Odin Hoff Gardå

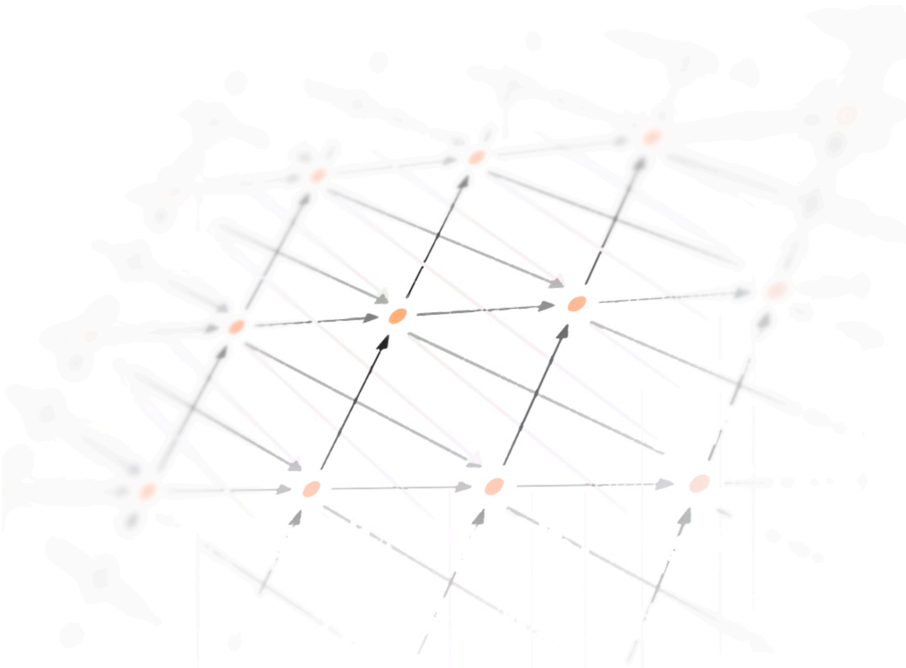
Multicomplexes and their spectral sequences

On the spectral sequence associated with a multicomplex over a field

Master's thesis in Mathematical Sciences
Supervisor: Markus Szymik
January 2022

Master's thesis

NTNU
Norwegian University of Science and Technology
Faculty of Information Technology and Electrical Engineering
Department of Mathematical Sciences



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I want to thank my supervisor Markus Szymik for his many suggestions and encouragement throughout this project. I also wish to thank my fellow students, family, and partner Elina for all their support.

> *"Oh, he seems like an okay person, except for being a little strange in some ways. All day he sits at his desk and scribbles, scribbles, scribbles. Then, at the end of the day, he takes the sheets of paper he's scribbled on, scrunges them all up, and throws them in the trash can."*
>
> — J. von Neumann's housekeeper, describing her employer

Abstract

Double complexes over a field are well-understood, and so are their associated spectral sequences. Multicomplexes generalise the notion of double complexes. We aim to understand the spectral sequence associated with a multicomplex over a field. The homotopy transfer theorem (HTT) equips the cohomology of the underlying cochain complex of a multicomplex with a transferred multicomplex structure. We characterise first-page degeneration in terms of these transferred differentials and then provide a method for computing the spectral sequence page-by-page by repeated application of the HTT.

Sammendrag

Multikomplekser generaliserer kjedekomplekser og dobbeltkomplekser. Vi ser hovedsaklig på spektralfølgen tilhørende et multikompleks over en kropp og gir betingelser for degenerasjon på første side i spektralfølgen. Videre ser vi på differensialene på senere sider i spektralfølgen og beskriver en fremgangsmåte for å regne ut disse eksplisitt ved hjelp av homotopioverføringsteoremet (homotopy transfer theorem).

Contents

Introduction

A multicomplex $(M, D_\bullet)$ over a field $\mathbb{K}$ is a bigraded $\mathbb{K}$-vector space M together with a family of endomorphisms $D_0, D_1, D_2, \ldots$ of degrees $|D_r| = (r, 1-r)$. These maps are required to satisfy the relations

$$D_0 D_n + D_1 D_{n-1} + \cdots + D_{n-1} D_1 + D_n D_0 = 0 \quad \text{for all } n \geqslant 0.$$

Set $n = 0$, and the relation above becomes $D_0 D_0 = 0$. Consequently, every multicomplex $(M, D_\bullet)$ has an underlying cochain complex (M, D_0) of graded vector spaces. Given a multicomplex M, we define the cohomology complex $H(M)$ by taking cohomology of the underlying cochain complex (M, D_0) degreewise.

$$H^{p,q}(M) = \frac{\ker(D_0 \colon M^{p,q} \longrightarrow M^{p,q+1})}{\operatorname{Im}(D_0 \colon M^{p,q-1} \longrightarrow M^{p,q})}$$

We equip $H(M)$ with the trivial differential so that it (trivially) becomes a cochain complex of graded vector spaces. Over a field, every cochain complex (M, D_0) is homotopy equivalent to its cohomology complex. There exists a slightly weaker notion of homotopy equivalence called a homotopy retract. If (M, D) and (N, D') are cochain complexes, then a homotopy retract data consists of cochain maps $\pi \colon M \longrightarrow N$, $\iota \colon N \longrightarrow M$ and a homotopy $h \colon M \longrightarrow M$ of degree -1 such that $\iota\pi - \mathrm{id}_M = Dh + hD$ and ι is a quasi-isomorphism.

$$\begin{tikzcd} (M, D) \arrow[loop left, "h"] \arrow[r, bend left, "\pi"] & (N, D'). \arrow[l, bend left, "\iota"] \end{tikzcd}$$

Over a field, every homotopy retract data can be extended to an equivalence, so the two notions are equivalent in this case. The homotopy transfer theorem (HTT) for multicomplexes tells us that if $(M, D_\bullet)$ is a multicomplex and we have a homotopy retract data as above, we can use the maps π, ι and h to define maps $D'_1, D'_2, \ldots \colon N \longrightarrow N$ compatible with $D'_0 := D'$ in the sense that $(N, D'_\bullet)$ becomes a multicomplex. In particular, since there always exists a homotopy retract data of (M, D_0) to the cohomology complex $H(M)$, we can transfer the multicomplex structure on M to one on $H(M)$. Said differently, the HTT allows us to lift the cohomology functor $H(-) \colon \mathbf{Ch}(\mathbf{Vect}^{\mathbb{Z}}) \to \mathbf{Ch}(\mathbf{Vect}^{\mathbb{Z}})$ to $\mathbf{MC}_{\mathbb{K}}$, where $\mathbf{Ch}(\mathbf{Vect}^{\mathbb{Z}})$ and $\mathbf{MC}_{\mathbb{K}}$ denote the categories of cochain complexes of graded vector spaces and multicomplexes over $\mathbb{K}$, respectively. We can think of this as the following commutative diagram, where the vertical arrows send a multicomplex to its underlying cochain complex.

$$\begin{tikzcd} \mathbf{MC}_{\mathbb{K}} \arrow[r, dotted, "H(-)"] \arrow[d] & \mathbf{MC}_{\mathbb{K}} \arrow[d] \\ \mathbf{Ch}(\mathbf{Vect}^{\mathbb{Z}}) \arrow[r, "H(-)"] & \mathbf{Ch}(\mathbf{Vect}^{\mathbb{Z}}) \end{tikzcd}$$

Main results

The spectral sequence $(E_\bullet(M), d_\bullet)$ associated with a multicomplex $(M, D_\bullet)$ is constructed similarly as the one associated with a double complex. Namely, the total complex of M enjoys a natural filtration by columns which gives us a spectral sequence by standard results. The E_0-page of this spectral sequence is the underlying cochain complex (M, D_0) and the E_1-page is the complex $H(M)$ equipped with the transferred differential D_1'. In [LWZ20] an alternative description of this spectral sequence is presented. They give a description in terms of witnessed cocycles and coboundaries. This description allows for explicit computation of the differentials in the spectral sequence whenever we have such a family of witnesses. This is the view we adopt for most of this text, and the proofs often boil down to finding the right witnesses. Our main goal is to understand the differentials in this spectral sequence. To do so, we introduce a family of multicomplexes $({}^sM, {}^sD_\bullet)$ indexed by $s \geqslant 1$ defined by repeated application of the HTT. Roughly speaking, the multicomplex $({}^1M, {}^1D_\bullet)$ is a shifted version of the cohomology complex $H(M)$ together with the transferred differentials. We then define $({}^sM, {}^sD_\bullet)$ inductively to be a shifted version of the multicomplex $(H({}^{s-1}M), {}^{s-1}D_\bullet')$. This re-indexing is inspired by the décalage functor introduced by Deligne in [Del71]. Put categorically, if we denote the re-indexing functor by ρ, then the functor ${}^1(-)$ is just the composition $\rho \circ H(-)\colon\ \mathbf{MC}_{\mathbb{K}} \to \mathbf{MC}_{\mathbb{K}}$ and ${}^s(-) = \underbrace{{}^1(-) \circ \cdots \circ {}^1(-)}_{s \text{ times}}$.

The essential observation is the following theorem relating the spectral sequence associated with M and the one associated with ${}^1M$.

Theorem A (Theorem 4.0.6)**.** The spectral sequence associated with ${}^1M$ is a shifted version of the spectral sequence associated with M in the sense that

$$E_r^{p,q}({}^1M) = E_{r+1}^{2p+q,-p}(M) \quad \text{and} \quad {}^1d_r = d_{r+1} \quad \text{for all} \quad r \geqslant 0.$$

What we truly are doing is pushing the multicomplex structure along while computing the pages in the spectral sequence. That is, we equip each page in the spectral sequence with a multicomplex structure coming from the previous page by choice of sections and the HTT. The following corollary of theorem A then tells us that the multicomplex structure on the E_r-page contains both the differential d_r (as the underlying cochain complex) and all the information (the higher differentials) needed to compute d_{r+1}.

Corollary B (Corollary 4.0.7)**.** We have

$$E_r^{p,q}(M) = {}^rM^{p-rn,\, q+rn} \quad \text{and} \quad d_r = {}^rD_0$$

for every $r \geqslant 1$ where $n = p + q$.

If we write $\mathbf{SpecSeq}_{\mathbb{K}}$ for the category of spectral sequences with vector spaces over $\mathbb{K}$ as entries, then theorem A and corollary B corresponds to the commutativity of the inner squares and the outer square in the following diagram, respectively.

$$\begin{tikzcd}
\mathbf{MC}_{\mathbb{K}} \arrow[r, "E(-)"] \arrow[d, "{}^1(-)"] \arrow[ddd, bend right=60, "{}^s(-)"'] & \mathbf{SpecSeq}_{\mathbb{K}} \\
\mathbf{MC}_{\mathbb{K}} \arrow[r, "E(-)"] \arrow[d, "{}^1(-)"] & \mathbf{SpecSeq}_{\mathbb{K}} \arrow[u, "\rho^{-1}"] \\
\vdots \arrow[d, "{}^1(-)"] & \vdots \arrow[u, "\rho^{-1}"] \\
\mathbf{MC}_{\mathbb{K}} \arrow[r, "E(-)"] & \mathbf{SpecSeq}_{\mathbb{K}} \arrow[u, "\rho^{-1}"] \arrow[uuu, bend right=60, "\rho^{-s}"']
\end{tikzcd}$$

Effectively, corollary B enables us to compute the spectral sequence associated with a multicomplex, page-by-page. We point out that a result similar to corollary B can be found in [Lap07, Proposition 3.1] but stated in a slightly different mathematical language. The following corollary, which also follows immediately from theorem A, completely characterises degeneration of the associated spectral sequence in terms of the transferred differentials.

Corollary C (Corollary 4.0.8)**.** The spectral sequence associated with a multicomplex M degenerates at the k-th page if and only if ${}^kD_r = 0$ for all $r \geqslant 0$.

First-page degeneration

The main inspiration which led to theorem A and its corollaries, and especially corollary C, was the following result on first-page degeneration appearing in [DSV15].

Theorem D (Theorem 3.0.1)**.** The spectral sequence associated with a multicomplex M degenerates at the first page if and only if all transferred differentials D_r' vanish.

Of course, once we have established theorem A, then theorem D follows from corollary C by setting $k = 1$. Still, section 3 is entirely dedicated to proving theorem D and is included because it inspires the techniques used to prove the generalisation in section 4.

Specialising to double complexes

Double complexes (also known as bicomplexes) are exactly the multicomplexes whose differentials D_r vanish for $r \geqslant 2$. Double complexes over a field are well understood as they decompose into direct sums of "squares" and "zig-zags".[1] Consequently, the spectral sequence associated with a double complex is also understood since the differentials involved can be computed by considering the zig-zags of different lengths appearing in the decomposition. Another approach to computing the spectral sequence associated with a double complex is applying the HTT and considering the transferred differentials on the cohomology complex. As pointed out in [LV12], this approach using the HTT gives us a "lifted version" of the spectral sequence. To be precise, if $(E_\bullet, d_\bullet)$ denotes the spectral sequence associated with the double complex (M, D_0, D_1), and $(D_r')_{r\geqslant 1}$ are the transferred differentials on $H(M, D_0)$, then we have the following result:

Theorem E. The map induced by D_r' on the E_r-page is exactly d_r.

The takeaway is that, in the case of double complexes, the transferred differentials on cohomology contain all the information of the associated spectral sequence (except the zeroth page, of course). We give two proofs of theorem E. The first one appears as proposition 2.3.1. Later, in example 4.0.9, we recover this result by applying corollary B. Theorem E fails in the general case where higher differential might be non-trivial. This is seen in example 4.0.3 where $D_3' = 0$ but d_3 is non-trivial. This suggests that there has to be more information contained in d_r than just the transferred differentials D_r' on $H(M, D_0)$ whenever $r \geqslant 3$.

[1]This decomposition of double complexes has been known as folklore for a long time. Recently, proofs of this fact have been given in [Ste21] and [KQ20].

Minimal models for multicomplexes

Similarly to how a cochain complex M can be decomposed into a direct sum $K \oplus H$, this is also true for multicomplexes. A multicomplex $(M, D_\bullet)$ is said to be *minimal* if $D_0 = 0$ and *acyclic trivial* if both $D_r = 0$ for $r \geqslant 1$ and the underlying cochain complex (M, D_0) is acyclic. In appendix B, we follow [DSV15] and show how every multicomplex M decomposes into a direct sum $K \oplus H$ where K is acyclic trivial and H is minimal.

Multicomplexes as homotopy algebras

As is remarked in [DSV15], [Val14] and [LV12], multicomplexes can be viewed as algebras over a certain operad. This is also true for double complexes, which are algebras over the operad $\mathscr{D}$ of dual numbers. First, we define an operad $\mathscr{M}$ and show that the category of algebras[2] over $\mathscr{M}$ is equivalent to the category of multicomplexes with the right definition of morphisms. We then explicitly compute the operad $\mathscr{D}_\infty$ which is the cobar construction on the Koszul dual cooperad of $\mathscr{D}$ and show that $\mathscr{D}_\infty = \mathscr{M}$. Then, by definition, multicomplexes are exactly the homotopy $\mathscr{D}$-algebras. Viewing multicomplexes as homotopy algebras allows us to apply results from the general theory of Koszul operads. For example, both the homotopy transfer theorem and minimal models for multicomplexes follows from more general results which can be found in [LV12].

Deformations of cochain complexes

We fix a cochain complex (M, D_0) and consider those formal power series

$$D(t) = D_0 + D_1 t + D_2 t^2 + \cdots \in \mathrm{End}(M)[\![t]\!]$$

which satisfy $D^2 = 0$. These are called *formal deformations* of (M, D_0). We prove that formal deformations of cochain complexes form a category equivalent to the category of multicomplexes. We also very briefly mention finite order deformations of cochain complexes. Deformations of cochain complexes might serve as a motivation for looking at multicomplexes and give us a compact notation for encoding them. Of course, the results established in the earlier sections can be translated to the category of deformations. This may be fruitful. However, this direction is not further investigated in this text.

[2]Every operad $\mathscr{P}$ of arity 1 is completely determined by the algebra $\mathscr{P}(1)$. Moreover, algebras over operads of arity 1 correspond to dg-modules over $\mathscr{P}(1)$

How to read it

If one is interested in the most general results to be found in this text, one can skip straight to section 4 and refer to section 1 and section 2 whenever necessary. On the other hand, if one wants the extended edition, including the motivation behind the general results, one should read it linearly from section 1. The appendices are not directly related to the main results but might motivate the notion of multicomplexes and reveal parts of the bigger picture they fit into.

Outline

Section 1 contains the preliminaries and prepares us for working with multicomplexes and the associated spectral sequence. First, we recall how a filtered cochain complex gives rise to a spectral sequence and then go on to define multicomplexes and the filtration on the total complex. We end this section with an exposition of witnessed cocycles and coboundaries closely following [LWZ20].

Section 2 consists of three parts. The first part discusses the notions of homotopy equivalence and homotopy retracts for cochain complexes over a field. We prove (the well-known fact) that every cochain complex is homotopy equivalent to its cohomology complex and explicitly write out the maps involved. The second part is dedicated to the homotopy transfer theorem. The third part discusses the special case of double complexes and their associated spectral sequences.

Section 3 is about first-page degeneration and contains the proof of theorem D.

Section 4 starts off by showing that the second differential d_2 in the spectral sequence is the map induced by D_2' on the E_2-page. We then give an example to show that this is not true for d_3. Inspired by how this example fails, we go on to prove theorem A and end the section by proving its two corollaries.

