Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences

Sigurd Gaukstad

The Balmer Spectrum of the Local Equivariant Stable Homotopy Category

Master's thesis in Mathematical Sciences
Supervisor: Drew Heard
December 2021

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Abstract. We introduce tensor triangulated geometry, with the goal of calculating the *Balmer spectrum* of the local G-equivariant stable homotopy category, for a finite group G. To start off, we define basic notions of tensor triangulated geometry. Then, we introduce the ∞-category of *local G-spectra* and see how it splits into a product of functor categories. We then compute the *Balmer spectrum* of each part. Finally, we will also see how our methods can, and cannot be used, to calculate the Balmer Spectra of *local G-spectra* for a general compact Lie group.

Sammendrag. Vi introduserer tensor triangulert geometri, med et mål for å regne ut *Balmer spektrumet* til den lokale G-ekvivariante stabile homotopi kategorien, for en endelig gruppe G. Vi starter ved å definere grunnleggende egenskaper for tensor triangulerte kategorier. Så introduserer ∞-kategorien av lokal G-spektra, og ser hvordan den splitter til et produkt av funktorkategorier. Videre, regner vi ut *Balmer spektrumet* til hver del. Avslutningsvis, vil vi også se hvordan våre metoder kan, og kan ikke, bli brukte til og regne *Balmer spektrumet* til lokal G-spektra for en generell Liegruppe.

Acknowledgements

This thesis concludes my master's degree in mathematics at NTNU. I want to thank my supervisor, Drew Heard, for suggesting and guiding me through the problem for this thesis and for taking his time for our weekly discussions on tensor triangulated geometry. It has been an enjoyable and instructive experience that has introduced me to many challenging and interesting fields of mathematics. Moreover, I would like to thank Cinco, Mørlenda, Elgeseter and Lesesal 395b. I also want to thank my mother and Melissa for proofreading.

The donut on the front page is not directly related to my thesis, but considering the number of different geometric shapes, it makes a great cover for a thesis in Algebraic Topology.

Sigurd Gaukstad
Trondheim, 2021

Outline

The goal of this thesis is to calculate the Balmer spectrum of the $\mathbb{Z}_P$-local G-equivariant stable homotopy category. The first three chapters, will introduce the setting where this calculation takes place. The last three chapters, will be used to compute the Balmer spectrum. In more detail:

Chapter 1 is used to give an informal introduction to tensor triangulated categories and the Balmer spectrum. We will introduce some known breakthroughs in the field and state the results from this thesis.

Chapter 2 is an introduction to tensor triangulated geometry. We will also see how the Balmer spectra of a product of tensor triangulated categories decomposes into a coproduct of Balmer spectra.

Chapter 3 is an introduction to the ∞-category of G-spectra. We will see how one can obtain ∞-categories from model categories. Moreover, we will define the $\mathbb{Z}_P$-local stable G-equivariant homotopy category. For different groups G, we will also introduce important functors between the ∞-categories of G-spectra.

Chapter 4 is used to prove the decomposition of the ∞-category of R-local G-spectra, into a finite product of functor categories $\mathrm{Fun}(BW_GH, \mathrm{Sp}_{\mathbb{Z}_P})$.

Chapter 5 is used to compute the Balmer spectra of $\mathrm{Fun}(BW_GH, \mathrm{SH}_{\mathbb{Z}_P})^{\mathrm{dual}}$. Moreover, we will describe the Balmer spectra of $\mathrm{SH}^c_{\mathbb{Z}_P}(G)$.

Chapter 6 is used to compute parts of the Balmer spectrum, of the rational stable homotopy category, $\mathrm{SH}^c_{\mathbb{Q}}(\mathrm{SO}(3))$.

Contents

Chapter 1

Introduction

This chapter is meant as an informal introduction to the Balmer spectrum of tensor triangulated categories. We will establish for the reader some of the known breakthroughs on the subject, and finally, present the results from this thesis.

1.1 Introducing the Balmer spectrum

Recall that for a commutative ring R with 1, we can define the Zariski topology on the set of prime ideals $\mathrm{Spec}(R)$ of R. The Zariski topology on $\mathrm{Spec}(R)$ is defined by the closed sets,

$$V(I) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \subset \mathfrak{p}\} = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \cap \mathfrak{p} = I\},$$

where $I \subset R$ is a subset of elements. This gives a connection between algebra and geometry and is an essential ingredient in modern algebraic geometry.
Tensor triangulated categories $(\mathcal{K}, \otimes, \oplus, \mathbb{1}, 0)$ behave a lot like rings, in the sense that they have a commutative product $\otimes$ with unit $\mathbb{1}$ and commutative addition $\oplus$ with identity 0, up to isomorphism. We will only care about objects up to isomorphism, and will therefore often treat isomorphism as identities. The Zariski topology on the Balmer spectra $\mathrm{Spc}(\mathcal{K})$ is defined similarly as for a ring. We define $\mathrm{Spc}(\mathcal{K})$ as the collection of prime tensor ideals in $\mathcal{K}$ and define the Zariski topology on $\mathrm{Spc}(\mathcal{K})$ by letting the closed sets be,

$$Z(\mathcal{S}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid S \cap \mathcal{P} = \emptyset\},$$

where $\mathcal{S} \subset \mathcal{K}$ is a subset of objects. One thing to note is that both $V(-)$ and $Z(-)$ reverse inclusions. However, the topology on $\mathrm{Spc}(\mathcal{K})$ differs from the one on $\mathrm{Spec}(R)$ as the *open subsets* of $\mathrm{Spc}(\mathcal{K})$ are defined as,

$$U(\mathcal{S}) := \mathrm{Spc}(\mathcal{K}) \setminus Z(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid S \cap \mathcal{P} \neq \emptyset\},$$

which gives the same formula as the *closed subsets* of $\mathrm{Spec}(R)$. Let $D(R^{\mathrm{perf}})$ be the derived category of *perfect* complexes over R, that is, complexes over R that are quasi-isomorphic to a bounded complex projective R-modules. Note also that $D(R^{\mathrm{perf}}) \cong D(R)^c$, i.e., the perfect complexes are precisely the compact objects of the unbounded derived category of R-complexes; $D(R)$. By [Bal05] there is a reassuring relation between the Balmer spectrum and the spectrum of a ring. Namely, there is a homeomorphism

$$\mathrm{Spc}(D(R^{\mathrm{perf}})) \cong \mathrm{Spec}(R), \tag{1.1}$$

of topological spaces. This homeomorphism further legitimize calling the Balmer spectrum a "spectrum". Moreover, as we might have suspected from the definitions of the different Zariski topologies on $\mathrm{Spc}(D(R)^c)$ and $\mathrm{Spec}(R)$, this map is inclusion reversing.

For the Balmer spectrum to actually define a set, we have to assume a certain smallness condition on the tensor triangulated categories we work with. The tensor triangulated category $\mathcal{K}$ should be essentially small, that is, $\mathcal{K}$ is equivalent to a small category. Consequently, $\mathcal{K}$ has a set of isomorphism classes of objects. This implies that $\mathrm{Spc}(\mathcal{K})$ is actually a set. We remark, as in [Bal05], that one could instead fix a universe to work in. Given a possibly "big" tensor triangulated category C, we can consider the full subcategory $C^c \subset C$, consisting of the compact objects of C. Under the assumption that the $\otimes$-unit of C is compact, C^c is an essentially small category, and a perfect example of an essentially small tensor triangulated category.

A natural question that arises when introducing a theory is the following: Why is it interesting to study? If one is not happy with the simple answer that tensor triangulated geometry gives a nice theory, one can hopefully be more pleased by knowing that certain subsets of Balmer spectrum of $\mathcal{K}$ classify the thick radical $\otimes$-ideals of $\mathcal{K}$.

THEOREM 1.1. *[**Theorem 4.10**; Bal05] Let $\mathcal{G}$ be the set of those subsets $Y \subset \mathrm{Spc}(\mathcal{K})$ of the form $Y = \bigcup_{i\in I} Y_i$ for closed subsets $Y_i \subset \mathrm{Spc}(\mathcal{K})$ with $\mathrm{Spc}(\mathcal{K}) \setminus Y_i$ quasi-compact for all $i \in I$. Let $\mathcal{R}$ be the set of radical thick $\otimes$-ideals of $\mathcal{K}$. Then there is an order preserving bijection $\mathcal{G} \xrightarrow{\cong} \mathcal{R}$.*

As further remarked in [Bal05], in the case that $\mathrm{Spc}(\mathcal{K})$ is Noetherian, one can replace the Thomason condition "$Y = \cup_i Y_i$ with $\mathrm{Spc}(\mathcal{K}) \setminus Y_i$ being quasi-compact" with the simpler condition that "Y is specialization closed in $\mathrm{Spc}(\mathcal{K})$". Then we can classify the thick $\otimes$-ideals of $\mathcal{K}$ via the specialization closed subsets of $\mathrm{Spc}(\mathcal{K})$:

$$\{Y \subset \mathrm{Spc}(\mathcal{K}) \mid Y \text{ is specialization closed}\} \leftrightarrow \{I \subset \mathcal{K} \mid \text{thick } \otimes\text{-ideal}\}$$

As an example, one can use Equation (1.1) to classify the thick $\otimes$-ideals of $D(R)^c$ via the specialization closed subsets of $\mathrm{Spec}(R)$.

1.2 Some known calculations

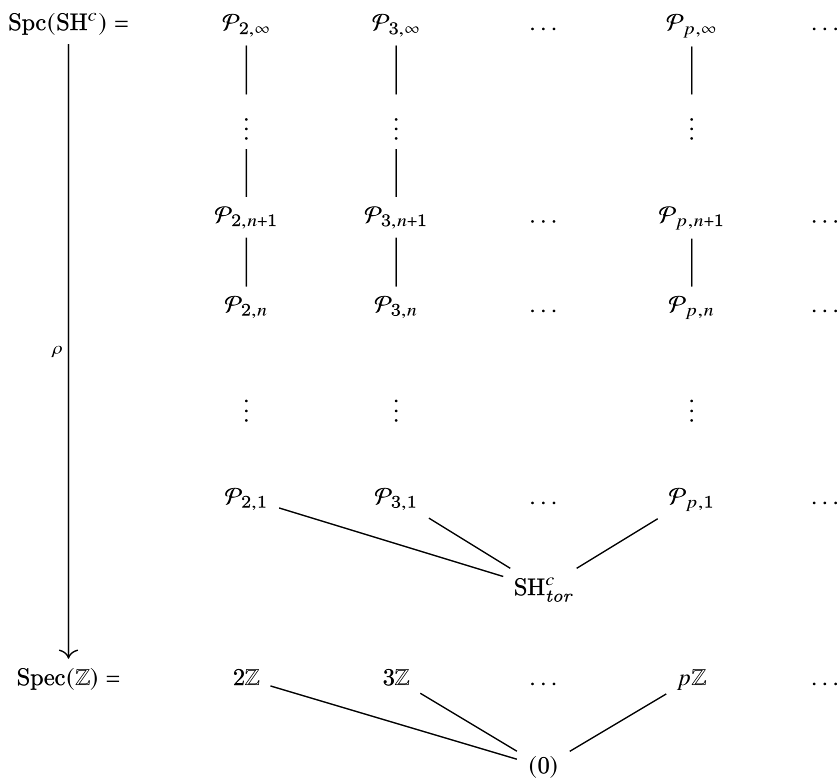
To help the reader get a feel of what is known and the complexity of the question of calculating $\mathrm{Spc}(\mathcal{K})$, we list some of the biggest results in the field. The list is greatly inspired by a similar one in [Bal20]. As mentioned above, the following example is known.

EXAMPLE 1.2. Let R be a commutative ring with 1, then by [Bal05]

$$\mathrm{Spc}(D(R^{\mathrm{perf}})) \cong \mathrm{Spec}(R)$$

Let G be a compact Lie group. An especially difficult tensor triangulated category to work with is the G-equivariant stable homotopy category, denoted $\mathrm{SH}(G)$. We will identify the non-equivariant stable homotopy category SH with $\mathrm{SH}(\{*\})$. The precise topology of $\mathrm{Spc}(\mathrm{SH}(G)^c)$ is still unknown, but some special cases are known. For $G = \{*\}$, Balmer described the topological space of $\mathrm{Spc}(\mathrm{SH}^c)$ in [Bal10], based on deep work of Hopkins-Smith, via *Morava K-theories.*

Corollary 1.3. *[***Corollary 9.5***; Bal10] The spectrum of* SH^c *is the following topological space:*



Here, the chain of prime ideals,

$$\mathcal{P}_{p,\infty} \subset \dots \mathcal{P}_{p,n+1} \subset \mathcal{P}_{p,n} \subset \dots \mathcal{P}_{p,1} \subset \mathrm{SH}^c_{tor},$$

is in the preimage of the prime ideal $p\mathbb{Z}$ *under* ρ*. Moreover,* SH^c_{tor} *denotes the subcategory of torsion spectra, and a line in the diagram indicates that a higher prime is in the closure of a lower one.*

Balmer and Sanders further completely describe the set $\mathrm{Spc}(\mathrm{SH}(G)^c)$ for a finite group G, using the *geometric fixed point functors* $\Phi^H\colon \mathrm{SH}(G) \to \mathrm{SH}$ for $H \leq G$. The induced map on the Balmer spectra $\mathrm{Spc}(\Phi^H)\colon \mathrm{Spc}(\mathrm{SH}^c) \to \mathrm{Spc}(\mathrm{SH}^c(G))$ cover $\mathrm{Spc}(\mathrm{SH}^c(G))$ as H varies over the conjugacy classes of $H \leq G$.

These define prime ideals, $\mathcal{P}(H, p, n) := \operatorname{Spc}(\Phi^H)(\mathcal{P}_{p,n})$. We will use the geometric fixed point functors to decompose the ∞-category of R-local G-spectra, in Theorem 4.1.

THEOREM 1.4 ([BS17]). *Let G be a finite group. Then every tt-prime in $\mathrm{SH}^c(G)$ is of the form $\mathcal{P}(H, p, n)$ for a unique subgroup $H \leq G$ up to conjugation and a chromatic tt-prime $\mathcal{P}_{n,p} \in \operatorname{Spc}(\mathrm{SH}^c)$. Furthermore, understanding the inclusions between tt-primes completely describes the topology on $\mathrm{SH}^c(G)$.*

The two main issues with $\mathrm{SH}(G)$ are equivariance and torsion. To simplify the question about $\mathrm{SH}(G)$, it is helpful to restrict to the R-localized stable homotopy category $\mathrm{SH}_R(G)$, for a commutative ring R. In other words, the category $\mathrm{SH}(G)$ where we invert maps $f: X \to Y \in \mathrm{SH}(G)$ that give an isomorphism $\pi_k^H(f) \otimes \mathrm{id}_R : \pi_k^H(X) \otimes R \to \pi_k^H(Y) \otimes R$ for all closed subgroups $H \leq G$ and $k \in \mathbb{Z}$. By choosing a suitable ring R, for example $\mathbb{Q}$, we are able to kill the torsion.
In Chapter 5 we will see the following:

PROPOSITION 1.5. *(Proposition 5.23) Let G be a finite group, then*

$$\operatorname{Spc}(\mathrm{SH}^c_{\mathbb{Q}}(G)) \cong \coprod_{(H) \leq G} \{*\}. \tag{1.2}$$

The previous proposition is a special case of [**Theorem 1.3**; Gre19] by Greenlees, when G is finite.
For a compact Lie group G, Greenlees describes the Balmer spectrum of the rational G-equivariant stable homotopy category using the *geometric fixed point functors* Φ^H. Every closed subgroup $H \leq G$ defines a tt-prime $\mathcal{P}_H \in \operatorname{Spc}(\mathrm{SH}^c_{\mathbb{Q}}(G))$, where $\mathcal{P}_H$ is the kernel of the geometric fixed point functor $\Phi^H : \mathrm{SH}^c_{\mathbb{Q}}(G) \to \mathrm{SH}^c_{\mathbb{Q}} \cong D(\mathbb{Q})^c$. If $H \sim_G \tilde{H}$ belong to the same conjugacy class of G, then their geometric fixed point functors coincide, and they yield the same prime. Recall that a closed subgroup $H \leq G$ is called cotoral if H is a normal subgroup of G and G/H is a torus, i.e., $G/H \cong (\mathbb{R}/\mathbb{Z})^k$ for some $k \geq 1$. Greenlees completely describes the Balmer spectra of the rational stable G-equivariant homotopy category, for a compact Lie group G, in the following theorem.

THEOREM 1.6. *[**Theorem 1.3**; Gre19] Let G be a compact Lie group. Then every tt-prime of the rational G-equivariant stable homotopy category $\mathrm{SH}^c_{\mathbb{Q}}(G)$ is equal to $\mathcal{P}_H$ for a closed subgroup $H \leq G$, that is unique up to conjugation. We have $\mathcal{P}_K \subset \mathcal{P}_H$ if and only if K is conjugate to a cotoral subgroup of H. The topology of $\operatorname{Spc}(\mathrm{SH}^c_{\mathbb{Q}}(G))$ corresponds to the f-topology from [**Section 8**; Gre98a].*

1.3 Results from this thesis

In this thesis, we will not only recover Greenless result [**Theorem 1.3**; Gre19] for a finite group G, but we are able to relax the assumptions on the ring $R \subset \mathbb{Q}$, and only invert the group order of G. Furthermore, we can calculate the Balmer spectra of $\mathrm{SH}_{\mathbb{Z}_P, \mathcal{F}}(G)$, where G is a compact Lie group and $\mathcal{F}$ is a collection of closed subgroups of G, with some further hard restrictions, stated below. We find that these restrictions on $\mathcal{F}$ make the result too hard to be particularly useful. Moreover, using Kędzioreks decomposition of

$\mathrm{SH}_{\mathbb{Q}}(SO(3))$, in [Kęd17a], we can calculate parts of the Balmer spectrum of $\mathrm{SH}^c_{\mathbb{Q}}(SO(3))$.

Main results

Let P be a set of primes, and define $\mathbb{Z}_P$ to be $\mathbb{Z}$ with every prime not in P inverted. Note that if P is empty, then $\mathbb{Z}_P = \mathbb{Q}$ and if P is the set of all primes, then $\mathbb{Z}_P = \mathbb{Z}$. Using Wimmer's decomposition of $\mathrm{Sp}_{\mathrm{R}}(G)$, in [Wim19], for a finite group G, we can prove:

THEOREM A (Theorem 4.1). *Let $\mathbb{Z}_P \subset \mathbb{Q}$ be a subring and let G be a finite group such that $|G|$ is invertible in $\mathbb{Z}_P$. Then we have a homeomorphism*

$$\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Z}_P}(G)) \cong \coprod_{(H)\leq G} \mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Z}_P}).$$

By [**Corlollary 9.5**; Bal10], we can fully describe $\mathrm{Spc}(\mathrm{SH}^c)$ and therefore $\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Z}_P})$. We look into this in Section 5.3 in Chapter 5.

Minor results

Using Wimmer's decomposition in its full capacity, we can prove:

THEOREM B (Theorem 5.29). *Let G be a compact Lie group, let $\mathbb{Z}_P \subset \mathbb{Q}$ be a subring and let $\mathcal{F}$ be a family of finite closed subgroups $H \leq G$, such that $|W_G H|$ is invertible $\mathbb{Z}_P$. If additionally, the subgroups of $\mathcal{F}$ only belong to finitely many conjugacy classes, then*

$$\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Z}_P,\mathcal{F}}(G)) \cong \coprod_{(H)\in\mathcal{F}} \mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Z}_P}).$$

Using Kędzioreks decomposition of $\mathrm{SH}_{\mathbb{Q}}(SO(3))$ in [Kęd17a], we can deduce the following result:

THEOREM C (Theorem 6.5). *Let $\mathcal{T}$ and $\mathcal{D}$ be the homotopy category of $L_{e_{\mathcal{T}}S_{\mathbb{Q}}}(\mathrm{OrthSpec}_{\mathbb{Q}}(\mathrm{SO}(3))$ and $L_{e_{\mathcal{D}}S_{\mathbb{Q}}}(\mathrm{OrthSpec}_{\mathbb{Q}}(\mathrm{SO}(3))$, respectively, then*

$$\mathrm{Spc}((\mathrm{SH}^c_{\mathbb{Q}}(\mathrm{SO}(3))) \cong \mathrm{Spc}(\mathcal{T}^{\mathrm{dual}}) \sqcup \mathrm{Spc}(\mathcal{D}^{\mathrm{dual}}) \coprod_{i=1}^{5}\{*\}.$$

Here, each i corresponds to one of the five conjugacy classes of exceptional subgroups of $SO(3)$.

Chapter 2

Tensor Triangulated Geometry

The goal of this chapter is to give an introduction to tensor triangulated geometry. Tensor triangulated geometry is the study of tensor triangulated categories, by methods from algebraic geometry. Being the main object of this theory, we will first remind the reader of what tensor triangulated categories are. Then, similarly as for rings, we will define the *spectrum* of prime ideals in tensor triangulated categories and endow this set with a topology. To not confuse this spectrum with the Zariski spectrum of a commutative ring, we will call this spectrum the *Balmer spectrum.* After seeing some basic definitions and results regarding the Balmer spectrum and tensor triangulated categories, we will prove an important result. Namely, that the Balmer spectrum commutes with taking finite products of tensor triangulated categories. This will later be used to compute the Balmer spectrum of $\mathrm{SH}^c_{\mathbb{Z}_p}(G)$.

2.1 Triangulated categories

We assume that the reader is familiar with triangulated categories, but we will briefly remind ourselves of some essential definitions and ideas.
A triangulated category is an additive category together with an auto-equivalence, often called the *shift functor* by algebraists, as it comes from shifting a complex, or the *suspension functor*, by topologists, as it comes from the suspension of a space. We also have a class of diagrams that we will call *triangles, distinguished triangles* or *exact triangles*, that satisfy some axioms that can be found at [Sta21, Definition 0145], if needed. We will work in a triangulated category $\mathcal{T}$ in the rest of the chapter, and we will denote the *Suspension* functor by $T\colon \mathcal{T} \to \mathcal{T}$.
First, let us remind ourselves what the correct definition of a triangulated subcategory is.

