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On sensitivity of exponentiality tests to data rounding: a Monte Carlo simulation study

N. G. Ushakov[a] and V. G. Ushakov[b,c]

[a] Department of Mathematical Sciences, Norwegian University of Science and Technology, Trondheim, Norway [b] Lomonosov Moscow State University, Moscow, Russia [c] Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences, Moscow, Russia

ABSTRACT

Different statistical procedures are differently sensitive to data rounding. It turns out that tests for exponentiality are more sensitive to the data rounding than many classical parametric tests or than nonparametric tests for normality. In this work we find out which exponentiality tests are more robust and which ones are less robust to the rounding. The main tool is Monte Carlo simulation. We estimate and compare the probability of Type I error of nineteen exponentiality tests for different rounding levels and different sample sizes.

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CONTACT N. G. Ushakov nikolai.ushakov@ntnu.no Department of Mathematical Sciences, Norwegian University of Science and Technology, Trondheim 7491, Norway.

1. Introduction

Experimental data in statistical analysis are always discrete, all "continuous data" is rounded to some extent. Especially in medical and biological experiments rounding errors are very significant due to their specificity. The rounding influences properties of statistical procedures. In some areas of statistics, the influence is well studied and workers are generally aware of the theoretical and practical problems involved. The influence is stronger or weaker for different procedures and, of course, depends on the level of discretization. In this work, we study sensitivity of exponentiality tests to the rounding. The study of this problem is motivated below.

There are several types of rounding: rounding up, rounding down, stochastic rounding, etc. For definiteness, we will study rounding to the nearest. Other rounding types are studied similarly. So, there is a rounding lattice $\{x : x = kh, k = 0, \pm 1, \pm 2, \ldots\}$, where $h > 0$ (call it the discretization step). For a random variable X, its rounding (or discretization) is the discrete random variable $X^{(h)}$ taking values on the lattice and the closest to X. More exactly, $X^{(h)} = kh$ when $kh - h/2 \le X < kh + h/2.$ We will use also the rounding down. It is the random variable $[X]_h$ such that $[X]_h = kh$ when $kh \le X < (k+1)h.$

For our purpose it is important to take into account not only the accuracy of the discretization (the discretization step) but also the spread of a rounded random variable. A natural measure of the level of rounding is the variable $r = h/\sigma,$ where h is the step of discretization and σ is the standard deviation of the rounded random variable.

The influence of the rounding on various statistical procedures (hypothesis testing and estimation) have been studied by a number of authors, see, in particular, Hall (1982), Tricker (1984a), Tricker (1990), Härdle and Scott (1992), Hall and Wand (1996), Tricker and Okell (1997), Minnotte (1998), Meintanis and Ushakov (2004), Bai et al. (2009), Schneeweiss, Komlos, and Ahmad (2010), Zhao and Bai (2020), and references therein.

Exponential distribution plays an important role in many fields of theoretical and applied statistics such as reliability, queuing theory, quality control, actuarial science, etc. For this reason, there is an extensive literature devoted to the exponential distribution and, in particular, to testing the hypothesis of exponentiality. There are a number of books and review papers describing and comparing various tests for exponentiality, see, in particular, Balakrishnan and Basu (1995), Ascher (1990), Henze and Meintanis (2005), and Rahman and Wu (2017).

As it was pointed out above, different statistical procedures are differently sensitive to data rounding. A simulation study shows that normality tests (at least those presented in R statistical packages “nortest” and “normtest”) are little sensitive to the rounding, if the rounding error is not too large. For example, estimated probabilities of Type I error for $r = 0.01$, the significance level 5% and different sample sizes are presented in Table 1. The table shows that the tests achieve perfect size control when the data is rounded. This is true for all tests under consideration and all sample sizes. In other words, the normality tests are robust to the data rounding. For a number of exponentiality tests, this does not hold true: they are not stable even for much slighter rounding. In this work we try to find out which tests for exponentiality are robust and which ones are not.