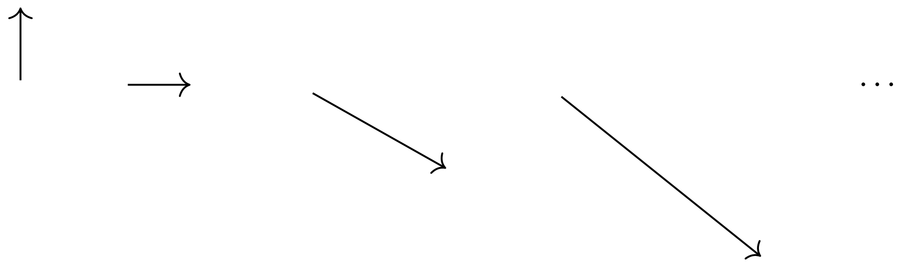
Appendix A explains how multicomplexes can be viewed as homotopy $\mathscr{D}$-algebras.

Appendix B is a short note on minimal models for multicomplexes.

Appendix C shows how (formal) deformations of cochain complexes form a category equivalent to the category of multicomplexes. We also briefly mention finite order deformations of cochain complexes at the end.

Conventions

We restrict ourselves to the case of (graded) vector spaces over some fixed field $\mathbb{K}$ of characteristic 0. Some results work in the more general setting. However, most of the results rely on the fact that every short exact sequence splits. We stick to cohomological grading for complexes, multicomplexes and spectral sequences. That is, differentials always increase the (total) degree by one. In the bigraded case, the differentials have bidegree $(r, 1-r)$ and can be visualised as follows:



We write J for multi-indices $(j_1, j_2, \ldots, j_k)$ of length $k \geqslant 1$ with all $j_s \geqslant 1$. When summing over $|J| = n$, we sum over all such multi-indices with $j_1 + \cdots + j_k = n$. For example, if (a_i) is some family indexed over the positive integers, then

$$\sum_{|J|=1} a_{j_1}a_{j_2}\cdots a_{j_k} = a_1, \quad \sum_{|J|=2} a_{j_1}a_{j_2}\cdots a_{j_k} = a_1a_1 + a_2,$$
$$\sum_{|J|=3} a_{j_1}a_{j_2}\cdots a_{j_k} = a_1a_1a_1 + a_1a_2 + a_2a_1 + a_3 \quad \text{and so on.}$$

1 Multicomplexes and spectral sequences

First, we briefly recall how a filtration on a cochain complex gives rise to a spectral sequence. More details and omitted proofs can be found in [Spa95], [Wei95] and [McC01]. Next, we clarify the notion of a multicomplex and morphisms of such. Under a mild boundedness assumption on a multicomplex M, we can construct the associated total complex $\operatorname{Tot} M$, equipped with a natural filtration. The spectral sequence arising from this filtration is what we will call *the spectral sequence associated with* M. In the last part, we follow [LWZ20] and describe the associated spectral sequence in terms of witnessed cocycles and coboundaries. This point of view will enable us to describe the differentials involved explicitly.

1.1 Filtered cochain complexes

Definition 1.1.1. A *filtered cochain complex* (C, F) is a cochain complex C with differential $D\colon C^n \to C^{n+1}$ together with a decreasing filtration $F = \{F^p C\}_p$

$$C \geqslant \cdots \geqslant F^p C \geqslant F^{p+1} C \geqslant \cdots \geqslant 0$$

such that the differential on C is compatible with the filtration in the sense that $D(F^p C^n) \leqslant F^p C^{n+1}$.

Given a filtered cochain complex (C, F) we define

$$Z_r^{p,\bullet} = F^p C \cap D^{-1}(F^{p+r} C), \quad B_0^{p,\bullet} = Z_0^{p+1,\bullet} \quad \text{and} \quad B_r^{p,\bullet} = Z_{r-1}^{p+1,\bullet} + D Z_{r-1}^{p-r+1,\bullet}$$

for $r \geqslant 1$. It can be shown that the quotients $E_r^{p,q} = Z_r^{p,q} / B_r^{p,q}$ are well-defined and that the differential on C induces differentials δ_r:

$$\begin{tikzcd} Z_r^{p,q} \arrow[r, "D"] \arrow[d, two heads] & Z_r^{p+r,q+1-r} \arrow[d, two heads] \\ E_r^{p,q} \arrow[r, dotted, "\delta_r"] & E_r^{p+r,q+1-r} \end{tikzcd}$$

where the vertical arrows are the quotient maps. The main point is that we have a spectral sequence. The following result is standard:

Proposition 1.1.2. There are isomorphisms $E_{r+1}^{p,q} \cong H^{p+q}(E_r, \delta_r)$ for $r \geqslant 0$.

If x is an element of $Z_r^{p,q}$, we denote its class in $E_r^{p,q} = Z_r^{p,q} / B_r^{p,q}$ by $[x]_r$. Using this notation, we have that $\delta_r([x]_r) = [Dx]_r$.

Definition 1.1.3. A filtration F on C is said to be *convergent* if $\bigcap_n F^n C = 0$ and $\bigcup_n F^n C = C$. We say that F is *bounded below* if for every n, there exists a $q(n)$ such that $F^{q(n)} C^n = 0$.

If (C, F) is a filtered cochain complex, then there is an induced filtration on the cohomology of C given by $F^p H(C) := \mathrm{Im}(H(F^p C) \to H(C))$. Requiring that F is bounded below ensures that the induced filtration on $H(C)$ is convergent as well.

Theorem 1.1.4. [Spa95, p.469] Let (C, F) be a filtered cochain complex with F convergent and bounded below. Then there is a convergent spectral sequence with

$$E_0^{p,q} = F^p C^{p+q} / F^{p+1} C^{p+q}, \quad E_1^{p,q} = H^{p+q}(F^p C / F^{p+1} C)$$

and E_∞ is isomorphic to the associated graded of the induced filtration on $H(C)$.

The following notion of degeneracy will be central throughout this text.

Definition 1.1.5. A spectral sequence $(E_\bullet, d_\bullet)$ is said to *degenerate at the k-th page* if $d_r = 0$ for all $r \geqslant k$.

Degeneration on some page merely means that nothing interesting will happen from that point on. That is, we have arrived at our target E_∞.

1.2 Multicomplexes and their associated spectral sequences

Definition 1.2.1. A *multicomplex* $(M, D_\bullet)$ over $\mathbb{K}$ consists of a bigraded $\mathbb{K}$-vector space $M = \{M^{p,q}\}$ together with a family of linear maps $\{D_r : M \to M\}_{r \geqslant 0}$ of bidegrees $|D_r| = (r, 1-r)$. These maps are required to satisfy the relation

$$\sum_{p+q=n} D_p D_q = 0 \quad \text{for every } n \geqslant 0.$$

The maps D_r when $r \geqslant 1$ are called *higher differentials* (or sometimes, even just *differentials*). This is a slight abuse of terminology as they in general do not square to zero. Multicomplexes generalise the notion of double complexes and cochain complexes. A multicomplex where the higher differentials $D_r = 0$ for all $r \geqslant 2$ is precisely a double complex. A multicomplex with only D_0 possibly non-trivial is a cochain complex of graded vector spaces. For every multicomplex $(M, D_\bullet)$, we have an underlying cochain complex (M, D_0) of graded vector spaces. One can visualise (M, D_0) as the following diagram:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots & M^{p-1,q+1} \arrow[u] & M^{p,q+1} \arrow[u] & M^{p+1,q+1} \arrow[u] & \cdots \\
\cdots & M^{p-1,q} \arrow[u] & M^{p,q} \arrow[u, "D_0"] & M^{p+1,q} \arrow[u] & \cdots \\
\cdots & M^{p-1,q-1} \arrow[u] & M^{p,q-1} \arrow[u] & M^{p+1,q-1} \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

We denote the cohomology of (M, D_0) in degree (p, q) by

$$H^{p,q}(M) := H^q(M^{p,\bullet}, D_0) = \frac{\ker(D_0 \colon M^{p,q} \to M^{p,q+1})}{\operatorname{Im}(D_0 \colon M^{p,q-1} \to M^{p,q})}$$

and endow $H(M)$ with trivial differential.

Remark 1.2.2. Multicomplexes (not necessarily over a field) appear in [Wal61] where resolutions for extensions of groups are constructed as the total complex of a multicomplex with $D_r = 0$ for $r \geqslant 3$. Such multicomplexes also appear in [Liu17] and [Liu14] under the name "homotopy double complexes", where they are used to construct resolutions of certain generalised Weyl algebras.

Definition 1.2.3. Let $(M, D_\bullet)$ and $(N, \tilde{D}_\bullet)$ be multicomplexes. A *morphism*

$$f \colon (M, D_\bullet) \to (N, \tilde{D}_\bullet)$$

of multicomplexes consists of a family $f = \{f_n \colon M \to N \mid f_n(M^{p,q}) \leqslant N^{p+n,q-n}\}_{n \geqslant 0}$ of linear maps . In addition, we require the maps to satisfy

$$\sum_{p+q=n} f_p D_q = \sum_{p+q=n} \tilde{D}_p f_q \quad \text{for all } n \geqslant 0. \tag{1}$$

For $n = 0$, eq. (1) amounts to f_0 being a cochain map $(M, D_0) \to (N, \tilde{D}_0)$. In the case where M and N are double complexes and $f_n = 0$ for $n \geqslant 1$, eq. (1) is to say that f_0 is a morphism of double complexes, i.e. f commutes with both differentials. On a multicomplex M, the *identity morphism* $\mathrm{id}_M \colon M \to M$ is given by $(\mathrm{id}_M)_0 = \mathrm{id}_{(M,D_0)}$ and $(\mathrm{id}_M)_n = 0$ for all $n \geqslant 1$. If f and g are morphisms of multicomplexes, we define their *composition* gf by $(gf)_n = \sum_{p+q=n} g_p f_q$ whenever it makes sense.

Remark 1.2.4. In the operadic language of [DSV15] morphisms of multicomplexes are called ∞-morphisms.

Definition 1.2.5. A morphism f of multicomplexes is an *isomorphism (quasi-isomorphism)* if f_0 is an isomorphism (quasi-isomorphism).

Proposition 1.2.6. A morphism f of multicomplexes is invertible if and only if f_0 is an isomorphism of cochain complexes.

Proof. If f has inverse g, then $(fg)_0$ is the identity on (M, D_0). Similarly, the same is true for gf. Conversely, suppose f_0 is an isomorphism and denote its inverse by g_0. It is now a matter of solving $(gf)_n = 0$ for each $n \geqslant 1$. Doing this, we obtain the following unique solution g:

$$g_n = \sum_{|I|=n} (-1)^k g_0 f_{i_1} g_0 f_{i_2} g_0 \cdots g_0 f_{i_k} g_0 \quad \text{for } n \geqslant 1.$$

$\square$

Throughout this text we will assume the following boundedness condition on multicomplexes: a multicomplex $(M, D_\bullet)$ is said to be *bounded below* if for each n there exists an integer $s(n)$ such that $M^{p,n-p} = 0$ whenever $p \geqslant s(n)$. In other words, each anti-diagonal eventually vanish going to the right. We associate to a multicomplex M the *total complex* denoted $\operatorname{Tot} M$ given in degree n by $\operatorname{Tot} M^n := \bigoplus_{a+b=n} M^{a,b}$. We make $\operatorname{Tot} M$ into a cochain complex by giving it the differential $D := \sum_{r \geqslant 0} D_r \colon \operatorname{Tot} M^n \to \operatorname{Tot} M^{n+1}$. It is easy to see that D is locally finite and hence well-defined whenever M is bounded below. There is a natural filtration by columns on $\operatorname{Tot} M$ defined by letting

$$F^p \operatorname{Tot} M^n := \bigoplus_{\substack{a+b=n \\ a \geqslant p}} M^{a,b}. \tag{2}$$

This filtration turns $(\operatorname{Tot} M, F)$ into a filtered complex.

Remark 1.2.7. Convergence in the case where we do not assume any finiteness condition on the multicomplex is discussed in [Boa98, Section 11].

Lemma 1.2.8. If $(M, D_\bullet)$ is bounded below, then the filtration defined in eq. (2) is bounded below.

Proof. By assumption, there exists for every n an integer $s(n)$ such that $M^{p,n-p} = 0$ whenever $p \geqslant s(n)$. By letting $q(n) := s(n)$, it is evident that $F^{q(n)}C^n = 0$. $\square$

The following corollary of theorem 1.1.4 now follows from the previous lemma:

Corollary 1.2.9. Let $(M, D_\bullet)$ be a multicomplex which is bounded below. Then the spectral sequence associated with the filtration on $\operatorname{Tot} M$ converges to $H(\operatorname{Tot} M)$.

The spectral sequence in the above corollary is called *the spectral sequence associated with the multicomplex M*. The zeroth page of this spectral sequence is given by

$$E_0^{p,q} = F^p M^{p+q} / F^{p+1} M^{p+q} = M^{p,q}$$

with differential $\delta_0 = D_0$. In other words, E_0 is just the underlying cochain complex (M, D_0). The first page is the given by $E_1^{p,q} = H^q(M^{p,\bullet}, D_0)$ and δ_1 is the map induced by D_1 on cohomology. We should be aware that this pattern does not generally hold for the differentials on later pages, as can be seen in the following example borrowed from [Hur10]:

Example 1.2.10. Consider the following multicomplex M consisting of one-dimensional vector spaces:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \mathbb{K}a \arrow[u] \arrow[r, "D_1"] & \mathbb{K}c \arrow[u] \arrow[r] & 0 & \\
 & 0 \arrow[u] \arrow[r] & \mathbb{K}b \arrow[u, "D_0"] \arrow[r, "D_1"] & \mathbb{K}d \arrow[u] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

with differentials $D_1(a) = c$, $D_1(b) = d$, $D_0(b) = c$ and $D_r = 0$ for all $r \geqslant 2$. The total complex

$$0 \to \mathbb{K}b \oplus \mathbb{K}a \xrightarrow{\begin{pmatrix} D_1 & 0 \\ D_0 & D_1 \end{pmatrix}} \mathbb{K}d \oplus \mathbb{K}c \to 0.$$

is exact because the differential $D = D_0 + D_1$ is an isomorphism. By corollary 1.2.9 this means that $E_\infty^{p,q} = 0$ for all p, q. Taking cohomology with respect to $\delta_0 = D_0$, we are left with two non-trivial entries at the E_1-page:

$$\begin{tikzcd} 0 \arrow[r, "\delta_1"] & \mathbb{K}\, a \arrow[r, "\delta_1"] \arrow[drr, "\delta_2 \neq 0"] & 0 & & \\ & & 0 \arrow[r, "\delta_1"'] & \mathbb{K}\, d \arrow[r, "\delta_1"'] & 0 \end{tikzcd}$$

Since $E_3 = E_\infty$ by degree reasons, the generators a and b must be killed by the δ_2 differential and hence $\delta_2 \neq 0$, whereas D_2 is zero.

1.2.1 Description in terms of witnessed cocycles and coboundaries

This part is essentially [LWZ20] with notation and grading conventions adapted to our setting. Let $(M, D_\bullet)$ be a multicomplex and $T = \operatorname{Tot} M$. The filtration on T defined as in eq. (2) is denoted by F. If $x \in F^pT$, we can write x uniquely as the sum

$$x = x_p + x_{p+1} + \cdots + x_{p+r-1} + x' \tag{3}$$

where $x_{p+j} \in M^{p+j,\bullet}$ is the projection of x to the column $M^{p+j,\bullet}$ and $x' \in F^{p+r}T$. Now, suppose that $x \in Z_r^{p,\bullet}$. This is to say that $x \in F^pT$ and $Dx \in F^{p+r}T$ where D is the differential on T. As a consequence of eq. (3), the parts of Dx which lie in $F^{p+j}T$ for $j = 0, 1, \ldots, r-1$ must vanish. In other words, the following equations are required to hold true.

$$\begin{aligned} D_0 x_p &= 0 \\ D_0 x_{p+1} + D_1 x_p &= 0 \\ &\vdots \\ D_0 x_{p+r-1} + D_1 x_{p+r-2} + \cdots + D_{r-1} x_p &= 0. \end{aligned} \tag{4}$$

Similarly, if we let $x = Dw$ for some $w \in F^{p-r+1}T$ we obtain another set of equations. These observations lead to the definition of the following two subspaces of $M^{p,\bullet}$:

$$\begin{aligned} \mathcal{Z}_r^{p,\bullet} = \Big\{\, x \in M^{p,\bullet} \mid \exists\, x_{p+j} \in M^{p+j,\bullet} \quad & \text{for } 1 \leqslant j \leqslant r-1 \text{ such that} \\ D_0 x = 0 \quad \text{and } D_n x + \sum_{i=0}^{n-1} D_i x_{p+n-i} = 0 \quad & \text{for } 1 \leqslant n \leqslant r-1 \Big\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_r^{p,\bullet} = \Big\{\, x \in M^{p,\bullet} \mid \exists\, w_{p-j} \in M^{p-j,\bullet} \quad & \text{for } 0 \leqslant j \leqslant r-1 \text{ such that} \\ x = \sum_{j=0}^{r-1} D_j w_{p-j} \quad \text{and } \sum_{j=l}^{r-1} D_{j-l} w_{p-j} = 0 \quad & \text{for } 1 \leqslant l \leqslant r-1 \Big\}. \end{aligned}$$

Proposition 1.2.11. [LWZ20, Proposition 2.7] We have that $\mathcal{B}_r^{p,\bullet} \leqslant \mathcal{Z}_r^{p,\bullet}$.

