$	$r_{TS,7} = 5.0000$ %

Figure 4: Miles-Ezzell Values and Discount Rates

We will now summarize the observations. Based on these observations we will develop the corresponding formulas in the next section.

(1) M&M and M&E coincide in single period settings.

(2) All discount rates are path independent.

(3) In the M&M tree the tax shield is always discounted with the risk-free rate. The reason for this is that debt is deterministically given for each period.

(4) In the M&E tree the required return on the tax shield is risk-free only in the last period. The more we go backwards in time (from future to present) the higher this discount rate becomes. The reason for this is that debt is not deterministically given, but path dependent. The same applies to the tax shields.

(5) In the M&M setup the required return on unlevered equity is decreasing from the future to the present. The reason for this is that the continuation value is deterministic, and if discounted separately, it would require a risk-free rate. Furthermore, the size of the continuation value V_t increases compared to the size of the cash flow CF_t.

(6) In the M&E tree the required return on the unlevered firm is constant for all nodes and time periods. The reason for this is that both the FCF as well as the continuation values are perfectly positively correlated.

(7) In the M&M setup the required return on levered equity is decreasing from the future to the present. Here the same reasoning as for the unlevered firm applies.

(8) In the M&E tree the required return on the levered equity $r_{\mathrm{E_L}}$ is constant for all nodes and time periods. The reason for this is that the flow to the equity and the continuation value of levered equity are perfectly positively correlated.

(9) In the M&M setup the discount rate in both the FCF method and CCF method are decreasing. The reason for this lies in the decreasing required return on the levered equity $r_{\mathrm{E_L}}$ beside the constancy of the remaining ingredients of r_{FCF} and r_{CCF}.

(10) In the M&E setup the discount rates in both the FCF method and the CCF method are constant. The reason for this lies in the constancy of all the ingredients of these discount rates.

(11) Although the non-conditional expected FCF was assumed to be constant (See assumption 4) the non-conditional expected flow to debt, flow to levered equity, CCF, interest payment, and tax shield are not constant (are not annuities) for both the settings of M&M and M&E.

These observations have the following immediate consequences:

(a) For both M&M and M&E, the discount rates are path independent. This allows for discounting the unconditional expectations of the cash flows and values. Hence, it is possible to use a deterministic backward induction scheme of the form (as opposed to a stochastic backward iteration shown in Becker, 2020 and 2021):

$$V_{i,t} = \frac{\mathbb{E}\left[CF_{i,t+1} + V_{i,t+1}\right]}{1 + r_i} \text{ for all } t = 0, \dots, T-1 \tag{5}$$

where $\mathbb{E}$ now represents the expectation under real probabilities and r_i is the risk-adjusted discount rate for $\mathbb{E}\left[CF_{i,t+1} + V_{i,t+1}\right]$.

(b) For M&M, none of the values in $t = 0$ can be computed directly by using the formula for the present value of an annuity: $\text{Present Value} = \text{Annuity} \times \text{Annuity Factor}$ with the annuity factor commonly being defined as:

$$\text{Annuity Factor} = \frac{(1+r)^T - 1}{(1+r)^T \cdot r}$$

where r is the discount rate and T represents the lifetime of the annuity. This is because either the expected cash flows, the discount rates or both are not constant.

(c) For M&E, the FCF method can be carried out by means of the formula: $Present\ Value = Annuity\ \times Annuity\ Factor$. The same applies to the value of the unlevered firm. For these valuations, both the FCF and the corresponding discount rates are constant across time. The value of levered equity, debt and tax shield cannot be computed by means of the annuity valuation formula. Here a backward induction of the form (5) needs to be applied. However, these values can always be deduced from the levered and unlevered firm value as follows:

$$EV_{\mathrm{L}} = FV_{\mathrm{L}} \cdot q, \qquad DV = FV_{\mathrm{L}} \cdot (1 - q), \qquad TSV = FV_{\mathrm{L}} - FV_{\mathrm{U}}$$

5 The Modigliani-Miller Annuity

In this section we develop the valuation formulas for an FCF annuity that can be represented by a stationary process as assumed in M&M. First, we consider the direct and recursive calculation of the unlevered firm. Then we determine the direct and recursive valuation of the levered firm. We then state the translation between the required return of the unlevered firm and the discount rate in the FCF method. Finally, we look at the valuation of the tax shield and the corresponding discount rate.

Direct Valuation of the Unlevered Firm:

Because of the strict stationarity of the stochastic FCF_{t+1} (assumption 16a) and the time-invariance of the pricing kernel (assumption 2), the expected FCF needs to be discounted with the same one-period discount rate r_{A} in each time period. Furthermore, the value of the unlevered firm $FV_{\mathrm{U},t}$ is the same in all states $s \in S_t$ of the same point in time t. This implies that from the perspective of point in time $t-1$ the continuing value $FV_{\mathrm{U},t}$ is deterministic. Deterministic values need to be discounted with the risk-free rate. For an FCF according to M&M, we can recursively calculate the value of the unlevered firm $FV_{\mathrm{U},t}$ as follows:

$$FV_{\mathrm{U},t} = \frac{\overline{FCF}}{1 + r_{\mathrm{A}}} + \frac{FV_{\mathrm{U},t+1}}{1 + r_{\mathrm{f}}}$$

Since we do not have a continuing value in $t = T$, the required return of the unlevered firm $r_{\mathrm{U},T-1}$ equals the discount rate r_{A} of the FCF:

$$r_{\mathrm{U},T-1} = r_{\mathrm{A}}$$

In a backward manner we will now look at the evolution of the value of the unlevered firm. At remaining maturity $v = 1$ (corresponding to point in time $T - 1$) we have:

$$FV_{\mathrm{U},[1]} = \frac{\overline{FCF}}{1 + r_{\mathrm{A}}}$$

In this and all subsequent formulas, we apply brackets in the form $[v]$ whenever a value, cash flow or discount rate is stated with respect to the remaining maturity. We apply no brackets whenever we refer to a point in time t. At remaining maturity $v = 2$ (corresponding to point in time $T - 2$) we have:

$$FV_{\mathrm{U},[2]} = \frac{\overline{FCF}}{1 + r_{\mathrm{A}}} + \frac{FV_{\mathrm{U},[1]}}{1 + r_{\mathrm{f}}} = \frac{\overline{FCF}}{1 + r_{\mathrm{A}}} + \frac{FCF}{(1 + r_{\mathrm{A}}) \cdot (1 + r_{\mathrm{f}})} = \frac{\overline{FCF}}{1 + r_{\mathrm{A}}} \cdot \left(1 + \frac{1}{(1 + r_{\mathrm{f}})}\right)$$

At remaining maturity $v = 3$ (corresponding to point in time $T - 3$) we have:

$$\begin{aligned} FV_{\mathrm{U},[3]} &= \frac{\overline{FCF}}{1 + r_{\mathrm{A}}} + \frac{FV_{\mathrm{U},[2]}}{1 + r_{\mathrm{f}}} \\ &= \frac{\overline{FCF}}{1 + r_{\mathrm{A}}} + \frac{\overline{FCF}}{(1 + r_{\mathrm{A}}) \cdot (1 + r_{\mathrm{f}})} + \frac{\overline{FCF}}{(1 + r_{\mathrm{A}}) \cdot (1 + r_{\mathrm{f}})^2} \\ &= \frac{\overline{FCF}}{1 + r_{\mathrm{A}}} \cdot \left(1 + \frac{1}{(1 + r_{\mathrm{f}})} + \frac{1}{(1 + r_{\mathrm{f}})^2}\right) \end{aligned}$$

