STRONG LOCALIZATION OF INVARIANT METRICS

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Abstract. A quantitative version of strong localization of the Kobayashi, Azukawa and Sibony metrics, as well as of the squeezing function, near a plurisubharmonic peak boundary point of a domain in $\mathbb{C}^n$ is given. As an application, the behavior of these metrics near a strictly pseudoconvex boundary point is studied. A weak localization of the three metrics and the squeezing function is also given near a plurisubharmonic antipeak boundary point.

1. Introduction

Denote by $\mathbb{D} \subset \mathbb{C}$ the unit disc. Let D be an open set in $\mathbb{C}^n$. The Kobayashi, Azukawa and Sibony metrics of D at $(z, X) \in D \times \mathbb{C}^n$ are defined in the following way:

$$K_D(z; X) = \inf\{|\alpha| : \exists f \in \mathcal{O}(\mathbb{D}, D),\ f(0) = z,\ \alpha f'(0) = X\};$$

$$A_D(z; X) = \limsup_{\lambda \to 0} \frac{g_D^*(z, z + \lambda X)}{|\lambda|},$$

where $g_D^* = \exp g_D$ and

$$g_D(z, w) = \sup\{u(w) : u \in \mathrm{PSH}(D),\ u < 0,\ u < \log|\cdot - z| + C\}$$

is the pluricomplex Green function of D with pole at z;

$$S_D(z; X) = \sup_v [L_v(z; X)]^{1/2},$$

where L_v is the Levi form of v, and the supremum is taken over all log-psh functions v on D such that $0 \le v < 1$, $v(z) = 0$, and v is C^2 near z (log=logarithm, (p)sh= (pluri)subharmonic).

It is well-known that $S_D \le A_D \le K_D$.

For various properties of these metric we refer the reader to [6].

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Denote now by $\mathbb{B}_n = \mathbb{B}_n(0,1)$ the unit ball in $\mathbb{C}^n$. For any holomorphic embedding $f : D \to \mathbb{B}_n$ with $f(z) = 0$, set

$$\sigma_D(f,z) = \sup\{r > 0 : r\mathbb{B}_n \subset f(D)\}.$$

The squeezing function of D is defined by $\sigma_D(z) = \sup_f \sigma_D(f,z)$ if such f's exist, and $\sigma_D(z) = 0$ otherwise (that is, if D is not biholomorphic to a bounded open set) – see e.g. [8, 9] and the references therein.

Recall now that a point $p \in \partial D$ is called a psh peak point (resp. antipeak) if there exists a psh function φ on D such that $\lim_{z\to p} \varphi(z) = 0$ and $\sup_{D\setminus U} \varphi < 0$ (resp. $\lim_{z\to p} \varphi(z) = -\infty$ and $\inf_{D\setminus U} \varphi > -\infty$) for any neighborhood U of p. Note that the notion of psh peak point has a local character, and such a point admits a negative psh antipeak function (see the proof of [5, Lemma 2.1.1]). Assuming that $\varphi = \log|f|$, where $f \in \mathcal{O}(D,\mathbb{D})$, we define the notion of holomorphic peak point.

Strong localization of invariant metrics as in (3) below plays crucial rule in many boundary problems in complex analysis. Such a localization for K_D near a psh peak point follows by [1, Proposition 2.1.a)] (see also [5, Lemma 2.1.1]). The same is true for A_D near a holomorphic peak point (see [7, Corollaries 1 & 2]). A quantitative version of strong localization for K_D near special holomorphic peak points is given in [4, Theorem 2.1 & Lemma 2.2].

The main aim of this note is to give a quantitative version of strong localization for $M_D \in \{K_D, A_D, S_D\}$ and σ_D near a psh peak point in terms of the respective psh peak function.