Proof. Pick some coboundary $x \in \mathcal{B}_r^{p,\bullet}$. First, we confirm that $D_0 x = 0$:

$$D_0 x = \sum_{j=0}^{r-1} D_0 D_j w_{p-j} = -\sum_{j=1}^{r-1}\sum_{l=0}^{j-1} D_{j-l} D_l w_{p-j} = -\sum_{l=1}^{r-1} D_l \underbrace{\sum_{j=l}^{r-1} D_{j-l} w_{p-j}}_{=0} = 0.$$

Next, we claim that the elements $x_{p+j} = \sum_{i=0}^{r-1} D_{j+i} w_{p-i} \in M^{p+j,\bullet}$ where $1 \leqslant j \leqslant r-1$ satisfy the required relations for x to be a cocycle. Again, we check this by direct computation. Let $x_p := x$ to simplify expressions.

$$\begin{aligned}
\sum_{i=0}^{n} D_i x_{p+n-i} &= \sum_{i=0}^{n} D_i \sum_{k=0}^{r-1} D_{n-i+k} w_{p-k} = \sum_{k=0}^{r-1}\sum_{i=0}^{n} D_i D_{n+k-i} w_{p-k} \\
&= -\sum_{k=1}^{r-1}\sum_{i=n+1}^{n+k} D_i D_{n+k-i} w_{p-k} = -\sum_{i=n+1}^{n+r-1} D_i \underbrace{\sum_{k=i-n}^{r-1} D_{k-i+n} w_{p-k}}_{=0} = 0.
\end{aligned}$$

□

Proposition 1.2.12. [LWZ20, Proposition 2.8] The map $\psi\colon E_r^{p,q} \to \mathcal{Z}_r^{p,\bullet}/\mathcal{B}_r^{p,\bullet}$ defined by letting $\psi([x]_r) = [x_p]$ is an isomorphism.

Proof. Consider the map $\hat{\psi}\colon Z_r^{p,\bullet} \to \mathcal{Z}_r^{p,\bullet}/\mathcal{B}_r^{p,\bullet}$ defined by $x \mapsto [x_p]$.

1. The map $\hat{\psi}$ is well-defined and surjective:

Let $x \in Z_r^{p,\bullet}$. From the decomposition in eq. (3) and the relations in eq. (4) we see that $x_p \in \mathcal{Z}_r^{p,\bullet}$, i.e., $\hat{\psi}(Z_r^{p,\bullet}) \leqslant \mathcal{Z}_r^{p,\bullet}$. Now, if the cocycle $x \in \mathcal{Z}_r^{p,\bullet}$ is witnessed by the elements $x_{p+1}, \dots, x_{p+r-1}$, then the image of $y := x + x_{p+1} + \cdots + x_{p+r-1}$ under $\hat{\psi}$ is exactly x. The only thing left to check is that $y \in Z_r^{p,\bullet}$, but this follows from eq. (4).

2. We have inclusion $\ker\hat{\psi} \leqslant B_r^{p,\bullet}$*:*

Suppose that $x \in \ker\hat{\psi} \leqslant Z_r^{p,\bullet}$. From eq. (3), it follows that we can write $x = x_p + w$ where $x_p \in M^{p,\bullet}$ and $w \in F^{p+1}T$. By assumption, $x_p \in \mathcal{B}_r^{p,\bullet}$ so there exist witnesses $w_p, w_{p-1}, \dots, w_{p-r+1}$ with $w_{p-j} \in M^{p-j,\bullet}$ satisfying

$$x_p = \sum_{j=0}^{r-1} D_j w_{p-j} \qquad \text{and} \tag{5}$$

$$0 = \sum_{j=l}^{r-1} D_{j-l} w_{p-j} \quad \text{for } 1 \leqslant l \leqslant r-1. \tag{6}$$

Define the element $c := \sum_{k=0}^{r-1} w_{p-k} \in F^{p-r+1}T$. From eq. (6) it follows that $Dc \in F^pT$ and so by definition we have that $c \in Z_{r-1}^{p-r+1,\bullet}$. Furthermore, eq. (5) implies that the part of Dc which lives in degree p is exactly x_p so $(x_p - Dc)_p = 0$ and hence $x_p - Dc \in F^{p+1}T$. Define the element $b := x_p - Dc + w \in F^{p+1}T$. We can now write $x = b + Dc$ and consequently $Dx = Db + D^2c = Db$. By assumption, $x \in Z_r^{p,\bullet}$ so $Dx = Db \in F^{p+r}T$ and therefore $b \in Z_{r-1}^{p+1,\bullet}$. We conclude that $x \in Z_{r-1}^{p+1,\bullet} + D(Z_{r-1}^{p-r+1,\bullet}) = B_r^{p,\bullet}$.

3. We have inclusion $B_r^{p,\bullet} \leqslant \ker \hat{\psi}$:

Let $x \in B_r^{p,\bullet}$ and write $x = b + Dc$ with $b \in Z_{r-1}^{p+1,\bullet}$ and $c \in Z_{r-1}^{p-r+1,\bullet}$. By definition, we have $b \in F^{p+1}T$ and $Dc \in F^pT$. Now, observe that $x_p = (Dc)_p = \sum_{j=0}^{r-1} D_j w_{p-j}$. Furthermore, $0 = (Dc)_{p-l} = \sum_{j=l}^{r-1} D_{j-l} c_{p-j}$ for each $l = 1, 2, \dots, r-1$. We conclude that $x_p \in \mathcal{B}_r^{p,\bullet}$ and hence $\hat{\psi}(x) = 0$.

It now follows from the first isomorphism theorem that ψ is an isomorphism.

$$\begin{tikzcd}
Z_r^{p,\bullet} \arrow[rr, "\hat{\psi}"] \arrow[dr, two heads] & & \mathcal{Z}_r^{p,\bullet}/\mathcal{B}_r^{p,\bullet} \\
& E_r^{p,\bullet} \arrow[ur, "\psi"] &
\end{tikzcd}$$

□

Proposition 1.2.13. [LWZ20, Theorem 2.10] Under the isomorphism in proposition 1.2.12 the differentials of the spectral sequence are given by

$$\begin{aligned} d_r \colon \mathcal{Z}_r^{p,\bullet}/\mathcal{B}_r^{p,\bullet} &\longrightarrow \mathcal{Z}_r^{p+r,\bullet}/\mathcal{B}_r^{p+r,\bullet} \\ [x] &\mapsto \left[D_r x + \sum_{i=1}^{r-1} D_i x_{p+r-i} \right] \end{aligned}$$

where $x_{p+1}, x_{p+2}, \dots, x_{p+r-1}$ are witnesses for $x \in \mathcal{Z}_r^{p,\bullet}$.

Proof. From the proof of proposition 1.2.12, we know that $x+x_{p+1}+\cdots+x_{p+r-1}$ lives in $Z_r^{p,\bullet}$. Thus, $[x+x_{p+1}+\cdots+x_{p+r-1}]_r \in E_r^{p,\bullet}$ and $\psi[x+x_{p+1}+\cdots+x_{p+r-1}]_r = [x]$. The result now follows by direct computation:

$$\begin{aligned}
d_r[x] &= \psi\delta_r[x+x_{p+1}+\cdots+x_{p+r-1}]_r \\
&= \psi[D(x+x_{p+1}+\cdots+x_{p+r-1})]_r \\
&= [(D(x+x_{p+1}+\cdots+x_{p+r-1}))_{p+r}] \\
&= \left[D_r x + \sum_{i=1}^{r-1} D_i x_{p+r-i}\right].
\end{aligned}$$

□

The following diagram illustrates how the image of x under d_r is computed from a family of witnesses.

$$\begin{tikzcd}
x \arrow[r] \arrow[rrrrddd, bend left, "D_r"] & \bullet & & & \\
& x_{p+1} \arrow[u] \arrow[r] \arrow[rrrdd, bend left, "D_{r-1}"] & \bullet & & \\
& & \ddots \arrow[u] \arrow[r] & \bullet & \\
& & & x_{p+r-1} \arrow[u] \arrow[r, "D_1"] & d_r[x_p]
\end{tikzcd}$$

Throughout the rest of this text, when we talk about the spectral sequence associated with a multicomplex, we will stick to this description in terms of witnessed cocycles and coboundaries. That is, we write $E_r^{p,\bullet} = \mathcal{Z}_r^{p,\bullet}/\mathcal{B}_r^{p,\bullet}$ and denote the differentials of the spectral sequence by d_r for $r \geqslant 0$. Moreover, for an r-cocycle $x \in \mathcal{Z}_r^{p,\bullet}$, we denote the class represented by x in $E_r^{p,\bullet}$ by $[x]_r$. Sometimes, we will leave out the subscript if it is clear from context where the classes live.

Example 1.2.14. If (M, D_0, D_1) is a double complex, then the differentials in the associated spectral sequence is given by $d_0 = D_0$ and $d_r[x] = [D_1 x_{p+r-1}]$ for $r \geqslant 1$ where $x_p := x$.

Proposition 1.2.15. We have inclusions $\mathcal{B}_r^{p,\bullet} \leqslant \mathcal{B}_{r+1}^{p,\bullet}$ and $\mathcal{Z}_{r+1}^{p,\bullet} \leqslant \mathcal{Z}_r^{p,\bullet}$ for all $r \geqslant 1$.

Proof. If $x \in \mathcal{Z}_{r+1}^{p,\bullet}$ is witnessed by the elements $x_{p+1}, \ldots, x_{p+r-1}, x_{p+r}$, then clearly the elements $x_{p+1}, \ldots, x_{p+r-1}$ witnesses that $x \in \mathcal{Z}_r^{p,\bullet}$. Similarly, if $x \in \mathcal{B}_r^{p,\bullet}$ is a coboundary witnessed by the elements $w_p, w_{p-1}, \ldots, w_{p-r+1}$, we can simply define $w_{p-r} := 0$. □

Corollary 1.2.16. Let $x, y \in \mathcal{Z}_r^{p,\bullet}$. If $[x]_{r_0} = [y]_{r_0}$ for some $r_0 \leqslant r$ then $[x]_s = [y]_s$ for all $r_0 \leqslant s \leqslant r$.

Proof. By assumption $x - y \in \mathcal{B}_{r_0}^{p,\bullet} \leqslant \cdots \leqslant \mathcal{B}_s^{p,\bullet} \leqslant \cdots \leqslant \mathcal{B}_r^{p,\bullet}$. □

In particular, if two r-cocycles represent the same class in cohomology, then they also represent the same class on the E_r-page.

2 Homotopy transfer

Let $(M, D_\bullet)$ be a multicomplex. Given a cochain complex (N, D'_0) which is quasi-isomorphic to the underlying cochain complex of M, we can transfer the higher differentials $D_1, D_2, \ldots$ to obtain a multicomplex $(N, D'_\bullet)$. This is the content of the homotopy transfer theorem (HTT) for multicomplexes. In general, we need a homotopy retract from M to N, but as we work over a field, a quasi-isomorphism (actually, even just isomorphic cohomology groups) turns out to be sufficient. We shall be most interested in the case where N is the cohomology complex $H(M, D_0)$ equipped with a trivial differential. Before that, we prove some elementary results about cochain complexes over a field.

2.1 Cochain complexes over a field

Definition 2.1.1. Let (M, D) and (N, D') be cochain complexes over $\mathbb{K}$. A *homotopy equivalence* (f, g, h, h') between M and N consists of cochain maps $f\colon M \leftrightarrow N : g$ together with maps $h\colon M \to M$ and $h'\colon N \to N$ of degree -1 such that

$$gf - \mathrm{id}_M = hD + Dh \quad \text{and } fg - \mathrm{id}_N = h'D' + D'h'.$$

In the graded setting, we require the maps above to be graded maps, i.e., respect the grading. Two cochain complexes are said to be *homotopy equivalent* if there exists a homotopy equivalence between them. A cochain complex is *contractible* if it is homotopy equivalent to the zero complex.

$$\begin{tikzcd} (M,D) \arrow[loop left, "h"] \arrow[r, bend left, "f"] & (N,D'). \arrow[l, bend left, "g"] \arrow[loop right, "h'"] \end{tikzcd}$$

We introduce an intermediate notion between quasi-isomorphism and homotopy equivalence following the terminology used in [LV12] and [DSV15]:

Definition 2.1.2. Let (M, D) and (N, D') be cochain complexes over $\mathbb{K}$. A *homotopy retract* (π, ι, h) of M to N consists of cochain maps $\pi\colon M \to N$, $\iota\colon N \to M$ and a homotopy $h\colon M \to M$ of degree -1 such that $\iota\pi - \mathrm{id}_M = Dh + hD$ and ι (or equivalently π) is a quasi-isomorphism. If in addition we have $\pi\iota = \mathrm{id}_N$, then (π, ι, h) is called a *deformation retract*.

$$\begin{tikzcd} (M,D) \arrow[loop left, "h"] \arrow[r, bend left, "\pi"] & (N,D') \arrow[l, bend left, "\iota"] \end{tikzcd}.$$

Every deformation retract extends to a homotopy equivalence by setting $h' = 0$ and every homotopy equivalence (f, g, h, h') restricts to a homotopy retract (f, g, h). More is true when working over a field: every quasi-isomorphism extends to a homotopy equivalence. Moreover, given the data (π, ι, h) of a homotopy retract, we can find a h' so that (π, ι, h, h') becomes a homotopy equivalence. Whenever (M, D) is a cochain complex and there is no room for confusion, we write $Z^n = Z^n(M)$ for the n-cocycles, $B^n = B^n(M)$ for n-coboundaries and $H^n = H^n(M)$ for the n-th cohomology space.

Proposition 2.1.3. Every cochain complex (M, D) over $\mathbb{K}$ is isomorphic to the cochain complex $K \oplus H$ where $K^n = B^n \oplus B^{n+1}$ with differential $D_K = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and H is the cohomology of M with trivial differential $D_H = 0$.

Proof. Working over a field, the short exact sequences

$$0 \to B^n \to Z^n \to H^n \to 0 \quad \text{and } 0 \to Z^n \to M^n \xrightarrow{D} B^{n+1} \to 0$$

split under a choice of sections. Thus, we can write $M^n = Z^n \oplus \tilde{B}^{n+1} = B^n \oplus \tilde{H}^n \oplus \tilde{B}^{n+1}$ where $\tilde{H}^n$ and $\tilde{B}^{n+1}$ are isomorphic to H^n and B^{n+1} respectively. Let $\tilde{\alpha} \colon \tilde{H}^n \to H^n$ and $\tilde{\beta} \colon \tilde{B}^{n+1} \to B^{n+1}$ be the isomorphisms we obtain from choosing sections. Define the isomorphism

$$\phi := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \tilde{\beta} \\ 0 & \tilde{\alpha} & 0 \end{pmatrix} \colon M^n \to B^n \oplus B^{n+1} \oplus H^n.$$

Clearly, the following diagram with $D' := D_K + D_H = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ commutes.

$$\begin{tikzcd} M^n \arrow[r, "D"] \arrow[d, "\phi"] & M^{n+1} \\ B^n \oplus B^{n+1} \oplus H^n \arrow[r, "D'"] & B^{n+1} \oplus B^{n+2} \oplus H^{n+1} \arrow[u, "\phi^{-1}"] \end{tikzcd}$$

In other words, we have that $M \cong K \oplus H$. Furthermore, the differential D is given by D' under this identification. □

Proposition 2.1.4. The decomposition in proposition 2.1.3 does not depend on the choice of sections.

Proof. Suppose we pick two different sets of sections so that

$$M^n = B^n \oplus \tilde{H}^n \oplus \tilde{B}^{n+1} = B^n \oplus \hat{H}^n \oplus \hat{B}^{n+1}.$$

Denote the isomorphisms coming from these sections by

$$\tilde{\alpha}\colon \tilde{H}^n \to H^n, \quad \tilde{\beta}\colon \tilde{B}^{n+1} \to B^{n+1}, \quad \hat{\alpha}\colon \hat{H}^n \to H^n \quad \text{and} \quad \hat{\beta}\colon \hat{B}^{n+1} \to B^{n+1}.$$

As H is equipped with a trivial differential, the isomorphism $\hat{\alpha}^{-1}\tilde{\alpha}\colon \tilde{H} \to \hat{H}$ is trivially an isomorphism of cochain complexes. Finally, it is easy to verify that the map

$$\begin{pmatrix} 1 & 0 \\ 0 & \hat{\beta}^{-1}\tilde{\beta} \end{pmatrix}\colon \tilde{K} \to \hat{K}$$

is an isomorphism of cochain complexes. ☐

Throughout this text, when we identify a cochain complex M with the decomposition $K \oplus H$, we will usually leave out the isomorphisms in the proofs above and simply write equality $M = K \oplus H$.

Proposition 2.1.5. Every cochain complex (M, D) over $\mathbb{K}$, admits a deformation retract (π, ι, h) of M to its cohomology.

$$\begin{tikzcd} (M,D) \arrow[loop left, "h"] \arrow[rr, bend left, "\pi"] & & (H(M),0). \arrow[ll, bend left, "\iota"] \end{tikzcd}$$

Proof. By proposition 2.1.3 we can write M as the decomposition $M = K^n \oplus H^n$ where the differential is given by

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\colon K^n \oplus H^n \to K^{n+1} \oplus H^{n+1}.$$

It is straightforward to check that the maps

$$\iota = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \pi = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \quad \text{and } h = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

form a deformation retract as claimed. ☐

The following remark will be essential for us later, as it allows us to apply the homotopy transfer theorem to get a multicomplex structure on the cohomology.