Without considering more time periods, we notice the factor: $\left(1+\frac{1}{(1+r_\mathrm{f})}+\frac{1}{(1+r_\mathrm{f})^2}+\cdots+\frac{1}{(1+r_\mathrm{f})^{v-1}}\right)$, which represents a geometric series that can be reduced to $(1+r_\mathrm{f})\cdot\frac{(1+r_\mathrm{f})^v-1}{(1+r_\mathrm{f})^v\cdot r_\mathrm{f}}$. Let us denote the annuity factor $\frac{(1+r_\mathrm{f})^v-1}{(1+r_\mathrm{f})^v\cdot r_\mathrm{f}}$ in this expression by $\theta_{\mathrm{f},[v]}$. Hence, the value of the unlevered firm can be written as:

$$FV_{\mathrm{U},[v]}=\overline{FCF}\cdot\frac{1+r_\mathrm{f}}{1+r_\mathrm{A}}\cdot\theta_{\mathrm{f},[v]}\quad\text{with}\qquad\theta_{\mathrm{f},[v]}=\frac{(1+r_\mathrm{f})^v-1}{(1+r_\mathrm{f})^v\cdot r_\mathrm{f}}\tag{6}$$

In terms of the elapsed time t, we can express the value of the unlevered firm at t of a cash flow with original maturity T as follows:

$$FV_{\mathrm{U},t}=\overline{FCF}\cdot\frac{1+r_\mathrm{f}}{1+r_\mathrm{A}}\cdot\theta_{\mathrm{f},t}\qquad\text{with}\qquad\theta_{\mathrm{f},t}=\frac{(1+r_\mathrm{f})^{T-t}-1}{(1+r_\mathrm{f})^{T-t}\cdot r_\mathrm{f}}\tag{7}$$

Recursive Valuation of the Unlevered Firm and the Corresponding Discount Rate:

Let us now turn to the recursive calculation of the unlevered firm, which is given by the following recursive expression:

$$FV_{\mathrm{U},[v]}=\frac{\overline{FCF}+FV_{\mathrm{U},[v-1]}}{1+r_{\mathrm{U},[v]}}$$

In what follows, we want to derive the correct discount rate $r_{\mathrm{U},[v]}$ to be applied in this valuation formula. Let us therefore rearrange this expression to:

$$FV_{\mathrm{U},[v]}\cdot\left(1+r_{\mathrm{U},[v]}\right)=\overline{FCF}+FV_{\mathrm{U},[v-1]}\tag{8}$$

The value of the unlevered firm with remaining maturity v was given by expression (6). Accordingly, the value of the unlevered firm with a remaining maturity of $v-1$ is:

$$FV_{\mathrm{U},[v-1]} = \overline{FCF} \cdot \frac{1+r_{\mathrm{f}}}{1+r_{\mathrm{A}}} \cdot \theta_{\mathrm{f},[v-1]} \tag{9}$$

Placing (9) and (6) into (8), we obtain:

$$\overline{FCF} \cdot \frac{1+r_{\mathrm{f}}}{1+r_{\mathrm{A}}} \cdot \theta_{\mathrm{f},[v]} \cdot \left(1+r_{\mathrm{U},[v]}\right) = \overline{FCF} + \overline{FCF} \cdot \frac{1+r_{\mathrm{f}}}{1+r_{\mathrm{A}}} \cdot \theta_{\mathrm{f},[v-1]}$$

In the first step, let us solve this expression for $r_{\mathrm{U},[v]}$. We obtain:

$$r_{\mathrm{U},[v]} = \frac{\dfrac{1+r_{\mathrm{A}}}{1+r_{\mathrm{f}}} + \theta_{\mathrm{f},[v-1]} - \theta_{\mathrm{f},[v]}}{\theta_{\mathrm{f},[v]}} \tag{10}$$

Note that: $\theta_{\mathrm{f},[v-1]} - \theta_{\mathrm{f},[v]} = -\frac{1}{(1+r_{\mathrm{f}})^{v-1}}$. After simplifying this expression for $r_{\mathrm{U},[v]}$ we obtain the desired formula that allows us to compute the required return on unlevered equity dependent of the remaining maturity and the required return r_{A} for discounting the FCF plus the continuing value from $t+1$ to t:

$$r_{\mathrm{U},[v]} = r_{\mathrm{f}} \cdot \frac{(1+r_{\mathrm{A}}) \cdot (1+r_{\mathrm{f}})^{v-1} - 1}{(1+r_{\mathrm{f}})^{v} - 1} \tag{11}$$

Using the elapsed time of a cash flow with original maturity T, this expression becomes:

$$r_{\mathrm{U},t} = r_{\mathrm{f}} \cdot \frac{(1+r_{\mathrm{A}}) \cdot (1+r_{\mathrm{f}})^{T-t-1} - 1}{(1+r_{\mathrm{f}})^{T-t} - 1}$$

Direct Valuation of the Levered Firm:

Now we will look at the derivation of the value of the levered firm. For this purpose, we will start with a recursive description of the flow-to-equity method:

$$EV_{\mathrm{L},t-1} = \frac{\overline{FCF} - \overbrace{DV_{t-1} \cdot r_{\mathrm{f}} \cdot (1-\tau)}^{\text{Interest \& Tax Shield}} \overbrace{- DV_{t-1} + DV_t}^{\text{Change of principal}} + \overbrace{EV_{\mathrm{L},t}}^{\text{Continuing value}}}{1 + r_{\mathrm{EL},t-1}}$$

We will rearrange this expression slightly to the following:

$$EV_{\mathrm{L},t-1} = \frac{\overline{FCF} - DV_{t-1} \cdot [1 + r_{\mathrm{f}} \cdot (1-\tau)] + FV_{\mathrm{L},t}}{1 + r_{\mathrm{EL},t-1}}$$

In the following step, we separate the stochastic terms (expressed by expectations) from the deterministic terms:

$$EV_{\mathrm{L},t-1} = \frac{\overline{FCF}}{1 + r_{\mathrm{A}}} - \frac{DV_{t-1} \cdot [1 + r_{\mathrm{f}} \cdot (1-\tau)]}{1 + r_{\mathrm{f}}} + \frac{FV_{\mathrm{L},t}}{1 + r_{\mathrm{f}}}$$

Finally, we replace $EV_{\mathrm{L},t-1}$ and DV_{t-1} by $FV_{\mathrm{L},t-1} \cdot q$ and $FV_{\mathrm{L},t-1} \cdot (1-q)$ respectively. This brings us to:

$$FV_{\mathrm{L},t-1} \cdot \big(1 + q \cdot r_{\mathrm{f}} + (1-q) \cdot r_{\mathrm{f}} \cdot (1-\tau)\big) = \overline{FCF} \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} + FV_{\mathrm{L},t}$$

Let us abbreviate the term $q \cdot r_{\mathrm{f}} + (1-q) \cdot r_{\mathrm{f}} \cdot (1-\tau)$ by r_{x}. The recursive procedure for determining the value of the levered firm is therefore:

$$FV_{\mathrm{L},t-1} = \overline{FCF} \cdot \frac{1 + r_{\mathrm{f}}}{(1 + r_{\mathrm{A}}) \cdot (1 + r_{\mathrm{x}})} + \frac{FV_{\mathrm{L},t}}{(1 + r_{\mathrm{x}})}$$

This recursive calculation can now be transformed into a closed form by looking at how the value of the levered firm $FV_{\mathrm{L},t}$ evolves when we increase the remaining lifetime of the FCF. Let us start with a remaining maturity of $v = 1$ (corresponding to $T - 1$) where the continuing value of the firm equals zero by assumption, i.e., $FV_{\mathrm{L},T} = FV_{\mathrm{L},[0]} = 0$:

$$FV_{\mathrm{L},[1]} = \overline{FCF} \cdot \frac{1 + r_{\mathrm{f}}}{(1 + r_{\mathrm{A}}) \cdot (1 + r_{\mathrm{x}})}$$

Let us now go one period backwards in time, more precisely to a remaining maturity of $v = 2$ (corresponding to point in time $t = T - 2$). Here we obtain:

$$FV_{\mathrm{L},[2]} = \overline{FCF} \cdot \frac{1 + r_{\mathrm{f}}}{(1 + r_{\mathrm{A}}) \cdot (1 + r_{\mathrm{x}})} + \overline{FCF} \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} \cdot \frac{1}{(1 + r_{\mathrm{x}})^2}$$