Definition 2.1. We say that $\mathcal{K}$ is a **triangulated subcategory** of a triangulated category $\mathcal{T}$, if $\mathcal{K}$ is a full subcategory of $\mathcal{T}$, containing 0, is closed under taking suspensions;

$$\forall n \in \mathbb{N} \quad x \in \mathcal{K} \Rightarrow T^n x \in \mathcal{K}$$

and closed under distinguished triangles; if $x \to y \to z \to Tx$ is a distinguished triangle in $\mathcal{T}$, and two out of x, y, z are in $\mathcal{K}$, then so is the third.

If additionally $\mathcal{K}$ is closed under taking summands;

$$x \oplus y \in \mathcal{K} \Rightarrow x, y \in \mathcal{K},$$

then we call $\mathcal{K}$ a **thick** triangulated subcategory of $\mathcal{T}$.

A triangulated subcategory is in particular additive, and is therefore closed under *finite* coproducts. We will now define triangulated subcategories that are closed under taking *arbitrary* coproducts.

DEFINITION 2.2. A triangulated subcategory $\mathcal{K} \subset \mathcal{T}$ is **localizing**, if it is closed under taking arbitrary coproducts. For a collection of objects $\mathcal{S} \subset \mathcal{K}$, we denote by $Loc(\mathcal{S})$, the smallest localizing subcategory containing all the objects in $\mathcal{S}$.

REMARK 2.3. Every localizing subcategory of a triangulated category $\mathcal{T}$ is thick: Suppose that $\mathcal{K} \subset \mathcal{T}$ is a localizing subcategory containing $x \oplus y$. Then, $(x \oplus y)^{\mathbb{N}}$ and $x \oplus (y \oplus x)^{\mathbb{N}}$ belong to $\mathcal{K}$. Let

$$\phi\colon (x \oplus y) \oplus (x \oplus y) \oplus \cdots \to x \oplus (y \oplus x) \oplus (y \oplus x) \oplus \ldots,$$

be the map that twists and injects $(x \oplus y)$ into $(y \oplus x)$. One can then observe that the cone of ϕ is x, proving that x belongs to $\mathcal{K}$.

The triangulated categories we will study in this thesis will be nice in the sense that they are *compactly generated* by a set of *compact* objects $\mathcal{G}$. That is, there is no proper *triangulated subcategory* containing the set of *compact* objects $\mathcal{G} \subset \mathcal{T}$, that is closed under taking arbitrary coproducts. In other words, the localizing subcategory generated by $\mathcal{G}$ is the whole category $\mathcal{T}$.
We will look at these important definitions more carefully:

DEFINITION 2.4. Let $y_i \in \mathcal{T}$ be a family of objects indexed by a set I, whose coproduct is in $\mathcal{T}$. An object $x \in \mathcal{T}$ is **compact** if for any $\phi\colon x \to \coprod_{i\in I} y_i \in \mathcal{T}$, there exists a finite subset $J \subset I$, such that ϕ factors through $\coprod_{i\in J} y_i$.

We note that this has another useful description.

PROPOSITION 2.5. *Let $y_i \in \mathcal{T}$ be a family of objects indexed by a set I, whose coproduct is in $\mathcal{T}$. An object $x \in \mathcal{T}$ in a triangulated category is compact if and only if for all families of objects $y_i \in \mathcal{T}$*

$$\mathrm{Hom}_{\mathcal{T}}(x, \coprod_{i\in I} y_i) \cong \coprod_{i\in I} \mathrm{Hom}_{\mathcal{T}}(x, y_i)$$

Compact objects can be remembered as having a certain finiteness condition, similar to a compact cover in topology.

NOTATION 2.6. We will let $\mathcal{T}^c$ denote the full subcategory of compact objects, but we warn that it is not necessarily closed under arbitrary coproducts.

DEFINITION 2.7. A set of objects $\mathcal{G}$ in a triangulated category $\mathcal{T}$ is a **set of generators** if for any $x \in \mathcal{T}$, such that for all $g \in \mathcal{G}$ and $m \in \mathbb{Z}$

$$\mathrm{Hom}_{\mathcal{T}}(T^m g, x) = 0$$

we must have that $x = 0$.

By Schwede and Shipley we have the following result.

LEMMA 2.8. *[**Lemma 2.2.1.**; SS03] Let $\mathcal{G} \subset \mathcal{T}$ be a set of compact objects. Then $\mathcal{G}$ is a set of compact generators if and only if $Loc(\mathcal{G}) = \mathcal{T}$.*

The result explains why we call them *compact generators*, as they are the collection of *compact* objects that *generate* the triangulated category as a localizing category.

2.2 Tensor triangulated categories

We will now introduce the language of tensor triangulated geometry, inspired by Balmer's paper [Bal05]. We often shorten "tensor triangulated" to just "tt".

DEFINITION 2.9. A **tensor triangulated category** (tt-category) is a triple $(\mathcal{K}, \otimes, \mathbb{1})$ where $\mathcal{K}$ is a triangulated category and $\otimes\colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is a *symmetric monoidal product* which is exact (hence additive) in each variable and with $\mathbb{1}$ as unit.

DEFINITION 2.10. A **lax symmetric monoidal functor** is a functor between tt-categories $F\colon \mathcal{C} \to \mathcal{D}$, together with a morphism $\epsilon\colon \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$ and a natural transformation $\mu_{x,y}\colon F(x) \otimes F(y) \to F(x \otimes y)$, satisfying the usual coherence axioms (See for instance [nLa21]). In the case that ϵ and $\mu_{x,y}$ are both isomorphisms, we call F a **tt-functor**.

DEFINITION 2.11. A **thick tensor ideal** $\mathcal{A}$ of $\mathcal{K}$ is a full subcategory containing 0 and such that the following conditions are met:

(i) $\mathcal{A}$ is **triangulated**: for any distinguished triangle $x \to y \to z \to Tx$ in $\mathcal{K}$, if two out of x, y, z are in $\mathcal{A}$, so is the third.

(ii) $\mathcal{A}$ is **thick**: if an object $a \in \mathcal{A}$ splits in $\mathcal{K}$ as $a \cong b \oplus c$, then both summands b and c belong to $\mathcal{A}$.

(iii) $\mathcal{A}$ is a **tensor ideal**: if $a \in \mathcal{A}$ and $b \in \mathcal{K}$, then $a \otimes b$ belongs to $\mathcal{A}$.

We note that (i) implies that $\mathcal{A}$ is closed under isomorphisms. If $\mathcal{K}$ is essentially small, i.e., it is equivalent to a small category, we only have a set of such subcategories. We also have that the intersection of any family of thick tensor ideals, is again a thick tensor ideal, and for a collection of objects $\mathcal{E} \in \mathcal{K}$, we let $\langle \mathcal{E} \rangle$ denote the smallest thick tensor ideal containing $\mathcal{E}$.

2.3 Prime ideals and the Zariski topology

Tensor triangulated categories are similar to commutative rings in the sense that we have two commutative operations $(\oplus, \otimes)$, with identities $(0, \mathbb{1})$ that satisfy the obvious distribution formulas. For the rest of the chapter, we let $\mathcal{K}$ be an essentially small triangulated category, unless otherwise stated. We will refer to non-essentially small tensor triangulated categories as "big" tensor triangulated categories. In the next sections we will imitate algebraic geometry, i.e., define prime ideals, spectra and the Zariski topology for this strange new "ring" $\mathcal{K}$.

After having the right notion of ideals, we define prime ideals, in a straightforward manner.

DEFINITION 2.12. A thick tensor ideal $\mathcal{P}$ of $\mathcal{K}$ is called **prime** if it is proper, $\mathcal{P} \neq \mathcal{K}$, and

$$x \otimes y \in \mathcal{P} \implies x \in \mathcal{P} \text{ or } y \in \mathcal{P}$$

DEFINITION 2.13. We let the **spectrum** or **Balmer spectrum** (to not confuse it with the ring spectrum) of $\mathcal{K}$, denoted $\mathrm{Spc}(\mathcal{K})$, be the set of prime ideals in $\mathcal{K}$.

DEFINITION 2.14. For any family of objects $\mathcal{S} \subset \mathcal{K}$ we denoted by $Z(\mathcal{S})$ the following subset of $\mathrm{Spc}(\mathcal{K})$:

$$Z(\mathcal{S}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{S} \cap \mathcal{P} = \emptyset\}$$

PROPOSITION 2.15. *The above definition defines the closed subsets of* $\mathrm{Spc}(\mathcal{K})$ *and this gives a topology on* $\mathrm{Spc}(\mathcal{K})$*, called the* ***the Zariski topology****.*

PROOF. We have that $\bigcap_i Z(\mathcal{S}_i) = Z(\bigcup_i \mathcal{S}_i)$, i.e., arbitrary intersection of closed subsets are closed, and $Z(\mathcal{S}_i) \cup Z(\mathcal{S}_j) = Z(\mathcal{S}_i \oplus \mathcal{S}_j)$, i.e., finite union of closed is closed. By $\oplus$ in the previous sentence, we mean the object-wise biproduct. Lastly, we have that $Z(\emptyset) = \mathrm{Spc}(\mathcal{K})$ and $Z(\mathcal{K}) = \emptyset$. □

NOTATION 2.16. We denote the open subsets in this topology by

$$U(\mathcal{S}) = \mathrm{Spc}(\mathcal{K}) \setminus Z(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{S} \cap \mathcal{P} \neq \emptyset\}.$$

LEMMA 2.17. *The assignment* $x \mapsto U(x) = \{\mathcal{P} \mid x \in \mathcal{P}\}$*, from objects of* $\mathcal{K}$ *to open subsets of* $\mathrm{Spc}(\mathcal{K})$*, satisfies the following properties:*

(i) $U(0) = \mathrm{Spc}(\mathcal{K})$ *and* $U(\mathbb{1}) = \emptyset$

(ii) $U(x \oplus y) = U(x) \cap U(y)$

(iii) $U(Tx) = U(x)$

(iv) $U(x) \supset U(y) \cap U(z)$ *for any distinguished triangle* $x \to y \to z \to Tx$

(v) $U(x \otimes y) = U(x) \cup U(y)$

PROOF. (i) Every prime contains 0, and no prime contains $\mathbb{1}$ as it is proper.

(ii) By thickness of prime ideals $x \oplus y \in \mathcal{P}$ implies that x and y are in $\mathcal{P}$. If x and y are in $\mathcal{P}$, then we can form the following triangle as the sum of two trivial triangles $x \to x \oplus y \to y \to Tx$, by the two out of three property for ideals $x \oplus y$ is in $\mathcal{P}$.

(iii) Use the trivial triangle to see that $x \in \mathcal{P}$ if and only if $Tx \in \mathcal{P}$.

(iv) Let $x \to y \to z \to Tx$ be a triangle in $\mathcal{K}$. For a prime $\mathcal{P} \subset \mathcal{K}$ containing y and z, then by the two out of three property, it must contain x.

(v) If $\mathcal{P} \in U(x \otimes y)$, then $x \otimes y \in \mathcal{P}$. By being a prime, $\mathcal{P}$ must contain x or y. Conversely, if $\mathcal{P}$ contain x, it must contain $x \otimes k$ for all $k \in \mathcal{K}$ as it is a tensor ideal.

□

REMARK 2.18. The complementary properties hold for the *support* of objects.

DEFINITION 2.19. For $x \in \mathcal{K}$ we let the **support** of the object x be the following closed subset of $\mathrm{Spc}(\mathcal{K})$

$$\mathrm{supp}(x) := Z(x) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid x \notin \mathcal{P}\}$$

An important fact about the Balmer spectrum is that it is functorial.

PROPOSITION 2.20. *The spectrum is functorial. For $F\colon \mathcal{K} \to \mathcal{L}$, a tt-functor, we get a well-defined continuous map*

$$\begin{aligned} \mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{L}) &\to \mathrm{Spc}(\mathcal{K}) \\ Q &\mapsto F^{-1}(Q). \end{aligned}$$

and for all $x \in \mathcal{K}$ we have that

$$(\mathrm{Spc}(F))^{-1}(\mathrm{supp}_{\mathcal{K}}(x)) = \mathrm{supp}_{\mathcal{L}}(F(x)).$$

This defines a contravariant functor

$$\mathrm{Spc}(-)\colon \{\textit{Essentially small tt-categories}\} \to \textit{Top}.$$

PROOF. First, we check that the map on the spectra is well-defined, i.e., $F^{-1}(Q) \in \mathrm{Spc}(\mathcal{K})$, when $Q \in \mathrm{Spc}(\mathcal{L})$. $F^{-1}(Q)$ is a full subcategory by definition. As an additive functor between additive categories must preserve the zero object, $F^{-1}(Q)$ must contain 0, as Q does. We claim that $F^{-1}(Q)$ is a thick $\otimes$-ideal:

(i) Let $x \to y \to z \to Tx$ be a triangle with $x, y \in F^{-1}(Q)$. Then $Fx, Fy \in Q$. As F is exact, $Fx \to Fy \to Fz \to FTx$ is a triangle in $\mathcal{L}$. As Q is a thick $\otimes$-ideal, it contains Fz, and hence $z \in F^{-1}(Q)$.

(ii) Let $x \oplus y \in F^{-1}(Q)$. This implies that $F(x \oplus y) = Fx \oplus Fy \in Q$ by additivity of F. As Q is a thick, it contains Fx and Fy. Therefore, x and y are is $F^{-1}(Q)$.

(iii) Let $x \in F^{-1}(Q)$ and let $y \in \mathcal{K}$. Then $F(x \otimes y) = Fx \otimes Fy \in Q$ as $Fx \in Q$ and Q is a thick $\otimes$-ideal. Hence, $x \otimes y \in F^{-1}(Q)$.

We need to check that $F^{-1}(Q)$ is prime. Let $x \otimes y \in F^{-1}(Q)$, then $F(x \otimes y) = Fx \otimes Fy \in Q$. Hence Fx or Fy belongs to Q. Therefore, x or y must be in $F^{-1}(Q)$ and $F^{-1}(Q)$ is a prime $\otimes$-ideal in $\mathcal{K}$. The map on spectra is therefore well-defined. It is also continuous:

$$\begin{aligned} \mathrm{Spc}(F)^{-1}(\mathrm{supp}_{\mathcal{K}}(x)) &= \{\mathcal{P} \in \mathrm{Spc}(\mathcal{L}) \mid x \notin F^{-1}(\mathcal{P})\} \\ &= \{\mathcal{P} \in \mathrm{Spc}(\mathcal{L}) \mid Fx \notin \mathcal{P}\} \\ &= \mathrm{supp}_{\mathcal{L}}(Fx). \end{aligned}$$

We finally note that for $\mathcal{K} \xrightarrow{F} \mathcal{L} \xrightarrow{G} \mathcal{M}$, we have that $\mathrm{Spc}(G \circ F) = (G \circ F)^{-1}(-) = F^{-1} \circ G^{-1}(-) = \mathrm{Spc}(F) \circ \mathrm{Spc}(G)$, which wraps up the functorial part. ∎

2.4 Local and rigid tensor triangulated categories

In this section, we develop some further language on tensor triangulated geometry from [Bal10]. In a commutative ring R with 1, then the ideal $\mathfrak{p} \subset R$ is prime if and only if $R/\mathfrak{p}$ is an integral domain. *Locality*, is a property of a tensor triangulated categories, that almost captures this idea.

DEFINITION 2.21. Let $\mathcal{K}$ be a tt-category. We call $\mathcal{K}$ **local** if every open cover of $\mathrm{Spc}(\mathcal{K})$ is trivial, i.e., if $\mathrm{Spc}(\mathcal{K}) = \bigcup_{i\in I} U_i$ is an open cover, then there exist a $j \in I$ such that $\mathrm{Spc}(\mathcal{K}) = U_j$.

As we will prove below, if $\mathcal{P} \subset \mathcal{K}$ is a prime ideal, then $\mathcal{K}/\mathcal{P}$ is local. For the converse statement to be true, we need an additional assumption on $\mathcal{K}$.

DEFINITION 2.22. We call the tt-category $\mathcal{K}$ **rigid** if there exists an exact functor, called the **dual**,

$$D(-)\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{K},$$

that induces a right adjoint to tensoring with any $x \in \mathcal{K}$

$$x \otimes -\colon \mathcal{K} \leftrightarrows \colon D(x) \otimes -.$$

We call such objects **rigid** or **strongly dualizable**.

REMARK 2.23. Note that tt-functors $F\colon \mathcal{K} \to \mathcal{L}$ preserve rigidity. Indeed, suppose that $x \in \mathcal{K}$ is rigid. Then one can use $F(D(x))$ as the dual of $F(x)$ and the image of units and counits of the adjunction serves again as units and counits, hence $F(x)$ is rigid. See for instance Proposition 3.10 in [FHM03] for more details.

EXAMPLE 2.24. As in [**Section 9.8**; HPS97], the category $\mathrm{SH}(G)$ is compact-rigidly generated, that is, compact objects coincide with rigid objects and the collection of these objects generates the category as a triangulated category.

OBSERVATION 2.25. If $\mathcal{K}$ is a rigid tensor triangulated category, then we immediately have some nice consequences. By [**Corollary 2.5**; Bal05], $\mathrm{supp}(x) = \emptyset$ if and only if $x \cong 0$, instead of x just being $\otimes$-nilpotent. By [**Corollary 2.8**; Bal05] $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$ implies that $\mathrm{Hom}_{\mathcal{K}}(x, y) = 0$. Furthermore, every $\otimes$-ideal $\mathcal{J}$ in $\mathcal{K}$ is radical by [**Proposition 2.4**; Bal05].

We will not use Proposition 2.26 in the later chapters, and it can be skipped, but it completes the discussion between the notions of *local* tensor triangulated categories and *integral domains* of rings.

PROPOSITION 2.26. *Let $\mathcal{K}$ be a rigid tensor triangulated category and let $\mathcal{J}$ be a thick $\otimes$-ideal. Then $\mathcal{J} \in \mathrm{Spc}(\mathcal{K})$ if and only if $\mathcal{K}/\mathcal{J}$ is local.*

PROOF. Assume first that $\mathcal{J} \in \mathrm{Spc}(\mathcal{K})$. Recall from [**Proposition 3.11**; Bal05], that $\mathrm{Spc}(\mathcal{K}/\mathcal{J}) \cong \{Q \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{J} \subset Q\}$. It is therefore sufficient to show that the latter subspace of $\mathrm{Spc}(\mathcal{K})$ is local. Let $\{U(\mathcal{S}_i) \mid i \in I\}$ be an open cover of $\{Q \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{J} \subset Q\}$. As $\mathcal{J}$ is prime, it is contained in $\{Q \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{J} \subset Q\}$ and therefore in $U(\mathcal{S}_j)$ for some

$j \in I$. The latter is equivalent to $\mathcal{J} \cap \mathcal{S}_j \neq \emptyset$. Hence, for any $\mathcal{Q} \in \operatorname{Spc}(\mathcal{K})$ such that $\mathcal{J} \subset \mathcal{Q}$, then $\mathcal{Q} \cap \mathcal{S}_j \neq \emptyset$, i.e., $\mathcal{Q} \in U(\mathcal{S}_j)$ and $U(\mathcal{S}_j) = \operatorname{Spc}(\mathcal{K})$.

Conversely, suppose $\mathcal{K}/\mathcal{J}$ is local. By [**Proposition 4.2**; Bal10], the zero-ideal of $\mathcal{K}/\mathcal{J}$ is prime, therefore $\mathcal{J} \in \operatorname{Spc}(\mathcal{K})$ is prime. □

We will now define a tt-ring. This is a ring object $A \in \mathcal{K}$ in a tensor triangulated category $\mathcal{K}$, such that the module category $\mathrm{Mod}_{\mathcal{K}}(A)$ over A, is again a tensor triangulated category. See [Bal14] and [Bal16] for more details. We remark as in [Bal16] the following:

Remark 2.27. A left A-module is a pair (x, ρ) where $x \in \mathcal{K}$ and where the *action* $\rho \colon A \otimes x \to x$ satisfies the usual associative and unit conditions. We denote by $\mathrm{Mod}_{\mathcal{K}}(A)$ the category of A-modules with A-linear morphisms.

Definition 2.28. A **ring object** $A \in \mathcal{K}$ in a tensor triangulated category is a monoid (A, μ, η), where $\mu \colon A \otimes A \to A$ is an associative multiplication and $\eta \colon \mathbb{1} \to A$ is a two-sided unit. We call the ring object **commutative** if the multiplication is commutative.

Definition 2.29. A ring object A is a **separable** if the multiplication $\mu \colon A \otimes A \to A$ has a (A, A)-bimodule section $\sigma \colon A \to A \otimes A$. That is,

$$\mu\sigma = \mathrm{id}_A \text{ and } (\mathbb{1} \otimes \mu) \circ (\sigma \otimes \mathbb{1}) = \sigma\mu = (\mu \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \sigma)$$

Remark 2.30. As remarked in [Bal16], this implies that A is a projective (A, A)-bimodule.

Definition 2.31. A ring object A is a **tt-ring** if it is commutative and separable.

An important property, shared by most of the tensor triangulated categories we will come across in this thesis, is the following:

Definition 2.32. Let $\mathcal{T}$ be a "big" tt-category, admitting arbitrary small colimits. We say that $\mathcal{T}$ is **compactly generated tensor triangulated category** if

1. $\mathcal{T}^c$ generates $\mathcal{T}$ as a localizing category. in symbols: $\mathcal{T} \cong Loc(\mathcal{T}^c)$.
2. $\mathcal{T}^c$ is essentially small, $\mathcal{T}^c$ is rigid and $\mathbb{1}$ is compact.

Remark 2.33. In the above setting, an object is *compact* if and only if it is *rigid*. See [HPS97] for details.

