Universal Zero Dynamics: The SISO Case

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***Abstract*—Given a single-input, single-output (SISO) system with a Chen-Fliess series representation whose generating series has a well defined relative degree, it is shown that there is a notion of universal zero dynamics that describes a set of dynamics evolving on a locally convex (infinite dimensional) Lie group so as to render the system's output exactly zero. Minimum phase in this setting is defined in terms of the boundedness of the applied input which zeros the output. As an application, it is shown that one can design a zero dynamics attack on cyber-infrastructure using only an estimate of the plant's generating series. That is, detailed knowledge of the plant's internal dynamics is not needed.**

***Keywords*—nonlinear control systems, zero dynamics, Chen-Fliess series, Lie groups, cybersecurity**

I. Introduction

Geometric nonlinear control theory in its most basic form rests on three fundamental concepts: relative degree, feedback linearization, and zero dynamics. As described, for example, in [17], [22], each of these notions assumes that the plant has a smooth local state space realization of finite dimension which is affine in the control. But beginning with [9] and further developed in [10], [12], it has been shown that the notion of relative degree can also be described in a purely input-output setting assuming the input-output map can be realized in terms of a Chen-Fliess series [4], [5]. Furthermore, it was shown in [10] that feedback linearization can also be described in this setting using concepts from combinatorial algebra to calculate the feedback linearizing law. That is, the availability of a finite dimensional state space realization is superfluous to solving the input-output linearization problem. The goal of this paper is to address in this setting the third element of this trifecta, the zero dynamics [18]. The claim is that given an input-output system having a (not necessarily convergent) Chen-Fliess series representation whose generating series has a well defined relative degree, there is a notion of *universal zero dynamics* that describes a set of dynamics evolving on a (infinite dimensional) locally convex Lie group so as to render the system's output exactly zero. This idea is inspired in part by the notion of a *universal control system* proposed by Kawski and Sussmann in [20], [26] and recently generalized by the authors in [11] for networks of systems. In this setting there is no a priori assumption that the input-output map has a finite dimensional state space realization. This is in sharp contrast to other algebraic methods such as those described in [25], [29] that work exclusively in such a state space setting using Gröbner bases and differential algebra, respectively. It is not immediately evident at present how the universal zero dynamics are related to the standard zero dynamics in the finite dimensional case. This will be a topic for future investigation. Nevertheless, from an input-output point of view, these concepts are indistinguishable in that they lead to the same nonlinear analogue of a transmission zero. As the title indicates, the treatment here will be restricted to the single-input, single-output (SISO) case mainly for brevity. There does not appear to be any technical barrier to addressing the full multivariable problem.

A potential application of universal zero dynamics is in the area of cybersecurity. It has been known for some time that when control systems are connected to the internet for monitoring and service, there is also the potential for malicious activity in the form of zero dynamics attacks [1], [19], [24]. The basic idea is to inject a specially designed input into the system that can not be detected externally but excites the internal zero dynamics. If the system is nonminimum phase, this can be catastrophic. The general belief at present is that such an attack requires detailed knowledge of a state space model for the plant. Usually it is assumed that the linearized dynamics are available. But it will be shown here that all that is needed to design a zero dynamics attack is some estimate of the plant's generating series, linearized or otherwise. Minimum phase in this setting will be defined in terms of the boundedness of the applied input which zeros the output. The idea is illustrated using a simple example from process control. While no algorithm is proposed in this paper to estimate the generating series, it should be mentioned that some aspects of this problem have appeared in the context of nonlinear system identification [13]. Therefore, system operators with sensitive assets connected to the internet should be aware of this potential threat.

The paper is organized as follows. In the next section some preliminaries are briefly reviewed to establish the terminology and notation. In Section III, the notion of universal zero dynamics is developed first from an algebraic point of view and then in a geometric setting. In Section IV, some detailed examples are presented. The conclusions are given in the final section.

II. Preliminaries

An *alphabet* $X = \{x_0, x_1, \ldots, x_m\}$ is any nonempty and finite set of noncommuting symbols referred to as *letters*. A *word* $\eta = x_{i_1} \cdots x_{i_k}$ is a finite sequence of letters from X. The number of letters in a word η, written as $|\eta|$, is called its *length*. The empty word, $\emptyset$, is taken to have length zero. The collection of all words having length k is denoted by

