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NTNU
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An introduction to classical tilting theory

Bachelor's thesis in BMAT
Supervisor: Mads Hustad Sandøy
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Kunnskap for en bedre verden

Tilting modules

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0.1 Abstract

This thesis will introduce torsion pairs. After this we will introduce tilting modules, which induces a torsion class, where the tilting module is a Morita progenerator. This yields an equivalence between a torsion pair in mod A and mod B, where B is the endomorphism algebra of a tilting module. The first part of the thesis will be show this equivalence. These tilting modules also gives an equivalence between the derived categories $D^b(A)$ and $D^b(B)$. The second part will be about this equivalence.

0.2 Sammendrag

Denne oppgaven vil introdusere torsjonspar. Etter dette vil vi introdusere vippemoduler, som er moduler induserer en torsjonsklasse, der vippemodulen er en Morita-progenerator. Dette gir en ekvivalens mellom et torsjonspar i mod A og mod B, der B er endomorfialgebraen til en vippemodul. Første del av oppgaven vil vise denne ekvivalensen. Disse vippemodulene gir også en ekvivalens mellom de deriverte kategoriene $D^b(A)$ og $D^b(B)$. Den andre delen vil handle om denne ekvivalensen.

Contents

Chapter 1

Preface

1.1 Introduction

This thesis will be an introduction to classical tilting theory, a very useful tool in describing the category of finitely generated right modules of an K-algebra. We will mainly be focusing on defining and the result about classical tilting modules.

Firstly we will introduce torsion pairs, a pair of subcategories $(\mathcal{T}, \mathcal{F})$ of mod A. These are the largest subcategories such that $\text{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}$ and $N \in \mathcal{F}$.

Tilting modules are modules that induces a torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$, where T is a tilting module. Tilting modules can be thought of as a generalization of a Morita progenerator. This is since T is a Morita progenerator in $\mathcal{T}(T)$. This hints towards there being an equivalence of torsion pairs in mod A and mod B. The result giving the equivalences of these subcategories is called the Brenner-Butler-theorem or the tilting theorem. The equivalences between the torsion pairs comes from the functors $\text{Hom}_A(T, -)$ and $- \otimes_B T$ and $\text{Ext}^1_A(T, -)$ and $\text{Tor}^1_B(-, T)$. This theorem is exactly what we will prove during the chapter about tilting modules. Moreover these tilting modules also induce an equivalence between the bounded derived categories $D^b(A)$ and $D^b(B)$. This will be the main focus of last chapter. The presentation of this thesis is based on Elements of the Representation Theory of Associative Algebras Volume 1 Techniques of Representation Theory[1] and Triangulated Categories in the Representation Theory of Finite Dimensional Algebras[2].

1.2 Preliminaries

Throughout this thesis it will be assumed the reader has some knowledge of homological algebra and representation theory. Most of the thesis will be formulated using the language of homological algebra. Especially the reader will be expected to be comfortable with Hom-functor and the tensor product. As well as their derived functors, Ext and Tor. These functor will be use frequently throughout this thesis. The fact that that Hom and $\otimes$ induces a long exact sequence in Ext and Tor respectively. Additionally the fact that $\mathrm{Ext}_A^1(M, N)$ has a one to one correspondence with equivalence classes of extensions of N by N. Here the 0 element in $\mathrm{Ext}_A^1(M, N)$ corresponds to the trivial extension.

It will also be assumed that the reader is familiar with representation theory of finite dimensional algebras. What we will mainly be needing is knowledge of $\mathrm{mod}\, A$, the category of finitely generated modules. In addition the thesis uses a bit of Auslander-Reiten-theory. The main results from Auslander-Reiten-theory that are used is the Auslander-Reiten-translation, defined as $\tau = DTr$, where D is $\mathrm{Hom}_A(-, K)$ and Tr is the transpose of a module. The Auslander-Reiten-formulas will also be used in some proofs, stating that

$$\mathrm{Ext}_A^1(N, M) \cong D\overline{\mathrm{Hom}}_A(M, \tau N) \cong D\underline{\mathrm{Hom}}_A(\tau^- 1M, N).$$

The Auslander-Reiten-quiver will also occur in examples. If the reader wants they can read more about these in Assem, Simson, Skowroński[1].

Chapter 2

Torsion

In this chapter we will introduce what is called a torsion pair. A pair of subcategories of mod A. These subcategories will end up being fruitful when trying to describe mod A and mod B, where B is the endomorphism algebra of a module. What this module is exactly we will discuss further in the next chapter. Although we hope to motivate its construction a bit in this chapter. The presentation of this chapter is based on Assem, Simson, Skowroński[1].

Definition 2.0.1. Let $\mathcal{T}$ and $\mathcal{F}$ be two full subcategories of mod A. We say that the pair ($\mathcal{T}$, $\mathcal{F}$) is a torsion pair if:

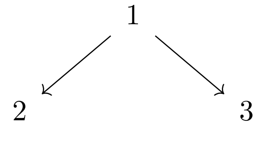
1. $\text{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}, N \in \mathcal{F}$.
2. $\text{Hom}_A(M, -)|_{\mathcal{F}} = 0 \implies M \in \mathcal{T}$.
3. $\text{Hom}_A(-, N)|_{\mathcal{T}} = 0 \implies N \in \mathcal{F}$.

$\mathcal{T}$ is then called a torsion class and $\mathcal{F}$ is called a torsion-free class.

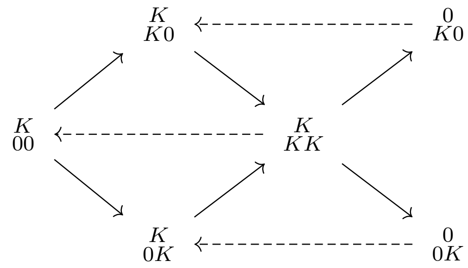
Example 2.0.2. In mod A we have the trivial torsion pairs $(\text{mod}\, A, 0)$ and $(0, \text{mod}\, A)$

Example 2.0.3. Let $\mathcal{C}$ be a class of A-modules. Then $\mathcal{C}$ induces a torsion pair ($\mathcal{T}$, $\mathcal{F}$), with $\mathcal{F} = \{N \mid \text{Hom}_A(-, N)|_{\mathcal{C}} = 0\}$ and $\mathcal{T} = \{M \mid \text{Hom}(M, -)|_{\mathcal{F}} = 0\}$.

Example 2.0.4. Let K be a field and A be the path algebra KQ of the quiver Q:



This gives rise to the AR-quiver of KQ.



Then $(\mathrm{Add}(\begin{smallmatrix}K\\KK\end{smallmatrix} \oplus \begin{smallmatrix}0\\K0\end{smallmatrix} \oplus \begin{smallmatrix}K\\0K\end{smallmatrix}), \mathrm{Add}(\begin{smallmatrix}K\\00\end{smallmatrix} \oplus \begin{smallmatrix}K\\K00\end{smallmatrix} \oplus \begin{smallmatrix}K\\0K\end{smallmatrix}))$ is a torsion pair.

We begin by establishing some basic properties of torsion classes and torsion-free classes. For this we will need to recall the notion of an idempotent radical. A idempotent radical is a functor t on mod A such that for every module M in mod A we have that $t(tM) = tM$ and $t(M/tM) = 0$.

Theorem 2.0.5.

(a) Let $\mathcal{T}$ be a full subcategory of mod A. *The following are equivalent:*

1. *$\mathcal{T}$ is a torsion class for some torsion pair $(\mathcal{T}, \mathcal{F})$.*
2. *$\mathcal{T}$ is closed under images, direct sums and extension.*
3. *There exists an idempotent radical such that $\mathcal{T} = \{M \mid tM = M\}$.*

b) Let $\mathcal{F}$ be a full subcategory of mod A. *The following are equivalent:*

1. *$\mathcal{F}$ is a torsion-free class in some torsion pair ($\mathcal{T}$, $\mathcal{F}$)*
2. *$\mathcal{F}$ is closed under submodules, direct products and extension*
3. *There exists an idempotent radical such that $\mathcal{F} = \{N \mid tN = 0\}$*

Proof.
(a) $(1. \implies 2.)$ $\mathcal{T}$ is closed under direct sums since the Hom-functor is additive. Thus we only need to prove it is closed under images and extensions. Let $L, M, N \in$ mod A such that there exists an short exact sequence.

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

Now let us take a $X \in \mathcal{F}$. Then there exists an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(N, X) \longrightarrow \mathrm{Hom}_A(M, X) \longrightarrow \mathrm{Hom}_A(L, X)$$

hence $M \in \mathcal{T} \iff N, L \in \mathcal{T}$. Thus $\mathcal{T}$ is closed under images and extensions. $(2. \implies 3.)$ Let tM denote the sum of images from all homomorphisms from modules in $\mathcal{T}$ to M. tM will be called the trace of M in $\mathcal{T}$. Since $\mathcal{T}$ is closed under images and direct sums, we get that tM is the largest submodule of M. We now take a brake to showed that the trace is an idempotent radical. Obviously we know $t(tM) = tM$. From the definition of the trace we know

$tM \subseteq M$, lets assume $t(M/tM) = M'/tM$, with $tM \subseteq M' \subseteq M$. Also from the definition of the trace we know $M'/tM \in \mathcal{T}$. Since $\mathcal{T}$ is closed under extensions we get, since $tM, M'/tM \in \mathcal{T}$ that $M' \in \mathcal{T}$. Thus $M' = tM$ and $t(M/tM) = 0$. We can then observe the clear equivalence $M \in \mathcal{T} \iff tM = M$. Thus there exists an radical idempotent such that $\mathcal{T} = \{M \mid tM = M\}$.
$(3. \implies 1.)$ Let $\mathcal{F} = \{N \mid tN = 0\}$, then we know $\mathrm{Hom}_A(M, -)|_{\mathcal{F}} = 0$ for all $M \in \mathcal{T}$. Next let us assume $\mathrm{Hom}_A(M, -)|_{\mathcal{F}} = 0$. Then $t(M/tM) = 0$ implies that $M/tM \in \mathcal{F}$. Thus we get that the cannonical projection $M \to M/tM$ is zero. This means $M/tM = 0$, thus $M = tM \in \mathcal{T}$. Similarly $\mathrm{Hom}_A(-, N)|_{\mathcal{F}} = 0$ implies $N \in \mathcal{F}$.
The proof of (b) is similar to (a). □

The previous theorem gives rise to the existence of an incredible useful type of short exact sequence.

