Daria Barjaktarevic

The Spanier-Whitehead category

Bachelor's thesis in Matematiske fag
Supervisor: Marius Thaule
Co-supervisor: Sebastian H. Martensen
May 2022

Bachelor's thesis

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1 Introduction

Stable homotopy theory and the stable homotopy category were originally motivated by stable phenomena that appear in homotopy theory. We have the Freudenthal Suspension Theorem which says that, given some connectivity restriction, we have isomorphisms of homotopy groups induced by the suspension. Furthermore, the suspension also induces isomorphisms in homology. Hence the initial motivation was to create a category in which these phenomena could be isolated. Afterwards it has been showed that stable homotopy theory is connected to several, often surprising, areas of mathemathics, which has increased it's motivation even further.

The Spanier–Whitehead category is one attempt to form the stable homotopy category. The way it is constructed is very direct and simple, and it is therefore relatively easy to understand, but this simplicity has its price. It lacks several of the properties we would like the stable homotopy category to have. Because of this, many books and articles use it to motivate other models for a stable homotopy category, but few devote time to the properties they claim that the Spanier–Whitehead category does have, for instance its triangulation.

One place where you can find an introduction to the Spanier–Whitehead category is in "Spectra and the Steenrod algebra" by Margolis [7]. However, even here the details are bygone. Margolis' proof of the triangulation on Spanier–Whitehead is largely based on properties of the homotopy category of CW-complexes, which are not proven. In addition, the octahedron axiom of triangulated categories is not mentioned in Margolis' proof. Therefore, this thesis is devoted to filling in the holes of the first chapter of [7], and we here prove that the Spanier–Whitehead category is a triangulated category.

We start by reviewing CW-complexes and their properties, after which we move on to the homotopy category of CW-complexes. In this section we define several essentials, like the suspension and the mapping cone, which are the basis on which we make triangles in both the homotopy category of CW-complexes and in Spanier–Whitehead.

Before we dive into the triangulation, we take a closer look at cofibration categories and their properties, and we show that the category of CW-complexes is a cofibration category. This section, and the reason for including it, is largely inspired by Schwede, and his article about topological triangulated categories [10], in which he proves that in fact all homotopy categories of a stable cofibration category are triangulated. We will not give such a generalized proof, but we will use the properties of cofibration categories in our proof.

After this we review the definition of a triangulated category, and make a triangulated structure on the homotopy category of CW-complexes. This will not make it a triangulated category, but we will see that this structure can be almost directly included into, and completed in, the Spanier–Whitehead category. This will prove that Spanier–

Whitehead is a triangulated category.

Conclusively we look at what Spanier–Whitehead lacks, and how this motivates further work with the stable homotopy category.

2 CW-complexes

2.1 CW-complexes and their properties

CW-complexes are the spaces we will restrict ourselves to throughout this thesis. They are a certain type of "nice" topological spaces, that is, they are compactly generated Hausdorff spaces. They are much used in algebraic topology as they are constructed from spheres and disks, spaces we have a lot of knowledge about topologically.

Definition 2.1. A **CW-complex** is a topological space X equipped with a filtration $\emptyset = X_{-1} \subseteq X_0 \subseteq ... \subseteq X_n \subseteq ... \subseteq \bigcup_n X_n = X$, where each X_n is a subspace of X_{n+1}. For each n we have an index set Γ_n and attaching maps $\{f_\alpha\}_{\alpha\in\Gamma_n}$ along with pushouts

$$\begin{array}{ccc} \coprod_{\alpha\in\Gamma_n} S^{n-1} & \longrightarrow & \coprod_{\alpha\in\Gamma_n} D^n \\ \Big\downarrow f_\alpha & & \Big\downarrow e_\alpha \\ X_{n-1} & \longrightarrow & X_n \end{array}$$

where the maps $\{e_\alpha\}_{\alpha\in\Gamma_n}$ are called the characteristic maps of the n-cells of X. An n-cell is an n-disk together with its characteristic map. A subset U of X is open if and only if $U \cap X_n$ is open in X_n for all n. The cell complex is called **pointed** if it has a basepoint.

Remark 2.2. We say that a CW-complex is **finite** if for some n we have $X_i = X_n$ for all $i > n$ and if Γ_n is a finite set for all n. This number n is then said to be the **dimension** of X. The subspace X_n of X is called the **n-skeleton** of the CW-complex.

We think of a CW-complex as a space inductively built from a set of points X_0 by attaching n-disks to X_{n-1}. The attaching maps tell us how the boundary of each n-disk is mapped into X_{n-1}, while the characteristic maps tell us how the disks map into X_n.

We say that we build a space as a cell complex, which means that we make a CW-structure for a space. But the same space would just be a topological space if no CW-structure was specified. For instance, S^1 with your preferred topology is a topological space, and it can be realized as a CW-complex, but it is not a CW-complex unless a filtration and attaching maps are specified. We will see that the filtration and attaching maps are not unique. There are, in fact, often several ways of realizing a space as a CW-complex.

Example 2.3. The n-sphere can be built from a single point $X_0 = \{*\}$ and one n-cell with the pushout

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & S^n \end{array}$$

where the attaching map collapses the boundary of D^n to the point. However, we can also build the n-sphere inductively from the $(n-1)$-sphere, like this:

$$\begin{array}{ccc} S^{n-1} \coprod S^{n-1} & \longrightarrow & D^n \coprod D^n \\ \downarrow & & \downarrow \\ S^{n-1} & \longrightarrow & S^n \end{array}$$

where $S^{n-1} \coprod S^{n-1} \longrightarrow S^{n-1}$ is the identity on their respective circles. So we map the two boundaries onto S^{n-1} such that the n-cells become the upper and lower hemisphere of S^n.

Example 2.4. The torus can be made from a 0-cell by attaching two 1-cells a and b to the basepoint like in Figure 5, and then attaching a 2-cell where its boundary has an identification like in Example 2.4.

The 2-cell is therefore attached to the circles along the loop $aba^{-1}b^{-1}$.

Definition 2.5. Let $f : X \longrightarrow Y$ be a map between two CW-complexes (X, x_0) and (Y, y_0), where X has filtration $\bigcup_n X_n$ and Y has filtration $\bigcup_n Y_n$. Then f is **cellular** if we have $f(X_n) \subseteq Y_n$ for all $n \in \mathbb{N}$. It is called **basepoint preserving** if $f(x_0) = y_0$.

To formalize everything, we define a category of based CW-complexes, which will be the category we work with from here on.

Definition 2.6. Let $\mathscr{CW}$ be the category with finite pointed CW-complexes as objects, and the morphisms are the cellular, basepoint preserving.

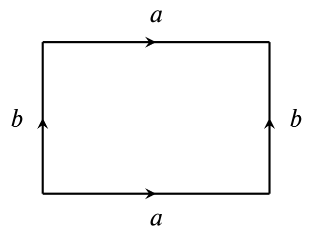


Figure 1: The identification on the 2-cell in the CW-complex T^2.

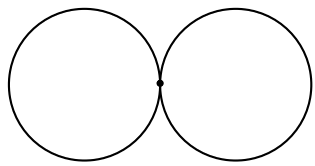


Figure 2: The 1-skeleton of T^2.

Remark 2.7. We often omit the basepoints in notation, and thus denote the sets of maps between two pointed CW-complexes as $\mathscr{CW}(X, Y)$.

We will be using pushouts a lot throughtout this thesis, and therefore include the following theorem.

Theorem 2.8. *The category $\mathscr{CW}$ is closed under pushouts.*

Definition 2.9. For X and Y in $\mathscr{CW}$ we define the **wedge** $X \vee Y$ as the subspace $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ of $X \times Y$, where $\{x_0\} \times \{y_0\}$ is the basepoint.

Remark 2.10. Geometrically, this looks like gluing two spaces together at their basepoints. See Figure 3 for an example.

In non-pointed topological spaces and non-pointed CW-complexes one can easily deduce that the coproduct is the disjoint union, but when we consider pointed spaces the coproduct of course needs to have a cannonical basepoint. The wedge gives us this.

Proposition 2.11. *The wedge is the categorical coproduct in $\mathscr{CW}$.*

Proof. Let X_1, X_2 and Y be pointed CW-complexes with basepoints x_1, x_2 and y, respectively. Let

$$f_1 : X_1 \longrightarrow Y, f_2 : X_2 \longrightarrow Y$$

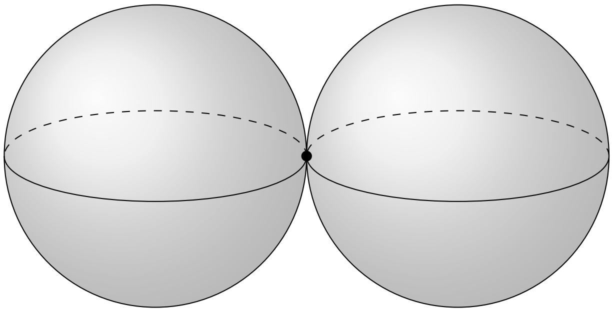
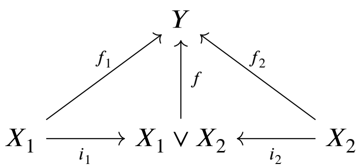


Figure 3: The wedge of two 2-spheres. The black bullet marks the new basepoint.

be two pointed maps. Then $f_1(x_1) = f_2(x_2) = y$, and therefore $f = (f_1, f_2)$, which is given by f_1 when restricted to the subspace X_1 in the wedge and f_2 when restricted to X_2, is well-defined, and makes the following diagram commute:



As both f_1 and f_2 preserve basepoints, it follows that f is unique.

□

We now use the wedge to construct a new space, the *smash product*.

Definition 2.12. The **smash product** of two pointed CW-complexes (X, x_0) and (Y, y_0) is given by

$$X \wedge Y = \frac{X \times Y}{X \vee Y}.$$

Here the wedge is collapsed to the basepoint.

Proposition 2.13. *The smash product is functorial in each variable on $\mathscr{CW}$.*

Proof. Let (V, v_0) be a based CW-complex. We prove the statement only for $V \wedge -$ as the proofs are similar.

Let (X, x_0) be another based CW-complex. Elements in $V \wedge X$ can be described with classes $[v, x]$, so for a map $f : X \longrightarrow Y$ we define

$$\begin{aligned} V \wedge f : V \wedge X &\longrightarrow V \wedge Y \\ [v, x] &\mapsto [v, f(x)] \end{aligned}$$

We need to check that this is well-defined. Let $[v, x]$ and $[v', x']$ be two elements such that $[v, x] = [v', x']$. Looking at the definition of the smash product we see that this happens in three cases. Either $v = v'$ and $x = x'$, or $v = v' = v_0$, or $x = x' = x_0$. That the functor is well-defined in the first case is trivial. For the second case we can note that classes $[v_0, f(x)]$ will all be identified as the basepoint in $V \wedge Y$, so we get $[v, f(x)] = [v', f(x')]$ in this case as well. The same argument can be used in the third case, as f is basepoint preserving. Hence, $V \wedge f$ is well-defined.

Now to functoriality: for two maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ in $\mathscr{CW}$ we get

$$(V \wedge g) \circ (V \wedge f)[v, x] = (V \wedge g)[v, f(x)] = [v, (g \circ f)(x)] = V \wedge (g \circ f)[v, x],$$

so

$$(V \wedge g) \circ (V \wedge f) = V \wedge (g \circ f).$$

For $id_X : X \longrightarrow X$ we get

$$V \wedge id_X[v, x] = [v, x],$$

so $V \wedge id_X = id_{V \wedge X}$. This proves functoriality. □

In general, a good geometric understanding of the smash product is difficult to obtain, but for certain spaces it can be visualized.

Example 2.14. The smash product of two circles is the 2-sphere: the subspace $S^1 \vee S^1$ of the torus $T^2 = S^1 \times S^1$ is the 1-skeleton of Example 2.4. Hence, collapsing this to a point gives us the CW-structure of a 2-sphere.