Example 2.34. As we mentioned, most of the "big" tt-categories we will encounter will be compactly generated:

1. Let R be a commutative ring. Then the derived category of chain complexes of modules over R, $D(R)$, is compactly generated by R. That is, the chain complex with R concentrated in degree zero and zero elsewhere.
2. The stable homotopy category SH is compactly generated by the sphere spectrum $\mathbb{S}$.

Note that in the above examples, R and $\mathbb{S}$ are the $\otimes$-units in the tt-categories $D(R)$ and SH, respectively.

3. Let G be a finite group, then the G-equivariant homotopy category $\mathrm{SH}(G)$ has the collection of suspension spectra $\{\Sigma^\infty(G/H)_+ \mid H \leq G\}$, as compact generators.

As a **non-example**: The stable homotopy category with an action from a group G, $\mathrm{Fun}(BG, \mathrm{SH})$, is not a compactly generated tensor triangulated category as the unit of $\mathrm{Fun}(BG, \mathrm{SH})$ is not compact.

2.5 The Balmer spectrum commutes with finite products

We will now prove that the Balmer spectrum commutes with taking a finite product of tensor triangulated categories. This lemma will play a big role in Chapter 5.

LEMMA 2.35. *Let $\{\mathcal{T}_i\}_{i=1}^n$ be a finite collection of non-zero tensor triangulated categories. Any prime of $\mathcal{T}_1 \times \cdots \times \mathcal{T}_n$ is of the form $\mathcal{T}_1 \times \cdots \times \mathcal{T}_{i-1} \times \mathcal{P}_i \times \mathcal{T}_{i+1} \times \cdots \times \mathcal{T}_n$ where $\mathcal{P}_i \in \mathrm{Spc}(\mathcal{T}_i)$.*

PROOF. By induction, it is sufficient to check the base case when $n = 2$. We first check that our candidate is prime:
Let $\mathcal{P} = \mathcal{P}_1 \times \mathcal{T}_2 \subset \mathcal{T}_1 \times \mathcal{T}_2$ with $\mathcal{P}_1 \in \mathrm{Spc}(\mathcal{T}_1)$. It follows immediately from the definition, that $\mathcal{P}$ is a thick $\otimes$-ideal, as everything is done component-wise. To see that $\mathcal{P}$ is prime, let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be such that $x \otimes y \in \mathcal{P} \iff (x_1, x_2) \otimes (y_1, y_2) \in \mathcal{P} \iff (x_1 \otimes y_1, x_2 \otimes y_2) \in \mathcal{P}$. By definition of $\mathcal{P}$, we have that $x_1 \otimes y_1 \in \mathcal{P}_1$ and $x_2 \otimes y_2 \in \mathcal{T}_2$. As $\mathcal{P}_1$ is prime, then x_1 or y_1 belongs $\mathcal{P}_1$. As x_2 and y_2 belongs to $\mathcal{T}_2$ by assumption, we have that $x \in \mathcal{P}$ or $y \in \mathcal{P}$. Hence, $\mathcal{P}$ is prime. The case where $\mathcal{P} = \mathcal{T}_1 \times \mathcal{P}_2$ is analogous, and we can conclude that our candidate is prime.

Now we must check that every prime $\mathcal{P} \subset \mathcal{T}_1 \times \mathcal{T}_2$ is of this form. Assume therefore that $\mathcal{P} \subset \mathcal{T}_1 \times \mathcal{T}_2$ is a prime ideal. Objects in the product of $\mathcal{T}_1 \times \mathcal{T}_2$ are tuples (t_1, t_2) with $t_i \in \mathcal{T}_i$. As $\mathcal{P}$ is a full subcategory of $\mathcal{T}_1 \times \mathcal{T}_2$, we can assume $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$, with $\mathcal{P}_i \subset \mathcal{T}_i$ being full subcategories. Both $\mathcal{P}_1$ and $\mathcal{P}_2$ have to be thick $\otimes$-ideals, as everything is done component-wise. We can immediately throw away the case where both $\mathcal{P}_1 \subset \mathcal{T}_1$ and $\mathcal{P}_2 \subset \mathcal{T}_2$ are proper ideals, since $(0,0) \in \mathcal{P}_1 \times \mathcal{P}_2$ as $\mathcal{P}_1 \times \mathcal{P}_2$ is prime, and hence

$$(\mathbb{1}_{\mathcal{T}_1}, 0) \otimes (0, \mathbb{1}_{\mathcal{T}_2}) = (0,0) \in \mathcal{P}_1 \times \mathcal{P}_2$$

We must therefore have that $(\mathbb{1}_{\mathcal{T}_1}, 0) \in \mathcal{P}_1 \times \mathcal{P}_2$ or $(0, \mathbb{1}_{\mathcal{T}_2}) \in \mathcal{P}_1 \times \mathcal{P}_2$. In other words, $\mathbb{1}_{\mathcal{T}_1} \in \mathcal{P}_1$ or $\mathbb{1}_{\mathcal{T}_2} \in \mathcal{P}_2$. Neither of them cannot be proper, as prime ideals are proper by definition.

This leaves the case where exactly one of them is proper: Assume without loss of generality that $\mathcal{P} = (\mathcal{P}_1, \mathcal{T}_2)$ where $\mathcal{P}_1$ is a proper thick $\otimes$-ideal in $\mathcal{T}_1$. We must check that $\mathcal{P}_1 \in \mathrm{Spc}(\mathcal{T}_1)$. Assume that $x_1 \otimes y_1 \in \mathcal{P}_1$ and that $x_1 \notin \mathcal{P}_1$. Take any $z \in \mathcal{T}_2$. Then $z \otimes z \in \mathcal{T}_2$. Therefore,

$$(x_1, z) \otimes (y_1, z) = (x_1 \otimes y_1, z \otimes z) \in \mathcal{P}_1 \times \mathcal{T}_2.$$

As $\mathcal{P}_1 \times \mathcal{T}_2$ is a prime, we must have that $(x_1, z) \in \mathcal{P}_1 \times \mathcal{T}_2$ or $(y_1, z) \in \mathcal{P}_1 \times \mathcal{T}_2$. By assumption $(x_1, z) \notin \mathcal{P}_1 \times \mathcal{T}_2$ as $x_1 \notin \mathcal{P}_1$, so we must have that $(y_1, z) \in \mathcal{P}_1 \times \mathcal{T}_2$, and in particular that $y_1 \in \mathcal{P}_1$. In other words, $\mathcal{P}_1 \in \mathrm{Spc}(\mathcal{T}_1)$. □

Lemma 2.36. *Let $\{\mathcal{T}_i\}_{i=1}^n$ be a finite collection of non-zero tensor triangulated categories. Then*

$$\coprod_i \mathrm{Spc}(\mathcal{T}_i) \cong \mathrm{Spc}(\prod_i \mathcal{T}_i).$$

Proof. We have $\otimes$-triangulated functors $\phi_j \colon \prod_i \mathcal{T}_i \to \mathcal{T}_j$ for every j, defined by sending $(x_1, \ldots, x_n)$ to x_j and $\{f_i \colon x_i \to y_i\}_i$ to $f_j \colon x_j \to y_j$. By Proposition 2.20 we get a continuous map $\mathrm{Spc}(\phi_j) \colon \mathrm{Spc}(\mathcal{T}_j) \to \mathrm{Spc}(\prod_i \mathcal{T}_i)$ defined by

$$\mathcal{Q} \to \phi_j^{-1}(\mathcal{Q}) = \mathcal{T}_1 \times \cdots \times \mathcal{Q} \times \cdots \times \mathcal{T}_n.$$

It is injective by [**Corollary 3.8**; Bal05], as ϕ_j is essentially surjective. By the universal property of the coproduct we get a continuous map

$$\phi \colon \coprod_i \mathrm{Spc}(\mathcal{T}_i) \to \mathrm{Spc}(\prod_i \mathcal{T}_i).$$

It is injective as it is injective on each component, and the intersection of images from different components is empty. This map is surjective by Lemma 2.35.

The last thing we need to check is that the inverse map is continuous. We will do so by showing that ϕ is a closed map. Let $Z(\mathcal{S}_j)$ be a closed set in $\mathrm{Spc}(\mathcal{T}_j)$. Then

$$\mathrm{Spc}(\phi_j)(Z(\mathcal{S}_j)) = \mathcal{T}_1 \times \cdots \times Z(\mathcal{S}_j) \times \cdots \times \mathcal{T}_n = Z(\{0\} \times \cdots \times \mathcal{S}_j \times \cdots \times Z\{0\})$$

which is closed, hence ϕ_j is closed. An arbitrary closed subset of $\coprod_i \mathrm{Spc}(\mathcal{T}_i)$ is of the form $\coprod_i Z(\mathcal{S}_i)$ for $\mathcal{S}_i \subset \mathcal{T}_i$. Then,

$$\begin{aligned}
\mathrm{Spc}(\phi)(\coprod_i Z(\mathcal{S}_i)) &= \coprod_i \mathrm{Spc}(\phi_i)(Z(\mathcal{S}_i)) \\
&= \coprod_i \phi_i^{-1}(Z(\mathcal{S}_i)) \\
&= \coprod_i Z(\{0\} \times \cdots \times \mathcal{S}_i \times \cdots \times Z\{0\}),
\end{aligned}$$

which is a finite union of closed subsets of $\mathrm{Spc}(\prod_i \mathcal{T}_i)$, hence is closed. □

Chapter 3

The ∞-category of G-spectra

In this chapter, we will introduce the ∞-category of G-spectra $\mathrm{Sp}(G)$, greatly inspired by [MNN17]. This is a stable ∞-category whose homotopy category $\mathrm{Ho}(\mathrm{Sp}(G))$, coincides with the classical G-equivariant stable homotopy category $\mathrm{SH}(G)$. $\mathrm{SH}(G)$ is closed symmetric monoidal with a product $- \wedge -$ induced by the smash product of based topological G-spaces. Moreover, $\mathrm{SH}(G)$ is triangulated, with a suspension functor induced from the suspension of based topological spaces. In other words, $\mathrm{SH}(G)$ is a "big" tensor triangulated category, with compact-rigid generators, G/H_+, for H running over the closed subgroups of G. Our goal is to calculate the Balmer spectrum for the tt-category $\mathrm{SH}^c(G)$ in as much generality as we can.
Two of the major difficulties with $\mathrm{SH}(G)$ are torsion and G-equivariance. Localizing at the right ring $R \subset \mathbb{Q}$, will kill the torsion, making it easier to handle. We will define the ∞-category of R-local G-spectra in Section 3.6. If we can find some algebraic model of $\mathrm{SH}_R(G)$, we could hope to reduce the calculation of $\mathrm{Spc}(\mathrm{SH}_R(G)^c)$ to the calculation of $\mathrm{Spc}(\mathrm{SH}^c_R)$, which is well known.

After getting to know the ∞-category of G-spectra, $\mathrm{Sp}(G)$, we will introduce some properties of the geometric fixed point functors, which allow us to prove the decomposition formula in Chapter 4. We will also introduce the Tate spectrum and the Tate spectral sequence, which will be used to calculate the Balmer spectrum in Chapter 5.

Finding algebraic models for $\mathrm{SH}(G)$ is an active research area, and it is conjectured by Greenlees in [Gre06], that for every compact Lie group G, there is an Abelian category $\mathcal{A}(G)$ and a Quillen equivalence

$$G - \mathrm{spectra}_{\mathbb{Q}} \cong \mathrm{dg}\mathcal{A}(G).$$

This conjecture is true in some cases. For instance;

1. For G finite
$$\mathcal{A}(G) = \prod_{(H) \leq_G} W_G H - \mathrm{Mod}_{\mathbb{Q}}$$

by [Gre06]. Furthermore, by Wimmer's result in [Wim19], which we will reprove in Chapter 4, this holds true if we relax the assumptions on the ring $R \subset \mathbb{Q}$, such that in R we only have to invert the order of G.

$$G-\text{spectra}_R \cong \prod_{(H)\leq G} W_G H - \text{Mod}_R$$

2. The conjecture holds true in some infinite cases of G as well. For instance; for $G = SO(2), SO(3), O(2)$ the conjecture holds true, see [Bar+17], [Gre01] and [Gre98b]. Another example is when $G = T$, the circle group, then the conjecture still holds true, see [Shi02].

In Chapter 6, we will see an algebraic model for $\text{SO}(3)-\text{spectra}_{\mathbb{Q}}$, given by Kędziorek in [Kęd17a].

3.1 Motivating the stable homotopy category

Although we will work with the stable homotopy category in an axiomatic way, we want to motivate it in such a way that it will not be too boring for those who have not come across it before. We will do so through reduced cohomology theories and sequential spectra. Our category of topological spaces, Top and the based version Top$_*$, will consist of compactly generated weak Hausdorff spaces. This category is a good choice as we have an adjunction

$$\text{Map}(X\times Y, Z) \cong \text{Map}(X, \text{Map}(Y,Z))$$

between $-\times-$ and $\text{Map}(-,-)$ and it avoids some pathological examples. Here $\text{Map}(X,Y)$ denotes the space of maps $X \to Y$, with the compact-open topology.
The following results are classic, but proofs can be found in the recent book by Barnes and Roitzheim [BR20]. They give the motivation for defining *sequential spectra*. We therefore recall the Brown Representability theorem, which classifies the reduced cohomology theories through connected pointed CW-complexes.

THEOREM 3.1. *(Brown Representability) Let $\tilde{E}^*$ be a reduced cohomology theory. Then for every $n \in \mathbb{Z}$ there exist a connected pointed CW-complex K_n such that for every connected pointed CW-complex X there is a natural isomorphism*

$$\widetilde{E}^n(X) \cong [X, K_n]_{Top*}$$

Furthermore, the spaces K_n are unique up to homotopy equivalence.

EXAMPLE 3.2. As is well known, singular cohomology with coefficients in an Abelian group G is represented by *Eilenberg MacLane* spaces. That is, for every $n \in \mathbb{N}$ there exists an Eilenberg MacLane space $K(n,G)$ and a natural isomorphism

$$H^n(X;G) \cong [X, K(n,G)]$$

for any based CW-complex X.

The Brown Representability theorem also has a converse. We therefore state both directions in the following corollary:

COROLLARY 3.3. *A reduced cohomology theory $\widetilde{E}^*$ determines, and is determined by, a sequence of connected pointed CW-complexes $\{K_n\}_\mathbb{Z}$ with pointed weak homotopy equivalences $\alpha_n \colon K_n \to \Omega K_{n+1}$. The spaces K_n and the maps α_n are unique up to homotopy equivalence.*

From this corollary, it is natural to make the following definitions:

DEFINITION 3.4. A **sequential spectrum X** is a sequence of pointed topological spaces X_n together with structure maps

$$\sigma_m^X \colon \Sigma X_n \to X_{n+1}$$

We denote the adjoints of the structure maps by

$$\widetilde{\sigma}_n^X \colon X_n \to \Omega X_{n+1}$$

and we call them the **adjoint structure maps**. A spectrum X is called an **Ω-spectrum** if the adjoint maps are weak homotopy equivalences. The **category of sequential spectra** SeqSpec, is given by the following: The objects are sequential spectra. The morphisms $f \colon X \to Y$ between sequential spectra X and Y are sequences of pointed topological maps $f_n \colon X_n \to Y_n$ with $n \in \mathbb{N}$ such that the following diagram commutes up to homotopy

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\mathrm{id} \wedge f_n} & \Sigma Y_n \\ \downarrow{\sigma_n^X} & & \downarrow{\sigma_n^Y} \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \end{array}$$

for every $n \in \mathbb{N}$.

Consequently, instead of studying the category of reduced cohomology theories, we can study the category of Ω-*spectra*. This, however, will exclude some important examples as the *sphere spectrum* as we will define below.
Some important sequential spectra are:

1. The **sphere spectrum** $\mathbb{S}$, defined by $\mathbb{S}_n := S^n$, with the canonical structure maps $\Sigma S^n \to S^{n+1}$.

2. The **suspension spectrum** of $K \in \mathrm{Top}_*$, $\Sigma^\infty K$, is defined by $(\Sigma^\infty K)_n := S^n \wedge K$ with the canonical structure maps. In particular, the sphere spectrum $\Sigma^\infty S^0 = \mathbb{S}$, is a suspension spectrum.

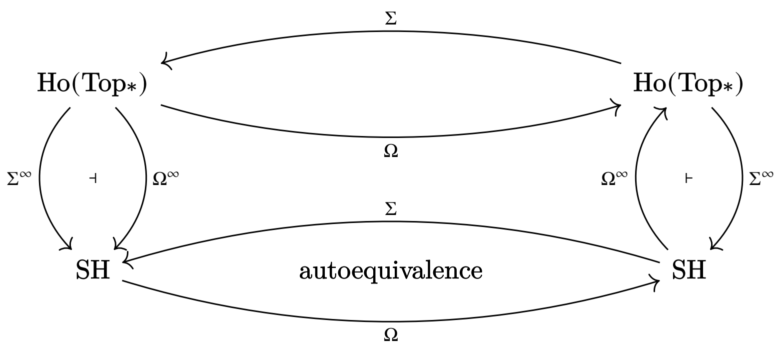
3. The **Eilenberg-Mac Lance spectrum** HG is defined for any Abelian group G as $(HG)_n := K(n, G)$ with the adjoint structure maps are given by a choice of weak homotopy equivalences
$$K(n, G) \to \Omega K(n+1, G).$$

EXAMPLE 3.5. $\mathbb{S}$ is not a Ω-spectrum as the canonical map $S^1 \to \Omega S^2$ is not a weak homotopy equivalence, since

$$\pi_2 S^1 \cong 0 \to \pi_2 \Omega S^2 \cong \pi_3 S^2 \cong \mathbb{Z}$$

cannot be an isomorphism.

The "right" category to consider is the category of sequential spectra. As discussed in chapter 2 of [BR20], one can give this category a level-wise model structure. It turns out that the category of sequential spectra is a stable model category, and therefore its homotopy category has a suspension functor with an adjoint called the loop functor, turning it into a triangulated category. The homotopy category of the model category of sequential spectra is called the stable homotopy category and denoted by SH. The stable homotopy category, SH, has a symmetric monoidal product, induced from the smash product on spaces, and is a "big" tensor triangulated category.
There is a functor $\Omega^\infty\colon \mathrm{SH} \to \mathrm{Ho}(\mathrm{Top}_*)$ (see [**Def. 2.4.1**; BR20] for an explicit description) that is left adjoint to $\Sigma^\infty\colon \mathrm{Ho}(\mathrm{Top}_*) \to \mathrm{SH}$. The relation between the suspension (loop space) of spaces and the suspension (loop space) of spectra can then be captured in the following commutative diagram.



As with spaces, we have a notion of homotopy groups of spectra. We define the homotopy group of a spectrum X as $\pi_n(X) := \mathrm{colim}_{k\to\infty} \pi_{n+k}(X_k)$. We note that the homotopy groups of sequential spectra are abelian groups.

EXAMPLE 3.6. The homotopy groups of the Eilenberg MacLane spectra HG are

$$\pi_n(HG) = \begin{cases} G \text{ if } n = 0 \\ 0 \text{ else.} \end{cases}$$

The homotopy groups of the sphere spectrum are, by definition, the *stable stems* or the *stable homotopy groups*, i.e., $\pi_n(\mathbb{S}) := \pi_n^{\mathrm{stable}}$. Importantly, the stable stems are finite Abelian groups for all n, except for $n = 0$. In this case $\pi_0^{\mathrm{stable}} \cong \mathbb{Z}$.

We remark that the homotopy groups of spectra are related to the homotopy groups of spaces in the following way

$$\pi_n(X) \cong \pi_n(\Omega^\infty X). \tag{3.1}$$

The homotopy groups of spectra can be non-zero for negative n, so Equation (3.1) only holds true for all $n \geq 0$. Note also that for every $X, Y \in \mathrm{SH}$, there exists a mapping spectrum $F(X,Y)$, that is, SH has an inner hom-object that we denote by $F(-,-)$. The mapping spectrum is left adjoint to the smash product in SH, making SH a closed symmetric monoidal category.

Remark 3.7. There are many models for the stable homotopy category, Sequential spectra, is just one of them. However, all the models have the same homotopy category SH. Another model for the stable homotopy category is *orthogonal spectra*, this approach is discussed in [Sch18].

We are interested in the G-equivariant analogue of the stable homotopy category SH, but as remarked in [GM95a], the entire framework of the non-equivariant homotopy category, works equally well in the G-equivariant homotopy category $\mathrm{SH}(G)$, where we let G act on every object sight. That is, instead of starting with the category of based topological spaces, we work with based G-spaces and G-equivariant maps. Another excellent introduction to the G-equivariant stable homotopy category is the lecture notes from "Topics in Algebraic Topology class" in Spring 2017, taught by Andrew Blumberg.

3.2 Unstable equivariant homotopy theory

In this section, we will briefly recall some basic definitions from the unstable homotopy theory. All of our spaces we work with will be pointed, if they are not, we will make them so, by adjoining a disjoint basepoint $X_+ := X \sqcup \{*\}$. If X is a G-space and $x \in X$, then an **orbit** of x is defined as

$$Gx := \{gx \mid g \in X\},$$

and we define the **orbit space** of X as the disjoint union of orbits,

$$X/G := \coprod_{x \in X} Gx,$$

with the induced topology from X. For subgroups $H, K \leq G$ and $g \in G$, we will write H^g to denote the g-conjugation of H, i.e., $H^g := \{g^{-1}hg \mid h \in H\}$. If $H^g \leq K$ for some $g \in G$, we write $H \leq_G K$ and say that H is **subconjugate** to K. We define the **normalizer**, $N_G(H)$, of the subgroup of H in G by

$$N_G(H) := \{g \in G \mid ghg^{-1} \in H \ \forall h \in H\}.$$

Definition 3.8. Let $H \leq G$ be a subgroup of G and let X be a G-space. We define the **H-fixed points** of X as

$$X^H := \{x \in X \mid \forall h \in H \text{ then } hx = x\}.$$

X^H is naturally a $W_G H$-space where $W_G(H) := N_G(H)/H$.

