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Symbolizing lines and planes as linear combinations in a dynamic geometry environment

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ABSTRACT

In this paper, we focus on students' symbolizing activity and mathematization relating to linear combination and span in the context of a task sequence designed with digital tools. Considering tools and functions of a digital environment (specifically of GeoGebra) with design heuristics of Realistic Mathematics Education, we introduce four tasks (in $\mathbb{R}^2$ and $\mathbb{R}^3$) and results of pilot studies. Data sources come from six individual task-based interviews with two linear algebra students. We analyzed these with a lens of mathematization, with a particular focus on tool use. We found students' use of sliders supported their reasoning about individual linear combinations; the trace function and slider animation supported their intuition for all possible linear combinations of vectors. After shifting from two to three dimensions, students symbolized all possible linear combinations with a parametric vector equation and reasoned about how this could represent particular points on the corresponding lines or planes.

1. Introduction

Learning linear algebra is a complex endeavor, and the learning challenges students encounter due to the nature of the content are prominent in the literature (Carlson, 1993; Dorier, Robert, Robinet, & Rogalski, 2000). There is a growing body of work documenting students' mathematical reasoning about linear algebra topics such as linear combinations, span, and linear independence (Plaxco & Wawro, 2015; Rasmussen, Wawro, & Zandieh, 2015; Stewart & Thomas, 2010). The notions of linear combination and span involve coordinating a number of mathematical representations and abstract mathematical objects (see Lay, 2006). Further, the binary operations of scalar multiplication and vector addition, are critical for making sense of vector spaces (Parraguez & Oktaç, 2010). As such, the notions of linear combination and span are the core elements of and needed for learning linear algebra (Harel, 2018; Turgut, 2018a).

Contextually driven pedagogical designs for linear combinations and span have been proposed. Some contexts lend themselves to geometry as a point of departure (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012). whereas others, such as the secure password generation setting proposed by Cárcamo, Fortuny, and Fuentealba (2018), are less inherently geometric in nature. Dogan

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(2018) found that students shown dynamic visualizations as part of instruction tended to integrate geometry into their abstract reasoning, whereas students shown static images during instruction tended to rely on numeric and computational reasoning. We interpret Dogan (2018) results to point to the pedagogical value of learning environments in which students engage in dynamic visualizations. We build on this work by exploring pedagogical designs for linear combinations and span in a particular digital geometry environment. Geometrical tools can be exploited by embedding in a context where the students explore “spatially-based linear algebraic concepts” (Harel, 2019, p. 1040).

The invention of sophisticated digital tools opened a new research plea for integrating technology into teaching linear algebra. Dynamic geometry environments (DGEs) have received particular attention from linear algebra educators (Gol Tabaghi & Sinclair, 2013; Oktaç, 2018; Turgut, 2019) as an opportunity to design meaningful contexts for exploring and developing students’ mathematical thinking. In any particular DGE, there are depictions of algebra and geometry in $\mathbb{R}^2$ and $\mathbb{R}^3$ (i.e., GeoGebra), so educational designers can create a context where the learners can manipulate objects and concurrently view the results of these manipulations both algebraically and geometrically. In this way, learners can move between different representations with different mathematical situations of linear algebra, which can support students’ meaning-making for spatially-based linear algebraic concepts (Turgut, 2018b). However, this requires a careful analysis of the tools and functions of the DGE in advance (Mariotti, 2013) so the learning environment can be designed to support students’ progressive mathematization and symbolization (Rasmussen, Zandieh, King, & Teppo, 2005). This presses us to ask: How can we design contexts for DGEs that support student learning and symbolization of linear combination and span?

In order to respond to this question, we draw on the instructional design heuristics of Realistic Mathematics Education (RME), which has recently informed the development of meaningful contexts for learning linear algebra (Andrews-Larson, Wawro, & Zandieh, 2017; Wawro et al., 2012; Wawro, Rasmussen, Zandieh, & Larson, 2013). RME builds on Freudenthal (1973, 1991) view of mathematics as an inherently human activity and frames students’ learning of mathematics in terms of students’ mathematical activity in the context of their opportunities for sense-making. Thus, RME highlights the importance of experientially real starting points (Gravemeijer, 1999) that support students’ own mathematical constructions, rather than beginning with mathematics concepts that have already been formalized by others. In this paper, we foreground the RME design heuristics of experientially real starting points and students’ mathematical activity as they progress through the sequence of tasks set in a DGE (Freudenthal, 1991; Gravemeijer, 1999). The lead author of this paper drew on RME to develop an experientially real task setting (Gravemeijer, 2004). In this setting we invited students to develop mathematics and/or reinvent geometric relationships behind concepts and notions, in our case, for the notions of linear combination and span.

In this paper, we first present a task sequence for sense-making about linear combination and span. We then analyze the reasoning of two linear algebra students with whom the task sequence was piloted, highlighting the ways in which this analysis informs refinements to the task sequence. What sets this paper apart from others is that we present a fully digital task sequence designed using RME. We look to see how this combination of technology and RME might present opportunities for students to reason differently than in pencil-and-paper tasks, with a particular focus on students’ symbolizing activity. In the next section, we briefly present a priori analysis regarding linear combination and span, which illustrates the ways in which tools and functions of a DGE (in particular, of GeoGebra) can be leveraged to support learning of these mathematical constructs. We then characterize the way in which we draw on RME as a theoretical framework in task design.

2. Related literature and theoretical framework

2.1. Student thinking about linear combination and span

In synthesizing recent research on the teaching and learning of linear algebra, Stewart, Andrews-Larson, and Zandieh (2019) identified student learning about span or linear independence as areas of recent linear algebra research that have received considerable attention, noting that 15 of 54 recent empirical pieces on linear algebra education focused on learning of these specific topics. However, student reasoning about span and linear combinations were often examined after students had already received formal instruction on the content, and often in relation to concepts such as vector spaces or subspaces (Parraguez & Oktaç, 2010; Wawro, Sweeney, & Rabin, 2011), basis (Stewart & Thomas, 2010), or linear dependence and independence (Bouhjar, Andrews-Larson, & Haider, 2021; Plaxco & Wawro, 2015). Those studies that did focus primarily on linear combinations and span tended to heavily foreground RME task sequences (Cárcamo et al., 2018; Wawro et al., 2012).

When reasoning about span, students often correctly describe the span of a set of vectors as the set of all possible linear combinations of those vectors (Stewart & Thomas, 2010). However, Bouhjar et al. (2021) found that the meaning students ascribe to these phrases can vary in systematic ways relative to their learning opportunities – with students in some instructional treatments more likely to correctly identify sums or scalar multiples of vectors in a given set as elements of the span of that set. In other words, for some students “all possible linear combinations” of a set of vectors explicitly includes sums and scalar multiples of vectors in the given set, whereas it does not for others – and these interpretations are systematically linked to students’ opportunities for exploration and sense-making as part of their introduction to the ideas about span and linear combinations. Stewart and Thomas (2010) also argued that students need more opportunities to explore embodied worlds (seeing and sensing properties of a concept) and symbolic worlds (symbolizations that emerge from an action) before moving to the formalization of a concept, something the activity in this study aims at doing.

Some authors have provided students with embodied and symbolic experiences for learning about linear combinations and span through RME task sequences (Cárcamo et al., 2018; Mauntel, Levine, Plaxco, & Zandieh, 2021, Wawro et al., 2012). Contexts in which students model password generators (Cárcamo et al., 2018) or describe travel in a video-game-like context (Mauntel et al., 2021; Wawro et al., 2012) have helped students develop productive ways of reasoning with and about linear combinations of vectors and span. For instance, Mauntel et al. (2021) detailed students' strategies using different aspects of a digital game-based environment in relation to their numeric and geometric reasoning (e.g., coordinating vector equations with graphic depections of vector addition, reasoning based on quadrants, focusing on a particular vector or on particular elements in a vector). As students progressed through the game, their strategy use became increasingly anticipatory (Mauntel et al., 2021). While this study particular student does lend itself well to geometric interpretations, none of these RME sequences were developed in a DGE context.

Our study addresses a gap in research focused specifically on student reasoning about linear combinations prior to formal instruction on span – and in the context of a DGE. The setting also allows for and lends to algebraic *and* geometric thinking, as both are displayed on the screen and attending to both is part of the tasks. The design of the task sequence presented in this paper is informed by RME in the context of a DGE. In alignment with this design, we theorize our exploration of student reasoning in terms of their mathematical activity and tool use.

2.2. Theorizing mathematical activity and tool use

We draw on Rasmussen et al.'s (2005) notion of advancing mathematical activity, which theorizes mathematical thinking and reasoning in terms of how students participate in practices that cut across mathematics, such as symbolizing, defining, and justifying. A central way of characterizing the advancement of mathematical activity through this lens draws on Treffers (1987) notions of horizontal and vertical mathematization. Horizontal mathematization "concerns organizing, translating, and transforming realistic problems into mathematical terms, in short, mathematizing reality" (Drijvers, 2003, p. 53). On the other hand, "vertical mathematization concerns reflection on the horizontal mathematization from a mathematical perspective, in short, mathematizing the mathematical activities and developing a framework of mathematical relations" (Drijvers, 2003, p. 53). When students shift from contextual reasoning (horizontal mathematization) to making mathematical connections (vertical mathematization), their model-of the contextual situations shifts to function as a model-for subsequent mathematical organization (Gravemeijer, 1999; Zandieh & Rasmussen, 2010).

One important form of mathematical activity that is of relevance to our work is the activity of symbolizing. Symbolizing refers to the notations and symbols students use when reasoning with a mathematics concept (Rasmussen et al., 2005). While symbolizing has been viewed as a detachment from context (Herscovics, 1996), Rasmussen and colleagues describe symbolizing as a way to connect context-dependent mathematical understandings to more formal ways of reasoning about a concept.

In the context of our study, DGE plays a central role in the design of our instructional sequence, and we conjecture it can play a central role in students' mathematical activity – particularly in relation to students' symbolization and generalization. As such, we aim to consider tool use in relation to students' mathematical activity. Tools "stand between the user and the phenomenon to be modelled, and shape activity structures" (Hoyles & Noss, 2003, p. 341). There exists a dialectical process between tool use and mathematical learning: "On the one hand, the user shapes the techniques for using the tool, but on the other hand the tool shapes and transforms the user's mathematical practice" (Drijvers, 2019, p. 15). In our study, we're interested in linking students' mathematical practices and conceptual elements with technical elements of the DGE (e.g., tool use such as dragging sliders, trace feature, animating sliders in a DGE, like GeoGebra).

