Thomas Agung Dibpa Anandita Thrane

Modular forms and Δ

Hovedoppgave i Bachelor i matematiske fag
Veileder: Kristian Seip
Medveileder: Andrii Bodarenko
Mai 2022

Hovedoppgave

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Norges teknisk-naturvitenskapelige universitet
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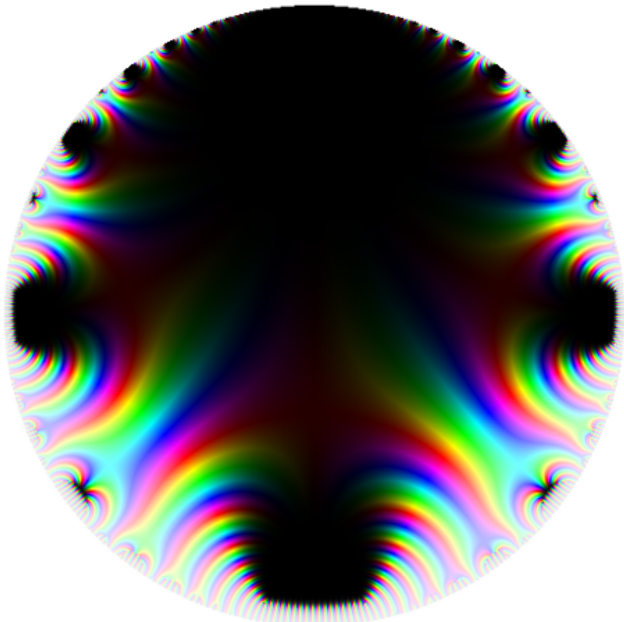

The modular discriminant plotted using SageMath.

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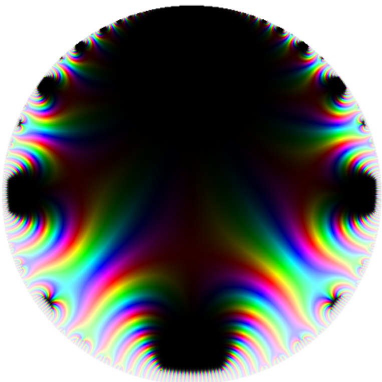

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Abstract

The theory of modular forms sits in the intersection of the mathematical branches: number theory, complex analysis, topology, algebraic geometry and group theory. For example, they play a part in the proof of Fermats last theorem by Andrew Wiles and have surprising connections to the Monster simple group via the j-invariant and Richard Brocherds' moonshine theory. In this bachelors project we investigate the simplest case of a modular form, level 1 and integer weight, using undergraduate level complex analysis with a sprinkle of group theory and linear algebra. We use the theory to prove that the modular discrimant, a special modular form, has multiplicative Fourier coefficients - a theorem conjectured by Ramanujan and proved by Mordell.

Sammendrag

Modulære former dukker opp i mange matematiske grener som topologi, tallteori, kompleksanalyse, gruppeteori, og algebraisk geometri. For eksempel, så ble de brukt i Andrew Wiles sitt bevis av Fermats siste teorem. De har til og med koblinger til den sporadiske monstergruppen via j-invarianten og måneskinnsteorien utviklet av Richard Brocherds. I dette bachelorprosjektet undersøker vi den enkleste varianten av modulære former, de med nivå 1 og heltallsvekt, ved å bruke kompleksanalyse, gruppeteori og lineær algebra. Vi bruker teorien for å vise at den modulære diskriminanten har multiplikative Fourier-koeffisienter, som var en av Ramanujans formodninger før Mordell beviste den i 1917.

Preface

Whenever I mention that I am writing a bachelor in mathematics, I always get the question: "What are you writing about?". Then I scratch my head and wonder how to explain modular forms to them. For my mathematician friends I say that modular forms are holomorphic functions with some symmetry. They are (almost) invariant under an action of a group. Sometimes I mention that these functions are used in the proof of Fermats last theorem by Andrew Wiles, and also have connections to the monster group via moonshine theory as shown by Richard Brocherds.

For my family and nonmathematician friends, I mumble some words which I know that they do not understand. Despite not knowing how to answer this question, I am grateful for the interest that people have in my work.

Thinking about this question, I wonder why I even decided to write a bachelor about modular forms. If I recall correctly, I think it was when I discovered Richard Brocherds' youtube channel right when he was making a series of lectures about modular forms. [1] This was around 1 year ago. I tried watching this series then, but although I understood some parts of it, there were a lot of missing details that I didn't know how to fill in. That was when I decided that I wanted to try to understand this topic.

I somehow managed to get two advisors for my project: Kristian Seip and Andrii Bodarenko. I thank Andrii Bodarenko for setting the goal for my bachelor project. I thank Kristian Seip for reading through my bachelor and giving me feedback.

I thank my mathematician friends who have expressed interest in reading my bachelor, even asking what the "prerequisites" are. I can tell you that I am assuming that the reader is familiar with linear algebra, number theory, group theory and especially complex analysis. It will be very helpful to be familiar with the terms: Fourier series, Laurent series, contour integrals, meromorphic functions, argument principle, pole, absolute convergence, ring, orbit.

Thomas Thrane
Trondheim, Norway
May 2022

CONTENTS

CHAPTER 1

INTRODUCTION

The goal of this bachelor will be to understand the basics of the theory of modular forms. In particular, we will aim to prove a theorem conjectured by Ramanujan and proved by Mordell [2].

We will prove that the Fourier coefficients of a special modular form, Δ, are multiplicative. That is to say if

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z},$$

then $\tau(mn) = \tau(m)\tau(n)$ for all coprime m, n. This was the goal set by my advisor Andrii Bodarenko. The extra challenge was to try to do this by myself without reading any literature.

When I first started working on this bachelor project, I knew what a modular form was, and I knew a little bit about this function Δ, the modular discriminant. This was because I had watched the youtube series on modular forms by Richard Brocherds [1]. From that series, I had gleaned the overall idea of how to prove that τ was multiplicative, even if I didn't understand any of the details. My strategy was then to try to understand the details.

These were the concepts mentioned in Brocherds' videos that I would have to understand:

1. The modular group and its fundamental domain.

2. Eisenstein series.

3. Finite dimensionality of the vector space of modular forms.
4. Hecke operator.
5. Eigenforms.

Trying the extra challenge at first, I managed to understand Eisenstein series, Hecke operators and Eigenforms on my own, using only the definitions found on wikipedia and the clues I remembered from Brocherds' videos. So chapter 3, chapter 6 and 7 follow my approach to understanding these concepts. That is why they lack references.

I think that doing this challenge helped me motivate the ideas of modular forms and present those motivations as well. Sometimes I feel that in a lot of mathematical literature, a magical definition or theorem just pops out of nowhere, with little to no motivation on how someone came up with it. I put in a little effort in chapter 3 and 6 to not do that, and instead tried to give the reader the same ideas and motivations that I had when I was proving the theory. I hope that I can make the reader believe that they could have come up with the theory given enough time and luck, when reading those chapters.

But alas, I could not understand why the space of modular forms were finite dimensional on my own, so I ended up needing to read the literature after all. The books by Serre [3] and Apostol [4] were a tremendous help when writing chapters 4 and 5, as well as for filling in some small details that I missed here and there.

That was some context on how I worked on this bachelor project, and I hope that you will enjoy learning about modular forms as much as I have.

CHAPTER 2

WHAT IS A MODULAR FORM?

What is a modular form? A concrete answer to this question is a bland definition.

A modular form f of weight $k \in \mathbb{Z}$ is a holomorphic function from the upper half plane $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ to the complex numbers $\mathbb{C}$ satisfying the following conditions:

1. For any $z \in \mathcal{H}$ and matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbb{Z})$,
$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$
2. $f(z)$ is bounded as $\mathrm{Im}(z) \to \infty$.

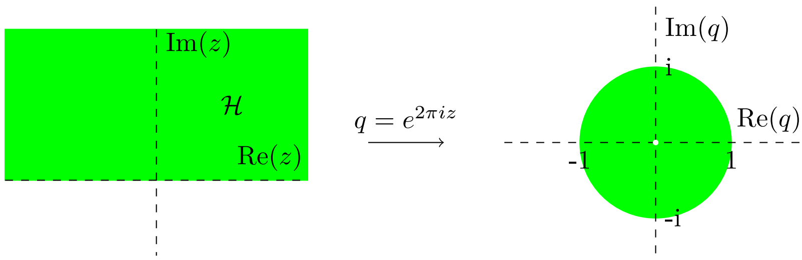
$\mathbf{SL}_2(\mathbb{Z})$ denotes the 2 by 2 matrices with integer coefficients and determinant 1. Note that $\mathbf{SL}_2(\mathbb{Z})$ forms a group under multiplication, because the inverse of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, which has integer coefficients.

Our main goal now is that we want to understand our definition of a modular form as much as possible. What would the first steps be? The simplest thing to do is to test the definition by choosing example matrices from $\mathbf{SL}_2(\mathbb{Z})$, and see what we get.

If we put $a = b = d = 1$ and $c = 0$, then we see that $f(z+1) = f(z)$, so f has period 1. This gives us the idea that modular forms may have a Fourier series representation. Suppose that $f(z) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n z}$ for some complex coefficients $\{a_n\}$. Then condition 2 would imply that $a_n = 0$ for all negative n, since $\lim_{\mathrm{Im}(z)\to\infty} e^{2\pi i n z} = \infty$ for all negative

n. In this case the Fourier series would only have nonnegatively indexed coefficients $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ and would have a limiting value $\lim_{\text{Im}(z)\to\infty} f(z) = a_0$.

What is really going on is that if we do a change of variable $q = e^{2\pi i z}$ and look at the function $\hat{f}(q) = f(\frac{\ln(q)}{2\pi i})$, we see that it is well-defined because f is periodic. Because f is holomorphic on the upper half plane $\mathcal{H}$ and f' is also periodic, we see that $\hat{f}$ is holomorphic with derivative $\hat{f}'(q) = f'(\frac{\ln(q)}{2\pi i})\frac{1}{2\pi i q}$ when $0 < |q| = |e^{2\pi i z}| = e^{-2\pi \text{Im}(z)} < 1$. So $\hat{f}$ admits a Laurent expansion $\hat{f}(q) = \sum_{n\in\mathbb{Z}} a_n q^n$ in the punctured disc $0 < |q| < 1$. Now condition 2 removes the possibility that $\hat{f}$ has any negative powers in its Laurent series, because $f(z)$ being bounded for $\text{Im}(z) > N$ translates to $\hat{f}(q)$ being bounded for $0 < |q| < e^{-2\pi N}$. So $\hat{f}(q)$ has a Taylor series expansion at zero, and is therefore holomorphic at zero. This translates to f having a Fourier series $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ valid for the entire upper half plane. Because $\hat{f}$ was analytically continued to the point $\hat{f}(0) = a_0$ which corresponds to the limit $\lim_{\text{Im}(z)\to\infty} f(z) = a_0$, one can informally consider that $\hat{f}$ extends f holomorphically to $i\infty$. This viewpoint is thanks to Serre [3].



We formalize this.

Lemma 2.1. *Let f be a modular form. Then $f(z+1) = f(z)$ and there exist complex coefficients $\{a_n\}_{n\geq 0}$ such that $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i z}$.*

Going back to plugging in values, the next one took me an embarrassingly long time to notice. If we set $a = b = -1, c = d = 0$, then we get the equation $f(z) = (-1)^k f(z)$. This means that for odd k, $f(z) = -f(z)$, which forces $f(z) = 0$.

Lemma 2.2. *The are no nonzero modular forms of odd weight.*

Finally, if we set $a = d = 0, b = -1, c = 1$, then we see that $f(-\frac{1}{z}) = z^k f(z)$. This may not seem like a very useful observation at first, but it does lead us to the more useful observation that this is a sufficient conditon. Namely we have the following lemma:

Lemma 2.3. *The modularity condition*

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \qquad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \boldsymbol{SL}_2(\mathbb{Z}),$$

is equivalent to the two conditions:

$$f(z+1) = f(z),$$
$$f\left(-\frac{1}{z}\right) = z^k f(z).$$

This is very useful for the lazy mathematician, since it is way easier to check two conditions than it is to check infinitely many.