Theorem 1. *Let D be a domain in $\mathbb{C}^n$. Suppose that there exists a psh peak function φ for $p \in \partial D$; φ is assumed C^2 near p if $M_D = S_D$. Then for any bounded neighborhood U of p there are a neighborhood $V \subset U$ of p and a constant $m > 0$ such that for $z \in D \cap V$ one has that*

$$M_D(z;X) \ge e^{m\varphi(z)} M_{D\cap U}(z;X), \quad X \in \mathbb{C}^n, \tag{1}$$

$$\sigma_{D\cap U}(z) \ge e^{m\varphi(z)} \sigma_D(z). \tag{2}$$

In particular,

(i) since $M_{D\cap U} \ge M_D$, then

$$\lim_{z\to p} \frac{M_D(z;X)}{M_{D\cap U}(z;X)} = 1 \text{ uniformly in } X \in (\mathbb{C}^n)_*; \tag{3}$$

(ii) since $\sigma_{D\cap U} \le 1$, then $\lim_{z\to p} \sigma_D(z) = 1$ implies $\lim_{z\to p} \sigma_{D\cap U}(z) = 1$.

$(*)$ *In addition, if D is bounded, then $V \Subset U$ can be chosen arbitrary.*

Remark. Theorem 1 (ii) is exactly [9, Proposition 2]. The inverse implication cannot be true without global assumptions about D; it is even possible $\sigma_D = 0$ but $\sigma_{D\cap U} = 1$.

When D is an unbounded domain in $\mathbb{C}^n$ and $p=\infty$, we use the same definition of a psh peak point as above (see e.g. [5, Definition 1.4. (a)]). Then we have the following counterpart of Theorem 1.

Proposition 2. *Let D be an unbounded domain in $\mathbb{C}^n$. Suppose that there exists a psh peak function φ for $p=\infty$. Then for any neighborhood U of ∞ there are a neighborhood $V\subset U$ of ∞ and a constant $m>0$ such that for $z\in D\cap V$ one has that*

$$K_D(z;X)\ge e^{m\varphi(z)}K_{D\cap U}(z;X),\quad X\in\mathbb{C}^n,$$

$$\sigma_{D\cap U}(z)\ge e^{m\varphi(z)}\sigma_D(z).$$

Note that if $p=\infty$ is a psh peak point of D, then any bounded subset of D is uniformly M_D-hyperbolic; more precisely:

Proposition 3. *Let $p=\infty$ be a psh peak point of an unbounded domain D in $\mathbb{C}^n$. Then for any $r>0$ there exists a constant $c>0$ such that*

$$M_D(z;X)\ge c|X|,\quad z\in D\cap r\mathbb{B}_n,\ X\in\mathbb{C}^n.$$

2. Proof of Theorem 1

The case $M_D=K_D$. By [10, Lemma 2], we have that

$$K_D(z;X)\ge K_{D\cap U}(z;X)\inf_{D\setminus U} l_D(z,\cdot),$$

where

$$l_D(z,w)=\inf\{|\alpha|:\exists f\in\mathcal{O}(\mathbb{D},D),\ f(0)=z,\ f(\alpha)=w\}.$$

Note that

$$l_D(z,w)=l_D(w,z)\ge g_D^*(w,z)\text{ and }\inf_{D\setminus U} l_D(z,\cdot)=\inf_{D\cap\partial U} l_D(z,\cdot)$$

($\ge$ follows by the Schwarz lemma for log-sh functions). Hence

$$K_D(z;X)\ge K_{D\cap U}(z;X)\inf_{D\cap\partial U} g_D^*(\cdot,z). \tag{4}$$

Let now $W=\mathbb{B}_n(p,1)$ and θ be a negative psh antipeak function for D at p. We may assume that $U=\mathbb{B}_n(p,r)$ $(r<1)$ and

$$\inf_{D\setminus W}\theta\ge c=1+\sup_{D\cap U}\theta$$

($\inf=c$ if $D\subset W$). Setting $\tilde\theta=1+(1-r^2)(\theta-c)$, then

$$\hat\theta=\begin{cases}|\cdot-p|^2, & D\cap U\\ \max\{|\cdot-p|^2,\tilde\theta\}, & D\cap W\setminus U\\ \tilde\theta, & D\setminus W\end{cases}$$

is a bounded psh function on D.