Remark 3.9. Let $H \leq G$ and X be a G-space. Then

$$X^H \cong \mathrm{Map}^G(G/H_+, X) \in \mathrm{Top}_*,$$

where $\mathrm{Map}^G(G/H_+, X)$ is the subspace of G-equivariant maps $G/H_+ \to X$, of the space of pointed maps $\mathrm{Map}(G/H_+, X)$. We also remark that

$$\mathrm{Map}^G(X,Y) = \mathrm{Map}(X,Y)^G.$$

Let us remind ourselves of the correct definition of equivariant homotopy groups of G-spaces, as we will later see the similar-looking definition of H-homotopy groups of G-spectra. For $H \leq G$, we define the **H-equivariant homotopy groups** of a G-space X to be

$$\pi_n^H(X) := \pi_n(X^H).$$

We will later see that *H-geometric fixed point functor* commutes with taking H-fixed points when restricted to suspension spectra. Knowing that the compact generators of $\mathrm{SH}(G)$ are the suspension spectra of the spaces G/K_+, calculating the H-fixed points of the quotient space G/H will be of great importance. We will come back to this in the next chapter.

3.3 Model categories and ∞-categories

In this section, we will explain how to go from a model category to an ∞-category as in [MNN17]. Our model for ∞-categories will be the quasi-categories, treated in detail in [Lur17].
A model category consists of three classes of maps, where one of the classes is often more awkward to write down. However, two out of the three classes determines the third.
We remind our self of a certain smallness condition that is often assumed and occurs for ∞-categories.

DEFINITION 3.10. We say that a ∞-category C is **presentable** if C contains all small colimits, and it is generated under small colimits by a set of compact objects.

To further agree on the language, we define a *span* of objects as the following subcategory.

DEFINITION 3.11. Let C be a category. For a subcollection of objects $\mathcal{S} \in C$, we define the subcategory **spanned** by $\mathcal{S}$ as the smallest full subcategory of C containing every object in $\mathcal{S}$.

We now have all the language to explain how one induces a presentable (symmetric monoidal) ∞-category from a (symmetric monoidal) model category.

CONSTRUCTION 3.12. [**Construction 5.1**; MNN17] Let $C^c \subset C$ be the full subcategory spanned by the cofibrant objects. The model category C *presents* a ∞-category $\underline{C}$ which, by definition ([**Def. 1.3.4.15**; Lur17]), is the ∞-categorical localization $\underline{C} := C^c[\mathcal{W}^{-1}]$, where $\mathcal{W}$ is the class of weak equivalences in C.
Similarly, if C is a symmetric monoidal model category, the ∞-categorical localization $\underline{C} := C^c[\mathcal{W}^{-1}]$ inherits a symmetric monoidal structure. See for instance [**Proposition 4.8.2.7**; Lur17].

REMARK 3.13. We should think of this construction as adding higher morphisms to the model category C, in a compatible way with the weak equivalences, such that the homotopy category of the ∞-category $\underline{C}$, coincide with the homotopy category of the model category of C. In symbols: $\mathrm{Ho}(\underline{C}) = C[\mathcal{W}^{-1}]$.

We will use this construction on the following examples:

1. Let Top denote the model category of compactly generated weakly Hausdorff spaces with the Quillen model structure, which is defined as follows:

   (a) Weak equivalences are the weak homotopy equivalences.

   (b) Fibrations are the Serre fibrations.

   (c) Induced cofibrations.

   Then $\mathcal{S} := \underline{\mathrm{Top}}$ is the ∞-category of spaces as in Construction 3.12.

2. Similarly, as Top, when taking the pointed version Top$*$ of pointed compactly generated weakly Hausdorff spaces considered as a model category with the Quillen Model structure. It is a symmetric monoidal category via the smash product. We end up with the symmetric monoidal ∞-category of pointed spaces, $\mathcal{S}_* := \underline{\mathrm{Top}*}$, as in Construction 3.12.

3. For a compact Lie group G, later assumed to be finite, let Top_G denote the category of compactly generated weakly Hausdorff spaces with a G-action. We give it the following model structure:

   (a) The weak equivalences are given by the maps $X \to Y$, such that for any $H \leq G$, $X^H \to Y^H$ is a weak homotopy equivalence.

   (b) Similarly, the fibrations are given by the maps $X \to Y$, such that for any $H \leq G$, $X^H \to Y^H$ is a Serre fibration.

   (c) Induced cofibrations.

   The category Top_G is also a symmetric monoidal model category, with the Cartesian product of G-spaces. We denote by $\mathcal{S}_{\mathrm{G}} := \underline{\mathrm{Top}_G}$, the symmetric monoidal ∞-category of G-spaces associated to the symmetric monoidal category as in Construction 3.12.

4. Similarly, as Top_G, consider the category of pointed compactly generated weakly Hausdorff spaces with a G-action that fixes the basepoints. It has a similar model structure, making it into a symmetrical monoidal model category. We denote by $\mathcal{S}_{\mathrm{G}*} := \underline{\mathrm{Top}*_G}$, the symmetric monoidal ∞-category of G-spaces associated to the symmetric monoidal category as in Construction 3.12.

We will further list two other important ∞-categories we will work with in later chapters

5. We will write BG, both to denote the classifying space of G, and its associated ∞-groupoid.

6. If C is an ∞-category, then $\mathrm{Fun}(BG, C)$ denotes the ∞-category of objects in C with an action of G.

3.4 The ∞-category of G-spectra

Let us briefly recall what a stable ∞-category is from [Lur17].

DEFINITION 3.14. Let C be an ∞-category. We say that C is a **pointed** ∞-category if it contains a **zero** object. That is, an object which is both initial and terminal.

DEFINITION 3.15. An ∞-category is **stable** if it is pointed and satisfy the following conditions:

1. Every morphism in C admits a fiber and a cofiber.
2. A triangle in C is a fiber sequence if and only if it is a cofiber sequence.

REMARK 3.16. By [**Theorem 1.1.2.14**; Lur17] the homotopy category $\mathrm{Ho}(C)$ of a stable ∞-category is triangulated. Moreover, if C is a symmetric monoidal stable ∞-category, then $\mathrm{Ho}(C)$ is a tt-category.

REMARK 3.17. The compact objects of a stable ∞-category C coincides with the compact objects of the triangulated category $\mathrm{Ho}(C)$. See [**Proposition 1.4.4.1**; Lur17]. Moreover, the dualizable objects in a stable ∞-category are exactly those objects that are dualizable in the triangulated category $\mathrm{Ho}(C)$.

NOTATION 3.18. For the rest of this chapter, we will assume that all groups are finite. However, most of what we do still holds makes sense and holds true, when G is a compact Lie group and every subgroup is closed in G.

DEFINITION 3.19. Let $\mathrm{OrthSpec}_G$ denote the symmetrical monoidal model category of orthogonal G-spectra. See for instance [MM02] for more details. We will let the **∞-category of G-spectra** $\mathrm{Sp}(G)$ denote the symmetrical monoidal stable ∞-category associated to $\mathrm{OrthSpec}_G$. Furthermore, we will define the **∞-category of spectra** Sp as $\mathrm{Sp}(*)$. The unit of $\mathrm{Sp}(G)$ will be denoted by $\mathbb{S}_G$ or $\mathbb{S}$ when the G is understood. We denote by $F_G(X,Y) \in \mathrm{Sp}(G)$ the inner hom-object in $\mathrm{Sp}(G)$.

REMARK 3.20. As remarked in [**Remark 1.4.3.2**; Lur17], we can identify the homotopy category of the stable ∞-category of spectra $\mathrm{Ho}(\mathrm{Sp})$ with the stable homotopy category SH. Similarly, for the ∞-category of G-spectra $\mathrm{Sp}(G)$, we can identify $\mathrm{Ho}(\mathrm{Sp}_G)$ with the G-equivariant stable homotopy category $\mathrm{SH}(G)$.

As $\mathrm{Sp}(G)$ is an ∞-category, it is naturally enriched in the category of topological spaces. We denote its mapping space by $\mathrm{Map}_{\mathrm{Sp}(G)}(X,Y)$. The morphisms in the homotopy category of the ∞-category $\mathrm{Sp}(G)$ will be denoted by $[X,Y]_{\mathrm{Sp}(G)}$. Purely from being a stable ∞-category, we have a natural identification.

$$[X,Y]_{\mathrm{Sp}(G)} \cong \pi_0 \mathrm{Map}_{\mathrm{Sp}(G)}(X,Y).$$

In fact, for any exact functor of stable ∞-categories $F\colon C \to \mathcal{D}$ we have a commutative diagram

$$\begin{array}{ccccc} \pi_k \operatorname{Map}_C(x,y) & & \longrightarrow & & \pi_k \operatorname{Map}_C(Fx,Fy) \\ \downarrow \cong & & & & \downarrow \cong \\ [\Sigma^k x, y]_C & \longrightarrow & [F\Sigma^k x, Fy]_C & \xrightarrow{\cong} & [\Sigma^k Fx, Fy]_C \end{array}$$

relating the suspension functor and the homotopy groups.
There is a left Quillen functor

$$\Sigma^\infty \colon \mathrm{Top}_{G*} \to \mathrm{OrthSpec}_G,$$

that yields a symmetric monoidal left adjoint functor Σ^∞, with right adjoint Ω^∞

$$\Sigma^\infty \colon \mathcal{S}_{G*} \rightleftarrows \mathrm{Sp}(G) \colon \Omega^\infty.$$

Notation 3.21. For simplicity, we will write we will X_+ instead of $\Sigma^\infty X_+$ for a G-space X.

An important subcategory of $\mathrm{Sp}(G)$, is Borel-equivariant spectra. The reason for this is that the full subcategory of Borel-equivariant spectra $\mathrm{Sp}(G)_{\mathrm{Borel}}$ is equivalent to the functor category $\mathrm{Fun}(BG, \mathrm{Sp})$ of the ∞-category of non-equivariant spectra with a G-action.

Definition 3.22. A G-spectrum $X \in \mathrm{Sp}(G)$ is said to be **Borel-equivariant** if the natural map

$$X \to F_G(EG_+, X)$$

is an equivalence in $\mathrm{Sp}(G)$. Here EG denotes the total space of the classifying space BG of the group G. We let $\mathrm{Sp}(G)_{\mathrm{Borel}}$ denote the full subcategory spanned by Borel-equivariant spectra.

Proposition 3.23. *[***Proposition 6.17***; MNN17] We have an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Sp}(G)_{\mathrm{Borel}} \cong \mathrm{Fun}(BG, \mathrm{Sp}).$$

Under this identification we can think of $\mathrm{Fun}(BG, \mathrm{Sp})$ as a full subcategory of $\mathrm{Sp}(G)$ of non-equivariant spectra with G-action.

3.5 Fixed point functors

We will now define the geometric fixed point functor in terms of ∞–categories, as in [MNN17]. The functors defined in this section are classical in the G-equivariant stable homotopy theory setting. See for instance [Lew+86], for details. In [MNN17], the authors are able to upgrade these functors from the homotopy category $\mathrm{Ho}(\mathrm{Sp}(G)) = \mathrm{SH}(G)$ to the ∞-category $\mathrm{Sp}(G)$.

Recall that a **tt-functor** is a symmetric monoidal triangulated functor. Every group homomorphism $\alpha \colon H \to G$ induces a tt-functor

$$\alpha^* \colon \mathrm{Sp}(G) \to \mathrm{Sp}(H)$$

that preserves compact objects, as compact objects coincide with rigid objects in $\mathrm{Sp}(G)$ and $\mathrm{Sp}(H)$, and any tt-functor preserves rigid objects. We now give an important example of such a functor.

DEFINITION 3.24. ([**Proposition 5.14.**; MNN17]) Let $H \le G$, then, the inclusion $H \hookrightarrow G$, induce a tt-functor called **restriction**

$$\mathrm{res}^G_H \colon \mathrm{Sp}(G) \to \mathrm{Sp}(H).$$

We note that the restriction functor has both left and right adjoints, called induction and coinduction. That is, we have two pairs of adjoint functors

$$\mathrm{ind}^G_H \colon \mathrm{Sp}(H) \rightleftarrows \mathrm{Sp}(G) \ : \mathrm{res}^G_H \ \text{ and } \ \mathrm{res}^G_H \colon \mathrm{Sp}(G) \rightleftarrows \mathrm{Sp}(H) \ : \mathrm{coind}^G_H \,. \tag{3.2}$$

Importantly, if G is a finite group, then $\mathrm{ind}^G_H \cong \mathrm{coind}^G_H$ are isomorphic, due to the Wirthmüller isomorphism.
As $\mathrm{Sp}(G)$ is a presentable symmetric monoidal ∞-category, then by [**Corollary 4.8.2.19.**; Lur17], it gets a canonical symmetric monoidal, colimit-preserving functor

$$i_* = i^G_* \colon \mathrm{Sp} \to \mathrm{Sp}(G).$$

By the adjoint functor theorem (see [**Corollary 5.5.2.9**; Lur09]) this therefore has a right adjoint. The right adjoint is lax monoidal, as i_* is monoidal, by [**Corollary 7.3.2.7**; Lur17].

DEFINITION 3.25. We call the lax symmetric monoidal functor $i^*_G = (-)^G \colon \mathrm{Sp}(G) \to \mathrm{Sp}$, right adjoint to i_*, for the **categorical fixed point functor**.

For any subgroup, $H \le G$. We can precompose the *categorical fixed point functor* with restriction, to get the H-categorical fixed point functor:

DEFINITION 3.26. Let $H \le G$ be any subgroup. Then we call the composition

$$\mathrm{Sp}(G) \xrightarrow{\mathrm{res}^G_H} \mathrm{Sp}(H) \xrightarrow{(-)^H} \mathrm{Sp}$$

the **categorical H-fixed point functor**. We will denote this composition again by $(-)^H \colon \mathrm{Sp}(G) \to \mathrm{Sp}$.

Similarly, as in the non-stable G-equivariant homotopy category, $\Sigma^\infty G/H_+$ represents categorical H-fixed points in the following sense:

PROPOSITION 3.27. *Let $X \in \mathrm{Sp}(G)$ then $\pi_0(X^H) \cong [G/H_+, X]_{\mathrm{Sp}(G)}$.*

PROOF. Suppose that $H \le G$, then, from the restriction-induction adjunction in Equation (3.2), [**Equation 5.17** ; MNN17] that states that $\operatorname{ind}_H^G \mathbb{S}_H \cong \Sigma^\infty G/H_+$ and as i_*^H is monoidal, so $i_*^H \mathbb{S} \cong \mathbb{S}_H$, we get that

$$\begin{aligned}
\pi_0(X^H) &:= [\mathbb{S}, (-)^H \operatorname{res}_H^G X)]_{\mathrm{Sp}} \\
&\cong \pi_0 \operatorname{Map}_{\mathrm{Sp}}(\mathbb{S}, (-)^H \operatorname{res}_H^G X) \\
&\cong \pi_0 \operatorname{Map}_{\mathrm{Sp}(H)}(i_*^H \mathbb{S}, \operatorname{res}_H^G X) \\
&\cong \pi_0 \operatorname{Map}_{\mathrm{Sp}(G)}(\operatorname{ind}_H^G \mathbb{S}_H, X) \\
&\cong \pi_0 \operatorname{Map}_{\mathrm{Sp}(G)}(\Sigma^\infty G/H_+, X) \\
&\cong [G/H_+, X]_{\mathrm{Sp}(G)}.
\end{aligned}$$

□

We define the *geometric fixed point functor* as in [**Def. 6.12**; MNN17]. We refer the reader to their paper for the construction.

DEFINITION 3.28. Let $\mathcal{P}_G$ denote the family of proper subgroups of G and $A_{\mathcal{P}_G} = \prod_{H \lneq G} F(G/H_+, \mathbb{S}_G)$. We define the **geometric fixed point functor** as the composition

$$\Phi^G \colon \mathrm{Sp}(G) \xrightarrow{(-)[A_{\mathcal{P}_G}^{-1}]} \mathrm{Sp}(G)[\mathcal{P}_G^{-1}] \subset \mathrm{Sp}(G) \xrightarrow{(-)^G} \mathrm{Sp}\,.$$

We want to define the geometric fixed point functor for every subgroup of H. We do so by pre-composing with restriction.

DEFINITION 3.29. Let $H \le G$ be any subgroup of G. We define the H-**geometric fixed point functor** as the following composition:

$$\Phi^H \colon \mathrm{Sp}(G) \xrightarrow{\operatorname{res}_H^G} \mathrm{Sp}(H) \xrightarrow{\Phi^H} \mathrm{Sp}\,.$$

We remark that the H-**geometric fixed point functor** is a colimit preserving tt-functor functor. Moreover, for any $X \in \mathrm{Top}_G$, $\Phi^H(\Sigma^\infty X_+) \cong \Sigma^\infty(X_+^H)$. Another important property of the geometric fixed point functor, is that it retains an action of $W_G H$. That is, it factors through the category $\mathrm{Fun}(BW_G H, \mathrm{Sp})$, by [**Remark 3.3.6**; Sch18], in the following way:

$$\begin{array}{ccc}
\mathrm{Sp}(G) & \xrightarrow{\quad\Phi^H\quad} & \mathrm{Sp} \\
& {\scriptstyle \Phi_c^H} \searrow \qquad \nearrow {\scriptstyle \text{forget action}} & \\
& \mathrm{Fun}(BW_G H, \mathrm{Sp}) &
\end{array}$$

Here Φ_c^H is Φ^H, with the codomain restricted to $\mathrm{Fun}(BG, \mathrm{Sp})$. For simplicity, we will write Φ^H for Φ_c^H.

3.6 R-local spectra

We will now discuss the ∞-category of R-local G-spectra $\mathrm{Sp}_{\mathrm{R}}(G)$. We want to localize $\mathrm{Sp}_{\mathrm{R}}(G)$ at a subring $R \subset \mathbb{Q}$. We will restrict ourselves to a specific type of subring of $\mathbb{Q}$, namely $R := \mathbb{Z}_P$. We denote the set of prime numbers by $\mathbb{P}$.

DEFINITION 3.30. Let $P \subset \mathbb{P}$ be a subset of prime numbers $\mathbb{P} \subset \mathbb{Z}$. Then

$$\mathbb{Z}_P := \{a/b \in \mathbb{Q} \mid \text{for all } p \in P,\ b \nmid p\}.$$

Let us note two edge cases, for this definition.

EXAMPLE 3.31. If $P = \emptyset$, we invert every prime, so $\mathbb{Z}_P = \mathbb{Q}$. On the contrary, if $P = \mathbb{P}$, the set of all primes. We invert no prime, so $\mathbb{Z}_P = \mathbb{Z}$.

NOTATION 3.32. Our ring R will be of the form $\mathbb{Z}_P$ for the rest of this thesis, as this gives us a *smashing* localization, as seen below.

DEFINITION 3.33. A map $f\colon X \to Y$ is a **π_*-isomorphism**, if the induced map

$$\pi_n(f)\colon \pi_n(X) \to \pi_n(Y),$$

is an isomorphism for each $n \in \mathbb{N}$. We say that spectra X and Y are **π_*-isomorphic** if there is a sequence of π_*-isomorphisms relating X and Y.

DEFINITION 3.34. Let $X \in \mathrm{Sp}(G)$, we define the **H-homotopy group** of X to be

$$\pi_*^H(X) := \pi_* \operatorname{Map}_{\mathrm{Sp}(G)}(\Sigma^\infty G/H_+, X).$$

REMARK 3.35. This definition restricts down to non-equivariant homotopy groups by letting $G = \{*\}$, that is, for $X \in \mathrm{Sp}$

$$\pi\{*\}_*(X) := \pi_* \operatorname{Map}_{\mathrm{Sp}}(\Sigma^\infty\{*\}_+, X) = \pi_* \operatorname{Map}_{\mathrm{Sp}}(\Sigma^\infty S^0, X).$$

DEFINITION 3.36. Let R be of the form $\mathbb{Z}_P$ and let $\mathcal{W}_R$ denote the class of maps in $\mathrm{Sp}(G)$ $f\colon X \to Y$, such that the induced maps

$$\pi_*^H(X) \otimes R \xrightarrow{\pi_*^H(f)\otimes \mathrm{id}_R} \pi_*^H(Y) \otimes R,$$

are isomorphisms for every closed subgroup $H \leq G$. We define the **∞-category of R-local G-spectra** $\mathrm{Sp}_{\mathrm{R}}(G)$ as

$$\mathrm{Sp}_{\mathrm{R}}(G) := \mathrm{Sp}(G)[\mathcal{W}_R^{-1}].$$

One can see section 5.2.7 and 5.5.4 of [Lur09], for further details on the localization of an ∞-category.

We will now see that the ∞-category of R-local G-spectra, gives us the R-local G-equivariant stable homotopy category, when passing to the homotopy category.
As one can see in chapter 7 of [BR20], the R-local G-equivariant stable homotopy category SH_R, is the *Bousfield localization* of SH at the *Moore spectrum* MR. See Example 7.4.7 in [BR20] for a construction of the Moore spectrum, MG, for an Abelian group G. The Moore spectrum is characterized by the following properties:

1. $\pi_{<0} MG = 0$
2. $\pi_0 MG = G$
3. $\pi_{>0}(MG \wedge H\mathbb{Z}) = 0$

Let $L_{MR}\colon \mathrm{SH} \to \mathrm{SH}_R$ be the Bousfield localization functor. Then, by [**Proposition 7.4.10**; BR20], every spectrum $L_{MR}(X) := X_R \in \mathrm{SH}_R$ is of the form $X_R \cong X \wedge MR$, for $X \in \mathrm{SH}$. We therefore say that L_{MR} is **smashing**. Furthermore,

$$\pi_*(X_R) = \pi_*(X) \otimes R, \tag{3.3}$$

and we have an isomorphism

$$\pi_*(X) \otimes R \cong \pi_*(X \wedge MR).$$

The unit in SH_R is the Moore spectrum $\mathbb{S}_R \cong MR$. We will often write X, instead of $X \wedge MR$, for objects is SH_R.
Let $L_R\colon \mathrm{Sp} \to \mathrm{Sp}_R$ denote the localization functor at the ring $R := \mathbb{Z}_P$ and
$L_{MR}\colon \mathrm{Ho}(\mathrm{Sp}) \to \mathrm{Ho}(\mathrm{Sp}_R)$ denote the Bousfield localization at the Moore spectrum MR. Then, as the Bousfield Localization is smashing, the order in which we invert morphisms does not matter. The following diagram therefore commutes:

$$\begin{array}{ccc} \mathrm{Sp} & \xrightarrow{L_R} & \mathrm{Sp}_R \\ \downarrow & & \downarrow \\ \mathrm{Ho}(\mathrm{Sp}) & \xrightarrow[L_{MR}]{} & \mathrm{Ho}(\mathrm{Sp}_R) \end{array}$$

REMARK 3.37. As the n-th homotopy group of the sphere spectrum is n-th stable stem, which is finite for $n > 0$, the unit of the rational stable homotopy category $\mathrm{SH}_{\mathbb{Q}}$, is the Eilenberg MacLane spectrum $H\mathbb{Q}$. In other words, the Eilenberg MacLane spectrum $H\mathbb{Q}$ coincides with the Moore spectrum $M\mathbb{Q}$ (see page 6 in [Ban10]).