Proof. The first thing to notice are that the maps

$$\phi_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} : \mathcal{H} \to \mathcal{H}, \quad z \mapsto \frac{az+b}{cz+d}$$

respect the group structure of $\mathbf{SL}_2(\mathbb{Z})$. Indeed if $\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_3 & b_3 \\ c_3 & d_3 \end{smallmatrix}\right)$, then

$$\begin{aligned}
\phi_{\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)} \circ \phi_{\left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)}(z) =& \frac{a_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right)+b_1}{c_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right)+d_1} = \frac{(a_1a_2+b_1c_1)z+(a_1b_2+b_1d_2)}{(c_1a_2+d_1c_2)z+(c_1b_2+d_1d_2)} \\
=& \frac{a_3z+b_3}{c_3z+d_3} = \phi_{\left(\begin{smallmatrix} a_3 & b_3 \\ c_3 & d_3 \end{smallmatrix}\right)}(z).
\end{aligned}$$

As a result we see that if the modularity condition is satisfied by $\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$, then it is also satisfied by their product:

$$\begin{aligned}
f\left(\phi_{\left(\begin{smallmatrix} a_3 & b_3 \\ c_3 & d_3 \end{smallmatrix}\right)}(z)\right) =& f\left(\phi_{\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)} \circ \phi_{\left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)}(z)\right) \\
=& (c_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right)+d_3)^k f\left(\phi_{\left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)}(z)\right) \\
=& (c_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right)+d_3)^k (c_2z+d_2)^k f(z) \\
=& (c_3z+d_3)^k f(z).
\end{aligned}$$

This result tells us that if the modularity condition is true for a generating set of $\mathbf{SL}_2(\mathbb{Z})$, then it is true for all of $\mathbf{SL}_2(\mathbb{Z})$! So we have reduced the problem to finding a generating set of $\mathbf{SL}_2(\mathbb{Z})$. It turns out that the matrices $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ generate $\mathbf{SL}_2(\mathbb{Z})$ as per future corollary 2.1. Since the modularity condition for these two matrices are precisely the two conditions above, we are done. □

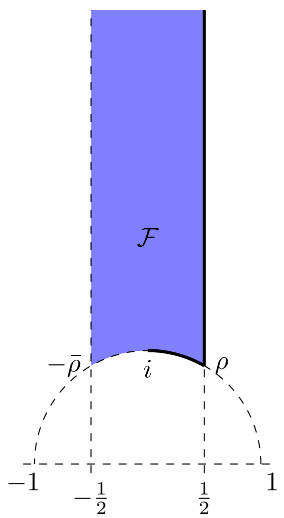
After this proof, we realize that we really want to study the group of transformations $\phi_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}$, which is called the modular group. Let us denote this group by G. G acts on the upper half plane by evaluation. To see that a point in $\mathcal{H}$ gets sent to another point in $\mathcal{H}$ under the action of G, we calculate that

$$\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{(ad-bc)\operatorname{Im}(z)}{|cz+d|^2} = \frac{\operatorname{Im}(z)}{|cz+d|^2} > 0.$$

If we fix a point $z_0 \in \mathcal{H}$, the orbit Gz_0 of the point z_0 under the action of G is defined as $Gz_0 = \{\frac{az_0+b}{cz_0+d} \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbb{Z})\}$. It is clear that the values of a modular form $f(z)$ at the points $z \in Gz_0$, are completely determined by the value $f(z_0)$. Therefore it begs the question: Can we find a subset of $\mathcal{H}$ such that each element represents a unique orbit, and that covers all orbits? Because then any modular form could be determined by the values it has on this subset. This idea is common enough that mathematicians have a name for it: a fundamental domain. There are of course many choices for a fundamental domain of G, but a nice candidate that is connected is the region:

$$\bar{\mathcal{F}} = \{z \in \mathcal{H} \mid |\operatorname{Re}(z)| \leq \frac{1}{2}, |z| \geq 1\}.$$

It is not exactly a fundamental domain yet, since there are a few problems at the boundary. The points on the line $-\frac{1}{2} + it$ are in the same orbit as $\frac{1}{2} + it$, and the ones on the arc $|z| = 1$ are in the same orbit as $-z^{-1} = -\bar{z}$. The solution is to choose only the points on the boundary where $\operatorname{Re}(z) \geq 0$. A picture gives the gist of it.



Lemma 2.4. *$\mathcal{F}$ as in the diagram is a fundamental domain of the modular group G. In addition the only points in $\mathcal{F}$ with nontrivial stabilizer group are:*

1. *i which is fixed by the subgroup $\left\langle \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \right\rangle$ of order 2.*

2. *$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ which is fixed by the subgroup $\left\langle \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right) \right\rangle$ of order 3.*

Proof. For each point $z_0 \in \mathbb{C}$, we look at the point in its orbit Gz_0 with maximal imaginary part. This point must exist because there are only a finite number of $c, d \in \mathbb{Z}$ such that $\mathrm{Im}(z_0) < \frac{\mathrm{Im}(z_0)}{|cz_0+d|^2}$. Translating this point with integer steps parallel to the real axis, we get a new point y with $|\mathrm{Re}(y)| \leq \frac{1}{2}$ and maximal imaginary part. Suppose that $|y| < 1$, then $\mathrm{Im}(-\frac{1}{y}) > \mathrm{Im}(y)$, contradicting the maximality of the imaginary part of y. Therefore $y \in Gz_0$ must lie in $\bar{\mathcal{F}}$, which reduces to the domain $\mathcal{F}$ by our previous observations.

Now suppose that $z_0 \in \mathcal{F}$ and $gz_0 \in \mathcal{F}$ for a $g \in G$. If $\mathrm{Im}(gz_0) \leq \mathrm{Im}(z_0)$, replace (z_0, g) with (gz_0, g^{-1}) instead so that we may assume that $\mathrm{Im}(gz_0) \geq \mathrm{Im}(z_0)$. Then if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we get that $\frac{\mathrm{Im}(z_0)}{|cz_0+d|^2} \geq \mathrm{Im}(z_0) \implies 1 \geq |cz_0 + d|$. If $|c| \geq 2$, then $1 \geq |cz_0 + d| \geq \mathrm{Im}(cz_0 + d) = c\mathrm{Im}(z_0) \geq 2\frac{\sqrt{3}}{2}$ which is a contradiction. So $c = 0, 1, -1$. If $c = 0$ then $ad - bc = 1 \implies a = d = \pm 1$, and so g is a translation parallel to the real axis which puts gz_0 outside of $\mathcal{F}$ unless g is the identity. So $c = \pm 1$.

If $|d| \geq 1$, then $1 \geq |cz_0 + d| = |z_0 + cd|$. This only works if $z_0 = \rho$ and $cd = -1$. We can calculate the stabilizer group of ρ. If $c = 1$, we must have $d = -1$ and $ad - bc = -a - b = 1$.

$$\frac{a\rho + (-1-a)}{\rho - 1} = g\rho = \rho \implies a + \frac{1}{1-\rho} = \rho \implies a = 0.$$

So $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ is a stabilizer. With $c = -1$ we can find the other nontrivial stabilizer: $\left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)^2$.

Now the only remaining case is $d = 0$. In this case $1 \geq |cz_0| = |z_0| \geq 1$, so $|z_0| = 1$. Since $\left(\begin{smallmatrix} a & \mp 1 \\ \pm 1 & 0 \end{smallmatrix}\right) z_0 = \pm a - \frac{1}{z_0}$, and $-\frac{1}{z_0}$ is just a reflection about the imaginary axis for points on the unit circle, a has to equal zero to stay in the fundamental domain. The only fixed point of $-\frac{1}{z_0}$ is i. This proof can be found in Serre's book [3, p. 78]. □

Corollary 2.1. *$\boldsymbol{SL}_2(\mathbb{Z})$ is generated by $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.*

Proof. Because $\phi_T(z) = z + 1$ and $\phi_S(z) = -\frac{1}{z}$, we notice that the maximal imaginary argument at the start of the previous proof works also for the subgroup $\langle S, T \rangle$. Namely for any point $z \in \mathcal{H}$, there exists a $\gamma \in \langle S, T \rangle$ such that $\gamma z \in \mathcal{F}$.

Let z_0 be in the interior $\mathcal{F}$ and let $\gamma \in \mathbf{SL}_2(\mathbb{Z})$. Then there exists a $\gamma' \in \langle S, T \rangle$ such that $\gamma' \gamma z_0 \in \mathcal{F}$, which forces $z_0 = \gamma' \gamma z_0$. So $\gamma' \gamma$ is in the stabilizer group of z_0 which is the trivial group because z_0 is not i or ρ. Therefore $\gamma = \gamma'^{-1} \in \langle S, T \rangle$ and so $\mathbf{SL}_2(\mathbb{Z}) = \langle S, T \rangle$.

□

Going back to inspecting the definition of a modular form, we notice that any linear combination of a modular form of weight k, is another modular form of weight k. So the modular forms of weight k form a vectorspace. We denote this by $\mathbb{M}_k$.

Multiplication also preserves the structure of modular forms.

Lemma 2.5. *The direct sum of all modular forms* $\oplus_{k=0}^{\infty}\mathbb{M}_k$ *has a graded ring structure. In other words, for modular forms* $f \in \mathbb{M}_k$ *and* $g \in \mathbb{M}_l$*:*

1. $fg \in \mathbb{M}_{k+l}$.
2. $\mathbb{M}_k$ *is an abelian group under addition.*

After exhausting all the straightforward observations that we can make, we now want to take the next step in understanding the subject: finding an example. If we can find concrete examples then we can calculate with them and discover new properties of modular forms.

CHAPTER 3

EISENSTEIN SERIES

Now our main goal is to find a nonzero modular form and calculate its Fourier coefficients. The reason why we want to find the Fourier series representation, is because it is easy to calculate with and work with algebraically.

The first thing that we can try is to see if we can turn a general Fourier series into a modular form.

So let us start with a Fourier series:

$$\phi(z) = \sum_{n=0}^{\infty} a_n e^{2\pi izn}.$$

ϕ is already periodic, so we only need to impose the condition $z^{-k}\phi(-\frac{1}{z}) = \phi(z)$ by lemma 2.3. We can maybe do this by trying to calculate the Fourier coefficients of the function $z^{-k}\phi\left(-\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^{-k} e^{-2\pi inz^{-1}}$ and set them equal to the coefficients of ϕ.

$$a_m = \int_0^1 \left(\sum_{n=0}^{\infty} a_n x^{-k} e^{-2\pi inx^{-1}} \right) e^{-2\pi imx} dx.$$

There is a slight problem. We cannot integrate from 0 to 1 on the real line, because modular forms are only defined on the upper half plane $\mathrm{Im}(z) > 0$. Luckily one can instead integrate along any contour that is parallel to the contour from 0 to 1, because when you use Cauchy's integral theorem on a parallelogram with 0 to 1 as a side, the two vertical sides cancel since they integrate over the same values ($\phi(z) = \phi(z+1)$). So we actually would need

$$a_m = \int_{0+i}^{1+i} \left(\sum_{n=0}^{\infty} a_n z^{-k} e^{-2\pi i n z^{-1}} \right) e^{-2\pi i m z} dz.$$

If convergence works then maybe we can hope that

$$a_m = \sum_{n=0}^{\infty} a_n \int_{0+i}^{1+i} z^{-k} e^{-2\pi i (nz^{-1} + mz)} dz.$$

But even so, there are just too many problems. The integral is ugly and hard to evaluate. And even if we solve the integral, we still have to solve an infinite system of linear equations. We must accept that this line of thinking is a dead end and give up.

We can still learn something from our endeavour. The main problem that we had before was that it was hard to satisfy the condition $z^{-k}\phi(-\frac{1}{z}) = \phi(z)$ after satisfying the periodicity condition. So perhaps in our next attempt, we will focus on satisfying this condition first. With some luck, one can discover that the function $g(z) = z^{-\frac{k}{2}}$ is almost what we want. Indeed

$$z^{-k} g\left(-\frac{1}{z}\right) = z^{-k} \frac{1}{(-\frac{1}{z})^{\frac{k}{2}}} = (-1)^{\frac{k}{2}} z^{-\frac{k}{2}} = (-1)^{\frac{k}{2}} g(z).$$

It does at least work for k divisible by 4. However it is definitely not periodic. But this may not be a problem, because there is a very general idea from group theory and representation theory that we may apply here.

If you want to make something invariant under an action, take the sum or average over the action.

We want to make it invariant under the action $f(z) \mapsto f(z+1)$, so we will sum over this action. We get a new candidate:

$$h(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^{\frac{k}{2}}}?$$

Of course this may not satisfy the original constraint. Sadly we get

$$h(-z^{-1}) = \sum_{n \in \mathbb{Z}} \frac{1}{(-z^{-1}+n)^{\frac{k}{2}}} = \sum_{n \in \mathbb{Z}} \frac{z^{\frac{k}{2}}}{(nz-1)^{\frac{k}{2}}},$$

which is not quite what we want. But it is very close. We see that we end up with an integer coefficient in front of the z, so perhaps we can sum over this integer coefficient as well.

$$\phi(z) = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} \frac{1}{(mz+n)^{\frac{k}{2}}}?$$

This works! However, it is in fact a modular form of weight $\frac{k}{2}$, so we might as well replace $\frac{k}{2}$ with k.

Theorem 3.1.

$$G_k(z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}$$

is a nonzero modular form of weight k for k an even integer larger than 2. The modular forms $G_k(z)$ are called Eisenstein series.

Proof. The modularity condition is satisfied since

$$\begin{aligned} G_k(z+1) &= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m(z+1)+n)^k} = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ m\neq 0}} \frac{1}{(mz+(m+n))^k} + \sum_{\substack{m=0 \\ n\neq 0}} \frac{1}{n^k} \\ &\overset{\hat{n}=m+n}{=} \sum_{\substack{(m,\hat{n})\in\mathbb{Z}^2 \\ m\neq 0}} \frac{1}{(mz+\hat{n})^k} + \sum_{\substack{m=0 \\ n\neq 0}} \frac{1}{n^k} = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k} = G_k(z) \end{aligned}$$

and

$$\begin{aligned} G_k(-z^{-1}) &= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(-mz^{-1}+n)^k} = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{z^k}{(-m+nz)^k} \\ &\overset{\substack{\hat{m}=n \\ \hat{n}=-m}}{=} z^k \sum_{\substack{(\hat{m},\hat{n})\in\mathbb{Z}^2 \\ (\hat{m},\hat{n})\neq(0,0)}} \frac{1}{(\hat{m}z+\hat{n})^k} = z^k G_k(z). \end{aligned}$$

Like most things in analysis, we actually need to check that this sum even converges, and that our interchange of variable is valid before we can celebrate. We will show that this sum converges absolutely and locally uniformly in the upper half plane. This is enough because Morera's theorem tells us that holomorphic functions uniformly converge to holomorphic functions, and Fubini's theorem tells us that change of summation order is justified by absolute convergence.