Theorem 2.0.6. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in* $\mathrm{mod}\, A$ *and let M be a right A-module. Then there exists a short exact sequence.*

$$0 \longrightarrow tM \longrightarrow M \longrightarrow M/tM \longrightarrow 0$$

such that $tM \in \mathcal{T}$ and $M/tM \in \mathcal{F}$. The sequence is unique, meaning that if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact with $M' \in \mathcal{T}$ and $M'' \in \mathcal{F}$, the two sequences are isomorphic.

A short exact sequence like the one in the theorem is called the canonical sequence of M

Proof. Since tM is the largest submodule of M in $\mathcal{T}$ we know $M' \subseteq tM$. We now get this commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & tM & \longrightarrow & M & \longrightarrow & M/tM & \longrightarrow & 0 \end{array}$$

From the snake lemma we get $tM/M' \cong \ker f$, but since $\mathcal{F}$ is closed under submodules, $\ker f \in \mathcal{F}$. Then the canonical projection $tM \to tM/M$ is the zero map. Thus we know $tM \cong M'$ and therefore $M'' \cong M/tM$ from the 5-lemma. □

For now let T be an arbitrary A-module. We will now introduce a subcategory of mod A called Gen T. This is the class of modules generated by T. In other words Cogen T is the class of modules, M such that there exist an integer $d \geq 0$ together with an epimorphism $T^d \to M$. Dually we define Cogen T, which is the class of modules cogenerated by T. In other words Cogen T is the class of modules N such that there exist an integer $d \geq 0$ together with a monomorphism $N \to T^d$. We want to know when Gen T is a torsion class and when Cogen T is a torsion-free class. In other words we want to show that Gen T is closed under images, direct sums and extensions and that Cogen T is closed under submodules, direct products and extensions.

Lemma 2.0.7. *Let $M \in \text{mod}\, A$ and denote $B = \text{End}_A(T)$.*

(a) Then $M \in \text{Gen}\, T$ if and only if the homomorphism

$$\varepsilon_M : \text{Hom}_A(T, M) \otimes_B T \longrightarrow M$$

with ε_M, given by $f \otimes t \mapsto f(t)$ is surjective.

(b) Then $M \in \text{Cogen}\, T$ if and only if the homomorphism

$$\eta_M : M \longrightarrow \text{Hom}_B(\text{Hom}_A(M, T), T)$$

with η_M, given by $x \mapsto (g \mapsto g(x))$is injective.

Proof. Firstly let $M \in \text{Gen}\, T$, then let us look at the basis $\{f_1, f_2, ..., f_d\}$ for $\text{Hom}_A(T, M)$ as a vector space. Then we will consider the function $f = [f_1\, f_2\, \cdots\, f_d] : T^d \to M$. Since $M \in \text{Gen}\, T$ we know there exists an $m \geq 0$ and a surjection $g : T^m \to M$. From the fact that g is a surjection and the definition of f we know there exists a homomorphism $h : T^m \to T^d$ such that $g = f \circ h$. Thus f is surjective. Next we will consider the short exact sequence

$$0 \longrightarrow \ker f \longrightarrow T^d \longrightarrow M \longrightarrow 0.$$

Applying $\text{Hom}_A(T, -)$ then gives us the short exact sequence

$$0 \longrightarrow \text{Hom}_A(T, \ker f) \longrightarrow \text{Hom}_A(T, T^d) \xrightarrow{\text{Hom}_A(T,f)} \text{Hom}_A(T, M) \longrightarrow 0$$

since $\text{Hom}_A(T, f)$ is surjective by the construction of f. Lastly applying $- \otimes_B T$ gives us this commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \to & \text{Hom}_A(T, \ker f) \otimes_B T & \to & \text{Hom}_A(T, T^d) \otimes_B T & \to & \text{Hom}_A(T, M) \otimes_B T & \to & 0 \\ & & \downarrow & & \downarrow \varepsilon_{T^d} & & \downarrow \varepsilon_M & & \\ 0 & \longrightarrow & \ker f & \longrightarrow & T^d & \xrightarrow{f} & M & \longrightarrow & 0 \end{array}$$

This gives us that ε_M is surjective since

$$\text{Hom}_A(T, T^d) \otimes_B T \cong B^d \otimes_B T \cong T^d.$$

The proof for (b) is similar to the proof of (a). □

This result will be used repeatedly throughout this thesis, beginning with giving a condition for when $\text{Gen}\, T$ is closed under extensions.

Lemma 2.0.8.

(a) If $\text{Ext}^1_A(T, -)|_{\text{Gen}\, T} = 0$ *then* $\text{Gen}\, T$ *is closed under extensions.*
(b) If $\text{Ext}^1_A(-, T)|_{\text{Cogen}\, T} = 0$ *then* $\text{Cogen}\, T$ *is closed under extensions.*

Proof. We will only prove (a) since the proof of (b) is similar. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence with $M', M'' \in \text{Gen}\, T$. The functor $\text{Hom}_A(T, -)$ then induces the exact sequence

$$0 \to \text{Hom}_A(T, M') \to \text{Hom}_A(T, M) \to \text{Hom}_A(T, M'') \to \text{Ext}^1_A(T, M') = 0.$$

We now want to use lemma 2.0.7 by applying $- \bigotimes_B T$ to the exact sequence, which gives us the commutative diagram below.

$$\begin{array}{ccccccccc} 0 \to & \text{Hom}_A(T, M') \otimes_B T & \to & \text{Hom}_A(T, M) \otimes_B T & \to & \text{Hom}_A(T, M'') \otimes_B T & \to 0 \\ & \downarrow{\scriptstyle \varepsilon_{M'}} & & \downarrow{\scriptstyle \varepsilon_{M'}} & & \downarrow{\scriptstyle \varepsilon_{M'}} & \\ 0 \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow 0 \end{array}$$

Since ε'_M and ε''_M are epimorphisms ε_M is also a epimorphism from the five-lemma. Hence M is in $\text{Gen}\, T$ and $\text{Gen}\, T$ is closed under extension. □

Now we have a condition for when $\text{Gen}\, T$ is a torsion class with $\{M \mid \text{Hom}_A(T, M) = 0\}$ as it's torsion free class. And we also have a condition for when $\text{Cogen}\, T$ is a torsion-free class with $\{M \mid \text{Hom}(M, T) = 0\}$ as its torsion class.

Definition 2.0.9. Let $\mathcal{C}$ be a full subcategory of $\text{mod}\, A$. A module $M \in \mathcal{C}$ is Ext-projective in $\mathcal{C}$ if $\text{Ext}^1_A(M, -)|_{\mathcal{C}} = 0$. And dually M is Ext-injective if $\text{Ext}^1_A(-, M)|_{\mathcal{C}} = 0$.

From this definition we reformulate the last result. If T is Ext-projective in $\mathsf{Gen}\,T$, then $\mathsf{Gen}\,T$ is a torsion class with $\{M \mid \mathsf{Hom}_A(T, M) = 0\}$ as it's torsion free class. Similarly if T is Ext-injective then $\mathsf{Cogen}\,T$ is a torsion-free class with $\{M \mid \mathsf{Hom}_A(M, T) = 0\}$ as it's torsion class.

Chapter 3

Tilting modules

In this chapter is split in to two parts. The first part gives an introduction to tilting modules and some results about the. The second part we prove the Brenner-Butler-theorem. Giving us an equivalence between torsion pairs in mod A and mod B. The presentation of this chaper is also based on Assem, Simson, Skowroński[1].

Definition 3.0.1. Let A be an algebra. An A-module T is a tilting module if:

T.1 The projective dimension of T is less then or equal to 1.
T.2 $\text{Ext}_A^1(T,T) = 0$.
T.3 There exists a short exact sequence

$$0 \longrightarrow A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$$

such that T' and T'' in Add T. That is the category of direct sums of direct summands of T.

A module satisfying the two first conditions is often called a partial tilting module.

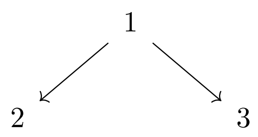
Example 3.0.2. The algebra A is a tilting module.