Theorem 2.15. *For X, Y and $Z \in \mathscr{CW}$ the smash product has natural isomorphisms (homeomorphisms):*

(i) $X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$

(ii) $X \wedge Y \cong Y \wedge X$

(iii) $S^0 \wedge X \cong X \cong X \wedge S^0$

Proof. We prove only (ii) and (iii). For (i) see 5.8.2 in [3].
(ii) Since the wedge is the coproduct, we have $X \vee Y \cong Y \vee X$ and so we get

$$X \wedge Y = \frac{X \times Y}{X \vee Y} \cong \frac{Y \times X}{Y \vee X} = Y \wedge X.$$

(iii) Since S^0 is two disjoint points where one of them is the basepoint, we get

$$S^0 \wedge X = \frac{S^0 \times X}{S^0 \vee X} \cong \frac{X \amalg X}{\{*\} \amalg X} \cong X.$$

The naturality of these homeomorphisms follows from the wedge and the cartesian product being functorial. □

As mentioned, smashing with simple spaces has simple geometric interpretations, and we use this to define two important functors on $\mathscr{CW}$.

Definition 2.16. For $X \in \mathscr{CW}$ we can build two new spaces with the smash product:

- The **suspension** of X, $\Sigma X = S^1 \wedge X$.
- The **cone** of X, $CX = I \wedge X$, where $I = [0, 1]$ has the basepoint $\{0\}$.

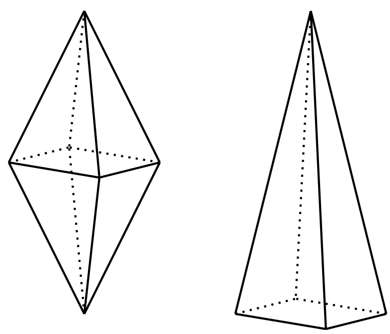


Figure 4: The suspension and cone of a square.

Remark 2.17. The suspension as defined here is often called the *reduced suspension* for based topological spaces, while the suspension is defined as a different, but similar construction on unbased spaces. However, in $\mathscr{CW}$, the two suspensions are homotopy equivalent under free homotopies, that is, homotopies that don't preserve basepoints.

Proposition 2.18. *As a direct consequence of Proposition 2.13 we have that the suspension and the cone are functorial on $\mathscr{CW}$.*

Remark 2.19. There are several equivalent ways of defining the suspension, and they are suitable for different purposes. Firstly, we can define it explicitly as a quotient space:

$$\Sigma X \cong \frac{I \times X}{\partial I \times X \cup I \times \{x_0\}}.$$

We can also define it as the pushout

$$\begin{array}{ccc} X & \xrightarrow{i_1^X} & CX \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

in which the left map sends all of X to the basepoint, and the above map is the inclusion of X as a subspace in CX at $t = 1$. We see from the definition of CX that $X \cong \{[1, x] : x \in X\} \subseteq CX$.

One of the most important properties of the suspension is that it takes n-spheres to $(n+1)$-spheres.

Proposition 2.20. *For all $n \in \mathbb{N}$ we have $\Sigma S^n \cong S^{n+1}$.*

We need the following technical lemma to prove this.

Lemma 2.21. *If X is a topological space and α is an equivalence relation on X, then there is a homeomorphism*

$$\frac{I \times X}{\sim} \longrightarrow I \times \frac{X}{\alpha}$$

where $\sim$ is the identity relation on I, i.e. it does nothing on I, and the relation α on X.

See [12, pp. 2] for a proof.

Proof of Proposition 2.20. Let s_0 denote the basepoint in S^n. Observe that the n-sphere can be constructed as the quotient $I^n/\partial I^n$, in which $s_0 = [t]$ for all $t \in \partial I^n$. We use Remark 2.19 to see that

$$\Sigma S^n \cong \frac{I \times S^n}{\partial I \times S^n \cup I \times \{s_0\}} \cong \frac{I \times \frac{I^n}{\partial I^n}}{\partial I \times \frac{I^n}{\partial I^n} \cup I \times \frac{\partial I^n}{\partial I^n}}.$$

Let $\sim$ denote the equivalence relation on $I \times I^n$ that is the identity relation on I, and is the quotient by ∂I^n on I^n. Then we can use Lemma 2.21 to see that the equivalence relation $\sim$ gives us a homeomorphism

$$\frac{I \times \frac{I^n}{\partial I^n}}{\partial I \times \frac{I^n}{\partial I^n} \cup I \times \frac{\partial I^n}{\partial I^n}} \cong \frac{I \times I^n}{\partial I \times I^n \cup I \times \partial I^n \cup \{0\} \times \partial I^n} = \frac{I \times I^n}{\partial I \times I^n \cup I \times \partial I^n}.$$

Using that the boundary of the n-cube can be written as $\partial I \times I^n \cup I \times \partial I^n$, we get

$$\Sigma S^n \cong \frac{I \times I^n}{\partial I \times I^n \cup I \times \partial I^n} \cong \frac{I^{n+1}}{\partial I^{n+1}} \cong S^{n+1}. \quad \square$$

Lastly, we present a technical lemma about pushouts that will be very useful in several proofs.

Lemma 2.22. *In any category $\mathscr{C}$, the following holds: If we have a commuting diagram of objects in $\mathscr{C}$ like the following*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' \end{array}$$

where the left square is a pushout, then the right square is a pushout if and only if the outer square is a pushout.

The similar result for pullbacks along with a proof can be found as 7.8.7 in [9]. The proof of Lemma 2.22 is dual to this.

2.2 The homotopy category of CW-complexes

Homotopy theory is, after all, the basis for the construction of a stable homotopy category, and although many of the terms probably are known to the reader, we review the most important ones in order to have a common base. The most prominent difference from the typical first introduction to homotopy theory is that the homotopies we consider will take basepoints into account.

Definition 2.23. For two maps $f, g \in \mathscr{CW}((X, x_0), (Y, y_0))$, we say that f is **homotopic** to g if there exists a continuous map

$$H : X \times I \longrightarrow Y$$

such that

$$H(x, 0) = f(x)$$
$$H(x, 1) = g(x)$$

for all $x \in X$. If we in addition have that $H(x_0, t) = y_0$ for all $t \in I$, we say that the homotopy is **based**. If the homotopy is not based we call it **free**.

Remark 2.24. We write $f \simeq g$ if there is a based homotopy between f and g. It is well known that this unbased homotopy is an equivalence relation.See for instance proposition 14.1.2 in [11]. The transfer from the unbased to the based case is trivial.

Definition 2.25. We define a new category of topological spaces, namely the homotopy category of CW-complexes, denoted $\mathscr{CW}_h$. The objects are the same as in $\mathscr{CW}$, but the maps are the equivalence classes $\mathscr{CW}(X, Y)/\simeq$ with respect to the homotopy relation for based homotopy. We denote the set of based homotopy classes of maps in $\mathscr{CW}_h((X, x_0, Y, y_0)) = [X, Y]$.

Among the most important concepts of homotopy theory and algebraic topology in general are the homotopy groups, which are defined to be $\pi_n(X, x_0) := [S^n, X]$ for $n > 0$, and set of path components of X for $n = 0$. The homotopy groups are actually groups for $n \geq 1$, and even abelian groups for $n \geq 2$. See chapter 2 in [12].

Definition 2.26. A based CW-complex (X, x_0) is ***n*-connected** if we have $\pi_i(X, x_0) = 0$ for all $i \leq n$.

The following theorem plays a big role in the history of stable homotopy categories. It gives us a connection between the homotopy classes of maps and the suspension, and as a consequence the homotopy groups and the suspension.

Theorem 2.27 (Freudenthal suspension theorem)**.** *Suppose that Y is a based n-connected CW-complex, and X a based CW-complex with dimension $\leq 2n$. Then the map*

$$\Sigma_* : [X, Y] \longrightarrow [\Sigma X, \Sigma Y]$$

of homotopy classes induced by the suspension is a bijection. The map is a surjection if X is of dimension $2n + 1$.

This theorem along with a proof can be found as corollary 3.2.3 in [6].

Remark 2.28. We know that Σ is well-defined on $\mathscr{CW}$, but we need to check that is also is well-defined on $\mathscr{CW}_h$ as well. That is, for two maps $f, g : X \longrightarrow Y$ with $f \simeq g$, we must have $\Sigma f \simeq \Sigma g$. Assuming we have a homotopy $H : X \times I \longrightarrow Y$ such that

$$H(x,0) = f(x),$$
$$H(x,1) = g(x),$$

we can define a homotopy $\tilde{H} : \Sigma X \times I \longrightarrow \Sigma Y$ by

$$\tilde{H}([s,x],t) = [s, H(x,t)].$$

This is continuous, and we have

$$\tilde{H}([s,x],0) = [s, f(x)]$$

and

$$\tilde{H}([s,x],1) = [s, g(x)].$$

So $\Sigma f \simeq \Sigma g$.

Corollary 2.29. *Suppose that X is a n-connected based CW-complex. Then the induced suspension map $\Sigma_* : \pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$ is a bijection for $i = 0$, an isomorphism for $i \leq 2n$, and a surjection for $i = 2n + 1$.*

Remark 2.30. We have that the suspension takes the i-sphere to the $(i+1)$-sphere, hence a class $[f] \in [S^i, X]$ is sent to $[\Sigma f] \in [\Sigma S^i, \Sigma X] = [S^{i+1}, \Sigma X]$. Since an n-sphere has dimension n, we see that the corollary follows from Theorem 2.27, where we let $X = S^n$.

From this we see that the suspension of a space increases the connectivity by 1: for an n-connected space X, we have $\pi_i(X) = 0$ for $i \leq n$, and the induced map Σ_* is an isomorphism $\pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$ for $i \leq 2n$, so $\pi_i(\Sigma X) = 0$ for $i \leq n+1$. But then $\pi_i(\Sigma X) \longrightarrow \pi_{i+1}(\Sigma^2 X)$ is an isomorphism for $i \leq 2(n+1)$. We see that we can continue this process, and since the level at which the induced suspension map is an isomorphism grows faster than the connectivity of X under the suspension, the homotopy groups will eventually stabilize. This leads to the following definition:

Definition 2.31. The nth **stable homotopy group** of X is

$$\pi_n^s(X) := \operatorname*{colim}_{r \to \infty} \pi_{n+r}(\Sigma^r X).$$

While homotopy groups give us a lot of interesting information about the spaces we want to study, the stable homotopy groups give another insight, and in a stable homotopy category we wish to make a framework in which these groups more easily can be computed. An important subproblem is the stable homotopy groups of spheres. Some of them have been computed, but while we for instance do have $\pi_n^s(S^n) \cong \mathbb{Z}$ for all n, the general pattern is unknown.

Lemma 2.32. *For any CW-complex X, ΣX is path-connected.*

Proof. From Remark 2.19 we have that $\Sigma X = I \times X/\sim$, which means we can describe elements in ΣX as equivalence classes $[t, x]$. Note that in ΣX all of $\{1\} \times X$ is collapsed to the basepoint.

Let $[t_1, x_1]$ and $[t_2, x_2]$ be two points in ΣX. We define two paths $p_1, p_2 : I \longrightarrow \Sigma X$

$$p_1(s) = [t_1(1-s) + s, x_1]$$
$$p_2(s) = [st_2 + (1-s), x_2],$$

where p_1 is a path from $[t_1, x_1]$ to $[1, x_1]$, and p_2 is a path from $[1, x_2]$ to $[t_2, x_2]$. Since $[1, x_1] = [1, x_2]$ we can concatenate these paths, and the concatenation is a path from $[t_1, x_1]$ to $[t_2, x_2]$. Hence, ΣX is path-connected. □

We introduce spaces which have certain convenient properties in $\mathscr{CW}_h$, and which we will use when constructing the Spanier–Whitehead category.