X^k. Define $X^* = \bigcup_{k\geq 0} X^k$, which is a monoid under the concatenation product. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. Often c is written as the formal sum $c = \sum_{\eta\in X^*}\langle c,\eta\rangle\eta$, where the *coefficient* $\langle c,\eta\rangle \in \mathbb{R}^\ell$ is the image of $\eta \in X^*$ under c. The *support* of c, $\mathrm{supp}(c)$, is the set of all words having nonzero coefficients. A series c is called *proper* if $\emptyset\notin\mathrm{supp}(c)$. The set of all noncommutative formal power series over the alphabet X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. The subset of series with finite support, i.e., polynomials, is represented by $\mathbb{R}^\ell\langle X\rangle$. Each set is an associative $\mathbb{R}$-algebra under the concatenation product and an associative and commutative $\mathbb{R}$-algebra under the *shuffle product*, that is, the bilinear product uniquely specified by the shuffle product of two words $(x_i\eta)\sqcup\!\sqcup(x_j\xi) = x_i(\eta\sqcup\!\sqcup(x_j\xi)) + x_j((x_i\eta)\sqcup\!\sqcup\xi)$, where $x_i, x_j \in X$, $\eta,\xi \in X^*$ and with $\eta\sqcup\!\sqcup\emptyset = \emptyset\sqcup\!\sqcup\eta = \eta$ [4]. For any letter $x_i \in X$, let x_i^{-1} denote the $\mathbb{R}$-linear *left-shift operator* defined by $x_i^{-1}(\eta) = \eta'$ when $\eta = x_i\eta'$ and zero otherwise. Higher order shifts are defined inductively via $(x_i\xi)^{-1}(\cdot) = \xi^{-1}x_i^{-1}(\cdot)$, where $\xi \in X^*$. It acts as a derivation on the shuffle product.

A. Chen-Fliess series and relative degree

Given any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ one can associate a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} :\ 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. Define inductively for each word $\eta = x_i\bar{\eta} \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The *Chen-Fliess series* corresponding to $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is

$$y(t) = F_c[u](t) = \sum_{\eta\in X^*} \langle c, \eta\rangle\, E_\eta[u](t, t_0) \tag{1}$$

[4]. If there exist real numbers $K_c, M_c > 0$ such that

$$|\langle c, \eta\rangle| \leq K_c M_c^{|\eta|}|\eta|!,\ \ \forall\eta \in X^*, \tag{2}$$

then F_c constitutes a well defined mapping from $B^m_\mathfrak{p}(R)[t_0,\ t_0 + T]$ into $B^\ell_\mathfrak{q}(S)[t_0,\ t_0 + T]$ for sufficiently small $R, T > 0$ and some $S > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1,\infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [14]. (Here, $|z| := \max_i |z_i|$ when $z \in \mathbb{R}^\ell$.) Any series c satisfying (2) is called *locally convergent*. The set of all locally convergent series is denoted by $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$, and F_c is referred to as a *Fliess operator*. Given Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$, the parallel and product connections satisfy $F_c + F_d = F_{c+d}$ and $F_cF_d = F_{c\sqcup\!\sqcup d}$, respectively [4]. It is also known that the composition of two Fliess operators F_c and F_d with $c \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ always yields another Fliess operator with generating series $c\circ d$, where this *composition product* is given by

$$c \circ d = \sum_{\eta\in X^*} \langle c, \eta\rangle\, \psi_d(\eta)(\mathbf{1})$$

[3]. Here ψ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R}\langle\langle X\rangle\rangle$ to the vector space endomorphisms on $\mathbb{R}\langle\langle X\rangle\rangle$, $\mathrm{End}(\mathbb{R}\langle\langle X\rangle\rangle)$, uniquely specified by $\psi_d(x_i\eta) = \psi_d(x_i) \circ \psi_d(\eta)$ with $\psi_d(x_i)(e) = x_0(d_i \sqcup\!\sqcup e)$, $i = 0, 1, \ldots, m$ for any $e \in \mathbb{R}\langle\langle X\rangle\rangle$, and where d_i is the i-th component series of d ($d_0 := \mathbf{1} := 1\emptyset$). By definition, $\psi_d(\emptyset)$ is the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. Finally, if $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ with d non proper, then their quotient is $c/d := c\sqcup\!\sqcup d^{\sqcup\!\sqcup -1}$ so that $F_c/F_d = F_{c/d}$ [9].

Observe that $c \in \mathbb{R}\langle\langle X\rangle\rangle$ can always be decomposed into its natural and forced components, that is, $c = c_N + c_F$, where $c_N := \sum_{k\geq 0}(c, x_0^k)x_0^k$ and $c_F := c - c_N$.

Definition 2.1: [9] Given $c \in \mathbb{R}\langle\langle X\rangle\rangle$ with $X = \{x_0, x_1\}$, let $r \geq 1$ be the largest integer such that $\mathrm{supp}(c_F) \subseteq x_0^{r-1}X^*$. Then c has *relative degree* r if the linear word $x_0^{r-1}x_1 \in \mathrm{supp}(c)$, otherwise it is not well defined.