REMARK 3.38. The functors in Section 3.5, naturally restricts to R-local G-spectra.

3.7 The Tate spectral sequence

In this section, we will briefly introduce two computational tools for calculating spectra. Namely, the *homotopy fixed point spectral sequence* and the *Tate spectral sequence*. We will also remind the reader of a classical result from group (co)homology, which will help us compute these spectral sequences.

Recall that $F_G(-,-)$ denote the mapping G-spectrum. Similarly, as for G-spaces, we have the notion of *homotopy orbits* and *homotopy fixed points*:

1. **The homotopy fixed point spectrum**
   $X^{hG} := F_G(EG_+, X)^G \in \mathrm{SH}$
   Warning! It is not true that $(\Sigma^\infty X_+)^{hG} \cong \Sigma^\infty(X_+^{hG})$.
2. **The homotopy orbit spectrum**
   $X_{hG} := (EG_+ \wedge X)^G \in \mathrm{SH}$
   It is true that $\Sigma^\infty(X_{hG})_+ \cong (\Sigma^\infty X_+)_{hG}$.

The **Tate Spectrum** is defined by; $X^{tG} := (\widetilde{EG}_+ \wedge F(EG_+, X))^G \in \mathrm{SH}$. Here $\widetilde{EG}$ denotes the mapping cone of the map $c\colon EG_+ \to S^0$ that collapses everything to the non-basepoint. As one can see in [GM95b], the Tate spectrum also appears as the homotopy cofiber of the norm map

$$N_X\colon X_{hH} \to X^{hH}$$

In other words, there exists a pushout diagram

$$\begin{array}{ccc} X_{hG} & \longrightarrow & X^{hH} \\ \downarrow & & \downarrow \\ * & \longrightarrow & X^{tH} \end{array}$$

defining X^{tH}.

REMARK 3.39. Similarly, as for categorical and geometric fixed points, we can extend the definitions of homotopy orbits, homotopy fixed points and Tate spectra to any subgroup $H \leq G$ by precomposing with restriction $\mathrm{res}^G_H : \mathrm{SH}(G) \to \mathrm{SH}(H)$.

The **Tate cohomology** sews together cohomology and homology and is defined for a finite group G and a $k[G]$-module M, where k is a commutative ring, as

$$\widehat{H}^i(G;M) := \begin{cases} H^i(G;M), & \text{if } i \geq 1 \\ H_{-i-1}(G;M), & \text{if } i \leq -2. \end{cases}$$

The cases for $i = 0, -1$ are defined by the exact sequence

$$0 \to \widehat{H}^{-1}(G;M) \to H_0(G;M) \xrightarrow{N} H^0(G;M) \to \widehat{H}^0(G;M) \to 0.$$

Here $N\colon H_0(G;M) = k \otimes_{k[G]} M = M_G \to H^0(G;M) = \mathrm{Hom}_{k[G]}(k, M) = M^G$ is induced by the norm map, defined by

$$[x] \mapsto \sum_{g \in G} gx.$$

Note that $\sum_{g\in G} gx$ actually is a G-fixed point. For $X \in \mathrm{SH}(G)$, we have the Tate spectral sequence:

THEOREM 3.40. *[GM95b] The Tate spectral sequence is the conditionally convergent spectral sequence with signature*

$$E^2_{p,q} = \widehat{H}^{-p}(G;\pi_q(X)) \Rightarrow \pi_{p+q}(X^{tG}). \tag{3.4}$$

We also have the *homotopy fixed point spectral sequence* for the homotopy fixed point spectrum of $X \in \mathrm{SH}(G)$:

THEOREM 3.41. *[GM95b] The homotopy fixed point spectral sequence is the spectral sequence with signature*

$$E^2_{p,q} = H^{-p}(G; \pi_q(X)) \Rightarrow \pi_{p+q}(X^{hG}). \tag{3.5}$$

To help us compute the two above spectral sequences, we are going to use the following fact, that can be found in any textbook, e.g., [Bro94], on group (co)homology:

THEOREM 3.42. *Let G be a finite group and let R be a commutative ring with unit and trivial G action such that $|G|$ is invertible in R. Then for all $n \in \mathbb{N}_{>0}$*

$$H_n(G; R) \cong 0 \text{ and } H^n(G; R) \cong 0.$$

EXAMPLE 3.43. Suppose that $G = S_4$, the symmetric group on 4 symbols. The order of S_4 is equal to $4! = 24$. By Theorem 3.42, we can conclude that $H_n(S_4; \mathbb{Z}_{\mathbb{P}\setminus\{2,3\}}) \cong 0$ for every $n > 0$.

Chapter 4

Decomposition of the Equivariant Stable Homotopy Category

In this chapter, we will reprove Theorem 3.10 in [Wim19], for $R := \mathbb{Z}_P$ and a finite group G, closely following Wimmer's footsteps, but hopefully making the proof more accessible for the non-experts. The proof should easily generalize to Wimmer's hypothesis on G being a compact Lie group and $\mathcal{F}$ a family of finite subgroups in G whose order is invertible in R. For our purposes, G being finite is almost as good as it gets, as the Balmer spectrum only commutes with a finite product of categories by Lemma 2.36. This will be further discussed in Section 5.4. There, we will also state the slightly more general result.
Therefore, in this chapter, we will let G be a finite group and $R := \mathbb{Z}_P$, a subring of $\mathbb{Q}$, such that $|G|$ is invertible in R.

4.1 Fixed points of homogeneous spaces

The geometric fixed point functor Φ^H sends the compact generator $\Sigma^\infty(G/K)_+$ of $\mathrm{Sp}(G)$ to the suspension spectra of the H-fixed points of the homogeneous space $(G/K)_+$. We therefore would like some better descriptions, of the W_GH-space, $(G/K)^H$. We know of three other descriptions for $(G/K)^K$, when $H, K \leq G$:

1. For any G-space X, we have that the homogeneous space G/H, corepresents H-fixed points, therefore, $\mathrm{Map}_G(G/H, G/K) \cong (G/K)^H$. This, in particular, implies that $W_GH \cong (G/H)^H$, via the isomorphism

$$W_GH \cong \mathrm{Map}_G(G/H, G/H) \text{ defined by } [g] \mapsto G/H \xrightarrow{(-)\cdot g^{-1}} G/H. \qquad (4.1)$$

2. As in [**Section 2(G)**; BS17], we have an isomorphism of W_GH-spaces $(G/K)^H \cong N(H,K)/K$, where

$$N(H,K) := \{g \in G \mid H^g \subset K\}$$

This has some immediate implications: If $H \not\leq_G K$, then $(G/K)^H = \emptyset$ and if $H \trianglelefteq G$ and $H \leq_G K$, then $(G/K)^H \cong G/K$.

3. From [Sch18][1]: Let $R_{H,K}$ denote the set of K-conjugacy classes of subgroups $L \leq K$ that are G-conjugate of H, in symbols:

$$R_{H,K} := \{(L) \leq_K K \mid L \sim_G H\}.$$

Then, for $(L) \in R_{H,K}$, we can choose an element $g_L \in G$ such that $H^{g_L} = L$. This implies that $g_L K \in (G/K)^H$, and the map

$$\coprod_{(L)\in R_{H,K}} W_G H / W_{g_L K} H \to (G/K)^H$$
$$[nH] \mapsto n g_L K,$$

is an isomorphism of $W_G H$-sets. As $W_K L = (W_K H)^{g_L} \cong W_{g_L K} H$, we can treat $W_K L$ as a subgroup of $W_G H$, and we can rewrite the above as

$$\coprod_{(L)\in R_{H,K}} W_G H / W_K L \cong (G/K)^H. \tag{4.2}$$

Here, $W_{g_L K} H$ denotes $(g_L)^{-1} K g_L$. This decomposition formula will be of major importance in the section to come.

4.2 The decomposition of $\mathrm{Sp}_{\mathrm{R}}(G)$

The goal of this section is to prove that the ∞-category of R-local G-spectra, $\mathrm{Sp}_{\mathrm{R}}(G)$, decomposes as a product of functor categories, in the following way:

THEOREM 4.1. *Let R be of the form $\mathbb{Z}_P$ and G be a finite group such that $|G|$ is invertible in R. Then the geometric fixed point functor induces an equivalence of symmetric monoidal stable ∞-categories*

$$\mathrm{Sp}_{\mathrm{R}}(G) \cong \prod_{(H)\leq G} \mathrm{Fun}(BW_G H, \mathrm{Sp}_{\mathrm{R}}).$$

REMARK 4.2. If $|G|$ is invertible in R, then for $H \leq G$ we have that $|W_G H|$ is invertible in R, as the order of H divides the order of G.

To prove this theorem, we will rely on the following proposition:

PROPOSITION 4.3. *[**Proposition 3.21**; PSW21] Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor between presentably symmetric monoidal stable ∞-categories, which admits a right adjoint G. If $\mathcal{C}$ is generated by a set of compact-rigid objects, then the functor F is an equivalence, if and only if*

1. *the functor F, sends a set of compact generators of $\mathcal{C}$, to a set of compact generators of $\mathcal{D}$.*

[1] See the beginning of the proof of (ii) in Theorem 3.4.22.

2. *the commutative algebra object* $G(\mathbb{1}_{\mathcal{D}})$*, is equivalent to the commutative algebra object* $\mathbb{1}_C$*.*

As one should expect, if F is an equivalence of symmetric monoidal stable ∞-categories, then F is a symmetric monoidal equivalence by the following lemma:

LEMMA 4.4. *[**Remark 2.1.3.8**; Lur17] Let* $F: C \to \mathcal{D}$ *be an exact monoidal functor of symmetric monoidal stable* ∞*-categories that induces an equivalence of stable* ∞*-categories. Then* F *is an equivalence of symmetric monoidal stable* ∞*-categories.*

We start by proving two lemmas, needed in using Proposition 4.3. First, we are going to see that taking the 0-th homotopy group commutes with fixed points, when working in the R-local Borel-equivariant setting.

LEMMA 4.5. *Let* $X \in \mathrm{Fun}(BG, \mathrm{Sp_R})$ *and* $H \leq G$*, then*

$$\pi_0(X^H) \cong (\pi_0 X)^H$$

PROOF. As $\pi_0 X$ is a module over R, then $H^i(H; \pi_0 X) = 0$ is immediate for $i > 1$ by using Theorem 3.42 and the universal coefficient theorem: Any subring of $\mathbb{Q}$ is a PID. Therefore, by the universal coefficient theorem

$$H^i(H, \pi_0 X) \cong \mathrm{Ext}^1_R(H_{i-1}(H, R), \pi_0 X) \oplus \mathrm{Hom}_R(H_i(X; R), \pi_0 X).$$

For $i > 1$, both summands are zero by Theorem 3.42. For the case $i = 1$, one must additionally note that $H_0(X; R)$ is a free and in particular, a projective R-module, hence $\mathrm{Ext}^1_R(H_0(X; R), \pi_0(R)) = 0$, which means that $H^i(H; \pi_0 X) = 0$ for all $i \neq 0$. The homotopy fixed point spectral sequence from Theorem 3.41 therefore collapses on the second page, with one non-zero column. We can therefore conclude that $H^0(H; \pi_0 X) \cong \pi_0(X^{hH})$. By definition $H^0(H; \pi_0(X)) := \pi_0(X)^H$. Putting the above together, we get

$$(\pi_0 X)^H =: H^0(H; \pi_0 X) \cong \pi_0(X^{hH}). \tag{4.3}$$

It is therefore sufficient to show that

$$\pi_0(X^{hH}) \cong \pi_0(X^H).$$

However, in the R-local Borel-equivariant setting, this is true on a stronger level.
By definition $X^H := (-)^H \circ \mathrm{res}^G_H(X)$ and by [**Proposition 6.16.**; MNN17] $\mathrm{res}^G_H(X)$ is again H-Borel-equivariant.
Recall from Proposition 3.23 that

$$\mathrm{Sp_R}(H)_{\mathrm{Borel}} \cong \mathrm{Fun}(BH, \mathrm{Sp_R}(H)),$$

and therefore

$$F_H(\Sigma^\infty EH_+, \mathrm{res}^G_H(X)) \cong \mathrm{res}^G_H(X). \tag{4.4}$$

By definition of H-homotopy fixed points,

$$(\mathrm{res}^G_H(X))^{hH} := F_H(\Sigma^\infty EH_+, \mathrm{res}^G_H(X))^H \cong (\mathrm{res}^G_H(X))^H,$$

by Equation (4.4). That is, $X^{hH} \cong X^H$, and in particular

$$\pi_0(X^{hH}) \cong \pi_0(X^H). \tag{4.5}$$

By combining Equation (4.3) and Equation (4.5) we are done. □

We will now see that morphisms in the homotopy category of the ∞-category Sp_{R}, can be realized as a coproduct of fixed-points of homotopy groups. This is a technical lemma, that allow us to use [**Corollary 3.4.28**; Sch18].

LEMMA 4.6. *[**Lemma 3.8**; Wim19] Let $H, K \leq G$ be subgroups of a finite group G. Then we have a natural isomorphism*

$$[\Sigma^\infty (G/K)^H_+, Y]_{\mathrm{Fun}(BW_G H, \mathrm{Sp}_{\mathrm{R}})} \cong \prod_{(L) \in R_{H,K}} (\pi_0 Y)^{W_K L}.$$

PROOF. Recall from Equation (4.2), that

$$(G/K)^H \cong \coprod_{(L) \in R_{H,K}} W_G H / W_K L.$$

Therefore,

$$\begin{aligned}[\Sigma^\infty (G/K)^H_+, Y]_{\mathrm{Fun}(BW_G H, \mathrm{Sp}_{\mathrm{R}})} &\cong [\Sigma^\infty (\coprod_{(L) \in R_{H,K}} W_G H / W_K L)_+, Y]_{\mathrm{Fun}(BW_G H, \mathrm{Sp}_{\mathrm{R}})} \\ &\cong \prod_{(L) \in R_{H,K}} [\Sigma^\infty (W_G H / W_K L)_+, Y]_{\mathrm{Fun}(BW_G H, \mathrm{Sp}_{\mathrm{R}})}.\end{aligned}$$

By Proposition 3.27 we have that

$$[\Sigma^\infty (W_G H / W_K L)_+, Y]_{\mathrm{Fun}(BW_G H, \mathrm{Sp}_{\mathrm{R}})} \cong \pi_0 (Y^{W_K L}).$$

We are now done, as the fixed point functor commutes with taking π_0, when working in the R-local Borel-equivariant setting, by Lemma 4.5.

□

LEMMA 4.7. *Let $F \colon \mathcal{C} \leftrightarrows \mathcal{D} : G$ be an adjunction of stable ∞-categories. If G preserves small colimits, then F preserves compact objects.*

PROOF. In the non ∞-world, this is straightforward. Assume that $\mathcal{C}$ and $\mathcal{D}$ are triangulated categories. Let $x \in \mathcal{C}$ be compact. By Proposition 2.5 this is the same as for any family $y_i \in \mathcal{C}$ indexed over the set I, we have that $\mathrm{Hom}(x, \coprod_{i \in I} y_i) \cong \coprod_{i \in I} \mathrm{Hom}(x, y_i)$. Assume therefore $x \in \mathcal{C}$ is a compact object and that z_i is any family of objects in $\mathcal{D}$ indexed over a set I. By adjunction

$$\mathrm{Hom}_{\mathcal{D}}(F(x), \coprod_{i \in I} z_i) \cong \mathrm{Hom}_{\mathcal{C}}(x, G(\coprod_{i \in I} z_i))$$

By assumption G commutes with small colimits, hence

$$\mathrm{Hom}_{\mathcal{C}}(x, G(\coprod_{i \in I} z_i)) \cong \mathrm{Hom}_{\mathcal{C}}(x, \coprod_{i \in I} G(z_i))$$

By combining that x is compact and the adjunction, we have that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(x, \coprod_{i\in I} G(z_i)) &\cong \coprod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(x, G(z_i)) \\ &\cong \coprod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(F(x), z_i)\end{aligned}$$

In the ∞-category world, we refer to [**Lemma 5.5.1.4**; Lur09]. □

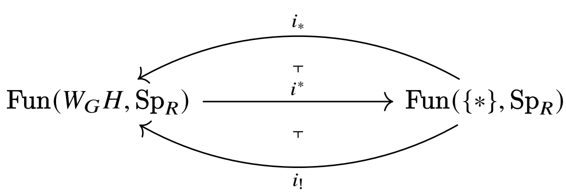
To prove the decomposition formula, we want to apply Proposition 4.3, we therefore need to check the following criteria for

$$\Phi = \prod_{(H)\leq G} \Phi^H : \mathrm{Sp}_{\mathrm{R}}(G) \to \prod_{(H)\leq G} \mathrm{Fun}(BW_G H, \mathrm{Sp}_{\mathrm{R}}).$$

1. Φ has a right adjoint G.
2. Φ takes compact generators to compact generators.
3. The commutative algebra object $G(\mathbb{1}_{\mathcal{D}})$, is equivalent to the commutative algebra object $\mathbb{1}_{\mathcal{C}}$.

PROOF. 1. Φ is colimit preserving as it is defined as a finite product of colimit preserving functors. It therefore has a right adjoint G, by the adjoint functor theorem [**Corollary 5.5.2.9**; Lur09].

2. The classifying space of G, BG, is a Kan complex for every group G. In particular, for a subgroup $H \leq G$, $BW_G H$ is a Kan complex. Let $i\colon \{*\} \to BW_G H$ be the inclusion of a vertex. Then we have an induced *restriction* functor $i^*\colon \mathrm{Fun}(BW_G H, \mathrm{Sp}_R) \to \mathrm{Fun}(\{*\}, \mathrm{Sp}_R) \cong \mathrm{Sp}_R$. As limits and colimits in $\mathrm{Fun}(BW_G H, \mathrm{Sp}_R)$ are computed pointwise in Sp_R, i^* preserves limits and colimits. Again, by the adjoint functor theorem [**Corollary 5.5.2.9**; Lur09] we have a left and right adjoints to i^*, which we denote by $i_!$ and i_*, respectively:



We need to figure out what the compact generators of $\mathrm{Fun}(BW_H G, \mathrm{Sp}_{\mathrm{R}})$ are:

Sp_R contain all small colimits. By [**Lemma 4.3.8**; HL13], $\mathrm{Fun}(BW_G H, \mathrm{Sp}_{\mathrm{R}})$, is therefore generated by objects of the form $i_!(c) \in \mathrm{Fun}(\{*\}, \mathrm{Sp}_{\mathrm{R}}) \cong \mathrm{Sp}_{\mathrm{R}}$, where $c \in \mathrm{Sp}_{\mathrm{R}}$. As $\mathbb{S}_R$ is a compact generator for Sp_{R}, $i_!(\mathbb{S}_R)$ generates $\mathrm{Fun}(BW_G H, \mathrm{Sp}_{\mathrm{R}})$. It can then be shown that $i_!(\mathbb{S}_R) = W_G H_+ \wedge \mathbb{S}_R$, by the pointwise formula for left Kan extension, see for instance Section 4.2 in [Hea21a]. Moreover, as i^* preserve small colimits, $e_!$ preserve compact objects by Lemma 4.7. Consequently, $\mathrm{Fun}(BW_G H, \mathrm{Sp}_{\mathrm{R}})$

is compactly generated by $W_GH_+ \wedge \mathbb{S}_R$.

$\mathrm{Sp}_{\mathrm{R}}(G)$ is compactly generated by objects of the form $\Sigma^\infty G/H_+ \wedge \mathbb{S}_R$, where H is a subgroup of G. The geometric fixed point functor commutes with taking the suspension, that is

$$\Phi^H(\Sigma^\infty G/H_+ \wedge \mathbb{S}_R) \cong \Sigma^\infty (G/H)^H_+ \wedge \Phi^H(\mathbb{S}_R) \cong \Sigma^\infty W_GH_+ \wedge \mathbb{S}_R$$

Here we have used Equation (4.1) to identify $(G/H)^H$ and W_GH, that H acts trivially on $\mathbb{S}_R$ and that Φ^H is monoidal. We can therefore conclude that Φ takes compact generators to compact generators.