Definition 2.33. For a map $f \in \mathscr{CW}(X, Y)$ we define the **mapping cone** of f as the pushout

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle i_1^X}\downarrow & & \downarrow{\scriptstyle i(f)} \\ CX & \xrightarrow[\;i_{CX}\;]{} & C(f) \end{array}$$

where i_1^X includes X as a subspace in CX at $t = 1$. Equivalently, it can be defined as the quotient space $CX \sqcup Y/\sim$, where $[1, x]_{CX} \sim y$ for $y \in f(X)$, i.e., we glue the base of the cone of X to Y along its image under f. We may denote equivalence classes in $C(f)$ by $[1, x]_{C(f)}$ if $x \in X$, and $[y]_{C(f)}$ if $y \in Y$.

The map $i(f)$ includes Y into $C(f)$ as a subspace, and likewise CX is included as a subspace in $C(f)$ under i_{CX}. Note that CX itself is contractible; since $[0, x]$ is identified with the basepoint in CX for all x, we can slide all points $[t, x]$ down to the basepoint along t. The mapping cone, however, is not necessarily contractible.

Example 2.34. For the simple map $i: S^1 \hookrightarrow D^2$, the inclusion of the circle as the boundary of the 2-disk, the mapping cone looks like an actual cone. Here we see that $C(i) \simeq S^2$, which is not contractible.

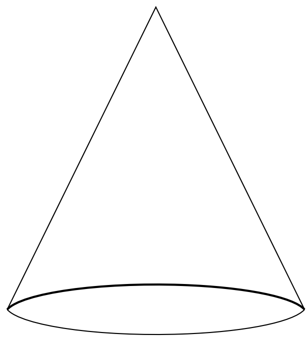


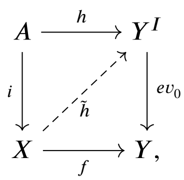
Figure 5: The mapping cone of $S^1 \hookrightarrow D^2$.

Definition 2.35. For a map $f: X \longrightarrow Y$ in $\mathscr{CW}(X,Y)$ we define the **mapping cylinder of** f, M_f, as the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_1}\downarrow & & \downarrow \\ X \times I & \longrightarrow & M_f \end{array}$$

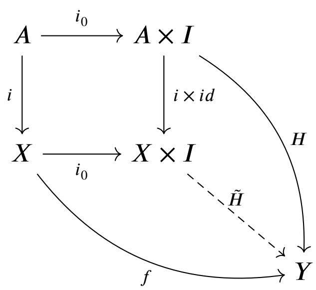
where i_1 is the inclusion of X into $X \times I$ at $\{1\}$. It is similar to the mapping cone, in the sense that we are gluing X to Y along its image, except now we have a "cylinder of X" instead of a cone. This makes $M_f \simeq Y$; M_f deformation retracts onto Y by sliding all points coming from $X \times I$, $[x,t]$, to their image in Y along t.

Definition 2.36. Let $\mathscr{E}$ be a class of topological spaces. A map $i: A \longrightarrow X$ has the **homotopy extension property** if the following extension problem has a solution: for every $Y \in \mathscr{E}$, for all f and h such that the solid part of the following diagram commutes



there is a map $\tilde{h}$ making the diagram commmute. The map ev_0 is the evaluation at 0, i.e., an element in F in $Y^I = \mathscr{CW}(I, Y)$ is sent to $F(0)$.

Remark 2.37. By the adjunction $\mathscr{CW}(X \times Y, Z) \cong \mathscr{CW}(X, Z^Y)$, where $Z^Y = \mathscr{CW}(Y, Z)$ is given the compact-open topology, the homotopy extension property for a map $i: A \longrightarrow X$ is equivalent to the following universal property: for all f and H such that the solid part of the diagram commutes, there exists an extension $\tilde{H}$ making the following diagram commute.



Definition 2.38. If we let $\mathscr{E} = \mathrm{Ob}(\mathscr{CW})$ in Definition 2.36 we call the map i a **cofibration**.

Example 2.39. The map i_1^X in Definition 2.33 is a cofibration. The map $j: X \longrightarrow M_f$ is a cofibration, where M_f is the mapping cylinder for $f: X \longrightarrow Y$. See [1, pp. 76] for a proof of the former.

We include the following lemmas for technical reasons.

Lemma 2.40. *If $i: A \longrightarrow B$ is a cofibration, then i is injective and homeomorphic into its image.*

This lemma can be found as Proposition 4H.1 in [5].

Lemma 2.41. *If $i: X \longrightarrow Y$ is an inclusion of connected CW-complexes, then X is a deformation retract of Y.*

This lemma can be found as Theorem 4.5 in [5].

3 Cofibration categories

We just defined a cofibration in $\mathscr{CW}_h$, but we now approach cofibrations from a categorical standpoint. We will see, however, that the two definitions coincide, and the properties from a cofibration category will be very useful later.

Definition 3.1. Let $\mathscr{C}$ be a category, and let there be two classes of maps in $\mathscr{C}$ called weak equivalences and cofibrations. The category $\mathscr{C}$ is a **cofibration category** if it satisfies the following axioms:

(CF1) All isomorphisms in $\mathscr{C}$ are weak equivalences and cofibrations. There exists an initial object $I \in \mathscr{C}$ such that for all $C \in \mathscr{C}$ the unique morphism $I \longrightarrow C$ is a cofibration. Cofibrations are stable under composition.

(CF2) Given two composable morphisms f and g, if two of f, g and $g \circ f$ are weak eqivalences, the last one is as well.

(CF3) Let $i : A \longrightarrow B$ be a cofibration. Then for any morphism $f : A \longrightarrow C$ there is a pushout square

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \longrightarrow & P \end{array} \tag{1}$$

in $\mathscr{C}$. The morphism j is a cofibration. If i is in addition a weak equivalence, then j is also a weak equivalence.

(CF4) All morphisms $f : A \longrightarrow B$ can be factored to $f = g \circ i$, where i is a cofibration and g is a weak equivalence.

The following lemma gives a particularly nice relation between pushouts and weak equivalences in a cofibration category.

Lemma 3.2 (Gluing lemma)**.** *Assume we have the following diagram in a cofibration category*

$$\begin{array}{ccccc} A & \xleftarrow{i} & B & \xrightarrow{f'} & C \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ A' & \xleftarrow{i'} & B' & \xrightarrow{f} & C' \end{array}$$

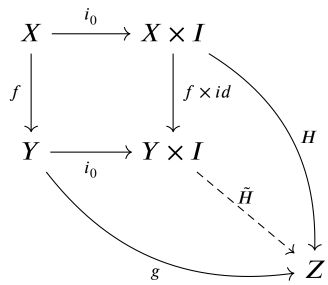
where i and i' are cofibrations, and all the vertical maps are weak equivalences. Then the induced morphism on pushouts, $A \sqcup_B C \longrightarrow A' \sqcup_{B'} C'$, is a weak equivalence.

The proof of the gluing lemma is not the most difficult, but it is rather long and technical, and we therefore leave it as a reference. See [8, pp. 7-10] for a proof.

Theorem 3.3. *The category $\mathscr{CW}$ is a cofibration category with weak equivalences the based homotopy equivalences of spaces, and cofibrations the maps in Definition 2.38.*

Remark 3.4. We could demand that the weak equivalences were weak homotopy equivalences, i.e., morphisms in $\mathscr{CW}$ inducing isomorphims on homotopy groups. However, by Whiteheads theorem (see Theorem 2.4.7 in [12] for full theorem and proof), weak equivalences on CW-complexes are homotopy equivalences. Hence we need only consider homotopy equivalences.

Proof. (CF1): Isomorphisms in $\mathscr{CW}$ are homeomorphisms, which are trivially homotopy equivalences. To see that all homeomorphims are cofibrations, look at the following diagram for a homeomorphism $f : X \longrightarrow Y$:

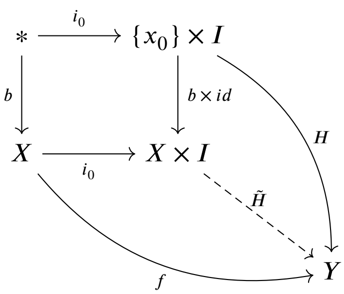


Here $\tilde{H}(y,t) := H(f^{-1}(y),t)$ is our extension. It is well defined, as f^{-1} exists and is itself a homeomorphism. Note that

$$\tilde{H}(y,0) = H(f^{-1}(y),0) = g \circ f \circ f^{-1}(y) = g(y),$$

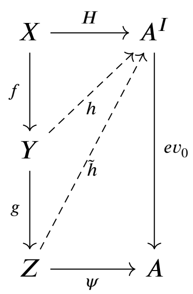
so the diagram commutes.

Let $(Y, y_0) \in \mathscr{CW}$. The initial object is the one-point space $*$, and since all morphisms in $\mathscr{CW}$ are basepoint-preserving the only possible map $* \longrightarrow (X, x_0)$ is the inclusion to the basepoint of X. Let b denote the inclusion of the basepoint. Then H in the following diagram



can only be $H(x_0, t) = y_0$, as we are working with based homotopies. Then the diagram commutes for $\tilde{H}(x, t) := f(x)$.

Lastly we check that that compostion of cofibrations are cofibrations. Let $f : x \longrightarrow Y$ and $g : Y \longrightarrow Z$ be cofibrations, i.e., both satisfy Definition 2.36. Look at the following composition:



where ψ and H are the arbitrary maps. Then there exists an h such that the diagram commutes, because of Definition 2.36 for f with $\psi \circ g$ and H as the arbitrary maps. Then, because of Definition 2.36 for g with h and ψ as the arbitrary maps, $\tilde{h}$ exists. In total $\tilde{h}$ solves the extension problem for $g \circ f$.

(CF2): Let $f \longrightarrow Y$ and $g : Y \longrightarrow Z$ be maps in $\mathscr{CW}$. We have three cases to show; f and g are homotopy equivalences, f and $g \circ f$ are homotopy equivalences and g and $g \circ f$ are homotopy eauivalences. However, the last two cases are similar, so we only prove the first two cases.

If f and g are homotopy equivalences, we have inverses f' and g' with composition homotopic to their respective identity morphisms. Then

$$f' \circ g' \circ g \circ f \simeq f' \circ id_Y \circ f \simeq id_X$$

and

$$g \circ f \circ f' \circ g' \simeq g \circ g' \simeq id_Z,$$

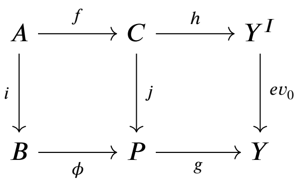
and so $f' \circ g'$ is the homotopy inverse to $g \circ f$.

If f and $(g \circ f)$ are homotopy equivalences, then we have inverses f' and h. We claim that $f \circ h$ is the homotopy inverse to g. We already know that $(g \circ f) \circ h \simeq id_Z$. Then

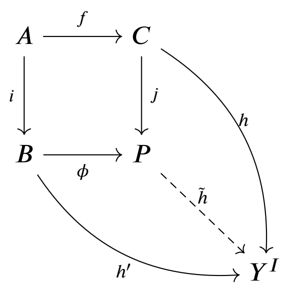
$$\begin{aligned} h \circ g \circ f &\simeq id_X \\ \implies f \circ h \circ (g \circ f) \circ f' &\simeq f \circ id_X \circ f' \\ \implies f \circ h \circ g &\simeq f \circ f' \simeq id_Y. \end{aligned}$$

(CF3): Pushouts exist in topological spaces, and from Theorem 2.8 we know that pushouts of CW-complexes are still CW-complexes. Given the pushout (1), we want to show that j is a cofibration.