Repeating the same computations as we go backwards in time, we observe the law:

$$FV_{\mathrm{L},[v]} = \overline{FCF} \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} \cdot \left(\frac{1}{1 + r_{\mathrm{x}}} + \frac{1}{(1 + r_{\mathrm{x}})^2} + \cdots + \frac{1}{(1 + r_{\mathrm{x}})^v} \right)$$

The geometric series on the right-hand side of this expression can be written as:

$$\theta_{\mathrm{x},[v]} = \frac{1}{1 + r_{\mathrm{x}}} + \frac{1}{(1 + r_{\mathrm{x}})^2} + \cdots + \frac{1}{(1 + r_{\mathrm{x}})^v} = \frac{(1 + r_{\mathrm{x}})^v - 1}{(1 + r_{\mathrm{x}})^v \cdot r_{\mathrm{x}}}$$

The closed form of the levered firm value is therefore:

$$FV_{\mathrm{L},[v]} = \overline{FCF} \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} \cdot \theta_{\mathrm{x},[v]} \qquad \text{with} \qquad \theta_{\mathrm{x},[v]} = \frac{(1 + r_{\mathrm{x}})^v - 1}{(1 + r_{\mathrm{x}})^v \cdot r_{\mathrm{x}}} \tag{12}$$

Again, we will express this value by means of the elapsed time:

$$FV_{\mathrm{L},t} = \overline{FCF} \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} \cdot \theta_{\mathrm{x},t} \qquad \text{with} \qquad \theta_{\mathrm{x},t} = \frac{(1 + r_{\mathrm{x}})^{T-t} - 1}{(1 + r_{\mathrm{x}})^{T-t} \cdot r_{\mathrm{x}}} \tag{13}$$

The values of debt and levered equity can be directly calculated by multiplying the value of the levered firm with $(1 - q)$ and q, respectively. The parameter q represents the equity-to-firm-value ratio q.

$$DV_t = (1 - q) \cdot FV_{\mathrm{L},t} = \overline{FCF} \cdot (1 - q) \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} \cdot \theta_{\mathrm{x},t}$$

$$EV_{\mathrm{L},t} = q \cdot FV_{\mathrm{L},t} = \overline{FCF} \cdot q \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} \cdot \theta_{\mathrm{x},t}$$

Recursive Valuation of the Levered Firm and the Corresponding Discount Rate:

We will now derive an expression for the discount rate in the FCF approach according to the following recursive expression:

$$FV_{\mathrm{L},[v]} = \frac{\overline{FCF} + FV_{\mathrm{L},[v-1]}}{1 + r_{\mathrm{FCF},[v]}}$$

Let us rearrange this expression to:

$$FV_{\mathrm{L},[v]} \cdot \left(1 + r_{\mathrm{FCF},[v]}\right) = \overline{FCF} + FV_{\mathrm{L},[v-1]} \tag{14}$$

The value of the levered firm at remaining maturity v was given by expression (12). Accordingly, the value of the levered firm at remaining maturity $v - 1$ is:

$$FV_{\mathrm{L},[v-1]} = \overline{FCF} \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} \cdot \theta_{\mathrm{x},[v-1]} \tag{15}$$

After placing (12) and (15) into (14) we obtain:

$$\overline{FCF} \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} \cdot \theta_{\mathrm{x},[v]} \cdot \left(1 + r_{\mathrm{FCF},[v]}\right) = \overline{FCF} + \overline{FCF} \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{A}}} \cdot \theta_{\mathrm{x},[v-1]}$$

After solving this expression for $r_{\mathrm{FCF},[v]}$ we obtain:

$$r_{\mathrm{FCF},[v]} = \frac{\frac{1 + r_{\mathrm{A}}}{1 + r_{\mathrm{f}}} + \theta_{\mathrm{x},[v-1]} - \theta_{\mathrm{x},[v]}}{\theta_{\mathrm{x},[v]}} \tag{16}$$

Knowing that $\theta_{\mathrm{x},[v-1]} - \theta_{\mathrm{x},[v]} = -\frac{1}{(1+r_{\mathrm{x}})^{v}}$, we can shorten this expression to the following:

$$r_{\mathrm{FCF},[v]} = r_{\mathrm{x}} \cdot \frac{(1 + r_{\mathrm{A}}) \cdot (1 + r_{\mathrm{x}})^{v} - (1 + r_{\mathrm{f}})}{[(1 + r_{\mathrm{x}})^{v} - 1] \cdot (1 + r_{\mathrm{f}})} \quad \text{or} \quad r_{\mathrm{FCF},t} = r_{\mathrm{x}} \cdot \frac{(1 + r_{\mathrm{A}}) \cdot (1 + r_{\mathrm{x}})^{T-t} - (1 + r_{\mathrm{f}})}{[(1 + r_{\mathrm{x}})^{T-t} - 1] \cdot (1 + r_{\mathrm{f}})}$$

Relationship between r_{FCF} and r_{U}:

Let us now look at the translation formula between the required return of the unlevered firm and the discount rate in the FCF method. Departing from (10) and (16), we can state:

$$r_{\mathrm{U},[v]} \cdot \theta_{\mathrm{f},[v]} + \frac{1}{(1+r_{\mathrm{f}})^{v}} = \frac{1+r_{\mathrm{A}}}{1+r_{\mathrm{f}}} \qquad \text{and} \qquad r_{\mathrm{FCF},[v]} \cdot \theta_{\mathrm{x},[v]} + \frac{1}{(1+r_{\mathrm{x}})^{v}} = \frac{1+r_{\mathrm{A}}}{1+r_{\mathrm{f}}}$$

Hence, we can immediately conclude the relationship between the discount factor in the FCF method and the required return on unlevered equity:

$$r_{\mathrm{FCF},[v]} = \frac{r_{\mathrm{U},[v]} \cdot \theta_{\mathrm{f},[v]} + \frac{1}{(1+r_{\mathrm{f}})^{v}} - \frac{1}{(1+r_{\mathrm{x}})^{v}}}{\theta_{\mathrm{x},[v]}}$$

or (17)

$$r_{\mathrm{FCF},t} = \frac{r_{\mathrm{U},t} \cdot \theta_{\mathrm{f},t} + \frac{1}{(1+r_{\mathrm{f}})^{T-t}} - \frac{1}{(1+r_{\mathrm{x}})^{T-t}}}{\theta_{\mathrm{x},t}}$$

Below, we will also show that this translation reduces to the formula proposed by M&M when the FCF becomes a perpetuity.

Direct Valuation of the Tax Shield and r_{TS}:

The value of the tax shield is the difference between the value of the levered firm $FV_{\mathrm{L},t}$ given by expression (13) and the value of the unlevered firm $FV_{\mathrm{U},t}$ given by expression (7). Using these two expressions leads immediately to:

$$TSV_t = FV_{\mathrm{L},t} - FV_{\mathrm{U},t} = \overline{FCF} \cdot \frac{1+r_{\mathrm{f}}}{1+r_{\mathrm{A}}} \cdot \left(\theta_{\mathrm{x},T-t} - \theta_{\mathrm{f},T-t}\right)$$

For the sake of completeness, we will here repeat that the required return of the tax shield in the recursive calculation equals the risk-free rate, i.e., $r_{\mathrm{TS}} = r_{\mathrm{f}}$. The recursive calculation of the value of the tax shield is therefore:

$$TSV_{t-1} = \frac{TS_t + TSV_t}{1 + r_{\mathrm{f}}} = \frac{DV_{t-1} \cdot r_{\mathrm{f}} \cdot \tau + TSV_t}{1 + r_{\mathrm{f}}}$$

6 The Miles-Ezzell Annuity

In this section we develop the valuation formulas for an FCF annuity that can be represented by a first-order autoregressive process that is assumed in M&E. We will apply the same structure as used for M&M: First we look at the direct and recursive calculation of the unlevered firm. Then we determine the direct and recursive valuation of the levered firm. We then state the translation between the required return of the unlevered firm and the discount rate in the FCF method. Finally, we look at the valuation of the tax shield and the corresponding discount rate.