Table 1. Probability of Type I error of normality tests, $\alpha = 0.05,$ $r = 0.01$. (Table view)

	$n = 25$	$n = 50$	$n = 100$	$n = 200$	$n = 500$
Pearson	0.0647	0.0535	0.0536	0.0571	0.061
Kolmogorov–Smirnov	0.0497	0.0476	0.0518	0.0524	0.0603
Cramer–von Mises	0.0498	0.0477	0.0477	0.0505	0.0498
Anderson–Darling	0.0474	0.0483	0.0477	0.0489	0.0499
Shapiro–Wilk	0.048	0.0493	0.0499	0.0506	0.0497
Shapiro–Francia	0.0499	0.0515	0.0515	0.0528	0.0496
Jarque–Bera	0.0468	0.0489	0.049	0.0531	0.0494
Adjusted Jarque–Bera	0.0472	0.0503	0.0482	0.0533	0.0489
Frosini	0.0478	0.048	0.0483	0.0497	0.0489
Geary	0.0506	0.0473	0.0522	0.0524	0.0504
Hegazy–Green (1)	0.0476	0.0495	0.0497	0.0504	0.0536
Hegazy–Green (2)	0.0479	0.0506	0.0486	0.0511	0.0496
Kurtosis	0.047	0.0491	0.0484	0.0512	0.0514
Skewness	0.0449	0.047	0.0522	0.0527	0.0486
Spiegelhalter	0.0485	0.0509	0.0485	0.0521	0.0474
Weisberg–Bingham	0.0479	0.0494	0.0496	0.05	0.0482

Thus, in what follows, observations will be independent random variables having the same exponential distribution and rounded to some level r. The goal is to control the size of various exponentiality tests. A brief description and preliminary analysis of the tests studied are given in the two next sections.

Throughout the paper we will accept the following convention. To simplify analysis of simulation results, we will choose a certain threshold value of the probability of Type I error and denote probabilities greater than this threshold by bold. Such values can be considered as unacceptable. Let us choose the threshold to be equal the double significance level, i.e., for example, 0.1 if the significance level is 5%. Of course, this agreement is very conditional and is made only for convenience.

2. Tests for exponentiality

There are a wide variety of tests for exponentiality. We study the nineteen tests shown in Table 2. All these tests are contained in the R package "exptest." We will list test statistics of these tests and make a brief analysis of them. A few tests of the package are excluded because the calculated probability of Type I error substantially differs from the required significance level even for pure (non-rounded) data.

Table 2. Tests studied and the R functions that were used to apply them. (Table view)

Test	Abbreviation	Function
Atkinson	At	atkinson.exp.test
Cox and Oakes	CO	co.exp.test
Deshpande	D	deshpande.exp.test
Epps and Pulley	EP	ep.exp.test
Epstein	E	epstein.exp.test
Frozini	F	frozini.exp.test
Gini statistic	Gi	gini.exp.test
Gnedenko *F*-test	Gn	gnedenko.exp.test
Harris modification of Gnedenko *F*-test	H	harris.exp.test
Hegazy–Green (1)	HG1	hegazy1.exp.test
Hegazy–Green (2)	HG2	hegazy2.exp.test
Hollander–Proshan	HP	hollander.exp.test
Kimber–Michael	KM	kimber.exp.test
Kolmogorov–Smirnov	KS	ks.exp.test
Moran	M	moran.exp.test
Pietra statistic	P	pietra.exp.test
Shapiro–Wilk	SW	shapiro.exp.test
WE	WE	we.exp.test
Wong and Wong	WW	ww.exp.test

Let $X_1, \ldots, X_n$ be a random sample, independent and identically distributed (iid) random variables with a distribution function $F(x)$. The sample mean, sample variance and empirical distribution function are denoted by $\overline{X}$, S^2, and $F_n(x)$, i.e.,

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I_{(X_i \le x)},$$

where I_A is the indicator of an event A. Denote also the ratio $X_i/\overline{X}$ by Y_i. Denote the empirical distribution function of the sample $Y_1, \ldots, Y_n$ by $G_n(x)$. The order statistics are denoted as usually by $X_{(1)}, \ldots, X_{(n)}$.

The Atkinson test, see, e.g., Mimoto and Zitikis (2008), is based on the test statistic

$$T = \sqrt{n}\left|\frac{\left(n^{-1}\sum_{i=1}^{n} X_i^p\right)^{1/p}}{\overline{X}} - (\Gamma(1+p))^{1/p}\right|,$$

where $p > 0$.