The proof of this fact is quite technical, so I will only give a rough outline. The first step in the proof is to see that we only need to look at the fundamental domain, as the other regions can be bounded using the modularity condition. For $z \in \mathcal{F}$, we know that $|z| \geq 1$ and $|\mathrm{Re}(z)| \leq \frac{1}{2}$, so

$$\frac{1}{|mz+n|^k} = \frac{1}{((mz+n)(m\bar{z}+n))^{\frac{k}{2}}} = \\ \frac{1}{(m^2|z|^2 + 2nm\mathrm{Re}(z) + n^2)^{\frac{k}{2}}} \leq \frac{1}{(m^2+mn+n^2)^{\frac{k}{2}}}.$$

The inequality obtains equality when $z = \rho = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, so in fact

$$\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{1}{|mz+n|^k} \leq \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{1}{|m\rho+n|^k}.$$

Now if we show that the right hand sum converges, then we get that $G_k(z)$ converges uniformly and absolutely on the fundamental domain. Convergence of the right hand side can be obtained by proving the following bound for some constant $C > 0$ and disc $\mathcal{D}_a = \{(x,y) : x^2 + y^2 < a\}$:

$$\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{1}{|m\rho+n|^k} \leq C \int\int_{\mathbb{R}^2\setminus\mathcal{D}_a} \frac{1}{(x^2+y^2)^{\frac{k}{2}}} dxdy$$
$$= C\int_0^{2\pi}\int_a^{\infty} \frac{1}{r^k} r dr d\theta = 2\pi C \frac{k-2}{a^{k-2}}.$$

Notice how the jacobian term $rdrd\theta$ makes it so that $k = 2$ just doesn't converge, while for $k > 2$ this integral does converge.

Finally the reason why we only consider k even, is that $G_k(z) = 0$ for odd k. □

We recall that modular forms should admit a Fourier series representation, so the next natural step is to find such a representation for our newly found modular forms. This will help us a lot. For example, it is not immediately obvious that $G_k(z) \neq 0$ for even k. However with the Fourier series representation, it will be clear.

The way we calculate the Fourier series representation will not be by calculating the Fourier coefficients using integrals, instead we will compare the series with known series representations of certain functions. For example, a natural first observation that we can make is that these series representations of the polygamma functions are quite similar to our Eisenstein series.

$$\psi^{(k)}(z) = (-1)^{k+1}k! \sum_{n=0}^{\infty} \frac{1}{(z+n)^{k+1}}.$$

With some manipulation we can get:

$$\frac{\psi^{(k-1)}(mz) + \psi^{(k-1)}(-mz)}{(k-1)!} - (mz)^{-k} = \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k}. \qquad (m \neq 0, k \text{ even})$$

If you are unfamiliar with the polygamma function, we will only use results found on WolframAlpha [5]. Using the recurrence relation $\psi^{(k)}(z+1) = \psi^{(k)}(z) + (-1)^k k!\, z^{-(k+1)}$, and the reflection formula $(-1)^k\psi^{(k)}(1-z) - \psi^{(k)}(z) = \pi \frac{\mathrm{d}^k}{\mathrm{d}z^k}\cot(\pi z)$ for the polygamma function we see that:

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^k} &= \frac{\psi^{(k-1)}(mz)+\psi^{(k-1)}(-mz)}{(k-1)!}-(mz)^{-k}\\ &= \frac{\psi^{(k-1)}(mz)+\psi^{(k-1)}(1-mz)}{(k-1)!}\\ &= \frac{-\pi\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\cot(\pi mz)}{m^{k-1}(k-1)!}. && (m\neq 0, k \text{ even})\end{aligned}$$

The amazing thing now is that we can calculate the Fourier series representation for $\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\cot(\pi mz)$. Observing that $|e^{2\pi imz}|<1$ when z is in the upper halfplane and m positive, we know that the series $\sum_{m=0}^{\infty}e^{2\pi imz}$ converges absolutely and uniformly to $(1-e^{2\pi imz})^{-1}$. So we can move the derivative across the sum in the following calculation:

$$\begin{aligned}\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\cot(\pi mz) =&\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\frac{i(e^{\pi imz}+e^{-\pi imz})}{e^{\pi imz}-e^{-\pi imz}}\\ =&\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\frac{i(e^{2\pi imz}+1)}{e^{2\pi imz}-1}\\ =&\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}-i(e^{2\pi imz}+1)\sum_{n=0}^{\infty}e^{2\pi inmz}\\ =&\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\left(-i+-2i\sum_{n=1}^{\infty}e^{2\pi inmz}\right)\\ =&-2i(2\pi i)^{k-1}\sum_{n=1}^{\infty}(nm)^{k-1}e^{2\pi inmz}. && (m>0, k \text{ even})\end{aligned}$$

And for m negative we use the fact that a derivative of an odd function is even and vice versa. So since $\cot(\pi mz)$ is odd, and we differentiate it an odd number of times, the function $\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \cot(\pi m z)$ is an even function. Putting everything together:

$$\begin{aligned}
G_k(z) &= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k} = \sum_{n\neq 0} n^{-k} + \sum_{m\neq 0}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k} \\
&= 2\sum_{n=0}^{\infty} n^{-k} + \frac{-\pi}{(k-1)!} \sum_{m\neq 0} m^{1-k} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \cot(\pi m z) \\
&= 2\sum_{n=0}^{\infty} n^{-k} + \frac{-2\pi}{(k-1)!} \sum_{m=1}^{\infty} m^{1-k} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \cot(\pi m z) \\
&= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{1-k}(nm)^{k-1} e^{2\pi i n m z} \\
&= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n m z} \\
&= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{\hat{n}=1}^{\infty}\sum_{d\mid\hat{n}} d^{k-1} e^{2\pi i \hat{n} z} && (\hat{n} = nm, d = n) \\
&= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}.
\end{aligned}$$

The divisor functions $\sigma_k(n) = \sum_{d|n} d^k$ and the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ are well known number theoretic functions, so it is quite strange that they suddenly appear here. The divisor functions take positive integer values and have the nice multiplicative property that $\sigma_k(ab) = \sigma_k(a)\sigma_k(b)$ for coprime a and b. It is genuinely surprising that the Eisenstein series have such beautiful Fourier coefficients.

By convention, it is common to rescale the Eisenstein series such that the constant term of the Fourier series is 1. Namely, we define $E_k(z) = \frac{1}{2\zeta(k)} G_k(z)$. For convenience, it is also common to use the implicit notation $q = e^{2\pi i z}$, and package the constants together. In conclusion:

Lemma 3.1. *For $k > 2$ even, define the Fourier series*

$$E_k(z) = 1 + c_k \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \qquad \text{where } c_k = \frac{(2\pi i)^k}{(k-1)!\zeta(k)},\ q = e^{2\pi i z}.$$

Then $E_k(z)$ is a modular form of weight k.

CHAPTER 4

THE MODULAR DISCRIMINANT

We will now introduce the special modular form that is the focus of this bachelor, namely the modular discriminant. One motivation for finding this modular form is by considering the following definition:

Definition 4.1. *A cusp form $\phi(z)$ is a modular form that vanishes as* $\mathrm{Im}(z) \to \infty$*. In other words, the constant term in the Fourier series of ϕ is zero. We denote the vector space of cusp forms of weight k by $\mathbb{S}_k$.*

It seems like it should be straightforward to find a nonzero cusp form: Take two different modular forms, scale them such that their constant terms match, and subtract. However if you attempt to do this using the Eisenstein series E_k, you quickly discover some interesting identities:

$$\begin{aligned} E_4^2 - E_8 &= 0 \\ E_4 E_6 - E_{10} &= 0 \\ E_4 E_{10} - E_{14} &= 0 \\ E_6 E_8 - E_{14} &= 0. \end{aligned}$$

No matter how hard you try, it doesn't seem possible to make a nonzero cusp form with weight 4, 6, 8, 10 or 14. It is only when you try weight 12, that you get something nonzero:

$$E_4^3 - E_6^2 = 1728(q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 + \ldots).$$

This series has a lot of weird coincidences. Firstly, the coefficients are integers, which is weird. Secondly, the second coefficent -24 multiplied by the third coefficient 252 is equal to the sixth coefficient -6048, which is even more weird. However, if you calculate more and more coefficients, it seems like all the coefficients are multiplicative just like the divisor functions, which means that something is definitely up.

These coefficients are also known as $\tau(n)$, the Ramanujan tau function. And the cusp form with these coefficients (up to a constant) is known as the modular discriminant.

Definition 4.2. *The modular discriminant is defined as*

$$\Delta(z) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z} = (2\pi)^{12} \frac{E_4^3 - E_6^2}{1728}.$$

The rest of this bachelor will aim to prove the following results about the modular discriminant, the Ramanujan tau function and modular forms in general.

Theorem 4.1. *Let $\Delta(z)$ be the modular discriminant, and $\tau(n)$ the Ramanujan tau function. Then the following holds:*

1. *$\Delta(z)$ is the discriminant of the polynomial $p(x) = 4x^3 - 60G_4 x - 140G_6$, the same polynomial in the differential equation of the Weierstrass elliptic function: $(\wp')^2 = p(\wp)$.*
2. *$\Delta(z) \neq 0$ for $z \in \mathbb{C}, \operatorname{Im}(z) > 0$.*
3. *$\Delta(z)$ is the only cusp form of weight 12 (up to a constant).*
4. *The function $\mathcal{T} : \mathbb{M}_k \to \mathbb{S}_{k+12}$ defined by $f \mapsto f \cdot \Delta$ is a bijection.*
5. *The spaces of modular forms $\mathbb{M}_k$ are finite dimensional. In particular:*
$$\dim_{\mathbb{C}} \mathbb{M}_k = \begin{cases} 0, & k \text{ odd or negative}, \\ \lfloor \frac{k}{12} \rfloor, & k \equiv 2 \mod 12, \\ \lfloor \frac{k}{12} \rfloor + 1, & k \not\equiv 2 \mod 12. \end{cases}$$
6. *$\Delta(z)$ is an eigenform. That is to say it is an eigenvector for all the linear Hecke operators T_m.*
7. $$\tau(a)\tau(b) = \sum_{d \mid \gcd(a,b)} d^{11} \tau\left(\frac{ab}{d^2}\right).$$
8. *In particular $\tau(a)\tau(b) = \tau(ab)$ when $\gcd(a,b) = 1$.*

The final goal of this bachelor is to prove point 8 in 4.1. To do this, we need to prove all 7 other points. So let us start with the first point.

The first point tells us why the modular discriminant is called a discriminant. It is simply a discriminant of this weird polynomial occuring in this weird differential equation. This is shown easily.

Theorem 4.2. *Δ is the discriminant of the polynomial $p(x) = 4x^3 - 60G_4x - 140G_6$.*

Proof. The discriminant of a depressed cubic ax^3+px+q is given by $\Delta = -4a^{-1}p^3 - 27q^2$. Applying this to our polynomial $p(x)$ we get:

$$\begin{aligned}
\Delta &= (60G_4)^3 - 3^3(140G_6)^2 \\
&= 2^6 3^3 5^3 G_4^3 - 3^3 2^4 5^2 7^2 G_6^2 \\
&= (2\pi)^{12}\left(\frac{3^3 5^3}{2^6\pi^{12}}G_4^3 - \frac{3^3 5^2 7^2}{2^8\pi^{12}}G_6^2\right) \\
&= (2\pi)^{12}\left(\frac{1}{2^9 3^3\zeta(4)^3}G_4^3 - \frac{1}{2^8 3^3\zeta(6)^2}G_6^2\right) \\
&= \frac{(2\pi)^{12}}{1728}\left(\frac{1}{2^3\zeta(4)^3}G_4^3 - \frac{1}{2^2\zeta(6)^2}G_6^2\right) \\
&= \frac{(2\pi)^{12}}{1728}\left(E_4^3 - E_6^2\right).
\end{aligned}$$

Here we used definition 3 and the specific values $\zeta(4) = \frac{\pi^4}{2\cdot 3^2\cdot 5}$ and $\zeta(6) = \frac{\pi^6}{3^3\cdot 5\cdot 7}$. ∎

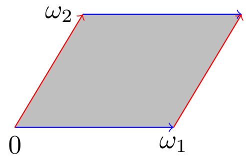
To prove the second statement of 4.1, we actually need to understand the whole story behind the differential equation of the Weierstrass elliptic function.[1] Thus we start a long detour of understanding elliptic functions. For this detour, we shall follow the approach of Apostol's book [4].

Definition 4.3. *Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent as $\mathbb{R}$ vectors. An elliptic function, or a doubly periodic function, is a meromorphic function f such that for all $z \in \mathbb{C}$:*

$$\begin{aligned}
f(z+\omega_1) &= f(z) \\
f(z+\omega_2) &= f(z).
\end{aligned}$$

Elliptic functions are thus invariant under the lattice $\Lambda = \{a\omega_1 + b\omega_2 \mid a, b \in \mathbb{Z}\}$. If we assume that ω_1 and ω_2 are the smallest vectors that generate the lattice, then the parallelogram spanned by ω_1 and ω_2, with opposite edges identified, is a fundamental domain. Just like Pac-man, the fundamental domain is topologically homeomorphic to a torus, so elliptic functions can be thought of as meromorphic functions from a torus to the complex numbers.

[1] I found out later that this is a lie. There is a much more elegant proof that $\Delta \neq 0$ using lemma 5.1, which can be found in Serre [3, p. 88]. But by the time I found out, I had already written the entire section on elliptic functions, so here it stays.