Direct and Recursive Valuation of the Unlevered Firm:

Contrary to M&M, the required return on the unlevered firm is constant across time. By assumption 4, the unconditional expected FCF is also constant. We can therefore apply the traditional annuity formula for directly calculating the value of the unlevered firm for any given remaining maturity or point in time as follows:

$$FV_{\mathrm{U},[v]} = \overline{FCF} \cdot \frac{(1 + r_{\mathrm{U}})^v - 1}{(1 + r_{\mathrm{U}})^v \cdot r_{\mathrm{U}}} \quad \text{or} \quad FV_{\mathrm{U},t} = \overline{FCF} \cdot \frac{(1 + r_{\mathrm{U}})^{T-t} - 1}{(1 + r_{\mathrm{U}})^{T-t} \cdot r_{\mathrm{U}}}$$

Because of the constancy of $r_{\mathrm{U}} = r_{\mathrm{A}}$ the recursive calculation is simply:

$$FV_{\mathrm{U},[v]} = \frac{\overline{FCF} + \overline{FV}_{\mathrm{U},v-1}}{1 + r_{\mathrm{U}}} \quad \text{or} \quad FV_{\mathrm{U},t} = \frac{\overline{FCF} + \overline{FV}_{\mathrm{U},t+1}}{1 + r_{\mathrm{U}}}$$

Since firm values evolve in a non-deterministic manner throughout time (see figure 4), it is useful to notice that these formulas can be used to calculate both expected firm values $\mathbb{E}\left[FV_{\mathrm{U},r} \mid \mathcal{F}_{t<r}\right]$ as

well as deterministic firm values $FV_{\mathrm{U},t} \mid \mathcal{F}_t$. Here $\mathcal{F}_t$ represents the available historical information at point in time t.

Direct and Recursive Valuation of the Levered Firm, Levered Equity and Debt:

In the M&E setup, the discount rate applied in the FCF method is constant across time. As in the case of the unlevered firm we can therefore apply the traditional annuity formula for directly calculating the value of the levered firm for any given remaining maturity:

$$FV_{\mathrm{L},[v]} = \overline{FCF} \cdot \frac{(1+r_{\mathrm{FCF}})^{v} - 1}{(1+r_{\mathrm{FCF}})^{v} \cdot r_{\mathrm{FCF}}} \quad \text{with} \quad r_{\mathrm{FCF}} = q \cdot r_{\mathrm{EL}} + (1-q) \cdot r_{\mathrm{f}} \cdot (1-\tau)$$

or

$$FV_{\mathrm{L},t} = \overline{FCF} \cdot \frac{(1+r_{\mathrm{FCF}})^{T-t} - 1}{(1+r_{\mathrm{FCF}})^{T-t} \cdot r_{\mathrm{FCF}}} \quad \text{with} \quad r_{\mathrm{FCF}} = q \cdot r_{\mathrm{EL}} + (1-q) \cdot r_{\mathrm{f}} \cdot (1-\tau)$$

Because of the constancy of r_{FCF} the recursive calculation is simply:

$$FV_{\mathrm{L},[v]} = \frac{\overline{FCF} + \overline{FV}_{\mathrm{L},[v-1]}}{1+r_{\mathrm{FCF}}} \quad \text{or} \quad FV_{\mathrm{L},t} = \frac{\overline{FCF} + \overline{FV}_{\mathrm{L},t+1}}{1+r_{\mathrm{FCF}}}$$

The value of the debt and levered equity can now be directly calculated by multiplying the value of the levered firm with $(1-q)$ and q, respectively. Again, q represents the equity-to-firm-value ratio.

$$DV_{\mathrm{L},[v]} = (1-q) \cdot FV_{\mathrm{L},[v]} \qquad \text{and} \qquad EV_{\mathrm{L},[v]} = q \cdot FV_{\mathrm{L},[v]}$$

As in the case of the unlevered firm value, these formulas can be used to calculate both deterministic and expected values.

Relationship between r_{FCF} and r_{U}:

The relationship between r_{FCF} and r_{U} is readily found in the literature. Since $r_{\mathrm{U},t}$ is constant throughout time, it will be the same for a single-period and an infinite FCF. For the single-period case the formula appears in Myers (1974, p.13). For a perpetual stream of cash flows this formula has been provided by Miles and Ezzell (1980, p. 726).

$$r_{\mathrm{FCF}} = r_{\mathrm{U}} - (1-q) \cdot r_{\mathrm{f}} \cdot \tau \cdot \frac{1+r_{\mathrm{U}}}{1+r_{\mathrm{f}}} \tag{18}$$

Valuation of the Tax Shield and the Corresponding Discount Rate:

The value of the tax shield is the difference between the value of the levered and the unlevered firm.

$$TSV_{[v]} = FV_{\mathrm{L},[v]} - FV_{\mathrm{U},[v]} = \overline{FCF} \cdot \left(\frac{(1+r_{\mathrm{FCF}})^{v} - 1}{(1+r_{\mathrm{FCF}})^{v} \cdot r_{\mathrm{FCF}}} - \frac{(1+r_{\mathrm{U}})^{v} - 1}{(1+r_{\mathrm{U}})^{v} \cdot r_{\mathrm{U}}} \right) \tag{19}$$

The recursive valuation and the discount rate for the tax shield are defined as follows:

$$TSV_{[v]} = \frac{DV_{[v]} \cdot r_{\mathrm{f}} \cdot \tau + \overline{TSV}_{[v-1]}}{1 + r_{\mathrm{TS},[v]}} \quad \rightarrow \quad r_{\mathrm{TS},[v]} = \frac{DV_{[v]} \cdot r_{\mathrm{f}} \cdot \tau + \overline{TSV}_{[v-1]}}{TSV_{[v]}} - 1 \tag{20}$$

The debt value can be calculated by means of the debt-to-firm-value ratio.

$$DV_{[v]} = (1-q) \cdot FV_{\mathrm{L},[v]} \tag{21}$$

Substituting (19) and (21) into (20) yields the following expression for the discount rate of the tax shield:

$$r_{\mathrm{TS},[v]} = \frac{(1-q) \cdot r_{\mathrm{f}} \cdot \tau \cdot \varphi_{\mathrm{FCF},[v]} - (1+r_{\mathrm{FCF}})^{-v} + (1+r_{\mathrm{U}})^{-v}}{\varphi_{\mathrm{FCF},[v]} - \varphi_{\mathrm{U},[v]}} \tag{22}$$

where $\varphi_{\mathrm{FCF},[v]}$ and $\varphi_{\mathrm{U},[v]}$ represent the annuity factors:

$$\varphi_{\mathrm{FCF},[v]} = \frac{(1+r_{\mathrm{FCF}})^{v} - 1}{(1+r_{\mathrm{FCF}})^{v} \cdot r_{\mathrm{FCF}}} \quad \text{and} \quad \varphi_{\mathrm{U},[v]} = \frac{(1+r_{\mathrm{U}})^{v} - 1}{(1+r_{\mathrm{U}})^{v} \cdot r_{\mathrm{U}}} \tag{23}$$

Note that r_{FCF} in this formula and in the factor $\varphi_{\mathrm{L},[v]}$ is related to r_{U} by means of expression (18). Hence, the discount rate of the tax shield can be seen as a function of the unlevered return r_{U}, the risk-free rate r_f, the tax rate τ, the leverage q, and the remaining cash flow maturity v. Stating the discount rate $r_{\mathrm{TS},[v]}$ in terms of r_U solely (without r_FCF) leads to a rather large-sized expression, which is omitted here.