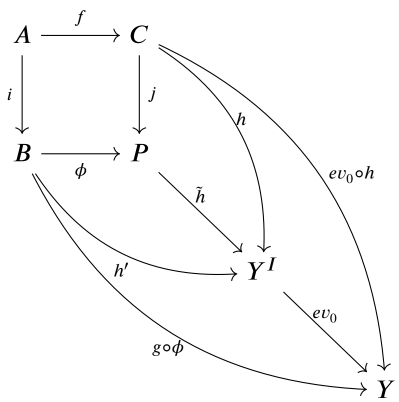
Consider the following diagram

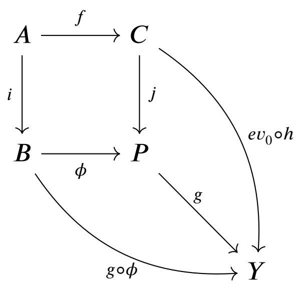


in which the left square is the pushout, and the right square is the homotopy extension problem for arbitrary maps g and h. Then there exists a map $h' : B \longrightarrow Y^I$ since i is a cofibration; h' is the solution to the extension problem for $g \circ \phi$. Now we use h' and h to see that the universal property of the following pushout



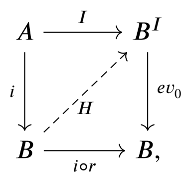
gives us a unique map $\tilde{h}$. Since $h = \tilde{h} \circ j$, we see that if $ev_0 \circ \tilde{h} = g$, $\tilde{h}$ solves the extension problem and j is a cofibration. Now compare the following diagrams:





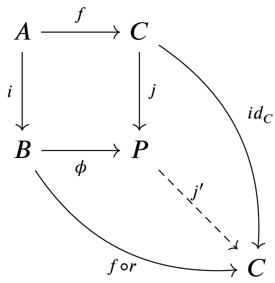
We have $ev_0 \circ h' = g \circ \phi$ since h' is the solution to extension problem with $f \circ h$ as the upper arbitrary map. From the uniqueness of the induced map in pushouts, $g = ev_0 \circ \tilde{h}$.

Now assume that i is a homotopy equivalence in addition to a cofibration. Furthermore, we will assume that A and B are connected CW-complexes. Then by Lemma 2.40 and Lemma 2.2 we have $A \cong i(A)$ and an $r : B \longrightarrow A$ with $r \circ i = id_A$, $i \circ r \simeq id_B$. Then we have a homotopy inverse $i' : B \longrightarrow A$, and $f \circ i' \circ i \simeq f$. Let $H : B \longrightarrow B^I$ be the homotopy between $i \circ r$ and id_B, i.e. H solves the following extension problem



where $I : A \longrightarrow B^I$ is the constant homotopy $ev_t \circ I = i$.

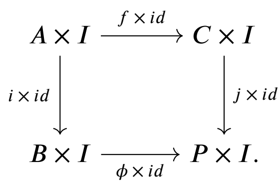
Using that $f \circ r \circ i = f$ we can make the following commutative diagram.



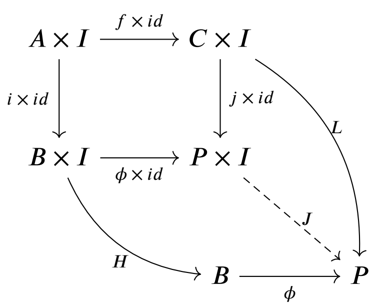
(2)

Then the universal property of the pushout gives us a map $j' : P \longrightarrow C$ with $j' \circ j = id_C$. Hence if $j \circ j' \simeq id_P$, j' is the homotopy inverse to j.

In Remark 2.37 we mentioned the adjunction $\mathscr{CW}(X \times Y, Z) \cong \mathscr{CW}(X, \mathscr{CW}(Y, Z))$, but more specifically $- \times X$ is a left adjoint functor, which means that it commutes with colimits. As pushouts are colimits, we see that the following diagram is a pushout:

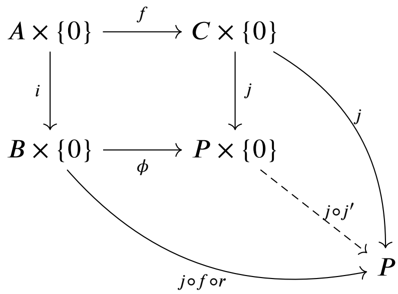


We already have a homotopy H with $H(i(a), t) = i(a)$. Let $L : C \times I \longrightarrow P$ be the constant homotopy $L(c, t) = j(c)$ for all t. Then looking at (2) we see that $\phi \circ H(i(a), t) = \phi \circ i(a) = j \circ f(a) = L(f(a), t)$ for all t, which means that the following diagram commutes for all t

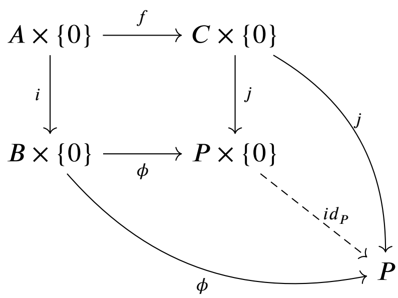


and the universal property of the pushout gives us a homotopy $J : P \times I \longrightarrow P$. The induced map is unique, we can evaluate it at $t = 0$ and $t = 1$ to see what it must be.

At $t = 0$ we get $\phi \circ H(b, 0) = \phi \circ i \circ r(b) = j \circ f \circ r$ because of (2). Then following diagram commutes for $J(p, 0) = j \circ j'$.



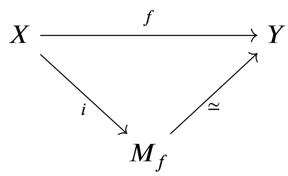
Likewise, at $t = 1$ we have $\phi \circ H(b, 1) = \phi \circ id_B$, so the diagram commutes for $J(p, 1) = id_P$.



In total, we have that J defines a homotopy from $j \circ j'$ to id_P, which means that j is a homotopy equivalence.

If A and B were not connected, we see that they would each be a disjoint union of CW-complexes. Furthermore, since we have a homotopy equivalence between them, this homotopy equivalence would have to distribute over the different components, i.e. if we have $A = \coprod A_i$ and $B = \coprod B_i$ we should have a homotopy equivalence $i_i : A_i \longrightarrow B_i$ for each i. Then we can repeat the above argument on the different components.

(CF4): As seen in Example 2.39, every map $f : X \longrightarrow Y$ factors as



in which i is a cofibration.

In total we see that $\mathscr{CW}$ is a cofibration category. □

Recall that we already know some cofibrations in $\mathscr{CW}_h$, and using the formalities of a cofibration category we can see the following.

Corollary 3.5. *Since the map $i_1 : X \longrightarrow CX$ is a cofibration, we have that $i(f) : Y \longrightarrow C(f)$ in Definition 2.33 is a cofibration by (CF3).*

4 Triangulated categories

Triangulated categories are categories with a class of "triangles", a sequence of three morphisms, that satisfies a set of axioms. They appear in several areas of mathematics, and one of the properties we want from a stable homotopy category is that it is triangulated. We are aiming to show that the Spanier–Whitehead category is triangulated.

In this section we will present the framework for and the definition of a triangulated category, after which we will make an attempt to define a triangulated structure on $\mathscr{CW}_h$. This attempt and its flaws leads us to the construction of the Spanier–Whitehead category.

4.1 Definition

All triangulated categories are constructed from additive categories, which we define here.

Definition 4.1. Let $\mathscr{C}$ be a category. We say that it is **additive** if it satisfies

(i) The set of morphisms $\mathscr{C}(A, B)$ is an abelian group for all $A, B \in \mathscr{C}$

(ii) Composition of morphisms is a bilinear operation.

(iii) There is a zero-object $0_{\mathscr{C}}$ such that $\mathscr{C}(0_{\mathscr{C}}, A)$ and $\mathscr{C}(A, 0_{\mathscr{C}})$ are trivial groups for all $A \in \mathscr{C}$.

(iv) For all $A, B \in \mathscr{C}$ there is a biproduct $A \oplus B$, i.e., morphisms

$$A \underset{\pi_A}{\overset{i_A}{\rightleftarrows}} A \oplus B \underset{\pi_B}{\overset{i_B}{\leftrightarrows}} B$$

such that $\pi_A \circ i_A = id_A$, $\pi_B \circ i_B = id_B$ and $id_{A\oplus B} = (i_A \circ \pi_A) + (i_B \circ \pi_B)$.

Definition 4.2. For two additive categories $\mathscr{C}$ and $\mathscr{D}$ a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is said to be **additive** if for all $A, B \in \mathscr{C}$ we have $F(f+g) = F(f)+F(g)$ for all $f, g \in \mathscr{C}(A, B)$.

Lemma 4.3. *If a category $\mathscr{A}$ satisfies (i) through (iii) in Definition 4.1 and $\mathscr{A}$ has all finite coproducts (products), then it also has all finite products (coproducts), which coincide such that $\mathscr{A}$ has a biproduct.*

See proposition 12.2.5 in [9] for a proof.

Having an additive category in mind, we move on to defining triangles. All triangles are defined for a specific functor Σ, often called the **suspension** or **shift**.

Definition 4.4. Let $\mathscr{C}$ be an additive category, and let $\Sigma : \mathscr{C} \longrightarrow \mathscr{C}$ be an additive automorphism, that is, an additive functor on $\mathscr{C}$ with an additive inverse Σ^{-1}, such that $\Sigma\circ\Sigma^{-1} = id_{\mathscr{C}}$ and $\Sigma^{-1}\circ\Sigma = id_{\mathscr{C}}$. A **triangle** in $\mathscr{C}$ with respect to Σ is a diagram in $\mathscr{C}$ of the form

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \xrightarrow{\ h\ } \Sigma A.$$

A **morphism of triangles** is a triple of morphisms (α, β, γ) such that the following diagram commutes

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma A'
\end{array}$$

We say that (α, β, γ) is an **isomorphism of triangles** if each of them are isomorphisms in $\mathscr{C}$. The triangles are then said to be **equivalent**.

The **left** and **right rotation** of $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is given by

$$B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$$

and

$$\Sigma^{-1} C \xrightarrow{-\Sigma^{-1} h} A \xrightarrow{f} B \xrightarrow{g} C$$

respectively.

Definition 4.5. Let $\mathscr{C}$ be an additive category, Σ an additive automorphsim on $\mathscr{A}$ and Δ a collection of triangles called **distinguished triangles**. We say that $(\mathscr{C}, \Sigma, \Delta)$ is a **triangulated category** if the following properties hold:

(TR1)
- Δ is closed under isomorphisms of triangles.
- The triangle $A \xrightarrow{id_A} A \xrightarrow{0} 0 \xrightarrow{0} \Sigma A \in \Delta$ for all $A \in \mathscr{C}$.
- For all $f : A \longrightarrow B$ in $\mathscr{C}$ there is a triangle starting with f.

(TR2) Δ is closed under left and right rotation.

(TR3) If we have the following commuting diagram of triangles

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma A'
\end{array}$$

then there is a $\gamma : C \longrightarrow C'$ making the diagram a morphism of triangles.

(TR4) Assume we have a commuting diagram with rows in Δ

$$
\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
\| & & \downarrow{\scriptstyle \beta} & & \vdots & & \| \\
A & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A \\
{\scriptstyle f}\downarrow & & \| & & \vdots & & \downarrow{\scriptstyle \Sigma f} \\
B & \xrightarrow{\beta} & B' & \xrightarrow{g''} & C'' & \xrightarrow{h''} & \Sigma B \\
 & & & & {\scriptstyle \Sigma g \circ h''}\downarrow & & \\
 & & & & \Sigma C & &
\end{array}
$$

where the solid diagram commutes. Then we have morphisms $\gamma : C \longrightarrow C'$ and $\gamma' : C' \longrightarrow C''$ such that the whole diagram commutes and the triangle $C \xrightarrow{\gamma} C' \xrightarrow{\gamma'} C'' \xrightarrow{\Sigma g \circ h''} \Sigma C$ is in Δ.

Example 4.6. A rather trivial example of a triangulated category is the category of vector spaces over a field k, whose maps are the k-linear maps of vector spaces. Here the suspension is the identity, and the distinguished triangles are isomorphic to $X \to Y \to Z \to X$, which is exact at X, Y and Z.

Other, more complicated examples are the homotopy category of an abelian category, and the derived category of an abelian category. However, these examples are what are known as *algebraic triangulated categories*. The Spanier–Whitehead category is one example of a *topological triangulated category*.