T.1 The algebra A itself is projective.
T.2 Since the algebra A is projective, we have that $\text{Ext}_A^1(A, A) = 0$
T.3 Clearly, the short exact sequence,

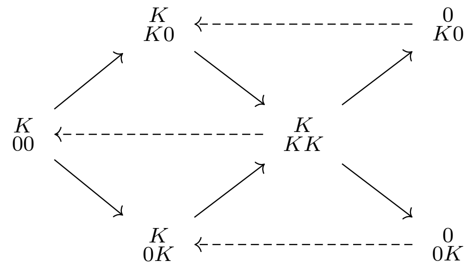
$$0 \longrightarrow A \xrightarrow{(1,0)} A \oplus A \xrightarrow{(0,1)} A \longrightarrow 0$$

satisfies the third condition.

Example 3.0.3. From a previous example we have seen that the quiver Q



gives rise to the AR-quiver:



Then we check that $\begin{smallmatrix} K \\ KK \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ 0K \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ 0K \end{smallmatrix}$ is a tilting module.

T.1 The algebra is hereditary, so pd $(\begin{smallmatrix} K \\ KK \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ 0K \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ 0K \end{smallmatrix}) = 1$.
T.2 From the Auslander-Reiten-formula we have that

$$\begin{aligned} &\mathrm{Ext}^1_A(\begin{smallmatrix} K \\ KK \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ 0K \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ K0 \end{smallmatrix} , \begin{smallmatrix} K \\ KK \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ 0K \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ 0K \end{smallmatrix}) \\ &\cong D\,\mathrm{Hom}_A(\begin{smallmatrix} K \\ KK \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ 0K \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ 0K \end{smallmatrix} , \begin{smallmatrix} K \\ 00 \end{smallmatrix} \oplus \begin{smallmatrix} k \\ 0K \end{smallmatrix} \oplus \begin{smallmatrix} K \\ K0 \end{smallmatrix}). \end{aligned}$$

T.3 There is a short exact sequence

$$0 \to A \to \begin{smallmatrix} K \\ KK \end{smallmatrix} \oplus \begin{smallmatrix} K \\ KK \end{smallmatrix} \oplus \begin{smallmatrix} K \\ KK \end{smallmatrix} \to \begin{smallmatrix} 0 \\ 0K \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ K0 \end{smallmatrix} \oplus (\begin{smallmatrix} 0 \\ 0K \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ K0 \end{smallmatrix}) \to 0.$$

Indeed we see that $\begin{smallmatrix} K \\ KK \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ 0K \end{smallmatrix} \oplus \begin{smallmatrix} 0 \\ 0K \end{smallmatrix}$ is a tilting module.

Another category we will consider is $\mathcal{T}(T)$, defined as

$$\mathcal{T}(T) = \{M \mid \mathrm{Ext}^1_A(T, M) = 0\}.$$

The following category is the largest full subcategory of mod A, such that T is Ext-projective. We can observe that every injective module is in $\mathcal{T}(T)$.

Theorem 3.0.4. *Let T be a partial tilting module. Then*

(a) Gen T *is a torsion class in which T is Ext-projective with* $\mathcal{F}(T) = \{M \mid \mathrm{Hom}_A(T, M) = 0\}$ *as its corresponding torsion-free class.*
(b) $\mathcal{T}(T)$ *is a torsion class, with Cogen τT as it's torsion-free class.*

Since Gen T is a torsion class where T is Ext-projective, we easily get that Gen $T \subseteq \mathcal{T}(T)$.

Proof.
(a) Let $M \in \operatorname{Gen} T$. This means there exists an epimorphism $T^d \twoheadrightarrow M$ with $d \leq 1$. Now applying $\operatorname{Hom}_A(T, -)$ induces an epimorphism $\operatorname{Ext}^1_A(T, T^d) \twoheadrightarrow \operatorname{Ext}^1_A(T, M)$, but since T is a partial tilting module $\operatorname{Ext}^1_A(T, T) = 0$ and using the fact that $\operatorname{Ext}^1_A(T, T^d) = \bigoplus_{i=1}^d \operatorname{Ext}^1_A(T, T)$ we have that $\operatorname{Ext}^1_A(T, M) = 0$, meaning that T is Ext-projective in $\operatorname{Gen} T$.
(b) First let us consider the following short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

applying $\operatorname{Hom}_A(T, -)$ induces the exact sequence,

$$\operatorname{Ext}^1_A(T, M') \longrightarrow \operatorname{Ext}^1_A(T, M) \longrightarrow \operatorname{Ext}^1_A(T, M'') \longrightarrow 0$$

Now if M' and M'' lies in $\mathcal{T}(T)$ then M also lies in $\mathcal{T}(T)$. And if $M \in \mathcal{T}(T)$, then M'' lies in $\mathcal{T}(T)$. And since $\operatorname{Ext}^1_A(T, -)$ is an additive functor, we get that $\mathcal{T}(T)$ is closed under extensions, images and direct sums. Thus $\mathcal{T}(T)$ is a torsion class.
To show $\operatorname{Cogen} \tau T$ is it's torsion free class we want to show that $\operatorname{Ext}^1_A(-, \tau T)|_{\operatorname{Cogen} \tau T} \cong 0$. Now from the Auslander-Reiten formulas we get that $\operatorname{Ext}^1_A(T, M) \cong D \operatorname{Hom}_A(M, \tau T)$ since pd $T \leq 1$ meaning that $M \in \mathcal{T}(T)$ if and only if $\operatorname{Hom}_A(M, \tau T) = 0$. Now let $N \in \operatorname{Cogen} \tau T$, then there exists an injection, $N \hookrightarrow \tau T^d$, which gives an injection $\operatorname{Hom}(T, N) \hookrightarrow \operatorname{Hom}(T, \tau T^d) \cong \operatorname{Ext}^1_A(T, T) = 0$. Then using the Auslander-Reiten- formulas we get that $\operatorname{Ext}^1_A(N, \tau T) \cong D\underline{\operatorname{Hom}}_A(T, N) = 0$.
Thus $\operatorname{Ext}^1_A(-, \tau T)|_{\operatorname{Cogen} \tau T} = 0$ and by lemma 2.0.8 $\operatorname{Cogen} \tau T$ is a torsion-free class and $\{M \mid \operatorname{Hom}_A(M, \tau T) = 0\} = \mathcal{T}(T)$ is it's torsion class. □

If you recall the definition of a tilting module, the third condition is often the hardest to show. Thus we want a way to know when a partial tilting module is a tilting module. The following theorem will gives us equivalent statements for a partial tilting module being a tilting module.

Theorem 3.0.5. *Let T be a partial tilting module, then the following are equivalent:*

a) *T is a tilting module.*
b) $\operatorname{Gen} T = \mathcal{T}(T)$.
c) *For every module $M \in \mathcal{T}(T)$ there exists a short exact sequence* $0 \longrightarrow L \longrightarrow T_0 \longrightarrow M \longrightarrow 0$ *with $L \in \mathcal{T}(T)$ and $T_0 \in \operatorname{Add} T$.*
d) *Let $X \in \operatorname{mod} A$. Then $X \in \operatorname{Add} T$ if and only if X is Ext-projective in $\mathcal{T}(T)$.*
e) $\mathcal{F}(T) = \operatorname{Cogen} \tau T$.

Proof.
(a) $\implies$ (b) From lemma 3.0.4 we have that Gen $T \subseteq \mathcal{T}(T)$, so we only need to show $\mathcal{T}(T) \subseteq$ Gen T. Now let $M \in \mathcal{T}(T)$. Then we have the canonical sequence

$$0 \to tM \to M \to M/tM \to 0$$

Then $\text{Hom}_A(T,-)$ induces the exact sequence,

$$0 \longrightarrow \text{Ext}^1_A(T,tM) \longrightarrow \text{Ext}^1_A(T,M) \longrightarrow \text{Ext}^1_A(T,M/tM) \longrightarrow 0.$$

but since $\text{Ext}^1_A(T,M) = 0$ we have that $\text{Ext}^1_A(T,M/tM) = 0$. Since T is a tilting module we have the short exact sequence:

$$0 \longrightarrow A \longrightarrow T' \longrightarrow T'' \longrightarrow 0,$$

which together with the functor $\text{Hom}_A(-,M/tM)$ gives us

$$0 \to \text{Hom}_A(T'',M/tM) \to \text{Hom}_A(T',M/tM) \to \text{Hom}_A(A,M/tM) \to 0$$

which implies $0 = \text{Hom}_A(A,M/tM) \cong M/tM$, and thus $M \cong tM$ and $M \in$ Gen T.
(b) $\implies$ (c) Let $M \in \mathcal{T}(T)$, then there exists an epimorphism, $\pi : T^d \twoheadrightarrow M$. We then get the short exact sequence,

$$0 \longrightarrow \ker\pi \longrightarrow T^d \xrightarrow{f} M \longrightarrow 0$$

Applying $\text{Hom}_A(T,-)$ induces this exact sequence,

$$0 \to \text{Hom}_A(T,\ker\pi) \to \text{Hom}_A(T,T^d) \to \text{Hom}_A(T,M) \to \text{Ext}^1_A(T,\ker\pi) \to 0.$$

But $\text{Hom}_A(T,f)$ is an epimorphism and thus $\text{Ext}^1_A(T,\ker\pi) = 0$.
(c) $\implies$ (d) Let $X \in$ Add T, then there exists an module, Q such that $X \oplus Q \cong T^d$ and hence we have the isomorphism.

$$0 = \text{Ext}^1_A(T^d,T) \cong \text{Ext}^1_A(X \oplus Q,T).$$

Since the Ext-functor is additive we have that X is Ext-projective in $\mathcal{T}(T)$. Now let X be Ext-projective in $\mathcal{T}(T)$. Then there exists a short exact sequence

$$0 \longrightarrow L \longrightarrow T_0 \longrightarrow X \longrightarrow 0$$

with $L \in \mathcal{T}(T)$ and $T_0 \in \mathrm{Add}\, T$. Now since X is Ext-projective in $\mathcal{T}(T)$ we know that $\mathrm{Ext}^1_A(X, L) = 0$. Hence the short exact sequence splits. Thus $X \in \mathrm{Add}\, T$.
(d) $\Longrightarrow$ (a) Let this short exact sequence correspond to an element of $\mathrm{Ext}^1_A(T, A)$

$$0 \longrightarrow A \longrightarrow E \longrightarrow T \longrightarrow 0.$$

Now we get that the partial tilting module T is a tilting module if $E \in \mathrm{Add}\, T$, which is equivalent to E being Ext-projective in $\mathcal{T}(T)$. Now let $M \in \mathcal{T}(T)$. Then applying $\mathrm{Hom}_A(-, M)$ gives us this exact sequence.

$$\cdots \longrightarrow 0 = \mathrm{Ext}^1_A(T, M) \longrightarrow \mathrm{Ext}^1_A(E, M) \longrightarrow \mathrm{Ext}^1_A(A, M) = 0$$

Thus we have that $E \in \mathrm{Add}\, T$, and T.3 is satisfied and T is a tilting module.