7 Transition to Perpetuities

Logically, the formulas stated above should collapse to the well-known formulas of M&M and M&E if the remaining maturity tends to infinity. The purpose of this section is to quickly show that this is indeed the case.

Modigliani & Miller: Departing from (6) and (12), we can determine the values of the unlevered and levered firm when the remaining maturity tends to infinity. The results are easily obtained as follows:

$$FV_{\mathrm{U},[\infty]} = \overline{FCF} \cdot \frac{1+r_\mathrm{f}}{1+r_\mathrm{A}} \cdot \frac{1}{r_\mathrm{f}} \quad \text{and} \quad FV_{\mathrm{L},[\infty]} = \overline{FCF} \cdot \frac{1+r_\mathrm{f}}{1+r_\mathrm{A}} \cdot \frac{1}{r_\mathrm{x}} \qquad (24)$$

$$\text{where } r_\mathrm{x} = q \cdot r_\mathrm{f} + (1-q) \cdot (1-\tau) \cdot r_\mathrm{f}$$

The relationship between the required return of the unlevered firm and the discount rate in the FCF method is given by expression (17). From this expression we can find the limit:

$$r_{\mathrm{FCF},[\infty]} = \lim_{v \to \infty} r_{\mathrm{FCF},[v]} = \frac{r_{\mathrm{U},[\infty]} \cdot r_\mathrm{x}}{r_\mathrm{f}} \quad \text{with} \quad r_\mathrm{x} = [q \cdot r_\mathrm{f} + (1-q) \cdot (1-\tau) \cdot r_\mathrm{f}]$$

which reduces to:

$$r_{\mathrm{FCF},[\infty]} = r_{\mathrm{U},[\infty]} - r_{\mathrm{U},[\infty]} \cdot (1-q) \cdot \tau$$

This is the well-known result of Modigliani & Miller (1963, p. 438, based on their formula 31.c). If we want to relate the discount rates $r_{\mathrm{FCF},[\infty]}$ or $r_{\mathrm{U},[\infty]}$ to the underlying stochasticity of the FCF, it can be useful to consider the relationship between $r_{\mathrm{U},[v]}$ and r_{A}, which can be retrieved as the limit of (11) or more directly from (24) as follows:

$$r_{\mathrm{U},[\infty]} = r_{\mathrm{f}} \cdot \frac{1 + r_{\mathrm{A}}}{1 + r_{\mathrm{f}}}$$

Miles & Ezzell: Let us start with the annuity factors stated in (23). The limits of these factors are:

$$\lim_{v \to \infty} \varphi_{\mathrm{L},[v]} = \frac{1}{r_{\mathrm{FCF}}} \quad \text{and} \quad \lim_{v \to \infty} \varphi_{\mathrm{U},[v]} = \frac{1}{r_{\mathrm{U}}}$$

For the values of the levered and unlevered firm this implies:

Value of levered firm

$$FV_{\mathrm{L},[\infty]} = \overline{FCF} \cdot \frac{1}{r_{\mathrm{FCF}}}$$

Valuation of unlevered firm

$$FV_{\mathrm{U},[\infty]} = \overline{FCF} \cdot \frac{1}{r_{\mathrm{U}}}$$

From these two expressions we can directly obtain the value of the tax shield as dependent on the FCF as follows:

$$TSV_{[\infty]} = \overline{FCF} \cdot \left(\frac{1}{r_{\mathrm{FCF}}} - \frac{1}{r_{\mathrm{U}}} \right)$$

The relationship between r_{FCF} and r_{U} is already stated in (18) and can readily be inserted into this expression.

For stating the dependency between the required return on unlevered equity and the tax shield we can depart from expression (22) and take the limit $v \to \infty$. This gives:

$$r_{\mathrm{TS},[\infty]} = \frac{(1 - q) \cdot r_{\mathrm{f}} \cdot \tau \cdot \frac{1}{r_{\mathrm{FCF}}}}{\frac{1}{r_{\mathrm{FCF}}} - \frac{1}{r_{\mathrm{U}}}}$$

After adding the relationship (18) into this expression we obtain:

$$r_{\mathrm{TS},[\infty]} = r_{\mathrm{U}} \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{U}}}$$

This result agrees with both Arzac & Glosten (2005, equation 13 with a growth rate $g = 0$) and Barbi (2012, equation 15).

8 Conclusions and Formula Overview

Those who do business valuation and capital budgeting in academia or practice want to evaluate streams of cash flows of firms or projects with appropriate and consistent methods. They also want to be able to analyze how leverage or changes in interest rates, tax rates, time to maturity, etc., affect the value of the project or firm. The literature provides a couple of valuation frameworks that practitioners can apply in capital budgeting or firm valuation. While in practice the structure and length of cash flows can be very different from project to project or firm to firm, the literature has developed complete frameworks only for a few possible types of cash flow streams. By complete we mean, that the valuation practitioner is able to consistently apply both the flow-to-equity method, the FCF method, the CCF method, and the adjusted present value method as well as to translate between these methods by means of appropriate formulas. Two such frameworks are these of Modigliani & Miller (1963, M&M) and Miles & Ezzell (1980, M&E). However, these two mutually exclusive frameworks are applicable only to perpetuities. The well-known formulas that correspond to these mutually exclusive approaches of M&M and M&E are summarized in table 1. A framework for arbitrary cash flows with a finite lifetime under prespecified debt financing is shown in Becker (2021).

In this paper, we focused on the establishment of a complete valuation framework for both autoregressive and stationary stochastic annuities. A summary of all formulas is shown in tables 2 to 5. More specifically, this paper's contributions relative to the existing literature are the following:

(1) For autoregressive stochastic annuities (cash flows according to M&E), the previous literature has not provided a formula for the required return on the tax shield. This formula is established here. This formula reveals that the required return on the tax shield depends on the remaining maturity of the FCF. This is not the case for stationary cash flows (cash flows according to M&M).

(2) Furthermore, we show the direct calculation of the value of the tax shield based on the FCF. This is a more convenient approach than an iterative backward valuation (like, for example, in Miles & Ezzell, 1985).

(3) We also show that autoregressive cash flow processes allow the application of the standard annuity formula for both the computation of the value of the unlevered firm and the valuation of the levered firm by means of the FCF method.

(4) Besides this, we confirm the translation formulas from r_{U} to r_{EL}, r_{U} to r_{FCF}, and r_{U} to r_{CCF} that have been developed in the previous literature for autoregressive cash flow processes. Contrary to stationary annuities, these translations are all independent of the remaining maturity of the FCF.

(5) For the stationary annuities the previous literature has not provided a consistent set of formulas. For this case we have established all the translation formulas, i.e., from r_{U} to r_{EL}, form r_{U} to r_{FCF} and from r_{U} to r_{CCF}). We observe that these formulas turn out differently than for perpetuities.

(6) In particular, we see that the required return on levered and unlevered equity and the discount rate in the FCF method vary with the remaining maturity of the FCF. This means that applying

the original formulas of M&M to cash flows with a finite lifetime will cause invalid and inconsistent valuation results.

Providing a framework for the valuation of both stationary and autoregressive annuities enriches the set of evaluation techniques for both academics and practitioners. This means that when working with annuities in capital budgeting or firm valuation, it is no longer necessary to regress to inappropriate or approximative formulas (like the formulas that apply to perpetuities). Nevertheless, one needs to be aware that translation formulas like the ones shown in tables 1 to 5 have only be established for time-invariant and deterministic pricing operators (see assumption 2 in section 2) and stationary or autoregressive cash flow processes (assumptions 16a or 16b). Applying these formulas to other stochastic environments will lead to valuation errors or inconsistencies between valuation methods.

Some of the strict assumptions presented in section 2 can be relaxed. Now that the basic framework for annuities exists, some interesting and immediate extensions are the inclusion of personal taxation, the consideration of growth, or the inclusion of risky debt.