2.3. Present study and research questions

In our case, we exploit the DGE context as a mediator for mathematization (and especially symbolization), where students interact and explore the given mathematical situation through the tools and functions of the DGE. Tool use (e.g., functions of DGE) intertwines with student thinking, in addition to refining and shaping students' reasoning (Drijvers, 2003, 2019).

In the present paper, we hypothesize that the potential of digital tools elaborated with the design heuristics of RME could help students transition from informal reasoning to sense-making for formal mathematics, in our case, regarding the notions of linear combination and span. Because such an approach includes a spatially-based view (Harel, 2019), it could provide an introductory point for learning the core concepts of linear algebra, where we strongly believe that geometry can be exploited to prepare students for abstract notions and abstract vector spaces (Harel, 2000). In sum, we focus on the following research questions:

1. How did participants reason about linear combinations and ideas about span in the context of this DGE-based RME task sequence?
2. What is the role of tool use in advancing students' symbolizing activity in the context of the DGE?

2.4. Linear combinations and span in GeoGebra context

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ be vectors in a vector space V. A vector $\boldsymbol{w}$ is called a linear combination of vectors if we can write it in the form

$$\boldsymbol{w} = \lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2 + \cdots + \lambda_r \boldsymbol{v}_r$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are scalars (Anton, 1981). Given the set of vectors $H = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$, the set of all linear combinations of the vectors in H is called the span of H. In order to construct a dynamic view, GeoGebra has a *slider* tool that can be built into any equation, figure, or function. For example, by defining two sliders as λ_1 and λ_2, if we consider $\mathbb{R}^2$ and take two vectors as $\boldsymbol{v}_1 = (1, 2)$ and $\boldsymbol{v}_2 = (-1, 3)$, we have the following interface (Fig. 1) (in the GeoGebra Classic 5.0 version).

Here, $\boldsymbol{w} = \lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2$ is set and when the user drags the sliders λ_1 and λ_2, the coordinates of $\boldsymbol{w}$ changes accordingly. Further, the user could activate the 'animate' function of the sliders, which changes the values of λ_1 and λ_2 systematically. In addition to the animate function, the software has a 'trace' function, which can be connected to any point, figure or vector. If we activate the animate function of the sliders λ_1 and λ_2 and activate trace function of the end point (A) of vector $\boldsymbol{w}$, we obtain the following figure (Fig. 2).

As seen from Fig. 2, the point A marks points on the plane when the scalars λ_1 and λ_2 begin to change. By changing slider values, the point A forms dots, jagged edges and line segments. Fig. 2 also provides a synchronized 3D Graphics window, which shows the variations in $\mathbb{R}^3$. In this paper, we hypothesized that the change of scalars and their variation (as described in Figs. 1 and 2) could evoke a *sense* for the notions of linear combination and span.

3. Methods

The present paper was extracted from a large-scale design-based research project (Bakker & van Eerde, 2015) aiming at designing several pedagogical tasks for learning linear algebra with digital tools. Following the nature of design-based research (Bakker & van Eerde, 2015), some analyses were conducted to discuss the potential of digital tools for creating sense-making. This was followed by task development and peer review (by two educational designers of mathematics and a linear algebra instructor), and piloting tasks with small groups and analysis of the pilot studies. The present paper is a part of a pilot study series focusing on the student understanding of linear combination and span with digital tools.

We note that other work done in the tradition of RME often heavily features classroom teaching experiments (e.g., Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003) or paired teaching experiments (e.g., Larsen, 2009). An affordance of this choice is the opportunity to document how students reason through tasks when working with peers, which is a central feature of the intended implementation of RME learning sequences in classrooms. Importantly, we consider both individual cognitive and socially situated aspects of learning in mathematics classrooms to be of central importance (Cobb & Yackel, 1996; Rasmussen et al., 2015). In this study, we focus on documentation of individual conceptions and their development across a sequence of RME inspired tasks in the context of a DGE.

3.1. Participants

The pilot studies were carried out with two (19-year-old) women, Maya and Sara (pseudonyms), who were sophomore level linear algebra students in a mathematics education program of a state university in central Turkey. The Linear Algebra course in their program covers Linear Systems, Matrix Algebra, Determinants, Vector Spaces, Eigenvalues and Eigenvectors, Inner Product Spaces and Diagonalization. The language of the course was Turkish, and the teacher developed teaching notes, written in Turkish, from the chapters of the textbook of Anton (1981). Pilot studies were conducted after the students learned Linear Systems (that was largely based on the geometry of solution types of linear systems), as well as Matrix Algebra and Determinants. In the Linear Systems part of the course, the students learned that a linear equation $ax + by + cz + d = 0$ (of three variables) corresponds to a plane equation in $\mathbb{R}^3$. Starting from this, the students learned the link between the number of free variables (i.e., parameters) in solving the system of linear

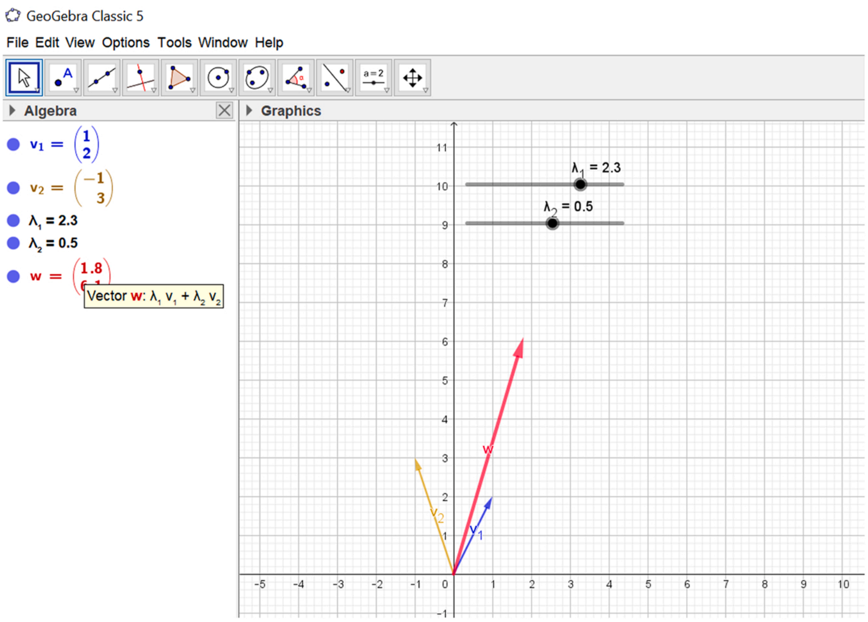

Fig. 1. GeoGebra interface regarding the scalars as sliders.

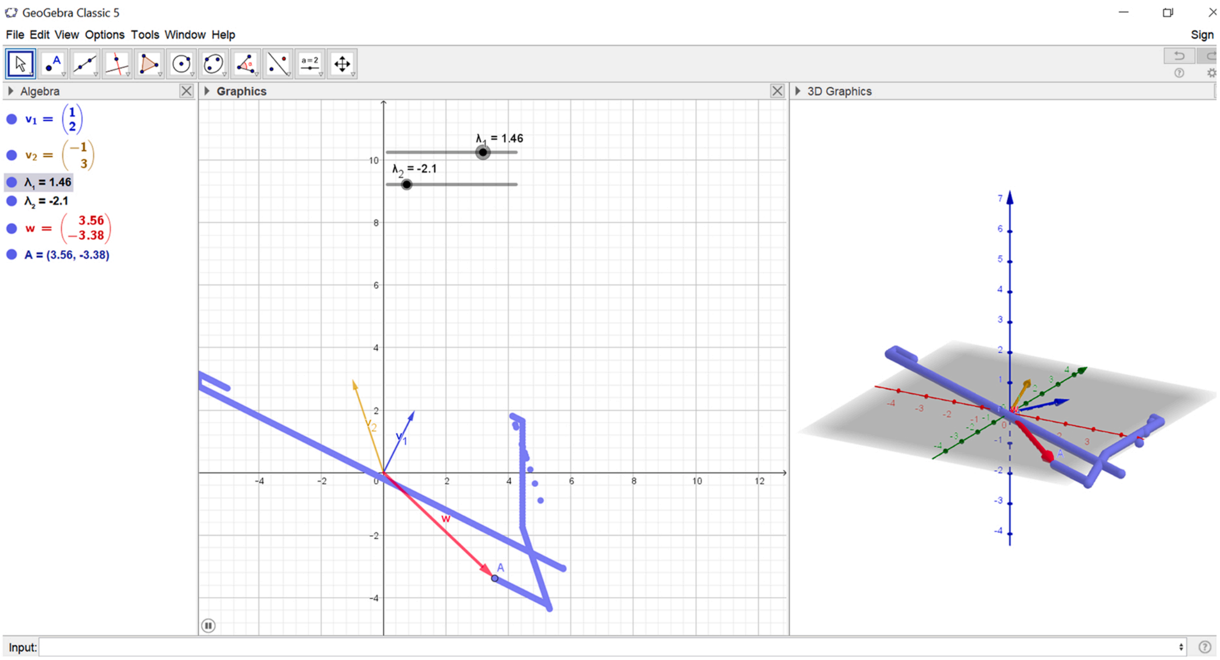

Fig. 2. An exemplary case for animate and trace functions of DGE.

equations and corresponding geometrical concepts. Six lecture hours were spent discussing free variables and geometric interpretations of solution sets. For example, in terms of various examples, they learned using a parameter(s) (when and how) while solving two or three linear equations with three unknowns, and through this, they learned how to formulate parametric line and plane equations in $\mathbb{R}^3$ (as an intersection of planes). Therefore, the students were familiar with parametric equations and the way of obtaining them in searching the geometry of the solution of the system of linear equations. These activities followed instruction on Matrices and Matrix Algebra and their link to solving linear systems. Along with the Linear Systems part, the teacher did not refer to any digital resource in the classroom.

At the time of the data collection, the students had not received formal instruction in the course focused on linear combination, span and linear independence. The implementation of the tasks took place separately from regular linear algebra lectures, which the participants of the pilot studies did not attend. So, the participants came to these sessions instead of their normal lectures on these topics.