□

Corollary 3.0.5.1. *Let $M \in \mathcal{T}(T)$ where T is a tilting module, then there exists an exact sequence*

$$\cdots \longrightarrow T_2 \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M$$

such that T_i's are all in $\mathrm{Add}\, T$

Proof. This follows from doing induction on part (c) of theorem 3.0.5. □

An exact sequence such as the one above will be referenced as a T-resolution of M. The exact sequences of this form are very useful. They will be used frequently in proofs throughout this chapter.

The next corollary will be a central part of proving the Brenner-Butler theorem.

Corollary 3.0.5.2. *Let T_A be a tilting module and $B = \mathrm{End}_A(T)$. Then $M \in \mathcal{T}(T)$ if and only if the canonical A- homomorphism $\varepsilon_M : \mathrm{Hom}_A(T, M) \otimes_B T \longrightarrow M$ is an isomorphism.*

Proof. From theorem 3.0.5 we know there exists short exact sequences

$$0 \longrightarrow L_1 \longrightarrow T_0 \longrightarrow X \longrightarrow 0$$

with $L_1 \in \mathcal{T}(T)$ and $T_0 \in \mathrm{Add}\, T$. Also from theorem 3.0.5 we know there exists a short exact sequence

$$0 \longrightarrow L_0 \longrightarrow T_1 \longrightarrow L_1 \longrightarrow 0$$

Next applying $\mathrm{Hom}_A(T, -)$ gives these exact sequences, since $\mathrm{Ext}^1_A(T, L_0) = 0$ and $\mathrm{Ext}^1_A(T, L_1) = 0$.

$$0 \longrightarrow \mathsf{Hom}_A(T, L_1) \longrightarrow \mathsf{Hom}_A(T, T_0) \longrightarrow \mathsf{Hom}_A(T, X) \longrightarrow 0$$
$$0 \longrightarrow \mathsf{Hom}_A(T, L_0) \longrightarrow \mathsf{Hom}_A(T, T_1) \longrightarrow \mathsf{Hom}_A(T, L_1) \longrightarrow 0$$

which combines to this right exact sequence.

$$\mathsf{Hom}_A(T, T_1) \longrightarrow \mathsf{Hom}_A(T, T_0) \longrightarrow \mathsf{Hom}_A(T, X) \longrightarrow 0$$

Now after applying $- \otimes T$, while also using corollary 2.0.9.1, we get this commutative diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathsf{Hom}_A(T, T_1) \otimes T & \longrightarrow & \mathsf{Hom}_A(T, T_0) \otimes T & \longrightarrow & \mathsf{Hom}_A(T, X) \otimes T \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \\ 0 & \longrightarrow & T_1 & \longrightarrow & T_0 & \longrightarrow & X \end{array}$$

Thus we get an isomorphism, $X \to \mathsf{Hom}_A(T, X) \otimes_B T$.

□

Example 3.0.6. Let M_P be a Morita progenerator M_P, then M_P is a tilting module. Obviously M_P has projective dimension less then or equal to 1 and $\mathsf{Ext}^1_A(M_P, M_P) = 0$. Hence M_P is a partial tilting module. And since M_P is a generator for mod A we know that Gen $M_P =$ mod A and from lemma 3.0.3 we have know that Gen $M_P \subseteq \mathcal{T}(M_P)$. Thus Gen $M_P = \mathcal{T}(M_P)$ and M_P is then a tilting module by theorem 3.0.5.

Many describe a tilting module as a module which is "close to" a Morita progenerators. In the sense that A is Morita equivalent to $\mathsf{End}_A(M_P)$, but being a tilting module is not as strong as being a progenerator. So a tilting module T does not induce a Morita equivalence between A and $\mathsf{End}_A(T)$. But it does induce an equivalence between torsion pairs of mod A and mod $\mathsf{End}_A(T)$. Hence tilting modules are tools we can use to compare mod A and mod $\mathsf{End}_A(T)$. We will now work our way towards these equivalences. To motivate these equivalences we look to theorem 3.0.5, stating that if T is a tilting module then Gen $T = \mathcal{T}(T)$. This means that T acts as a projective module in Gen T. Thus it is a Morita progenerator for this subcategory. This hints towards there being an equivalence of subcategories in mod A and mod $\mathsf{End}_A(T)$.

3.0.1 The Brenner-Butler theorem

In this section we want to compare mod A and mod B by using tilting modules. Therefore we will need to show that an right A-tilting module also is a left B-tilting module. With this we will know that T also induces a torsion pair in mod B. As one might expect since a tilting module is a Morita progenerator in $\mathcal{T}(T)$, this gives us an equivalence between these torsion pairs. This equivalence is described by the Brenner-Butler-theorem, and is exactly what we will prove in this section.

We will start out by showing some lemmas which we will need later.

Lemma 3.0.7. *Let T be any right A-module, and $B = \mathrm{End}_A(T)$. Then for each module $T_0 \in \mathrm{Add}\, T$ we have the functorial isomorphism*

$$\mathrm{Hom}_A(T_0, M) \cong \mathrm{Hom}_B(\mathrm{Hom}_A(T, T_0), \mathrm{Hom}_A(T, M)).$$

Proof. Let $T_0 = T$. Then we have

$\mathrm{Hom}_B(\mathrm{Hom}_A(T, T), \mathrm{Hom}_A(T, M)) = \mathrm{Hom}_B(B, \mathrm{Hom}_A(T, M)) \cong \mathrm{Hom}_A(T, M).$

The rest of the proof follows since the Hom-functor is a additive. □

Now we can observe that the functor $\mathrm{Hom}_A(T, -)$ gives an equivalence between $\mathrm{Add}\, T$ in mod A and the projective modules in B, denoted $\mathcal{P}_B$. Since $\mathrm{Hom}_A(T, -)$ is obviously dense, and it is full and faithful from lemma 3.0.7.

Lemma 3.0.8. *Let $M, N \in \mathcal{T}(T)$. Then there exists natural isomorphisms*

(a) $\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_B(\mathrm{Hom}_A(T, M), \mathrm{Hom}_A(T, N))$
(b) $\mathrm{Ext}^1_A(M, N) \cong \mathrm{Ext}^1_B(\mathrm{Hom}_A(T, M), \mathrm{Hom}_A(T, N))$

Proof.
(a) Using theorem 3.0.5 c) we get the right exact sequence

$$T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} M \longrightarrow 0$$

Then applying $\mathrm{Hom}_A(-, N)$ we get this left exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_A(T_0, N) \longrightarrow \mathrm{Hom}_A(T_1, N)$$

which together with lemma 3.0.7 gives us this commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Hom}_A(M, N) & \dashrightarrow & \mathrm{Hom}_B(\mathrm{Hom}_A(T, M), \mathrm{Hom}_A(T, N)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_A(T_0, N) & \xrightarrow{\cong} & \mathrm{Hom}_B(\mathrm{Hom}_A(T, T_0), \mathrm{Hom}_A(T, N)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_A(T_1, N) & \xrightarrow{\cong} & \mathrm{Hom}_B(\mathrm{Hom}_A(T, T_1), \mathrm{Hom}_A(T, N))
\end{array}$$

(b) Taking the short exact sequence

$$0 \longrightarrow \operatorname{Im} d_1 \xrightarrow{\iota} T_0 \xrightarrow{d_0} M \longrightarrow 0$$

and applying $\operatorname{Hom}_A(-, N)$, we get

$$0 \to \operatorname{Hom}_A(M,N) \to \operatorname{Hom}_A(T_0,N) \to \operatorname{Hom}_A(L,N) \to \operatorname{Ext}^1_A(M,N) \to 0.$$

Thus we get that $\operatorname{Ext}^1_A(M,N) \cong \operatorname{Coker}\operatorname{Hom}_A(\iota, N)$, which when considering the T-resolution $T_\bullet$ of M is isomorphic to the first cohomology group of the complex $\operatorname{Hom}_A(T_\bullet, N)$.
As we know from lemma 3.0.7, $\operatorname{Hom}_A(T, T_\bullet)$ yields a projective resolution of $\operatorname{Hom}_A(T, M)$. Thus by the definition of the Ext^n_B-functor, we have $\operatorname{Ext}^1_B(\operatorname{Hom}_A(T,M), \operatorname{Hom}_A(T,N))$ is the first cohomology group of the complex $\operatorname{Hom}_B(\operatorname{Hom}_A(T,T_\bullet), \operatorname{Hom}_A(T,N))$. Hence since the complex $\operatorname{Hom}_B(\operatorname{Hom}_A(T,T_\bullet), \operatorname{Hom}_A(T,N))$ is isomorphic to $\operatorname{Hom}_A(T, T_\bullet)$ by (a), thus we are done. □

Recall that a module M is said to be faithful if the right annihilator is zero. Or in other word

$$\operatorname{Ann} M = \{a \in A \mid Ma = 0\} = 0$$

Lemma 3.0.9. *A tilting module is faithful.*

Proof. Let T be a tilting module. Then we know from the definition, that there exists an short exact sequence

$$0 \longrightarrow A \xrightarrow{f} T' \longrightarrow T'' \longrightarrow 0.$$

Assume $a \in \operatorname{Ann} M$. We then know $T'a = 0$ since $a \in \operatorname{Ann} T$, but then $f(a) = f(1)a = 0$, since f is injective. This means $a \in \ker f = 0$. Thus we get that T is faithful. □

Now we have all the tools to prove that a right tilting A-module is also a left tilting B-module. We will also prove that we can go back and forth between mod A and mod B.