It is immediate that c has relative degree r if and only if there exists some proper $e \in \mathbb{R}\langle\langle X\rangle\rangle$ with $x_1 \notin \mathrm{supp}(e)$ such that

$$c = c_N + c_F = c_N + Kx_0^{r-1}x_1 + x_0^{r-1}e$$

with $K \neq 0$. This notion of relative degree coincides with the usual definition given in a state space setting [10].

B. Formal state space realizations

For any finite $T > 0$, $u \in L^m_1[0, T]$ and fixed $t \in [0, T]$, one can associate the formal power series in $\mathbb{R}\langle\langle X\rangle\rangle$

$$P[u](t) = \sum_{\eta\in X^*} \eta\, E_\eta[u](t, 0),$$

which is usually called a *Chen series*. In general, $P[u]$ is the solution to the formal differential equation

$$\frac{d}{dt}P[u] = \left(\sum_{i=0}^{m} x_iu_i\right)P[u],\ \ P[u](0) = \mathbf{1}, \tag{3}$$

so that $P[u]$ is always the exponential of some Lie element over X. That is, if $\mathcal{L}(X)$ is the free Lie algebra generated by X, then any $d \in \mathbb{R}\langle\langle X\rangle\rangle$ is a *Lie series* if it can be written in the form $d = \sum_{n\geq 1} p_n$, where each polynomial $p_n \in \mathcal{L}(X)$ has support residing in X^n. The set of all Lie series will be denoted by $\widehat{\mathcal{L}}(X)$. An *exponential Lie series* is any series $e = \exp(d) := \sum_{n=0}^{\infty} d^n/n!$, where d is a Lie series. In general, (3) has a solution of the form $P[u] = \exp(U)$ with $U(t) \in \widehat{\mathcal{L}}(X)$ for fixed $t \geq 0$. As a consequence of the Baker–Campbell–Hausdorff formula, the set of all exponential Lie series forms a group, $\mathcal{G}(X)$, under the Cauchy product with unit $\mathbf{1}$.

Following the approach of Kawski and Sussmann in [20], [26], $\mathcal{G}(X)$ can be viewed as a *formal Lie group* with $\widehat{\mathcal{L}}(X)$ as its corresponding Lie algebra. A commutative algebra of real-valued functions on $\mathcal{G}(X)$ is defined using the shuffle algebra on the $\mathbb{R}$-vector space $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$. Specifically, for any fixed $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ define $f_c : \mathcal{G}(X) \to \mathbb{R}$ by

$$z \mapsto f_c(z) = \sum_{\eta\in X^*} \langle c, \eta\rangle\langle z, \eta\rangle =: \langle c, z\rangle,$$

so that via Friedrich's criterion

$$f_c(z)f_d(z) = \langle c, z\rangle\langle d, z\rangle = \langle c\sqcup\!\sqcup d, z\rangle = f_{c\sqcup\!\sqcup d}(z).$$

Convergence follows from the fact that the shuffle product is known to preserve local convergence [28]. Often $f_c(z)$ will be abbreviated as $c(z)$. Analogous to standard Lie group theory, the *formal tangent space* at the unit $\mathbf{1}$, $T_{\mathbf{1}}\mathcal{G}(X)$, is identified with $\widehat{\mathcal{L}}(X)$. Thus, for any fixed $p \in \widehat{\mathcal{L}}(X)$, there is a corresponding tangent vector at $\mathbf{1}$ written as the linear functional $V_p(\mathbf{1}) : \mathbb{R}_{LC}\langle\langle X\rangle\rangle \to \mathbb{R}$, $c \mapsto V_p(\mathbf{1})(c) := \langle c, p\mathbf{1}\rangle$ and satisfying the Leibniz rule

$$V_p(\mathbf{1})(c \sqcup\!\sqcup d) = V_p(\mathbf{1})(c)\, d(\mathbf{1}) + c(\mathbf{1})V_p(\mathbf{1})(d).$$

In turn, the tangent space at $z \in \mathcal{G}(X)$, denoted $T_z\mathcal{G}(X)$, is defined via right translation to be the vector space of linear functionals $V_p(z) : \mathbb{R}_{LC}\langle\langle X\rangle\rangle \to \mathbb{R}$, $c \mapsto V_p(z)(c) := \langle c, pz\rangle$, $p \in \widehat{\mathcal{L}}(X)$ satisfying

$$\begin{aligned} V_p(z)(c \sqcup\!\sqcup d) = \langle c \sqcup\!\sqcup d, pz\rangle &= \langle c, pz\rangle\langle d, z\rangle + \langle c, z\rangle\langle d, pz\rangle \\ &= V_p(z)(c)\, d(z) + c(z)V_p(z)(d). \end{aligned}$$