Remark 2.1.6. If $(M, D_\bullet)$ is a multicomplex, we can decompose the underlying cochain complex (M, D_0) as follows. Denote the coboundaries in bidegree (p,q) by $B^{p,q}$. That is, $B^{p,q} = \mathrm{Im}(D_0\colon M^{p,q-1} \to M^{p,q})$ and let $H = H(M)$. By proposition 2.1.5 we can now write M degreewise as the sum

$$M^{p,q} = B^{p,q} \oplus B^{p,q+1} \oplus H^{p,q}.$$

Under this identification, we obtain the deformation retract

$$\begin{tikzcd} (M, D_0) \arrow[loop left, "h"] \arrow[r, shift left, "\pi"] & (H(M), 0) \arrow[l, shift left, "\iota"] \end{tikzcd}$$

$$\text{where } \pi = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \quad \iota = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad D_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

Proposition 2.1.7. Every quasi-isomorphism $f\colon (M, D) \to (N, D')$ extends to a homotopy equivalence (f, g, h, h').

Proof. By decomposing M and N as in proposition 2.1.5 we obtain homotopy retracts (π, ι, h) and (π', ι', h') of M and N to their respective cohomology complexes. Let $g := \iota H(f)^{-1}\pi'\colon N \to M$. We compute

$$gf = \iota H(f)^{-1}\pi' f = \iota H(f)^{-1} H(f)\pi = \iota\pi = Dh + hD + \mathrm{id}_M\,.$$

In a similar way, we see that $fg - \mathrm{id}_N = D'h' + h'D'$. $\square$

Example 2.1.8. A cochain complex (M, D) of vector spaces is contractible if and only if it is exact. This follows from the above proposition applied to the zero map $M \to 0$.

Proposition 2.1.9. Given a homotopy retract data (π, ι, h) of (M, D) to (N, D'), we can always find some $h'\colon N^\bullet \to N^{\bullet-1}$ such that (π, ι, h, h') is a homotopy equivalence.

Proof. Since π is a quasi-isomorphism, we use proposition 2.1.7 to find maps

$$\hat{\iota}\colon N \to M \quad \text{and} \quad \hat{h}\colon N^\bullet \to N^{\bullet-1}$$

such that $\mathrm{id}_N - \pi\hat{\iota} = D'\hat{h} + \hat{h}D'$. The required homotopy is given by the map defined as $h' := \pi h\hat{\iota} + \pi\iota\hat{h} - \hat{h}$ as the following calculation shows:

$$\begin{aligned} D'h' + h'D' &= D'\pi h\hat{\iota} + D'\pi\iota\hat{h} - D'\hat{h} + \pi h\hat{\iota}D' + \pi\iota\hat{h}D' - \hat{h}D' \\ &= \pi(Dh + hD)\hat{\iota} + \pi\iota(D'\hat{h} + \hat{h}D') - (D'\hat{h} + \hat{h}D') \\ &= \pi\iota\pi\hat{\iota} - \pi\hat{\iota} + \pi\iota - \pi\iota\pi\hat{\iota} + \pi\hat{\iota} - \mathrm{id}_N \\ &= \pi\iota - \mathrm{id}_N\,. \end{aligned}$$

□

2.2 Homotopy Transfer Theorem (HTT)

We now show that the data of a homotopy retract (π, ι, h) is sufficient to transfer the higher differentials. As we have seen, a homotopy retract can always be extended to a homotopy equivalence when working over a field. However, only the maps constituting the homotopy retract will be involved in defining the transferred structure. Therefore, we will only assume the data of a homotopy retract to be specified.

Theorem 2.2.1 (Homotopy transfer theorem)**.** [DSV15, Proposition 1.3] Let $(M, D_\bullet)$ be a multicomplex and let (N, D'_0) be a cochain complex of graded vector spaces. If (π, ι, h) is a homotopy retract of (M, D_0) to N and we define the maps $D'_n\colon N \to N$ by

$$D'_n := \sum_{|J|=n} \pi D_{j_1} h D_{j_2} h \cdots h D_{j_k} \iota \quad \text{for } n \geqslant 1,$$

then $(N, D'_\bullet)$ is a multicomplex. Moreover, the maps ι and π extend to morphisms of multicomplexes ι_∞ and π_∞ respectively. These maps are explicitly given by setting $\iota_0 := \iota$, $\pi_0 := \pi$,

$$\iota_n := \sum_{|J|=n} h D_{j_1} h D_{j_2} h \cdots h D_{j_k} \iota \quad \text{and} \quad \pi_n := \sum_{|J|=n} \pi D_{j_1} h D_{j_2} h \cdots h D_{j_k} h \quad \text{for } n \geqslant 1.$$

Proof. Repeatedly using the two relations

$$D_0 D_k = -\sum_{i=1}^{k} D_i D_{k-i} \quad \text{and} \quad \iota\pi - \mathrm{id}_M = D_0 h + h D_0,$$

one can see that

$$
\begin{aligned}
D'_0 D'_n &= -\pi\left(\sum_{|J|=n}\sum_{s=1}^{k-1}(D_{j_1}h\cdots hD_{j_s}\iota)(\pi D_{j_{s+1}}h\cdots hD_{j_k}) + \sum_{|J|=n} D_{j_1}h\cdots hD_{j_k}D_0\right)\iota \\
&= -\sum_{k=1}^{n-1} D'_k D'_{n-k} - D'_n D'_0.
\end{aligned}
$$

It then immediately follows that $(N, D'_\bullet)$ is a multicomplex. Next, one has to show that π_∞ is indeed a morphism of multicomplexes. That is, we need to show that

$$\sum_{p+q=n} \pi_p D_q = \sum_{p+q=n} D'_p \pi_q \quad \text{holds for all } n \geqslant 0.$$

The required computation again relies on the two relations mentioned above and the definition of D'_r. A similar argument also works for ι_∞. □

The following toy example demonstrates the homotopy transfer theorem where we set $N = H(M)$ and $D'_0 = 0$.

Example 2.2.2. Consider the multicomplex M from example 1.2.10 where the differentials map generators to generators and $D_r = 0$ for all $r \geqslant 2$.

$$
\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \mathbb{K}a \arrow[u] \arrow[r, "D_1"] & \mathbb{K}c \arrow[u] \arrow[r] & 0 & \\
 & 0 \arrow[u] \arrow[r] & \mathbb{K}b \arrow[u, "D_0"] \arrow[r, "D_1"] & \mathbb{K}d \arrow[u] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

Taking cohomology of the underlying cochain complex (M, D_0), we obtain

$$
\begin{tikzcd}
H^0(M^{-1,\bullet}) = \mathbb{K}a \arrow[r, "D'_1"] \arrow[dr, "D'_2"] & 0 & \\
 & 0 \arrow[r, "D'_1"] & H^{-1}(M^{1,\bullet}) = \mathbb{K}d.
\end{tikzcd}
$$

Let (π, ι, h) be a deformation retract as in remark 2.1.6. We get the following transferred multicomplex structure on $H(M)$:

$$D'_1 = 0 \quad \text{and } D'_2 = \pi D_1 h D_1 \iota \colon a \mapsto -d.$$

If we modify M by adding a non-trivial $D_2\colon \mathbb{K}\,a \to \mathbb{K}\,d, a \mapsto d$, we see that all the transferred differentials vanish on $H(M)$.

Lemma 2.2.3. Every homotopy retract (π, ι, h) of M to $H(M)$ is a deformation retract.

Proof. This follows from proposition 2.1.9 as the differential on $H(M)$ is trivial. □

Proposition 2.2.4. If (π, ι, h) and $(\hat{\pi}, \hat{\iota}, \hat{h})$ are any two deformation retracts of M to $H(M)$, then the transferred multicomplex structures on $H(M)$ coming from each of these retracts are isomorphic.

Proof. Extend the maps π, ι, $\hat{\pi}$ and $\hat{\iota}$ to morphisms of multicomplexes as in theorem 2.2.1. We claim that the map $\hat{\pi}_\infty \iota_\infty\colon H(M) \to H(M)$ is an isomorphism of multicomplexes with inverse $\pi_\infty \hat{\iota}_\infty$. By proposition 1.2.6, it is enough to check that $\hat{\pi}\iota$ is invertible. Using lemma 2.2.3 and the fact that the involved maps are cochain maps, we see that

$$(\hat{\pi}\iota)(\pi\hat{\iota}) = \hat{\pi}(\mathrm{id} + D_0 h + h D_0)\hat{\iota} = \hat{\pi}\hat{\iota} + (\hat{\pi} D_0) h \hat{\iota} + \hat{\pi} h (D_0 \hat{\iota}) = \hat{\pi}\hat{\iota} = \mathrm{id}_{H(M)}\,.$$

The other composition can be checked similarly. □

Remark 2.2.5. In [DSV15], a homotopy retract

$$\begin{tikzcd} (M, D_0) \arrow[loop left, "h"] \arrow[r, bend left, "\pi"] & (H(M), 0) \arrow[l, bend left, "\iota"] \end{tikzcd}$$

with the property that all transferred differentials $(D'_r | r \geqslant 1)$ from theorem 2.2.1 vanish on $H(M)$ is called a *Hodge-to-de-Rham degeneration data* of M. We will refer to such a homotopy retract as a *degeneration data*.

Let us illustrate the homotopy transfer theorem applied to the cohomology complex in the cases $n = 1, 2$ and 3 with diagrams. The first transferred map is just $\pi D_1 \iota$ which agrees with the first differential $d_1 = H(D_1)$ in the spectral sequence associated with M.

$$\begin{tikzcd} M^{p,q} \arrow[r, "D_1"] & M^{p+1,q} \arrow[d, "\pi"] \\ H^q(M^{p,\bullet}) \arrow[u, "\iota"] \arrow[r, dotted, "D'_1"] & H^q(M^{p+1,\bullet}) \end{tikzcd}$$

The second case is the sum of the two different ways we can walk down the stairs:

$$
\begin{tikzcd}
M^{p,q} \arrow[r, "D_1"] \arrow[drr, bend left, "D_2"] & M^{p+1,q} \arrow[d, "h"] & \\
& M^{p+1,q-1} \arrow[r, "D_1"] & M^{p+2,q-1} \arrow[d, "\pi"] \\
H^q(M^{p,\bullet}) \arrow[uu, "\iota"] \arrow[rr, dotted, "D_2'"] & & H^{q-1}(M^{p+2,\bullet})
\end{tikzcd}
$$

$$D_2' = \pi(D_1 h D_1 + D_2)\iota.$$

We will see later that D_2' induce the second differential d_2 on the E_2-page in the spectral sequence. In the third case, we sum the four different possibilities:

$$
\begin{tikzcd}
M^{p,q} \arrow[r, "D_1"] \arrow[drr, bend left, "D_2"] \arrow[ddrrr, bend left, "D_3"] & M^{p+1,q} \arrow[d, "h"] & & \\
& M^{p+1,q-1} \arrow[r, "D_1"] \arrow[drr, bend right, "D_2"'] & M^{p+2,q-1} \arrow[d, "h"] & \\
& & M^{p+2,q-2} \arrow[r, "D_1"] & M^{p+3,q-2} \arrow[d, "\pi"] \\
H^q(M^{p,\bullet}) \arrow[uuu, "\iota"] \arrow[rrr, dotted, "D_3'"] & & & H^{q-2}(M^{p+3,\bullet})
\end{tikzcd}
$$

$$D_3' = \pi(D_1 h D_1 h D_1 + D_1 h D_2 + D_2 h D_1 + D_3)\iota.$$

As we have already mentioned, the pattern stops here as D_3' does *not* in general induce the differential d_3 in the spectral sequence.

2.3 HTT and the spectral sequence associated with a double complex

This section treats the particular case where M is a double complex. That is, we have two potentially non-trivial differentials on M: the vertical differential D_0 and the horizontal differential D_1. These are subject to the relations

$$D_0 D_0 = 0, \quad D_1 D_1 = 0 \quad \text{and} \quad D_1 D_0 + D_0 D_1 = 0.$$

Let (π, ι, h) be a deformation retract of M to $H(M, D_0)$ obtained by choosing sections as in remark 2.1.6 and let $D'_\bullet$ be the transferred differentials on $H(M)$. Since $D_r = 0$ for $r \geqslant 2$, the transferred differentials all consist of exactly one term:

$$
\begin{aligned}
D'_1 &= \pi D_1 \iota \\
D'_2 &= \pi D_1 h D_1 \iota \\
&\vdots \\
D'_r &= \pi D_1 h D_1 h \cdots h D_1 \iota.
\end{aligned}
$$

We show that the differential d_r on the E_r-page of the spectral sequence associated with M is exactly the map induced by the transferred differential D'_r. Put differently, the multicomplex $(H(M), D'_\bullet)$ is a "lifted version" of the spectral sequence. This fact is already claimed in [LV12, p. 385] and [Val14, p. 36]. Recall that the differential d_r on the E_r-page of the spectral sequence is given by $d_r[x]_r = [D_1 x_{p+r-1}]_r$. Both [Ste21] and [KQ20] give proofs of the (previously folklore) result stating that double complexes can be written as a sum of squares and zig-zags. For simplicity, let us write $\bullet$ for a one-dimensional $\mathbb{K}$-vector space and ± 1 for the map sending generator to $\pm$ generator.

$$
\begin{tikzcd}
\bullet \arrow[r, "-1"] & \bullet & & & & \bullet & \bullet \arrow[r, "1"] & \bullet & & \\
& & \bullet & \bullet \arrow[r, "1"] & \bullet & & & & & \cdots \\
\bullet \arrow[uu, "1"] \arrow[r, "1"] & \bullet \arrow[uu, "1"'] & & & & \bullet \arrow[uu, "1"] & & \bullet \arrow[uu, "1"] \arrow[r, "1"] & \bullet &
\end{tikzcd}
$$

Squares and the zig-zags starting and ending in a vertical arrow die when passing to the E_1-page. Single points survive to the E_∞-page. The starting (ending) point of a zig-zag starting with a horizontal (vertical) arrow and ending with a vertical (horizontal) arrow also survives to the E_∞-page. That leaves us with the zig-zags with an even number of $\bullet$'s. These are the ones corresponding to the differentials in the spectral sequence.

Proposition 2.3.1. If (M, D_0, D_1) is a double complex, then the map induced by D'_r on the E_r-page is exactly d_r.

Proof. Let x denote the generator in degree (p, q) and let $x^{p+i, q-j}$ denote the the other generators in degrees $(p + i, q - j)$ in the following zig-zag diagram:

$$\begin{tikzcd}
\underset{(p,q)}{\bullet} \arrow[r, "1"] & \underset{(p+1,q)}{\bullet} & & & \\
& \underset{(p+1,q-1)}{\bullet} \arrow[u, "1"] \arrow[r, "1"] & \cdots & & \\
& & \ddots & & \\
& & \cdots \arrow[r, "1"] & \underset{(p+r-1,q-r+2)}{\bullet} & \\
& & & \underset{(p+r-1,q-r+1)}{\bullet} \arrow[u, "1"] \arrow[r, "1"] & \underset{(p+r,q-r+1)}{\bullet}
\end{tikzcd}$$

Choosing sections as in remark 2.1.6 we obtain a deformation retract (π, ι, h) where h maps $x^{p+i,q-j}$ to $-x^{p+i,q-j-1}$. Define the elements

$$x_{p+1} := hD_1x, \quad x_{p+2} := hD_1hD_1x, \quad x_{p+3} := hD_1hD_1hD_1x \quad \text{and so on.}$$

Clearly, we have $D_1x_{p+i} + D_0x_{p+i+1} = 0$ so these elements defines witnesses for x and we can compute the differential

$$d_r[x]_r = [D_1x_{p+r-1}]_r = [D_1hD_1x_{p+r-2}]_r = \cdots = [D_1hD_1h\cdots hD_1x]_r = [D'_r[x]]_r.$$

Since we assume x to be an r-cycle, we only have to consider zig-zags as the one above, so we are done. $\square$

3 First-page degeneracy

In the case of double complexes, we saw that the differentials in the spectral sequence are precisely the maps induced by the transferred differentials on cohomology. Thus, for the spectral sequence associated with a double complex, degeneration at the k-th page is equivalent to the vanishing of the maps induced by the transferred differentials D'_r for all $r \geqslant k$. This is not true in general when there are non-trivial differentials D_r with $r \geqslant 2$ in play. We will give examples of this failure in section 4. However, for first-page degeneration, we prove the following theorem which appears in [DSV15]. Recall that a degeneration data is just a homotopy retract with the property that all transferred differentials vanish on $H(M)$.

Theorem 3.0.1. The spectral sequence associated with a multicomplex M degenerates at the first page if and only if there exists a degeneration data of M.

We split this theorem into two propositions, one for each direction.

Proposition 3.0.2. If there exists a degeneration data of the multicomplex M, then the spectral sequence associated with M degenerates at the first page.

Proof. Suppose (π, ι, h) is a degeneration data of M and let $x \in \mathcal{Z}_r^{p,\bullet}$ with $r \geqslant 1$ be an arbitrary r-cocycle. By definition, $D_0 x = 0$ meaning x is an 1-cocycle and hence represents a class in cohomology. Consider the 1-cocycle $x_p := \iota\pi(x)$. Observe that $[x_p] = [x]$ because $\iota\pi$ is homotopic to the identity. The idea is to construct a family of elements witnessing that x_p is in fact an r-cocycle and consequently $[x_p]_r = [x]_r$ by corollary 1.2.16. In the end, we compute the image of $[x]_r$ under d_r to be zero using the constructed witnesses.