The test statistic of the Cox and Oakes test, see e.g., Henze and Meintanis (2005), is

$$T = n + \sum_{i=1}^{n} (1 - Y_i)\ \log Y_i.$$

The Deshpande test was proposed in Deshpande (1983). The test statistic is

$$T = \frac{1}{n(n-1)} \sum_{i \neq j} I_{(X_i > bX_j)},$$

where $0 < b < 1$.

The test statistic of the Epps–Pulley test, see, e.g., Henze and Meintanis (2005), is

$$T = \sqrt{48n}\left(\frac{1}{n}\sum_{i=1}^{n} e^{-Y_i} - \frac{1}{2}\right).$$

The Epstein test, see, e.g., Ascher (1990), is based on the test statistic

$$T = \frac{2n\left(\log\left(n^{-1}\sum_{i=1}^{n} D_i\right) - n^{-1}\sum_{i=1}^{n} \log\left(D_i\right)\right)}{1 + (n+1)/(6n)},$$

where $D_i = (n - i + 1)(X_{(i)} - X_{(i-1)})$, $X_{(0)} = 1$.

The test statistic of the Frozini test, see Frozini (1987), is

$$T = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left|1 - e^{-X_{(i)}/\overline{X}} - \frac{i - 1/2}{n}\right|.$$

The Gini test, see, e.g., Gail and Gastwirth (1978), has the test statistic

$$T = \frac{\sum_{i,j=1}^{n} \left| X_i - X_j \right|}{2n(n-1)\overline{X}}.$$

The Gnedenko F-test, see, e.g., Ascher (1990), is based on the test statistic

$$T = \frac{r^{-1} \sum_{i=1}^{r} D_i}{(n-r)^{-1} \sum_{i=r+1}^{n} D_i},$$

where r is an integer such that $1 < r < n.$

The Harris test is a modification of the Gnedenko F-test, see, e.g., Ascher (1990). It is based on the test statistic

$$T = \frac{(2r)^{-1} \left(\sum_{i=1}^{r} D_i + \sum_{i=n-r+1}^{n} D_i \right)}{(n-2r)^{-1} \sum_{i=r+1}^{n-r} D_i},$$

where $1 < r < n/2.$

There are two variants of the Hegazy–Green test, see Hegazy and Green (1975). One is based on the test statistic

$$T = \frac{1}{n} \sum_{i=1}^{n} \left| X_{(i)} + \ln \left(1 - \frac{i}{n+1} \right) \right|.$$

Another one has the test statistic

$$T = \frac{1}{n} \sum_{i=1}^{n} \left[X_{(i)} + \ln \left(1 - \frac{i}{n+1} \right) \right]^2.$$

The test statistic of the Hollander–Proshan test, see Hollander and Proshan (1972), is

$$T = \frac{1}{n(n-1)(n-2)} \sum_{i \neq j,k; j<k} I_{(X_i > X_j + X_k)}.$$

The Kimber and Michael test, see Michael (1983) and Kimber (1985), has the test statistic

$$T = \frac{2}{\pi} \max_i \left| \arcsin \sqrt{1 - e^{-X_{(i)}/\overline{X}}} - \arcsin \sqrt{(i - 1/2)/n} \right|.$$

The Kolmogorov–Smirnov test, see, e.g., Henze and Meintanis (2005), is based on the test statistic

$$T = \sup_{x \geq 0} \left| G_n(x) - (1 - e^{-x}) \right|.$$

The test statistic of the Moran test, see Moran (1951) and Tchirina (2005), is

$$T = \gamma + \frac{1}{n} \sum_{i=1}^{n} \log \frac{X_i}{\overline{X}},$$

where γ is the Euler–Mascheroni constant.

The Pietra test, see, e.g., Ascher (1990), has the test statistic

$$T = \sum_{i=1}^{n} \frac{|X_i - \overline{X}|}{2n\overline{X}}.$$

The test statistic of the Shapiro–Wilk test, see Shapiro and Wilk (1972), is

$$T = \frac{n(\overline{X} - X_{(1)})^2}{(n-1)\sum_{i=1}^{n} (X_i - \overline{X})^2}.$$

The WE test, see, e.g., Ascher (1990), is based on the test statistic

$$T = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{(\sum_{i=1}^{n} X_i)^2}.$$

The Wong–Wong test, see, e.g., Ascher (1990), is based on the test statistic

$$T = \frac{X_{(n)}}{X_{(1)}}.$$

For the tests listed above, the null hypothesis of exponentiality is rejected for large values of the test statistic. The p-values of the tests are computed by Monte Carlo simulation.