For any $p \in \widehat{\mathcal{L}}(X)$, the mapping

$$V_p : \mathcal{G}(X) \to T_z\mathcal{G}(X),\ z \mapsto V_p(z) := pz$$

is a formal right-invariant vector field on $\mathcal{G}(X)$. Here $\mathcal{X}$ will denote the set of all such right-invariant vector fields. In addition, the *formal Lie derivative* is defined to be the mapping

$$L_p : \mathbb{R}_{LC}\langle\langle X\rangle\rangle \to \mathbb{R}_{LC}\langle\langle X\rangle\rangle,\ c \mapsto L_p c := p^{-1}c$$

so that $L_p c(z) = \langle L_p c, z\rangle = \langle p^{-1}c,\ z\rangle = \langle c, pz\rangle = V_p(z)(c)$. The following definition is used in the next section.

Definition 2.2: For any $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $V_i \in \mathcal{X}$, $i = 0, 1, \ldots, m$ the *formal state space realization* is

$$\dot{z} = \sum_{i=0}^{m} V_i(z)zu_i, \quad z(0) = z_0 \tag{4a}$$

$$y_k = \langle c_k, z\rangle, \quad k = 1, 2, \ldots, \ell, \tag{4b}$$

where c_k denotes the k-th component of c and $u_0 = 1$.

Note that (1) can be written componentwise as $y_k(t) = \langle c_k, z(t)\rangle$, $k = 1, 2, \ldots, \ell$, where $z(t) = P[u](t)$. Thus, $y = F_c[u]$ has a formal realization of the form (4), where $V_i(z) = x_i$, $i = 0, 1, \ldots, m$. Further, observe that

$$\begin{aligned} L_{x_i} c_k(\mathbf{1}) &= x_i^{-1}c_k(\mathbf{1}) = \langle x_i^{-1}c_k, \mathbf{1}\rangle = \langle c_k, x_i\rangle \\ L_{x_j}L_{x_i} c_k(\mathbf{1}) &= x_j^{-1}x_i^{-1}c_k(\mathbf{1}) = \langle x_j^{-1}x_i^{-1}c_k, \mathbf{1}\rangle = \langle c_k, x_ix_j\rangle, \end{aligned}$$

so that the coefficients of c_k can always be written in terms of formal Lie derivatives as

$$\langle c_k, \eta\rangle = \langle c_k, x_{i_1}\cdots x_{i_k}\rangle = L_{x_{i_k}}\cdots L_{x_{i_1}} c_k(\mathbf{1}) =: L_\eta c_k(\mathbf{1}).$$

These particular realizations were called *universal control systems* by Kawski and Sussmann in [20]. Also note in the SISO case that if $c \in \mathbb{R}\langle\langle X\rangle\rangle$ has relative degree r, then

$$\begin{aligned} \langle c, x_0^k x_1\rangle &= L_{x_1}L_{x_0}^k c(\mathbf{1}) = 0, \quad k = 0, 1, \ldots, r-2, \\ \langle c, x_0^{r-1} x_1\rangle &= L_{x_1}L_{x_0}^{r-1} c(\mathbf{1}) \neq 0, \end{aligned}$$

which is analogous to the usual definition of relative degree for state space realizations [17], [22].

III. UNIVERSAL ZERO DYNAMICS FOR SISO SYSTEMS

A. Algebraic Approach

Assume henceforth that $X = \{x_0, x_1\}$ and define $X_0 = \{x_0\}$. Let $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ with relative degree r. In light of the identity $\dot{y} = F_{x_0^{-1}(c)}[u] + uF_{x_1^{-1}(c)}[u]$, it follows that

$$\begin{aligned} y &= F_c[u] \\ y^{(1)} &= F_{x_0^{-1}(c)}[u] \\ &\ \ \vdots \\ y^{(r-1)} &= F_{(x_0^{r-1})^{-1}(c)}[u] \\ y^{(r)} &= F_{(x_0^{r})^{-1}(c)}[u] + uF_{(x_0^{r-1}x_1)^{-1}(c)}[u], \end{aligned}$$

where having well defined relative degree ensures that $F_{(x_0^{r-1}x_1)^{-1}(c)}[u](0) = \langle c, x_0^{r-1}x_1\rangle \neq 0$ for any admissible u, and furthermore $F_{(x_0^{r-1}x_1)^{-1}(c)}[u](t) \neq 0$ over some interval $[0, T)$, $T > 0$ [9].[1] Setting $y^{(r)} = 0$ it follows that the corresponding $u = u^*$ satisfies the equation

$$0 = u + F_{(x_0^r)^{-1}(c)/(x_0^{r-1}x_1)^{-1}(c)}[u].$$

It is shown in [8] that this equation can be solved for u^* uniquely by computing the composition inverse of a generating series d, denoted here by $d^{\circ -1}$, so that

$$u^* = F_{((x_0^r)^{-1}(c)/(x_0^{r-1}x_1)^{-1}(c))^{\circ -1}}[0].$$

Given the uniqueness of generating series and using the definition of the formal Lie derivative gives

$$u^*(t) = \sum_{k=0}^{\infty} \langle c_{u^*}, x_0^k\rangle \frac{t^k}{k!}, \tag{6}$$

where

$$c_{u^*} = \left(\frac{L_{x_0}^r c(\mathbf{1})}{L_{x_1}L_{x_0}^{r-1}c(\mathbf{1})}\right)_N^{\circ -1}. \tag{7}$$