1. The 1*-cocycle x_p is an r-cocycle:*

Define the following family of elements:

$$x_{p+j} = h \sum_{|I|=j} D_{i_1} h D_{i_2} h \cdots h D_{i_k} x_p \quad \text{for } 1 \leqslant j \leqslant r-1. \tag{8}$$

To see that $x_{p+1}, \ldots x_{p+r-1}$ are witnesses for x_p being an r-cocycle, we need to verify that the equation

$$D_n x_p + \sum_{i=0}^{n-1} D_i x_{p+n-i} = 0 \tag{9}$$

holds for all $n = 1, 2, \ldots, r-1$. First, we verify the base case $n = 1$:

$$\begin{aligned} D_0 x_{p+1} + D_1 x_p &= D_0 h D_1 x_p + D_1 x_p = (D_0 h + \mathrm{id}) D_1 x_p \\ &= (\iota\pi - h D_0) D_1 x_p = \iota \underbrace{(\pi D_1 \iota)}_{=0} \pi(x) + h D_1 \underbrace{(D_0 x_p)}_{=0} = 0. \end{aligned}$$

Now, suppose that $\sum_{i=0}^{k} D_i x_{p+n-i} = 0$ holds for all $k = 1, 2, \ldots, n-1$. We want to show that it also holds for $k = n$. First, observe that we can rewrite

$$\sum_{i=0}^{n} D_i x_{p+n-i} = (D_0 h + \mathrm{id}) \sum_{|I|=n} D_{i_1} h D_{i_2} h \cdots h D_{i_k} x_p.$$

Using this observation, we complete the induction step:

$$\begin{aligned} \sum_{i=0}^{n} D_i x_{p+n-i} &= (D_0 h + \mathrm{id}) \sum_{|I|=n} D_{i_1} h \cdots h D_{i_k} x_p \overset{(1)}{=} (\iota\pi - h D_0) \sum_{|I|=n} D_{i_1} h \cdots h D_{i_k} x_p \\ &= \iota \underbrace{\pi \sum_{|I|=n} D_{i_1} h \cdots h D_{i_k} \iota}_{= D_n' = 0} \pi(x) - h D_0 \sum_{|I|=n} D_{i_1} h \cdots h D_{i_k} x_p \\ &\overset{(2)}{=} h \sum_{|I|=n} (-D_0 D_{i_1}) h \cdots h D_{i_k} x_p \\ &\overset{(3)}{=} h \sum_{|I|=n} \left(\sum_{r=1}^{i_1} D_r D_{i_1 - r} \right) h D_{i_2} h \cdots h D_{i_k} x_p \\ &= h \sum_{i_1=1}^{n} \sum_{r=1}^{i_1} D_r D_{i_1 - r} \sum_{|J|=n-i_1} h D_{j_1} h \cdots h D_{j_k} x_p \\ &\overset{(4)}{=} h \sum_{j=1}^{n} \sum_{r=1}^{j} D_r D_{j-r} x_{p+n-j} \\ &\overset{(5)}{=} h \sum_{r=1}^{n-1} D_r \sum_{j=r}^{n} D_{j-r} x_{p+n-j} \quad + \quad h D_n \underbrace{D_0 x_p}_{=0} \\ &= h \sum_{r=1}^{n-1} D_r \underbrace{\sum_{i=0}^{n-r} D_i x_{p+n-r-i}}_{=0 \text{ by induction hyp.}} = 0. \end{aligned}$$

In (1) we use that $\iota\pi - \mathrm{id} = D_0 h + h D_0$. In (2) we use the assumption that (π, ι, h)

is a degeneration data, and in (3) we use the multicomplex relations. In (4) we relabel the index i_1 to j and use the definition of x_{p+n-1} to replace the innermost sum. In (5) we switch the order of summation by re-indexing and separate the last term which is always zero. In the last step we re-index by letting $i = j - r$.

2. The differential d_r is trivial:

This is now a straightforward computation using the witnesses from the previous part.

$$\begin{aligned} d_r([x]_r) = d_r([x_p]_r) &= \left[\sum_{i=1}^{r} D_i x_{p+r-i} \right]_r = \left[\sum_{i=1}^{r} \sum_{|J|=r-i} D_i h D_{j_1} h D_{j_2} h \cdots h D_{j_k} x_p \right]_r \\ &= \left[\sum_{|J|=r} D_{j_1} h D_{j_2} h \cdots h D_{j_k} \, x_p \right]_r = \left[\iota \, \underbrace{\pi \sum_{|J|=r} D_{j_1} h D_{j_2} h \cdots h D_{j_k} \, \iota}_{= D'_r = 0} \pi(x) \right]_r = 0. \end{aligned}$$

□

We now prove the second part of theorem 3.0.1.

Proposition 3.0.3. If the spectral sequence associated with a multicomplex M degenerates at the first page, then there exists a degeneration data of M.

Proof. Let (π, ι, h) be a deformation retract as in remark 2.1.6 and suppose the associated spectral sequence degenerates at the first page, i.e., $d_r = 0$ and $E_r = H(M)$ for $r \geqslant 1$. Now, fix some $r \geqslant 1$ and let x represent a class in $H^{p,q}$. Define the element $x_p := \iota[x]$. The restriction of π to coboundaries is zero, so it is sufficient to show that

$$\sum_{|I|=r} D_{i_1} h D_{i_2} h \cdots h D_{i_k} \iota[x] \in B^{p+r, q-r+1}$$

for D'_r to vanish.

1. The element x_p is an r-cycle:

Since $D_0 \iota = 0$ we have that $D_0 x_p = 0$. By assumption, $d_1 = 0$, so $D_1 x_p \in B^{p+1,q}$ and consequently $D'_1[x] = \pi D_1 \iota[x] = 0$. Furthermore, if we define $x_{p+1} := h D_1 x_p$, then $D_0 x_{p+1} + D_1 x_p = (D_0 h + \mathrm{id}) D_1 x_p = 0$ since $D_1 x_p \in B^{p+1,q}$ and $D_0 h|_{B^{p+1,q}} = -\mathrm{id}$. Continuing in this fashion, we prove by induction that x_p is an r-cocycle witnessed by the elements $x_{p+1}, \ldots, x_{p+r-1}$ defined as in eq. (8). Suppose x_p is an m-cocycle witnessed by the elements $x_{p+1}, x_{p+2}, \dots, x_{p+m-1}$. By assumption, we have that

$$d_m[x_p] = \left[\sum_{|I|=m} D_{i_1} h D_{i_2} h \cdots h D_{i_k} \ x_p \right] = 0$$

and hence the sum lives in $B^{p+m,q-m+1}$. Using that the restriction of $D_0 h$ to $B^{p+m,q-m+1}$ is multiplication by -1 we see that

$$\sum_{i=0}^{m} D_i x_{p+m-i} = \underbrace{(D_0 h + \mathrm{id})}_{-1+1} \sum_{|I|=m} D_{i_1} h D_{i_2} h \cdots h D_{i_k} x_p = 0,$$

which is to say that x_p is an $(m+1)$-cocycle witnessed by the elements $x_{p+1}, \dots, x_{p+m}$. This completes our inductive argument.

2. The transferred differential $D'_r = 0$:

Using the witnesses above and the assumption $d_r = 0$, we see that

$$0 = d_r[x_p] = \left[\sum_{i=1}^{r} D_i x_{p+r-i} \right] = \left[\sum_{|I|=r} D_{i_1} h D_{i_2} h \cdots h D_{i_k} \ i[x] \right]$$

or in other words:

$$\sum_{|I|=r} D_{i_1} h D_{i_2} h \cdots h D_{i_k} \iota[x] \in B^{p+r,q-r+1}.$$

Hence, $D'_r = 0$ for all $r \geqslant 1$. □

This concludes the proof of theorem 3.0.1. Moreover, the proof shows that it is enough to consider any deformation retract as in remark 2.1.6. This is not surprising as any two deformation retracts induce isomorphic multicomplex structures on cohomology by proposition 2.2.4.

Example 3.0.4. Let M be the following multicomplex with $2n$ non-zero entries for some $n \geqslant 2$:

$$
\begin{tikzcd}
\mathbb{K} \arrow[r, "1"] \arrow[rrrrddd, bend left, "(-1)^n"] & \mathbb{K} & & & \\
& \mathbb{K} \arrow[u, "1"] \arrow[r, "1"] & \mathbb{K} & & \\
& & \ddots \arrow[u, "1"] \arrow[r, "1"] & \mathbb{K} & \\
& & & \mathbb{K} \arrow[u, "1"] \arrow[r, "1"] & \mathbb{K}
\end{tikzcd}
$$

The spectral sequence associated with this multicomplex degenerates at the first page since the only possible non-trivial transferred differential on $H(M)$ is D_n', which is easily computed to be zero (cf. last part of example 2.2.2).

4 Differentials on later pages

This section establishes the relationship between the transferred differential D_2' and the differential d_2 in the spectral sequence. Namely, d_2 agrees with the map induced by D_2' on the E_2-page. Next, we provide an example where $D_3' = 0$, but d_3 is non-trivial. This suggests that there is more to d_3 than just D_3'. Lastly, we prove that applying the homotopy transfer theorem repeatedly gives us the differentials on later pages in the spectral sequence.

Example 4.0.1. In this example, we have that $d_2 = 0$ in spite of D_2' being non-trivial. Consider the following multicomplex M.

$$\begin{tikzcd}
\mathbb{K}a \arrow[drr, "D_2"] & 0 & \mathbb{K}c & 0 \\
0 & \mathbb{K}b \arrow[r, "D_1"] & \mathbb{K}d \oplus \mathbb{K}e \arrow[u, "D_0"] \arrow[r, "D_1"] & \mathbb{K}f
\end{tikzcd}$$

$$\begin{array}{lll} D_0 d = 0 & D_0 e = c & D_1 b = d \\ D_1 d = 0 & D_1 e = f & D_2 a = d \end{array}$$

Computing cohomology of M with respect to D_0 gives us the E_1-page.

$$\begin{tikzcd}
\mathbb{K}a & 0 & 0 & 0 \\
0 & \mathbb{K}b \arrow[r] & \mathbb{K}d \arrow[r] & \mathbb{K}f
\end{tikzcd}$$

The differential $d_1 \colon \mathbb{K}b \to \mathbb{K}d$ is an isomorphism as it is the map induced on cohomology by $D_1 \colon b \mapsto d$. The other differential, $d_1 \colon \mathbb{K}d \to \mathbb{K}f$ is trivial. This gives us the following E_2-page.

$$\begin{array}{cccc} \mathbb{K}a & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{K}f \end{array}$$

For degree reasons, we must have $d_2 = 0$. The transferred differential $D_2' \colon \mathbb{K}a \to \mathbb{K}d$ is non-trivial as

$$D_2'[a] = \pi(D_1 h D_1 a + D_2 a) = 0 + d = d.$$

But this is okay since d represents zero on the E_2-page so the map *induced* by D_2' does agree with d_2.

In fact, what happened in the above example was not accidental, as the following proposition shows:

Proposition 4.0.2. The map induced by D_2' on the E_2-page is agrees with d_2.

Proof. Let $x \in \mathcal{Z}_2^{p,q}$ be a 2-cocycle witnessed by $x_{p+1} \in M^{p+1,q-1}$ and let $\tilde{x}_{p+1}$ denote the projection of x_{p+1} onto $B^{p-1,q}$. Since we have $D_0\tilde{x}_{p+1} = D_0 x_{p+1}$, it follows that $D_0\tilde{x}_{p+1} + D_1 x = 0$. Moreover, because $hD_0\tilde{x}_{p+1} = -\tilde{x}_{p+1}$ we see that $\tilde{x}_{p+1} = hD_1 x$. Now, we simply compute d_2 to conclude our proof:

$$d_2[x]_2 = [D_2 x + D_1\tilde{x}_{p+1}]_2 = [D_2 x + D_1 h D_1 x]_2 = [D_2'[x]]_2.$$

□

The above proposition will follow as a special case of what is to come. We now look at an example where D_3' vanish but d_3 is non-trivial.

Example 4.0.3. Consider the following multicomplex M

$$\begin{tikzcd}
\mathbb{K}\,x_1 \arrow[r, "D_1"] \arrow[drr, "D_2"] \arrow[ddrrr, bend right, "D_3"'] & \mathbb{K}\,y_1 & & \\
& \mathbb{K}\,x_2 \oplus \mathbb{K}\,x_3 \arrow[u, "D_0"] \arrow[r, "D_1"] \arrow[drr, "D_2"] & \mathbb{K}\,y_2 \oplus \mathbb{K}\,y_3 \oplus \mathbb{K}\,y_4 & \\
& & \mathbb{K}\,x_4 \oplus \mathbb{K}\,x_5 \arrow[u, "D_0"] \arrow[r, "D_1"'] & \mathbb{K}\,y_5 \oplus \mathbb{K}\,y_6
\end{tikzcd}$$

where the differentials are given in the table below.

	x_1	x_2	x_3	x_4	x_5
D_0	0	y_1	0	y_2	y_3
D_1	y_1	y_2	$y_3 + y_4$	y_5	0
D_2	$-y_4$	$-y_5 - y_6$	y_6	0	0
D_3	$-2y_5 - y_6$	0	0	0	0

Taking cohomology with respect to D_0, we compute the E_1-page to be

$$\mathbb{K}\,x_1 \xrightarrow{d_1} 0$$

$$\mathbb{K}\,x_3 \xrightarrow{d_1} \mathbb{K}\,y_4$$

$$0 \xrightarrow{d_1} \mathbb{K}\,y_5 \oplus \mathbb{K}\,y_6$$

where the differential d_1 is the isomorphism $x_3 \mapsto y_4$ induced by D_1. Choosing sections gives us a deformation retract from M to the E_1-page which allows us to compute the transferred differential

$$D_3'x_1 = [D_1hD_1hD_1x_1 + D_2hD_1x_1 + D_1hD_2x_1 + D_3x_1] = [y_5 + y_5 + y_6 + -2y_5 - y_6] = 0.$$

Passing to the E_2-page, only the generators x_1, y_5 and y_6 survives. It is readily seen that the total complex $\operatorname{Tot} M$ is (isomorphic to) $\mathbb{K}^5 \xrightarrow{D} \mathbb{K}^6$ where

$$D = D_0 + D_1 + D_2 + D_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

The differential D is injective and hence the cohomology of $\operatorname{Tot} M$ is 0 and $\mathbb{K}$ in degree 0 and 1, respectively. For degree reasons, d_3 is the only possible nontrivial differential in the spectral sequence and by convergence we must have $d_3 \neq 0$. For demonstration purposes, we compute d_3x_1 directly: it is easy to check that the elements $-x_2 + x_3$ and $x_4 - x_5$ witness x_1 being a 3-cycle. Consequently, we can compute $d_3x_1 = [D_3x_1 + D_2(-x_2 + x_3) + D_1(x_4 - x_5)] = y_6 \neq 0$. The point is that d_3 can be non-trivial even though D_3' is trivial and this suggests that there is more to d_3 than just D_3'.

Construction 4.0.4. Given a multicomplex $(M, D_\bullet)$ over $\mathbb{K}$, we can always define another multicomplex $({}^1M, {}^1D_\bullet)$ by letting ${}^1M^{p,q} := H^{2p+q,-p}(M)$ and ${}^1D_r := D_{r+1}'$. That is, ${}^1M$ is a shifted version of $H(M)$ with the transferred differentials from the HTT (theorem 2.2.1). It is clear that ${}^1M$ is a multicomplex as we have

$$\sum_{p+q=r} {}^1D_p\,{}^1D_q = \sum_{i=1}^{r+1} D_i'D_{r+2-i}' = -D_0'D_{r+2}' - D_{r+2}'D_0' = 0 \quad \text{for all } r \geqslant 0.$$

We apply this very construction on ${}^1M$ to obtain another multicomplex $({}^2M, {}^2D_\bullet)$. In other words, ${}^2M^{p,q} := H^{2p+q,-p}({}^1M, {}^1D_0)$ and ${}^2D_r := {}^1D_{r+1}'$. Continuing in this manner, we obtain a family of multicomplexes $({}^sM, {}^sD_\bullet)$ where

$$ {}^sM^{p,q} := H^{2p+q,-p}({}^{s-1}M, {}^{s-1}D_0) \quad \text{and} \quad {}^sD_r := {}^{s-1}D'_{r+1}. $$

Of course, it is understood that there is a choice of sections involved in each step, giving rise to a deformation retract. However, the construction above is independent of choice in the sense that different choices of sections give isomorphic multicomplexes by proposition 2.2.4.

Let us revisit example 4.0.3 and compute the multicomplexes ${}^sM$ and their differentials.