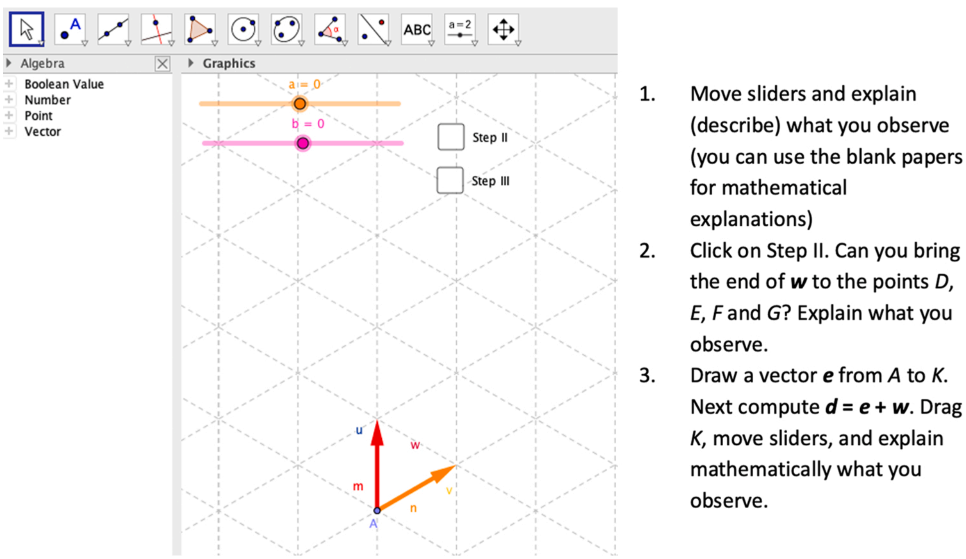

Fig. 3. GeoGebra interface and steps of Task 1.

The students had previously taken the following mathematical courses at the freshman level: abstract mathematics (which includes logic, theory of sets, relations and functions), calculus, and geometry (where they learned 2D and 3D geometry). At the same time as they were taking linear algebra, the participants were enrolled in Physics, where they learned the notion of a (geometric) vector and its fundamental operations such as scalar multiplication and vector addition, and application in physics. Students learned the "parallelogram rule" in Physics in which it was referred to simplify the resultant vector of two vectors. They were also taking an elective course for integrating digital tools to mathematics teaching. In this course, they were learning the main tools and functions (like dragging, slider etc.) of DGEs and discussing particularly how digital tools could be used in teaching lower secondary school geometry. The students were familiar with the DGE context due to their elective course.

3.2. Task sequence and its design

The first author designed a task sequence (including four tasks) for the notions of linear combination and span. In this section, we first describe the task sequence, highlighting the role of GeoGebra in its design. We then explain how RME instructional design heuristics are reflected in the task sequence. The aim of the first task is to create a sense for linear combination, i.e., a sense for obtaining a vector in terms of (weights and sums of two and three) other vectors in a specific grid plane, which depicts a non-standard coordinate system. The task is divided into three steps. Fig. 3 represents the task and GeoGebra interface.

Here, $\boldsymbol{u} = a \bullet \boldsymbol{m}$ and $\boldsymbol{v} = b \bullet \boldsymbol{n}$. So, the dragging sliders change $\boldsymbol{u}$ and $\boldsymbol{v}$ accordingly and a third vector $\boldsymbol{w}$ appears as the resultant vector of $\boldsymbol{u}$ and $\boldsymbol{v}$ when the sliders a and b are non-zero. By clicking on Step II, points are visible. Because of (set) increments of the sliders, the end of $\boldsymbol{w}$ can meet F and G, while it cannot exactly meet D and E. When the user clicks on Step III, then the point K appears, so that the user will be able to draw a new vector as requested.

The aim of the second task is to create a sense for variation of linear combinations of the vectors and to approach linear combinations as a set of vectors. In other words, we aim to prepare an infrastructure for the notion of span in terms of digital tools such as dragging, sliding, and tracing functions of the DGE. Fig. 4 summarizes the task.

Here, when the user drags the sliders a and b, vectors $\boldsymbol{u}$ (pink) and $\boldsymbol{v}$ (purple) lengthen and shorten accordingly. Here $\boldsymbol{w}$ (red) is the resultant vector (of summing $\boldsymbol{u}$ and $\boldsymbol{v}$), and the end point A marks red points as a result of dragging sliders. So, changing the sliders forms different red points on the Cartesian plane. In the task, the focus is on 'the meaning of red points', to evoke a sense for the span of $\{\boldsymbol{u}, \boldsymbol{v}\}$.

The aim of Task 3 is to move up to an $\mathbb{R}^3$ context (particularly for looking at the span of a set consisting of a single nonzero vector) and recreate a sense for – this time – 3D vectors, which are not easy to comprehend in a paper-and-pencil format. Here, through Task 3, we consider a line as a *span* of linear combinations of vectors. Fig. 5 shows the interface and the steps of Task 3.

Here, the first slider, λ, lengthens and shortens vector $\boldsymbol{u} = (2, 1, 0)$ while the second slider (that appears in the second step) changes vector $\boldsymbol{v} = (1,1,1)$. Because $\boldsymbol{u}$ lies on the grid plane, it is visible both in Graphics and 3D Graphics windows and both trace the vectors on them. We anticipate that the user will explore cases one by one and at the end arrive at a conclusion that the lines can be considered (i.e., generated) as a set of points marked (traced) by the movement of the sliders.

The aim of the fourth task is to shift to the context of a span (set of two vectors in $\mathbb{R}^3$) as planes. Here, the notion of span is conceived as a set of points generated by linear combinations of (given) two vectors. The main aim of this task was to discuss the mathematics behind the task sequence, connecting the grid plane to the 3D context. Fig. 6 summarizes the task.

In Task 4, when the user drags the given sliders, the end of the resultant vector marks red points and all these points form a rectangle on the screen. However, the teacher's role has crucial importance here, as what would happen if the sliders moved along the

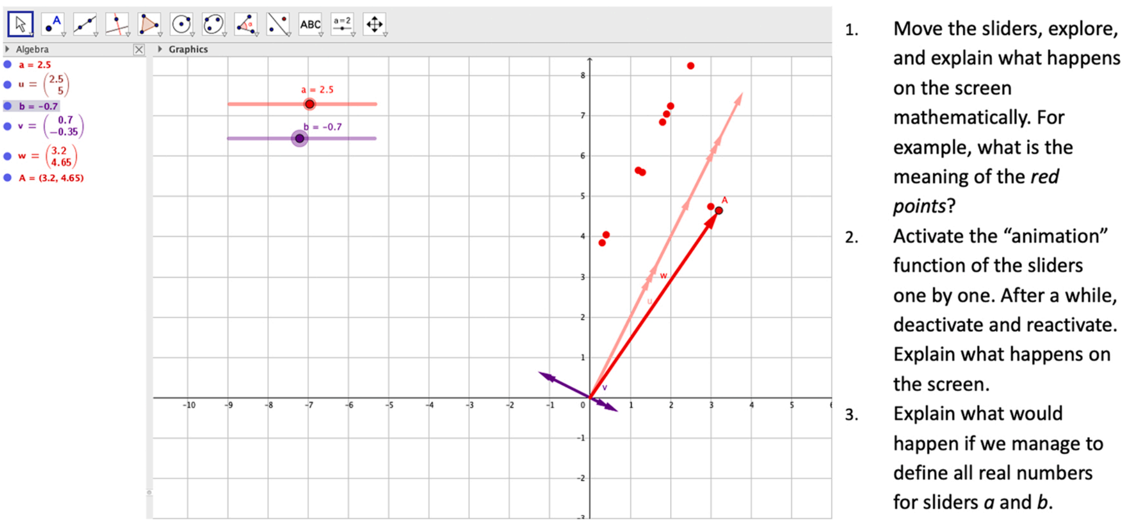

Fig. 4. GeoGebra interface and steps of Task 2.

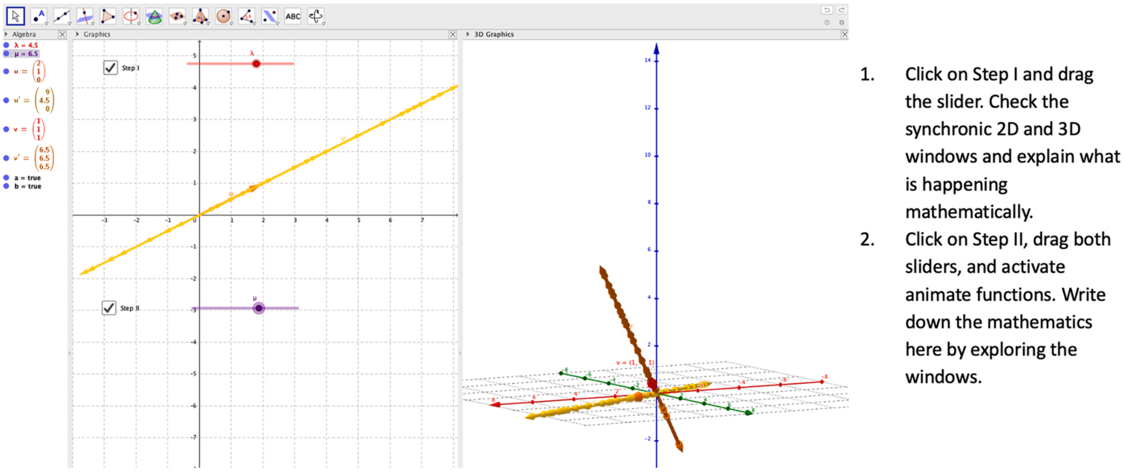

Fig. 5. GeoGebra interface and steps of Task 3.

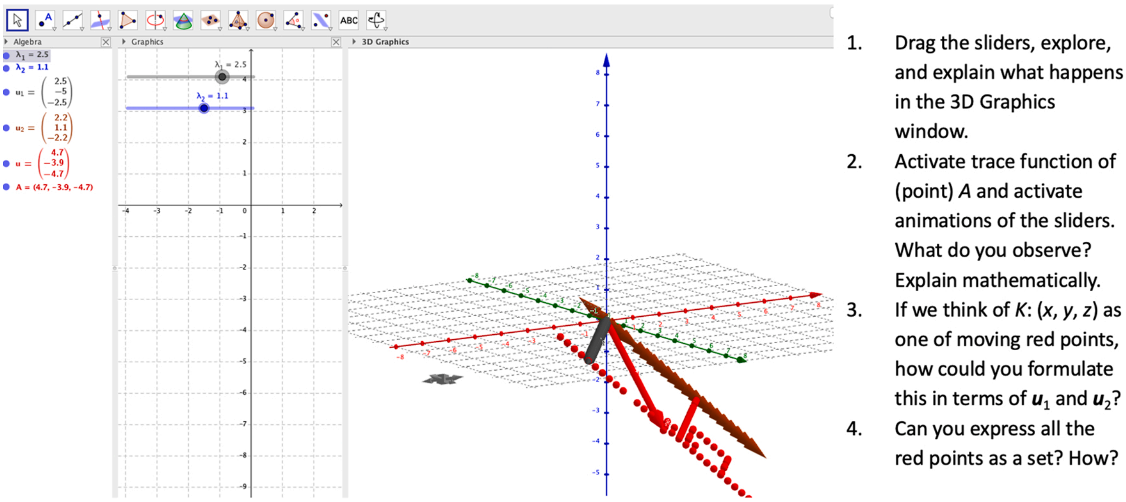

Fig. 6. GeoGebra interface and steps of Task 4.

real number line needs to be discussed.