It is known in this case that u^* is analytic at $t = 0$ [9], therefore this series has a nonzero radius of convergence. It can be shown directly that the zero output is in the range of F_c if $\mathrm{supp}(c_N) \subseteq x_0^r X_0^*$. Recalling that $y = F_c[u]$ has a formal realization (4) with $V_i(z) = x_i$, $i = 0, 1$ produces the following definition.

Definition 3.1: Suppose $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ has relative degree r and $\mathrm{supp}(c_N) \subseteq x_0^r X_0^*$. Its *universal zero dynamics* are defined as

$$\dot{z} = x_0 z + x_1 z u^*, \quad z(0) = \mathbf{1}, \tag{8}$$

where u^* is given by (6)-(7). If u^* is entire, then (8) is called *minimum phase* when u^* is uniformly bounded.

The solution to (8) is the Chen series $z^* = P[u^*] = \exp(U^*)$, where $U^* = \log(z^*)$. Therefore, $y = \langle c, \exp(U^*)\rangle = 0$ on $[0, T]$. The bilinear structure of (8) gives immediately a Volterra series representation of its solution

$$\begin{aligned} z^*(t) = \mathrm{e}^{x_0 t}\mathbf{1} + \sum_{k=1}^{\infty} \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_2} \mathrm{e}^{x_0(t-\tau_k)} x_1 \mathrm{e}^{x_0(\tau_k - \tau_{k-1})} \\ \cdots x_1 \mathrm{e}^{x_0\tau_1}\ u^*(\tau_k)\cdots u^*(\tau_1)\, d\tau_1\cdots d\tau_k. \end{aligned} \tag{9}$$

[1]In the SISO case, for brevity $u_0 = 1$ and $u_1 = u$.

(see, for example, [4], [17]). Equivalently,

$$z^*(t) = \mathrm{e}^{x_0 t}\left[\mathbf{1} + \sum_{k=1}^{\infty}\int_0^t\int_0^{\tau_k}\cdots\int_0^{\tau_2}\mathrm{Ad}_{x_0\tau_k}(x_1)\cdots \right.$$
$$\left. \mathrm{Ad}_{x_0\tau_1}(x_1)\,u^*(\tau_k)\cdots u^*(\tau_1)\,d\tau_1\cdots d\tau_k\right],$$

where $\mathrm{Ad}_{x_0\tau_k}(x_1) := \mathrm{e}^{-x_0\tau_k}x_1\mathrm{e}^{x_0\tau_k}$.

In the event that $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is not locally convergent, and thus the Chen-Fliess series may not converge in any sense, it is still possible to characterize the universal zero dynamics using *formal* Fliess operators [15]. Namely, $u \mapsto y = F_c[u]$ is replaced with the always well defined mapping $c\circ : \mathbb{R}[[X_0]] \to \mathbb{R}[[X_0]], c_u \mapsto c \circ c_u$. In this context, u^* is defined to be the formal input whose generating series c_{u^*} satisfies

$$c_y = c \circ c_{u^*} = (c_N + c_F) \circ c_{u^*} = c_N + c_F \circ c_{u^*} = 0. \quad (10)$$

The solution c_{u^*} and the corresponding formal dynamics still have the forms given in (7) and (8), respectively, but there is no obvious notion of minimum phase in this context.

B. Geometric Approach

In this section, differentiable structures are constructed for which the formal objects from the last section (tangent spaces, Lie derivatives, etc.) become objects of a differential geometry. To describe the evolution of the zero dynamics on an infinite dimensional space, some essential definitions are needed in this setting.

Recall that a topological vector space is locally convex if every 0-neighborhood contains a convex 0-neighborhood. Locally convex spaces generalize normed spaces as their topology can be described by a family of seminorms. For mappings between open subsets of these spaces, differentiability can be defined by requiring the existence and continuity of iterated directional derivatives. Due to local convexity, the resulting calculus, known as Bastiani calculus, behaves similarly to finite dimensional calculus. It admits a chain rule, and thus, manifolds, tangent spaces, and Lie derivatives can be defined as in the finite dimensional case. Finally, a locally convex Lie group is a group which is a manifold modelled on a locally convex space such that the group operations are smooth in the Bastiani sense [21].