Theorem 3.0.10. *Let T_A be a tilting module and $B = \operatorname{End}_A(T_A)$. Then the following holds:*

- *a)* $D({}_BT) = \operatorname{Hom}_A({}_BT, DA)$
- *b)* ${}_BT$ *is a left tilting B module.*
- *c) The canonical K-algebra homomorphism* $\varphi : A \to \operatorname{End}({}_BT)^{op}$ *defined by* $a \mapsto (t \mapsto t(a))$ *is an isomorphism.*

Proof.
(a) Using the fact that ${}_BT \otimes_A A \cong_B T$ and hom-tensor adjunction we get that

$$\begin{aligned} D({}_BT) &\cong \mathrm{Hom}_K({}_BT, K) \\ &\cong \mathrm{Hom}_K(T_B \otimes_A A, K) \\ &\cong \mathrm{Hom}_A({}_BT, \mathrm{Hom}_K(A, K)) \\ &\cong \mathrm{Hom}_A({}_BT, DA). \end{aligned}$$

(b)

T.1 Since T is a tilting module we know there exists a short exact sequence.

$$0 \longrightarrow A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$$

with $T', T'' \in \mathrm{Add}\, T$. Then using $\mathrm{Hom}_A(-,_B T_A)$ we get this short exact sequence:

$$0 \to \mathrm{Hom}_A(T'', T) \to \mathrm{Hom}_A(T', T) \to \mathrm{Hom}_A(A,_B T_A) \to 0$$

with $\mathrm{Hom}_A(A,_B T_A) \cong_B T$ and $\mathrm{Hom}_A(T', T)$ and $\mathrm{Hom}_A(T'', T)$ projective from lemma 3.0.7. So pd ${}_BT \leq 1$.

T.2 Using part a) we have that,

$$\mathrm{Ext}^1_B(DT, DT) \cong \mathrm{Ext}^1_B(\mathrm{Hom}_A(T, DA), \mathrm{Hom}_A(T, DA)).$$

Then using Lemma 3.0.8 we get that

$$\mathrm{Ext}^1_B(\mathrm{Hom}_A(T, DA), \mathrm{Hom}_A(T, DA)) \cong \mathrm{Ext}^1_A(DA, DA) = 0$$

since DA is injective. Thus $\mathrm{Ext}^1_B(T, T) = 0$.

T.3 Take the projective resolution of T_A:

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow T_A \longrightarrow 0$$

then using $\mathrm{Hom}_A(-,_B T_A)$ we get

$$0 \to \mathrm{Hom}_A(T, T) = B \to \mathrm{Hom}_A(P_0, T) \to \mathrm{Hom}_A(P_1,_B T_A) \to 0$$

Then using the fact that every projective module is a direct summand of A^n , the additivity of $\mathrm{Hom}(-, T)$ together with the isomorphism $\mathrm{Hom}_A(A, T) \cong T$. We get that $\mathrm{Hom}_A(P_1,_B T_A), \mathrm{Hom}_A(P_0,_B T_A) \in \mathrm{Add}\, T$.

(c) Let $a \in \ker \varphi$, then $Ta = 0$, but from lemma 3.0.9 we know that a tilting module is always faithful, meaning $a = 0$. Thus φ is injective.
Next we know that

$$A \cong D(DA \otimes A) \cong \mathrm{Hom}_A(DA, DA)$$

Now using lemma 3.0.7, we get an isomorphism.

$$\mathsf{Hom}_A(DA, DA) \cong \mathsf{Hom}_B(\mathsf{Hom}(T, DA), \mathsf{Hom}_A(T, DA)) \cong \mathsf{End}_B({}_BT)$$

Thus we get that $A \cong \mathsf{End}_B({}_BT)$, meaning that $\dim_K(A) = \dim_K(\mathsf{End}_B({}_BT))$, meaning φ is a vector space surjection since φ is injective. Thus φ is an isomorphism. □

Corollary 3.0.10.1. $D({}_BT) \cong \mathsf{Hom}_A(T, T \otimes_B DT)$

Proof. This follows from the fact that $D({}_BT) \cong \mathsf{Hom}_A(T, DA)$ and that $A \cong \mathsf{End}({}_BT)^{op}$. □

We now know that a right tilting A-module is also a left tilting B-module, we can start building a similar theory in mod B as we did in mod A. Together with a way to move between the two. More specifically we will prove the Brenner-Butler theorem, which gives equivalences between torsion pairs induced by a tilting module in mod A and mod B.

As we know ${}_BT$ induces a torsion pair $(\mathcal{T}({}_BT), \mathcal{F}({}_BT))$ in the category of left B-modules, given by

$$\mathsf{Gen}_B\, T = \mathcal{T}({}_BT) = \{U \mid \mathsf{Ext}^1_B(T, U) = 0\}$$

$$\mathsf{Cogen}\, \tau({}_BT) = \mathcal{F}({}_BT) = \{L \mid \mathsf{Hom}_B(T, L) = 0\}$$

But the goal here is to compare the categories of right A and B modules. Therefore we will rather consider the torsion pair $(D\mathcal{F}({}_BT), D\mathcal{T}({}_BT))$ of right B-modules.

Corollary 3.0.10.2. *Let A be an algebra and T be a tilting module. Then T induces a torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in mod B, where $B = \mathsf{End}_A(T)$ and*
$\mathcal{X}(T) = \{X_B \mid \mathsf{Hom}_B(X, DT) = 0\} = \{X_B \mid X \otimes_B T = 0\}$
$\mathcal{Y}(T) = \{Y_B \mid \mathsf{Ext}^1_B(Y, DT) = 0\} = \{Y_B \mid \mathsf{Tor}^B_1(Y, T) = 0\}$

Proof. Firstly we know that $\mathsf{Hom}_B(X, DT) \cong D(X \otimes_B T)$ from Hom and $\otimes$ being adjoint functors. For the second part let us take a projective resolution of X.

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

Then applying $- \otimes_B T \cong D\,\mathsf{Hom}_B(-, DT)$ to the resolution yields $\mathsf{Tor}^B_1(X, T) = H_1(P_\bullet \otimes T) \cong H_1(D\,\mathsf{Hom}_B(P_\bullet, DT)) = D\,\mathsf{Ext}^1_B(X, DT)$, where H_1 denotes the first homology group.

□

We can observe that similarly to $\mathcal{T}(T)$, which contains all injective A-modules. $\mathcal{Y}(T)$ contaions all projective B-modules. Since we now are considering the dual of the tilting module. We need to show some of the analogous results already show previously. We will need the analogous result to to part (c) of theorem 3.0.5 and its corollaries. To start is part (c) of theorem 3.0.5.

Lemma 3.0.11. *Let T be a right tilting A-module, $B = \text{End}_A(T)$ and $Y_B \in \mathcal{Y}(T)$. Then there exists a short exact sequence*

$$0 \longrightarrow {}_BY \longrightarrow T' \longrightarrow {}_BZ \longrightarrow 0$$

With $T^{'} \in \text{Add}\, DT$ and $Z_B \in \mathcal{Y}({}_AT)$.

Proof. (a) ${}_BT$ is a tilting module, we have that $DY_B \in D\mathcal{Y}({}_AT) = \mathcal{T}(T_B)$. Hence from theorem 3.0.5 part c) we know there exists a short exact sequence

$$0 \longrightarrow L \longrightarrow T'_B \longrightarrow D_BY \longrightarrow 0$$

with $L, D_BY \in \mathcal{T}(T_B)$ and $T'_B \in \text{Add}\, T_B$. Now dualizing the sequence gives us

$$0 \longrightarrow {}_BY \longrightarrow DT'_B \longrightarrow DL \longrightarrow 0$$

Where $DT'_B \in \text{Add}\, DT_B$ and $DL \in \mathcal{Y}({}_AT)$. □

The next is the analogous result to corollary 3.0.5.1. Which follow directly from induction on lemma 3.0.11.

Corollary 3.0.11.1. *Let $Y_B \in \mathcal{Y}({}_AT)$, then there exists an exact sequence*

$$Y_B \longrightarrow T_0^* \longrightarrow T_1^* \longrightarrow \cdots$$

Lastly we will need the corollary 3.0.5.2.