Table 1: Valuation frameworks for perpetual FCF by Modigliani & Miller (1963) and Miles & Ezzell (1980, 1985)

	Modigliani/Miller Annuity	***Miles/Ezzell Annuity***
Flow-to-equity method:	$EV_{\mathrm{L}} = \frac{(EBIT - I) \cdot (1-\tau)}{r_{\mathrm{EL}}}$ where $I = DV \cdot r_f$	
FCF method:	$FV_{\mathrm{L}} = \frac{EBIT \cdot (1-\tau)}{r_{\mathrm{FCF}}}$ where $r_{\mathrm{FCF}} = q \cdot r_{\mathrm{EL}} + (1-q) \cdot (1-\tau) \cdot r_{\mathrm{D}}$	
CCF method:	$FV_{\mathrm{L}} = \frac{(EBIT - I) \cdot (1-\tau) + I}{r_{\mathrm{CCF}}}$ where $r_{\mathrm{CCF}} = q \cdot r_{\mathrm{EL}} + (1-q) \cdot r_{\mathrm{D}}$	
APV method:	$FV_{\mathrm{L}} = FV_{\mathrm{U}} + TSV \quad FV_{\mathrm{U}} = \frac{EBIT \cdot (1-\tau)}{r_{\mathrm{U}}} \quad TSV = \frac{TS}{r_{\mathrm{TS}}} = \frac{DV \cdot \tau \cdot r_{\mathrm{f}}}{r_{\mathrm{TS}}}$	
Tax shield	$r_{\mathrm{TS}} = r_{\mathrm{f}}$ $TSV = \frac{DV \cdot r_{\mathrm{f}} \cdot \tau}{r_{\mathrm{f}}} = DV \cdot \tau$	$r_{\mathrm{TS}} = r_{\mathrm{U}} \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{U}}}$ $TSV = DV \cdot \tau \cdot \frac{r_{\mathrm{f}} \cdot (1 + r_{\mathrm{U}})}{r_{\mathrm{U}} \cdot (1 + r_{\mathrm{f}})}$
Translation $\boldsymbol{r_{\mathrm{U}} \rightarrow r_{\mathrm{EL}}}$**:**	$r_{\mathrm{EL}} = r_{\mathrm{U}} + (1-\tau) \cdot (r_{\mathrm{U}} - r_{\mathrm{f}}) \cdot \frac{(1-q)}{q}$	$r_{\mathrm{EL}} = \frac{r_{\mathrm{U}} - r_{\mathrm{f}} \cdot (1-q) \cdot \left[\frac{\tau \cdot (r_{\mathrm{U}} - r_{\mathrm{f}})}{1 + r_{\mathrm{f}}} + 1\right]}{q}$
Translation $\boldsymbol{r_{\mathrm{U}} \rightarrow r_{\mathrm{FCF}}}$**:**	$r_{\mathrm{FCF}} = r_{\mathrm{U}} - r_{\mathrm{U}} \cdot \tau \cdot (1-q)$	$r_{\mathrm{FCF}} = r_{\mathrm{U}} - r_{\mathrm{f}} \cdot \tau \cdot (1-q) \cdot \frac{1 + r_{\mathrm{U}}}{1 + r_{\mathrm{f}}}$
Translation $\boldsymbol{r_{\mathrm{U}} \rightarrow r_{\mathrm{CCF}}}$**:**	$r_{\mathrm{CCF}} = r_{\mathrm{U}} + (r_{\mathrm{f}} - r_{\mathrm{U}}) \cdot \tau \cdot (1-q)$	$r_{\mathrm{CCF}} = r_{\mathrm{U}} - r_{\mathrm{f}} \cdot (1-q) \cdot \frac{\tau \cdot (r_{\mathrm{U}} - r_{\mathrm{f}})}{1 + r_{\mathrm{f}}}$

Table 2: Recursive calculation of values

<table>
<tr><th></th><th>Modigliani/Miller Annuity</th><th>Miles/Ezzell Annuity</th></tr>
<tr><td>$FV_{\mathrm{U},t}$ by means of $r_{\mathrm{U},t}$</td><td colspan="2">$$FV_{\mathrm{U},t} = \frac{\overline{FCF} + FV_{\mathrm{U},t+1}}{1 + r_{\mathrm{U},t}}$$
Note that $FV_{\mathrm{U},t}$ changes over time. Furthermore, $r_{\mathrm{U},t}$ is time-invariant in the case of M&E, but will change across time in the case of M&M.</td></tr>
<tr><td>$FV_{\mathrm{L},t}$ by means of $r_{\mathrm{FCF},t}$ (free-cash-flow method)</td><td colspan="2">$$FV_{\mathrm{L},t} = \frac{\overline{FCF} + FV_{\mathrm{L},t+1}}{1 + r_{\mathrm{FCF},t}}$$
with $r_{\mathrm{FCF},t} = q_t \cdot r_{\mathrm{EL},t} + (1 - q) \cdot r_{\mathrm{f}} \cdot (1 - \tau)$
Note that $FV_{\mathrm{L},t}$ changes over time. Furthermore, $r_{\mathrm{EL},t}$ and $r_{\mathrm{FCF},t}$ are time-invariant in the case of M&E, but these discount rates change across time in the case of M&M.</td></tr>
<tr><td>$FV_{\mathrm{U},t}$ by means of r_{A}</td><td>$$FV_{\mathrm{U},t} = \frac{\overline{FCF}}{1 + r_{\mathrm{A}}} + \frac{FV_{\mathrm{U},t+1}}{1 + r_{\mathrm{f}}}$$</td><td>$$FV_{\mathrm{U},t} = \frac{\overline{FCF} + \overline{FV}_{\mathrm{U},t+1}}{1 + r_{\mathrm{A}}}$$
Note that in M&E: $r_{\mathrm{U},t} = r_{\mathrm{A}}$</td></tr>
<tr><td>$FV_{\mathrm{L},t}$ by means of r_{A}</td><td>$$FV_{\mathrm{L},t} = \frac{\overline{FCF}}{1 + r_{\mathrm{A}}} \cdot \frac{1 + r_{\mathrm{f}}}{1 + r_{\mathrm{x}}} + \frac{FV_{\mathrm{L},t+1}}{1 + r_{\mathrm{x}}}$$
where $r_{\mathrm{x}} = q \cdot r_{\mathrm{f}} + (1 - q) \cdot r_{\mathrm{f}} \cdot (1 - \tau)$</td><td>$$FV_{\mathrm{L},t} = \frac{\overline{FCF} + \overline{FV}_{\mathrm{L},t+1}}{1 + r_{\mathrm{A}} - (1 - q) \cdot r_{\mathrm{f}} \cdot \tau \cdot \frac{1 + r_{\mathrm{A}}}{1 + r_{\mathrm{f}}}}$$
Note, that in M&E: $r_{\mathrm{U},t} = r_{\mathrm{A}}$</td></tr>
<tr><td>TSV_t</td><td>$$TSV_t = \frac{TS_{t+1} + TSV_{t+1}}{1 + r_{\mathrm{f}}}$$</td><td>$$TSV_t = \frac{TS_{t+1} + \overline{TSV}_{t+1}}{1 + r_{\mathrm{TS},t}}$$
where $r_{\mathrm{TS},t}$ is defined in table 4</td></tr>
</table>