Example 4.0.5. Let M be the multicomplex in example 4.0.3. We get the first multicomplex ${}^1M$ from re-indexing $E_1 = H(M)$ and computing the transferred differentials:

$$\begin{tikzcd} \mathbb{K}\,x_1 \arrow[r, "{}^1D_1"] & \mathbb{K}\,y_4 & \\ & \mathbb{K}\,x_3 \arrow[u, "{}^1D_0"] \arrow[r, "{}^1D_1"'] & \mathbb{K}\,y_5 \oplus \mathbb{K}\,y_6 \end{tikzcd}$$

	x_1	x_3
${}^1D_0$	0	y_4
${}^1D_1$	$-y_4$	y_6

We have already computed the differential ${}^1D_2 = D_3'\colon \mathbb{K}\,x_1 \to \mathbb{K}\,y_5 \oplus \mathbb{K}\,y_6$ to be trivial in example 4.0.3. All higher differentials are trivial by degree reasons. Computing cohomology with respect to ${}^1D_0$ and re-indexing we obtain ${}^2M$:

$$ \mathbb{K}\,x_1 \xrightarrow{{}^2D_1} \mathbb{K}\,y_5 \oplus \mathbb{K}\,y_6 $$

The differential ${}^2D_0$ is trivial and ${}^2D_1 = {}^1D_2'$ is computed to be the map $x_1 \mapsto y_6$. Since the zeroth differential is trivial, taking cohomology does nothing, and ${}^3M$ is just the multicomplex

$$\begin{tikzcd} \mathbb{K}\,y_5 \oplus \mathbb{K}\,y_6 \\ \mathbb{K}\,x_1 \arrow[u, "{}^3D_0"] \end{tikzcd}$$

with ${}^3D_0\colon x_1 \mapsto y_6$. Note that $d_3 = {}^3D_0$ by our calculation in example 4.0.3. Lastly, the multicomplex ${}^4M$ is just $\mathbb{K}\,y_5$ concentrated in a single point. In particular, all the differentials will be trivial from this point on and ${}^4M$ can be thought of as the limit term. In the spirit of spectral sequences, let us write ${}^4M = {}^\infty M$.

We write $(E_\bullet({}^kM), {}^kd_\bullet)$ for the spectral sequence associated with the multicomplex ${}^kM$. The following result relates the spectral sequence $(E_\bullet({}^1M), {}^1d_\bullet)$ to the spectral sequence $E(M)$ associated with M.

Theorem 4.0.6. The spectral sequence associated with ${}^1M$ is a shifted version of the spectral sequence associated with M in the sense that

$$E_r^{p,q}({}^1M) = E_{r+1}^{2p+q,-p}(M) \quad \text{and} \quad {}^1d_r = d_{r+1} \quad \text{for all} \quad r \geqslant 0.$$

Proof. We suppress the indices for clarity. The case where $r = 0$ holds by construction as $E_0({}^1M) = {}^1M = H(M) = E_1(M)$ and ${}^1d_0 = {}^1D_0 = D_1' = d_1$. Consequently, $E_1({}^1M) = E_2(M)$ and the equality ${}^1d_1 = d_2$ follows from proposition 4.0.2. We now assume $E_r({}^1M) = E_{r+1}(M)$ for some $r \geqslant 0$ and show that ${}^1d_r = d_{r+1}$. This will imply that $E_{r+1}({}^1M) = E_{r+2}(M)$ so we can conclude by induction. Let x_p represent a cycle in $E_r({}^1M)$ witnessed by the family $\{[x_{p+i}]\}_{i=1,2,\ldots,r-1}$. Define the elements

$$\begin{aligned}\hat{x}_{p+i} &:= \sum_{n=0}^{i} \sum_{|J|=n} hD_{j_1}h\cdots hD_{j_k}\iota[x_{p+i-n}] \\ &= \iota[x_{p+i}] + h\left(D_1\iota[x_{p+i-1}] + (D_1hD_1 + D_2)\iota[x_{p+i-2}] + \cdots + \sum_{|J|=i} D_{j_1}h\cdots hD_{j_k}\iota[x_p]\right)\end{aligned}$$

for $i = 1, 2, \ldots, r-1, r$ where we set $[x_{p+r}] := 0$. Next, we prove that the family $\{\hat{x}_{p+i}\}_{i=1,\ldots,r}$ defines witnesses for $\hat{x}_p := \iota[x_p]$ being an $(r+1)$-cycle in $E_{r+1}(M)$. We have that

$$\sum_{i=0}^{n} {}^1D_i[x_{p+n-i}] = 0 \quad \text{for } n = 1, 2, \ldots, r-1 \tag{10}$$

and we want to show that

$$\sum_{i=0}^{n} D_i\hat{x}_{p+n-i} = 0 \quad \text{for } n = 1, 2, \ldots, r \tag{11}$$

since these are precisely the relations required, by definition of $\mathcal{Z}_{r+1}(M)$. We verify eq. (11) by induction. In the base case, we will use that ${}^1D_0[x_p] = 0$ and $D_0\iota = 0$:

$$\begin{aligned} D_0\hat{x}_{p+1} + D_1\hat{x}_p &= D_0\iota[x_{p+1}] + D_0hD_1\iota[x_p] + D_1\iota[x_p] = (D_0h + \mathrm{id})D_1\iota[x_p] \\ &= (\iota\pi - hD_0)D_1\iota[x_p] = \iota\,{}^1D_0[x_p] + hD_1D_0\iota[x_p] = 0. \end{aligned}$$

For the induction step, assume

$$\sum_{i=0}^{k} D_i\hat{x}_{p+k-i} = 0 \quad \text{for } k = 1, 2, \ldots, n-1. \tag{12}$$

Using the induction hypothesis and eq. (10), we show that the relation above also holds in the case $k = n$. The required verification is an exercise in the manipulation of sums.

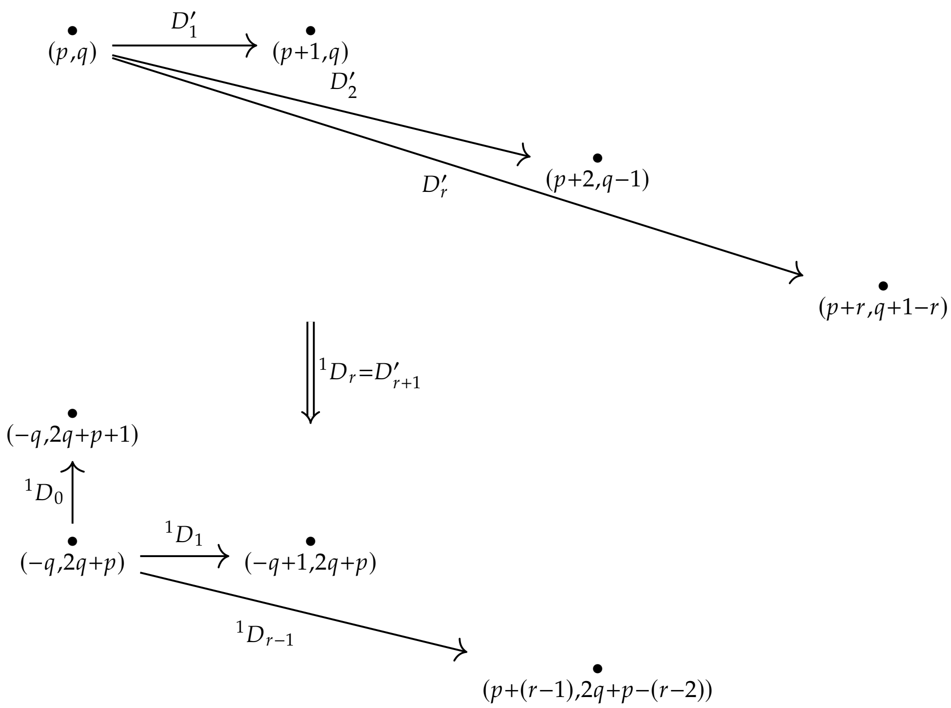
$$\begin{aligned} \sum_{i=0}^{n} D_i\hat{x}_{p+n-i} &= (D_0h + \mathrm{id})\sum_{i=1}^{n}\sum_{|J|=i} D_{j_1}hD_{j_2}h\cdots hD_{j_k}\iota[x_{p+n-i}] \\ &= (\iota\pi - hD_0)\sum_{i=1}^{n}\sum_{|J|=i} D_{j_1}hD_{j_2}h\cdots hD_{j_k}\iota[x_{p+n-i}] \\ &= \iota\sum_{i=1}^{n} D_i'[x_{p+n-i}] + h\sum_{i=1}^{n}\sum_{|J|=i}(-D_0D_{j_1})hD_{j_2}h\cdots hD_{j_k}\iota[x_{p+n-i}] \\ &= \underbrace{\iota\sum_{i=0}^{n-1} {}^1D_i[x_{p+n-1-i}]}_{=0 \quad \text{by eq. (10)}} + h\sum_{n\geqslant i\geqslant s\geqslant l\geqslant 1} D_lD_{s-l}\sum_{|J|=i-s} hD_{j_1}h\cdots hD_{j_k}\iota[x_{p+n-i}] \\ &= h\sum_{i=1}^{n-1} D_i(D_0h + \mathrm{id})\sum_{l=1}^{n-i}\sum_{|J|=l} D_{j_1}h\cdots hD_{j_k}\iota[x_{p+n-i-j}] \\ &= h\sum_{i=1}^{n-1} D_i\underbrace{\sum_{j=0}^{n-i} D_j\hat{x}_{p+(n-i)-j}}_{=0 \quad \text{by eq. (12)}} = 0. \end{aligned}$$

Using the witnesses $\{\hat{x}_{p+i}\}_{i=1,2,\ldots,r}$ and the fact that $[\hat{x}_p]_{r+1} = [x_p]_{r+1}$ (which follows from corollary 1.2.16), we see that d_{r+1} and ${}^1d_r$ agree.

$$
\begin{aligned}
d_{r+1}[x_p] &= \left[\sum_{i=1}^{r+1} D_i \hat{x}_{p+r+1-i}\right] = \left[\sum_{i=1}^{r} \sum_{|J|=i+1} D_{j_1} h D_{j_2} h \cdots h D_{j_k} \iota[x_{p+r-i}]\right] \\
&= \left[\sum_{i=1}^{r} D'_{i+1}[x_{p+r-i}]\right] = \left[\sum_{i=1}^{r} {}^1D_i[x_{p+r-i}]\right] = {}^1d_r[x_p].
\end{aligned}
$$

Here the outer brackets denote the equivalence class in $E_r({}^1M) = E_{r+1}(M)$. ◻

The following picture illustrates the re-indexing when going from $E_1 = H(M)$ to ${}^1M$:



We now get the following corollary which gives us an alternative approach to computing the spectral sequence associated with a multicomplex $(M, D_\bullet)$:

Corollary 4.0.7. We have

$$
E_r^{p,q} = {}^rM^{p-rn,\,q+rn} \quad \text{and} \quad d_r = {}^rD_0
$$

for every $r \geqslant 1$ where $n = p + q$.

Proof. From theorem 4.0.6, we have that $E_r({}^1M) = E_{r+1}(M)$ for every $r \geqslant 0$. Since we obtain ${}^sM$ from ${}^{s-1}M$ in exactly the same way as we obtain ${}^1M$ from M we can conclude that

$$^{r}M = E_0(^{r}M) = E_1(^{r-1}M) = E_2(^{r-2}M) = \cdots = E_{r-1}(^{1}M) = E_r(M).$$

Similarly, for the differentials, we get

$$^{r}D_0 = {}^{r}d_0 = {}^{r-1}d_1 = {}^{r-2}d_2 = \cdots = {}^{1}d_{r-1} = d_r.$$

□

It is immediate to see that theorem 3.0.1 follows as a corollary of theorem 4.0.6. Even better, we now have a more general result characterising k-th-page degeneracy:

Corollary 4.0.8. The spectral sequence associated with a multicomplex M degenerates at the k-th page if and only if $^{k}D_r = 0$ for all $r \geqslant 0$.

Proof. The spectral sequence $E(M)$ degenerates at the k-th page $\iff$ $d_r = 0$ for all $r \geqslant k$ $\iff$ The spectral sequence $E(^{k-1}M)$ degenerates at the first page (by corollary 4.0.7) $\iff$ For all $r \geqslant 0$, we have $0 = {}^{k-1}D'_{r+1} = {}^{k}D_r$ (by theorem 3.0.1). □

Example 4.0.9. Let us try to recover the differentials in the spectral sequence associated with a double complex using corollary 4.0.7. To keep things simple, let us first consider a 3-cycle x. Recall that by the classification of double complexes (see, for example, [Ste21]), x has to be on the top of a zig-zag like the one below with differentials ± 1.

$$\begin{tikzcd}
x \in \bullet \arrow[r] & \bullet & & \\
& \bullet \arrow[u, "D_0"] \arrow[r, "D_1"] & \bullet & \\
& & \bullet \arrow[u] \arrow[r] & \bullet
\end{tikzcd}$$

By corollary 4.0.7 we have

$$\begin{aligned}
d_3 = {}^{3}D_0 &= \pi_2\, {}^{2}D_1 \iota_2 = \pi_2\pi_1({}^{1}D_1 h_1\, {}^{1}D_1 + {}^{1}D_2)\iota_1\iota_2 \\
&= \pi_2\pi_1(D'_2 h_1 D'_2 + D'_3)\iota_1\iota_2 = [D'_2 h_1 D'_2]_3 + [D'_3]_3 = [D'_3]_3.
\end{aligned}$$

The term $D'_2 h_1 D'_2$ is zero because the cohomology of the zig-zag with respect to D_0 is zero everywhere but in the start and end points as we see in the following picture:

$$
\begin{array}{cccc}
\bullet & 0 & & \\
 & 0 & 0 & \\
 & & 0 & \bullet
\end{array}
$$

An identical argument can be carried out for arbitrary r-cycles. That is, the only term in ${}^{r}D_0$ which does not factor through zero, is the term $\pi_{r-1}\pi_{r-2}\cdots\pi_1 D'_r \iota_1\iota_2\cdots\iota_{r-1}$ which is exactly the map induced by D'_r on the E_r-page. Consequently, we have recovered proposition 2.3.1 as a corollary of corollary 4.0.7.

Appendix A Multicomplexes as homotopy algebras

This section is a brief comment on how multicomplexes fit into the general theory of operads. In this section, we aim to show that double complexes correspond to algebras over the operad $\mathscr{D}$ of dual numbers and that multicomplexes are the algebras over some operad $\mathscr{M}$. This corresponds to theorem A.1.1 and theorem A.2.2, respectively. We then show that $\mathscr{M}$ is in fact equal to a certain operad which will be denoted $\mathscr{D}_\infty$ called the Koszul resolution of $\mathscr{D}$. This is the content of theorem A.3.1. It turns out that we do not need to go much into details about operads as remark A.0.2 allows us to stay in the setting of (co)algebras.

We restrict ourselves to non-symmetric dg-operads, which are operads in the (symmetric monoidal) category of cochain complexes of graded vector spaces. Consequently, we allow dg-algebras and dg-modules to be bigraded. For example, if A is a dg-algebra and M is a dg-module over A, we require the action to respect the grading, i.e., $A^{p,q} \cdot M^{p',q'} \leqslant M^{p+p',q+q'}$. Moreover, if x is an element of bidegre (p,q) we denote the *total degree* of x by $|x| = p+q$. The main references used here are [LV12] and [Val14] with grading conventions adapted to our situation.

Definition A.0.1. A *(non-symmetric dg-) operad* $\mathscr{P}$ is a family $\{\mathscr{P}(n)\}_{n\in\mathbb{N}}$ of cochain complexes with an element $I \in \mathscr{P}(1)$ and composite maps

$$\gamma_{i_1,i_2,\dots,i_k}\colon \mathscr{P}(k)\otimes\mathscr{P}(i_1)\otimes\mathscr{P}(i_2)\otimes\cdots\otimes\mathscr{P}(i_k)\to\mathscr{P}(i_1+i_2+\cdots+i_k)$$

satisfying certain unital and associativity axioms.

An operad $\mathscr{P}$ with $\mathscr{P}(n) = 0$ for all $n \neq 1$ is said to be of *arity 1*. Such operads encode operations with exactly one input and one output. This is the case with double complexes and multicomplexes when considering the maps D_r as the operations.

Remark A.0.2. If $\mathscr{P}$ is an operad of arity 1, then $\gamma_1\colon \mathscr{P}(1)\otimes\mathscr{P}(1)\to\mathscr{P}(1)$ is the only composite map which can be non-trivial. Now, let $\mathcal{P}$ be a dg-algebra with multiplication μ and unit $1_{\mathcal{P}}$. We see that the operads of arity 1 are the same as dg-algebras under the identification

$$\mathcal{P}\leftrightarrow\mathscr{P}(1)\quad \mu\leftrightarrow\gamma_1\quad I\leftrightarrow 1_{\mathcal{P}}.$$

Furthermore, as remarked in [LV12, p. 551], an algebra over an operad $\mathscr{P}$ of arity 1 is the same as a dg-module over $\mathcal{P}$. Consequently, we can more or less restrict our focus to the algebras defining the operads of interest. In general, operads of arity 1 in some category $\mathscr{C}$ corresponds to monoids in $\mathscr{C}$.

A.1 The operad of dual numbers

Let $\mathcal{D} = \mathbb{K}[\epsilon]$ (with the relation $\epsilon^2 = 0$) denote the *dual numbers over* $\mathbb{K}$ considered as a dg-algebra with trivial differential $\partial = 0$, the generator 1 in bidegree $(0, 0)$ and the generator ϵ in bidegree $(1, 0)$. If (M, D_0) is a cochain complex of graded vector spaces with $D_0\colon M^{\bullet,q} \to M^{\bullet,q+1}$, then making M into a double complex is the same as specifying a linear map $D_1\colon M^{p,\bullet} \to M^{p+1,\bullet}$ such that $D_1D_1 = 0$ and $D_1D_0 + D_0D_1 = 0$. Making M into a $\mathcal{D}$-module is the same as specifying an action of ϵ that is compatible with the differential D_0 in the sense that the graded Leibniz rule holds:

$$D_0(\epsilon x) = \partial(\epsilon)x + (-1)^{|\epsilon|}\epsilon D_0 x = -\epsilon D_0 x \quad \text{for all } x \in M.$$

If we identify D_1 with multiplication by ϵ, then $D_1D_1 = \epsilon^2 = 0$ and the Leibniz rule becomes $D_0D_1 + D_1D_0 = 0$. In other words, $\mathcal{D}$-modules are the same thing as double complexes over $\mathbb{K}$. We summarise the above discussion in the following theorem:

Theorem A.1.1. The category of $\mathscr{D}$-algebras, where $\mathscr{D}$ is the operad of arity 1 defined by $\mathscr{D}(1) := \mathcal{D}$, is equivalent to the category of double complexes.