Elliptic functions have a lot of nice properties. Assuming that ω_1 and ω_2 span a fundamental domain, we have the following results:

Lemma 4.1. *Elliptic functions form a division ring with differentiation as a linear operator. Namely if f is doubly periodic, then $\frac{1}{f}$ are f' also doubly periodic.*

Proof. It is clear that the elliptic functions form a subring of the division ring of meromorphic functions. Differentiating the equations in 4.3, gives us the second part. □

Lemma 4.2. *An entire elliptic function is constant.*

Proof. An entire elliptic function f takes all its value on the fundamental domain. Because the fundamental domain is compact, the holomorphic f has a maximum in this domain, ergo it is bounded everywhere. By Louville's theorem, bounded entire functions are constant functions, so f is constant. □

Lemma 4.3. *The integral of any elliptic function along the contour around the parallelogram spanned by ω_1 and ω_2 is zero.*

Proof. Because the elliptic function takes the same values on the opposite sides of the parallelogram by double periodicity, the value of the integral along the top/left side cancels with the integral along the bottom/right side and we get zero. □

Lemma 4.4. *An elliptic function has an equal number of zeroes and poles in the fundamental domain.*

Proof. If f is an elliptic function, then $\frac{f'}{f}$ is elliptic by 4.1. Let Z equal the number of zeroes (including degrees) in the fundamental domain, and P the number of poles. Then by the argument principle and lemma 4.3, we have that

$$Z - P = \frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)} dz = 0. \tag{4.1}$$

A small point to note is if there are any zeroes or poles on the border, then we adjust the contour to circle around them, and let those circle arcs shrink to zero. Because $\frac{f'}{f}$ has at most a simple pole at p, $\frac{f'(z)}{f(z)} - \frac{\mathrm{Ord}_{z=p}(f)}{z}$ is holomorphic at p, so these circular arcs $\mathcal{C}_r$ converge to partial residues as the radius $r \to 0$.

$$\begin{aligned}\int_{\mathcal{C}_r} \frac{f'(z)}{f(z)} dz &= \mathrm{Ord}_{z=p}(f) \int_{\mathcal{C}_r} \frac{1}{z} dz + \int_{\mathcal{C}_r} \frac{f'(z)}{f(z)} - \frac{\mathrm{Ord}_{z=p}(f)}{z} dz \\ &= \theta i \mathrm{Ord}_{z=p}(f) + \mathcal{O}(r).\end{aligned}$$

Here θ is the angle of $\mathcal{C}_r$ and $\mathrm{Ord}_{z=p}(f)$ is the order of the zero/pole of f at p. A zero/pole on the border will appear multiple times (either 2 or 4), but the sum of the angles of the residues for each time it appears will equal 2π, so in the end the zeroes/poles on the border will be counted correctly. □

The Weierstrass elliptic function $\wp$, as one might expect, is an example of an elliptic function. It is in fact constructed with much the same motivation and idea as the Eisenstein series. Namely, we want our function to be invariant under the "group action" or "addition" of the lattice Λ, so the Weierstrass elliptic function is made by summing over this lattice.

Definition 4.4. *The Weierstrass elliptic function $\wp$ is defined by*

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}.$$

It is not hard to see that $\wp$ could be double periodic if we let it be equal to $\sum_{\lambda \in \Lambda}(z-\lambda)^{-2} - \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-2}$. However this doesn't work, because in this form it doesn't converge.

If we recall from the proof of theorem 3.1, the sum

$$\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} (mz+n)^{-k}$$

converged only for $k > 2$. But this is just equal to $\sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-k}$ for the lattice $\Lambda = \{mz + n1 \mid m, n \in \mathbb{Z}\}$. Setting $z = \frac{\omega_1}{\omega_2}$ and multiplying the sum with ω_2^{-k}, we see that this convergence rule generalizes for all lattices.

$$\sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-k} \text{ converges absolutely} \Leftrightarrow k > 2. \tag{4.2}$$

Theorem 4.3. *The Weierstrass elliptic function converges absolutely and uniformly in compacts, except for each lattice point, where there is a double pole.*

Proof. Assume that z lies in a compact and is not a lattice point. Then we notice that $|z|$ is bounded and that $|z-\lambda| \geq |\lambda| - |z| > \frac{1}{2}|\lambda|$ for large λ, so

$$\left|\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right| = \left|\frac{z^2 - 2z\lambda}{\lambda^2(z-\lambda)^2}\right| < 4\left|\frac{z^2 - 2z\lambda}{\lambda^4}\right| < \frac{C}{|\lambda|^3}$$

for a constant C independent of λ and z. So summing over the lattice, we see that the $\sum_{\lambda\in\Lambda\setminus\{0\}}\left|\frac{1}{(z-\lambda)^2}-\frac{1}{\lambda^2}\right|$ is bounded by $C\sum_{\lambda\in\Lambda\setminus\{0\}}|\lambda|^{-3}$ which converges by 4.2. □

Corollary 4.1. *$\wp$ is meromorphic.*

Theorem 4.4. *The Weierstrass elliptic function is an elliptic function.*

Proof. Let $\hat\lambda\in\Lambda$.

$$\begin{aligned}
\wp(z-\hat\lambda) &= \frac{1}{(z-\hat\lambda)^2}+\sum_{\lambda\in\Lambda\setminus\{0\}}\frac{1}{(z-(\hat\lambda+\lambda))^2}-\frac{1}{\lambda^2}\\
&= \frac{1}{(z-\hat\lambda)^2}+\sum_{\lambda\in\Lambda\setminus\{\hat\lambda\}}\frac{1}{(z-\lambda)^2}-\frac{1}{(\lambda-\hat\lambda)^2}\\
&= \frac{1}{(z-\hat\lambda)^2}+\frac{1}{z^2}-\frac{1}{\hat\lambda^2}+\sum_{\lambda\in\Lambda\setminus\{0,\hat\lambda\}}\frac{1}{(z-\lambda)^2}-\frac{1}{\lambda^2}+\frac{1}{\lambda^2}-\frac{1}{(\lambda-\hat\lambda)^2}\\
&= \frac{1}{z^2}+\sum_{\lambda\in\Lambda\setminus\{0\}}\frac{1}{(z-\lambda)^2}-\frac{1}{\lambda^2}+\sum_{\lambda\in\Lambda\setminus\{0,\hat\lambda\}}\frac{1}{\lambda^2}-\frac{1}{(\lambda-\hat\lambda)^2}\\
&= \wp(z).
\end{aligned}$$

The last sum must equal zero because the bijection $\lambda\mapsto\hat\lambda-\lambda$ of the index set $\Lambda\setminus\{0,\hat\lambda\}$ changes the sign of the sum without changing its value.

We could split the sums because they converged absolutely. □

Now we will assume that our lattice is of the form $\Lambda=\{m\omega+n\mid m,n\in\mathbb{Z}\}$. If we also assume that $|z|<\inf_{\lambda\in\Lambda\setminus\{0\}}|\lambda|$, then we can calculate the Laurent series of $\wp$ and observe something interesting:

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2}+\sum_{\lambda\in\Lambda\setminus\{0\}}\frac{1}{(z-\lambda)^2}-\frac{1}{\lambda^2}=\frac{1}{z^2}+\sum_{\lambda\in\Lambda\setminus\{0\}}\frac{1}{\lambda^2}\left(\frac{1}{1-(\frac{z}{\lambda})^2}-1\right)\\
&= \frac{1}{z^2}+\sum_{\lambda\in\Lambda\setminus\{0\}}\frac{1}{\lambda^2}\left(\left(\sum_{n=0}^{\infty}(n+1)\left(\frac{z}{\lambda}\right)^n\right)-1\right)\\
&= \frac{1}{z^2}+\sum_{\lambda\in\Lambda\setminus\{0\}}\frac{1}{\lambda^2}\sum_{n=1}^{\infty}(n+1)\left(\frac{z}{\lambda}\right)^n=\frac{1}{z^2}+\sum_{n=1}^{\infty}(n+1)\left(\sum_{\lambda\in\Lambda\setminus\{0\}}\frac{1}{\lambda^{n+2}}\right)z^n\\
&= \frac{1}{z^2}+\sum_{n=1}^{\infty}(n+1)G_{n+2}(\omega)z^n=\frac{1}{z^2}+\sum_{n=1}^{\infty}(2n+1)G_{2n+2}(\omega)z^{2n}.
\end{aligned}$$

Indeed our favorite Eisenstein series show up. This is not as surprising as one might think. The coefficients of $\wp$ have to be modular forms, because they are invariant under the choice of generators for the lattice Λ. In fact the lattice generated by $(a\omega+b, c\omega+d)$ is equal to the one generated by $(\omega, 1)$ if and only if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbb{Z})$. And so

$$\begin{aligned} G_{2k+2}(\omega) &= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\omega+n)^{k+2}} = \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{\lambda^{k+2}} \\ &= \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m(a\omega+b)+n(c\omega+d))^{k+2}} = (c\omega+d)^{-(k+2)} G_{2k+2}\left(\frac{a\omega+b}{c\omega+d}\right). \end{aligned}$$

This is an interesting perspective on our definition of modular forms. We can actually consider them as holomorphic functions from the set of lattices to $\mathbb{C}$. If $f : \mathcal{H} \to \mathbb{C}$ is a modular form as we have previously defined, then we can define the function F for an input $\Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$.

$$F(\Lambda) = f\left(\frac{\omega_1}{\omega_2}\right).$$

In this context, the modularity condition reduces to F being well defined (independent of the choice of ω_1, ω_2).

Anyway now that we have the Laurent series of $\wp$, we can calculate the Laurent series of its derivative $\wp'(z)$,

$$\wp'(z) = -\frac{2}{z^3} + \sum_{n=1}^{\infty} 2n(2n+1)G_{2n+2}z^{2n-1}, \tag{4.3}$$

and figure out where the Weierstrass function's differential equation comes from.

Theorem 4.5.
$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6.$$

Proof. One can calculate the Laurent series of $\wp'(z)^2$ and $\wp(z)^3$:

$$\begin{aligned} \wp'(z)^2 &= 4z^{-6} - 24G_4z^{-2} - 80G_6 + \mathcal{O}(z^2) \\ \wp(z)^3 &= z^{-6} + 9G_4z^{-2} + 15G_6 + \mathcal{O}(z^2) \\ \wp(z) &= z^{-2} + \mathcal{O}(z^2). \end{aligned}$$

Then it is not hard to verify that

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_4\wp(z) + 140G_6 = \mathcal{O}(z^2).$$

By lemma 4.1, this $\mathcal{O}(z^2)$ is a doubly periodic entire function. By lemma 4.2, it must be constant. This constant is zero, since $\mathcal{O}(z^2)$ vanishes at zero. $\square$

Next we will find the zeroes of $\wp'(z)$, in order to find the zeroes of the polynomial $4x^3 - 60G_4x - 140G_6$.

Lemma 4.5. *The half periods $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$ and $\frac{\omega_1+\omega_2}{2}$ are simple zeroes of $\wp'(z)$.*

Proof. Observing the Laurent series of $\wp'(z)$ in 4.3, we see that $\wp'(z)$ is odd. Then for $\frac{\omega_1}{2}$, we have

$$\begin{aligned} \wp'\left(\frac{\omega_1}{2}\right) = -\wp'\left(-\frac{\omega_1}{2}\right) = -\wp'\left(-\frac{\omega_1}{2} + \omega_1\right) = -\wp'\left(\frac{\omega_1}{2}\right), \\ \implies \wp'\left(\frac{\omega_1}{2}\right) = 0 \text{ or is a pole.} \end{aligned}$$

We can conclude the cases $\frac{\omega_2}{2}$ and $\frac{\omega_1+\omega_2}{2}$ similarly. From theorem 4.3, it follows that $\wp'$ only has poles of degree 3 at each lattice point, meaning that none of the three are poles of $\wp'$, so they must all be zeroes. Theorem 4.3 also implies that the degree 3 pole at zero is the only pole of $\wp'$ in the fundamental domain, so by lemma 4.4, there are exactly three zeroes in the fundamental domain. Thus all of them are simple. $\square$

Theorem 4.6. *The values $e_1 = \wp\left(\frac{\omega_1}{2}\right)$, $e_2 = \wp\left(\frac{\omega_2}{2}\right)$ and $e_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$ are pairwise distinct zeroes of $p(x) = 4x^3 - 60G_4x - 140G_6$.*

Proof. By theorem 4.5 and the previous lemma, we know that e_1, e_2, e_3 are zeroes of $p(x)$. So we only need to show that they are distinct. Define $W(z) = \wp(z) - e_1$. $W(z)$ has a double zero at $\frac{\omega_1}{2}$ because the derivative vanishes too: $W'\left(\frac{\omega_1}{2}\right) = \wp'\left(\frac{\omega_1}{2}\right) = 0$. Similarly $\wp(z) - e_2$ has a double zero at $\frac{\omega_2}{2}$. Suppose that $e_1 = e_2$. Then $W(z) = \wp(z) - e_2$, implying that $W(z)$ has a total of four zeroes in the fundamental domain. But $W(z)$ is just a constant shift away from $\wp$, so it only have a pole of degree 2 in the fundamental domain. These two facts contradict lemma 4.4, so $e_1 \neq e_2$. This works similarly for the other pairs. $\square$

Now we are finally ready to conclude this side quest into elliptic functions.

Corollary 4.2. *The modular discriminant Δ does not vanish for any $z \in \mathcal{H}$.*

Proof. We know from theorem 4.2, that Δ is the discriminant of the polynomial $p(x)$, as long as the lattice is of the form $\Lambda = \{mz + n1 \mid m, n \in \mathbb{Z}\}, z \in \mathcal{H}$. But the discriminant of a polynomial vanishes if and only if its roots are not pairwise distinct. But we have just showed that these roots are always pairwise distinct, no matter the lattice chosen. Therefore the modular discriminant is never zero in the upper half plane. $\square$

CHAPTER 5

FINITE DIMENSIONALITY

Now we will tackle what is arguably the most important theorem of modular forms.