Let $\chi : [0,\infty) \to [0,1]$ be C^∞ such that $\chi = 1$ on $[0,(1-r)/2]$ and supp $\chi \in [0, 1-r]$. For any neighborhood $\hat{U} \Subset U$ of p, we may choose first $\hat{m}$ and then m such that

$$\psi = \chi(|\cdot - w|)\log|\cdot - w| + \hat{m}(\hat{\theta} - \sup_D \hat{\theta} - 1)$$

and

$$\hat{\varphi} = \begin{cases} \max\{\psi, m\varphi\}, & D \cap \hat{U} \\ \psi, & D \setminus \hat{U} \end{cases}$$

to be psh functions on D, when $w \in D \cap \partial U$. Then

$$g_D(w,\cdot) \geq \hat{\varphi} \geq m\varphi \quad \text{on} \quad D \cap \hat{U}$$

which implies (1) for $M_D = K_D$.

The case $M_D = A_D$. Let $a = \sup_{D\setminus U} \varphi$. We may choose a neighborhood $W \subset U$ of p such that

$$0 < c = \inf_{D\cap W} \varphi - a.$$

Let $V \Subset W$ be a neighborhood of p and

$$m = -c^{-1} \inf\{g_{D\cap U}(z,w) : z \in D \cap V,\ w \in D \cap U \setminus W\}.$$

For $z \in D \cap V$ and $w \in D \cap U$, set $v_z(w) = g_{D\cap U}(z,w) + m\varphi(w)$ and

$$u_z = \begin{cases} v_z, & D \cap W \\ \max\{v_z, ma\}, & D \cap U \setminus W \\ ma, & D \setminus U \end{cases}.$$

Then $u_z < 0$ is psh on D which implies (1) for $M_D = A_D$.

The case $M_D = S_D$. Since p is a psh antipeak point, an obvious modification in the proof of [7, Theorem 1] (see also the construction of $\hat{\theta}$ above) implies that one may find a ball $W = \mathbb{B}_n(p,r)$ and a constant $c > 0$ such that for any $z \in D \cap W$ there exists a psh function $\theta_z < c$ on D with

$$\theta_z(w) = \log|w - z|, \quad w \in D \cap W.$$

The rest of the proof is similar to that of [3, Lemma 5]. We may assume that $U = \mathbb{B}_n(p, r/5)$. Let $\varepsilon \in (0, r^{-2}/2]$, $z \in U$ and v_z be a competitor for $S_{D\cap U}(z;X)$. Setting

$$\tilde{\theta}_z = 3\theta_z - 3\log r + \log 9, \quad \tilde{v}_z = v_z + \varepsilon e^{2\theta_z},$$

then

$$\hat{v}_z = \begin{cases} \max\{\tilde{\theta}_z, \tilde{v}_z\}, & D \cap \mathbb{B}_n(z, r/2) \\ \tilde{\theta}_z, & D \setminus \mathbb{B}_n(z, r/2) \end{cases}.$$

is a psh function and $\hat{v}_z = \tilde{v}_z$ near z.

If $w \in D \cap U$, then $|w - z| < 2r/5$ and hence $\tilde{\theta}_z(w) < 0$. So, we may choose a number m such that $\tilde{\theta}_z + m\varphi < 0$ for any $z \in D \cap U$. Then $(1 + \varepsilon e^{2c})^{-1} e^{\hat{v}_z + m\varphi}$ is a competitor for $S_D(z, X)$ which implies that

$$(1 + \varepsilon e^{2c}) S_D(z;X) e^{m\varphi(z)} S_{D\cap U}(z;X).$$

It remains to let $\varepsilon \to 0$.

The case σ_D. The proof of [9, Proposition 2] implies that

$$\sigma_{D\cap U}(z) \geq \sigma_D(z) \inf_{D\setminus U} l_D(z,\cdot) \geq \sigma_D(z) \inf_{D\cap\partial U} g_D^*(\cdot,z). \tag{5}$$

Then (2) follows as (1) in the case $M_D = K_D$.