A.2 The operad encoding multicomplexes

Let $\mathcal{M}$ be the dg-algebra defined as follows: For $p \geqslant 0$ and $q \leqslant 0$, we define

$$\mathcal{M}^{p,q} = \bigoplus \mathbb{K}\, \delta^{i_1}\delta^{i_2}\cdots\delta^{i_k}$$

where we take the sum over multi-indices of length $k = p + q$ which sums to p. We equip $\mathcal{M}$ with the differential $\partial\colon \mathcal{M}^{p,q} \to \mathcal{M}^{p,q+1}$ defined on generators by

$$\partial(\delta^n) = -\sum_{i=1}^{n-1}(-1)^i\delta^i\delta^{n-i}.$$

In other words, $\mathcal{M}$ is the non-commutative polynomial algebra in the variables $\{\delta^i\}_{i\geqslant 1}$ equipped with the differential ∂ and a suitable grading. As an algebra, $\mathcal{M}$ is clearly generated by the elements $\delta^1, \delta^2, \ldots$. Observe that each basis element in $\mathcal{M}^{p,q}$ (as a vector space) is determined by some partition of the integer p

of length $p + q$. Multiplication is just concatenation:

$$(\delta^{i_1} \cdots \delta^{i_k})(\delta^{i_{k+1}} \cdots \delta^{i_{k+l}}) = \delta^{i_1} \cdots \delta^{i_{k+l}}.$$

To see that the multiplication respects the grading on $\mathcal{M}$, suppose $(i_1, i_2, \ldots, i_{p+q})$ and $(j_1, j_2, \ldots, j_{p'+q'})$ are partitions of p and p' respectively. Then the partition $(i_1, \ldots, i_{p+q}, j_1, \ldots, j_{p'+q'})$ is of length $(p + p') + (q + q')$ and sums to $p + p'$ so the product ends up in $\mathcal{M}^{p+p',q+q'}$. The dimension in degree (p, q) can be seen to be $\dim_{\mathbb{K}} \mathcal{M}^{p,q} = \binom{p-1}{-q}$. The following picture shows the basis elements of $\mathcal{M}$ in the different degrees:

$$\begin{array}{ccccccc}
q\backslash p & (0) & (1) & (2) & (3) & (4) & \cdots \\
(0) & 1 & \delta^1 & \delta^1\delta^1 & \delta^1\delta^1\delta^1 & \delta^1\delta^1\delta^1\delta^1 & \cdots \\
 & & & {\scriptstyle\partial}\uparrow & \uparrow & \uparrow & \\
(-1) & & & \delta^2 & \delta^1\delta^2, \delta^2\delta^1 & \delta^1\delta^1\delta^2, \delta^1\delta^2\delta^1, \delta^2\delta^1\delta^1 & \cdots \\
 & & & & \uparrow & \uparrow & \\
(-2) & & & & \delta^3 & \delta^1\delta^3, \delta^2\delta^2, \delta^3\delta^1 & \cdots \\
 & & & & & \uparrow & \\
(-3) & & & & & \delta^4 & \cdots \\
\vdots & & & & & & \ddots
\end{array}$$

Figure 1: The grading on $\mathcal{M}$.

Remark A.2.1. When defining double complexes one has the choice between requiring either commutative squares, or anti-commutative squares. That is, we can choose to have $D_1D_0 - D_0D_1 = 0$ or $D_1D_0 + D_0D_1 = 0$. The choice does not matter in the sense that we get equivalent categories either way. The same is true for multicomplexes where we can replace

$$\sum_{p+q=n} D_pD_q = 0 \quad \text{by the relation} \quad \sum_{p+q=n} (-1)^p D_pD_q = 0$$

and obtain equivalent categories. In this case, one will also have to introduce signs in the definition of morphisms.

Suppose that (M, D_0) is a dg-module over $\mathcal{M}$. The action of $\mathcal{M}$ is determined by how the generators act on M. In addition, the action has to satisfy the Leibniz rule

$$\begin{aligned} D_0(\delta^n \cdot x) &= \partial(\delta^n) \cdot x + (-1)^{|\delta^n|}\delta^n D_0(x) \\ &= -\sum_{i=1}^{n-1}(-1)^i \delta^i \delta^{n-i} \cdot x + (-1)^{1-n}\delta^n D_0(x). \end{aligned}$$

If we let $D_n \colon M \to M$ denote multiplication by δ^n, then the Leibniz rule can be rewritten as

$$\sum_{p+q=n} (-1)^p D_p D_q = D_0 D_n - D_1 D_{n-1} + \ldots + (-1)^n D_n D_0 = 0.$$

In other words, $\mathcal{M}$-modules are precisely multicomplexes by remark A.2.1. Let us denote by $\mathscr{M}$ the operad of arity 1 with $\mathscr{M}(1) = \mathcal{M}$. This time, we need to be a bit more careful when specifying the morphisms of $\mathscr{M}$-algebras. We define the morphisms to be the analogue to the morphisms of multicomplexes given in definition 1.2.3. In other words, we require

$$\tilde{D}_0 f_n + \delta^1 f_{n-1} + +\delta^2 f_{n-2} + \cdots + \delta^n f_0 = f_n D_0 + f_{n-1}\delta^1 + f_{n-2}\delta^2 + \cdots f_0 \delta^n$$

to hold for all $n \geqslant 0$. Note that this requirement is weaker than having morphisms of dg-modules. We conclude this section by summarising everything into the following theorem:

Theorem A.2.2. The category of $\mathscr{M}$-algebras (with morphisms defined as above) is equivalent to the category of multicomplexes.

A.3 Multicomplexes are homotopy double complexes

We now want to introduce the notion of a homotopy algebra over an operad. To do this in its full generality, one have to introduce the Koszul dual cooperad $\mathscr{P}^{¡}$ of a quadratic operad $\mathscr{P}$ and the cobar construction $\Omega\mathscr{C}$ on a cooperad $\mathscr{C}$. One then proceeds to define the Koszul resolution $\mathscr{P}_\infty := \Omega\mathscr{P}^{¡}$ of $\mathscr{P}$. By definition, a *homotopy $\mathscr{P}$-algebra* is an algebra over the Koszul resolution $\mathscr{P}_\infty$ of $\mathscr{P}$. Recall that we will only be working with operads of arity 1, so we can restrict our attention to the setting of (co)algebras. The precise definitions of the aforementioned constructions for (co)algebras will be given later throughout this section. Loday and Vallette [LV12] serves as a comprehensive reference for the general setting of Koszul operads and homotopy algebras over these. The rest of this section is dedicated to proving the following theorem:

Theorem A.3.1. Multicomplexes are homotopy double complexes in the sense that $\mathscr{M} = \mathscr{D}_\infty$.

We will prove theorem A.3.1 by computing the Koszul resolution $\mathscr{D}_\infty$ on the level of (co)algebras. That is, we first realise the algebra of dual numbers $\mathscr{D}(1) = \mathcal{D}$ as a quadratic algebra and compute its Koszul dual coalgebra $\mathcal{D}^{¡}$. In the last step, we compute the cobar construction $D_\infty := \Omega\mathcal{D}^{¡}$. This defines the operad $\mathscr{D}_\infty$ by remark A.0.2.

The tensor (co)algebra

Let V be a graded vector space over $\mathbb{K}$. We form the tensor vector space (or tensor module) over V, denoted $T(V)$, by letting $T(V)^n := V^{\otimes n}$. The *tensor algebra* over V is $T(V)$ together with multiplication defined by the concatenation of tensors:

$$(v_1 \otimes \cdots \otimes v_k) \cdot (v_{k+1} \otimes \cdots \otimes v_{k+l}) := v_1 \otimes \cdots \otimes v_{k+l}.$$

We can also equip $T(V)$ with a comultiplication $\Delta \colon T(V) \to T(V) \otimes T(V)$ given by deconcatenation of tensors to obtain the *tensor coalgebra* denoted by $T^c(V)$.

$$\Delta(v_1 v_2 \cdots v_k) := \sum_{i=0}^{n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n \quad \text{and} \quad \Delta(1) := 1 \otimes 1.$$

Remark A.3.2. The product and coproduct defined above are not compatible in the sense that we can not consider $T(V)$ as a bialgebra with this definition of the coproduct.

We equip the tensor coalgebra with counit $\epsilon \colon v \mapsto 0$ and it is coaugmented by the inclusion of $\mathbb{K}$ into degree 0. In general, given a coalgebra C with counit ϵ and coaugmented by u, we have that $\ker \epsilon \to C \to \mathbb{K}$ splits since $\epsilon u = \mathrm{id}$ and hence $C = \ker \epsilon \oplus \mathbb{K}$. We define $\bar{C} := \ker \epsilon$ and equip $\bar{C}$ with the *reduced coproduct* $\bar{\Delta}(x) := \Delta(x) - 1 \otimes x - x \otimes 1$. For the tensor coalgebra, the reduced coproduct is

$$\bar{\Delta}(v_1 \cdots v_n) := \sum_{i=1}^{n-1} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n.$$

Example A.3.3. If $V = \mathbb{K}x_1 \oplus \mathbb{K}x_2 \oplus \cdots \oplus \mathbb{K}x_n$, then $T(V) = \mathbb{K}\langle x_1, x_2, \ldots, x_n \rangle$, the non-commutative (for $n \geqslant 2$) polynomial algebra in n variables. Similarly, if $V = \bigoplus_{i\geqslant 1} \mathbb{K}\, x_i$, then $T(V)$ is the non-commutative polynomial algebra in infinitely many variables.

Some conventions

Given two cochain complexes V and W of graded vector spaces, we define the tensor product $V \otimes W$ as follows:

$$(V\otimes W)^n := \bigoplus_{p+q=n} V^p \otimes_{\mathbb{K}} W^q \quad \text{and} \quad d_{V\otimes W}(v\otimes w) := d_V(v)\otimes w + (-1)^{|v|} v\otimes d_W(w).$$

Let $\mathbb{K}\, s$ denote the graded vector space generated by the element s in degree -1. If V is a graded vector space, define the *suspension* of V to be the graded vector space $sV := \mathbb{K}\, s \otimes V$. Similarly, we let $\mathbb{K}\, s^{-1}$ be the graded vector space generated by the element s^{-1} in degree 1 and define the *desuspension* of V to be the graded vector space $s^{-1}V := \mathbb{K}\, s^{-1} \otimes V$. Clearly, we have that $(sV)^n = \mathbb{K}\, s \otimes V^{n-1}$ and similarly $(s^{-1}V)^n = \mathbb{K}\, s \otimes V^{n+1}$. Thus, this is nothing but the usual suspension functor with the addition of a bookkeeping variable s. If V is a cochain complex of graded vector spaces, i.e. V has a differential, we see that $d_{sV} = -s d_V$. Note that our convention fits the setting of cohomological grading and therefore is opposite to the one used in [LV12]. The natural symmetry isomorphism $\tau\colon V\otimes W \to W\otimes V$ is given by

$$vw \mapsto (-1)^{|w||v|} wv \quad \text{for } v \in V, w \in W.$$

In other words, the cost of permuting two elements in a product is a sign. This is known as the Koszul sign convention.

Quadratic (co)algebras

A *quadratic data* is a pair (V, R) where V is a graded vector space and R is a graded subspace of $V \otimes V$. Given a quadratic data, we can associate with it an algebra. The *quadratic algebra* $A(V, R)$ associated to a quadratic data (V, R) is defined as

$$A(V,R) := T(V)/(R)$$

where (R) is the ideal generated by R. Explicitly, $A(V, R)$ can be described as follows:

$$A(V,R) = \mathbb{K}\cdot 1 \oplus V \oplus V^{\otimes 2}/R \oplus \cdots \oplus \left(V^{\otimes n} / \sum_{i+j+2=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} \right) \oplus \cdots .$$

The quadratic algebra $A(V,R)$ is universal among quotient algebras of $T(V)$ with respect to the property that the composition

$$R \hookrightarrow T(V) \twoheadrightarrow A'$$

is zero. In other words, it is the cokernel of the inclusion of (R). Some familiar examples of quadratic algebras include:

R	**A(V,R)**
0	Tensor algebra $T(V)$
$< vw - wv >$	Symmetric algebra $S(V)$
$< v^2 >$	Exterior algebra $\Lambda(V)$

In particular, if V is the one-dimensional vector space generated by ϵ in degree 0 and $R =< \epsilon^2 >= V \otimes V$, then $A(V,R)$ is the dual numbers $\mathcal{D} = \mathbb{K} \oplus \mathbb{K}\,\epsilon$.

There is also a way to associate a coalgebra to any given quadratic data. Define the *quadratic coalgebra* $C(V,R)$ to be the sub-coalgebra of $T^c(V)$ which is universal among the sub-coalgebras C' such that the composition

$$C' \hookrightarrow T^c(V) \twoheadrightarrow V^{\otimes 2}/R$$

is zero. That is, any such map factors uniquely through $C(V,R)$. The coalgebra structure on $C(V,R)$ is the restriction of the one on $T^c(V)$. The quadratic coalgebra $C(V,R)$ can be described explicitly as follows:

$$C(V,R) = \mathbb{K}\cdot 1 \oplus V \oplus R \oplus \cdots \oplus \left(\bigcap_{i+j+2=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} \right) \oplus \cdots .$$

Endowed with the trivial differential, $C(V,R)$ is a differential graded coalgebra.

The Koszul dual coalgebra

Given a quadratic data (V,R), we define the *Koszul dual coalgebra* $A^{¡}$ of the quadratic algebra $A = A(V,R)$ to be $A^{¡} := C(sV, s^2R)$. Here, the notation s^2R denotes the image of R under the map $V \otimes V \longrightarrow sV \otimes sV, v \otimes w \mapsto sv \otimes sw$.

Again, consider the graded vector space V generated by ϵ in degree 0 so that $A(V,R)$ is the dual numbers. Let $\delta := s\epsilon$ be the generator of sV in degree -1 and write $\delta^n = \underbrace{\delta \otimes \cdots \otimes \delta}_{n\text{-times}}$. The Koszul dual coalgebra of $\mathcal{D}$ is then easily seen to be

$$\mathcal{D}^{¡} = C(sV, s^2R) = C(\mathbb{K}\,\delta, \mathbb{K}\,\delta^2) = \mathbb{K}\,1 \oplus \mathbb{K}\,\delta \oplus \mathbb{K}\,\delta^2 \oplus \cdots = T^c(\delta)$$

with coproduct $\Delta\colon \mathcal{D}^{¡} \to \mathcal{D}^{¡} \otimes \mathcal{D}^{¡}$ being

$$\Delta(\delta^n) = \sum_{i=0}^{n} \delta^i \otimes \delta^{n-i}, \quad \Delta(1) = 1 \otimes 1.$$

The reduced coproduct on $\bar{\mathcal{D}}^{¡} = \bigoplus_{n\geqslant 1} \mathbb{K}\,\delta^n$ is given by $\bar{\Delta}(\delta^n) = \sum_{i=1}^{n-1} \delta^i \otimes \delta^{n-i}$. Equipped with trivial differential, $(\bar{\mathcal{D}}^{¡}, \bar{\Delta})$ is a coaugmented dg-coalgebra.

The cobar construction

The cobar construction is a functor Ω from the category of coaugmented dg-coalgebras to the category of augmented dg-algebras. Let $\Delta_s\colon\ \mathbb{K}\,s^{-1} \to \mathbb{K}\,s^{-1} \otimes \mathbb{K}\,s^{-1}$ denote the diagonal map $s^{-1} \mapsto -s^{-1} \otimes s^{-1}$. Given a coaugmented dg-coalgebra C, we define the map $f\colon s^{-1}\bar{C} \to s^{-1}\bar{C} \otimes s^{-1}\bar{C}$ as the composition

$$\mathbb{K}\,s^{-1} \otimes \bar{C} \xrightarrow{\Delta_s \otimes \bar{\Delta}} \mathbb{K}\,s^{-1} \otimes \mathbb{K}\,s^{-1} \otimes \bar{C} \otimes \bar{C} \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} \mathbb{K}\,s^{-1} \otimes \bar{C} \otimes \mathbb{K}\,s^{-1} \otimes \bar{C}.$$

By proposition 1.1.2 in [LV12], f has a unique extension to a derivation ∂_2 on $T(s^{-1}\bar{C})$. Explicitly, the derivation is given as

$$\begin{aligned} \partial_2\colon T(s^{-1}\bar{C}) &\to T(s^{-1}\bar{C}) \\ v_1 \cdots v_n &\mapsto \sum_{i=1}^{n} v_1 \cdots f(v_i) \cdots v_n \quad \text{and} \quad \partial_2(1) = 0. \end{aligned}$$

Proposition A.3.4. [LV12, Proposition 2.2.4] The coassociativity of $\bar{\Delta}$ implies that ∂_2 is a differential on $T(s^{-1}\bar{C})$, i.e., $\partial_2\partial_2 = 0$.