Crucial to the approach is the concept of a continuous inverse algebra (CIA). A unital algebra $(A, m, 1_A)$ is called a CIA if A is a locally convex space, m is continuous, the unit group $A^\times \subseteq A$ is open, and inversion $A^\times \to A^\times$ is continuous. This construction generalizes matrix and Banach algebras. It is well known that the unit group of a CIA is an analytic locally convex Lie group [6].

Proposition 3.1: The group $\mathcal{G}(X)$ is a locally convex, infinite dimensional and analytic Lie group. Its Lie algebra is $\widehat{\mathcal{L}}(X)$.

Proof: First note that $\mathbb{R}\langle\langle X\rangle\rangle$ is a complete metrisable locally convex space which becomes a continuous inverse algebra (CIA) with respect to the concatenation product [16, Lemma 2.4]. Using the grading by word length, one can show that the exponential series is analytic on its domain. Indeed, using the logarithm, the exponential gives rise to an analytic diffeomorphism exp on the set $\mathcal{I}$ of all proper series [16, Section 2.3]. Since $\widehat{\mathcal{L}}(X) \subseteq \mathcal{I}$ is a closed Lie subalgebra of $\mathbb{R}\langle\langle X\rangle\rangle$, $\exp(\widehat{\mathcal{L}}(X)) = \mathcal{G}(X)$ is an analytic manifold and a closed subgroup of $\mathbb{R}\langle\langle X\rangle\rangle^\times$. Therefore, [21, Theorem IV.3.3.] implies that $\mathcal{G}(X)$ is a locally convex (infinite dimensional) Lie group with Lie algebra $\widehat{\mathcal{L}}(X)$. ∎

The Volterra type series representation (9) for z^* solves the evolution equation (8) on $\mathcal{G}(X)$ with $z^*(t) \in \mathcal{G}(X)$ and $(z^*(t))^{-1}\dot{z}^*(t) \in \widehat{\mathcal{L}}(X)$ for all $t \geq 0$. The regularity of the Lie group $\mathcal{G}(X)$ ensures that this type of ordinary differential equation has a unique solution (which is not automatic as the usual ODE solution theory breaks down beyond Banach spaces) [6], [21]. A potential advantage of the geometric approach is that it may lead to a more *global* description of z^* and u^* than what the power series methods can provide given their finite radius of convergence in most instances.

IV. Examples

Three examples are presented in this section. First, the linear time-invariant case is presented as a point of reference. The next example involves a linear system whose generating series is not required to be convergent. Finally, a physical nonlinear system is treated.

Example 4.1: Consider a linear, time-invariant system with irreducible transfer function

$$H(s) = K\frac{b(s)}{a(s)} = K\frac{b_0 + b_1 s + \cdots + b_{n-r-1}s^{n-r-1} + s^{n-r}}{a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n},$$

where $K \neq 0$ and with relative degree $1 \leq r < n$. This corresponds to having a generating series $c = c_F = \sum_{k\geq r} h_k x_0^{k-1}x_1 \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$, where $H(s) = \sum_{k\geq r} h_k s^{-k}$ and $h_r = K$. Divide $b(s)$ into $a(s)$ so that $a(s) = b(s)p(s) + r(s)$ with $(r(s), b(s))$ being a coprime pair of polynomials

$$p(s) = p_0 + p_1 s + \cdots + p_{r-1}s^{r-1} + s^r$$
$$r(s) = r_0 + r_1 s + \cdots + r_{n-r-2}s^{n-r-2} + r_{n-r-1}s^{n-r-1}$$

and $\deg(r(s)) < \deg(b(s))$. It is shown in [7] that there exists a realization (A, b, c) with the Byrnes-Isidori normal form

$$\begin{aligned}
&\dot{\xi}_1 = \xi_2,\ \dot{\xi}_2 = \xi_3,\ \ldots,\ \dot{\xi}_{r-1} = \xi_r \\
&\dot{\xi}_r = P\xi + R\eta + Ku \\
&\dot{\eta} = S\xi + Q\eta \\
&y = \xi_1,
\end{aligned}$$

where $\xi = [\xi_1 \cdots \xi_r]$, $\eta = [\eta_1 \cdots \eta_{n-r}]$, $P = -[p_0 \cdots p_{r-1}]$, $R = -[r_0 \cdots r_{n-r-1}]$, $S = e_{n-r}(n-r)e_1^T(r)$, and

$$Q = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-r-1} \end{bmatrix}.$$

(Here $e_i(j) \in \mathbb{R}^j$ has a one in the i-th position and zero elsewhere. If j is understood then the notation is abbreviated to e_i.) In this case, the zero dynamics correspond to choosing $\xi(0) = 0$ and $\dot{\xi}_r = 0$ so that $\xi(t) = 0$ for $t \geq 0$, $u^* = -R\eta/K$, and $\dot{\eta} = Q\eta$, $\eta(0) = \eta_0$. Therefore,