Lemma 3.0.12. *The canonical K-algebra homomorphism $Y_B \to \text{Hom}_A(T, Y \otimes_B T)$ given by $y \mapsto (t \mapsto y \otimes t)$ is an isomorphism.*

Proof. From 3.0.5 c) we know there exists short exact sequences

$$0 \longrightarrow Y \longrightarrow T_0^{'} \longrightarrow Z_0 \longrightarrow 0$$

With $Z_0 \in \mathcal{Y}(T)$ and $T_0^{'} \in \text{Add}\, T$. Also

$$0 \longrightarrow Z_0 \longrightarrow T_1^{'} \longrightarrow Z_1 \longrightarrow 0$$

Next applying $-\otimes_B T$ gives these exact sequences, since $\text{Tor}^1_B(Z_0, T) = 0$ and $\text{Tor}^1_B(Z_0, T) = 0$.

$$0 \longrightarrow Y \otimes T \longrightarrow T_0^{'\otimes T} \longrightarrow Z_0 \otimes T \longrightarrow 0$$
$$0 \longrightarrow Z_0 \otimes T \longrightarrow T_1^{'\otimes T} \longrightarrow Z_1 \otimes T \longrightarrow 0$$

which combines to this left exact sequence.

$$0 \longrightarrow Y \otimes T \longrightarrow T_0^{'} \otimes T \longrightarrow T_1 \otimes T$$

Now after applying $\text{Hom}_A(T, -)$, while also using corollary 0.0.9.1. We get this commutative diagram.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}_A(T, Y \otimes T) & \longrightarrow & \text{Hom}_A(T, T_0^{'} \otimes T) & \longrightarrow & \text{Hom}_A(T, T_1 \otimes T) \\
 & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & Y & \longrightarrow & T_0^{'} & \longrightarrow & T_1^{'}
\end{array}$$

Thus we get an isomorphism, $Y_B \to \text{Hom}_A(T, Y \otimes_B T)$.

□

Now we have all the tool to prove the main and final theorem of this section, the Brenner-Butler theorem.

Theorem 3.0.13 (Brenner-Butler theorem)**.** *Let A be an algebra, T_A a tilting module, $B = \text{End}_A(T_A)$, and let $(\mathcal{T}(T), \mathcal{F}(T))$ and $(\mathcal{X}(T), \mathcal{Y}(T))$ be the induced torsion pairs in* $\text{mod}\, A$ *and* $\text{mod}\, B$ *respectively. Then the following is true:*

- *a) ${}_BT$ is a tilting module and the canonical homomorphism $A \to \text{End}({}_BT)^{op}$ is an isomorphism.*
- *b) The functors $\text{Hom}_A(T, -)$ and $-\otimes_B T$ induce quasi-inverse equivalences between $\mathcal{T}(T)$ and $\mathcal{Y}(T)$.*
- *c) The functors $\text{Ext}^1_A(T, -)$ and $\text{Tor}^B_1(-, T)$ induce quasi-inverse equivalences between $\mathcal{F}(T)$ and $\mathcal{X}(T)$.*

Proof.
(a) This follow directly form part (b) and (c) of theorem 3.0.10.
(b) Let $M \in \mathcal{T}(T)$. Then taking the dual of $\text{Hom}_A(T, M)$, we get $D\,\text{Hom}_A(T, M) \cong_B T_A \otimes_A DM$ from Hom and tensor being adjoint. We then know there exist an epimorphism ${}_BT_A \otimes_A P \twoheadrightarrow_B T_A \otimes_A DM$, and we also know

there exists another projective module Q, such that $P \oplus Q \cong A^d$. Hence there exists an epimorhism,

$$T \cong_B T_A \otimes_A (P \oplus Q) \twoheadrightarrow_B T_A \otimes_A DM.$$

Thus $D\,\mathsf{Hom}_A(T, M) \in \mathsf{Gen}\,T$. Meaning $\mathsf{Hom}_A(T, M) \in D\,\mathsf{Gen}\,T = \mathcal{Y}(T)$.
Next let $Y \in \mathcal{Y}$, then $Y \otimes_B T \in \mathsf{Gen}\,T = \mathcal{T}(T)$. From corollary 3.0.5.2 and lemma 3.0.12 we have that $\mathsf{Hom}_A(T, M) \otimes_B T \cong M$ and
$M \cong \mathsf{Hom}_A(T, M \otimes_B T)$. Thus $\mathsf{Hom}_A(T, -)$ and $- \otimes_B T$ induce quasi-inverse equivalences between $\mathcal{T}(T)$ and $\mathcal{Y}(T)$.
(c) Let $N \in \mathcal{F}(T)$, then we can make a short exact sequence

$$0 \longrightarrow N \longrightarrow I \longrightarrow L \longrightarrow 0$$

with I injective, and thus $I \in \mathcal{T}(T)$, since $\mathcal{T}(T)$ is closed under images we also get $L \in \mathcal{T}(T)$. Then $\mathsf{Hom}_A(T, -)$ induces this short exact sequence:

$$0 = \mathsf{Hom}_A(T, N) \to \mathsf{Hom}_A(T, I) \to \mathsf{Hom}_A(T, L) \to \mathsf{Ext}^1_A(T, N) \to 0.$$

Then using $- \otimes_B T$, the isomorphism in lemma 3.0.12 and the fact that
$L \in \mathcal{T}(T)$ gives $\mathsf{Hom}_A(T, L) \in \mathcal{Y}$, implying $\mathsf{Tor}^1_B(\mathsf{Hom}_A(T, N), T) = 0$, we get this commutative diagram.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathsf{Tor}^1_B(\mathsf{Ext}^1_A(T, N)) & \dashrightarrow & N \\
\downarrow & & \downarrow \\
\mathsf{Hom}_A(T, I) \otimes T & \xrightarrow{\cong} & L \\
\downarrow & & \downarrow \\
\mathsf{Hom}_A(T, L) \otimes T & \xrightarrow{\cong} & I \\
\downarrow & & \downarrow \\
\mathsf{Ext}^1_A(T, N) \otimes T & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Now since the two middle arrows are isomorphisms we get that
$\mathsf{Tor}^1_B(\mathsf{Ext}^1_A(T, N)) \cong N$ and $\mathsf{Ext}^1_A(T, N) \otimes T = 0$, giving us that
$\mathsf{Ext}^1_A(T, N) \in \mathcal{X}(T)$.

Let $X \in \mathcal{X}(T)$, then we can make a short exact sequence

$$0 \longrightarrow Y \longrightarrow P \longrightarrow X \longrightarrow 0$$

with P projective, and thus $P \in \mathcal{Y}(T)$, since $\mathcal{Y}(T)$ is closed under submodules we get $Y \in \mathcal{Y}(T)$. Then $- \otimes_B T$ induces this short exact sequence:

$$0 = \mathsf{Tor}^1_B(P,T) \to \mathsf{Tor}^1_B(X,T) \to Y \otimes_B T \to P \otimes_B T \to 0$$

Then using $\mathsf{Hom}_A(T,-)$, the isomorphism in corollary 3.0.5.2 and that $Y \in \mathcal{Y}(T)$ gives $Y \otimes T \in \mathcal{T}(T)$ implying $\mathsf{Ext}^1_A(T, Y \otimes T) = 0$, we get this commutative diagram.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathsf{Hom}_A(T, \mathsf{Tor}^1_B(X,T)) \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\cong} & \mathsf{Hom}_A(T, Y \otimes_B T) \\
\downarrow & & \downarrow \\
P & \xrightarrow{\cong} & \mathsf{Hom}_A(T, P \otimes_B T) \\
\downarrow & & \downarrow \\
X & \dashrightarrow & \mathsf{Ext}^1_A(T, \mathsf{Tor}^1_B(X,T)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Now since the two middle arrows are isomorphisms we get that, $X \cong \mathsf{Ext}^1_A(T, \mathsf{Tor}^1_B(X,T))$ and $\mathsf{Hom}_A(T, \mathsf{Tor}^1_B(X,T)) = 0$, giving us $\mathsf{Tor}^1_B(X,T) \in \mathcal{F}(T)$

□

Chapter 4

Derived categories

The main point of this chapter is to quickly introduce some concepts and fix some notation for the next chapter. In this section we will introduce a type of category called triangulated category. We will especially look at the triangulated categories $K^b(A)$, the bound homotopy category and $D^b(A)$, derived category of mod A. For a "nice enough" algebra the derived category can be quite accessible, and can gives information about mod A, although we will not discuss this in this thesis. The reason for introducing the derived category, is because a tilting module induces a triangle equivalence between $D^b(A)$ and $D^b(B)$, where $B = \text{End}_A(T)$.

Let T be a additive category and let Σ be an automorphism on T.

Definition 4.0.1. A triangle in T is a diagram of the form:

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

A morphism of triangles is triple (α, β, γ) such that this diagram commutes

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\Sigma\alpha} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma A' \end{array}$$

A morphism (α, β, γ) is an isomorphism of triangles if α, β and γ are isomorphisms.