The design heuristics of RME informed the development of this task sequence as follows. We considered geometric vectors as an experientially real for students since they had worked with these in physics lessons. It was our intent that students would create a "model-of" identifying what points they could reach using linear combinations of vectors, and that this would come to function as a "model-for" their subsequent mathematical activity as the constraints for taking linear combinations of vectors varied across the sequence of tasks.

3.3. Data collection and analysis

The data was collected through six task-based (individual) interviews with Maya and Sara (three interviews each) in front of the teacher and a laptop installed with GeoGebra. In order to pilot task sequence and understand the dialectics between tool use and student thinking/individual conceptions in depth, we referred to individual interviews. Each task entailed an average length of 35 min of interview time. The first author of the paper was the interviewer in the pilot study and had a twofold role in the interviews. The first was a researcher's role, where was trying to understand students' reasoning. At the same time, having a teacher's role, where he was trying to guide student's sense making regarding the notions of linear combination and span.

A video-camera looking at the students' working area, screen recorder software and the student productions were referred to employ data triangulation. Collected data was analyzed with the perspective of mathematization and symbolizing activity throughout two phases. Within the first phase, coding was focused on students' tool use (with eye toward mathematization) by the first author for each task. This included inductive and theory driven development of codes (DeCuir-Gunby, Marshall, & McCulloch, 2011) as well as

Table 1
Codes for DGE (tool) use and mathematization.

Code (s)	*Code Description*	*Examples*
Explore/ Experiment	● The use of tools and functions of DGE in order to explore (the given) situations mathematically. For example, dragging sliders and/or making extra drawings. ● Speaking about the situation in terms of the DGE context, but not *mathematically*. For example, speaking about the function of tools, in particular, what is happening on the screen. But not speaking and/or describing the mathematics of the situation.	● Dragging the sliders and speaking about the effects on the screen, like "moving", "overlapping", "movement of sliders", "red points generated by." etc. ● Implicit expressions regarding tool use, not explicitly for linear combination and span. ● Making extra drawings for the points and/or resultant vector.
Describe	● Description of the (dynamic) effects of the functions and tools of DGE context in terms of mathematical symbols/ notions/expressions. ● Referring to verbal and/or mathematical expressions regarding the mathematical situation by relating it to associated pre-knowledge (come from previous lectures etc.). But not a *conceptual* articulation and not establishing a connection between the concepts.	● Expressing initial observations mathematically (but this is *primitive*), like the relation between the slider values and the movement of the vectors. ● Referring to the resulting vector (comes from a Physics lesson) and summing up two vectors like $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ or formulating such an equation in terms of slider values.
Connect/Relate	● Connect and relate the mathematical situation with existing mathematical concepts, but not arriving at a *generalization* regarding the situation and/or connection.	● Making connections between the obtained (red) dots and set theory. ● Connecting the mathematical situation with the geometry of lines/ planes.

(*continued on next page*)

Table 1 (*continued*)

Code (s)	*Code Description*	*Examples*
	● Connecting *locally* - connecting the situation based on the results coming from the DGE context.	● Making connections between the movement of sliders, resultant vector and vector equations, system of linear equations, matrix algebra.
Formalize	● Decontextualization of the mathematical situation from a DGE-based task context: Generalizing the situation independent from the task situation. ● Informal way of reasoning when the formal definitions are unknown: Sense-making and building up new knowledge.	● Generalizing the movement(s) of the resultant vector into obtaining entire lines and planes as a set of red dots (i.e., arriving at infinite sets). ● Combining geometry and algebra to formulate the set of red dots. ● Genesis of the (informal) ideas of linear combination and span with other related mathematical notions.

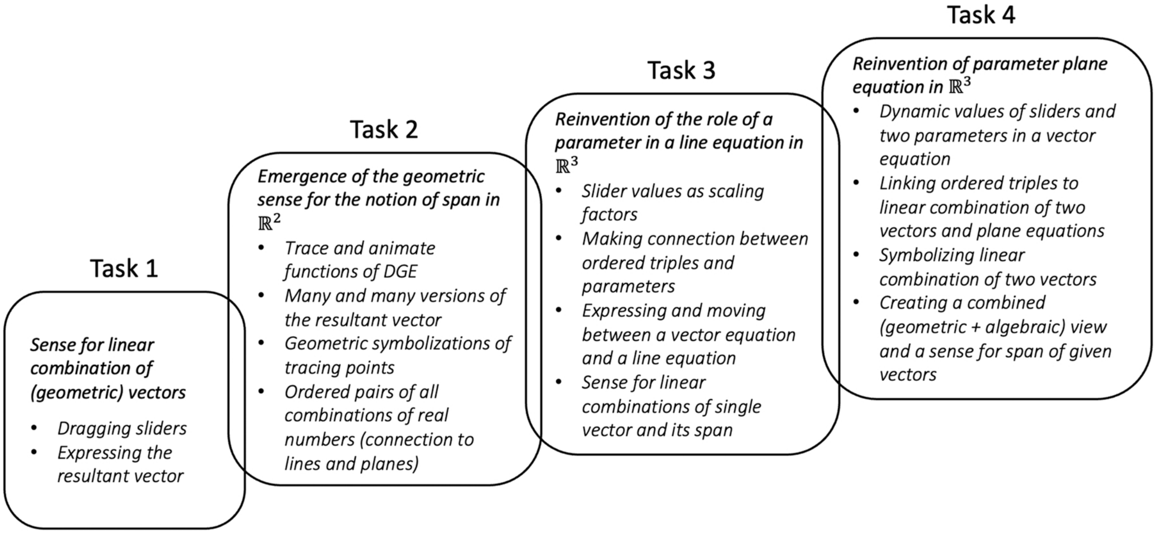

Fig. 7. An overview of student reasoning in the task sequence.

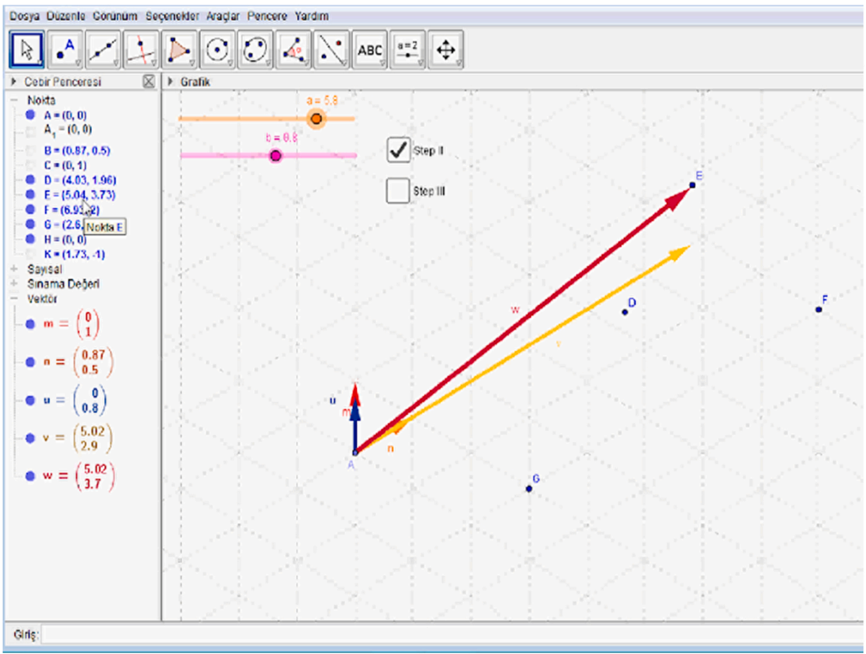

Fig. 8. Maya's exploration for the point E (screenshot from screen recordings).

checks by an external coder to improve inter-rater reliability. The first coder defined three codes (Explore, Explain and Building) at the beginning, while the external coder decided on two initial codes (Thinking with Digital Tools and Thinking without Digital Tools). Two coders again watched the whole pilot study (in Turkish) and then compared, contrasted, and articulated their findings (within three meetings) regarding the task sequence. Final decisions (regarding four codes) were made on consensus in the final meeting. The codebook described in Table 1 contained final codes which were theory driven but also informed by the initial pass through the data.

In the second phase of analysis, the lead author summarized all sessions and provided summaries and translated transcripts for key mathematical exchanges to the rest of the author team. Then the descriptive accounts of students' mathematical activity were developed for each task. The team analyzed these through the lens of mathematization with particular reference to students' symbolizing activity and the role of DGE in this activity.

The main distinction between Explore/Experiment and Describe is (through the exploration and experimentation) moving from speaking about the situation in terms of tools to the use of mathematical symbols/notions/expressions regarding the mathematical

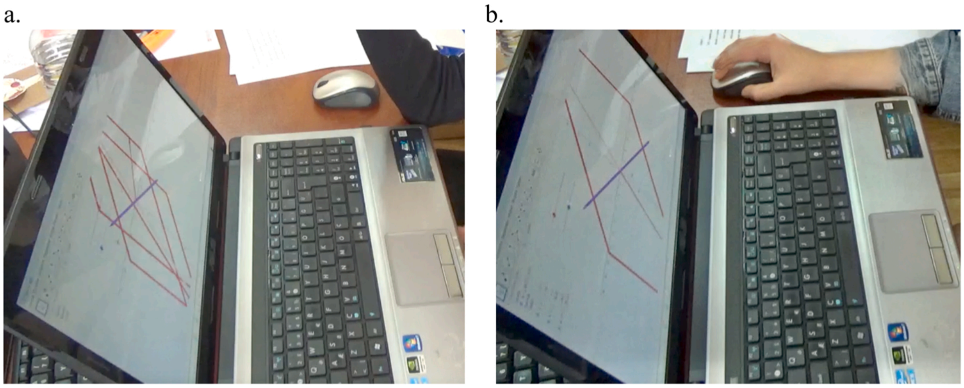

Fig. 9. (a) Maya's figure, (b) Sara's parallelogram.

situation. In other words, Explore/Experiment is strongly based on speaking about what is seen, while the "Describe" is relating the experiences that come from the screen with mathematical symbols/notions/expressions (which is heavily based on student pre-knowledge and/or phenomenological experiences). We note that (still) there is no explicit evidence of connections among mathematical concepts in "Describe." Connect/relate is something connecting the situation with existing phenomenology *without* making a generalization, which can be described as "connecting locally" since a connection here may be due to the DGE context. However, "Formalize" refers to decontextualization of the mathematical situation from the proposed context by generalizing and/or making informal inferences and finally building up new knowledge.