3. Analysis

First we obtain an upper bound for the distance between the empirical distribution function found from original observations and the empirical distribution function found from rounded observations. Then we conduct a brief preliminary analysis of the test statistics listed in the previous section. Recall that the concentration function of a random variable X is defined as

$$Q(X;l) = \sup_a P(a \leq X \leq a+l),\ l \geq 0.$$

Lemma 3.1. *For any random variable X and any real a and h > 0*

$$P(a < [X]_h \leq a+h) \leq Q(X;h).$$

Proof. Let integer k be such that $a < kh \leq a+h$. Then

$$P(a < [X]_h \leq a+h) = P([X]_h = kh) = P\left(kh \leq X < (k+1)h\right) \leq Q(X;h).$$

□

Suppose now that there is a random sample (iid random variables) $X_1, \ldots, X_n$ and rounded observations $X_1^{(h)}, \ldots, X_n^{(h)}$. We are interested in the distance between the empirical distribution function $F_n(x)$ constructed from the original sample and the empirical distribution function $F_n^{(h)}(x)$ constructed from the rounded observations. The distance we use is the mean absolute distance.

Theorem 3.1. $\sup_x E\left|F_n(x) - F_n^{(h)}(x)\right| \leq Q(X_1, h).$

Proof. Since

$$\{X_i \leq x\} \subseteq \{[X_i]_h \leq x\},\ \{[X_i]_h + h \leq x\} \subseteq \{X_i \leq x\}$$

and

$$\left\{X_i^{(h)} \leq x\right\} \subseteq \left\{[X_i]_h \leq x\right\},\ \left\{[X_i]_h + h \leq x\right\} \subseteq \left\{X_i^{(h)} \leq x\right\},$$

the following inequalities hold:

$$I_{([X_i]_h \leq x-h)} \leq I_{(X_i \leq x)} \leq I_{([X_i]_h \leq x)},\ I_{([X_i]_h \leq x-h)} \leq I_{(X_i^{(h)} \leq x)} \leq I_{([X_i]_h \leq x)}.$$

Therefore

$$\left|I_{(X_i \leq x)} - I_{(X_i^{(h)} \leq x)}\right| \leq I_{([X_i]_h \leq x)} - I_{([X_i]_h \leq x-h)} = I_{(x-h < [X_i]_h \leq x)}. \tag{1}$$

Using inequality (1) and Lemma 3.1, we obtain that for any x

$$E\left|F_n(x) - F_n^{(h)}(x)\right| = E\left|\frac{1}{n}\sum_{i=1}^n I_{(X_i\le x)} - \frac{1}{n}\sum_{i=1}^n I_{(X_i^{(h)}\le x)}\right|$$
$$\le \frac{1}{n}\sum_{i=1}^n E\left|I_{(X_i\le x)} - I_{(X_i^{(h)}\le x)}\right| \le \frac{1}{n}\sum_{i=1}^n E\left|I_{([X_i]_h\le x)} - I_{([X_i]_h\le x-h)}\right|$$
$$= \frac{1}{n}\sum_{i=1}^n EI_{(x-h<[X_i]_h\le x)} = \frac{1}{n}\sum_{i=1}^n P(x-h<[X_i]_h\le x) \le \frac{1}{n}\sum_{i=1}^n Q(X_i;h) = Q(X_1;h).$$

Since x is arbitrary this completes the proof. □

Now suppose that $X_1,\dots,X_n$ is a sample from an exponential distribution, and the observations $X_1,\dots,X_n$ are rounded to the level r (the discretization step h). If λ is the parameter of the distribution, then $r=\lambda h$. For the concentration function we get

$$Q(X_1;h) = P(0\le X_1\le h) = 1-e^{-\lambda h} \le 1-(1-\lambda h) = r.$$

Using Theorem 3.1, finally obtain

$$\sup_x E\left|F_n(x) - F_n^{(h)}(x)\right| \le r,$$

i.e., the empirical distribution function is robust to the rounding.