Finally, to prove $(*)$, note that a weak localization for M_D holds near any boundary point (see e.g. [10, Lemma 2] or [3, Lemma 3], [7, Remark] and [3, Lemma 5] for $M_D = K_D$, $M_D = A_D$ and $M_D = S_D$, respectively; see also Proposition 7). Then a compactness argument provides a constant $c > 0$ such that

$$M_D \geq cM_{D\cap U} \text{ on } (D\cap V)\times\mathbb{C}^n, \tag{6}$$

and now $(*)$ easily follows.

Remark. The proof of the case $M_D = A_D$ implies that if $\varphi < 0$ is a psh function on a bounded domain D in $\mathbb{C}^n$, K and L are disjoint compacts in $\mathbb{C}^n$, and $\sup_{D\cap\partial K} \varphi < 0$, then there exists a constant $m > 0$ such that

$$g_D(z,w) \geq m\varphi(w), \quad z \in D\cap K, w \in D\cap L.$$

For example, this can be applied to any compact subset K of a bounded hyperconvex domain D with an exhaustion function φ (that is, $\varphi \in \mathrm{PSH}(D)$, $\varphi < 0$, and $\lim_{z\to\partial D} \varphi(z) = 0$).

3. Proofs of Propositions 2 and 3

Proof of Proposition 3. We will assume that z, w and $|z| < r$.

Let φ be a psh peak function at ∞. Choose $s > r+1$ such that

$$\inf_{|w|>s} \varphi(w) =: \alpha > \beta := \sup_{|w|<r+1} \varphi(w).$$

Set $\theta_z(w) = \log|w-z| + \frac{\beta}{\alpha-\beta}\log(s+r)$, $\eta = \frac{\log(s+r)}{\alpha-\beta}\varphi$,

$$\psi_z(w) = \begin{cases} \theta_z(w), & |w-z| \leq 1 \\ \max\{\theta_z(w), \eta(w)\}, & |w-z| > 1,\ |w| < s \\ \eta(w), & |w| \geq s \end{cases}.$$

Then we may take $e^{2\psi_z}$ as a candidate in the definition of $S_D(z;X)$ which implies that

$$M_D(z;X) \geq S_D(z;X) \geq (s+r)^{\frac{\beta}{\alpha-\beta}}|X|. \quad \square$$

Proof of Proposition 2. We may assume that $U = W_r := \{z \in \mathbb{C}^n : |z| > r\}$. Keeping the notations from the previous proof and setting $m = \frac{\log(s+r)}{\alpha-\beta}$, $V = W_s$, it follows that

$$g_D(w,z) \geq m\varphi(z), \quad w \in D\setminus U,\ z \in D\cap V.$$

Then (4) and (5) complete the proof. $\square$

4. Further results

The next proposition is known (with a different proof) in the case, when D is bounded and $M_D = K_D$ (see [4, p. 244, Remark]).

Set $d_D = \operatorname{dist}(\cdot, \partial D)$, $\delta_D = -d_D$ on D, and $\delta = d_D$ otherwise.

Recall that a point $p \in \partial D$ is said to be strictly pseudoconvex if ∂D is C^2 near p and $L_\delta(p; X) > 0$ for any $X \in T_p^{\mathbb{C}}(\partial D)$, $X \neq 0$.

Proposition 4. *Let p be a strictly pseudoconvex boundary point of a domain D in $\mathbb{C}^n$. Then for any neighborhood U of p there are a neighborhood $V \subset U$ of p and a constant $c > 0$ such that for $z \in D \cap V$ one has that*

$$M_D(z; X) \geq (1 - c d_D(z)) M_{D\cap U}(z; X), \quad X \in \mathbb{C}^n,$$
$$\sigma_{D\cap U}(z) \geq (1 - c d_D(z)) \sigma_D(z).$$

In addition, if D is bounded, then $V \Subset U$ can be chosen arbitrary.