3. For readability, we let $\mathcal{D} := \coprod_{(H)\leq G} \mathrm{Fun}(BW_GH, \mathrm{Sp}_{\mathrm{R}})$. The unit

$$\eta\colon \mathrm{id}_{\mathrm{Sp}_{\mathrm{R}}(G)} \Rightarrow G \circ \Phi$$

is a natural isomorphism if and only if Φ is fully faithful, see [Sta21, Lemma 07RB]. As Φ is monoidal, we have that $\Phi(\mathbb{1}_{\mathrm{Sp}_{\mathrm{R}}(G)}) \cong \mathbb{1}_{\mathcal{D}}$. Suppose for a second that Φ is fully faithful, then the unit η, is a natural isomorphism. In particular, this implies that

$$\mathbb{1}_{\mathrm{Sp}_{\mathrm{R}}(G)} \cong G \circ \Phi(\mathbb{1}_{\mathrm{Sp}_{\mathrm{R}}(G)}) \cong G(\mathbb{1}_{\mathcal{D}})$$

as objects. By naturality of the unit, η, $\mathbb{1}_{\mathrm{Sp}_{\mathrm{R}}(G)}$ and $G(\mathbb{1}_{\mathcal{D}})$ are equivalent as algebra objects. It is therefore sufficient to check that Φ is fully faithful. We can do so by checking fully faithfulness on the set of compact generators of $\mathrm{Sp}_{\mathrm{R}}(G)$. Therefore, let $X \in \mathrm{Sp}_{\mathrm{R}}(G)$ be a compact generator of $\mathrm{Sp}_{\mathrm{R}}(G)$ and $Y \in \mathrm{Sp}_{\mathrm{R}}(G)$ be any object. We need to verify that

$$[X, Y]_{\mathrm{Sp}_{\mathrm{R}}(G)} \to [\Phi X, \Phi Y]_{\prod_{(H)\leq G} \mathrm{Fun}(BW_GH, \mathrm{Sp}_{\mathrm{R}})}$$

is an isomorphism. By assumption, X is a compact generator, that is, X is of the form $\Sigma^\infty G/K_+$ for some subgroup $K \leq G$. By definition of Φ, we have the map

$$[\Sigma^\infty G/K_+, Y]_{\mathrm{Sp}_{\mathrm{R}}(G)} \to [\Phi\Sigma^\infty G/K_+, \Phi Y]_{\mathcal{D}} \tag{4.6}$$
$$\cong \prod_{(H)\leq G} [\Phi^H \Sigma^\infty_+(G/K), \Phi^H Y]_{\mathrm{Fun}(BW_GH, \mathrm{Sp}_{\mathrm{R}})} \tag{4.7}$$

The geometric fixed point functor commutes with taking H-fixed points; $\Phi^H\Sigma^\infty(G/H)_+ \cong \Sigma^\infty(G/H)^H_+$. Hence,

$$[\Phi^H\Sigma^\infty(G/K)_+, \Phi^H Y]_{\mathrm{Fun}(BW_GH, \mathrm{Sp}_{\mathrm{R}})} \cong [\Sigma^\infty(G/K)^H_+, \Phi^H Y]_{\mathrm{Fun}(BW_GH, \mathrm{Sp}_{\mathrm{R}})}.$$

By Lemma 4.6, we can identify Equation (4.6) with

$$\prod_{(H)\leq G} \Phi^H : \pi_0^K(Y) \to \prod_{(L)\in R_{H,K}} (\pi_0 \Phi^H Y)^{W_K L}. \tag{4.8}$$

As $H \leq G$ runs over the H-conjugacy classes of G, L hits every K-conjugacy class in $R_{H,K}$ exactly once, therefore Equation (4.8) is reduced to

$$\Phi^H : \pi_0^K(Y) \to \prod_{(H)\leq G} \pi_0(\Phi^H Y)^{W_K H}.$$

As $|G|$ is invertible in R, we can apply [**Corollary 3.4.28**; Sch18][2] to get that Equation (4.6) is an isomorphism. In particular, Φ is fully faithful when restricted to the full subcategory of compact objects.

We can now safely apply Proposition 4.3 to see that Φ is an equivalence of symmetric monoidal stable ∞-categories

$$\mathrm{Sp}_{\mathrm{R}}(G) \cong \prod_{(H)\leq G} \mathrm{Fun}(BW_G H, \mathrm{Sp}_{\mathrm{R}}).$$

Moreover, as Φ is exact monoidal, this will be an equivalence of symmetric monoidal stable ∞-categories by Lemma 4.4 □

[2] Note that Schwede defines $\Phi_k^H := \pi_k \circ \Phi^H$

Chapter 5

Calculation of the Balmer spectrum

In this chapter, we will calculate the Balmer spectrum of the tt-category $\mathrm{SH}^c_R(G)$. Recall that $\mathrm{SH}_R(G)$ is the homotopy category of the symmetric monoidal stable ∞-category $\mathrm{Sp}_R(G)$. By the previous chapter, we know that that $\mathrm{Sp}_\mathrm{R}(G)$ decomposes into finite product of functor categories $\mathrm{Fun}(BW_GH, \mathrm{Sp}_\mathrm{R})$, we can therefore reduce the calculation of the Balmer spectrum of $\mathrm{SH}^c_R(G)$, to the calculation of the Balmer spectrum of $\mathrm{Fun}(BW_GH, \mathrm{SH}_R)^{\mathrm{dual}}$. This calculation will occupy us for most of the chapter. We will not be able to use Wimmer's decomposition in its full capability, we will address why, and give a stronger version of our main result, in Section 5.4.

Notation 5.1. As before, let $P \subset \mathbb{P}$ be a subset of the prime numbers and $R := \mathbb{Z}_P$. We will identify $\mathrm{Ho}(\mathrm{Fun}(BG, \mathrm{Sp}_R))$ with $\mathrm{Fun}(BG, \mathrm{SH}_R)$, i.e., spectra in the R-local stable homotopy category, with an action of G.

Recall that C^{dual} denotes the dualizable objects in C and C^c denotes the compact objects. In $\mathrm{SH}_R(G)$, these notions coincide. These, however, do not coincide in $\mathrm{Fun}(BW_GH, \mathrm{SH}_R)$. In fact, $\mathrm{Fun}(BW_GH, \mathrm{SH}_R)^c$ is not even a tt-category, as the unit is not compact. Instead, by restricting to the dualizable objects of $\mathrm{Fun}(BW_GH, \mathrm{SH}_R)$, yields us a rigid tt-category $\mathrm{Fun}(BW_GH, \mathrm{SH}_R)^{\mathrm{dual}}$.

We now reap the harvest of previous chapters, in the following corollary:

Lemma 5.2. *From the monoidal equivalence of symmetric monoidal stable ∞-categories in Theorem 4.1, we have an equivalence of tt-categories*

$$\mathrm{SH}^c_R(G) = \mathrm{SH}^{\mathrm{dual}}_R(G) \cong \coprod_{(H)\leq G} \mathrm{Fun}(BW_GH, \mathrm{SH}_R)^{\mathrm{dual}}.$$

In light of Lemma 2.36, we have an induced homeomorphism of Balmer spectra

$$\mathrm{Spc}(\mathrm{SH}^c_R(G)) \cong \coprod_{(H)\leq G} \mathrm{Spc}(\mathrm{Fun}(BW_GH, \mathrm{SH}_R)^{\mathrm{dual}}).$$

The rest of this chapter will be used on computing

$$\mathrm{Spc}(\mathrm{Fun}(BW_GH, \mathrm{SH}_R)^{\mathrm{dual}}).$$

NOTATION 5.3. In this chapter, we let $C := \mathrm{Fun}(BW_GH, \mathrm{Sp}_R)$. For $X \in \mathrm{Sp}_R(G)$, we will let $\mathbb{D}_R X := F_G(X, \mathbb{S}_R)$ denote the dual of X.

5.1 Outline of the proof

We will rely on [**Corollary 9.5.**; Bal10], by Balmer. In this corollary, he describes the topological space $\mathrm{Spc}(\mathrm{SH}^c)$. We want to reduce the computation of $\mathrm{Spc}(\mathrm{Fun}(BW_GH, \mathrm{SH})^{\mathrm{dual}})$ to computing the Balmer spectrum of SH^c_R. Then we can compare the Balmer spectrum of SH^c_R with the Balmer spectrum of SH^c via the localization functor. In more detail; by Proposition 5.5, we can identify Sp_R and $\mathrm{Mod}_C(\mathbb{D}_R W_GH_+)$. This will restrict to an identification of $\mathrm{Sp}_\mathrm{R}{}^{\mathrm{dual}}$ and $\mathrm{Mod}_{C^{\mathrm{dual}}}(\mathbb{D}_R W_GH_+)$. We will prove that $\mathbb{D}_R W_GH_+$ is descendable in C^{dual} in Corollary 5.11, a property that is sufficient, by Corollary 5.15, for the *extension of scalar functor* to induce a surjection on the Balmer spectra

$$\mathrm{Spc}(\mathrm{Mod}_{\mathrm{Ho}(C^{\mathrm{dual}})}(\mathbb{D}_R W_GH_+)) \to \mathrm{Spc}(\mathrm{Ho}(C)^{\mathrm{dual}}).$$

It will turn out that this is, in fact, a homeomorphism, proving that

$$\mathrm{Spc}(\mathrm{SH}_R^{\mathrm{dual}}) \cong \mathrm{Spec}(\mathrm{Fun}(BW_GH, \mathrm{SH}_R)^{\mathrm{dual}}).$$

5.2 The proof

DEFINITION 5.4. Let $L\colon C \leftrightarrows \mathcal{D} :R$ be adjoint functors. We say that the adjunction (L, R), satisfies the **projection formula** if, for $X \in C$, $Y \in \mathcal{D}$ the natural map

$$R(Y) \otimes X \to R(Y \otimes L(X)),$$

which is adjoint to the map

$$L(R(Y) \otimes X) \cong LR(Y) \otimes L(X) \xrightarrow{\mathrm{counit}\otimes \mathbb{1}_{L(X))}} Y \otimes L(X),$$

is an equivalence in C.

We want to apply the following proposition to identify Sp_R and $\mathrm{Mod}_C(\mathbb{D}_R W_GH_+)$:

PROPOSITION 5.5. *[**Proposition 5.29**; MNN17] Suppose we have an adjunction (L, R) : $C \leftrightarrows \mathcal{D}$ between presentable, symmetric monoidal stable ∞-categories such that the tensor structure on each commutes with colimits in each variable, and L is a symmetric monoidal functor. Suppose the adjunction has the following three properties:*

1. *The adjunction (L, R) satisfies the projection formula.*
2. *The right adjoint R commutes with arbitrary colimits.*

3. *The right adjoint R is conservative.*

Then the induced adjunction $(\overline{L}, \overline{R})\colon \mathrm{Mod}_{C}(R(\mathbb{1}_{\mathcal{D}})) \leftrightarrows \mathcal{D}$ *is an inverse equivalence of symmetric monoidal ∞-categories.*

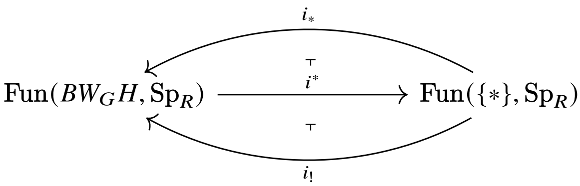
LEMMA 5.6. *Let C and $\mathcal{D}$ be closed symmetric monoidal categories. Let $i^*\colon C \to \mathcal{D}$ be closed symmetric monoidal, i.e., $i^*F_C(X,Y) \cong F_{\mathcal{D}}(i^*X, i^*Y)$, with left adjoint $i_!$ and right adjoint i_*. If $i_!$ coincides with i_*, then (i^*, i_*) satisfies the projection formula.*

PROOF. We aim to apply the Yoneda Lemma, so let $T \in \mathcal{D}$ be our test object. Then

$$\begin{aligned}
\mathrm{Hom}_C(i_*D \otimes C, T) &\cong \mathrm{Hom}_C(i_*D, F_C(C,T)) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(D, i^*F_C(C,T)) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(D, F_{\mathcal{D}}(i^*C, i^*T)) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(D \otimes i^*C, i^*T) \\
&\cong \mathrm{Hom}_C(i_*(D \otimes i^*C), T).
\end{aligned}$$

By the Yoneda Lemma, we can conclude that $i_*D \otimes C \cong i_*(D \otimes i^*C)$. □

Recall that for the inclusion of a vertex, $i\colon \{*\} \to BW_GH$, we have induced adjunctions:



THEOREM 5.7. *The adjunction (i^*, i_*), from above, satisfies the criteria of Proposition 5.5, therefore we have an inverse equivalence of symmetric monoidal ∞-categories:*

$$\overline{i^*}\colon \mathrm{Mod}_{\mathrm{Fun}(BW_GH,\mathrm{Sp}_{\mathrm{R}})}(i_*(\mathbb{S}_R)) \cong \mathrm{Sp}_{\mathrm{R}} \ :\overline{i_*}.$$

PROOF. We want to use Proposition 5.5, we must therefore check that the adjunction satisfies its conditions:

1. The adjunction (i^*, i_*) satisfy the projection formula: This follows from [**Proposition 5.14.**; MNN17], but we also give another approach: By the pointwise formula for Kan extension, i^* is closed monoidal, see e.g., [And+09]. We are therefore in the *Wirthmüller context*, as in [FHM03]. By applying Lemma 5.6, we get that the adjunction (i^*, i_*) satisfy the projection formula.

2. Coinduction i_* commutes with arbitrary colimits, as i^* has a left adjoint $i_!$, which coincides with i_* by the Wirthmüller isomorphism.

3. Coinduction i_* is conservative: Recall that we write $C := \mathrm{Fun}(BW_GH, \mathrm{Sp}_{\mathrm{R}})$. i^* preserve compact objects by Lemma 4.7, as $i_!$ coincide with i_*. Moreover, as i^*

is essentially surjective, then, by Lemma 2.8, i^* preserve compact generators. Let $Y \in \mathrm{Sp}_R$ such that $i_*(Y) \cong 0$ and X be a compact generator of C. Then

$$\mathrm{Hom}_C(X, i_*(Y)) \cong 0,$$

and by adjunction

$$\mathrm{Hom}_C(X, i_*(Y)) \cong \mathrm{Hom}_{\mathrm{Sp}_R}(i^*(X), Y).$$

As $i^*(X)$ is a compact generator, we can conclude that $Y \cong 0$.

We can therefore apply Proposition 5.5 and identify

$$\begin{aligned} \mathrm{Sp}_{\mathrm{R}} &\cong \mathrm{Mod}_C(i_*(\mathbb{S}_R)) && (5.1)\\ &\cong \mathrm{Mod}_C(F(W_G H_+, \mathbb{S}_R)) && (5.2)\\ &= \mathrm{Mod}_{\mathrm{Fun}(BW_G H, \mathrm{Sp}_{\mathrm{R}})}(\mathbb{D}_R\, W_G H_+), && (5.3)\end{aligned}$$

as symmetric monoidal stable ∞-categories. □

Definition 5.8. Let $\mathcal{T}$ be tt-category and let A be an algebra object. An object of $\mathcal{T}$ is said to be **A-nilpotent** if it belongs to the thick $\otimes$-ideal generated by A. Furthermore, we say A is **descendable**, if the $\otimes$-unit of C is A-nilpotent.

Remark 5.9. Let $\mathcal{T}$ be a tt-category and let A be an algebra object. Then A is descendable if and only if the thick $\otimes$-ideal generated by A is $\mathcal{T}$.

Theorem 5.10. *[**Theorem 4.9**; Mat18] Let C be a presentably symmetric monoidal stable ∞-category and let G be a finite group. Let $R \in \mathrm{Fun}(BG, C)$ be an algebra object. If the $\otimes$-unit $\mathbb{1} \in C$ is compact, then the following are equivalent:*

1. *The Tate spectrum vanishes, i.e., $R^{tG} = 0$*
2. *R is $\mathbb{D}_R\, G_+$-nilpotent.*

Corollary 5.11. *$\mathbb{D}_R\, W_G H_+ \in \mathrm{Fun}(BW_G H_+, \mathrm{Sp}_{\mathrm{R}})$ is descendable.*

Proof. By Theorem 5.10, it suffices to show that the Tate spectrum of $\mathbb{S}_R$ vanishes. We will do so by showing that the Tate cohomology $\widehat{H}^p G(W_G H; \pi_q(\mathbb{S}_R)) \cong 0$ for all p and q. This is sufficient for the Tate spectrum of $\mathbb{S}_R$ to vanish by Theorem 3.40.
By Equation (3.3), $\pi_q(\mathbb{S}_R) \cong \pi_q(\mathbb{S}) \otimes R$ which is canonically isomorphic to the *group ring* $R[\pi_q(\mathbb{S})]$, by sending the elementary tensor, $g \otimes r$, to $gr \in R[G]$. All our group actions on the coefficients in group (co)homology are trivial. Recall that for a finite group G and a commutative ring R with unit, the **group ring** $R[G]$ is a ring with elements $\sum_g r_g g$, with $r_g \in R$ and $g \in G$ and with addition and multiplication defined by

1. $\sum_g r_g g + \sum_g s_g g := \sum_g (r_g + s_g) g$
2. $(\sum_g r_g g)(\sum_h s_h h) := \sum_{g,h} (r_g s_h)(gh)$

One can observe that if 1_R denotes the unit in R and e denotes the identity element in G, then $1_R e$ is the unit in $R[G]$.

We will now prove that the Tate spectrum of $\mathbb{S}_R$ vanishes. Suppose first that $p \geq 1$, then

$$\widehat{H}^p(W_G H; \pi_q(\mathbb{S}_R)) := H^p(W_G H; \pi_q(\mathbb{S}_R))$$

By assumption $|G|$ and therefore $|W_G H|$ is invertible in R (see Remark 4.2). As $|W_G H|$ is invertible in R, it is invertible in $R[\pi_q(\mathbb{S})]$. That is, if $|W_G H| = n$, then $n \cdot 1_R$ is invertible in R, and especially $(n \cdot 1_R)e(n \cdot 1_R)^{-1}e = 1_R e \in R[\pi_q(\mathbb{S}_R)]$. Hence, $H^p G(W_G H; \pi_q(\mathbb{S}_R)) \cong 0$ by Theorem 3.42.
Similarly, if $p \leq -2$ then

$$\widehat{H}_p(W_G H; \pi_q(\mathbb{S}_R)) := H_{-p-1}(W_G H; \pi_q(\mathbb{S}_R)) \cong 0$$

by Theorem 3.42
Finally, we have two more Tate cohomology groups appearing in the exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & \widehat{H}^{-1}(W_G H; \pi_q(\mathbb{S}_R)) & \longrightarrow & H_0(W_G H; \pi_q(\mathbb{S}_R)) \\ & & & \overset{N}{\swarrow} & \\ H^0(W_G H; \pi_q(\mathbb{S}_R)) & \longrightarrow & \widehat{H}^0(W_G H; \pi_q(\mathbb{S}_R)) & \longrightarrow & 0 \end{array}$$

As $W_G H$ act trivially on $\pi_q(\mathbb{S}_R)$, the *norm map* N is multiplication with the order of $W_G H$, and as $|W_G H|$ is invertible in $\pi_q(\mathbb{S}_R) \cong R[\pi_q(\mathbb{S})]$, the norm map N is an invertible map of $R[W_G H]$-modules. We conclude that the Tate cohomology $\widehat{H}^p(W_G H; \pi_q(\mathbb{S}_R))$ vanishes for all p. Now we can use the Tate spectral sequence: Theorem 3.40

$$E^2_{p,q} = \widehat{H}^{-p}(W_G H; \pi_q(\mathbb{S}_R)) \Rightarrow \pi_{p+q}(\mathbb{S}_R^{tW_G H}).$$

By the above discussion, the second page is zero, therefore, $\pi_{p+q}(\mathbb{S}_R^{tW_G H}) \cong 0$ for all p, q, and the Tate spectrum $\mathbb{S}_R^{tW_G H}$ of $\mathbb{S}_R$ vanishes. □

We will now see that descendable algebra objects induce a surjective map on the Balmer spectra, via the *extension of scalars functor.*

Definition 5.12. Let C be a symmetric monoidal ∞-category (or tt-category) and let A be a commutative algebra object in C. We call the canonical functor

$$F_A : C \to \mathrm{Mod}_C(A)$$

the **extension of scalars functor**. It is defined by sending $X \in C$ to $X \otimes A$, with the canonical A-module structure, induced by tensoring with A.

Consider the following diagram, where $\overline{i^*}$ is the induced functor that yield an equivalence in Proposition 5.5:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F_{\mathbb{D}(W_G H)}} & \mathrm{Mod}_{\mathcal{C}}(\mathbb{D}(W_G H_+)) \\ & \searrow^{i^*} & \downarrow \cong \, \overline{i^*} \\ & & \mathrm{Sp_R} \end{array}$$

Note that if $X \otimes \mathbb{D}_R(W_G H_+) \in \mathrm{Mod}_{\mathcal{C}}(\mathbb{D}_R(W_G H_+))$ is in the image of $F_{\mathbb{D}_R(W_G H_+)}$, then by construction (see [**Example 5.25.**; MNN17])

$$\overline{i^*}(X \otimes \mathbb{D}_R(W_G H_+)) \cong i^*(X).$$

The diagram therefore commutes. Consequently, as i^* is essentially surjective, so is $F_{\mathbb{D}_R(W_G H)}$. We can consider $\mathrm{Sp_R}$ as a full subcategory of $\mathcal{C}$ endowed with the trivial G action, hence, can find $\mathrm{Sp_R}^{\mathrm{dual}}$ as a full subcategory of $\mathcal{C}^{\mathrm{dual}}$. Moreover, by restricting the above diagram to dualizable objects, we can therefore conclude that $F_{\mathbb{D}_R(W_G H)} \colon \mathcal{C}^{\mathrm{dual}} \to \mathrm{Mod}_{\mathcal{C}}(\mathbb{D}_R(W_G H))^{\mathrm{dual}}$ is essentially surjective. In particular, this implies that

$$\mathrm{Mod}_{\mathcal{C}}(\mathbb{D}_R(W_G H))^{\mathrm{dual}} = \mathrm{Mod}_{\mathcal{C}^{\mathrm{dual}}}(\mathbb{D}_R(W_G H)).$$

DEFINITION 5.13. Let $\mathcal{C}$ and $\mathcal{D}$ be tt-categories and $F \colon \mathcal{C} \to \mathcal{D}$ be a tt-functor. We say F *detects $\otimes$-nilpotent morphisms* if for any $f \colon X \to Y \in \mathcal{C}$ such that $F(f) \cong 0$, then $f^{\otimes n} \cong 0$ for some $n > 0$.