4. Results

Our main finding is that, at first, student reasoning was based on a geometric sense for linear combinations of vectors (Task 1 and Task 2), with critical support provided by the DGE context and tools (e.g., dragging sliders, animation, and trace functions). However, students later shifted to an algebraic view focusing on the notion of parameter (Task 3). In the end, students argued that taking all linear combinations of given sets of one and two vectors in three-dimensional space would form lines and planes by combining geometric and algebraic views (Fig. 7).

As shown in Fig. 7, we highlight four trends in student reasoning in the tasks; the emergence of geometric sense for the linear combination of vectors, geometric sense for the notion of span (conceived as the set of all resultant vectors when one allows the weight on the vectors to vary through all real numbers), conceiving of all linear combinations of one and two given vectors as lines and planes in $\mathbb{R}^3$, and coordinating algebraic and geometric views corresponding to the notion of span. It also describes the roles of tool use related to developing these ways of reasoning. We will present findings by recalling the task situation, highlighting the role of tool use and teacher intervention, and detail the nature of students' mathematical activity. In particular, we use our coding scheme to capture the

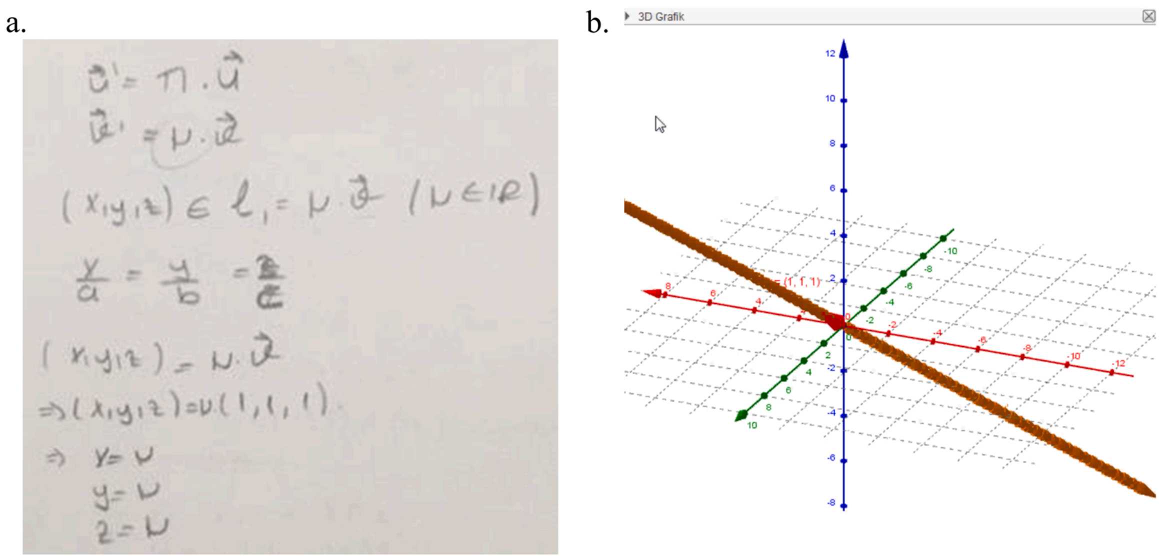

Fig. 10. (a) Maya's findings regarding Task 3, (b) Maya's 3D Graphics window.

ways in which students leveraged DGE tools in their mathematical activity, and particularly highlight the ways the DGE supported students' symbolizing activity.

4.1. Task 1: emerging sense for linear combinations of vectors

In Task 1, both students came to reason about how specific points (vectors) in the plane $\mathbb{R}^2$ could be described in terms of slider values (scalars) and a given pair of vectors. In this section, we detail the ways in which the two students organized their reasoning about the vectors and sliders in the DGE across the subparts of Task 1.

Students were first asked to drag sliders and form points using (a linear combination of) vectors in GeoGebra. Sliders were limited to specific values of real numbers so some points could not be reached. In their separate interviews, both students' initial interactions with GeoGebra involved dragging the sliders for a and b. For example, both explored the effects of sliders for several specific (integer) values such as $a = 1, 2, \cdots$ and so on. We coded this part as Explore/Experiment because they first dragged both sliders spontaneously and then explored the screen to understand simultaneous effects of dragging on the given vectors.

Both students observed that $\boldsymbol{w}$ is the resultant vector of $\boldsymbol{u}$ and $\boldsymbol{v}$ and the sliders change the vectors (i.e., the vectors get longer/shorter when the sliders were dragged). This was coded as Describe because students used the sliders in their explanations for what was happening mathematically with the vectors. For example, Maya stated ". this is like parallelogram rule, w is the resultant vector for u and v. Here the sliders change u and v and therefore w …". This was followed by a particular focus on given points (as it was the focus of the next question in the task statement given to the students) – specifically considering which points can be obtained through dragging the sliders and which cannot. Students both dragged sliders and explored the screen for a while and concluded that the points *F* and *G* could be obtained by dragging sliders, while *D* and *E* could not be obtained. In her exploration, Maya referred to the Algebra window to compare coordinates of the given points (as listed "Nokta" in Turkish) with the vector description of $\boldsymbol{w}$. In this way, Maya showed that point E cannot be exactly obtained by dragging the sliders (Fig. 8).

After the teacher asked for her justification, Maya brought the cursor on the point *E* and stated that the vector $\boldsymbol{w}$ and the coordinates of *E* are always different. Even the end of $\boldsymbol{w}$ seems to be on the point *E* on the screen, which was due to the sliders' values (as she thinks). She summarized her conclusions (which we coded as Describe as she is describing a reason point *E* cannot be reached due to the intentional slider constraints):

> Maya: "… I could be able to pass [vector w] over D and E as well … but umm … that is due to sliders' decimal values that I cannot change, but normally can be obtained, I think. All points and vectors are on the plane …"

In her interview, Sara emphasized that the points *D* and *E* cannot be obtained because they are not located on the intersections of grid corners. She observed (Describe):

> Sara: "… when I want to make slider a as four and a half, it only precedes as zero point two, so it is impossible to get every real number value here, as the components of vectors …"

Regarding the final step of the task, both students arrived at a common view that all given vectors are interconnected (coded as Describe because of their explanations for why the sliders are related to a specific vector). For example, Maya stated that vector $\boldsymbol{d}$ was connected to the sliders due to vector $\boldsymbol{w}$, and therefore vector $\boldsymbol{d}$ was dependent on $\boldsymbol{w}$ and can be expressed in terms of sliders and given vectors (meaning $\boldsymbol{m}$, $\boldsymbol{n}$, and $\boldsymbol{e}$).

We note that there were a few specific functionalities (tools) provided by the DGE that aided the students as they worked through this task sequence, namely: synchronous views of Algebra and Graphics window, grid function (aligning with the two given vectors), and sliders that could be dragged to adjust the weights on the two given vectors. At the beginning, the students immediately identified $\boldsymbol{w}$ as a resultant vector; recall they had received instruction on this topic in their Physics course but not their linear algebra course. However, the DGE helped the students connect the sliders (weights on vectors $\boldsymbol{u}$ and $\boldsymbol{v}$) and components of their weighted sum. Next, they determined that some points could be written as linear combinations of given vectors, but that other points could not unless the sliders were to have smaller increments. Such a view opened a door to focusing on the components of the resultant vector. Both students ultimately concluded that the vectors on the screen were interconnected to each other due to slider values. In other words, they developed ways of organizing their activity around "expressing vectors in terms of other vectors", which seems an initial understanding for the notion of (geometric) linear combination of vectors.

Although students didn't engage in the creation of new mathematical symbolizations as they worked on the first task, we highlight two ways in which their interaction with the DGE in the context of the task may have supported their reasoning about span as well as their subsequent symbolizing activity. First, through their activity of exploring/experimenting (dragging sliders to see that they shorten and lengthen vectors and determining which points are and are not reachable), they interacted with the algebraic and geometric symbolizations presented in the DGE in ways that helped them organize their reasoning about how scalars (sliders) interact with individual vectors. Second, there is evidence that both students also reasoned productively about linear combinations of the vectors to explain why certain points are not reachable based on the values of the scalars (sliders). Taken together, students' exploration/experimentation in the DGE and verbal descriptions of their reasoning about their work suggest that they have developed productive ways for coordinating scalars, vectors, and linear combinations of vectors as symbolized algebraically and geometrically in the DGE.

4.2. Task 2: geometric sense for the notion of span in $\mathbb{R}^2$

In Task 2, students were first given two (linearly independent) vectors in $\mathbb{R}^2$, told to manipulate the sliders, and asked to interpret the red points (which correspond to the linear combination of the vectors with the weights that have been traced). They are then asked to animate the sliders and predict what would happen if the sliders managed to trace through all possible real numbers for slider *a* and *b* (which corresponds to taking all possible linear combinations and forming the span of a plane). The students concluded that the red dots were created from the resultant vector made by linear combinations of two vectors. Both students identified the red dots as an infinite collection of points arising from allowing the weights on each of the two vectors to vary through all real numbers. One student reasoned that the red dots create a parallelogram that could be taken out to an infinite length, whereas the other concluded that the red dots would eventually create the whole plane. In this section, we detail the ways in which the students come to reason about the notion of span using the red dots made by the resultant vector, and how they represented this algebraically.

Students dragged sliders and both quickly realized that the red points were generated by the movement of the resultant vector (coded as Explore/Experiment as the students were describing the effects of tool use with explicit mathematical connections). For example, after the teacher asked Maya for her interpretation regarding the red points, she stated:

> Maya: w is here the resultant vector of u and v, if I sum up them, I immediately find w.
> Teacher: What happens when you drag the sliders?
> Maya: … a changes u, that is, such as a times u and v is something b times v**.**

This was coded as Describe because Maya expressed a mathematical description verbally (which comes directly from DGE context as displayed in the Algebra window). Maya then activated the animation function of the sliders at different times so that she had line segments and an "interesting figure" (Fig. 9a) as she characterized.