There are a number of results, see Tricker (1984b), Janson (2006), and Schneeweiss, Komlos, and Ahmad (2010), showing that moments of a rounded random variable are close to the corresponding moments of the original variable provided that the rounding level r is not too large (for the exponential distribution, for the first and second moments, if $r < 0.25$).

Thus, the empirical distribution function and empirical moments are robust to the rounding. Therefore, test statistics based only on empirical moments and the empirical distribution function must be robust, i.e., critical regions constructed from original and from rounded observations must be close, that must lead to close values of the probability of Type I error. Among the tests studied here the following tests have tests statistics based only on the empirical distribution function and empirical moments: At, D, EP (including exponential moments), Gi, KS, P, SW, WE. The analysis above shows that there is a hope that at least these tests (but maybe some others too) are robust to the rounding. In the next section we find out if this is so.

4. Estimated probability of Type I error

In this section we provide some Monte Carlo experiments to study the finite sample properties of the tests listed above in the case of rounded data. We estimate and compare the probability of Type I error for different tests, different sample sizes and different levels of the rounding. The main subject is the dependence of the probability of Type I error on r.

Since the Type I error is the rejection of a true null hypothesis, its probability is estimated as follows. If there are N random samples of a given size from the distribution satisfying the null hypothesis, and if K is the number of samples for which the null hypothesis is rejected, then it is natural to approximate the probability of Type I error by the ratio K/N. Due to the Central limit theorem K/N has approximately normal distribution with expectation p and variance $p(1-p)/N$, where p is the true value of the probability of Type I error. If the significance level is α, then the null hypothesis is rejected if the p-value is less than α. For tests for exponentiality, the p-values can be obtained using R package “exptest.”

In our case, using Monte Carlo simulation, 100,00 random samples are generated from an exponential distribution for each sample size $n = 25$, $n = 50$, $n = 100$. The observations are rounded. All the tests are applied to the same samples. The significance level is 5%. For each sample and each of the 19 tests, the hypothesis of exponentiality is rejected if the p-value obtained is less than 0.05. For a given test and fixed r and n, the proportion of 10,000 samples for which the hypothesis is rejected is the estimated probability of Type I error. The results are presented in Tables 3–5.

Table 3. Simulated probability of Type I error, $n = 25$. (Table view)

	$r = 0$	$r = 10^{-3}$	$r = 0.01$	$r = 0.1$	$r = 0.2$	$r = 0.5$	$r = 0.7$	$r = 1$
At	0.0376	0.0393	0.0361	0.0383	0.0461	**0.1236**	**0.2441**	**0.4848**
CO	0.0449	0.0522	**0.1521**	**0.7314**	**0.9311**	**0.9995**	**0.9999**	**1**
D	0.0716	0.068	0.0656	0.082	**0.137**	**0.387**	**0.5447**	**0.765**
EP	0.0464	0.0461	0.0472	0.0437	0.0489	**0.1**	**0.1855**	**0.4046**
E	0.0443	**0.1624**	**0.8064**	**1**	**1**	**1**	**1**	**1**
F	0.0526	0.053	0.0518	0.0547	0.0795	**0.5432**	**0.926**	**0.9992**
Gi	0.0529	0.0528	0.052	0.0493	0.0568	**0.104**	**0.1548**	**0.2837**
Gn	0.0499	0.0527	0.049	0.0528	0.0636	0.0862	**0.1021**	**0.5228**
H	0.0522	0.0438	0.0451	0.0595	0.0735	**0.4728**	**0.6647**	**0.5632**
HG1	0.0472	0.0495	0.0505	0.0528	**0.5345**	**1**	**1**	**0.153**
HG2	0.0469	0.0495	0.0498	0.0518	0.0737	**0.9989**	**1**	0.0706
HP	0.0684	0.0625	0.0618	0.0989	**0.1803**	**0.4871**	**0.5989**	**0.7245**
KM	0.0549	0.0503	0.0564	**0.3978**	**0.8049**	**1**	**1**	**1**
KS	0.0517	0.0529	0.0491	0.0685	**0.1544**	**0.8155**	**0.9864**	**1**
M	0.0461	0.0497	**0.1459**	**0.7287**	**0.9297**	**0.9995**	**0.9999**	**1**
P	0.0525	0.0508	0.0527	0.049	0.0522	0.0831	**0.1235**	**0.2868**
SW	0.0512	0.0567	0.0475	0.047	0.0425	0.0497	0.0728	**0.1487**
WE	0.0498	0.0529	0.0523	0.0487	0.048	0.0559	0.0656	**0.12**
WW	0.0561	0.0487	**0.1227**	**0.7163**	**0.92**	**0.9976**	**0.9995**	**1**