Given a coaugmented dg-coalgebra (C, Δ, d_C) we define the *cobar construction* of C to be

$$\Omega C := (T(s^{-1}\bar{C}), \partial = \partial_1 + \partial_2)$$

where ∂_1 is the differential induced by d_C. In particular, if $A = A(V, R)$ is a quadratic algebra, then we define $A_\infty := \Omega A^{¡} = T(s^{-1}\bar{A}^{¡})$ and $\partial = \partial_2$ since $\partial_1 = 0$ by definition. The cobar construction of C is given two gradings called the *weight degree* and the *syzygy degree*. In our case where $C = \mathcal{D}^{¡}$, the weight grading is given by $\omega(\delta^{i_1}\cdots\delta^{i_k}) = i_1 + i_2 + \cdots + i_k$ and the syzygy degree is given by $\sigma(\delta^{i_1}\cdots\delta^{i_k}) = k - \omega(\delta^{i_1}\cdots\delta^{i_k}) = k - (i_1 + i_2 + \cdots + i_k)$. The differential ∂ increases the syzygy degree by one. We take the vertical grading to be the syzygy degree and the horizontal grading to be the weight degree. The differential $\partial = \partial_2$ is explicitly computed from $\bar{\Delta}$ to be

$$\partial_2(\delta^n) = (\mathrm{id}\otimes\tau\otimes\mathrm{id})(\Delta_s\otimes\overline{\Delta})(\delta^n) = -\sum_{i=1}^{n-1}(-1)^i\delta^i\delta^{n-i}$$

where the signs come from the Koszul sign convention. We see that $\mathcal{D}_\infty = \mathcal{M}$ which concludes the proof of theorem A.3.1. □

Remark A.3.5. The morphisms we described for $\mathscr{M}$-algebras right before theorem A.2.2 are precisely what is known as ∞-morphisms of $\mathscr{P}_\infty$-algebras in the case where $\mathscr{P} = \mathscr{D}$.

Remark A.3.6. The homotopy transfer theorem for multicomplexes (theorem 2.2.1) follows from the homotopy transfer theorem for operads (See theorem 10.3.1 in [LV12]).

Appendix B Minimal models for multicomplexes

A cochain complex is said to be *minimal* if its differential is zero. A cochain complex is *acyclic* (or *exact*) if it has trivial cohomology in all degrees. As we already have seen in proposition 2.1.3, every cochain complex decomposes as a direct sum $K \oplus H$ with K acyclic and H minimal. We extend the notions above to multicomplexes in the following way:

Definition B.0.1. A multicomplex $(M, D_\bullet)$ is called *minimal* if $D_0 = 0$ and *acyclic* if the underlying cochain complex (M, D_0) is acyclic. If $D_r = 0$ for all $r \geqslant 1$, we say that M is *trivial*.

We note that a trivial multicomplex is nothing else than a cochain complex of graded vector spaces, i.e., it has no higher differentials. It is also evident that a multicomplex $(M, D_\bullet)$ is minimal and acyclic if and only if M is the zero multicomplex. In this section, we prove the following result from [DSV15] which is the analogue of proposition 2.1.3 for multicomplexes:

Theorem B.0.2. [DSV15, Theorem 1.6] Every multicomplex M can be decomposed into a direct sum $K \oplus H$ where K is acyclic trivial and H is minimal.

Recall that we can decompose the underlying cochain complex of a multicomplex M as $M^{p,q} = K^{p,q} \oplus H^{p,q}$ where $K^{p,q} := B^{p,q} \oplus B^{p,q+1}$. Moreover, we have a deformation retract (π, ι, h) of (M, D_0) to H. By the homotopy transfer theorem (theorem 2.2.1) we can turn H into a minimal multicomplex $(H, D'_\bullet)$ using (π, ι, h). The differential on K is $D_0^K = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) : K^{p,q} \to K^{p,q+1}$, so clearly K is acyclic. By letting $D_r^K = 0$ for all $r \geqslant 1$, we have that $(K, D_\bullet^K)$ is an acyclic trivial multicomplex. Now, $K \oplus H$ is a multicomplex with differentials $D_n^{K\oplus H} = D_n^K \oplus D'_n$. In matrix form, the differentials are

$$D_0^{K\oplus H} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and } D_n^{K\oplus H} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D'_n \end{pmatrix} \quad \text{for } n \geqslant 1.$$

What is left to show is that $(M, D_\bullet)$ and $(K \oplus H, D_\bullet^{K\oplus H})$ are isomorphic as multicomplexes. Define $\pi_0 := \pi$ and $\pi_n := \sum_{|I|=n} \pi D_{i_1} h \cdots h D_{i_k} h$ for $n \geqslant 1$. Similarly, let $q\colon M \to K$ be the projection to K and define $q_0 := q$ and $q_n := -q h D_n$ for $n \geqslant 1$.

Lemma B.0.3. The maps $\pi_\infty = \{\pi_n\}\colon M \to H$ and $q_\infty = \{q_n\}\colon M \to K$ are morphisms of multicomplexes.

Proof. For π_∞, this is part of the homotopy transfer theorem (theorem 2.2.1). For q_∞, we have

$$\sum_{i=0}^{n} q_i D_{n-i} = qD_n + qhD_0D_n \quad \text{and} \quad \sum_{i=0}^{n} D_i^K q_{n-i} = D_0^K q_n = -D_0^K qhD_n.$$

Multiplying matrices, we see that $q + qhD_0 = -D_0^K qh$ and hence q_∞ is a morphism of multicomplexes. □

Let $f\colon M \to N$ and $g\colon M \to Q$ be morphisms of multicomplexes. We define the sum of f and g denoted $f + g\colon M \to N \oplus Q$ by letting $(f+g)_n = f_n + g_n$. We claim that $f + g$ is a morphism of multicomplexes as it is straightforward to check that

$$\sum_{i=0}^{n} (f+g)_i D_{n-i}^M = \sum_{i=0}^{n} D_i^{N\oplus Q}(f+g)_{n-i} \quad \text{holds for all } n \geqslant 0.$$

Proof of Theorem B.0.2. Define the map $r = \{r_n\}\colon M \to K \oplus H$ where $r_0 := q + \pi$ and

$$r_n := q_n + \pi_n = -qhD_n + \sum_{|I|=n} \pi D_{i_1} h \cdots hD_{i_k} h \quad \text{for } n \geqslant 1.$$

By lemma B.0.3, r defines a morphism of multicomplexes. Since r_0 is an isomorphism of cochain complexes, it follows from proposition 1.2.6 that r is invertible. □

Remark B.0.4. The notions of being minimal, acyclic and trivial can be translated into the more general setting of Koszul operads. Precise definitions and results can be found in detail in [LV12, Chapter 10.4.2]. In particular, as is remarked in [DSV15], theorem B.0.2 follows from an application of [LV12, Theorem 10.4.3] to the operad $\mathscr{D}$ of dual numbers.

Example B.0.5. We consider the toy example where M is the multicomplex

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "1"] & \mathbb{K} & \\
& \mathbb{K} \arrow[u, "1"] \arrow[r, "1"] & \mathbb{K}
\end{tikzcd}$$

and compute the multicomplexes K and H such that $M = K \oplus H$. The acyclic trivial multicomplex K can be seen to be

$$\begin{tikzcd}
0 & \mathbb{K} \oplus 0 & \\
& 0 \oplus \mathbb{K} \arrow[u, "\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)"] & 0
\end{tikzcd}$$

The transferred differentials are all trivial, except for D_2' which is multiplication by -1. (We computed this in example 2.2.2.) Thus, the minimal multicomplex H looks as follows:

$$\begin{tikzcd}
\mathbb{K} \arrow[drr, "-1"] & 0 & \\
& 0 & \mathbb{K}
\end{tikzcd}$$

Appendix C Deformations of cochain complexes

If V is a vector space over $\mathbb{K}$, we define $V[\![t]\!] := V \otimes_{\mathbb{K}} \mathbb{K}[\![t]\!]$. In other words, $V[\![t]\!]$ is the vector space of formal power series with coefficients in V. Similarly, we define $V[t]$ to be the vector space of polynomials with coefficients in V. A formal deformation of a cochain complex (M, D_0) is a $\mathbb{K}[\![t]\!]$-linear map $D\colon M[\![t]\!] \to M[\![t]\!]$ such that $D^2 = 0$. In addition, we require that D evaluated at $t = 0$ is exactly the underlying differential D_0. One might think of such deformations as curves in the "space" of cochain complexes passing through M at $t = 0$. We prove the following theorem later in this section:

Theorem C.0.1. The category $\mathbf{Dfm}_{\mathbb{K}}$ of formal deformations is equivalent to the category $\mathbf{MC}_{\mathbb{K}}$ of multicomplexes.

Proposition C.0.2. If V and W are vector spaces over $\mathbb{K}$, then we have an isomorphism

$$\mathrm{Hom}_{\mathbb{K}[\![t]\!]}(V[\![t]\!], W[\![t]\!]) \cong \prod_{i\geqslant 0} \mathrm{Hom}_{\mathbb{K}}(V, W).$$

Proof. Let $f\colon V[\![t]\!] \to W[\![t]\!]$ be a $\mathbb{K}[\![t]\!]$-linear map. For every v in V, we can write $f(v)$ as a sum $f(v) = w_0 + w_1 t + w_2 t^2 + \cdots$ with coefficients $w_0, w_1, \ldots$ in W. The family $(f_i\colon v \mapsto w_i)_{i\geqslant 0}$ defines an element in $\prod_{i\geqslant 0} \mathrm{Hom}_{\mathbb{K}}(V, W)$. Conversely, given a family of linear maps $(f_i)_{i\geqslant 0}$, we define $f\colon V[\![t]\!] \to W[\![t]\!]$ to be the map given by $v \mapsto f_0(v) + f_1(v)t + f_2(v)t^2 + \cdots$ on V. □

Let us introduce the notion of a differential vector space which is nothing more than a non-graded analogue of (co)chain complexes of vector spaces. We will introduce the appropriate grading before proving theorem C.0.1.

Definition C.0.3. A *differential vector space* (V, d) consists of a vector space V together with a square-zero linear map $d\colon V \to V$. A *morphism of differential vector spaces* $f\colon (V, d) \to (W, d')$ is a linear map $f\colon V \to W$ which commutes with the differentials.

Definition C.0.4. Let (V, d) be a differential vector space. A *(formal) deformation* of V is a $\mathbb{K}[\![t]\!]$-linear map $D\colon V[\![t]\!] \to V[\![t]\!]$ which satisfies

$$D^2 = 0 \quad \text{and} \quad D \equiv d \pmod{(t)}.$$

By proposition C.0.2, we can think of D as a family of maps $D_0, D_1, D_2, \ldots$ such that $D(v) = D_0(v) + D_1(v)t + D_2(v)t^2 + \cdots$. The condition $D \equiv d \pmod{(t)}$ then becomes equivalent to having $D_0 = d$. The condition $D^2 = 0$ can be written as follows:

$$0 = D^2(v) = D\left(\sum_{r\geqslant 0} D_r(v)t^r\right) = \sum_{r\geqslant 0} D(D_r(v))t^r = \sum_{r\geqslant 0}\sum_{s\geqslant 0} D_s D_r(v)t^{r+s}.$$

By comparing coefficients, we see that $D^2 = 0$ if and only if

$$\sum_{i=0}^{n} D_i D_{n-i} = 0 \quad \text{for all } n \geqslant 0. \tag{13}$$

Let (M, D_0) be a cochain complex of graded vector spaces. To comply with our grading convention, we equip $M[\![t]\!]$ with the grading

$$M[\![t]\!]^{p,q} := M^{p,q}\cdot 1 \oplus M^{p+1,q-1}\cdot t \oplus \cdots \oplus M^{p+n,q-n}\cdot t^n \oplus \cdots .$$

That is, every homogeneous element of degree (p, q) is of the form $f(t) = \sum_{n\geqslant 0} c_n t^n$ with $c_n \in M^{p+n,q-n}$. In this case, where (M, D_0) is a cochain complex of graded vector spaces, we define a *(formal) deformation D of M* to be a $\mathbb{K}[\![t]\!]$-linear map

$$D\colon M[\![t]\!] \to M[\![t]\!]$$

of degree $|D| = (0, 1)$ which satisfies the two conditions from definition C.0.4. In this graded setting, we impose the same boundedness condition on M as we did with multicomplexes. That is, we require that for each n there exists an integer $s(n)$ such that $M^{p,n-p} = 0$ whenever $p \geqslant s(n)$. Note that this ensures that every formal deformation D is locally finite. We can visualise a deformation D as follows:

$$\begin{tikzcd}
M[\![t]\!]^{p,q+1} & M^{p,q+1}\cdot 1 & M^{p+1,q}\cdot t & M^{p+2,q-1}\cdot t^2 & M^{p+3,q-2}\cdot t^3 & \cdots \\
M[\![t]\!]^{p,q} \arrow[u, "D"] & M^{p,q}\cdot 1 \arrow[u, "D_0"] \arrow[ur, "D_1 t"] \arrow[urr, "D_2 t^2"] \arrow[urrr, "D_3 t^3"] & M^{p+1,q-1}\cdot t & M^{p+2,q-2}\cdot t^2 & M^{p+3,q-3}\cdot t^3 & \cdots
\end{tikzcd}$$

Definition C.0.5. Let D and D' be deformations of (V, D_0) and (W, D'_0) respectively.

1. A *morphism $f\colon (V, D) \to (W, D')$ of deformations* is the data of a $\mathbb{K}[\![t]\!]$-linear map $f\colon V[\![t]\!] \to W[\![t]\!]$ which satisfies $D'f = fD$. In the graded setting, we require f to have degree 0.

2. If $g\colon (W, D') \to (U, D'')$ is a morphism of deformations, we define the composition $gf\colon (V, D) \to (U, D'')$ by letting $(gf)(v) := g(f(v))$.

Definition C.0.6. The category $\mathbf{Dfm}_{\mathbb{K}}$ consists of objects (M, D) where D is a deformation of (M, D_0) and with morphisms and composition defined as above.

Let (M, D_0) and (N, D_0') be cochain complexes of graded vector spaces and let $f\colon (M, D) \to (N, D')$ be a morphism of deformations. By proposition C.0.2, f consists of a family $(f_n)_{n\geqslant 0}$ of linear maps $f_n\colon M^{p,q} \to N^{p+n,q-n}$ (the degrees of these maps follow from our grading convention). Furthermore, by $K[\![t]\!]$-linearity, the requirement $D'f = fD$ can be rewritten as

$$\sum_{n\geqslant 0}\sum_{p+q=n} f_q D_q = \sum_{n\geqslant 0}\sum_{p+q=n} D_p f_q. \tag{14}$$

Proof of Theorem C.0.1. Define the functor $F\colon \mathbf{Dfm}_{\mathbb{K}} \to \mathbf{MC}_{\mathbb{K}}$ by $(M, D) \mapsto (M, D_\bullet)$ and $f \mapsto (f_n)_{n\geqslant 0}$. That $F(M)$ is a multicomplex follows from eq. (13). By comparing coefficients in eq. (14), we see that $F(f)$ defines a morphism of multicomplexes. An inverse $G\colon \mathbf{MC}_{\mathbb{K}} \to \mathbf{Dfm}_{\mathbb{K}}$ to F is given by mapping a multicomplex $(M, D_\bullet)$ to the deformation D on (M, D_0) given by $D(v) = \sum_{n\geqslant 0} D_n(v)t^n$. On morphisms, we let $G(f)$ be the map defined by $G(f)(v) := \sum_{n\geqslant 0} f_n(v)t^n$. Again, by comparing coefficients, one can easily see that the given definition of composition in $\mathbf{Dfm}_{\mathbb{K}}$ ensures that F and G are indeed functors. $\square$

C.1 Finite order deformations

Replacing $\mathbb{K}[\![t]\!]$ by the polynomial ring $\mathbb{K}[t]/(t^{n+1})$ in definition C.0.4, we obtain the notion of *n-th order deformations*. To simplify notation, let us write $V[t]/(t^{n+1})$ for the vector space $V \otimes \mathbb{K}[t]/(t^{n+1})$. We adopt the same grading convention as before when considering deformations of cochain complexes of graded vector spaces. We now describe the cases where $n = 0, 1, 2$ for a fixed cochain complex of graded vector spaces.

Example C.1.1 ($n = 0$)**.** Since $\mathbb{K}[t]/(t) = \mathbb{K}$, a zeroth order deformation of (M, D_0) is a map $D\colon M \to M$ such that $D(x) = D_0(x)$ for all $x \in M$. In other words, the only zeroth order deformation of (M, D_0) is the constant one with $D = D_0$.

Example C.1.2 ($n = 1$)**.** Suppose $D\colon M[t]/(t^2) \to M[t]/(t^2)$ is a first order deformation of M. That is, for every x in M we can write $D(x) = D_0(x) + D_1(x)t$. The condition $D^2 = 0$ is equivalent to having $D_0D_0 = 0$ and $D_0D_1 + D_1D_0 = 0$. The first equation is always true of course. Thus, the first order deformations of (M, D_0) are in one-to-one correspondence with linear maps $D_1 \colon M \to M$ satisfying $D_0 D_1 + D_1 D_0 = 0$. In other words (and symbols),

$$\text{First order deformations of } (M, D_0) \cong \prod_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Ch}\,\mathbb{K}}(M^{p,\bullet}, M^{p+1,\bullet}).$$

Example C.1.3 ($n = 2$)**.** Similarly to the previous example, second-order differentials are determined by two linear maps D_1 and D_2 such that

$$0 = D_0 D_0 = D_0 D_1 + D_1 D_0 = D_0 D_2 + D_1 D_1 + D_2 D_0 = D_1 D_2 + D_1 D_2.$$

In the spirit of theorem C.0.1 we see that double complexes appear as the second order deformations $D = D_0 + D_1 t + D_2 t^2$ with $D_2 = 0$.

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