Table 3: Direct calculation means of the FCF

	Modigliani/Miller Annuity	*Miles/Ezzell Annuity*
$FV_{\mathrm{U},t}$	$FV_{\mathrm{U},t} = \overline{FCF} \cdot \frac{1+r_{\mathrm{f}}}{1+r_{\mathrm{A}}} \cdot \theta_{\mathrm{f},t}$ with $\theta_{\mathrm{f},t} = \frac{(1+r_{\mathrm{f}})^{T-t}-1}{(1+r_{\mathrm{f}})^{T-t} \cdot r_{\mathrm{f}}}$	$FV_{\mathrm{U},t} = \overline{FCF} \cdot \varphi_{\mathrm{U},t}$ with $\varphi_{\mathrm{U},t} = \frac{(1+r_{\mathrm{U}})^{T-t}-1}{(1+r_{\mathrm{U}})^{T-t} \cdot r_{\mathrm{U}}}$
$FV_{\mathrm{L},t}$ (FCF or CCF method)	$FV_{\mathrm{L},t} = \overline{FCF} \cdot \frac{1+r_{\mathrm{f}}}{1+r_{\mathrm{A}}} \cdot \theta_{\mathrm{x},t}$ with $\theta_{\mathrm{x},t} = \frac{(1+r_{\mathrm{x}})^{T-t}-1}{(1+r_{\mathrm{x}})^{T-t} \cdot r_{\mathrm{x}}}$ and $r_{\mathrm{x}} = q \cdot r_{\mathrm{f}} + (1-q) \cdot r_{\mathrm{f}} \cdot (1-\tau)$	$FV_{\mathrm{L},t} = \overline{FCF} \cdot \varphi_{\mathrm{L},t}$ with $\varphi_{\mathrm{L},t} = \frac{(1+r_{\mathrm{FCF}})^{T-t}-1}{(1+r_{\mathrm{FCF}})^{T-t} \cdot r_{\mathrm{FCF}}}$ and $r_{\mathrm{FCF}} = q \cdot r_{\mathrm{EL},t} + (1-q) \cdot r_{\mathrm{f}} \cdot (1-\tau)$
TSV_t from $\overline{FCF}$ (part of APV method)	$TSV_t = \overline{FCF} \cdot \frac{1+r_{\mathrm{f}}}{1+r_{\mathrm{A}}} \cdot \left(\theta_{\mathrm{x},t} - \theta_{\mathrm{f},t}\right)$ $\theta_{\mathrm{f},t}$ and $\theta_{\mathrm{x},t}$ are defined above.	$TSV_t = \overline{FCF} \cdot \left(\varphi_{\mathrm{L},t} - \varphi_{\mathrm{U},t}\right)$ $\varphi_{\mathrm{U},t}$ and $\varphi_{\mathrm{L},t}$ are defined above.

Table 4: Transition formulas between discount rates (These discount rates only apply to recursive calculations in table 2)

	Modigliani/Miller Annuity	***Miles/Ezzell Annuity***
$r_{\mathrm{U},t}$ from r_{A}	$r_{\mathrm{U},t} = r_{\mathrm{f}} \cdot \frac{(1+r_{\mathrm{A}}) \cdot (1+r_{\mathrm{f}})^{T-t-1} - 1}{(1+r_{\mathrm{f}})^{T-t} - 1}$	$r_{\mathrm{U}} = r_{\mathrm{A}}$
$r_{\mathrm{FCF},t}$ from r_{A}	$r_{\mathrm{FCF},t} = r_{\mathrm{x}} \cdot \frac{(1+r_{\mathrm{A}}) \cdot (1+r_{\mathrm{x}})^{T-t} - (1+r_{\mathrm{f}})}{[(1+r_{\mathrm{x}})^{T-t} - 1] \cdot (1+r_{\mathrm{f}})}$	$r_{\mathrm{FCF}} = r_{\mathrm{A}} - (1-q) \cdot r_{\mathrm{f}} \cdot \tau \cdot \frac{1+r_{\mathrm{A}}}{1+r_{\mathrm{f}}}$
$r_{\mathrm{FCF},t}$ from $r_{\mathrm{U},t}$	$r_{\mathrm{FCF},t} = \frac{r_{\mathrm{U},t} \cdot \theta_{\mathrm{f},t} + \frac{1}{(1+r_{\mathrm{f}})^{T-t}} - \frac{1}{(1+r_{\mathrm{x}})^{T-t}}}{\theta_{\mathrm{x},t}}$	$r_{\mathrm{FCF}} = r_{\mathrm{U}} - (1-q) \cdot r_{\mathrm{f}} \cdot \tau \cdot \frac{1+r_{\mathrm{U}}}{1+r_{\mathrm{f}}}$
$r_{\mathrm{EL},t}$ from $r_{\mathrm{U},t}$	$r_{\mathrm{EL},t} = \frac{r_{\mathrm{U},t} \cdot \theta_{\mathrm{f},t} - (1-q) \cdot r_{\mathrm{f}} \cdot (1-\tau) \cdot \theta_{\mathrm{x},t} + \frac{1}{(1+r_{\mathrm{f}})^{T-t}} - \frac{1}{(1+r_{\mathrm{x}})^{T-t}}}{q \cdot \theta_{\mathrm{x},t}}$	$r_{\mathrm{EL}} = \frac{r_{\mathrm{U}} - (1-q) \cdot r_{\mathrm{f}} \cdot \left[1 + \tau \cdot \frac{r_{\mathrm{U}} - r_{\mathrm{f}}}{1+r_{\mathrm{f}}}\right]}{q}$
Discount Rate of Tax Shield	$r_{\mathrm{TS}} = r_f$	$r_{\mathrm{TS},t} = \frac{(1-q) \cdot r_{\mathrm{f}} \cdot \tau \cdot \varphi_{\mathrm{L},t} - (1+r_{\mathrm{FCF}})^{-(T-t)} + (1+r_{\mathrm{U}})^{-(T-t)}}{\varphi_{\mathrm{L},t} - \varphi_{\mathrm{U},t}}$

Table 5: Component wise calculations of the values

	Modigliani/Miller Annuity	*Miles/Ezzell Annuity*
$FV_{\mathrm{U},t}$	$FV_{\mathrm{U},t} = \sum_{i=t+1}^{T} \left(\frac{(\overline{EBIT}) \cdot (1-\tau)}{(1+r_{\mathrm{f}})^{i-t-1} \cdot (1+r_{\mathrm{A}})} \right)$	$FV_{\mathrm{U},t} = \sum_{i=t+1}^{T} \left(\frac{(\overline{EBIT}) \cdot (1-\tau)}{(1+r_{\mathrm{A}})^{i-t}} \right)$
TSV_t from $\overline{TS}$ (as part of APV method)	$TSV_t = \sum_{i=t+1}^{T} \frac{TS_i}{(1+r_{\mathrm{f}})^{i-t}}$ Note that tax shield is deterministic. Note time-dependence of TS_i; annuity formula is not applicable.	$TSV_t = \sum_{i=t+1}^{T} \frac{\overline{TS}_i}{(1+r_{\mathrm{A}})^{i-t-1} \cdot (1+r_{\mathrm{f}})}$ See also Miles and Ezzell (1985) Note that tax shield is stochastic. Note time-dependence of TS_i; annuity formula is not applicable. Note that $r_{\mathrm{U}} = r_A$.
Capital-cash-flow method	$FV_{\mathrm{L},t} = \sum_{i=t+1}^{T} \left(\frac{(\overline{EBIT}) \cdot (1-\tau)}{(1+r_{\mathrm{f}})^{i-t-1} \cdot (1+r_{\mathrm{A}})} + \frac{I_{t+1} \cdot \tau}{(1+r_{\mathrm{f}})^{i-t}} \right)$ where the I_{t+1} will vary with time. Note that $r_{\mathrm{A}} \neq r_{\mathrm{U},t}$.	$FV_{\mathrm{L},t} = \sum_{i=t+1}^{T} \left(\frac{(\overline{EBIT}) \cdot (1-\tau)}{(1+r_{\mathrm{A}})^{i-t}} + \frac{\bar{I}_{t+1} \cdot \tau}{(1+r_{\mathrm{A}})^{i-t-1} \cdot (1+r_{\mathrm{f}})} \right)$ where the $\bar{I}_{t+1}$ will vary with time, and $r_{\mathrm{U}} = r_A$.