PROPOSITION 5.14. *[**5.19. Proposition.**; Hea21b] Suppose A is a descendable commutative algebra object in a tt-category $\mathcal{C}$. Then the extension of scalar functor,*

$$F_A \colon \mathcal{C} \to \mathrm{Mod}_{\mathcal{C}}(A),$$

detects $\otimes$-nilpotent morphism.

COROLLARY 5.15. *As before, let* $\mathrm{Ho}(\mathcal{C}) := \mathrm{Fun}(BW_G H, \mathrm{SH}_R)$. *The extension of scalar functor,*

$$F_{\mathbb{D}_R(W_G H_+)} \colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Mod}_{\mathrm{Ho}(\mathcal{C})}(\mathbb{D}_R(W_G H_+)),$$

detects $\otimes$-nilpotent morphisms, and hence, so does its restriction to dualizable objects.

PROOF. Combine Theorem 5.10 and Proposition 5.14. □

THEOREM 5.16. *[**Theorem 1.3**; Bal18] Let $F \colon \mathcal{K} \to \mathcal{L}$ be a tt-functor between tt-categories, where $\mathcal{K}$ is also rigid. If F detects $\otimes$-nilpotent morphisms, then the induced map on the Balmer spectra,* $\mathrm{Spc}(F) \colon \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$, *is surrjective.*

NOTATION 5.17. Let $F_A\colon \mathcal{K} \to \mathrm{Mod}_{\mathcal{K}}(A)$ be an extension of scalar functor. To not overload the notation, we will denote the induced map on the Balmer spectrum $\phi_A := \mathrm{Spc}(F_A)$.

COROLLARY 5.18. *As before, let* $\mathrm{Ho}(C) := \mathrm{Fun}(BW_GH_+, \mathrm{SH}_R)$. *Then the extension of scalars functor,* $F\colon \mathrm{Ho}(C)^{\mathrm{dual}} \to \mathrm{Mod}_{\mathrm{Ho}(C)^{\mathrm{dual}}}(W_GH_+)$, *induces a bijective map on the Balmer spectra*

$$\phi_{\mathbb{D}_R(W_GH_+)}\colon \mathrm{Spc}(\mathrm{Mod}_{\mathrm{Ho}(C)^{\mathrm{dual}}}(W_GH_+)) \to \mathrm{Spc}(\mathrm{Ho}(C)^{\mathrm{dual}}). \tag{5.4}$$

PROOF. Combine Corollary 5.15 and Theorem 5.16 to get a surjective continuous map $\phi_{\mathbb{D}_R(W_GH_+)}$. As $F_{\mathbb{D}_R(W_GH_+)}\colon \mathrm{Ho}(C)^{\mathrm{dual}} \to \mathrm{Mod}_{\mathrm{Ho}(C)^{\mathrm{dual}}}(\mathbb{D}_R(W_GH))$ is essentially surjective, $\phi_{\mathbb{D}_R(W_GH_+)}$ is injective by [**Corollary 3.8.**; Bal05]. We can therefore conclude that,

$$\phi_{\mathbb{D}_R(W_GH_+)}\colon \mathrm{Spc}(\mathrm{Mod}_{\mathrm{Ho}(C)^{\mathrm{dual}}}(\mathbb{D}_R(W_GH_+))) \to \mathrm{Spc}(\mathrm{Ho}(C)^{\mathrm{dual}}), \tag{5.5}$$

is a continuous bijection. ◻

The last thing to check is that the inverse map of Equation (5.4) in continuous. We will do so by showing that $\phi_{\mathbb{D}_R(W_GH_+)}$ is a closed map. This follows from [**Theorem 3.4 b)**; Bal16], when one assumes that $\mathbb{D}_R(W_GH_+)$ is a tt-ring (which it is, see Remark 5.19). As footnoted on that page, if $\mathcal{K}$ is additionally assumed to be rigid, then

$$\phi_A(\mathrm{supp}_{\mathrm{Mod}_{\mathcal{K}}(R)}(A)) = \mathrm{supp}_{\mathcal{K}}(U_R(A)). \tag{5.6}$$

We will show that under another assumption, namely if F_A is essentially surjective, then Equation (5.6) is also true, and hence (5.4) is closed.

REMARK 5.19. By [BDS15] in the proof of Theorem 1.1 one can see that G/H_+ is a separable ring object. As G is finite, G/H_+ is self dual, and $\mathbb{D}_R(G/H_+)$ is also a separable ring object. Consequently, $\mathbb{D}_R(W_GH_+)$ is a tt-ring.

LEMMA 5.20. *Let* $\mathcal{K}$ *be a tt-category and let* R *be a tt-ring in* $\mathcal{K}$ *such that the extension of scalars functor* $F_R\colon \mathcal{K} \to \mathrm{Mod}_{\mathcal{K}}(R)$ *is essentially surjective. Then, the induced map on the Balmer spectra, sends support to support:*

$$\phi_R(\mathrm{supp}_{\mathrm{Mod}_{\mathcal{K}}(R)}(A)) = \mathrm{supp}_{\mathcal{K}}(U_R(A)),$$

and is consequently closed.

PROOF. We will let $U_R\colon \mathrm{Mod}_{\mathcal{K}}(R) \to \mathcal{K}$ denote the forgetful functor. Let us first recall some facts:

1. By [**Proposition 1.2**; Bal14], for all $X \in \mathrm{Mod}_{\mathcal{K}}(R)$ and $Y \in \mathcal{K}$, we have the projection formula $U_R(X \otimes F_A(Y)) \cong U_R(X) \otimes Y$.
2. By [Bal11], the forgetful functor U_R is exact.
3. By [Bal11], for any $X \in \mathrm{Mod}_{\mathcal{K}}(R)$, we have that X is a direct summand of $F_RU_R(X)$.
4. By construction, for any $X \in \mathcal{K}$, we have that $U_RF_R(X) \cong X \otimes R$

Let $A \in \mathrm{Mod}_{\mathcal{K}}(R)$, we claim that

$$\phi_R(\mathrm{supp}_{\mathrm{Mod}_{\mathcal{K}}(R)}(A)) = \mathrm{supp}_{\mathcal{K}}(U_R(A))$$

Suppose that $\mathcal{P} \in \mathrm{supp}_{\mathrm{Mod}_{\mathcal{K}}(R)}(A)$, that is, $A \notin \mathcal{P}$. If

$$\phi_R(\mathcal{P}) \notin \mathrm{supp}_{\mathcal{K}}(U_R(A)) \iff U_R(A) \in \phi_R(\mathcal{P}) = F_R^{-1}(\mathcal{P}),$$

this implies that $F_R(U_R(A)) \in \mathcal{P}$, but as A is a direct summand of $F_R(U_R(A))$ by (3) and $\mathcal{P}$ is closed under summands, A must lie in $\mathcal{P}$. This is a contradiction. We can therefore conclude that $\phi_R(\mathcal{P}) \in \mathrm{supp}_{\mathcal{K}}(U_R(A))$, proving the first containment:

$$\phi_R(\mathrm{supp}_{\mathrm{Mod}_{\mathcal{K}}(R)}(A)) \subset \mathrm{supp}_{\mathcal{K}}(U_R(A)).$$

For the converse containment, assume that $\mathcal{P} \in \mathrm{supp}_{\mathcal{K}}(U_R(A))$, that is, $U_R(A) \notin \mathcal{P}$. By unwinding what $\phi_R(\mathrm{supp}_{\mathrm{Mod}_{\mathcal{K}}(R)}(A))$ is, we observe that

$$\phi_R(\mathrm{supp}_{\mathrm{Mod}_{\mathcal{K}}(R)}(A)) = \bigcup_{Q \in \mathrm{supp}_{\mathrm{Mod}_{\mathcal{K}}(R)}(A)} \{K \in \mathcal{K} \mid F_R(K) \in Q\}.$$

Define $\mathcal{L}$ to be the full subcategory of $\mathrm{Mod}_{\mathcal{K}}(R)$, spanned by the objects

$$\mathcal{L} := \{X \in \mathrm{Mod}_{\mathcal{K}}(R) \mid U_R(X) \in \mathcal{P}\}.$$

Suppose for a second that $\mathcal{L}$ is a prime in $\mathrm{Mod}_{\mathcal{K}}(R)$. Then A cannot belong to $\mathcal{L}$, as this would imply that $U_R(A) \in \mathcal{P}$. Therefore, $\{K \in \mathcal{K} \mid F_R(K) \in \mathcal{L}\} \subset \phi_R(\mathrm{supp}_{\mathrm{Mod}_{\mathcal{K}}(R)}(A))$, and it suffices to show that $\mathcal{P} \subset \{K \in \mathcal{K} \mid F_R(K) \in \mathcal{L}\}$. If $X \in \mathcal{P}$, then $F_R(X) \in \mathcal{L} \iff U_R F_R(X) \in \mathcal{P}$, which is the case by (4). Proving the other containment. We are left by verifying that $\mathcal{L}$ actually defines a prime in $\mathrm{Mod}_{\mathcal{K}}(R)$:

1. $\mathcal{L}$ is triangulated: Let $X \to Y \to Z \to TX$ be a triangle in $\mathrm{Mod}_{\mathcal{K}}(R)$, with X and Y belonging to $\mathcal{L}$, that is, $U_R(X)$ and $U_R(Y)$ belong to $\mathcal{P}$. As U_R is exact by (2), we get an exact triangle

$$U_R(X) \to U_R(Y) \to U_R(Z) \to U_R(TX).$$

   By assumption, $U_R(X)$ and $U_R(Y)$ belong to $\mathcal{P}$, and as $\mathcal{P}$ is triangulated it must also contain $U_R(Z)$. By definition, Z therefore belongs to $\mathcal{L}$.

2. $\mathcal{L}$ is thick: Suppose that $X \oplus Y \in \mathcal{L}$. Then $U_R(X \oplus Y) \cong U_R(X) \oplus U_R(Y) \in \mathcal{P}$, since (2), in particular, implies that U_R is additive. As $\mathcal{P}$ is thick, $U(X)$ and $U(Y)$ belong to $\mathcal{P}$, and consequently, X and Y belong to $\mathcal{L}$.

3. $\mathcal{L}$ is a tensor ideal: Suppose that $X \in \mathcal{L}$ and S be any object in $\mathrm{Mod}_{\mathcal{K}}(R)$. By assumption F_R is essentially surjective, and we can therefore assume there exists $\overline{S} \in \mathcal{K}$ such that $F_R(\overline{S}) \cong S$. Hence,

$$U_R(X \otimes S) \cong U_R(X \otimes F_R(\overline{S})) \cong U_R(X) \otimes \overline{S}$$

   by the projection formula from (1). As $U_R(X) \in \mathcal{P}$ and $\mathcal{P}$ is a $\otimes$-ideal, $U_R(X) \otimes \overline{S}$ belongs to $\mathcal{P}$ and consequently $X \otimes S$ belongs to $\mathcal{L}$.

4. $\mathcal{L}$ is prime: Suppose that $X \otimes Y \in \mathcal{L}$. Again, as F_R is essentially surjective, we can assume there exist $\overline{Y} \in \mathcal{K}$ such that $F_R(\overline{Y}) \cong Y$. Then by the projection formula from (1) again,
$$U_R(X \otimes Y) \cong U_R(X \otimes F_R(\overline{Y})) \cong U_R(X) \otimes \overline{Y}$$
belongs to $\mathcal{P}$. If $U_R(X)$ belongs to $\mathcal{P}$, we are done. If not, $\overline{Y}$ must belong to $\mathcal{P}$, this, in particular, implies that $\overline{Y} \otimes R \in \mathcal{P}$, as $\mathcal{P}$ is a $\otimes$-ideal. Moreover, as
$$U_R(Y) \cong U_R F_R(\overline{Y}) \cong \overline{Y} \otimes R$$
by (4). We must have that $U_R(Y) \in \mathcal{P}$, and consequently, $Y \in \mathcal{L}$.

□

THEOREM 5.21. *Let G be a finite group whose order is invertible in $\mathbb{Z}_P$. The Balmer spectrum of* $\mathrm{Fun}(BG_+, \mathrm{SH}_{\mathbb{Z}_P})^{\mathrm{dual}}$ *is homeomorphic to the Balmer spectrum of* $\mathrm{Spc}(\mathrm{SH}_{\mathbb{Z}_P})$.

PROOF. Combine Corollary 5.18 and Lemma 5.20. □

COROLLARY 5.22. *Let G be a finite group whose order is invertible in $\mathbb{Z}_P$. Then we have a homeomorphism*
$$\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Z}_P}(G)) \cong \coprod_{(H) \leq G} \mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Z}_P}).$$

PROOF. Follows from Lemma 5.2 and Theorem 5.21. □

We are now capable of recovering Greenlees result, [**Theorem 1.3**; Gre19], when G is finite.

PROPOSITION 5.23. *Let G be a finite group, then*
$$\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Q}}(G)) \cong \coprod_{(H) \leq G} \{*\}$$

PROOF. By Corollary 5.22, we have that
$$\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Q}}(G)) \cong \coprod_{(H) \leq G} \mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Q}}).$$
It is well known that $\mathrm{SH}_{\mathbb{Q}} \cong D(\mathbb{Q})$, see for instance [**Theorem 7.1.2.13**; Lur17]. When we restrict this equivalence to the compacts objects, we get that $\mathrm{SH}^c_{\mathbb{Q}} \cong D(\mathbb{Q})^c \cong D(\mathbb{Q}^{\mathrm{perf}})$. By [**Corollary 4.4**; Bal20], $\mathrm{Spc}(D(\mathbb{Q}^{\mathrm{perf}})) \cong \mathrm{Spec}(\mathbb{Q}) \cong \{*\}$. Therefore, $\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Q}}) \cong \{*\}$. By combining the above, we are done. □

A downside of going to rational G-equivariant stable homotopy category is that the Balmer spectra cannot differentiate different certain G-equivariant stable homotopy categories, as we will see in the following example.

EXAMPLE 5.24. Let Q_8 denote *Quaternion* group and let S_5 denote the symmetry group on 5 symbols. Both the Q_8 and S_5 have five conjugacy classes of subgroups. We therefore have that $\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Q}}(Q_8) \cong \mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Q}}(S_5))$.

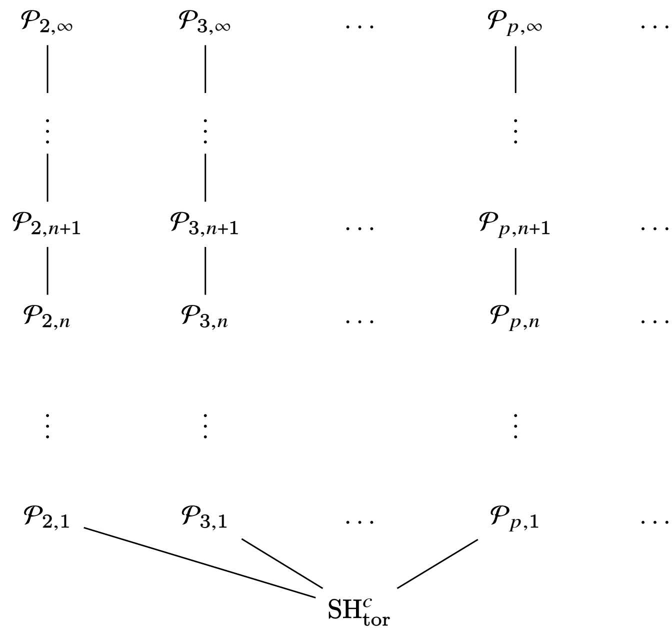
5.3 The comparison of $\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Z}_P})$ and $\mathrm{Spc}(\mathrm{SH}^c)$

We will now compare $\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Z}_P})$ and $\mathrm{Spc}(\mathrm{SH}^c)$ as in [Bal10]. By a deep result by Hopkins-Smith in [HS98] we can describe the Balmer spectrum of the p-local stable homotopy category as the chain of prime ideals:

$$0 := \mathcal{P}_{p,\infty} \subsetneq \cdots \subsetneq \mathcal{P}_{p,n+1} \subsetneq \mathcal{P}_{p,n} \subsetneq \cdots \subsetneq \mathcal{P}_{p,1} \subsetneq \mathrm{SH}^c_{\mathrm{tor}}\,.$$

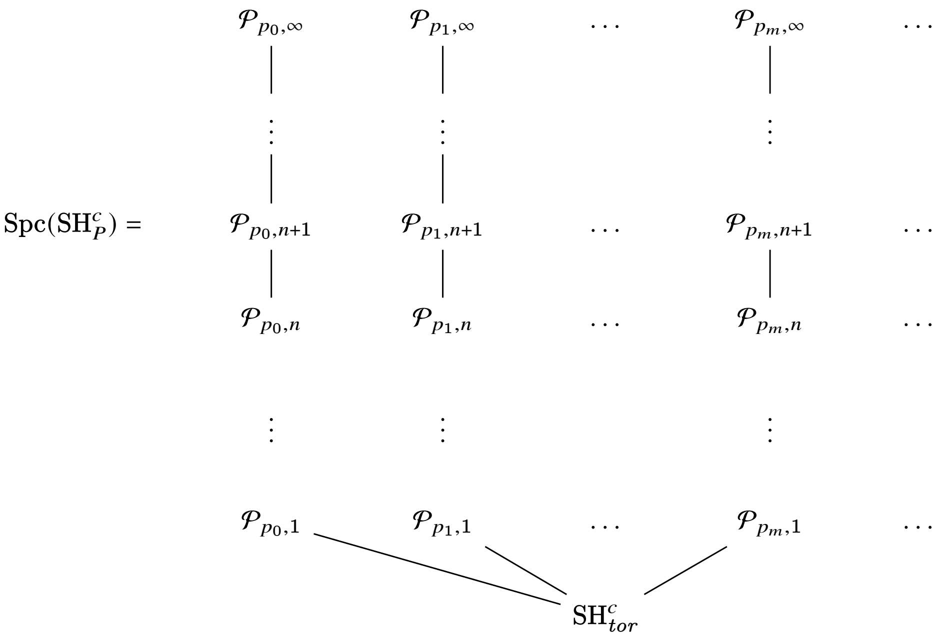
Based on this, Balmer describes the Balmer spectrum of SH^c as the following topological space in [**Corollary 9.5.**; Bal10]:

$\mathrm{Spc}(\mathrm{SH}^c) =$



Here $\mathrm{SH}^c_{\mathrm{tor}}$ denotes the dense point of torsion spectra, see [Bal10] for more detail. The Balmer spectrum of $\mathrm{SH}^c_{\mathbb{Z}_{\{p\}}}$, is the Balmer spectrum of SH^c with every column except the p-th one killed. By letting $P \subset \mathbb{Z}$ be a set of primes. Then the Balmer spectrum of $\mathrm{SH}^c_{\mathbb{Z}_{\{p\}}}$ is what one would expect. We get a chain of primes for every prime $p \in P$. If P is empty, i.e., we invert every prime, then we would only get one point in the spectra as in Proposition 5.23.

THEOREM 5.25. *Let $P = \{p_0, p_1, \ldots\}$ be a set of primes in $\mathbb{Z}$. The Balmer spectrum of $\mathrm{SH}^c_{\mathbb{Z}_P}$ is the following subspace of* $\mathrm{Spc}(\mathrm{SH}^c)$*:*



A line, in the above diagram, symbolizes that the higher prime is in the closure of the lower prime.

PROOF. If P is empty, then we are done by [**Corollary 9.5**; Bal10]. Suppose therefore that P is non-empty. The localization functor $L_P\colon \mathrm{SH}^c \to \mathrm{SH}^c_{\mathbb{Z}_P}$ naturally induces an injection on the Balmer spectrum via [**Corollary 3.8.**; Bal05]. For any prime p_i in P, the localization functor $L_{p_i}\colon \mathrm{SH}^c \to \mathrm{SH}^c_{\mathbb{Z}_{p_i}}$, factors uniquely through $\mathrm{SH}^c_{\mathbb{Z}_P}$. Let ϕ_P denote $\mathrm{Spc}(F_P)$. By [**Corollary 9.5**; Bal10], the image of $\phi_P \circ \phi_{p_i}$ is the chain of prime ideals

$$0 := \mathcal{P}_{p_i,\infty} \subsetneq \cdots \subsetneq \mathcal{P}_{p_i,n+1} \subsetneq \mathcal{P}_{p_i,n} \subsetneq \cdots \subsetneq \mathcal{P}_{p_i,1} \subsetneq \mathrm{SH}^c_{\mathrm{tor}}\,.$$

Moreover, by taking the union over all chains of prime ideals, defined by $p_i \in P$, we get $\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Z}_P})$. □

5.4 A slight generalization

We follow [Wim19] in order to define the ∞-category of $\mathrm{Sp}_{\mathrm{R},\mathcal{F}}$. We can then make a slight generalization of Corollary 5.22. We will generalize to the case that G is a compact Lie group, but with hard restrictions on the set of subgroups $\mathcal{F}$. In this section G will be a compact Lie group, $\mathcal{F}$ a subset of closed subgroups, $H \leq G$, such that $W_G H$ is invertible in the ring $R = \mathbb{Z}_P$. Moreover, the subgroups of $\mathcal{F}$ can only belong to a finite set of conjugacy classes.

A morphism of G-spectra is an **R-local $\mathcal{F}$-equivalence** if it induces an isomorphism $(\pi_*^H X) \otimes R \to (\pi_*^H Y) \otimes R$ for all $H \in \mathcal{F}$. We will denote the collection of R-local $\mathcal{F}$-equivalences by $W_{R,\mathcal{F}}$.

DEFINITION 5.26. We define the symmetric monoidal ∞-category $\mathrm{Sp}_{\mathrm{R},\mathcal{F}}$ by

$$\mathrm{Sp}_{\mathrm{R},\mathcal{F}}(G) := \mathrm{Sp}(G)[(W_{R,\mathcal{F}})^{-1}]$$

We note that $\mathrm{Sp}_{\mathrm{R},\mathcal{F}}(G)$ is similarly defined as $\mathrm{Sp}_R(G)$, but with possibly fewer isomorphism classes of objects, as we invert more morphisms.