At this point, the teacher asked the role of the animations (i.e., animated sliders). The following excerpt is extracted from the discussion between the teacher and Maya.

> Teacher: What would happen if [*slider*] a changes along real number line?
> Maya: … we would have a line then. Sure, a line. If [*sliders*] a and b change along with real numbers, then we would have a plane because all the red points mark (or travel) on the x-y plane [*dragging sliders*] … I would obtain many and many versions of the resultant vector as a result of the sum of two dynamic vectors … I think, the resultant vector of any two vectors in R-two always gives us the x-y plane…

In the end, Maya not only identified the link between slider values and line segments on the screen, but also related her findings to a line and a plane when *a* and *b* change along real numbers (coded as Connect/Relate because of her connection between the context and a different *specific* example). After dragging the sliders, she characterized movement of the resultant vector and generalized to movement of a resultant vector of two vectors in $\mathbb{R}^2$, which can be considered as vertical mathematization (coded as Formalize because she extended her previous reasoning to *any* resultant vector). We interpret this as an informal way of reasoning about why the span (or set of all possible linear combinations) of the two given vectors is the entire plane (further noting that if only one slider is changed, a line is created).

In Sara's case, while exploring the steps of the task, she activated the animation function of the sliders only once, so that she had a single parallelogram on the screen (Fig. 9b). The teacher asked her to look again at the steps of the task which underlined to activate-deactivate sliders to observe different values. After dragging the sliders and activating/deactivating the animation function, she stated that different line segments could be obtained and concluded:

> Sara: [*deactivates-reactivates animations*] … if I activate-deactivate again and again, then it would be an infinite thing [*gestures*] I mean the red points would mark the area here [*showing screen*] all ordered pairs here by having all combinations of real numbers…

Similar to Maya, in the end, Sara connected the task situation to the geometry by addressing line segments as generated by red points (Connect/Relate as she used the red dots created by various tool functions to come to a mathematical conclusion). Sara also

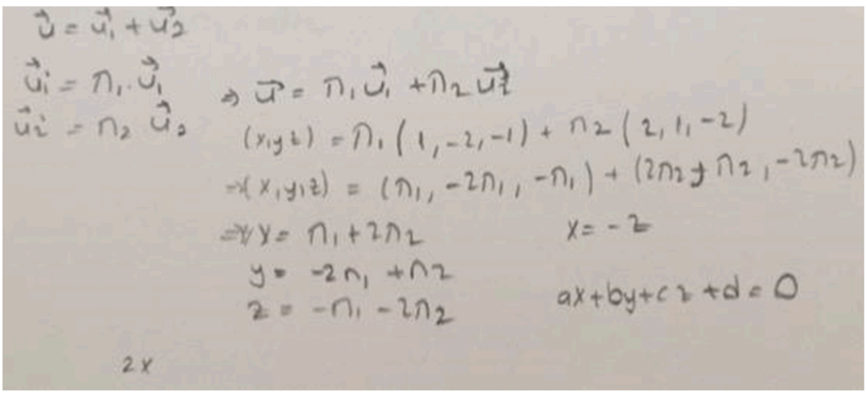

Fig. 11. Maya's findings regarding Task 4.

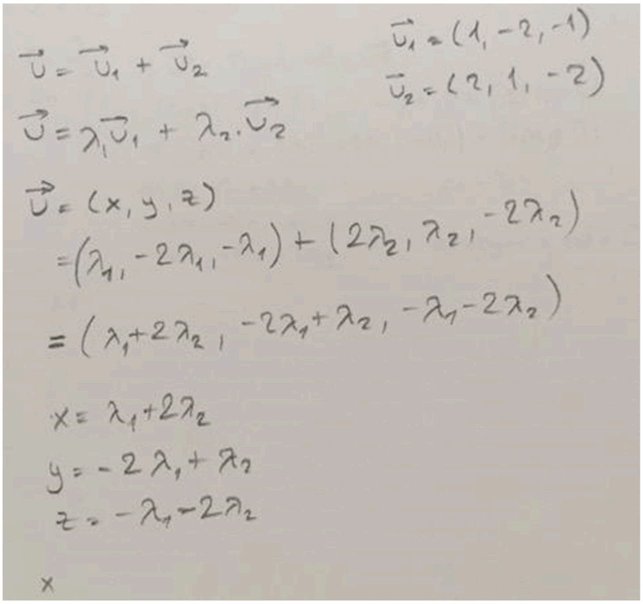

Fig. 12. Sara's findings regarding Task 4.

conjectured that she obtained an "infinite thing" and generalized that this was a result of taking all combinations of real numbers for sliders (scalar weights) a and b (Formalize).

In this task, students' interaction with the DGE—using the trace and animate functions—resulted in the generation of a geometric symbolization that provided a tool for them to reason about many possible linear combinations at the same time. More specifically, Maya animated the sliders at different times; this interaction with the DGE resulted in the creation of a geometric symbolization of the points that were traced out in which multiple parallel lines of varying lengths were depicted on her screen (one set of which lie along the direction of one vector in the linear combination, the other set of which lie along the direction of the other vector). When asked by the interviewer, she noted that if just one slider was animated, a line would be traced out. She extended this reasoning to conclude that if both sliders were animated, the entire plane would be traced out.

Sara, on the other hand, initially animated the two sliders together and the geometric symbolization that resulted appeared as a parallelogram. When pressed by the interviewer to animate and reanimate the sliders, she extended her reasoning to consider that the whole plane might be traced out by larger and larger parallelograms. Importantly, the DGE environment supported the students in leveraging their horizontal mathematization (e.g., through the geometric symbolizations generated via their interaction with the DGE) toward vertical mathematization. Specifically, this vertical mathematization is evidenced by students' description of and connection to their prior knowledge of lines and planes as a way of organizing their reasoning about ALL possible linear combinations (e.g., what is traced out when the sliders are animated through all real numbers).

4.3. *Task 3: interpreting linear combinations of a single vector as a line in* $\mathbb{R}^3$

In Task 3, students were asked to drag (but not initially animate) two sliders, λ and μ, which corresponded to scaling two different vectors in $\mathbb{R}^3$ separately. They viewed these simultaneously in the x-y plane and in $\mathbb{R}^3$. One of the vectors they were scaling was in the x-y plane so they could see it being scaled in both places; the other vector was not in the x-y plane so when they drag the corresponding slider it could only be seen being scaled in $\mathbb{R}^3$ view. The goal was to get them to think about scalar multiples of vectors in $\mathbb{R}^3$. Both students identified the set of combinations of a single vector as tracing out a line and worked to express the equations of the lines they observed algebraically. In this section, we highlight how students used parameters to symbolize equations for lines in both settings and the role sliders played in the development of students' reasoning.

After first dragging and then animating sliders while using the Rotate 3D Graphics tool (coded as Explore/Experiment because students were observing effects of manipulating different tools), the students – individually – observed that the given slider values λ and μ are weights for the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, where $\boldsymbol{u}$ was only moving on x-y plane and $\boldsymbol{v}$ was moving in $\mathbb{R}^3$ (coded as Describe because students connected their tool use to graphs of the vectors). Maya described $\boldsymbol{u}'$ and $\boldsymbol{v}'$ as moving vectors and started to write her conclusions algebraically (coded as Describe). To describe the role of the sliders, Maya wrote the first and second lines of Fig. 10a.

Maya stated that the animation of slider μ produced a line in $\mathbb{R}^3$. At this point, the teacher asked how to formulate this line mathematically. Looking at the right part of the screen (Fig. 10b), she expressed the third line of Fig. 10a, indicating a point is on the line, which can be expressed as all the possible scalar multiples of vector $\boldsymbol{v}$. But she was unsure how to proceed. After the teacher asked whether she tried to formulate a line equation, she wrote the fourth line of her findings. She searched for a, b and c for a while, and finally in terms of the 3D Graphics window, which shows a vector $\boldsymbol{v} = (1, 1, 1)$, she arrived at the parametric line equation $(x, y, z) =$

$\mu(1,1,1)$ (Fig. 10a) (coded as Connect/Relate because she used the Graphics window in developing her equation). Maya started to connect two different (object) representations, dynamic vectors in the DGE context on the one hand, a vector equation corresponding to a parametric line equation on the other. However, she reflected that it was not easy for her to remember a line equation. Maya characterized the role of sliders as follows:

> Maya: We can say parameters, but at the same time a factor for a given vector. They are generators for the lines, since they form vectors on it. In terms of their movements of vectors so that the points, I managed to obtain lines…

In the end, Maya established and generalized a link between the sliders (i.e., slider values) as parameters and "generators of lines". In other words, Maya reasoned about the role of parameter in a line equation in $\mathbb{R}^3$ by expressing them as "generators", where sliders acted as mediators to arrive at such a generalization. We consider the genesis of this informal idea regarding span as vertical mathematization (coded as Formalize).

When the teacher asked Sara about the role of the sliders, she identified two roles of sliders: "as a parameter of the line" and acting as factors in the "vector equations" (coded Connect/Relate as she is connecting sliders to specific mathematical examples). After dragging and animating both sliders, Sara similarly expressed what was happening algebraically by writing the equation $\mu \bullet (1,1,1) = \boldsymbol{v}'$, and $x = \mu, y = \mu, z = \mu$. Further, she emphasized that sliders were coefficients, they produced line segments, and if they had been changed along all real numbers, then it would be an exact line in $\mathbb{R}^3$. Finally, she conjectured that every line in $\mathbb{R}^3$ can be obtained with a vector and a slider (coded as Formalize because she expands on her previous notions to general cases).