Theorem 5.1. *The spaces of modular forms $\mathbb{M}_k$ are finite dimensional. In particular:*

$$\dim_{\mathbb{C}} \mathbb{M}_k = \begin{cases} 0, & k \text{ odd or negative}, \\ \lfloor \frac{k}{12} \rfloor, & k \equiv 2 \mod 12, \\ \lfloor \frac{k}{12} \rfloor + 1, & k \not\equiv 2 \mod 12. \end{cases}$$

This theorem will be vital to our understanding of modular forms. For example it explains all the weird coincidences we found in chapter 4. All of them are true because the dimension of $\mathbb{M}_k$ is 1 for $k = 8, 10, 14$.

So how will we prove this? Well, the key is to realize that we haven't used the true power of complex analysis on modular forms yet. Just like we did for elliptic functions, the simple idea is to calculate a contour integral around the fundamental domain. This idea was found in the book A Course in Arithmetic by Serre [3], and we will follow his approach by first defining a new class of functions.

Definition 5.1. *A modular function of weight k is a meromorphic function defined on the upper half plane $\mathcal{H}$ such that for any $z \in \mathcal{H}$ and matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \boldsymbol{SL}_2(\mathbb{Z})$,*

1.
$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$

2. The Laurent series expansion of $\hat{f}(q) = f\left(\frac{\ln(q)}{2\pi i}\right)$ around zero has a finite lowest term.

Essentially a modular function is a weaker version of a modular form, as it is no longer required to be holomorphic everywhere, only meromorphic. We notice that the second point is equivalent to saying that the Fourier series expansion of f has a finite number of negative powers. It can also be thought of as removing the possibility that f has an essential singularity at infinity.

We recall what the order of a meromorphic function at a point is.

Definition 5.2. *The order $\vartheta_p(f)$ of a meromorphic function f at a point p is the lowest degree appearing in the Laurent expansion of f at p. For modular functions, the second condition allows us to define $\vartheta_{i\infty}(f)$ to be equal to $\vartheta_0(\hat{f})$.*

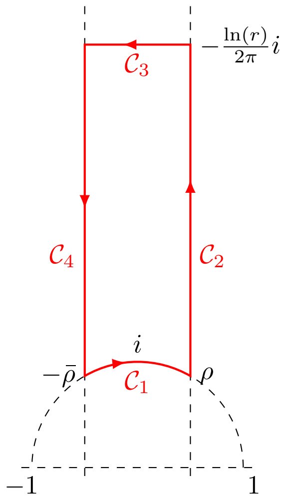
For example if f has a simple zero at p, then $\vartheta_p(f) = 1$. If f has a pole of order 3 at p then $\vartheta_p(f) = -3$. In particular, we showed in chapter 4 that $\vartheta_p(\Delta) = 0$ for all $p \in \mathcal{H}$.

Lemma 5.1. *Let f be a nonzero modular function of weight k, and let G be the modular group with fundamental domain $\mathcal{F}$. Let e_p denote the order of the subgroup of G that fixes the point p. Then*

$$\vartheta_{i\infty}(f) + \sum_{p \in \mathcal{F}} \frac{1}{e_p} \vartheta_p(f) = \frac{k}{12}.$$

Proof. We first need to make sure that the sum above is well defined. Since f is a modular function, the function $\hat{f}(q) = f\left(\frac{\ln(q)}{2\pi i}\right)$ is meromorphic around zero. Then, there exists an $r > 0$ such that $\hat{f}$ has no poles or zeroes for $0 < |q| \leq r$, because zeroes and poles are isolated. This translates to the fact that there are no poles or zeroes of $f(z)$ in the region $\text{Im}(z) \geq -\frac{\ln(r)}{2\pi}$. This means that there are finitely many poles and zeroes in the fundamental domain, because the region $\mathcal{F}* \cap \{z \mid \text{Im}(z) \leq -\frac{\ln(r)}{2\pi}\}$ is compact. Therefore the sum $\sum_{p \in \mathcal{F}} \frac{1}{e_p} \vartheta_p(f)$ is finite and well defined.

We will now calculate $\frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)} dz$ along the following contour:



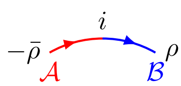
Because f has period 1, f' also has period 1. So immediately we see that the integrals of $\frac{f'}{f}$ over $\mathcal{C}_2$ and $\mathcal{C}_4$ cancel each other.

Looking at $\mathcal{C}_3$ we notice that

$$\frac{1}{2\pi i}\int_{\mathcal{C}_3}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\mathcal{C}_3}\frac{\hat{f}'(e^{2\pi iz})(2\pi i e^{2\pi iz})}{\hat{f}(e^{2\pi iz})}dz = \frac{1}{2\pi i}\int_{\mathcal{C}_3'}\frac{f'(q)}{f(q)}dq, \qquad q = e^{2\pi iz},$$

where $\mathcal{C}_3'$ is the clockwise circle with radius r and center 0, parameterized by $re^{2\pi xi}$ from $x = \frac{1}{2}$ to $x = -\frac{1}{2}$. So $\frac{1}{2\pi i}\int_{\mathcal{C}_3}\frac{f'(z)}{f(z)}dz$ is equal to the negative residue of $\frac{\hat{f}'}{\hat{f}}$ at zero, because there are no other zeroes or poles of f within a radius of r. By the argument principle this is equal to $-\vartheta_0(\hat{f}) = -\vartheta_{i\infty}(f)$.

To tackle $\mathcal{C}_1$, we will split it into $\mathcal{A}$ and $\mathcal{B}$.



Using the fact that $f(-\frac{1}{z}) = z^k f(z)$, and that $-\frac{1}{z}$ reflects the path $\mathcal{B}$ to the path $-\mathcal{A}$, we get:

$$\frac{f'\left(-\frac{1}{z}\right)\frac{1}{z^2}}{f\left(-\frac{1}{z}\right)} = \frac{k}{z} + \frac{f'(z)}{f(z)}, \tag{5.1}$$

$$\begin{aligned}
\implies \frac{1}{2\pi i}\int_{\mathcal{B}}\frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_{\mathcal{B}}\left(\frac{f'\left(-\frac{1}{z}\right)}{f\left(-\frac{1}{z}\right)}-kz\right)\frac{1}{z^2}dz \\
&= \frac{1}{2\pi i}\int_{-\mathcal{A}}\left(\frac{f'\left(\tau\right)}{f\left(\tau\right)}+k\frac{1}{\tau}\right)d\tau \qquad \left(\tau=-\frac{1}{z}\right) \\
&= -\frac{1}{2\pi i}\int_{\mathcal{A}}\frac{f'\left(\tau\right)}{f\left(\tau\right)}d\tau+\frac{k}{2\pi i}\int_{-\mathcal{A}}\frac{1}{\tau}d\tau \\
&= -\frac{1}{2\pi i}\int_{\mathcal{A}}\frac{f'\left(\tau\right)}{f\left(\tau\right)}d\tau+\frac{k}{12}, \\
\implies \frac{1}{2\pi i}\int_{\mathcal{C}_1}\frac{f'(z)}{f(z)}dz &= \frac{k}{12}.
\end{aligned}$$

So we conclude that

$$\frac{1}{2\pi i}\oint\frac{f'(z)}{f(z)}dz = -\vartheta_{i\infty}(f)+\frac{k}{12}. \tag{5.2}$$

This answer is not entirely accurate though, because here we actually assumed that there were no zeroes or poles of f on the contour. For $\mathcal{C}_3$ this is no problem, because we chose r to exclude this case. But for the other contours, we don't have that guarantee. Luckily, we can use the technique we used before where for each zero or pole on the contour, we can take a small clockwise circular arc around it, and let the radius go to zero (see lemma 4.4). For each pole/zero p, we add the residue $\vartheta_p(f)$ multiplied by the counterclockwise angle of the contour at p divided by 2π to our value in 5.2.

For each pole/zero p on $\mathcal{C}_2$, excluding ρ, the counterclockwise angle we get is $-\pi$, so we add $-\frac{1}{2}\vartheta_p(f)$ to our sum. But for each $p\in\mathcal{C}_2$, $p-1\in\mathcal{C}_4$ corresponds to the same point, so we add the residues of $\mathcal{C}_4$: $-\frac{1}{2}\vartheta_{p-1}(f) = -\frac{1}{2}\vartheta_p(f)$ as well.

$$\begin{aligned}
\frac{1}{2\pi i}\oint\frac{f'(z)}{f(z)}dz &= -\vartheta_{i\infty}(f)+\frac{k}{12}-\frac{1}{2}\sum_{p\in\mathcal{C}_2\setminus\{\rho\}}\left(\vartheta_p(f)+\vartheta_{p-1}(f)\right) \\
&= -\vartheta_{i\infty}(f)+\frac{k}{12}-\sum_{p\in\mathcal{C}_2\setminus\{\rho\}}\vartheta_p(f).
\end{aligned}$$

For each pole/zero $p\in\mathcal{A}$, p is a pole of $\frac{f'}{f}$ but not a pole of $\frac{k}{z}$ or $\frac{1}{z^2}$. Therefore by equation 5.1, $-\frac{1}{p}\in\mathcal{B}$ is a pole/zero of f with the same order. $-\frac{1}{p}$ is distinct from p, except for when $p=i$, so we have to make sure to not double count i. Except for ρ and $-\bar{\rho}$, the counterclockwise angle of the contour at $p\in\mathcal{C}_1$ approaches $-\pi$ as the radius of the circular arc goes to zero. For ρ and $-\bar{\rho}$, this angle can be seen geometrically to be equal to $\frac{\pi}{3}$. Let us use the notation $\tilde{\mathcal{A}},\tilde{\mathcal{B}}$ to denote the contours $\mathcal{A},\mathcal{B}$ without the endpoints. Thus we get the final equation:

$$\begin{aligned}\frac{1}{2\pi i}\oint\frac{f'(z)}{f(z)}dz &= -\vartheta_{i\infty}(f)+\frac{k}{12}-\sum_{p\in\mathcal{C}_2\setminus\{\rho\}}\vartheta_p(f)-\frac{1}{2}\sum_{p\in\tilde{\mathcal{A}}}\vartheta_p(f)-\frac{1}{2}\sum_{p\in\tilde{\mathcal{B}}}\vartheta_p(f)\\ &\qquad -\frac{1}{2}\vartheta_i(f)-\frac{1}{6}\vartheta_\rho(f)-\frac{1}{6}\vartheta_{-\bar{\rho}}(f)\\ &= -\vartheta_{i\infty}(f)+\frac{k}{12}-\sum_{p\in\mathcal{C}_2\setminus\{\rho\}}\vartheta_p(f)-\sum_{p\in\tilde{\mathcal{A}}}\vartheta_p(f)-\frac{1}{2}\vartheta_i(f)-\frac{1}{3}\vartheta_\rho(f).\end{aligned}$$

If $\tilde{\mathcal{F}}$ is the region inside the contour, we know by the argument principle that the closed contour integral is equal to the sum of the residues in $\tilde{\mathcal{F}}$, namely $\sum_{p\in\tilde{\mathcal{F}}}\vartheta_p(f)$. Recalling lemma 2.4, the only points in the fundamental domain with nontrivial stabilizer group are i and $\rho=\frac{1}{2}+\frac{\sqrt{3}}{2}$, with $e_i=2$ and $e_\rho=3$. Therefore we can group all the residues in one sum:

$$\vartheta_{i\infty}(f)+\sum_{p\in\mathcal{F}}\frac{1}{e_p}\vartheta_p(f)=\vartheta_{i\infty}(f)+\sum_{p\in\tilde{\mathcal{F}}\cup\mathcal{A}\cup\mathcal{C}_2}\frac{1}{e_p}\vartheta_p(f)=\frac{k}{12}.$$

□

After working through this difficult theorem, we will now go through all the amazing results that can be derived from it, including theorem 5.1. The first lemmas will aim to prove theorem 5.1 for special cases, which will help us form the base case of an induction argument.

We first make the trivial observation that modular forms are modular functions with no poles, even at infinity:

Lemma 5.2. *If $f\in\mathbb{M}_k$, then $\vartheta_p(f)\geq 0$ for all $p\in\mathcal{H}$ and $p=i\infty$.*

Corollary 5.1. *There are no nonzero modular forms of negative weight.*

Proof. By lemma 5.1 and 5.2 we would need to have that

$$0\leq\vartheta_{i\infty}(f)+\sum_{p\in\mathcal{F}}\frac{1}{e_p}\vartheta_p(f)=\frac{k}{12}<0,$$

which is a contradiction. □

Theorem 5.2. *There are no nonzero cusp forms of weight less than 12. In other words, $\dim_{\mathbb{C}}\mathbb{S}_k=0$ for $k<12$.*

Proof. Let f be a nonzero cusp form of weight 0. Then $\vartheta_{i\infty}\geq 1$. So by lemma 5.2 and 5.1:

$$1\leq\vartheta_{i\infty}(f)+\sum_{p\in\mathcal{F}}\frac{1}{e_p}\vartheta_p(f)=\frac{k}{12}.$$

Therefore $k\geq 12$. □

Corollary 5.2. *There are no nonconstant modular forms of weight 0.*

Proof. Let a be the constant term in the Fourier expansion of f. Because constants are modular forms of weight 0 and $\mathbb{M}_0$ is a vector space, $f(z) - a$ is a nonzero cusp form of weight 0, of which there are none. So $f(z) = a$. $\square$

Lemma 5.3. *Assume that* $\dim_{\mathbb{C}} \mathbb{S}_k < \infty$ *and that* $k \geq 4$ *is even, then*

$$\dim_{\mathbb{C}} \mathbb{M}_k = \dim_{\mathbb{C}} \mathbb{S}_k + 1.$$

Proof. We use the same idea as before, except we use the Eisenstein series instead of a constant. Let $f \in \mathbb{M}_k$ be a modular form and a be the constant term in the Fourier series of f. Because the constant term of E_k is 1, and $\mathbb{M}_k$ is a vector space, $f(z) - aE_k(z)$ is a cusp form of weight k. So every $f \in \mathbb{M}_k$ can be written as a linear combination of something in $\mathbb{S}_k$ and E_k. Because $\mathbb{S}_k$ is a subspace of $\mathbb{M}_k$ and $E_k \notin \mathbb{S}_k$, E_k is linearly independent to a basis of $\mathbb{S}_k$. Thus we conclude the statement. $\square$

Corollary 5.3. *The dimension of* $\mathbb{M}_k$ *over* $\mathbb{C}$ *is 1 for* $k = 4, 6, 8, 10$.