Definition 4.0.2. A triple (T, Σ, Δ), where Δ is a collection of triangles, is a triangulated category if the following holds:

Tr.1 (a) Δ is closed under isomorphisms of triangles.
(b) For every object $A \in T$ there is a triangle in Δ

$$A \xrightarrow{id} A \longrightarrow 0 \longrightarrow \Sigma A$$

(c) For every morphism $f : A \to B$ there exists a triangle in Δ

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \Sigma A$$

Tr.2 For every triangle $(A, B, C, f, g, h) \in \Delta$. The triangles

$$\Sigma^{-1}C \xrightarrow{g} A \longrightarrow B \longrightarrow C$$

$$B \xrightarrow{f} C \longrightarrow \Sigma A \longrightarrow \Sigma B$$

are in Δ.

Tr.3 Let (A, B, C, f, g, h) and (A', B', C', f', g', h'), be two triangles in Δ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A' & \xrightarrow{f'} & B' \end{array}$$

commutes. Then there exists a morphism $\gamma : C \to C'$ such that (α, β, γ) is a morphism of triangles between (A, B, C, f, g, h) and (A', B', C', f', g', h').

Tr.4 Let (A, B, C, f, g, h), (A, B', C', f', g', h') and $(B, B', C'', \beta, g'', h'')$ be triangles in Δ, such that the diagram below commutes.

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \| \\ A & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A \\ \downarrow{\scriptstyle f} & & \| & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\ B & \xrightarrow{\beta} & B' & \xrightarrow{g''} & C'' & \xrightarrow{h''} & \Sigma B \\ & & & & \downarrow{\scriptstyle \Sigma g \circ h''} & & \\ & & & & \Sigma C & & \end{array}$$

Then $(C, C', C'', \gamma, \gamma', \gamma') \in \Delta$.

With this new type of categories, we want to introduce triangle functors. This being functors that preserves the triangulated structure of the categories. This means a triangulated functor is an additive functor that commutes with Σ, and preserves distinguished triangles.

Definition 4.0.3. Let $\mathcal{A}$ be an abelian category. A chain complex, often just called a complex, is a sequence of morphisms in $\mathcal{A}$,

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

such that $d_n \circ d_{n+1} = 0$

A morphism between complexes is a collection of morphisms $\{f_n : A_n \to B_n\}_{n\in\mathbb{Z}}$ such that $f_n \circ d_{n+1} = d_n \circ f_{n+1}$. The category of complexes in $\mathcal{A}$ is an abelian category.

Definition 4.0.4. A complex gives rise to an exact sequence

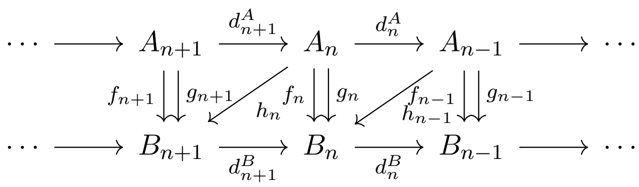
$$0 \longrightarrow \text{Im}\, d_{n+1} \xrightarrow{h_n} \ker d_n \longrightarrow \text{Coker}\, h_n$$

We then say the nth homology of $A_\bullet$, denoted $H_n = \text{Coker}\, h_n$.

Taking the nth homology gives rise to a functor $H_n : C(\mathcal{A}) \to \mathcal{A}$.

Definition 4.0.5. Let $f_\bullet : A_\bullet \to B_\bullet$ be a morphism of chain complexes. If $H_n(f_\bullet)$ is an isomorphism we say $f_\bullet$ is a quasi-isomorphism.

Definition 4.0.6. We say a two morphism in f and g in $C(\mathcal{A})$ are homotopic if there exists a collection of morphisms h_n such that
$f - g = d^B_{n+1} \circ h_n + h_{n+1} \circ d^A_n$



Being homotopic is an equivalence relation.

We will now introduce the homotopy category of $\mathcal{A}$, $K(\mathcal{A})$. The objects of this category are the chain complexes of $\mathcal{A}$, and the morphisms are morphisms of complexes, but up to homotopy. In other words
$\text{Hom}_{K(\mathcal{A})}(A, B) = \text{Hom}_{C(\mathcal{A})}(A, B)/\backsim$, where $\backsim$ is the homotopoy relation.

We can observe that the homotopy category $K(\mathcal{A})$ is a triangulated category $(K(\mathcal{A}), \Sigma, \Delta)$, where Σ is shifting the complex to the left. And Δ is the collection of all triangles of the form.

$$A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{\iota} Cone(f_\bullet) \xrightarrow{\pi} \Sigma A_\bullet$$

Now we will define $D(\mathcal{A})$, the derived category of $\mathcal{A}$. The main idea here is taking the homotopy category and inverting the quasi-isomorphisms. The quasi-isomorphism are inverted by localizing with regards to the set of quasi-isomorphisms. This gives rise to the localization functor $Q_{\mathcal{A}} : K^b(\mathcal{A}) \to D^b(\mathcal{A})$, such that $Q(f_\bullet)$ is an isomorphism whenever $f_\bullet$ is a quasi-isomorphism. Any functor that inverts quasi-isomorphisms factors through Q. It is exactly from this functor that $D^b(\mathcal{A})$ gets its triangulated structure. The distingushed triangles in $D^b(\mathcal{A})$ are exactly the triangles that are isomorphic to images of the distingushed triangles in $K^b(\mathcal{A})$.

Proposition 4.0.7. *Let A be a finite dimensional algebra with finite global dimension. Then $K^b({}_A\mathcal{P}) \cong D^b(A)$ as triangulated categories.*

Recall that we in section 3 called the full subcategory of projective A-modules $\mathcal{P}_A$.

Proposition 4.0.8. *Let A be a finite dimensional algebra. Then*
$\mathsf{Ext}^i_A(M, N) \cong \mathsf{Hom}_{D^b(A)}(M, \Sigma^i N)$

These propositions will in this thesis be regarded as facts and will not be proven here. For proof of these see the last chapter of Iversen[3].

Chapter 5

Tilting in the derived categoy

Throughout this section let A be a finite dimensional K-algebra with finite global dimension. And let ${}_AT$ be a tilting module. In this section we will prove that a tilting module gives rise to a triangle equivalence between $D^b(A)$ and $D^b(B)$ where B is the endomorphism algebra of T. The way we will be doing this is by showing that $K^b(\text{Add}\,T) \cong D^b(A)$ with the functor $L : K^b(\text{Add}\,T) \to D^b(A)$. This is the composition of the embedding functor of $K^b(\text{Add}\,T)$ into $K^b(\text{mod}\,A)$ and the localization functor $Q : K^b(\text{mod}\,A) \to D^b(A)$. This chapter is based of Happel[2].

Firstly we need to reformulate the third condition of a tilting module.

Lemma 5.0.1. *Let T be a tilting A module and P an indecomposible projective module. Then there exists an exact sequence,*

$$0 \longrightarrow P \longrightarrow T' \longrightarrow T'' \longrightarrow 0$$

with $T', T'' \in \text{Add}\,T$.

Proof. Since P is an indecomposible projective module we know there is another projective module Q such that $P \oplus Q \cong A$. Then since T is a tilting module we know from T.3, that there exists an short exact sequence

$$0 \longrightarrow A \xrightarrow{f} T' \longrightarrow T'' \longrightarrow 0$$

with $T', T'' \in \text{Add}\,T$. Letting $f = [f_1\, f_2] : P \oplus Q \to T'$ gives us these short exact sequences.

$$0 \longrightarrow P \xrightarrow{f_1} T' \longrightarrow \text{Coker}\, f_1 \xrightarrow{\pi_1} 0$$
$$0 \longrightarrow Q \xrightarrow{f_2} T' \longrightarrow \text{Coker}\, f_2 \xrightarrow{\pi_2} 0$$

Combining theses sequences to

$$0 \longrightarrow P \oplus Q \xrightarrow{[f_1\, f_2]} T' \longrightarrow \text{Coker}\, f_1 \oplus \text{Coker}\, f_2 \xrightarrow{[\pi_1\, pi_2]} 0$$

which is isomorphic to our first sequence. This means that Coker f_1 and Coker f_2 are isomorphic to modules in Add T. Thus there exists a short exact sequence like the one described by this lemma. □

With this reformulation of the third condition of a tilting module, we can now start showing the equivalence between $K^b(\text{Add}\, T)$ and $D^b(A)$. Firstly we will show the functor is dense.

Lemma 5.0.2. *Let $_AT$ be a tilting module. The functor $L : K^b(\text{Add}\, T) \to D^b(A)$ is dense.*

Proof. Since A has finite global dimension, we have that $K^b(_A\mathcal{P})$ is triangle equivalent to $D^b(A)$. Also since $_AT$ is a tilting module we know that there exists an exact sequence,

$$0 \longrightarrow P \longrightarrow T' \longrightarrow T'' \longrightarrow 0$$

with the kernel of $T' \to T'' = P$. Thus there exists a morphism of chain complexes:

$$\begin{array}{ccccccccc} & \cdots & \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & 0 & \longrightarrow \cdots \\ & & & \downarrow & & \downarrow & & \downarrow & \\ T_* = & \cdots & \longrightarrow & 0 & \longrightarrow & T' & \longrightarrow & T'' & \longrightarrow \cdots \end{array}$$

this is a quasi-isomorphism, meaning $P \cong_{D^b(A)} L(T_*)$. This gives us by shifting and taking direct sums and extensions, that $_A\mathcal{P}$ lies in the images of L. Thus L is dense since $_A\mathcal{P}$ generates $D^b(A)$, from proposition 4.0.7. □

Now We will show that the functor is full and faithfull. Although before that we need to recall that what the width of a complex $M^\bullet$ is. Say $M^i = 0$ for all $i < b$ and $i > a$, but $M^a, M^b \neq 0$. The width of $M^\bullet$, notated as follows $w(M^\bullet) = a - b + 1$.