This is the first task in which the students explicitly engaged in developing their own algebraic symbolization. Here, after experimenting with the sliders, Maya described $\boldsymbol{u}'$ and $\boldsymbol{v}'$ as the "moving vectors," and algebraically symbolized their relation to the other two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and sliders (λ and μ) as $\boldsymbol{u}'=\lambda\boldsymbol{u}$ and $\boldsymbol{v}'=\mu\boldsymbol{v}$. Notably, the current values of $\boldsymbol{u}, \boldsymbol{u}', \boldsymbol{v}$ and $\boldsymbol{v}'$ were represented as vertical vectors whose entries were listed vertically as a 3-tuple in the Algebra window — and Maya's algebraic symbolization explicitly captured the multiplicative relationship between appropriate pairs of vectors and the parameter (scalar) by which those vectors were multiplicatively related. Further, the third line of Maya's inscription suggests that she imagines each (x, y, z) 3-tuple should be a point on the line which should correspond to $\mu\boldsymbol{v}$ as μ varies through all real numbers. However, Maya then turns to a previously known form for expressing equations of lines in 3-space that she seems to find unhelpful in resolving her need to link the vector equations she had written to her ideas for expressions representing lines in 3-space. The realization that $\boldsymbol{v}$ is in fact just the 3-tuple (1,1,1) appears to have been instrumental for resolving this tension, as she then expressed $\boldsymbol{v}'=\mu(1,1,1)$ which could then be written as a set of general equations for x, y, and z in the parameter μ. We argue that the algebraic and geometric symbolizations presented in the DGE, paired with the exploration afforded by the DGE and the tasks, supported Maya in algebraically symbolizing vectors in 3-space being dynamically scaled to form a line — and thus connect to the standard form of a parametric vector equation for a line.

While students did have prior knowledge about parametric vector equations based on the instruction, they received on systems of linear equations, it is non-trivial that they connected this to ideas emerging from a focus on linear combinations of vectors rather than solutions to linear systems. We see this as vertical mathematization because students are coming to reorganize how they coordinate, via algebraic symbolization, vectors, scalars, lines, and points on a line. Meaning, students used their broad notions of linear combinations to reason about linear combinations of a single vector as a line in $\mathbb{R}^2$ or $\mathbb{R}^3$, which they had only previously done in a linear systems context. Interestingly, they reason about the parameter as the entity that is "generating" the line, which is consistent with their previous activity wherein lines are generated when sliders are animated.

For both students, the notion of parameter linked to students' existing knowledge of a line equation, so they needed to establish a

Table 2
Horizontal and Vertical Mathematization observed in the task sequence.

Task no	Horizontal Mathematization	Vertical Mathematization
1	Explore/Experiment → Describe → Describe ● Dragging and referring to sliders to explore the mathematical situation ● Noticing the sliders' roles and expressing resultant vector in terms of given vectors	—
2	Explore/Experiment → Describe ● Red points are generated through the movement of resultant vector $\boldsymbol{w}$ ● Expressing $\boldsymbol{w}$ as linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$	Connect/Relate → Formalize ● Extending the case to lines and planes, when the sliders change infinitely ● $\mathbb{R}^2$ plane as all combinations of resultant vector $\boldsymbol{w}$
3	Explore/Experiment → Describe ● Noticing the slider values as (scaling) factors for the given vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ ● Animation of sliders produce line	Connect/Relate → Formalize ● Arriving at a vector equation (as linear combination of a single vector) and parametric description of lines ● Reasoning on obtaining a line through a vector and a factor
4	Explore/Experiment → Describe ● Dynamic values of sliders in a vector equation ● Formulation of linear combination of (given) two vectors	Connect/Relate → Formalize ● Noticing two parameters in a vector equation and transforming it to system of linear equations ● Arriving at set of red points (generated by two dynamic vectors) as a plane in $\mathbb{R}^3$

link between the traced points generated by sliders which were *ordered triples*, and the parameter in a line equation. Their thinking was based on the link between the notion of parameter and scalar multiplies (as ordered triples) of a single vector. We argue that this indicates students had a sense for all possible linear combinations (scalar multiples) of a single vector — this time — in $\mathbb{R}^3$ and span of the given vector.

4.4. Task 4: emergence of algebraic and geometric views for span in $\mathbb{R}^3$

In Task 4, students were asked to drag two sliders (λ_1 and λ_2) to scale two (linearly independent) vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in $\mathbb{R}^3$. Next, students were asked to activate the trace function of a given point A which was the end of the resultant vector $\boldsymbol{u} = \lambda_1 \bullet \boldsymbol{u}_1 + \lambda_2 \bullet \boldsymbol{u}_2$. The dragging and animation of sliders would mark a set of red points, in the present case, in a plane in $\mathbb{R}^3$. Students were requested to consider a point $K:(x,y,z)$ as a moving (arbitrary) red point and formulate K in terms of given vectors. The goal was expressing all red points as a set. Both students built on the geometric ideas developed in Task 2 and the algebraic ideas developed in Task 3, writing parametric equations using the given vectors and concluding that the set of all linear combinations of the two vectors traced out a plane. In this section, we discuss students' methods for connecting their algebraic and geometric understandings of vectors, parameters, and the linear combinations of two vectors through their tool use.

After the students used the trace tool and animated the sliders (coded as Explore/Experiment because there was no explicit mathematical connection), they used two parameters to formulate the resultant vector; both students offered an algebraic formulation without connection to tool use (coded as Describe). Maya initially summarized her findings by the left-hand side of Fig. 11. She initially represented u as the sum of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ before shifting her representation to capture the sum the two vectors when weighted by the scalars λ_1 and λ_2.

Maya then used the Rotate 3D Graphics tool to explore what kind of figure the set of red points produced. She emphasized that it was impossible to obtain a 3D figure, since there were only two given vectors, so that it was a plane (coded as Connect/Relate because she connected her use of the 3D graphics tool to a a plane). The teacher then asked her for an algebraic formulation of the plane. Maya expressed that it would be in the form of $ax + by + cz + d = 0$, but indicated she did not know how to associate the resultant vector to the equation. After referring to the Algebra window and writing for a while (as shown in the right side of Fig. 11), Maya wrote a parametric equation for the given plane and seemed surprised. The teacher suggested she could consider parametric equations as a system of linear equations, and Maya stated "… *well this is a system of linear equations, yes, I can solve this. If I sum of the first and the third equations, I have*…" and wrote $x = -z$. She generalized the changing of parameters to the generation of a plane (coded as Formalize because she extended to general case):

> Maya:. In fact, this is a very good way inventing myself… A vector equation includes two parameters like lambda-one and lambda-two, they change, and we have different resultant vector and they mark red points and generate a plane and also its equation… [*laughs*]…

Similar to Maya's work, Sara noted her findings (Fig. 12) regarding expressing the mathematics initially observed (coded as Describe because it was not directly connected to tool use).

At first Sara summed the vectors as shown in the first and second line of Fig. 12. She then expressed the vector $\boldsymbol{u}$ as an ordered triple as (x,y,z) and algebraically manipulated the given vectors using λ_1 and λ_2 as factors. Thereafter, Sara found a system of parametric equations. Because a parametric equation (representation) of a plane was new for Sara, she also was unsure how to proceed. After a while, the teacher suggested she could consider this as a system of linear equations; she summed up the first and the third parametric equations and eliminated the parameters (coded as Connect/Relate due to connecting parametric equations and a system). Sara summarized her conclusion:

> Sara: … two parameters. I am not sure I can always eliminate them for example x plus z equals to zero, as a simple equation. Now, the ordered triples are connected to the sliders which are parameters. Due to parameters, we obtain a plane, they [*sliders*] generate, since we take all possible points as ordered triples on the plane… This is something obtaining an arbitrary plane's equation by moving vectors…

Sara indicated that the plane was generated by parameters, and parameters were connected to the sliders. And due to changing slider values, ordered triples produced the plane and, by moving vectors, it was possible to obtain an arbitrary plane (coded as Formalize). However, she did not seem fully convinced about the generalizability of the method she used for linking the system of parametric equations for a plane to the equation for a plane that takes the form $ax + by + cz + d = 0$.

Both students' reasoning started with the existing knowledge coming from the previous tasks (i.e., sliders as parameters and weights). In this way, they offered an algebraic formulation of the linear combination of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in $\mathbb{R}^3$. Interestingly, possibly due to their prior work in Physics lessons and because the Algebra window does not specifically show the factors, both students first wrote $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$, in a static way without the parameters. However, next, Maya argued that it was impossible to obtain a 3D figure, because there was no movement to a third direction on the screen; rather she found it was a two-dimensional figure when she explored the red points through Rotate 3D Graphics tool. In other words, the DGE helped convey her thinking for expressing linear combination of – only – two vectors, since the figure was two-dimensional.

In terms of students' symbolizing activity, both students concluded based on their exploration in the DGE that the geometric object to be characterized (by taking linear combinations of the two given vectors in $\mathbb{R}^3$ by animating sliders) was a plane. They also readily symbolized reachable points as a parametric vector equation (initially inscribed as a sum, then as a weighted sum) using the two given vectors. However, based on the students' prior knowledge, they anticipated that the equation of a plane in three-dimensional space should be algebraically symbolized in the form $ax+by+cz+d=0$. The teacher-researcher encouraged both students to reinterpret their parametric vector equations through a system of equations lens. Under this encouragement, although both students identified a connection to the closed form expression for the equation of a plane, they expressed different levels of satisfaction with this resolution. Maya seemed satisfied, noting that it made sense that the geometric interpretation would correspond to a plane due to the fact that there were two parameters for the (linear) expressions describing x, y, and z. Sara, on the other hand, acknowledged the algebraic connection but did not seem convinced of the generalizability of the approach — indicating that she was not sure she could always eliminate the parameters to arrive at the closed form equation in x, y, and z. (Notably, neither student commented on the role of the second equation in their manipulation.) Taken together, there is evidence that both students were coordinating the geometric symbolization of a plane shown in the DGE with the algebraic symbolization of a plane as a parametric vector equation — and that students are working to resolve this new algebraic representation of a plane with the closed form expression for representing a plane that they knew from their prior experience. As such, we interpret this as evidence of vertical mathematization in students' symbolizing activity as they are working to reorganize and coordinate their symbolic representations for planes in three-dimensional space. However, we conjecture that additional support may be needed for students to connect this new conception of planes (as generated by taking linear combinations of vectors) to their previous conceptions of planes (as solutions to linear equations of a particular form).

5. Conclusions and discussion

In this work, we address the questions: *How did participants reason about linear combinations and ideas related to span in the context of this DGE-based RME task sequence? What is the role of tool use in advancing students' symbolizing activity in the context of the DGE?* We found that student reasoning was not linear and mostly interplayed in an amalgamated context including tool use, referencing mathematical expressions, exploiting vector algebra and geometry knowledge, and building up senses (toward reinventing mathematical formulation) for lines and planes as a set of points. In other words, after the task sequence, students created a sense for the notions of linear combination and span and extended their knowledge, specifically on the link between vector algebra and geometry of lines and planes, as a set of points. Here the notion of "parameter" had a core role. Researchers have identified the critical role of the notion of parameters in solving and reasoning about systems of linear equations (Sandoval & Possani, 2016; Turgut & Drijvers, 2021; Zandieh & Andrews-Larson, 2019), and in this work we have shown how a pre knowledge for the notion of parameter was also critical to students' coordination of algebraic and geometric symbolization of ideas regarding linear combination and span.