$$u^*(t) = \sum_{k=0}^{\infty} -\frac{R}{K}Q^k\eta_0\frac{t^k}{k!}. \quad (11)$$

The system is clearly minimum phase in the sense of Definition 3.1 if and only if it minimum phase by the standard definition, namely, the roots of $b(s)=\det(sI-Q)$ all have strictly negative real parts. To compute u^* directly from the generating series c, apply (10), where $c_F\circ c_{u^*}$ reduces in the present context to series convolution

$$\langle c_F\circ c_{u^*},x_0^k\rangle=\sum_{j=r-1}^{k-1}\langle c,x_0^jx_1\rangle\langle c_{u^*},x_0^{k-1-j}\rangle,\;\; k\geq r.$$

Deconvolution can be done inductively using the fact that $\langle c,x_0^{r-1}x_1\rangle=K\neq 0$ to yield

$$\langle c_{u^*},x_0^k\rangle=-\frac{1}{K}\left[\langle c,x_0^{k+r}\rangle+\sum_{j=r}^{k+r-1}\langle c,x_0^jx_1\rangle\langle c_{u^*},x_0^{k+r-1-j}\rangle\right] \tag{12}$$

for $k\geq 0$.

As a specific example, consider the minimum phase system

$$H(s)=\frac{4+5s+s^2}{1+2s+3s^2+s^3}$$

with relative degree $r=1$. Since $a(s)=1+2s+3s^2+s^3=(4+5s+s^2)(-2+s)+(9+8s)$, its normal form is

$$\dot z=\left[\begin{array}{c|cc}2&-9&-8\\\hline 0&0&1\\1&-4&-5\end{array}\right]z+\begin{bmatrix}1\\0\\0\end{bmatrix}u$$
$$y=\left[\begin{array}{c|cc}1&0&0\end{array}\right]z.$$

If $z(0)=[0\;1\;-2]^T$, then the generating series is $c=c_N+c_F$, where

$$c_N=7x_0-16x_0^2+34x_0^3-77x_0^4+179x_0^5-417x_0^6+\cdots$$
$$c_F=x_1+2x_0x_1-4x_0^2x_1+7x_0^3x_1-15x_0^4x_1+35x_0^5x_1-\cdots$$

From either (11) or (12) it follows that

$$u^*(t)=-7+30t-122\frac{t^2}{2!}+490\frac{t^3}{3!}-1962\frac{t^4}{4!}+7850\frac{t^5}{5!}-\cdots$$

□

Example 4.2: Any series in $\mathbb{R}\langle\langle X\rangle\rangle$ with relative degree r is known to be affine feedback equivalent to a series of the form $c=c_N+Kx_0^{r-1}x_1$, where K is a nonzero real number [10]. Applying (12) gives

$$u^*(t)=-\frac{1}{K}\sum_{k=r}^{\infty}\langle c,x_0^k\rangle\frac{t^{k-r}}{(k-r)!}$$

under the assumption that $\mathrm{supp}(c_N)\subseteq x_0^rX_0^*$. As there is no a priori state space model to compare against, one can instead verify that $z^*(t)$ as given in (9) has the desired property. Observe

$$\begin{aligned}&\langle c,z^*(t)\rangle=\langle c_N+Kx_0^{r-1}x_1,z^*(t)\rangle\\&=\langle c_N,\mathrm{e}^{x_0t}\mathbf{1}\rangle+K\left\langle x_0^{r-1}x_1,\int_0^t\mathrm{e}^{x_0(t-\tau)}x_1\,u^*(\tau)\,d\tau\right\rangle\\&=\langle c,\mathrm{e}^{x_0t}\mathbf{1}\rangle-\left\langle x_0^{r-1}x_1,\sum_{k=r}^{\infty}\langle c,x_0^k\rangle\right.\\&\qquad\left.\sum_{n=0}^{\infty}\int_0^t\frac{(t-\tau)^n}{n!}\frac{\tau^{k-r}}{(k-r)!}\,d\tau\,x_0^nx_1\right\rangle\\&=\sum_{k=r}^{\infty}\langle c,x_0^k\rangle\frac{t^k}{k!}-\sum_{k=r}^{\infty}\langle c,x_0^k\rangle\int_0^t\frac{(t-\tau)^{r-1}}{(r-1)!}\frac{\tau^{k-r}}{(k-r)!}\,d\tau\\&=0,\end{aligned}$$

using the identity for $k\geq r\geq 1$

$$\int_0^t\frac{(t-\tau)^{r-1}}{(r-1)!}\frac{\tau^{k-r}}{(k-r)!}\,d\tau=\frac{t^k}{k!}.$$