Lemma 5.0.3. *The functor $L : K^b(\text{Add}\, T) \to D^b(A)$ is full and faithfull.*

Proof. Let $M_1^\bullet, M_2^\bullet \in K^b(\text{Add}\, T)$. Now we will do double induction on the widths of $M_1^\bullet$ and $M_2^\bullet$. First let $w(M_1^\bullet), w(M_2^\bullet) = 1$. We may assume $M_2^i = 0$ for $i < 0$, using Σ if needed. Since $w(M_1^\bullet) = 1$ and its lower bound is $i = 0$ we know there exists some $i \in \mathbb{Z}$, such that $M_1^\bullet = \Sigma^i M_1$ for some stalk complex

M_1. Thus if $i = 0$ we get $\mathrm{Hom}_{K^b(\mathrm{Add}\,T)}(M_1^\bullet, M_2^\bullet) = \mathrm{Hom}_{D^b(A)}(M_1^\bullet, M_2^\bullet)$. Otherwise $\mathrm{Hom}_{K^b(\mathrm{Add}\,T)}(M_1^\bullet, M_2^\bullet) = 0$ and $\mathrm{Hom}_{D^b(A)}(M_1^\bullet, M_2^\bullet) = 0$ since $\mathrm{Ext}^i_A(M_1, M_2) = 0$. Now assume this holds for $w(M_1^\bullet) = 1$ and $w(M_2^\bullet) = r - 1$. So now let $w(M_1^\bullet) = 1$ and $w(M_2^\bullet) = r$, then we can consider the triangle.

$$\Sigma^{-1} M^2_\bullet \longrightarrow M_2^* \longrightarrow M_2^\bullet \longrightarrow M_2^\bullet$$

Where M_2^* is the truncated complex of $M_2^\bullet$. Meaning we make the $M_2^0 = 0$. Now applying both $\mathrm{Hom}_{K^b(\mathrm{Add}\,T)}(M_1^\bullet, -)$ and $\mathrm{Hom}_{D^b(A)}(M_1^\bullet, -)$ we get, from the induction hypothesis and the 5-lemma, that

$$\mathrm{Hom}_{K^b(\mathrm{Add}\,T)}(M_1^\bullet, M_2^\bullet) \cong \mathrm{Hom}_{D^b(A)}(M_1^\bullet, M_2^\bullet).$$

Now the remaining part of the double induction is the dual argument.

□

We now have a triangle equivalence between $K^b(\mathrm{Add}\,T)$ and $D^b(A)$. We know from 4.0.7 that if B has finite global dimension that we get a triangle equivalence between $K^b(\mathcal{P}_B) \cong D^b(B)$. Thus we will need show that B has finite global dimension. In fact we will even be able to bound its global dimension. Although to show this we are going to need two lemmas first.

Firstly we will show a lemma connecting the projective dimensions in a short exact sequence.

Lemma 5.0.4. *Let*

$$0 \longrightarrow X \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence. And let $\mathrm{pd}\, M < \mathrm{pd}\, N$*, then* $\mathrm{pd}\, X \leq \mathrm{pd}\, N - 1$.

Proof. For this proof let us call $\mathrm{pd}\, N = n$. Now let us start by taking the short exact sequence and applying $\mathrm{Hom}_A(-, Y)$. This yields the exact sequence

$$\cdots \longrightarrow 0 = \mathrm{Ext}^n_A(M, Y) \longrightarrow \mathrm{Ext}^n_A(X, Y) \longrightarrow \mathrm{Ext}^{n+1}_A(N, Y) = 0 \longrightarrow \cdots.$$

This gives us that $\mathrm{Ext}^n_A(X, Y) = 0$, meaning that $\mathrm{pd}\, X \leq N - 1$. □

This lemma is a specialization of a more general result, giving a bound for X, M and N. Though in this thesis we only need this case.

Lemma 5.0.5. *Let* T_A *be a tilting module and* $M \in \mathcal{T}(T)$*. Then* $\mathrm{pd}\,\mathrm{Hom}_A(T, M) \leq \mathrm{pd}\, M$.

Proof. We will do induction on pd M. To start if pd $M = 0$, then M is projective, and thus Ext-projective in $\mathcal{T}(T)$ and is by theorem 3.0.5 in Add T. Thus $\text{Hom}_A(T, M)$ is projective in B.
Now let pd $M \leq 1$, then using theorem 3.0.5 we have a short exact sequence

$$0 \longrightarrow X \longrightarrow T' \longrightarrow M \longrightarrow 0$$

with $X \in \mathcal{T}(T)$ and $T' \in \text{Add}\, T$. Then applying $\text{Hom}_A(-, T)|_{\mathcal{T}(T)}$ gives us this exact sequence,

$$0 = \text{Ext}^1_A(T', -)|_{\mathcal{T}(\mathcal{T})} \longrightarrow \text{Ext}^1_A(X, -)|_{\mathcal{T}(\mathcal{T})} \longrightarrow \text{Ext}^2_A(M, -)|_{\mathcal{T}(\mathcal{T})} = 0$$

Then we observe that X is Ext-projective in $\mathcal{T}(T)$, and thus $X \in \text{Add}\, T$. Thus by using $\text{Hom}_A(T, -)$ to the sequence, we get this projective resolution of $\text{Hom}_A(T, M)$.

$$0 \longrightarrow \text{Hom}_A(T, L) \longrightarrow \text{Hom}_A(T, T') \longrightarrow \text{Hom}_A(T, M) \longrightarrow 0$$

Giving us pd $\text{Hom}_A(T, M) \leq 1$.
Now for the last part of the proof, let pd $M \geq 2$. We then know, since pd $T' \leq 1$ that pd $X \leq$ pd $M - 1$. Giving us, from the induction that
pd $\text{Hom}_A(T, X) \leq$ pd $M - 1$. Hence from lemma 5.0.5 the third sequence gives us pd $\text{Hom}_A(T, M) \leq 1 +$ pd $\text{Hom}_A(T, X)$. Thus

$$\text{pd}\, \text{Hom}_A(T, M) \leq 1 + \text{pd}\, \text{Hom}_A(T, X) \leq 1 + \text{pd}\, X \leq \text{pd}\, M.$$

□

Now we can use the bound given by lemma 5.0.6 to bound the global dimension of B.

Lemma 5.0.6. *Let T_A be a tilting module of an algebra A with finite global dimension. Then $B = \text{End}\, T$ also has finite global dimension.*

Proof. Let N be any B-module, then there exists a short exact sequence

$$0 \longrightarrow Y \longrightarrow P \longrightarrow N \longrightarrow 0$$

such that P is projective. Since P is projective, we have that $P \in \mathcal{Y}(T)$, thus $Y \in \mathcal{Y}(T)$. Now since $Y \in \mathcal{Y}(T)$ we know there exists some $M \in \mathcal{T}(T)$ such that $\text{Hom}_A(T, M) \cong Y$. Thus from the previous lemma we know that
pd $Y \leq$ pd M. Which now gives us that

$$\text{pd}\, X \leq 1 + \text{pd}\, Y \leq 1 + \text{pd}\, M \leq 1 + \text{gl. dim}\, A.$$

Finally giving us that gl. dim $B \leq 1 +$ gl. dim A. □

From lemma 3.1.1 we have the equivalence $\text{Hom}_A(T,-)$ between $\text{Add}\,T$ and $\mathcal{P}_B$. This equivalence extends to a triangle equivalence between $K^b(\text{Add}\,T)$ and $K^b(\mathcal{P}_B)$. Since $\text{Hom}_A(T,-)$ is an additive functor, it preserves the homotopy relation and also sends distinguished triangles to distinguished triangles. This is because the distinguished triangles in the homotopy category is of the form

$$T_0 \xrightarrow{f} T_1 \xrightarrow{\iota} Cone(f) \xrightarrow{\pi} \Sigma T_0$$

This gets sent to

$$P_0 \xrightarrow{\text{Hom}(T,f)} P_1 \xrightarrow{\iota} \Sigma P_0 \oplus P_1 \xrightarrow{\pi} \Sigma P_0$$

Where $P_0 \cong \text{Hom}_A(T,T_0)$ and $P_1 \cong \text{Hom}_A(T,T_1)$. This is isomorphic to a triangle in $K^b(\mathcal{P}_B)$.

Now we have all the ingredients to show a tilting module ${}_AT$ induces a triangle equivalence between $D^b(A)$ and $D^b(B)$, for $B = \text{End}\,T$. That is whenever A is a finite dimensional algebra of finite global dimension.

Theorem 5.0.7. *Let ${}_AT$ be a tilting module and $B = \text{End}_A\,T$. Then the derived categories $D^b(A)$ and $D^b(B)$ are triangle equivalent.*

Proof. From lemma 5.0.2 and 5.0.3 we know that $K^b(\text{Add}\,T)$ and $D^b(A)$ are triangle equivalent. Moreover the equivalence $\text{Hom}_A(T,-) : \text{Add}\,T \to \mathcal{P}_B$ extends to a triangle equivalence between $K^b(\text{Add}_A\,T)$ and $K^b(\mathcal{P}_B)$. Finally since B has finite global dimension we know that $K^b(\mathcal{P}_B)$ and $D^b(B)$ are triangle equivalent. Thus giving us $D^b(A) \cong D^b(B)$. □

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