5.1. Students' horizontal and vertical mathematization

In order to look back across the set of tasks and identify students' key ways of reasoning about linear combinations, we briefly sketch an overview of the instances of our coding that correspond to horizontal and vertical mathematization (see Table 2). In this overview, we foreground themes relating to students' tool use and sense-making. Notably, aspects of both horizontal and vertical mathematization were observed in Tasks 2, 3, and 4. The arrows in Table 2 represent the progression and interplay between codes. For example, in Task 1, students made a number of dragging practices, explored the proposed mathematical situation (Explore/Experiment), and described sliders' roles in that mathematical situation (Describe). They finally expressed the resultant vector through in terms of the slider values (the given vectors) (Describe).

In the first task, students engaged in horizontal mathematization (exploration/ experimentation and description) as they worked to express given points as linear combinations of two vectors in $\mathbb{R}^2$. In particular, students had to develop models of adjusting slider values (which weighted the two given vectors) so that the sum of the weighted vectors could meet the desired points when possible. Importantly, some points intentionally could not be reached due to the increments of the sliders — which pressed students to reason more carefully about the role of the sliders and their values in identifying which points could be reached.

In the second task, students' horizontal mathematization (exploration and description) was driven by a shift in both the tasks and the tools in the DGE: they had to make sense of red points left as traces of ordered pairs of values that had been taken on by the resultant vector. Each student referred to the mathematical formulation of linear combinations of vectors even though they were not familiar with the notion, and both considered the geometry of lines. Geometric elaboration of the situation (e.g., set of red points, line, plane) was intertwined with the algebraic formulation and students reasoned about the link between the x-y plane and linear combinations of the given vectors. This facilitated vertical mathematization in which students considered the set of all points that could be reached when both sliders vary through all real numbers – a context in which their models of adjusting slider values to express points as linear combinations of vectors came to function as a model-for reasoning that any point in the plane could be reached, connecting to their ideas about lines and planes. Specifically, we interpreted the students' conclusion that all points in the plane would be traced out by varying the slider values (weights) for the two vectors as a form of formalization of the idea of span.

In the third task, students' horizontal mathematization was similar to that noted in the previous two tasks, but the new setting (shifting to a three-dimensional setting) afforded connections to students' prior work and knowledge. The view that all linear combinations of a single vector in $\mathbb{R}^3$ form a line seemed natural for both students (presumably supported by tools and previous tasks). Students' ideas about (models-of) taking all possible linear combinations of vectors in $\mathbb{R}^2$ (e.g., by animating sliders) served as a model for doing so in $\mathbb{R}^3$ – and both students denoted this line with a parametric vector equation by relating points on the line to the vector that was being weighted by the slider.

In the fourth and final task analyzed in this paper, students' horizontal mathematization was again similar to that observed in Tasks 1–3. Both students leveraged aspects of models from their prior mathematical activities of taking linear combinations of vectors by animating sliders (in both two and three dimensions), and vertically extended their reasoning from Task 2 in which they reasoned that a plane would be formed by taking all linear combinations of the two given vectors. Additionally, their model-of expressing all linear combinations of a single vector in three dimensions to describe a line using a parametric vector equation, vertically became a model-for expressing all linear combinations of the two given vectors in three-dimension to describe a plane using a parametric vector equation. However, both students struggled to resolve this representation of a plane with their prior knowledge of other algebraic representations of a plane.

5.2. The function of DGE in symbolizing linear combination and span

The DGE functioned to support students' progressive symbolization, which functioned to help students' construct and coordinate meaning for linear combinations of vectors in important ways. In the first two tasks, students' mathematization was primarily horizontal in nature as they reasoned about linear combinations of two (linearly independent) vectors in $\mathbb{R}^2$. Further, their reasoning was primarily geometric in nature — and their symbolizing activity consisted primarily of using (exploring/experimenting with) and interpreting (describing) symbolizations that appeared in the DGE. Importantly, these interactions provided feedback that helped students develop productive geometric interpretations of vectors, scalars, individual sums of scaled vectors, and multiple sums of scaled vectors. In Task 1, the activity of dragging sliders helped students develop geometric meaning for the way in which scalars and vectors interact, as well as the way in which sums of scaled vectors (e.g., linear combinations) can be formed to represent some points in the plane (and fail to represent others when scalar or slider values are restricted). In Task 2, the trace and animate functions allowed students to co-construct geometric symbolizations of multiple linear combinations to reason toward coordinating ALL possible linear combinations of the two given vectors in the plane (and link these to lines and planes in an informal geometric way).

Thus, we see a sort of "micro shift" from Task 1 to Task 2 wherein students' (primarily geometric) model-of individual linear combinations comes to function as a model-for reasoning about multiple linear combinations. The trace and animate functions are also critical in supporting this shift. This shift and sequencing are similar to the design employed in Wawro et al.'s (2012) Magic Carpet Ride task, where students first reason about how linear combinations of vectors can be used to arrive at a single location in the first task, before reasoning about whether there is any location in the plane that can NOT be reached (e.g., essential reasoning about whether the whole plane can be reached) in the second tasks. An important distinction of the work in this study is that the DGE provided feedback to students rather than requiring students to immediately construct, interpret, and test their own symbolic representations against the context. This form of support might be of particular importance for students who come in with limited experience or confidence in developing and manipulating their own symbolic representations of a mathematical situation (whether those are geometric or algebraic in nature.).

In the third and fourth tasks, students' mathematization was primarily vertical in nature as they leveraged the geometric imagery and organization of reasoning about linear combinations of vectors (e.g., in relation to lines and planes) developed in the first two tasks and shifted into the context of $\mathbb{R}^3$. Further, students' symbolizing activity in these latter tasks was heavily algebraic in nature but was critically organized around the geometric intuitions linking linear combinations of vectors to lines and planes developed in the first two tasks in the setting of $\mathbb{R}^2$ (prominently featuring connect/relate codes that linked exploration/experimentation and description toward formalization). The increased use of the Algebra window was particularly important as students worked to link algebraic symbolization of vector equations in $\mathbb{R}^3$ to their geometric intuitions for lines and planes (from Tasks 1 and 2) as well as their prior knowledge about symbolizing lines and planes in $\mathbb{R}^3$ (from previous course content focused on systems of linear equations).

Task 3 heavily drew on the connections students developed for geometrically coordinating scalars and vectors (by dragging sliders, and through the trace/animate functions). Importantly, this task emphasizes linear combinations (e.g., scalings) of just a single vector in $\mathbb{R}^3$ which seems productive for helping them make sense of how the individual points on the line traced out can be represented through manipulations of the parametric vector equation (e.g., an expression for the x, y, and z component can be written in terms of a single parameter). Task 4 leveraged both interpretations from Task 2 (tracing out a plane with a pair of vectors) and Task 3 (representing objects in $\mathbb{R}^3$ using parametric vector equations). Interestingly, both students considered sums of vectors before weighted sums of vectors in their symbolization – and both additionally experienced some disconnect between the parametric vector equation of a plane and their expectation that the plane should be captured in a particular closed form expression.

5.3. The function of "tool use" in symbolizing linear combination and span

When considering this work in relation to the literature, this work documents important shifts from $\mathbb{R}^2$ to $\mathbb{R}^2$ as students consider linear combinations of sets of vectors prior to any formal instruction on the notion of span — and the rich potential of DGEs for supporting students' pre-symbolizing activity; connections between geometric and algebraic interpretations of vectors, scalars, and linear combinations; and shifts from two to three-dimensional settings. Further, the drag function emerged as particularly critical for supporting students' coordination of scalars with vectors (in forming initial conceptions of vector scaling and summing involved in linear combinations). The trace and animate functions emerged particularly useful for supporting students' reasoning about large sets of linear combinations of vectors, which is of critical importance as students come to coordinate ALL possible linear combinations to reason about span. As a result, we have replicated that careful use of dynamic geometry contexts could provide meaningful learning environments, which was in parallel to the existing literature (Gol Tabaghi & Sinclair, 2013; Oktaç, 2018; Turgut, 2018b, 2019).

5.4. The function of RME perspective

In this research, the experientially real task setting for (geometric) vectors seemed to work well. But it is important to note that even though students' movements were somehow smooth in teacher-student interaction, we conjecture this was largely based on their experience in DGE use and their prior knowledge of the geometry of lines and planes. In the present case, for example, knowledge for the parametric equation for a line carried the discussion to characterizing planes as a set of points obtained by dynamic vectors, which can be considered as (mathematical) sense-making for the notion of span. A second key topic was a system of linear equations and the notion of a parameter, which helped students to arrive at a general equation for planes. Note that these might be needed in the implementation for task sequence, and it may not be similarly smooth in large groups since it is important to know student's background knowledge, as well as their prior DGE experience.

In our case, an RME perspective assigned a specific role to the teacher as a guide for student learning and it was of crucial importance. The teacher's prepared and spontaneous questions in the interviews carried the discussion to the critical points for student learning. For example, one student concluded that the obtained figure was going to be a plane, but the student had no idea how to link the vector equation with it. However, the teacher was aware that the student knew the plane equation and asked a question to this end. This contributed to the emergence of a combined view for algebra and geometry to approach the notion of span. However, this is consistent with research documenting the critical role of instructors in classroom interventions in teaching linear algebra (Andrews-Larson et al., 2017; Stewart et al., 2019).

Given the critical role of linear combinations and span as the conceptual structure at the heart of linear algebra (e.g., viewing vector spaces and subspaces as non-empty sets of vectors that are closed under linear combinations) (Parraguez & Oktaç, 2010), we are excited about the potential of this task sequence to contribute to our understanding of the potential of digital tools for supporting students' symbolizing and mathematizing activity toward coordinated conceptions of linear combinations.

CRediT authorship contribution statement

Melih Turgut: Conceptualization, Methodology, Resources, Investigation, Formal analysis, Visualization, Writing – original draft, Writing – review & editing. **Jessica Lynn Smith:** Conceptualization, Methodology, Formal analysis, Visualization, Writing – original draft, Writing – review & editing. **Christine Andrews-Larson:** Conceptualization, Methodology, Formal analysis, Visualization, Writing – original draft, Writing – review & editing.

Declarations of interest

None.

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