4.2 A triangulated structure on $\mathscr{CW}_h$

We can motivate the construction of the Spanier–Whitehead category by looking at how $\mathscr{CW}_h$ is almost a triangulated category, and how it fails to be a triangulated category. Note that the homotopy category is not additive, meaning that it fails to be triangulated before we have even tried to define some triangulated structure. However, we will see that our attempt to define triangles on $\mathscr{CW}_h$ will largely translate into the Spanier–Whitehead category, and the properties of these triangles will be very useful to us later. The following definitions are due to [7].

Definition 4.7. For an $f \in \mathscr{CW}(X, Y)$ we define the **mapping sequence** of f as

$$X \xrightarrow{f} Y \xrightarrow{i(f)} C(f) \xrightarrow{j(f)} \Sigma X$$

We say that a sequence of CW-complexes

$$U \xrightarrow{f'} V \xrightarrow{g} W \xrightarrow{h} \Sigma U$$

is an **unstable distinguished triangle** if it is equivalent in $\mathscr{CW}_h$ to a mapping sequence. This means that there exists an isomorphism of triangles (α, β, γ) in $\mathscr{CW}_h$ such that the following diagram commutes up to homotopy

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{i(f)} & C(f) & \xrightarrow{j(f)} & \Sigma X \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\Sigma\alpha} \\ U & \xrightarrow{f'} & V & \xrightarrow{g} & W & \xrightarrow{h} & \Sigma U. \end{array}$$

Remark 4.8. The map $i(f)$ from Y into the mapping cone is given by the pushout in Definition 2.33. Recall that this includes Y as a subspace into $C(f)$. Then Y as a subspace is mapped to the basepoint under $j(f)$, while classes $[t, x]_{CX}$ in CX are sent to their classes $[t, x]_{CX}$ in ΣX, where $[0, x]_{\Sigma X} = [1, x]_{\Sigma X}$.

What happens in the mapping sequences is that the mapping cone is constructed for some map, and then all of the codomain of the map, which is a subspace of the mapping cone, is collapsed, to reveal the suspension of our original space. Consider the simple map which includes the circle into the 1-disk as its boundary. We have seen in Example 2.34 that the mapping cone of this looks like a cone, and collapsing the bottom disk of this space gives a space that is homotopy equivalent to the 2-sphere. (In fact, the mapping cone in this example is also homotopy equivalent to the 2-sphere. This is because Y in this case is the 2-disk, which again is homotopy equivalent to a point).

We are now going to see that, as far as the category allows us, $\mathscr{CW}_h$ does satisfy several of the axioms of Definition 4.5. The construction and proof given for the attempted triangulated structure on $\mathscr{CW}_h$ is a very geometric one, and differs a lot from other proofs of the triangulation of stable homotopy categories, which are often more based on category theory. However, it can be nice to have an understanding of what the triangles in the stable homotopy categories "look like".

Theorem 4.9. *The homotopy category of CW-complexes, $\mathscr{CW}_h$ satisfies* *(i)* *The set of unstable distinguished triangles is closed under isomorphisms of triangles in $\mathscr{CW}_h$.*

(ii) *The triangle $* \} :\to X \to X \to *$, where $*$ is the one-point space is an unstable distinguished triangle.*

(iii) *For every $f \in [X, Y]$, there is an unstable distinguished triangle starting with f.*

(iv) *If we have the following diagram of unstable distinguished triangles:*

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\
U & \xrightarrow{f'} & V & \xrightarrow{g'} & W & \xrightarrow{h'} & \Sigma U
\end{array}$$

such that the left, right and outer squares commute, then there is a map $\gamma : Z \longrightarrow W$ such that the whole diagram commutes up to homotopy.

(v) *If we have the following commuting diagram of unstable distinguished triangles*

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
\| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \| \\
X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X \\
\downarrow{\scriptstyle f} & & \| & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \Sigma f} \\
Y & \xrightarrow{\beta} & Y' & \xrightarrow{g''} & Z'' & \xrightarrow{h''} & \Sigma Y \\
 & & & & \downarrow{\scriptstyle \Sigma g \circ h''} & & \\
 & & & & \Sigma Z & &
\end{array}$$

then there exists maps γ and γ' such that the triangle $[\gamma, \gamma', \Sigma g \circ h'']$ is an unstable distinguished triangle.

Proof. We see that (i) and (iii) follows from the definition of the unstable distinguished triangles.

(ii) This triangle is equivalent to the mapping sequence of the inclusion $\imath : * \hookrightarrow X$. To see this, note that the cone of a point is the interval, which again is glued to X at the basepoint, so the mapping cone of $\imath$ is homeomorphic to the wedge $I \vee X$. So, the mapping sequence of $\imath$ is

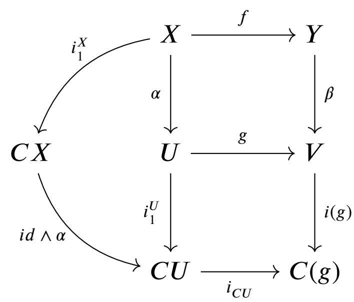
$$* \longrightarrow X \longrightarrow I \vee X \longrightarrow I, \tag{3}$$

since $j(\imath)$ sends all of X to the basepoint in $\Sigma * \cong I$. Because $I \simeq *$, (3) equivalent to $* \longrightarrow X \longrightarrow X \longrightarrow *$.

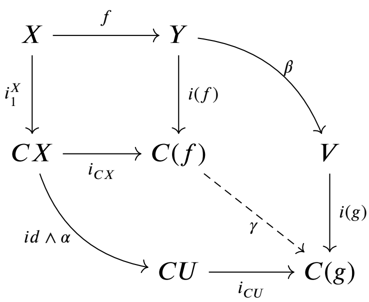
(iv) This reduces to showing that the statement holds for mapping sequences. We want there to be a γ such that the following diagram commutes:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{i(f)} & C(f) & \xrightarrow{j(f)} & \Sigma X \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\ U & \xrightarrow{g} & V & \xrightarrow{i(g)} & C(g) & \xrightarrow{j(g)} & \Sigma U \end{array} \tag{4}$$

Both $C(f)$ and $C(g)$ are defined by the pushout in Definition 2.33. Consider the following diagram where the bottom square is the pushout defining $C(g)$, and the upper square comes from the above diagram:



Because of the functoriality of the smash product, the outer square commutes. Now we can use this in the diagram defining $C(f)$ to see that the universal property of the pushout gives us $\gamma : C(f) \longrightarrow C(g)$.



From this diagram we see that γ commutes with the middle square in (4). Thus, what remains is to show that the right square also commutes.

The space $C(f)$ is the cone of X glued to Y along the image of f. Hence, points in the mapping cone can either be described as some class $[y]_{C(f)}$ where $y \in Y$, or by some class $[t, x]_{C(f)}$ where $x \in X$, such that $[1, x]_{C(f)} = [f(x)]_{C(f)}$. From the definition of $j(f)$ we know that all points $[y]_{C(f)}$ in the mapping cone are sent to the basepoint in ΣU under the composition $\Sigma\alpha \circ j(f)$. Likewise, $j(g)$ sends all $[v]_{C(g)}$ to the basepoint in ΣU. Since the map $i(f) : Y \longrightarrow C(f)$ is the inclusion and the composition $j(g) \circ i(g) \circ \beta$ sends everything to the basepoint in ΣU, we see that $j(g) \circ \gamma$ also must send all $[y]_{C(f)}$ to the basepoint.

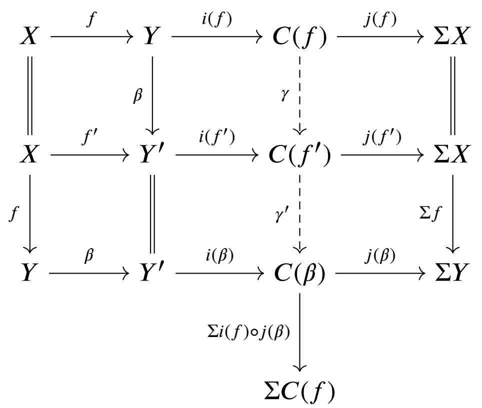
Elements in $C(f)$ on the form $[t, x], t \neq 1$, are just points in CX, which are sent to $[t, \alpha(x)]$ in $C(g)$ under $i_{CU} \circ (id \wedge \alpha)$. These elements are just included into $C(f)$, and so by functoriality of the smash product, they are sent to $[t, \alpha(x)]$ in ΣU under $j(g) \circ \gamma$.

This concludes the proof of *(iv)*, but for the sake of the next part we write an explicit decription for γ, based on the above argument. Indeed, looking at the diagrams we see that we must have

$$\gamma([t, z]_{C(f)}) = \begin{cases} [t, \alpha(z)]_{C(g)}, & 0 \leq t \leq 1, z \in X \\ [\beta(z)]_{C(g)}, & z \in Y \end{cases}$$

where the subscripts denotes where the classes are.

(v) Again, we can consider all the unstable distinguished triangles to be mapping sequences, in which case the diagram looks like the following.



(5)

From (iv) we know that such maps γ and γ' exit, and that the diagram commutes. Hence, to prove the statement we need to show that the triangle

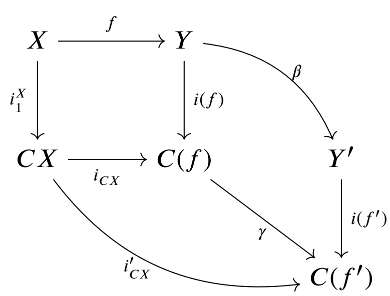
$$C(f) \xrightarrow{\gamma} C(f') \xrightarrow{\gamma'} C(\beta) \xrightarrow{(\Sigma i(f')) \circ j(\beta)} \Sigma C(f') \tag{6}$$

is an unstable distinguished triangle. We will construct $C(\gamma)$ and $C(\beta)$ as pushouts from Y, after which Lemma 3.2 will give us a homotopy equivalence between them. Then we check that the mapping sequence of γ,

$$C(f) \xrightarrow{\gamma} C(f') \xrightarrow{i(\gamma)} C(\gamma) \xrightarrow{j(\gamma)} \Sigma C(f), \tag{7}$$

is equivalent to (6).

We already know that $C(\beta)$ is a pushout from Y, the difficult part is to show that $C(\gamma)$ also is a pushout from Y. Recall how γ was constructed; it is (in this case) the unique map that makes the following diagram commute:



where i'_{CX} is the inclusion of CX as a subspace of $C(f')$. Most importantly, we have $\gamma \circ i_{CX} = i'_{CX}$ and $f' = \beta \circ f$, which means that when we look at the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{\beta} & Y' \\ {\scriptstyle i_1^X}\big\downarrow & & \big\downarrow & & \big\downarrow \\ CX & \xrightarrow{i_{CX}} & C(f) & \xrightarrow[\gamma]{} & C(f') \end{array}$$

the outer square is the pushout defining $C(f')$, and the left square is the pushout defining $C(f)$. Then Lemma 2.22 implies that the right square is a pushout as well. In order to simplify the notation we denote the cone of $C(f)$, which is $C(C(f))$, by C, and recall that the mapping cone of γ is defined by the pushout of

$$C \xleftarrow{i_1^C} C(f) \xrightarrow{\gamma} C(f'),$$

where i_1^C is the inclusion of $C(f)$ into C at $t = 1$. Now we can consider the following:

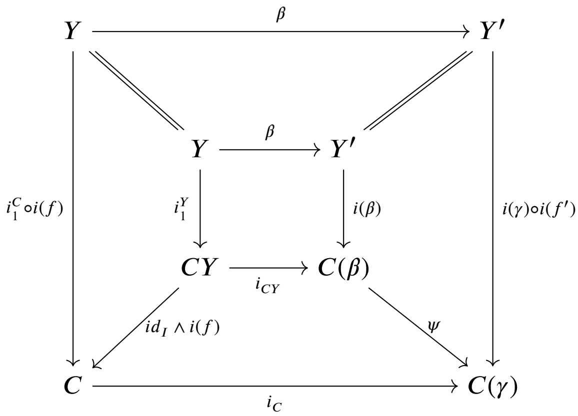
$$\begin{array}{ccccc} Y & \xrightarrow{i(f)} & C(f) & \xrightarrow{i_1^C} & C \\ {\scriptstyle \beta}\big\downarrow & & \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle i_C} \\ Y' & \xrightarrow[i(f')]{} & C(f') & \xrightarrow[i(\gamma)]{} & C(\gamma) \end{array}$$

Again by Lemma 2.22 we see that the outer square is a pushout, and hence we are all set to use Lemma 3.2:

$$\begin{array}{ccccc} CY & \xleftarrow{i_1^Y} & Y & \xrightarrow{\beta} & Y' \\ {\scriptstyle id_I \wedge i(f)}\big\downarrow & & \big\| & & \big\| \\ C & \xleftarrow[i_1^C \circ i(f)]{} & Y & \xrightarrow[\beta]{} & Y' \end{array}$$

We have that all the vertical maps are homotopy equivalences; the identities are trivially so, and for $id_I \wedge i(f) \colon CY \longrightarrow C$ we can let its inverse be the map sending everything to the basepoint. Since both C and CY are contractible, their respective identities are homotopic to the maps collapsing everything to their respective basepoints. Furthermore, $i_1^Y : Y \longrightarrow CY$ and $i_1^C \circ i(f) : Y \longrightarrow C$ are cofibrations by Example 2.39, Corollary 3.5 and **(CF1)** in Definition 3.1. In total, we have a homotopy equivalence $\psi : C(\beta) \longrightarrow C(\gamma)$.