Remark. The estimate for the squeezing function is optimal. Indeed, let $D = \mathbb{B}^n$ and U be a neighborhood of $p \in \partial D$ such that $D \cap U$ is not biholomorphic to D. Then $\sigma_D = 1$ and, by [2, Theorem 1.2], $\sigma_{D\cap U} \leq 1 - c' d_D$ near p for some $c' > 0$.

Proof. It is well-known that there exist a constant $c' > 0$, a neighborhood U' of p and a continuous function h in the closure of $(\partial D \cap U') \times (D \cap U')$ such that for any $q \in \partial D \cap U'$:
(i) $h(q; \cdot)$ is a holomorphic peak function for $D \cap U'$ at q;
(ii) $|1 - h(q; z)| \leq c' d_D(z)$, $z \in D \cap U' \cap n_q$, where n_q is the inner normal to ∂D at q.

Setting $\tilde{\varphi} = \log |h|$, it remains to repeat the proof of Theorem 1 for q near p. □

Corollary 5. *Let p be a strictly pseudoconvex point of a domain D in $\mathbb{C}^n$. Then there exist a neighborhood V of p and a constant $c > 0$ such that*

$$1 \geq \frac{A_D(z; X)}{K_D(z; X)} \geq \frac{S_D(z; X)}{K_D(z; X)} \geq 1 - c d_D(z), \quad z \in D \cap V,\ X \in \mathbb{C}^n.$$

Proof. There exists a bounded neighborhood U of p such that $D \cap U$ is biholomorphic to a convex domain. Then Lempert's theorem implies that $K_{D\cap U} = A_{D\cap U} = S_{D\cap U}$. It remains to apply Proposition 4. □

Corollary 6. *Let $\varepsilon \in (0, 1]$, $k \in \{0, 1\}$, $\varepsilon_k = \frac{k+\varepsilon}{2}$, and p be a $C^{k+2,\varepsilon}$-smooth strictly pseudoconvex boundary point of a domain D in $\mathbb{C}^n$. Then there exist a neighborhood V of p and a constant $c > 0$ such that*

$$(1 - C d_D(z)^{\varepsilon_k}) \Big(\frac{|\langle \partial d_D(z), X\rangle|^2}{d_D^2(z)} + \frac{L(z; X)}{d_D(z)} \Big)^{1/2} \leq M_D(z; X)$$
$$\leq (1 + C d_D(z)^{\varepsilon_k}) \Big(\frac{|\langle \partial d_D(z), X\rangle|^2}{d_D^2(z)} + \frac{L(z; X)}{d_D(z)} \Big)^{1/2}, \quad z \in D \cap V,\ X \in \mathbb{C}^n,$$

where L is the Levi form of $-d_D$.

Proof. Corollary 5 implies that it is enough to prove Corollary 6 for $M_D = K_D$ which is exactly [8, Theorem 2]. □

Corollary 6 remains true in the C^2-smooth case, replacing the term $Cd_D(z)^{\varepsilon_k}$ by any positive number.

Finally, we claim the following weak localization for M_D which can be easily derived from the proof of Theorem 1.

Proposition 7. *Let p be a psh antipeak boundary point of a domain D in $\mathbb{C}^n$.*[1] *Then for any neighborhood U of p there are a neighborhood $V \subset U$ of p and a constant $c > 0$ such that*

$$M_D(z;X) \geq cM_{D\cap U}(z;X), \quad \sigma_{D\cap U}(z) \geq c\sigma_D(z), \quad z \in D,\ X \in \mathbb{C}^n.$$

In particular, if D is bounded, this holds for any $p \in \partial D$ and any $V \Subset U$.

Note that Proposition 7 for A_D is claimed in [7, Remark, pp. 70–71].

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[1] In fact, we need a weaker assumption on the respective antipeak function φ : $\limsup_{z\to p} \varphi(z) < \inf_{D\setminus U} \varphi$ for any neighborhood U of p.

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