PROPOSITION 5.27. *[**Proposition 3.4**; Wim19] The following is true in the ∞-category $\mathrm{Sp}_{\mathrm{R},\mathcal{F}}(G)$:*

1. *It is stable and admits all small limits and colimits.*
2. *The R-local homotopy groups*

$$[\Sigma^\infty (G/H)_+, X]_{\mathrm{Sp}_{\mathrm{R},\mathcal{F}}(G)} \cong (\pi_0^H X) \otimes R$$

are corepresented in the homotopy category by the suspension spectra of transitive G-sets. In particular, $\{\Sigma^\infty (G/H)_+ \mid H \in \mathcal{F}\}$ forms a set of compact generators.

Wimmer proves a more general version of Theorem 4.1 in [**Theorem 3.10**; Wim19] for $\mathrm{Sp}_{\mathrm{R},\mathcal{F}}$.

THEOREM 5.28. *[**Theorem 3.10**; Wim19] Let G be a compact Lie group and $\mathcal{F}$ a subset of finite closed subgroups $H \le G$, such that $|H|$ is invertible in the subring $R := \mathbb{Z}_P$ of $\mathbb{Q}$. Then the geometric fixed point functor Φ induces an equivalence of ∞-categories*

$$\mathrm{Sp}_{\mathrm{R},\mathcal{F}}(G) \cong \prod_{(H) \in \mathcal{F}} \mathrm{Fun}(BW_G H, \mathrm{Sp}_R). \tag{5.7}$$

By Lemma 4.4, this will be an equivalence of symmetric monoidal stable ∞-categories. Let R and G be as above, and $\mathrm{SH}_{R,\mathcal{F}} := \mathrm{Ho}(\mathrm{Sp}_{R,\mathcal{F}})$. By applying Lemma 2.36 and Theorem 5.21 to the induced equivalence on the homotopy category of Equation (5.7), we get the following theorem:

THEOREM 5.29. *Let G be a compact Lie group, let $R := \mathbb{Z}_P$ be a subring of $\mathbb{Q}$ and let $\mathcal{F}$ be a family of finite closed subgroups $H \le G$ such that $|W_G H|$ is invertible in the ring R. Additionally, if the subgroups of $\mathcal{F}$, only belong to a finite set of conjugacy classes. Then,*

$$\mathrm{Spc}(\mathrm{SH}^c_{R,\mathcal{F}}(G))) \cong \coprod_{(H) \in \mathcal{F}} \mathrm{Spc}(\mathrm{SH}^c_R).$$

We cannot use [**Theorem 3.10**; Wim19] calculate the Balmer spectra of $\mathrm{Sp}_{\mathrm{R},\mathcal{F}}$ in the generality of Wimmer's paper for two reasons:

1. In Wimmer's splitting of $\mathrm{Sp}_{\mathrm{R},\mathcal{F}}$:

$$\mathrm{Sp}_{\mathrm{R},\mathcal{F}}(G) \cong \prod_{(H)\in\mathcal{F}} \mathrm{Fun}(BW_G H, \mathrm{Sp}_R),$$

   $\mathcal{F}$ is allowed to have an infinite number of subgroups that belong to possibly infinite different conjugacy classes of G. We are therefore taking a possibly infinite product of tt-categories. In Lemma 2.36, we only showed that the Balmer spectrum commutes with finite products of tt-categories. We also conjecture that it is not true that the Balmer spectra commute with an infinite product of tt-categories, as in the analog of rings, this is not the case.

2. Wimmer also allows for the finite subgroup $H \leq G$ to have Weyl groups with infinite order. However, if G is a compact Lie group, and $\mathcal{F}$ consist of finite subgroups of G. Then we have no control over the orders of the Weyl groups $W_G H$. Take for instance the cyclic subgroups C_n of $SO(2)$, the normalizer of C_n is $SO(2)$, and therefore $W_{C_n} SO(2) \cong SO(2)$, which is not finite. The Weyl groups are therefore possibly infinite, and then we have no notion of the Tate spectrum and the order of $W_G H$ is certainly not invertible in $\mathbb{Z}_P$.

All hope of saying something, in the more general case of a compact Lie group, is not over. In the next chapter, we will see that we are able to calculate parts of the Balmer spectra of $\mathrm{SH}_{\mathbb{Q}}(SO(3))$.

Chapter 6

An example when G is infinite

6.1 The Balmer spectra of $\mathrm{SH}_{\mathbb{Q}}(\mathrm{SO}(3))$

As we have seen in Section 5.4, there is little hope for fully applying the methods of Chapter 5 to calculate $\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Q}}(G))$, for a general Lie group G. However, Kędziorek gave in [Kęd17a], an algebraic description of rational orthogonal SO(3)-spectra in terms of a product of a *toral* part $\mathcal{T}$, *dihedral* part $\mathcal{D}$ and an *exceptional* part $\mathcal{E}$. That is, Kędziorek divides the closed subgroups of SO(3) into three parts, the toral part consists of all *tori* in SO(3) and all cyclic subgroups of these tori, the dihedral part consisting of all dihedral subgroups D_{2n} of SO(3) and an *exceptional* part consisting of exceptional subgroups. With our machinery, we are able to calculate the Balmer spectra of the *exceptional* part. We define the notion of an exceptional subgroup as in [Kęd17b].

Definition 6.1. Let G be a compact Lie group. We say that a closed subgroup, $H \leq G$, is **exceptional** in G if $W_G H$ is finite, there exists an idempotent $e_{(H)_G}$ in the rational Burnside ring of G corresponding to the conjugacy class of H in G, and H does not contain any subgroup cotoral in H, where a subgroup $K \leq H$ is cotoral in H if H/K is a non-trivial torus.

Remark 6.2. An exceptional subgroup therefore has a finite Weyl group. This is one of our criterions for our machinery, from Chapter 5, to work. We further remark as in [Kęd17a] that there are five conjugacy classes of subgroups which are exceptional in SO(3): the rotation group of a cube Σ_4 with normalizer itself, the rotation group of a tetrahedron A_4 with normalizer Σ_4, the rotation group of a dodecahedron A_5 with normalizer itself and the dihedral group of order 4, D_4, with normalizer Σ_4.

Importantly for us, there are only a finite number of different conjugate classes of exceptional subgroups.
We now follow [Kęd17a], to decompose the category of rational orthogonal SO(3)-spectra.

PROPOSITION 6.3. *[**Proposition 2.6**; Kęd17a] The following adjunction*

$$\begin{array}{c} \mathrm{OrthSpec}_{\mathbb{Q}}(\mathrm{SO}(3)) \\ \prod \Big\uparrow \Big\downarrow \Delta \\ L_{e_{\mathcal{T}}\mathcal{S}_{\mathbb{Q}}}(\mathrm{OrthSpec}_{\mathbb{Q}}(\mathrm{SO}(3))) \times L_{e_{\mathcal{D}}\mathcal{S}_{\mathbb{Q}}}(\mathrm{OrthSpec}_{\mathbb{Q}}(\mathrm{SO}(3))) \times L_{e_{\mathcal{E}}\mathcal{S}_{\mathbb{Q}}}(\mathrm{OrthSpec}_{\mathbb{Q}}(\mathrm{SO}(3))), \end{array}$$

is a strong monoidal Quillen equivalence, where $\mathrm{OrthSpec}_{\mathbb{Q}}(\mathrm{SO}(3))$ *denotes the category of rational* $\mathrm{SO}(3)$*-orthogonal spectra, the left adjoint is the diagonal functor and the right adjoint is the product.*

By Barnes in [**Theorem 4.4**; Bar09], we have a symmetric monoidal Quillen equivalence between the exceptional part and the product over the conjugacy classes of exceptional subgroups:

$$\Delta\colon L_{e_{\mathcal{E}}\mathcal{S}_{\mathbb{Q}}}(\mathrm{OrthSpec}_{\mathbb{Q}}(\mathrm{SO}(3))) \leftrightarrows \prod_{(H)\in\mathcal{E}} L_{e_H\mathcal{S}_{\mathbb{Q}}}(\mathrm{OrthSpec}_{\mathbb{Q}}(\mathrm{SO}(3))) \ : \prod \tag{6.1}$$

We can therefore apply the following lemma, to each component of the product, in the exceptional part.

THEOREM 6.4. *[**Theorem 1.1**; Kęd17b] Suppose G is any compact Lie group and H is an exceptional subgroup of G. Then there is a zigzag of symmetric monoidal Quillen equivalences from rational orthogonal G-spectra over H to*

$$\mathrm{Ch}(\mathbb{Q}[W_G H] - \mathrm{Mod}).$$

By [**Lemma 5.7**; Kęd17b], we have monoidal Quillen Equivalence between $\mathrm{Ch}(\mathbb{Q}[W_G H] - \mathrm{Mod})$ and rational spectra with $W_G H$-action. Therefore, when passing to the homotopy categories, we get by Theorem 6.4, an equivalence of tt-categories

$$\mathrm{Ho}(L_{e_H\mathcal{S}_{\mathbb{Q}}}(\mathrm{OrthSpec}_{\mathbb{Q}}(\mathrm{SO}(3)))) \cong \mathrm{Fun}(BW_G H, \mathrm{SH}_{\mathbb{Q}}), \tag{6.2}$$

for every $H \in \mathcal{E}$. By combining Proposition 6.3, Equation (6.2) and Equation (6.1), we get a symmetric monoidal equivalence

$$\begin{array}{c} \mathrm{SH}_{\mathbb{Q}}(\mathrm{SO}(3)) \\ \cong \Big\downarrow \\ L_{e_{\mathcal{T}}\mathcal{S}_{\mathbb{Q}}}(\mathrm{SH}_{\mathbb{Q}}(\mathrm{SO}(3)) \times L_{e_{\mathcal{D}}\mathcal{S}_{\mathbb{Q}}}(\mathrm{SH}_{\mathbb{Q}}(\mathrm{SO}(3)) \times \prod_{(H)\in\mathcal{E}} \mathrm{Fun}(BW_G H, \mathrm{SH}_{\mathbb{Q}}). \end{array}$$

To avoid the cumbersome notation, we rename $\mathcal{T} \coloneqq L_{e_{\mathcal{T}}\mathcal{S}_{\mathbb{Q}}}(\mathrm{SH}_{\mathbb{Q}}(\mathrm{SO}(3)))$ and $\mathcal{D} \coloneqq L_{e_{\mathcal{D}}\mathcal{S}_{\mathbb{Q}}}(\mathrm{SH}_{\mathbb{Q}}(\mathrm{SO}(3)))$. By restricting to dualizable objects, we get an equivalence of essentially small tt-categories:

$$\mathrm{SH}^{\mathrm{dual}}_{\mathbb{Q}}(\mathrm{SO}(3)) \cong \mathcal{T}^{\mathrm{dual}} \times \mathcal{D}^{\mathrm{dual}} \times \prod_{(H)\in\mathcal{E}} \mathrm{Fun}(BW_G H, \mathrm{SH}_{\mathbb{Q}})^{\mathrm{dual}}.$$

By Lemma 2.36, we have that

$$\begin{aligned}
\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Q}}(\mathrm{SO}(3))) &\cong \mathrm{Spc}(\mathrm{SH}^{\mathrm{dual}}_{\mathbb{Q}}(\mathrm{SO}(3))) \\
&\cong \mathrm{Spc}(\mathcal{T}^{\mathrm{dual}} \times \mathcal{D}^{\mathrm{dual}} \times \prod_{(H)\in\mathcal{E}} \mathrm{Fun}(BW_GH, \mathrm{SH}_{\mathbb{Q}})^{\mathrm{dual}}) \\
&\cong \mathrm{Spc}(\mathcal{T}^{\mathrm{dual}}) \sqcup \mathrm{Spc}(\mathcal{D}^{\mathrm{dual}}) \coprod_{(H)\in\mathcal{E}} \mathrm{Spc}(\mathrm{Fun}(BW_GH, \mathrm{SH}_{\mathbb{Q}})^{\mathrm{dual}}).
\end{aligned}$$

By Theorem 5.21, we have that $\mathrm{Spc}(\mathrm{Fun}(BW_GH, \mathrm{SH}_{\mathbb{Q}})^{\mathrm{dual}}) \cong \mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Q}})$, and by Proposition 5.23 we have that $\mathrm{Spc}(\mathrm{SH}_{\mathbb{Q}}) \cong \{*\}$. We can therefore conclude that

$$\mathrm{Spc}(\mathcal{T}^{\mathrm{dual}}) \sqcup \mathrm{Spc}(\mathcal{D}^{\mathrm{dual}}) \coprod_{(H)\in\mathcal{E}} \{*\} \cong \mathrm{Spc}(\mathcal{T}^{\mathrm{dual}}) \sqcup \mathrm{Spc}(\mathcal{D}^{\mathrm{dual}}) \coprod_{(H)\in\mathcal{E}} \{*\}.$$

We sum up the above discussion into the following theorem:

THEOREM 6.5. *Following the notation as above, then we have a homeomorphism*

$$\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Q}}(\mathrm{SO}(3))) \cong \mathrm{Spc}(\mathcal{T}^{\mathrm{dual}}) \sqcup \mathrm{Spc}(\mathcal{D}^{\mathrm{dual}}) \coprod_{i=1}^{5} \{*\}.$$

Here, each i corresponds to one of the five conjugacy classes of exceptional subgroups of $SO(3)$.

In plain English; we see that the Balmer spectrum of the rational SO(3)-equivariant stable homotopy category, decomposes as a disjoint union, corresponding to the toral, dihedral and exceptional part. Moreover, the exceptional part consist of a disjoint union of points, each one corresponding to the conjugacy classes of the exceptional subgroups.

6.2 Summary

For a general Lie group G, calculating the Balmer spectrum of the G-equivariant homotopy category is a challenging task. We have seen how one can approach this, when G is finite, by localizing at a ring R where the order of G is invertible. By letting R be a proper subring of $\mathbb{Q}$, we get richer spectrum. Moreover, with our approach, we are able to calculate parts of the Balmer spectrum when G is infinite.
Greenlees has calculated the Balmer spectrum of the rational G-equivariant stable homotopy category, for a general Lie group G (see Theorem 1.6). As a future project, knowing the topology on $\mathrm{Spc}(\mathrm{SH}^c_{\mathbb{Q}}(\mathrm{SO}(3)))$, it could be interesting to study the f-topology to find our disjoint points, given by the exceptional subgroup of SO(3), in Theorem 6.5.

Bibliography

[And+09] M. Ando et al. *Units of ring spectra and Thom spectra.* 2009.

[Bal05] P. Balmer. "The spectrum of prime ideals in tensor triangulated categories". In: *J. Reine Angew. Math.* 588 (2005), pp. 149–168.

[Bal10] P. Balmer. "Spectra, spectra, spectra—tensor triangular spectra versus Zariski spectra of endomorphism rings". In: *Algebr. Geom. Topol.* 10.3 (2010), pp. 1521–1563.

[Bal11] P. Balmer. "Separability and triangulated categories". In: *Adv. Math.* 226.5 (2011), pp. 4352–4372.

[Bal14] P. Balmer. "Splitting tower and degree of tt-rings". In: *Algebra Number Theory* 8.3 (2014), pp. 767–779.

[Bal16] P. Balmer. "Separable extensions in tensor-triangular geometry and generalized Quillen stratification". In: *Ann. Sci. Éc. Norm. Supér. (4)* 49.4 (2016), pp. 907–925.

[Bal18] P. Balmer. "On the surjectivity of the map of spectra associated to a tensor-triangulated functor". In: *Bull. Lond. Math. Soc.* 50.3 (2018), pp. 487–495.

[Bal20] P. Balmer. "A guide to tensor-triangular classification". In: *Handbook of homotopy theory.* CRC Press/Chapman Hall Handb. Math. Ser. CRC Press, Boca Raton, FL, [2020] ©2020, pp. 145–162.

[BDS15] P. Balmer, I. Dell'Ambrogio, and B. Sanders. "Restriction to finite-index subgroups as étale extensions in topology, KK-theory and geometry". In: *Algebr. Geom. Topol.* 15.5 (2015), pp. 3025–3047.

[BS17] P. Balmer and B. Sanders. "The spectrum of the equivariant stable homotopy category of a finite group". In: *Invent. Math.* 208.1 (2017), pp. 283–326.

[Ban10] M. Banagl. "Rational generalized intersection homology theories". In: *Homology Homotopy Appl.* 12.1 (2010), pp. 157–185.

[Bar09] D. Barnes. "Splitting monoidal stable model categories". In: *Journal of Pure and Applied Algebra* 213.5 (May 2009), pp. 846–856.

[BR20] D. Barnes and C. Roitzheim. *Foundations of stable homotopy theory.* Vol. 185. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2020, pp. vi+423.

[Bar+17] D. Barnes et al. “Rational SO(2)-equivariant spectra”. In: *Algebr. Geom. Topol.* 17.2 (2017), pp. 983–1020.

[Bro94] K. S. Brown. *Cohomology of groups.* Vol. 87. Graduate Texts in Mathematics. Corrected reprint of the 1982 original. Springer-Verlag, New York, 1994, pp. x+306.

[FHM03] H. Fausk, P. Hu, and J. P. May. “Isomorphisms between left and right adjoints”. In: *Theory Appl. Categ.* 11 (2003), No. 4, 107–131.

[Gre98a] J. P. C. Greenlees. “Rational Mackey functors for compact Lie groups. I”. In: *Proc. London Math. Soc. (3)* 76.3 (1998), pp. 549–578.

[Gre98b] J. P. C. Greenlees. “Rational O(2)-equivariant cohomology theories”. In: *Stable and unstable homotopy (Toronto, ON, 1996).* Vol. 19. Fields Inst. Commun. Amer. Math. Soc., Providence, RI, 1998, pp. 103–110.

[Gre01] J. P. C. Greenlees. “Rational SO(3)-equivariant cohomology theories”. In: *Homotopy methods in algebraic topology (Boulder, CO, 1999).* Vol. 271. Contemp. Math. Amer. Math. Soc., Providence, RI, 2001, pp. 99–125.

[Gre19] J. P. C. Greenlees. “The Balmer spectrum of rational equivariant cohomology theories”. In: *J. Pure Appl. Algebra* 223.7 (2019), pp. 2845–2871.

[GM95a] J. P. C. Greenlees and J. P. May. “Equivariant stable homotopy theory”. In: *Handbook of algebraic topology.* North-Holland, Amsterdam, 1995, pp. 277–323.

[GM95b] J. P. C. Greenlees and J. P. May. “Generalized Tate cohomology”. In: *Mem. Amer. Math. Soc.* 113.543 (1995), pp. viii+178.

[Gre06] J. Greenlees. “Triangulated Categories of Rational Equivariant Cohomology Theories”. In: (Jan. 2006).

[Hea21a] D. Heard. “Rational Local Systems and Connected Finite Loop Spaces”. In: *Glasgow Mathematical Journal* (2021), pp. 1–29.

[Hea21b] D. Heard. *The $Sp_{k,n}$-local stable homotopy category.* 2021.

[HL13] M. Hopkins and J. Lurie. “Ambidexterity in K(n)-Local Stable Homotopy Theory.” In: (2013).

[HS98] M. J. Hopkins and J. H. Smith. “Nilpotence and stable homotopy theory. II”. In: *Ann. of Math. (2)* 148.1 (1998), pp. 1–49.

[HPS97] M. Hovey, J. H. Palmieri, and N. P. Strickland. “Axiomatic stable homotopy theory”. In: *Mem. Amer. Math. Soc.* 128.610 (1997), pp. x+114.

[Kęd17a] M. Kędziorek. “An algebraic model for rational SO(3)-spectra”. In: *Algebr. Geom. Topol.* 17.5 (2017), pp. 3095–3136.

[Kęd17b] M. Kędziorek. “An algebraic model for rational G-spectra over an exceptional subgroup”. In: *Homology Homotopy Appl.* 19.2 (2017), pp. 289–312.

[Lew+86] L. G. Lewis Jr. et al. *Equivariant stable homotopy theory.* Vol. 1213. Lecture Notes in Mathematics. With contributions by J. E. McClure. Springer-Verlag, Berlin, 1986, pp. x+538.

[Lur09] J. Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925.

[Lur17] J. Lurie. *Higher Algebra*. Harvard University, 2017.

[MM02] M. A. Mandell and J. P. May. "Equivariant orthogonal spectra and S-modules". In: *Mem. Amer. Math. Soc.* 159.755 (2002), pp. x+108.

[Mat18] A. Mathew. "Examples of descent up to nilpotence". In: *Geometric and topological aspects of the representation theory of finite groups*. Vol. 242. Springer Proc. Math. Stat. Springer, Cham, 2018, pp. 269–311.

[MNN17] A. Mathew, N. Naumann, and J. Noel. "Nilpotence and descent in equivariant stable homotopy theory". In: *Adv. Math.* 305 (2017), pp. 994–1084.

[nLa21] nLab authors. *monoidal functor*. `http://ncatlab.org/nlab/show/monoidal+functor`. Revision 46. Sept. 2021.

[PSW21] I. Patchkoria, B. Sanders, and C. Wimmer. *The spectrum of derived Mackey functors*. 2021.

[Sch18] S. Schwede. *Global homotopy theory*. Vol. 34. New Mathematical Monographs. Cambridge University Press, Cambridge, 2018, pp. xviii+828.

[SS03] S. Schwede and B. Shipley. "Stable model categories are categories of modules". In: *Topology* 42.1 (2003), pp. 103–153.

[Shi02] B. Shipley. "An algebraic model for rational S^1-equivariant stable homotopy theory". In: *Q. J. Math.* 53.1 (2002), pp. 87–110.

[Sta21] T. Stacks project authors. *The Stacks project*. `https://stacks.math.columbia.edu`. 2021.

[Wim19] C. Wimmer. *A model for genuine equivariant commutative ring spectra away from the group order*. 2019.

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