We write the full diagram induced by the gluing lemma to understand what ψ does.



(8)

Using (8) we construct the following explicit description of ψ:

$$\psi([t,z]_{C(\beta)}) = \begin{cases} i_C[t, i(f)(z)]_C, & 0 \le t \le 1, z \in Y \\ i(\gamma)\circ i(f')([z]_{C(\beta)}), & t = 1, z \in Y' \end{cases}$$

Furthermore, from the proof of *(iv)*, we have the following descriptions of γ and γ':

$$\gamma([t,z]_{C(f)}) = \begin{cases} [t,z]_{C(f')}, & 0 \le t \le 1, z \in X \\ [\beta(z)]_{C(f')}, & t = 1, z \in Y \end{cases}$$

$$\gamma'([t,z]_{C(f')}) = \begin{cases} [t, f(z)]_{C(\beta)}, & 0 \le t \le 1, z \in X \\ [z]_{C(\beta)}, & t = 1, z \in Y' \end{cases}$$

which we will use to see that the following diagram commutes up to homotopy

$$\begin{array}{ccccccc} C(f) & \xrightarrow{\gamma} & C(f') & \xrightarrow{\gamma'} & C(\beta) & \xrightarrow{\Sigma i(f')\circ j(\beta)} & \Sigma C(f) \\ \| & & \| & & \| & & \| \\ C(f) & \xrightarrow{\gamma} & C(f') & \xrightarrow{i(\gamma)} & C(\gamma) & \xrightarrow{j(\gamma)} & \Sigma C(f). \end{array} \qquad (9)$$

If so, we have an equivalence of the triangles (6) and (7), which means that the former is an unstable distinguished triangle.

We start with the middle square in (9), and we will see that $\psi \circ \gamma' \simeq i(\gamma)$. Checking this is an exercise in keeping track of where the elements lie. We split it in two cases; $0 \leq t < 1$ and $t = 1$. For $0 \leq t < 1$ the classes $[t, z]_{C(f')}$ are included from CX, i.e., $z \in X$. We get

$$(\psi \circ \gamma')([t, z]_{C(f')}) = \psi([t, f(z)]_{C(\beta)}) = i_C([t, i(f)(f(z))]_C)$$

where

$$i(f)(f(z)) = [f(z)]_{C(f)} = [1, z]_{C(f)},$$

so we get

$$i_C([t, i(f)(f(z))]_C = [t, [1, z]_{C(f)}]_{C(\gamma)}.$$

On the other hand we have

$$i(\gamma)([t, z]_{C(f)}) = [1, [t, z]_{C(f')}]_{C(\gamma)}.$$

Then $H : C(f') \times I \longrightarrow C(\gamma)$ given by

$$H([t, z]_{C(f')}, s) = [s + (1 - s)t, [st + (1 - s), z]_{C(f')}]_{C(\gamma)}$$

is a continuous map which we claim to be a homotopy from $\psi \circ \gamma'$ to $i(\gamma)$.

Since $[t, x]_{C(f)} \mapsto [t, x]_{C(f')}$ under γ, they identify as the same class in $C(\gamma)$. We therefore get

$$H(x, 0) = [t, [1, z]_{C(f')}]_{C(\gamma)} = [t, [1, z]_{C(f)}]_{C(\gamma)} = (\psi \circ \gamma)([t, z])$$

and

$$H(x, 1) = [1, [t, z]_{C(f')}]_{C(\gamma)} = i(\gamma)([t, z]_{C(f')}),$$

and so $\psi \circ \gamma' \simeq i(\gamma)$ for $t < 1$. For $t = 1$ the diagram commutes directly. We start with a class $[z]_{C(f')}$ where $z \in Y'$, and get

$$\psi \circ \gamma'([z]_{C(f')}) = \psi([z]_{C(\beta)}) = (i(\gamma) \circ i(f'))([z]_{C(\beta)}) = i(\gamma)([z]_{C(f')}).$$

Now we show that $j(\gamma) \circ \psi = \Sigma i(f') \circ j(\beta)$. Again we split it into $0 \leq t < 1$ and $t = 1$. For $0 \leq t < 1$ we start with a class $[t, z]_{C(\beta)}$, where $z \in Y$. We then have

$$\begin{aligned}(j(\gamma) \circ \psi)([t, z]_{C(\beta)}) &= (j(\gamma) \circ i_C)([t, i(f)(z)]_C \\ &= j(\gamma)([t, [z]_{C(f)}]_{C(\gamma)}) = [t, [z]_{C(f)}]_{\Sigma C(f)}.\end{aligned}$$

Furthermore we have

$$(\Sigma i(f)\circ j(\beta))([t,z]_{C(\beta)}) = \Sigma i(f)([t,z]_{\Sigma Y}) = [t,[z]_{C(f)}]_{\Sigma C(f)}.$$

For $t = 1$ we have that both $j(\beta)$ and $j(\gamma)$ send all classes to the basepoint in ΣY and $\Sigma C(f)$ respectively, and it follows that the diagram commutes.

This concludes the proof. □

A lot of these properties are similar to those of Definition 4.5. What we are lacking is, as mentioned before, an additive structure on $[X,Y]$, and of course that Σ is an automorphism. For now, left and right rotation does not exist in $\mathscr{CW}_h$. We will see that the homotopy classes of maps almost have an additive structure, and using this we can also prove that left rotation holds in $\mathscr{CW}_h$.

The following lemma is often used to show that the homotopy groups are abelian for $n \geq 2$, and will also be used to show that we have an abelian groups structure on $[X,Y]$.

Lemma 4.10 (Eckmann–Hilton)**.** *Let S be a set and let $*, \times$ be two binary operations on S with unit elements. If we have $(a*b)\times(c*d) = (a\times c)*(b\times d)$ for all $a,b,c,d \in S$, then the two operations coincide and the two units coincide. Furthermore, the operation is commutative and associative.*

See [4, pp. 10–11] for a proof.

Theorem 4.11. *In $\mathscr{CW}_h$ we have that $[\Sigma X, Y]$ are groups, and $[\Sigma^2 X, Y]$ are abelian groups for all X and Y in $\mathscr{CW}$.*

Sketch of proof. We present only a sketch of the proof, as there are many details, and most can easily be verified by the reader. For a proof see proposition 2.3.4 and 2.3.8 in [1].

We begin by showing that there is a group structure on $[\Sigma X, Y]$. As before, we describe an element in ΣX by an equivalence class $[t,x]$. For two maps $f,g : \Sigma X \longrightarrow Y$ we can define a binary operation on them as

$$(f+g)[t,x] = \begin{cases} f[2t,x] & 0 \leq t \leq \frac{1}{2}, \\ g[2t-1,x] & \frac{1}{2} \leq t \leq 1. \end{cases}$$

This is a continuous function since $f[1,x] = g[0,x] = y_0$, the basepoint in Y. We claim that $[\Sigma X, Y]$ is a group under this operation, with the constant map sending everything to the base-point in Y as the unit. Denote this map by c_{y_0}.

Left/right unit: Consider the homotopy $H : \Sigma X \times I \longrightarrow Y$ given by

$$H([t,x],s) = \begin{cases} f([2t(1-s)+st,x] & 0 \le t \le \frac{1}{2}(1+s) \\ c_{y_0}([t,x],s) & \frac{1}{2}(1+s) \le t \le 1. \end{cases}$$

This gives us $f + c_{y_0} \simeq f$. Likewise we can define a homotopy to see that $c_{y_0} + f \simeq f$.

Inverse: For a map f the inverse is given by $f[1-t,x]$ and we get:

$$(f-f)([t,x]) = \begin{cases} f[2t,x] & 0 \le t \le \frac{1}{2}, \\ f[2-2t,x] & \frac{1}{2} \le t \le 1, \end{cases}$$

i.e., $f - f \simeq c_{y_0}$.

Associativity: For three maps $f, g, h \in [\Sigma X, Y]$ we get

$$(f+(g+h))([t,x]) = \begin{cases} f[2t,x] & 0 \le t \le \frac{1}{2}, \\ (g+h)[2t-1,x] & \frac{1}{2} \le t \le 1, \end{cases}$$
$$= \begin{cases} f[2t,x] & 0 \le t \le \frac{1}{2}, \\ g[4t-2,x] & \frac{1}{2} \le t \le \frac{3}{4}, \\ h[4t-3,x] & \frac{3}{4} \le t \le 1, \end{cases}$$

so associativity follows by stretching the three intervals for t appropriately. Hence, $[\Sigma X, Y]$ is a group.

To prove that $[\Sigma^2 X, Y]$ is abelian, we use that the double suspension can be written as a quotient space $(I \times I \times X)/\sim$. So an element in $\Sigma^2 X$ is an equivalence class $[t,s,x]$. We can now define two binary operations on the set, $+_t$ and $+_s$ as follows

$$(f +_t g)[t,s,x] = \begin{cases} f[2t,s,x] & 0 \le t \le \frac{1}{2}, \\ g[2t-1,s,x] & \frac{1}{2} \le t \le 1, \end{cases}$$

$$(f +_s g)[t,s,x] = \begin{cases} f[t,2s,x] & 0 \le s \le \frac{1}{2}, \\ g[t,2s-1,x] & \frac{1}{2} \le s \le 1. \end{cases}$$

The reader can check that the two operations satisfy Lemma 4.10 up to homotopy. It follows that the operations agree and are commutative, and thus $[\Sigma^2 X, Y]$ is an abelian groups. □

Corollary 4.12. *For two maps $f, g\colon \Sigma X \longrightarrow Y$ we get*

$$\Sigma(f+g)([s,t,x]) = \Sigma(f+g)([s,[t,x]] = [s,(f+g)([t,x])]$$

$$\begin{aligned} &= \begin{cases} [s, f[t,x]], & 0 \le s \le \frac{1}{2} \\ [s, g[t,x]], & \frac{1}{2} \le s \le 1 \end{cases} \\ &= \begin{cases} f[s,[t,x]], & 0 \le s \le \frac{1}{2} \\ g[s,[t,x]], & \frac{1}{2} \le s \le 1 \end{cases} \\ &= \begin{cases} f[s,t,x], & 0 \le s \le \frac{1}{2} \\ g[s,t,x], & \frac{1}{2} \le s \le 1 \end{cases} \\ &= (\Sigma f + \Sigma g)[s,t,x]. \end{aligned}$$

i.e., the suspension is an additive functor when restricted to doubly suspended spaces.