Table 4. Simulated probability of Type I error, $n = 50$. (Table view)

	$r = 0$	$r = 10^{-3}$	$r = 0.01$	$r = 0.1$	$r = 0.2$	$r = 0.5$	$r = 0.7$	$r = 1$
At	0.0445	0.0423	0.0389	0.0493	0.0537	**0.1791**	**0.3808**	**0.7142**
CO	0.0469	0.0654	**0.2514**	**0.9254**	**0.9958**	**1**	**1**	**1**
D	0.0634	0.0564	0.0545	0.0756	**0.1277**	**0.4358**	**0.6606**	**0.9047**
EP	0.0504	0.0483	0.0436	0.0506	0.0503	**0.1247**	**0.2681**	**0.5929**
E	0.0407	**0.4686**	**0.9987**	**1**	**1**	**1**	**1**	**1**
F	0.0539	0.0479	0.0483	0.0606	**0.1023**	**0.976**	**1**	**1**
Gi	0.0518	0.0508	0.0465	0.0542	0.0554	**0.1092**	**0.1875**	**0.3655**
Gn	0.054	0.0451	0.0482	0.0558	0.0797	**0.221**	**0.0436**	**0.5419**
H	0.0498	0.048	0.0471	0.0613	0.0991	**0.6536**	**0.8934**	**0.693**
HG1	0.049	0.0476	0.0518	0.0534	**0.9825**	**1**	**1**	**0.443**
HG2	0.0489	0.0498	0.0471	0.0529	**0.7234**	**1**	**1**	0.0854
HP	0.0608	0.0549	0.0552	**0.1131**	**0.262**	**0.7326**	**0.8642**	**0.9655**
KM	0.0524	0.0491	0.0523	**0.5609**	**0.985**	**1**	**1**	**1**

	$r = 0$	$r = 10^{-3}$	$r = 0.01$	$r = 0.1$	$r = 0.2$	$r = 0.5$	$r = 0.7$	$r = 1$
KS	0.0569	0.0505	0.0497	**0.1067**	**0.3481**	**0.9998**	**1**	**1**
M	0.0476	0.066	**0.2482**	**0.9264**	**0.9964**	**1**	**1**	**1**
P	0.0553	0.05	0.0448	0.0523	0.0512	0.0918	**0.1569**	**0.3348**
SW	0.0529	0.0493	0.0445	0.045	0.0435	0.0621	0.0956	**0.1948**
WE	0.0497	0.0521	0.0456	0.0515	0.0509	0.0561	0.0711	**0.1385**
WW	0.0505	0.0434	**0.2221**	**0.9146**	**0.9928**	**1**	**1**	**1**

Table 5. Simulated probability of Type I error, $n = 100$. (Table view)