Proof. We combine theorem 5.2 with our previous lemma. $\square$

Lemma 5.4. *The dimension of* $\mathbb{M}_2$ *over* $\mathbb{C}$ *is zero.*

Proof. Observing the equality

$$\vartheta_{i\infty}(f) + \frac{1}{2}\vartheta_i(f) + \frac{1}{3}\vartheta_\rho(f) + \sum_{p \in \mathcal{F} \setminus \{\rho, i\}} \vartheta_p(f) = \frac{1}{6}, \tag{5.3}$$

we notice that none of the ϑ's can be greater or equal to 1, otherwise the sum on the left would exceed $\frac{1}{6}$. So all of the ϑ's must be equal to zero, which also doesn't work [1]. $\square$

If we include our lemma 2.2 from chapter 2, we have now shown 5.1 for all integers $k < 12$. To tackle $k = 12$, we notice that we need another observation:

Lemma 5.5. *The space of modular functions form a graded division ring. In particular, reciprocals of modular functions of weight* k *are modular functions of weight* $-k$.

Proof. As modular functions form a graded ring [2], we only need to show that reciprocals of modular functions are modular functions. We take the reciprocal of the functional

[1] This comes from the ingenious observation that $0 \neq \frac{1}{6}$.

[2] We can easily adjust lemma 2.5 to suite modular functions.

equation in definition 5.1, and see that $\frac{1}{f}$ satisfies it for weight $-k$:

$$\frac{1}{f\left(\dfrac{az+b}{cz+d}\right)} = (cz+d)^{-k}\frac{1}{f(z)}$$

Now $\frac{1}{f}$ is meromorphic as well, even at $i\infty$, because $(\hat{\frac{1}{f}}) = \frac{1}{\hat{f}}$. Therefore $\frac{1}{f}$ is a modular function of weight k. □

In chapter 4 we showed that the modular discriminant Δ doesn't vanish for all points except for $i\infty$. In other words $\vartheta_p(\Delta) = 0$ for all $\rho \in \mathcal{F}$. To show that $\vartheta_{i\infty}(\Delta) = 1$, we simply state the fact that $\tau(1) = 1$ and that Δ is a cusp form. We will use this to prove part 4 of theorem 4.1.

Theorem 5.3. *The function $\mathcal{T} : \mathbb{M}_k \to \mathbb{S}_{k+12}$ defined by $f \mapsto f \cdot \Delta$ is a linear bijection.*

Proof. Since $\lim_{\mathrm{Im}(z)\to\infty} \mathcal{T}(f)(z) = \hat{f}(0)\hat{\Delta}(0) = 0$, it is clear that $\mathcal{T}(f) \in \mathbb{S}_{k+12}$ by lemma 2.5. So $\mathcal{T}$ is well defined. Since $(af + bg)\Delta = a(f\Delta) + b(g\Delta)$, $\mathcal{T}$ is a linear map.

We now want to show that the inverse function $\mathcal{T}^{-1}$ defined by $f(z) \mapsto \frac{f(z)}{\Delta(z)}$ is a valid function from $\mathbb{S}_{k+12} \to \mathbb{M}_k$. Let $f \in \mathbb{S}_{k+12}$ be nonzero, then $\vartheta_{i\infty}(f) \geq 1$. Because the modular functions form a graded division ring, $\frac{f(z)}{\Delta(z)}$ is a modular function of weight $k + 12 - 12 = k$. For $p \in \mathcal{F}$, we observe the following:

$$\begin{aligned}
\vartheta_p\left(\frac{f(z)}{\Delta(z)}\right) &= \vartheta_p(f) - \vartheta_p(\Delta) = \vartheta_p(f) \geq 0, \\
\vartheta_{i\infty}\left(\frac{f(z)}{\Delta(z)}\right) &= \vartheta_{i\infty}(f) - \vartheta_{i\infty}(\Delta) = \vartheta_{i\infty}(f) - 1 \geq 0.
\end{aligned}$$

So $\frac{f(z)}{\Delta(z)}$ is holomorphic, and is therefore a modular form of weight k. □

Part 3 of theorem 4.1 follows as a corollary.

Corollary 5.4. *Δ spans the one dimensional space $\mathbb{S}_{12}$.*

Proof. Let $f \in \mathbb{S}_{12}$ be nonzero. Then $\mathcal{T}^{-1}(f) = \frac{f}{\Delta}$ is an element of $\mathbb{M}_0$. By corollary 5.2, this must be a constant, implying the desired statement. □

We conclude this chapter by proving our very important theorem 5.1.

Proof. We want to show that

$$\dim_{\mathbb{C}} \mathbb{M}_k = \begin{cases} 0, & k \text{ odd or negative,} \\ \lfloor \frac{k}{12} \rfloor, & k \equiv 2 \mod 12, \\ \lfloor \frac{k}{12} \rfloor + 1, & k \not\equiv 2 \mod 12. \end{cases}$$

Corollary 5.1 and lemma 2.2 from chapter 2, proves the theorem for the first case. In addition:

1. Corollary 5.2 proves the case $k = 0$.
2. Lemma 5.4 proves the case $k = 2$.
3. Corollary 5.3 proves the case $k = 4, 6, 8, 10$.

Theorem 5.3 implies that $\dim_{\mathbb{C}} \mathbb{M}_k = \dim_{\mathbb{C}} \mathbb{S}_{k+12}$. Combining this with lemma 5.3, we see that

$$\dim_{\mathbb{C}} \mathbb{M}_k = \dim_{\mathbb{C}} \mathbb{S}_{k+12} = \dim_{\mathbb{C}} \mathbb{M}_{k+12} - 1.$$

Because we have covered all cases $k < 12$, it is not hard to see that what remains is a simple induction argument. $\square$

CHAPTER 6

WHAT THE HECKE IS A HECKE OPERATOR?

Now that we have proved the powerful theorem that modular forms are finite dimensional, we are now ready to tackle proving the multiplicative property of the τ function: $\tau(ab) = \tau(a)\tau(b)$ for coprime a, b.

The first step on this endeavor is to introduce the notion of a Hecke operator. But before we look at the definition, we will try to understand the motivation and idea behind the Hecke operator. We start with a seemingly innocent question:

We know that the functional equation

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

is valid only for matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *in* $\boldsymbol{SL}_2(\mathbb{Z})$*, but what happens when we look at any integer matrix instead? Can we get another modular form from this?*

At first it seems like there is no reason for why this would work, but then we notice something interesting. If $\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right)$ is an integer matrix with determinant m and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbb{Z})$, then both $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are integer matrices with determinant m. Let M_m denote the 2 by 2 integer matrices with determinant m. Because $\mathbf{SL}_2(\mathbb{Z})$ is a group under multiplication, our previous observation is equivalent to saying that $\mathbf{SL}_2(\mathbb{Z})$ has a left and right group action on the set M_m. This gives us an idea. What if we sum over all matrices in M_m? Then we can create a new function $\tilde{f}$, that is also invariant under the transformation $\tilde{f}(z) \mapsto (cz+d)^{-k}\tilde{f}\left(\dfrac{az+b}{cz+d}\right)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbb{Z})$. We will use the same notation $\phi_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(z) = \dfrac{az+b}{cz+d}$ as in the proof of lemma 2.3, and recall that ϕ

converts composition to matrix multiplication. Indeed if everything converges:

$$\begin{aligned}
\tilde{f}(z) = &\sum_{\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right)\in M_m} (uz+v)^{-k} f\left(\frac{sz+t}{uz+v}\right)\\
\mapsto \quad & (cz+d)^{-k}\tilde{f}\left(\frac{az+b}{cz+d}\right)\\
= &\sum_{\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right)\in M_m} (cz+d)^{-k}\left(u\frac{az+b}{cz+d}+v\right)^{-k} f\circ\phi_{\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right)}\circ\phi_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(z)\\
= &\sum_{\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right)\in M_m} ((ua+cv)z+ub+dv)^{-k} f\circ\phi_{\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(z)\\
= &\sum_{\left(\begin{smallmatrix} s' & t' \\ u' & v' \end{smallmatrix}\right)\in M_m} (u'z+v')^{-k} f\circ\phi_{\left(\begin{smallmatrix} s' & t' \\ u' & v' \end{smallmatrix}\right)}(z)\\
= &\tilde{f}(z).
\end{aligned}$$

Here we used the fact that $\left(\begin{smallmatrix} s' & t' \\ u' & v' \end{smallmatrix}\right) = \left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a bijection of M_m, and showed that this new function $\tilde{f}$ would be a modular form if it converged.

Well, it turns out that it doesn't converge. So do we give up? No. We notice that in our calculation, we never used the fact that f was a modular form. We also only used the right action of $\mathbf{SL}_2(\mathbb{Z})$ on the set M_m. So this motivates the next step. What happens to the summand $(uz+v)^{-k} f\left(\dfrac{sz+t}{uz+v}\right)$ when we consider a left action of $\mathbf{SL}_2(\mathbb{Z})$? If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\mathbf{SL}_2(\mathbb{Z})$ and $\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} s' & t' \\ u' & v' \end{smallmatrix}\right)$, we get

$$\begin{aligned}
&(uz+v)^{-k} f\left(\frac{sz+t}{uz+v}\right)\\
&= (uz+v)^{-k} f\circ\phi_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} s' & t' \\ u' & v' \end{smallmatrix}\right)}(z)\\
&= (uz+v)^{-k} f\circ\phi_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}\circ\phi_{\left(\begin{smallmatrix} s' & t' \\ u' & v' \end{smallmatrix}\right)}(z)\\
&= (uz+v)^{-k}\left(c\phi_{\left(\begin{smallmatrix} s' & t' \\ u' & v' \end{smallmatrix}\right)}(z)+d\right)^{k} f\circ\phi_{\left(\begin{smallmatrix} s' & t' \\ u' & v' \end{smallmatrix}\right)}(z) \qquad (f \text{ is a modular form})\\
&= ((cs'+du')z+ct'+dv')^{-k}\left(\frac{c(s'z+t')+d(u'z+v')}{u'z+v'}\right)^{k} f\circ\phi_{\left(\begin{smallmatrix} s' & t' \\ u' & v' \end{smallmatrix}\right)}(z)\\
&= (u'z+v')^{-k} f\left(\frac{s'z+t'}{u'z+v'}\right).
\end{aligned}$$

What this shows is that the summand $(uz+v)^{-k} f\left(\dfrac{sz+t}{uz+v}\right)$ is invariant under the left action of $\mathbf{SL}_2(\mathbb{Z})$. So in fact, when we were summing over all matrices in M_m, we were repeating the same value for each element in the same left orbit. This gives us the idea to sum only over each left orbit of M_m instead of all of M_m. Let $\mathbf{SL}_2(\mathbb{Z})\backslash M_m$ denote the orbits of M_m under the left group action of $\mathbf{SL}_2(\mathbb{Z})$.

Definition 6.1. *For a positive integer m and modular form f of weight k, we define the m'th Hecke operator T_m as*

$$T_m f(z) = m^{k-1} \sum_{\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \in \boldsymbol{SL}_2(\mathbb{Z})\backslash M_m} (uz+v)^{-k} f\left(\frac{sz+t}{uz+v}\right).$$

Note that we index the sum by choosing a single representative from each orbit. The reason why the sum stays well-defined despite this, is because the summand is the same for any two elements in the same orbit. We will show that there are only finitely many orbits of M_m over the left action of $\mathbf{SL}_2(\mathbb{Z})$, so this sum is in fact finite and converges without problems. We will also see that the constant term m^{k-1} is there to make it so that the Hecke operator of a modular form with integer Fourier coefficients, also has integer Fourier coefficients.

Our idea from before now works.