Using this additive structure, we can at least look at what happens when we rotate triangles in $\mathscr{CW}_h$ to the left.

Theorem 4.13. *If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is an unstable distinguished triangle, then so is $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$.*

Proof. To prove this we need only consider the left rotation of a mapping sequence, $X \xrightarrow{f} Y \xrightarrow{i(f)} C(f) \xrightarrow{j(f)} \Sigma X$. Then the proof reduces to showing that the mapping cone of $i(f)$, denoted $C(i(f))$, is homotopy equivalent to ΣX, and that the following diagram commutes up to homotopy:

$$\begin{array}{ccccccc} Y & \xrightarrow{i(f)} & C(f) & \xrightarrow{i(i(f))} & C(i(f)) & \xrightarrow{j(i(f))} & \Sigma Y \\ \| & & \| & & \downarrow{\scriptstyle \gamma} & & \| \\ Y & \xrightarrow{i(f)} & C(f) & \xrightarrow{j(f)} & \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y. \end{array} \tag{10}$$

We will use Lemma 3.2 to show this. Then we need both $C(i(f))$ and ΣX to be defined as pushouts from X. We know from Remark 2.19 that ΣX can be identidied as the pushout

$$CX \xleftarrow{i_1^X} X \rightarrow *,$$

where i_1^X is a cofibration. To see that $C(i(f))$ is a pushout from X, consider the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\quad f \quad} & Y & \xrightarrow{\quad i_1^Y \quad} & CY \\ \Big\downarrow{\scriptstyle i_1^X} & & \Big\downarrow{\scriptstyle i(f)} & & \Big\downarrow{\scriptstyle i_{CY}} \\ CX & \xrightarrow{\quad i_{CX} \quad} & C(f) & \xrightarrow{\quad i(i(f)) \quad} & C(i(f)) \end{array} \tag{11}$$

where we again have that i_1^Y is a cofibration. The left and right square are the construction of the mapping cones of f and $i(f)$ respectively, i.e., both squares are pushouts. Then by Lemma 2.22 the outer square is a pushout as well. We then have the following commutative diagram

$$\begin{array}{ccccc} CX & \xleftarrow{\quad i_1^X \quad} & X & \xrightarrow{\quad i_1^Y \circ f \quad} & CY \\ \Big\| & & \Big\| & & \Big\downarrow{\scriptstyle \simeq} \\ CX & \xleftarrow{\quad i_1^X \quad} & X & \xrightarrow{\qquad\quad} & *, \end{array}$$

on which Lemma 3.2 applies, and we get a homotopy equivalence $\gamma : C(i(f)) \longrightarrow \Sigma X$.

Before proceeding, we look a bit more closely at $C(i(f))$. We now know that it can be defined as the pushout of $CX \xleftarrow{i_1^X} X \xrightarrow{i_1^Y \circ f} CY$. This tells us quite a bit about the space $C(i(f)$; while we could consider it to be the space $CY \sqcup_{i(f)} C(f)$, where Y is glued to $C(f)$ along $i(f)$ as in our original construction of a mapping cone, we see that it can also be identified as the two cones CX and CY glued together along $[1, x] \sim [1, f(x)]$.

To make the following part, in which we check the commutativity of (10), more comprehensive, we make a more explicit identification on the space $C(i(f))$. We want to make an identification like in Figure 6, where the whole space is given by classes $[t, z]$ for $t \in I$, with the two cones glued together at $t = \frac{1}{2}$. This is clearly homeomorphic to $C(i(f))$, it is just a matter of shrinking the intervals.

We have homeomorphisms

$$CX = I \wedge X \cong \left[0, \frac{1}{2}\right] \wedge X =: \tilde{CX}, CY = I \wedge Y \cong \left[\frac{1}{2}, 1\right] \wedge Y =: \tilde{CY},$$

where $\tilde{CX}$ has basepoint at $t = 0$, and $\tilde{CY}$ has basepoint at $t = 1$. Note that these spaces are *not* constructed with the functor in Definition 2.16, only a new space homeomorphic

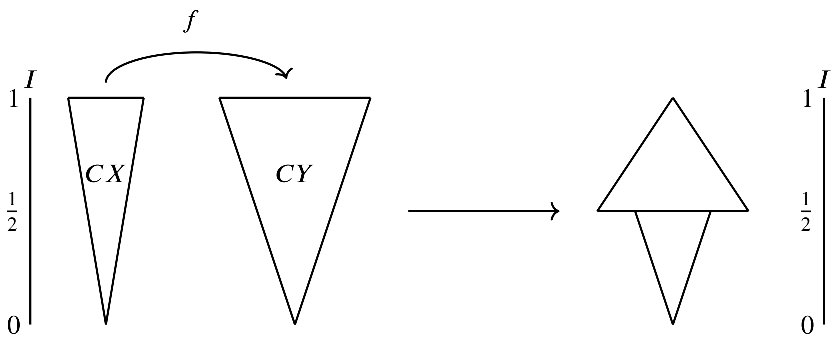


Figure 6: We have the cone of $X = [0,1]$ and $Y = [-1,2]$ on the left, where our map $f : X \longrightarrow Y$ is the inclusion. Then we want an identification on $C(i(f))$ as above, where CY is the upper half and CX is the lower half. We see that CY must be rotated before we can glue the cones together.

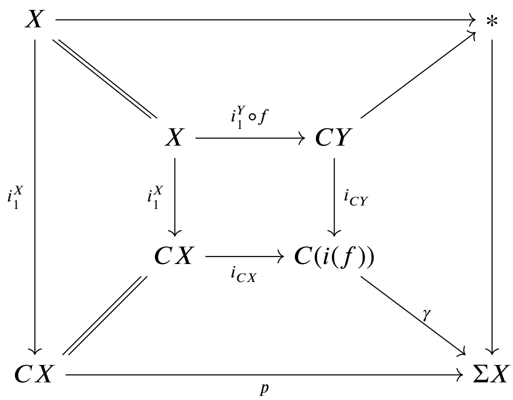
to the cone of X and Y. Furthermore, when $\tilde{CY}$ is glued to $\tilde{CX}$, $\tilde{CY}$ is both a rotated and shrinked version of CY, so the map $CY \longrightarrow \tilde{CY}$ is given by $[t,y] \mapsto \left[1 - \frac{1}{2}t, y\right]$.

Then $C(i(f)) \cong CX' \sqcup_X CY'$, which is exacly the space that Figure 6 describes. We see that $C(f) \cong \{[t,z]_{C(i(f)} : 0 \leq t \leq \frac{1}{2}\}$, so $C(f)$ lives in $C(i(f))$ as a subspace.

Under this identification, we see that the following maps are altered:

$$i_{CX} : CX \longrightarrow C(i(f)), [t,z]_{CX} \mapsto \left[\frac{t}{2}, z\right]_{C(i(f))}$$
$$i_{CY} : CY \longrightarrow C(i(f)), [t,z]_{CY} \mapsto \left[1 - \frac{1}{2}t, z\right]_{C(i(f))}$$
$$i(i(f)) : C(f) \longrightarrow C(i(f)), [t,z]_{C(f)} \mapsto \left[\frac{t}{2}, z\right]_{C(i(f))}$$

By writing out the diagram from the gluing lemma, we can explicitly say what γ does.



(12)

The map $p\colon CX \longrightarrow \Sigma X$ is the quotient map collapsing $X \times \{1\}$ to the basepoint. Looking at (12) we see that γ sends all classes $[t, z] \in C(i(f))$ for $\frac{1}{2} \leq t \leq 1$ to the basepoint, while for $0 \leq t \leq \frac{1}{2}$ they are sent to their corresponding class in ΣX. We also need γ to commute, i.e, we need $\gamma \circ i_{CX} = p$. Therefore, an explicit description of γ for our identification on $C(i(f))$ is given by

$$\gamma([t, z]_{C(i(f)}) = \begin{cases} [2t, z]_{\Sigma X}, & 0 \leq t \leq \frac{1}{2} \\ *, & \frac{1}{2} \leq t \leq 1, \end{cases}$$

which is well-defined since $[1, z]_{\Sigma X} \sim *$ for all z. We see also that

$$\gamma \circ i_{CX}([t, z]_{CX}) = \gamma([\frac{t}{2}, z]_{C(i(f))}) = [t, z]_{\Sigma X},$$

so (12) commutes.

Now we use this to check commutativity of (10). Firstly, we need $\gamma \circ i(i(f)) \simeq j(f)$. We have

$$j(f)([t, z]_{C(f)}) = [t, z]_{\Sigma X},$$

so we get

$$(\gamma \circ i(i(f)))([t, z]_{C(f)}) = \gamma([\frac{t}{2}, z]_{C(i(f))}) = [t, z]_{\Sigma X} = j(f)([t, z]_{\Sigma X},$$

i.e., the middle square in (10) commutes directly.

Now we check that $(-\Sigma f \circ \gamma) \simeq j(i(f))$. Since $j(i(f))$ collapses the subspace $C(f) \subseteq C(i(f))$ to the basepoint in ΣY, we see that $j(i(f))$ is described by

$$j(i(f))([t,z]_{C(i(f))}) = \begin{cases} *, & 0 \leq t \leq \frac{1}{2} \\ [2-2t, z]_{\Sigma Y}, & \frac{1}{2} \leq t \leq 1, \end{cases}$$

while

$$(-\Sigma f \circ \gamma)([t,z]_{C(i(f))}) = \begin{cases} [1-2t, f(z)]_{\Sigma Y}, & 0 \leq t \leq \frac{1}{2} \\ *, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then we claim that $H : C(i(f)) \times I \longrightarrow \Sigma Y$ given by

$$H([t,z]_{C(i(f))}, s) = \begin{cases} [(1-s)(1-2t), f(z)]_{\Sigma Y}, & 0 \leq t \leq \frac{1}{2} \\ [s(2-2t), z]_{\Sigma Y}, & \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a homotopy from $(-\Sigma f \circ \gamma)$ at $s = 0$ to $j(i(f))$ at $s = 1$. This holds because we have $[0, z]_{\Sigma Y} = *$ for all z.

This concludes the proof. □

Remark 4.14. Looking back at Figure 6 we see that we rotated CY when making our identification on $C(i(f))$. Yet, ΣY does *not* have this rotated identification in the triangle

$$Y \xrightarrow{i(f)} C(f) \xrightarrow{i(i(f))} C(i(f)) \xrightarrow{j(i(f))} \Sigma Y. \tag{13}$$

This explains why we need the minus in $-\Sigma f$. The map γ does not take into consideration that CY has been rotated, so we need $-\Sigma f$ to give ΣY the same identification as it has in (13).

The conclusion of this subsection is that if we restricted ourselves to doubly suspended (or more) spaces in CW_h, and Σ had an inverse which made right rotation compatible with our definition of triangles, we would have a triangulated category. This is, roughly speaking, the idea behind the Spanier–Whitehead category; we try to fix these flaws directly by defining some structure on $\mathscr{CW}_h$ which makes Σ invertible.

5 The Spanier–Whitehead category

5.1 Definition and triangulation

Definition 5.1. The **Spanier–Whitehead category**, $\mathscr{SW}$, consists of objects (X, n) where X is an object from $\mathscr{CW}_h$ and $n \in \mathbb{Z}$. The set of maps between objects is given by

$$\mathscr{SW}((X,m),(Y,n)) = \operatorname*{colim}_{r\to\infty}[\Sigma^{r+m}X, \Sigma^{r+n}Y].$$

Remark 5.2. Theorem 2.27 tells us that the colimit in this definition will be attained for some finite number r. Let r be a number such that $r-m$ and $r-n$ are positive. This is finite, since m and n are integers. Then, assume that $X' = \Sigma^r X$ is N dimensional, and that $Y' = \Sigma^r Y$ is k-connected. We have already seen that the suspension increases the connectivity and dimension of a space by 1. Then Theorem 2.27 tells us that $[\Sigma^l X', \Sigma^l Y'] \longrightarrow [\Sigma^{l+1}X', \Sigma^{l+1}Y']$ is a bijection if $N + l \leq 2(k+l)$. Equivalently, if $N - 2k \leq l$. Since we are restricting ourselves to finite CW-complexes, k and N are finite, we have that there is a large l such that for all $l' \geq l$ $[\Sigma^{l'}X', \Sigma^{l'}Y'] \longrightarrow [\Sigma^{l'+1]}X', \Sigma^{l'+1}Y']$ is an isomorphism.