	$r = 0$	$r = 10^{-3}$	$r = 0.01$	$r = 0.1$	$r = 0.2$	$r = 0.5$	$r = 0.7$	$r = 1$
At	0.0484	0.0425	0.0477	0.0499	0.0611	**0.2792**	**0.5886**	**0.9266**
CO	0.0516	0.086	**0.418**	**0.9948**	**1**	**1**	**1**	**1**
D	0.0545	0.0521	0.0529	0.0702	**0.127**	**0.5127**	**0.7948**	**0.9816**
EP	0.051	0.0457	0.0502	0.0513	0.0537	**0.1679**	**0.399**	**0.8174**
E	0.0392	**0.9222**	**1**	**1**	**1**	**1**	**1**	**1**
F	0.051	0.048	0.051	0.0722	**0.2094**	**1**	**1**	**1**
Gi	0.0529	0.0454	0.052	0.052	0.0557	**0.1217**	**0.2344**	**0.5166**
Gn	0.0503	0.0498	0.0541	0.0612	0.0824	**0.5493**	0.0268	**0.7866**
H	0.0481	0.0517	0.0448	0.0839	**0.2092**	**0.8311**	**0.9816**	**0.9041**
HG1	0.0535	0.0461	0.0511	0.0527	**1**	**1**	**1**	**1**
HG2	0.0551	0.0508	0.0466	0.0495	**0.9985**	**1**	**1**	**0.165**
HP	0.0547	0.0497	0.0532	**0.1616**	**0.4298**	**0.9568**	**0.9946**	**1**
KM	0.053	0.0481	0.0626	**0.9921**	**1**	**1**	**1**	**1**
KS	0.0745	0.0477	0.0524	**0.1918**	**0.7699**	**1**	**1**	**1**
M	0.0517	0.0858	**0.4152**	**0.9958**	**1**	**1**	**1**	**1**
P	0.0527	0.0468	0.0535	0.0514	0.0516	0.0997	**0.2178**	**0.3741**
SW	0.0497	0.0502	0.0454	0.0434	0.0442	0.0757	**0.1206**	**0.2881**
WE	0.0507	0.0483	0.0462	0.0495	0.05	0.0583	0.0846	**0.2109**
WW	0.0483	0.0456	**0.3908**	**0.9928**	**1**	**1**	**1**	**1**

From the tables it is seen that in general, the simulation confirms (with rare exceptions) the assumption of the previous section that methods based on simple statistics (the empirical distribution function and empirical moments) are more robust than tests based on more complicated statistics. The exceptions are: the Kolmogorov–Smirnov test (it is based on only the empirical distribution function but is less robust than a number of tests based on complicated statistics), the Gnedenko test and the Harris test (they are based not only on the simple statistics but are very robust). Note also that, although the behavior of the probability of Type I error is generally regular (almost monotone), there are some sharp deviations. For example, the Hegazy–Green tests.

Since the Epstein test is most sensitive to the data rounding, it would be useful to investigate on its example other aspects of the effect of the rounding on the probability of Type I error. In concluding this section we make some simulation study of its behavior when the sample size increases. The results are presented in Table 6. It is seen that non-stability increases sharply with an increase of the sample size.

Table 6. Simulated probability of Type I error, the Epstein test. (Table view)

	$r = 0$	$r = 10^{-8}$	$r = 10^{-7}$	$r = 10^{-6}$	$r = 10^{-5}$	$r = 10^{-4}$	$r = 10^{-3}$

	$r = 0$	$r = 10^{-8}$	$r = 10^{-7}$	$r = 10^{-6}$	$r = 10^{-5}$	$r = 10^{-4}$	$r = 10^{-3}$
$n = 25$	0.0443	0.0417	0.0407	0.0456	0.0433	0.0507	**0.1624**
$n = 50$	0.0407	0.041	0.0402	0.0404	0.0433	0.0884	**0.4686**
$n = 100$	0.0392	0.0402	0.0408	0.0392	0.0602	**0.2402**	**0.9222**
$n = 250$	0.0375	0.0387	0.0374	0.0474	**0.1618**	**0.7921**	**1**
$n = 500$	0.0319	0.0329	0.0363	0.0886	**0.4723**	**0.9983**	**1**
$n = 10^3$	0.0266	0.0309	0.0496	**0.2378**	**0.918**	**1**	**1**
$n = 10^4$	0.0166	**0.2315**	**0.9189**	**1**	**1**	**1**	**1**

5. Conclusions

The tests for exponentiality studied in this work can be divided into two groups. The first group consists of the tests whose test statistics are continuous functions of simple sample statistics: the empirical distribution function and empirical moments. These are At, D, EP, Gi, KS, P, SW, WE tests. Tests statistics of the second group are based on more complicated functions of the sample. These are CO, E, F, Gn, H, HG1, HG2, HP, KM, M, WW tests. With a rare exception the tests of the first group are essentially more robust to the data rounding than the tests of the second group. The sensitivity of exponentiality test to the rounding can be very high. It is the case, for example, for the Epstein test where the probability of Type I error is close to one even for the rounding level $r = 0.001$ (the significance level $\alpha = 0.05$, the sample size $n = 100$). It should be pointed out also that the sensitivity to the data rounding increases substantially as the sample size increases.

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