Theorem 6.1. *$T_m f$ is a modular form of weight k.*

Proof. It is first clear that $T_m f(z)$ is holomorphic in the upper half plane if the sum is finite, because all potential new poles would have to lie on the real axis. We will postpone showing the fact that $T_m f(z)$ is holomorphic at infinity, to when we show that $T_m f(z)$ has an explicit Fourier series. We show that it satisfies the functional equation:

$$\begin{aligned}
&(cz+d)^{-k} T_m f\left(\frac{az+b}{cz+d}\right) \\
&= m^{k-1} \sum_{\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbb{Z})\backslash M_m} (cz+d)^{-k} \left(u\frac{az+b}{cz+d}+v\right)^{-k} f \circ \phi_{\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right)} \circ \phi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) \\
&= m^{k-1} \sum_{\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbb{Z})\backslash M_m} ((ua+cv)z+ub+dv)^{-k} f \circ \phi_{\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(z) \\
&= m^{k-1} \sum_{\left(\begin{smallmatrix} s' & t' \\ u' & v' \end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbb{Z})\backslash \in M_m} (u'z+v')^{-k} f \circ \phi_{\left(\begin{smallmatrix} s' & t' \\ u' & v' \end{smallmatrix}\right)}(z) \\
&= T_m(z).
\end{aligned}$$

The first two steps are exactly as before. The only step that we should squint our eyes at is the third step, when we change our index. To justify this step, we need to prove that the right action of $\mathbf{SL}_2(\mathbb{Z})$ permutes the left orbits $\mathbf{SL}_2(\mathbb{Z})\backslash M_m$. Let $[A]$ denote the left orbit of $\gamma \in M_m$. We observe that for any $A_1, A_2 \in M_m$ and $\gamma \in \mathbf{SL}_2(\mathbb{Z})$,

$$\begin{aligned} & [A_1] = [A_2] \\ \iff \quad & A_1 = \beta A_2, \quad \text{for a } \beta \in \mathbf{SL}_2(\mathbb{Z}), \\ \iff \quad & A_1\gamma = \beta A_2, \gamma \quad \text{for a } \beta \in \mathbf{SL}_2(\mathbb{Z}), \\ \iff \quad & [A_1\gamma] = [A_2\gamma]. \end{aligned}$$

Therefore $[A] \cdot \gamma = [A\gamma]$ is a well-defined right action on the left orbits $\mathbf{SL}_2(\mathbb{Z})\backslash M_m$. In fact it is a right group action:

$$[A] \cdot (\gamma_1\gamma_2) = [A(\gamma_1\gamma_2)] = [(A\gamma_1)\gamma_2] = [A\gamma_1] \cdot \gamma_2 = ([A] \cdot \gamma_1) \cdot \gamma_2.$$

Group actions permute their underlying set, so our change of index is justified. $\square$

So we have a family of Hecke operators $T_m : \mathbb{M}_k \to \mathbb{M}_k$. But from the definition, it is not clear how we can handle or calculate the Hecke operator. The next step will be to simplify the summation by finding and choosing simple representatives for each left orbit.

If you are familiar with the Smith normal form of a matrix, then you should see that our problem of finding a simple representative is very similar.

Definition 6.2. *For R a principle ideal domain and a matrix ring M with coefficients in R, a matrix $A \in M$ has a Smith normal form. I.e. there exist invertible matrices P and Q such that*

$$PAQ = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & a_n \end{pmatrix},$$

and $a_1 | a_2 | \cdots | a_n$.

In our case $R = \mathbb{Z}$ is a principle ideal domain, so we can apply it here. However the Smith normal form is not good enaugh. It only gives us a representative of the orbit under both left and right actions. We need a similar theorem where we only consider the left action. Namely we want to find a simple matrix form in the set $\{PA \mid P \in \mathbf{SL}_2(\mathbb{Z})\}$.

Luckily we can adapt the proof of the existence of the Smith normal form to get the following theorem:

Theorem 6.2. *Let R be a principle ideal domain and M a matrix ring with coefficients in R, For every $A \in M$, there exists an invertible $P \in M$ such that PA is upper triangular.*

Proof. We will only prove it for the case $R = \mathbb{Z}$ and M the set of 2 by 2 integer matrices, because that is all we need. We start by establishing which elementary row/column operations we have with only left multiplication.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} z & w \\ x & y \end{pmatrix} \qquad \text{(switch rows)},$$
$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x + az & y + aw \\ z & w \end{pmatrix} \qquad \text{(add rows to another)}.$$

We notice that we can execute the Euclidean algorithm on the first column by always placing the smallest number in the second row and subtracting the second number from the first an appropriate number of times. When we have completed the Euclidean algorithm, we end up with the greatest common divisor in the first column and second row.

$$E_1 \cdot E_2 \cdots E_l \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} r & * \\ \gcd(x, z) & * \end{pmatrix}.$$

Because the greatest common divisor of the first column is also invariant under these row operations, we must have that $\gcd(x, z) = \gcd(\gcd(x, z), r)$. This is only possible if $\gcd(x, z) \mid r \iff r = s \gcd(x, z)$. Applying the second row operation with $a = s$, we get a zero in the top left entry which we can place in the bottom left entry with a row switch. The resulting matrix is upper triangular. $\square$

Corollary 6.1. *The number of left orbits $|\boldsymbol{SL}_2(\mathbb{Z})\backslash M_m|$ is $\sigma(m)$ for $m > 0$. In fact,*

$$\mathcal{A} = \left\{ \begin{pmatrix} \alpha & \gamma \\ 0 & \frac{m}{\alpha} \end{pmatrix} : \alpha > 0,\, \alpha \mid m,\, 0 \le \gamma \le \frac{m}{\alpha} - 1 \right\}$$

is a complete list of left orbit representatives.

Proof. From the previous theorem, every left orbit contains an upper triangular element $\left(\begin{smallmatrix} \alpha & \gamma \\ 0 & \beta \end{smallmatrix}\right)$. Because this matrix is in M_m, it has determinant $\alpha\beta = m$. Multiplying by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ if necessary, we can assume that $\alpha > 0$. Observing that

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \alpha & \gamma + \beta y \\ 0 & \beta \end{pmatrix}, \tag{6.1}$$

we see that we can reduce γ modulo β. Therefore each orbit contains an element in the following set:

$$\mathcal{A} = \left\{ \begin{pmatrix} \alpha & \gamma \\ 0 & \frac{m}{\alpha} \end{pmatrix} : \alpha > 0,\, \alpha \mid m,\, 0 \le \gamma \le \frac{m}{\alpha} - 1 \right\}.$$

We can also show that these are in distinct orbits. Suppose that $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \left(\begin{smallmatrix} \alpha & \gamma \\ 0 & \frac{m}{\alpha} \end{smallmatrix}\right)$ is another element of $\mathcal{A}$, for some $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbb{Z})$. Then by matrix multiplication, we must have $\alpha z = 0 \implies z = 0$. Then as an upper triangular element of $\mathbf{SL}_2(\mathbb{Z})$, we must have determinant $xw = 1$ which implies that $x = w = \pm 1$. We can exclude the case $x = w = -1$, because the diagonal terms of the product must be positive. So we have reduced it to the same case as in equation 6.1, where it is clear that y must equal to zero. Therefore $\mathcal{A}$ is a complete list of left orbit representatives.

The cardinality of $\mathcal{A}$ is easy to count. For each positive $\alpha \mid m$, we have $\frac{m}{\alpha}$ choices for γ, so

$$|\mathcal{A}| = \sum_{\alpha|m} \frac{m}{\alpha} = \sum_{\alpha|m} \alpha = \sigma(m).$$

□

Corollary 6.2.

$$T_m f(z) = m^{k-1} \sum_{ad=m} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right). \tag{6.2}$$

Proof. Use the set of representatives $\mathcal{A}$ as index in the sum over the left orbits in definition 6.1. □

Now we will use this simplified formula to calculate what happens to the Fourier coefficients of a modular form when we apply the Hecke operator.

Theorem 6.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i z}$ be a modular form of weight k. Then*

$$T_m f(z) = \sum_{n=0}^{\infty} \sum_{d|\gcd(m,n)} a_{\frac{mn}{d^2}} d^{k-1} e^{2\pi i z n}.$$

Proof. Applying equation 6.2 directly:

$$\begin{aligned}
T_m f(z) &= m^{k-1} \sum_{ad=m} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right) \\
&= m^{k-1} \sum_{n=0}^{\infty} a_n \sum_{ad=m} d^{-k} e^{2\pi i a z \frac{n}{d}} \sum_{b=0}^{d-1} e^{2\pi i b \frac{n}{d}} \\
&= m^{k-1} \sum_{n=0}^{\infty} a_n \sum_{ad=m} d^{-k} e^{2\pi i a z \frac{n}{d}} \begin{cases} \dfrac{(e^{2\pi i \frac{n}{d}})^d - 1}{e^{2\pi i \frac{n}{d}} - 1} & \frac{n}{d} \notin \mathbb{Z} \\ \displaystyle\sum_{b=0}^{d-1} 1 & \frac{n}{d} \in \mathbb{Z} \end{cases} \\
&= m^{k-1} \sum_{n=0}^{\infty} a_n \sum_{ad=m} d^{-k} e^{2\pi i a z \frac{n}{d}} \begin{cases} 0 & \frac{n}{d} \notin \mathbb{Z} \\ d & \frac{n}{d} \in \mathbb{Z} \end{cases} \\
&= m^{k-1} \sum_{n=0}^{\infty} a_n \sum_{\substack{d|m \\ d|n}} d^{1-k} e^{2\pi i z \frac{m}{d} \frac{n}{d}}.
\end{aligned}$$

We will now do a change of variable. We set $\tilde{n} = \frac{mn}{d^2}$ and $\tilde{d} = \frac{m}{d}$. This is equivalent to $n = \frac{m\tilde{n}}{\tilde{d}^2}, d = \frac{m}{\tilde{d}}$. Since the inverse change of variable is exactly the same we see that we can easily switch d, n with $\tilde{d}, \tilde{n}$ in the following implications.

$$d \in \mathbb{N}, n \in \mathbb{Z}, d \mid m, d \mid n \implies \tilde{d} \in \mathbb{N}, \tilde{d} \mid m, \tilde{n} = \frac{m}{d}\frac{n}{d} \in \mathbb{Z} \text{ and } \frac{\tilde{n}}{\tilde{d}} = \frac{n}{d} \in \mathbb{Z} \implies \tilde{d} \mid \tilde{n}.$$

This means that that we have the following bijection.

$$d \in \mathbb{N}, n \in \mathbb{Z}, d \mid m, d \mid n \Longleftrightarrow \tilde{d} \in \mathbb{N}, \tilde{n} \in \mathbb{Z}, \tilde{d} \mid m, \tilde{d} \mid \tilde{n}.$$

Therefore we can apply the change of variable $\tilde{n} = \frac{mn}{d^2}$, $\tilde{d} = \frac{m}{d}$ with the conditions $\tilde{d} \mid m, \tilde{d} \mid \tilde{n}$ to our sum.

$$\begin{aligned}
T_m f(z) &= m^{k-1} \sum_{n=0}^{\infty} a_n \sum_{\substack{d \mid m \\ d \mid n}} d^{1-k} e^{2\pi i z \frac{m}{d}\frac{n}{d}} \\
&= m^{k-1} \sum_{\tilde{n}=0}^{\infty} \sum_{\substack{\tilde{d} \mid m \\ \tilde{d} \mid \tilde{n}}} a_{\frac{m\tilde{n}}{\tilde{d}^2}} \left(\frac{m}{\tilde{d}}\right)^{1-k} e^{2\pi i z \tilde{n}} \\
&= \sum_{\tilde{n}=0}^{\infty} \sum_{\tilde{d} \mid \gcd(m, \tilde{n})} a_{\frac{m\tilde{n}}{\tilde{d}^2}} \tilde{d}^{k-1} e^{2\pi i z \tilde{n}}.
\end{aligned}$$

□

We end this chapter with a couple remarks.

1. $T_m : \mathbb{M}_k \to \mathbb{M}_k$ is a linear transformation. Since $\mathbb{M}_k$ is finite dimensional, the Hecke operators have matrix representations in some basis.
2. T_m sends cusp forms to cusp forms. This is because the constant term in the Fourier series is
$$\widehat{T_m f}(0) = \sum_{d \mid \gcd(m,0)} a_0 d^{k-1} = a_0 \sigma_{k-1}(m).$$

CHAPTER 7

EIGENFORMS

Now that we have an explicit formula for calculating the Hecke operator, it is natural to try to calculate the Hecke operator on the modular forms that we know of: The Eisenstein series.

Theorem 7.1. *For even $k \geq 4$,*

$$T_m E_k = \sigma_{k-1}(m) E_k.$$

For the cases $k = 4, 6, 8, 10, 14$, this theorem follows from the fact that $\mathbb{M}_k$ is 1-dimensional in these cases, and that the constant term of $T_m f(z)$ is $a_0 \sigma_{k-1}(m)$. However the fact that it works for all other Eisenstein series is not obvious.

Proof. This proof was moved to the appendix (A.1), because it is really long and we will not use this result. □

This result is very peculiar, because it essentially says that the Eisenstein series E_k is an eigenvector for all the linear transformations $T_m : \mathbb{M}_k \to \mathbb{M}_k$. This observation motivates the definition of an eigenform.

Definition 7.1. *An eigenform f is a modular form of weight k that is an eigenvector for all Hecke operators $T_m : \mathbb{M}_k \to \mathbb{M}_k$. I.e. there exist eigenvalues $\lambda_m \in \mathbb{C}$ for each integer $m \geq 1$, such that*

$$T_m f(z) = \lambda_m f(z), \qquad \forall m \geq 1.$$

The reason we want to interpret it in this way, is because 1 dimensional vectorspaces force the existence of eigenvectors.

Theorem 7.2. *The modular discriminant Δ is an eigenform.*

Proof. Δ spans the 1 dimensional space of weight 12 cusp forms $\mathbb{S}_{12}$. From the final remark in chapter 6, we know that the Hecke operators T_m send cusp forms to cusp forms. Therefore we can consider them as a linear maps $T_m : \mathbb{S}_{12} \to \mathbb{S}_{12}$. Because $\mathbb{S}_{12} = \text{span}_{\mathbb{C}}\{\Delta\}$ is 1 dimensional and $T_m\Delta \in \mathbb{S}_{12}$, there must exist a $\lambda_m \in \mathbb{C}$ such that $T_m\Delta = \lambda_m\Delta$ for each Hecke operator T_m. □

What is even more exciting is that these eigenvalues are known values.