□

Example 4.3: Consider a first order, exothermic, irreversible reaction of a reactant in a product substance carried out in a well mixed continuous stirred chemical reactor (CSTR). The mass and energy balances give the dynamics (in dimensionless form):

$$\dot z=\begin{bmatrix}-z_1+\alpha(1-z_1)\mathrm{e}^{\frac{z_2}{1+z_2/\gamma}}\\-(\beta+1)z_2+\kappa\alpha(1-z_1)\mathrm{e}^{\frac{z_2}{1+z_2/\gamma}}\end{bmatrix}+\begin{bmatrix}0\\\beta\end{bmatrix}u \tag{13a}$$

$$y=z_2. \tag{13b}$$

Here z_1 is the reactant concentration, z_2 is the reactor temperature, and u is the cooling reactor jacket temperature [2], [27]. The physical constants α, β, γ, and κ are all set to unity for convenience. For $z(0)=0$ the corresponding generating series for the input-output map $y=F_c[u]$, either computed from (13) or determined by direct measurement, is

$$\begin{aligned}c=&\,x_0+x_1-2x_0^2-x_0x_1-2x_0^2x_1-2x_0x_1x_0\\&-x_0x_1^2+22x_0^4+15x_0^3x_1+11x_0^2x_1x_0+4x_0^2x_1^2\\&+6x_0x_1x_0^2+2x_0x_1x_0x_1+2x_0x_1^2x_0+x_0x_1^3+\cdots,\end{aligned}$$

which has relative degree $r=1$ and satisfies $\mathrm{supp}(c_N)\subseteq x_0^rX_0^*$. A direct application of (6)-(7) (with the help of Mathematica package NonCommutative Formal Power Series [23]) gives

$$u^*(t)=-1+t-2\frac{t^2}{2!}+4\frac{t^3}{3!}-8\frac{t^4}{4!}+16\frac{t^5}{5!}-32\frac{t^6}{6!}+\cdots. \tag{14}$$

This result can also be computed using the given local state representation by noting that if $y=z_2=0$ and $\dot z_2=0$, then the physical zero dynamics are

$$\dot z_1=-2z_1+1,\;\; z_1(0)=0$$

with $u^*=-(1-z_1)$. As these dynamics are linear, solving directly gives $u^*=-(1+\mathrm{e}^{-2t})/2$, $t\geq 0$, which is the closed form of (14). The system is clearly minimum phase by either the classical definition or that given here for universal zero dynamics. The output and states of (13) when $u=u^*$ up to sixth order were computed via MatLab and shown in Figure 1. Observe that the reactant level can be brought to steady state using the approximated u^* without tripping any temperature sensing system, provided of course that this input can be physically implemented. As the system is minimum phase, there is no chance of unbounded behavior. But the ability to run the reactor to steady-state while zeroing a key monitoring variable could leave the system vulnerable to bad actors with remote access to the system. In addition, this can be

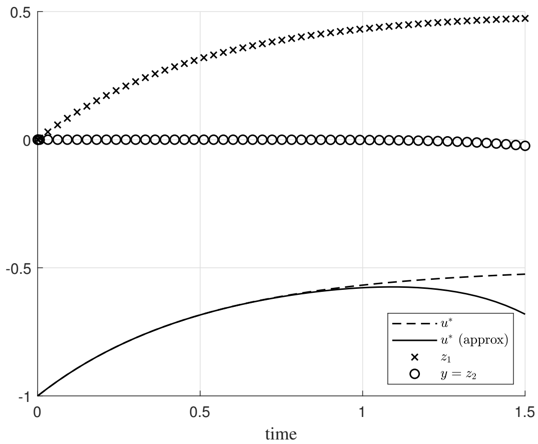

Fig. 1. Approximate zero dynamics of the CSTR system in Example 4.3 using only knowledge of the plant's generating series.

accomplished without using detailed knowledge of the plant's state space model, only its generating series. □

V. Conclusions

Given a single-input, single-output system with a Chen-Fliess series representation whose generating series has a well defined relative degree, it was shown that there is a notion of universal zero dynamics which describes a set of dynamics evolving on a locally convex (infinite dimensional) Lie group so as to render the system's output exactly zero. The system was said to be minimum phase when the applied input which zeros the output is entire and uniformly bounded. The formal case, i.e., where the plant's generating series is not convergent, was also treated. Here there is less analytic structure available, but the algebraic definition of universal zero dynamics still applies. As an application, it was shown that one can design a zero dynamics attack on cyber-infrastructure using only an estimate of the plant's generating series. That is, detailed knowledge of the plant's internal dynamics is not needed.

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