Remark 5.3. The sets of maps in $\mathscr{SW}$ are constructed with the colimit operation, which is not completely unambiguous. For, which category is the colimit taken in? We have seen that $[X,Y]$ in $\mathscr{CW}_h$ are abelian groups if X is an (at least) doubly suspended space, so for $r \geq 2$ we can consider this a colimit in the category of abelian groups. However for $r = 0$ and $r = 1$, $[\Sigma^r X, Y]$ is a set and a group, respectively. Hence $\mathscr{SW}((X,n),(Y,m))$ is not *completely* well-defined. This could be mended by ignoring it, i.e., we first let the suspension act on the sets two times, and then do the colimit in abelian groups. However, through the colimit we identify elements in $[X,Y]$ and $[\Sigma X, \Sigma Y]$ with elements in $[\Sigma^2 X, \Sigma^2 Y]$ where we do have an abelian groups structure. Thus, we have an induced abelian group structure in $[X,Y]$ and $[\Sigma X, \Sigma Y]$ which we can use to make the colimit in Definition 5.1 more well-defined.

We see then that two objects (X,n) and (Y,m) in $\mathscr{SW}$ are isomorphic if and only if after some finite number r, $\Sigma^{n+r}X$ and $\Sigma^{m+r}Y$ are homotopy equivalent.

As mentioned before, this definition is the most direct way to mend the shortcomings of $\mathscr{CW}_h$. The sets $\mathscr{SW}((X,n),(Y,m))$ trivially abelian groups, just make a restriction on r in Definition 5.1 such that $\min\{r+n, r+m\} \geq 2$. Furthermore, this definition allows us to define an obvious automorphism on $\mathscr{SW}$.

Definition 5.4. The **suspension** in $\mathscr{SW}$, $s : \mathscr{SW} \longrightarrow \mathscr{SW}$, is given by $s(X,n) = (X, n+1)$.

Remark 5.5. The inverse of s, s^{-1} is of course given by $s^{-1}(X,n) = (X, n-1)$, and we see that $s \circ s^{-1} = id_{\mathscr{SW}}$ and $s^{-1} \circ s = id_{\mathscr{SW}}$. So s is an automorphism.

We also have an obvious functor from $\mathscr{CW}_h \longrightarrow \mathscr{SW}$ where $X \mapsto (X,0)$, known as the **stabilization**.

The suspension in the homotopy category of CW-complexes extends naturally to the Spanier Whitehead category, where we let $\Sigma(X,n) = (\Sigma X, n)$.

Proposition 5.6. *There is a natural equivalence from s to Σ.*

Proof. We have that $s(X,n) \cong \Sigma(X,n) \cong (\Sigma X, n)$ if for some r we have $\Sigma^r(s(X,n)) \cong \Sigma^r(\Sigma X, n)$ in $\mathscr{CW}_h$. This holds because in $\mathscr{SW}$ we have

$$\mathscr{SW}(s(X,n),(\Sigma X,n)) = \operatorname*{colim}_{r\to\infty}[\Sigma^{r+n+1}X, \Sigma^{r+n}(\Sigma X)] = \operatorname*{colim}_{r\to\infty}[\Sigma^{r+n+1}X, \Sigma^{r+n+1}X]$$

which means that in $[\Sigma^{r+n+1}X, \Sigma^{r+n+1}X]$ for some large enough r we find the identity, which corresponds to the desired natural isomorphism in $\mathscr{SW}(s(X,n),(\Sigma X,n))$. □

This means that we can bring a lot of the structure from $\mathscr{CW}$ and $\mathscr{CW}_h$ into $\mathscr{SW}$. We see, for instance, that s is additive, as a consequence of Proposition 5.6 and Corollary 4.12. Which means that Σ is an additive automorphsim, as we would like it to be. Note that the inverse suspension Σ^{-1} does not have any meaningful geometric interpretation, it is only a technical tool to enable the triangulated structure.

For there to be a triangulated structure on $\mathscr{SW}$ it needs to be additive. Now, the coproduct is inherited from $\mathscr{CW}$, where it is $X \vee Y$. Then in $\mathscr{SW}$ the coproduct of (X,n) and (Y,m) is

$$(X,n) \oplus (Y,m) = (\Sigma^{l-m}(X) \vee \Sigma^{l-n}(Y), n+m-l),$$

where $l \geq \max\{n,m\}$.

We also inherit the zero-object from $\mathscr{CW}_h$: look at $[\Sigma^r X, *]$ and $[*, \Sigma^r X]$ in $\mathscr{CW}_h$. For all r they can only consist of one map, namely the map sending everything to the point, and the inclusion to the basepoint, respectively. Hence, the zero-object in $\mathscr{SW}$ is the one-point space $(*,0)$.

Proposition 5.7. *The category $\mathscr{SW}$ is additive.*

Proof. This is a direct consequence of Lemma 4.3 and Theorem 4.11, where we have seen that there is an additive structure on $\mathscr{CW}_h$. In fact, it would be enough to restrict ourselves to doubly-suspended spaces in $\mathscr{CW}_h$ for there to be a biproduct. The bilinearity of compositions can be checked by the reader. □

Hence we are all set to define the triangulated structure on $\mathscr{SW}$.

Definition 5.8. For a sequence $(X,n) \longrightarrow (Y,m) \longrightarrow (Z,k) \longrightarrow (X,n+1)$ of objects in $\mathscr{SW}$ we say that it is a **distinguished triangle** if for some even number l the following sequence

$$\Sigma^{l+n}X \xrightarrow{f} \Sigma^{m+l}Y \xrightarrow{g} \Sigma^{k+l}Z \xrightarrow{h} \Sigma^{n+l+1}X$$

of objects in $\mathscr{CW}_h$ is an unstable distinguished triangle.

The restriction on l to even numbers will become obvious in the following proof.

Theorem 5.9. *The Spanier–Whitehead category is triangulated.*

Proof. For a distinguished triangle

$$(X,n) \longrightarrow (Y,m) \longrightarrow (Z,k) \longrightarrow (X,n+1) \tag{14}$$

we can find an unstable distinguished triangle representing (14). Using these representations we see that we can repeat all of the arguments from Theorem 4.9 and Theorem 4.13, and so $\mathscr{SW}$ satisfies almost all of Definition 4.5. There are only two properties left to show: right rotation and that the trivial triangle $(X,n) \xrightarrow{id_{(X,n)}} (X,n) \longrightarrow (*,0) \longrightarrow (X,n+1)$ is distinguished. However, the latter follows from the former and part (i) in Theorem 4.9. So we are left with only right rotation.

Let $(X,n) \longrightarrow (Y,m) \longrightarrow (Z,k) \longrightarrow (X,n+1)$ be a distinguished triangle. Then we have a large, even l such that the following triangle in $\mathscr{CW}_h$ is an unstable distinguished triangle.

$$\Sigma^{n+l}X \xrightarrow{f} \Sigma^{m+l}Y \xrightarrow{g} \Sigma^{k+l}Z \xrightarrow{h} \Sigma^{n+l+1}X. \tag{15}$$

Rotating this triangle five times to the left gives us

$$\Sigma^{k+l+1} \xrightarrow{-\Sigma h} \Sigma^{n+l+2}X \xrightarrow{\Sigma^2 f} \Sigma^{m+l+2}Y \xrightarrow{\Sigma^2 g} \Sigma^{k+l+2}Z. \tag{16}$$

Then $(Z,k-1) \longrightarrow (X,n) \longrightarrow (Y,m) \longrightarrow (Z,k)$ kan be represented by (16), an unstable distinguished triangle in $\mathscr{CW}_h$, and hence the right rotation of (15) is a distinguished triangle. We see here why we restrict ourselves to even numbers in the definition of triangles in $\mathscr{SW}$ – it is forced by the signs that appear in the rotations.

This completes the proof. □

Remark 5.10. The Spanier–Whitehead category is sometimes defined with objects included from $\mathscr{CW}$ and $\mathscr{SW}(X,Y) = \operatorname{colim}_{r\to\infty}[\Sigma^r X, \Sigma^r Y]$. See for instance [2, pp. 28]. This does seem similar at first sight, but with this definition we have, at least according to Margolis, no formal desuspension.

5.2 $\mathscr{SW}$ as a stable homotopy category

Up until now the triangulated structure on the stable homotopy category has been the main focus, and we have indeed shown that $\mathscr{SW}$ is triangulated. However, it is not all we want from a stable homotopy category. There are several models for the stable homotopy category, and they do vary slightly. Here, we will base ourselves on [2, pp. 26 – 27], where a list of the wanted properties is presented. We will see that our naive attempt is, in fact, naive, and falls short in several aspects.

Firstly, we have restricted ourselves to finite CW-complexes, which also means that we have restricted ourselves to finite coproducts, as a countable coproduct of finite CW-complexes rarely is a finite CW-complex. This is a disadvantage because it leaves out spaces like the infinite sphere S^∞, and the infinite projective spaces, $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$. Were we, however, to include infinite CW-complexes in our definition we would see that our definition of coproducts in $\mathscr{SW}$ does not extend to the infinite case. Take for instance

$$\mathscr{SW}\left(\bigvee_{i\geq 0}(S^i,0),(S^0,0)\right) = \operatorname*{colim}_{r\to\infty}\left[\bigvee_{i\geq 0}\Sigma^r S^i, \Sigma^r S^0\right] \cong \operatorname*{colim}_{r\to\infty}\prod_{i\geq 0}\left[S^{r+i}, S^r\right]$$
$$\neq \prod_{i\geq 0}\operatorname*{colim}_{r\to\infty}\left[S^{r+i}, S^r\right] = \prod_{i\geq 0}\operatorname*{colim}_{r\to\infty}\left[\Sigma^r S^i, \Sigma^r S^0\right] = \prod_{i\geq 0}\mathscr{SW}\left((S^i,0),(S^0,0)\right),$$

which must hold for coproducts.

Another important notion is **Brown Representability**. This is a result about reduced cohomology theories; for a reduced cohomology theory $\tilde{E}^*$ we have that for all n there is a connected CW-complex K_n such that for all connected CW-complexes X there is a natural isomorphism $\tilde{E}^n(X) \cong [X, K_n]$. What holds in the stable homotopy category, and fails in $\mathscr{SW}$, is the opposite. Namely, that every element in the stable homotopy category defines a reduced cohomology theory on pointed CW-complexes. In $\mathscr{SW}$ that would mean that every $(X,0)$ defines a reduced cohomology theory via $\tilde{H}^i(Y) = \mathscr{SW}((Y,0),(X,i))$. As this is zero whenever i is greater than the dimension of Y, and hence we can not represent for example K-theory, which is periodic and therefore has negative dimension cohomologies.

We should also mention that $\mathscr{SW}$ is a symmetric monoidal category, as we would like the stable homotopy category to be, with the smash product as the symmetric monoidal product, and the 0-sphere as the unit. It has not been given much attention through this thesis, but it does follow from the naturality of the morphisms in Theorem 2.15.

One of the interesting things to notice, is that the category $\mathscr{SW}$ as defined here, with finite CW-complexes, should be contained in every stable homotopy category as a subcategory of compact objects. This tells a great deal about $\mathscr{SW}$ - it is the right idea, but it is just too small. We need more objects, for the reasons described above.

The overall takeaway from this thesis is that the Spanier–Whitehead category is a great place to start when learning about the stable homotopy category. It is simple, and it has a lot of geometric intuition, while it tells you clearly where to continue in the search for the stable homotopy category.

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