Continuation of table 5: Component wise calculations of the values

Flow-to-equity method	$EV_{\mathrm{L},t} = \sum_{i=t+1}^{T}\left(\frac{(\overline{EBIT})\cdot(1-\tau)}{(1+r_{\mathrm{f}})^{i-t-1}\cdot(1+r_{\mathrm{A}})} - \frac{I_{t+1}\cdot(1-\tau)+\Delta DV_{t+1}}{(1+r_{\mathrm{f}})^{i-t}}\right)$ where the I_{t+1} and $\Delta DV_{t+1} = DV_t - DV_{t+1}$ will vary with time. Note that $r_{\mathrm{A}} \neq r_{\mathrm{U},t}$.	$EV_{\mathrm{L},t} = \sum_{i=t+1}^{T}\left(\frac{(\overline{EBIT})\cdot(1-\tau)+\overline{DV}_{t+1}}{(1+r_{\mathrm{A}})^{i-t}} - \frac{\bar{I}_{t+1}\cdot(1-\tau)+DV_t}{(1+r_{\mathrm{A}})^{i-t-1}\cdot(1+r_{\mathrm{f}})}\right)$ where the $\bar{I}_{t+1}$, $\overline{DV}_{t+1}$ and DV_t will vary with time. We cannot apply $\Delta DV_{t+1} = DV_t - DV_{t+1}$ because different discount rates apply. Note that $r_{\mathrm{A}} = r_{\mathrm{U}}$
Debt valuation	$DV_{\mathrm{L},t} = \sum_{i=t+1}^{T}\left(\frac{I_{t+1}+\Delta DV_{t+1}}{(1+r_{\mathrm{f}})^{i-t}}\right)$ where the I_{t+1} and ΔDV_{t+1} will vary with time. Note that $r_{\mathrm{A}} \neq r_{\mathrm{U},t}$.	$DV_{\mathrm{L},t} = \sum_{i=t+1}^{T}\left(\frac{\bar{I}_{t+1}+DV_t}{(1+r_{\mathrm{A}})^{i-t-1}\cdot(1+r_{\mathrm{f}})} - \frac{DV_{t+1}}{(1+r_{\mathrm{A}})^{i-t}}\right)$ where the $\bar{I}_{t+1}$, $\overline{DV}_{t+1}$ and DV_t will vary with time. We cannot apply $\Delta DV_{t+1} = DV_t - DV_{t+1}$ because different discount rates apply. Note that $r_{\mathrm{A}} = r_{\mathrm{U}}$

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Appendix 1 – The Discount Rate in the FCF method for $\boldsymbol{DN \neq DV}$

This appendix shows, how a violation of assumption 15 affects the calculation of the discount rate in the free cash flow method (weighted average cost of capital with taxes). Assume that the nominal (contractual) interest rate r_{nom} is different from the risk-free rate r_{f} (the discount rate applied to risk-free debt). The interest payment I_{t+1} at point of time $t+1$ is calculated as $I_{t+1} = DN_t \cdot r_{\text{nom}}$, where DN_t is the outstanding amount of debt at point of time t. For simplicity, we will now focus on the final point of time T of the cash flow (firm, project). At this point the outstanding amount of debt will be fully paid down. Hence, the cash flow to the debt holders is:

$$FtD_T = DN_{T-1} + DN_{T-1} \cdot r_{\text{nom}} = DN_{T-1} \cdot (1 + r_{\text{nom}})$$

The value of debt DV_{T-1} is then:

$$DV_{T-1} = \frac{DN_{T-1} \cdot (1 + r_{\text{nom}})}{1 + r_f} \tag{25}$$

We immediately see that $DV_{T-1} = DN_{T-1}$ only if $r_{\text{nom}} = r_f$, or vice versa. Solving (25) for DN_{T-1} gives:

$$DN_{T-1} = DV_{T-1} \cdot \frac{1 + r_f}{1 + r_{\text{nom}}} \tag{26}$$

The flow to equity method can be written as follows:

$$EV_{\text{L},T-1} = \frac{(EBIT_T - DN_{T-1} \cdot r_{\text{nom}}) \cdot (1 - \tau) - DN_{T-1}}{1 + r_{\text{EL},T-1}}$$

where $EBIT_T$ are the earnings before interest and taxes, and $r_{\text{EL},T-1}$ is the required return on equity.

We can now substitute (26) together with $EV_{\mathrm{L},T-1} = q \cdot FV_{\mathrm{L},T-1}$ and $DV_{\mathrm{L},T-1} = (1-q) \cdot FV_{\mathrm{L},T-1}$ into the flow to equity method. Then we solve for $FV_{\mathrm{L},T-1}$ and obtain the free cash flow method:

$$FV_{\mathrm{L},T-1} = \frac{EBIT_T \cdot (1-\tau)}{1 + q_T \cdot r_{\mathrm{EL}} + (1-q_T) \cdot \left[\frac{1+r_f}{1+r_{\mathrm{nom}}} \cdot \left(1 + r_{\mathrm{nom}} \cdot (1-\tau)\right) - 1\right]}$$

If the debt value does not coincide with the outstanding principal, or if the nominal (contractual) interest rate does not correspond to the risk-free rate (required return on debt) then the discount rate in the FCF method will be different from what is known in the literature. The same reasoning applies to all points in time $t < T$.

Appendix 2 – Assumption of symmetric taxation

In this appendix, we want to illustrate that symmetric taxation is an important assumption in the valuation models studied in this paper and its preceding literature. If this assumption does not hold, the FCF in the unlevered firm valuation and the FCF in the FCF method (WACC method) will be subject to different average tax rates. In addition, the average tax rate applied in the calculation of the interest tax shield will be different. Figure 5 shows two tables. The first table contains calculations for the case of symmetric taxation. The second table contains calculations for the case where negative income does not imply negative (reversed) taxes. Each of this tables shows the EBIT for five different states of the world. Note that the EBIT is negative in the fifth state. In the case of the unlevered firm, the EBIT coincide with the income before taxes (we neglect any changes in working capital, investments, etc.). In the case of the levered firm, this income is calculated as EBIT minus interest. Taxes are calculated based on this income. The tax shield is calculated as the interest payment times the tax rate. Each table also contains the average EBIT, income, interest payment, taxes, and tax shield. Finally, for these two cases, the average tax rates are calculated. As can be seen in the case of tax symmetry, all average tax rates are equal. In the second case, negative taxes are prohibited, and here we see that the average tax rates are different for the income in the unlevered firm, the income in the levered firm and the tax shield. Traditional firm valuation methods do not deal with differences in these rates. Also, this paper assumes symmetric taxation and a single tax rate.

Tax Rate on Income	**30 %**	*(same for all cases)*

Case 1: Negative taxes are allowed in the case of negative income

Unlevered firm:

	EBIT = Income	*Taxes unlevered*
State 1	1000	300
State 2	700	210
State 3	400	120
State 4	100	30
State 5 *(negative EBIT)*	-200	-60
Average	**400**	**120**

Average Tax Rate	**30,00 %**	
	% of Income = EBIT	

Levered firm:

EBIT	*Interest*	*Income*	*Taxes*	*Tax Shield*
1000	10	990	297	3
700	10	690	207	3
400	10	390	117	3
100	10	90	27	3
-200	10	-210	-63	3
400	**10**	**390**	**117**	**3**

Average Tax Rate	**30,00 %**	**30,00 %**
	% of Income	*% of Interest*

Case 2: Negative taxes are NOT allowed in the case of negative income

Unlevered firm:

	EBIT = Income	*Taxes unlevered*
State 1	1000	300
State 2	700	210
State 3	400	120
State 4	100	30
State 5 *(negative EBIT)*	-200	0
Average	**400**	**132**

Average Tax Rate	**33,00 %**	
	% of Income = EBIT	

Levered firm:

EBIT	*Interest*	*Income*	*Taxes*	*Tax Shield*
1000	10	990	297	3
700	10	690	207	3
400	10	390	117	3
100	10	90	27	3
-200	10	-210	0	0
400	**10**	**390**	**129,6**	**2,4**

Average Tax Rate	**33,23 %**	**24,00 %**
	% of Income	*% of Interest*

Figure 5: The requirement of tax symmetry in DCF models