Theorem 7.3. *Let f be an eigenform and cusp form of weight k with Fourier series $f(z) = \sum_{n=1}^{\infty} \hat{f}(n)e^{2\pi inz}$. If $\hat{f}(1) = 1$, then the eigenvalues λ_m are the Fourier coefficients $\hat{f}(m)$:*

$$T_m f(z) = \hat{f}(m)f(z).$$

Proof. The proof is surprisingly simple. We only need to look at first Fourier coefficient of $T_m f(z)$.

$$\begin{aligned} T_m f(z) &= \lambda_m f(z) & &\implies \widehat{T_m f}(1) = \lambda_m \hat{f}(1) = \lambda_m. \\ \widehat{T_m f}(n) &= \sum_{d|\gcd(m,n)} \hat{f}\left(\frac{mn}{d^2}\right) d^{k-1} & &\implies \widehat{T_m f}(1) = \sum_{d|\gcd(m,1)} \hat{f}\left(\frac{m}{d^2}\right) d^{k-1} = \hat{f}(m). \end{aligned}$$

□

Corollary 7.1. *Let f be a cusp eigenform with first coefficient $\hat{f}(1) = 1$ as above. Then we have the following.*

$$\hat{f}(m)\hat{f}(n) = \sum_{d|\gcd(m,n)} \hat{f}\left(\frac{mn}{d^2}\right) d^{k-1}.$$

In particular $\hat{f}$ is multiplicative.

$$\hat{f}(m)\hat{f}(n) = \hat{f}(mn), \qquad \gcd(m,n) = 1.$$

Proof. Comparing coefficients in the equation $T_m f(z) = \hat{f}(m)f(z)$, we see that both of these terms are equal to $\widehat{T_m f}(n)$. If $\gcd(m,n) = 1$, we see that the right hand side is equal to

$$\sum_{d|1} \hat{f}\left(\frac{mn}{d^2}\right) d^{k-1} = \hat{f}\left(\frac{mn}{1^2}\right) 1^{k-1} = \hat{f}(mn).$$

□

We recall that we wanted to prove some special properties of the τ function.

$$\tau(a)\tau(b) = \sum_{d|\gcd(a,b)} d^{11}\tau\left(\frac{ab}{d^2}\right).$$

Especially that τ is multiplicative:

$$\tau(a)\tau(b) = \tau(ab), \qquad \gcd(a,b) = 1.$$

These conditions would follow immediately from corollary 7.1, if it happened to be that $\sum_{n=1}^{\infty} \tau(n)e^{2\pi i n z}$ was an eigenform and cusp form of weight 12 with first coefficient $\tau(1) = 1$. But it couldn't be that convenient, could it?

Theorem 7.4. *$\sum_{n=1}^{\infty} \tau(n)e^{2\pi i n z}$ is an eigenform and cusp form of weight 12 with first coefficient $\tau(1) = 1$.*

Proof. The τ function was defined thus.

$$\Delta(z) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n)e^{2\pi i n z}.$$

By construction, Δ is a cusp form of weight 12. Theorem 7.2 tells us that Δ is also an eigenform. As a scalar multiple, these facts translate easily to $(2\pi)^{-12}\Delta(z) = \sum_{n=1}^{\infty} \tau(n)e^{2\pi i n z}$. So all that remains is to show that $\tau(1) = 1$.

Let $A = \sum_{n=1}^{\infty} \sigma_3(n)q^n$ and $B = \sum_{n=1}^{\infty} \sigma_5(n)q^n$, then

$$E_4(z) = 1 + \frac{(2\pi i)^4}{3!\zeta(4)}A = 1 + \frac{2^4\pi^4 \cdot 90}{3!\pi^4}A = 1 + 240A, \qquad \zeta(4) = \frac{\pi^4}{90},$$
$$E_6(z) = 1 + \frac{(2\pi i)^6}{5!\zeta(6)}B = 1 - \frac{2^6\pi^6 \cdot 945}{5!\pi^6}B = 1 - 504B, \qquad \zeta(6) = \frac{\pi^6}{945}.$$

Because $A = \mathcal{O}(q)$ and $B = \mathcal{O}(q)$, we see that

$$\begin{aligned} E_4(z)^3 - E_6(z)^2 &= (1 + 3\cdot 240A + 3^2 240^2 A^2 + 240^3 A^3) \\ &\quad - (1 - 2\cdot 504B + 504^2 B^2) \\ &= 3\cdot 240q + 2\cdot 504q + \mathcal{O}(q^2) \\ &= 1728q + \mathcal{O}(q^2). \end{aligned}$$

Therefore the first coefficient of $(2\pi)^{-12}\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728}$ is 1 as desired. The proof of $\tau(1) = 1$ was inspired by a proof in Apostol [4, p. 20]. $\square$

CHAPTER 8

CONCLUSION

That was it. We managed to prove that the Ramanujan tau function is multiplicative. Are we done?

With reading this bachelor: yes.

With learning about modular forms: no.

There is a lot of interesting theory about modular forms that I didn't cover in this bachelor. Firstly, I only covered modular forms of level 1: I didn't consider modular forms where the modularity condition is true for a subgroup of $\mathbf{SL}_2(\mathbb{Z})$. Modular forms defined using a subgroup $\Gamma \subset \mathbf{SL}_2(\mathbb{Z})$ that contains the principle congruence subgroup of level N,

$$\Gamma_N = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathbf{SL}_2(\mathbb{Z}) \mid a, d \equiv 1 \mod N,\, b, c \equiv 0 \mod N \right\},$$

are called modular forms of level N.

To study these, you would need to understand the theory of Riemann surfaces. This is because these modular forms turn out to be differential forms defined on the Riemann surface $Y(\Gamma) = \Gamma \backslash \mathcal{H}$. These quotient spaces $\Gamma \backslash \mathcal{H}$ can be made compact in the topological sense, by adding points at the cusps, just like we added a point at $i\infty$ for $\Gamma = \mathbf{SL}_2(\mathbb{Z})$ and our fundamental domain $\mathcal{F} = \Gamma \backslash \mathcal{H}$. If we denote the compactification of $Y(\Gamma)$ by $X(\Gamma)$, we see that modular forms are meromorphic functions on a compact Riemann surface $X(\Gamma)$. This means that we can apply the Riemann Roch theorem, which connects the dimension of the vector space of meromorphic functions on a compact Riemann surface X with specific zeroes and poles, to the genus of X. Essentially we can use it to compute the (finite) dimension of the modular forms of level N just like in lemma 5.1.

If you want to learn more about the geometrical and topological side of modular forms, I would recommend reading the notes written by Milne [6]. He gives a good overview of the geometrical understanding of modular forms. These notes do skim over a lot of the details and is fairly advanced, so perhaps I will read them again later once I am more familiar with topology, manifolds, and algebraic geometry.

I also didn't talk about the j-invariant - the non-zero modular function of weight 0. It is the j-invariant whose coefficients are close to the dimensions of the irreducible representations of the monster group [7]. The j-invariant and the Weierstrass elliptic function together help form the connection between elliptic curves over $\mathbb{C}$ and complex torii [6, p. 46, 47] [4, p. 42]. It also has interesting results with class field theory, creating the theory of complex multiplication [6, p. 121]. To begin understanding the j-invariant, I would recommend reading chapter 2 in Apostol's book [4] as it goes into more detail.

I think that now you will also be able to watch and understand a lot of Brocherds' youtube series [1]. I am proud to say that I certainly did! A lot of things that I had no clue of before now make sense.

I hope that you enjoyed learning about modular forms!

APPENDIX A

APPENDIX

A.1 Proof that the Eisenstein series are eigenforms

Proof. From theorem 6.3, we know that for $n > 0$,

$$\widehat{T_m E_k}(n) = \sum_{d|\gcd(m,n)} a_{\frac{mn}{d^2}} d^{k-1} = \sum_{d|\gcd(m,n)} \sigma_{k-1}\left(\frac{mn}{d^2}\right) d^{k-1} = \sum_{d|\gcd(m,n)} \sum_{e|\frac{mn}{d^2}} d^{k-1}e^{k-1}.$$

We want to show that this sum is just equal to $\sigma_{k-1}(m)\sigma_{k-1}(n)$. To do this we will use the power of multiplicativity. The idea is simple yet elegant. We see that the result we want is multiplicative in both m, n: $\sigma_{k-1}(m)\sigma_{k-1}(n)$, so we will try to prove something similar for $\widehat{T_m E_k}(n)$, and see if we can reduce the problem to prime powers.

Assume that $m = m_1 m_2$ where $\gcd(m_1, m_2) = 1$. We have the following facts.

1. $\gcd(m, n) = \gcd(m_1, n)\gcd(m_2, n), \quad \forall n \in \mathbb{Z}$.
2. $\gcd(\gcd(m_1, n), \gcd(m_2, n)) = 1, \quad \forall n \in \mathbb{Z}$.
3. $\sum_{d|m} f(d) = \sum_{\substack{d_1|m_1\\ d_2|m_2}} f(d_1 d_2) = \sum_{d_1|m_1} \sum_{d_2|m_2} f(d_1 d_2)$ for any function f.

For each prime p let $s_p, t_p \geq 0$ be integers such that $m = p^{s_p} m'$ and $n = p^{t_p} n'$ and m', n' are coprime to p. Then $\gcd(m, n) = \gcd(p^{s_p} m', p^{t_p} n') = \gcd(p^{s_p}, p^{t_p} n')\gcd(m', p^{t_p} n') =$

$\gcd(p^{s_p}, p^{t_p}) \gcd(m', n')$. We also see that we can factor out this prime from the sum.

$$
\begin{aligned}
\widehat{T_m E_k}(n) &= \sum_{d \mid \gcd(m,n)} \sum_{e \mid \frac{mn}{d^2}} d^{k-1} e^{k-1} = \sum_{d \mid \gcd(p^{s_p}, p^{t_p}) \gcd(m',n')} \sum_{e \mid \frac{mn}{d^2}} d^{k-1} e^{k-1} \\
&= \sum_{d_1 \mid p^{\min(s_p,t_p)}} \sum_{d_2 \mid \gcd(m',n')} \sum_{e \mid \frac{p^{s_p+t_p}}{d_1^2} \frac{m'n'}{d_2^2}} d_1^{k-1} d_2^{k-1} e^{k-1} \\
&= \sum_{i=0}^{\min(s_p,t_p)} \sum_{d \mid \gcd(m',n')} \sum_{e \mid \frac{m'n'}{d^2} p^{s_p+t_p-2i}} p^{i(k-1)} d^{k-1} e^{k-1} \\
&= \sum_{i=0}^{\min(s_p,t_p)} \sum_{d \mid \gcd(m',n')} \sum_{e_1 \mid \frac{m'n'}{d^2}} \sum_{e_2 \mid p^{s_p+t_p-2i}} p^{i(k-1)} d^{k-1} e_1^{k-1} e_2^{k-1} \\
&= \left(\sum_{i=0}^{\min(s_p,t_p)} \sum_{e_2 \mid p^{s_p+t_p-2i}} p^{i(k-1)} e_2^{k-1} \right) \left(\sum_{d \mid \gcd(m',n')} \sum_{e_1 \mid \frac{m'n'}{d^2}} d^{k-1} e_1^{k-1} \right) \\
&= \left(\sum_{i=0}^{\min(s_p,t_p)} \sum_{j=0}^{s_p+t_p-2i} p^{(i+j)(k-1)} \right) \widehat{T_{m'} f}(n').
\end{aligned}
$$

It easily follows from inducting over the primes that:

$$
\widehat{T_m E_k}(n) = \prod_p \left(\sum_{i=0}^{\min(s_p,t_p)} \sum_{j=0}^{s_p+t_p-2i} p^{(i+j)(k-1)} \right).
$$

We observe that if both s_p and t_p are zero, then the left sum is equal to $p^0 = 1$, so this is a finite product.

Now let us look at the sum $\sum_{i=0}^{\min(s,t)} \sum_{j=0}^{s+t-2i} p^{(i+j)(k-1)}$. Without loss of generality we can assume that $s \le t$. So $\min(s,t) = s$.

$$\begin{aligned}
\sum_{i=0}^{s} \sum_{j=0}^{s+t-2i} p^{(i+j)(k-1)} &= \sum_{i=0}^{s} \sum_{u=i}^{s+t-i} p^{u(k-1)} && (u = i + j) \\
&= \sum_{i=0}^{s} \sum_{u=i}^{s} p^{u(k-1)} + \sum_{i=0}^{s} \sum_{u=s}^{s+t-i} p^{u(k-1)} \\
&= \sum_{i=0}^{s} \sum_{u=i}^{s} p^{u(k-1)} + \sum_{j=0}^{s} \sum_{u=s}^{j+t} p^{u(k-1)} && (j = s - i) \\
&= \sum_{i=0}^{s} \sum_{u=i}^{i+t} p^{u(k-1)} \\
&= \sum_{i=0}^{s} \sum_{v=0}^{t} p^{(i+v)(k-1)} && (v = u - i) \\
&= \sigma_{k-1}(p^s)\sigma_{k-1}(p^t).
\end{aligned}$$

Therefore it follows that

$$\widehat{T_m E_k}(n) = \prod_p \sigma_{k-1}(p^{s_p})\sigma_{k-1}(p^{t_p}) = \prod_p \sigma_{k-1}(p^{s_p}) \prod_p \sigma_{k-1}(p^{t_p}) = \sigma_{k-1}(m)\sigma_{k-1}(n).$$

$\square$

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