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Bachelor's project in BMAT
Supervisor: Stensønes Berit
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NTNU
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Kunnskap for en bedre verden

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Abstract

In preparation of constructing so called peak functions we are studying properties of subharmonic polynomials.

1. Introduction

We work in $\mathbb{C}^n$ with coordinates $(z_1, ..., z_n)$ where two standard domains in $\mathbb{C}^n$ are the unit polydisc $\Delta^n = \Delta(0,1) \times ... \times \Delta(0,1)$ and the unit open ball $\mathbb{B}^n$. By definition $\Delta(0,1) = \{z \in \mathbb{C} :\mid z \mid< 1\}$ is the unit disc in $\mathbb{C}$ with center 0 and radius 1, and the unit open ball is defined by $\mathbb{B}^n = \{(z_1, ..., z_n) \mid \sqrt{\mid z_1 \mid^2 + ... + \mid z_n \mid^2} < 1\}$. Some additional definitions are needed in order to talk about peak functions.

Definition 1. *Let Ω be a domain in $\mathbb{C}^n$ and $f : \Omega \to \mathbb{C}$ a function. Then f is holomorphic in Ω if $z_j \mapsto f(z_1, ..., z_n)$ is analytic for every $j = 1, ..., n$. This is written as $f \in H(\Omega)$.*

A holomorphic function is a key ingredient in the definition of a peak function, but first we need to define the kind of domain we want to find peak functions on.

Definition 2. *Ω is pseudoconvex if for all $p \in \partial\Omega$ there exists a neighbourhood $U(p)$ of p and if V is a connected component of $U \cap \Omega$, then there exists $f \in H(\Omega)$ such that $f \mid_V$ does not extend holomorphically to any set W that contains p.*

Not all domains are pseudoconvex however.
Example of non-pseudoconvex domain: Consider the domain $A \times \Delta$ where we have the annulus $A = \{1/2 <\mid z \mid< 1\}$ and unit disc $\Delta = \Delta(0,1)$. Choose the function f such that $f \in H(A \times \Delta)$, then $f(z,w) = \Sigma_{j=-\infty}^{\infty} a_j(w) z^j$ where $a_j(w)$ is analytic in Δ and $\Sigma_{j=0}^{\infty} a_j(w) z^j$ converges when $\mid z \mid< 1$ while $\Sigma_{-\infty}^{-1} a_j(w) z^j$ converges when $\mid z \mid> 1/2$. Thus $\Sigma_{j=0}^{\infty} a_j(w) z^j$ converges in $\Delta \times \Delta$.

Consider the open neighbourhood U of a point in Δ of the variable w and define $\Omega = (A \times \Delta) \cup (\Delta \times U)$. Choose the forementioned function f such that $f \in H(\Omega)$. Hence we find that $a_{j \leq -1}(w) = 0$ for $w \in U$ in order to make the sum $\Sigma_{-\infty}^{-1} a_j(w) z^j$ converge in $\Delta \times \Delta$. Being analytic and zero in an open set, we find that $a_j(w) \equiv 0$ in Δ of the variable w. Thus $f = \Sigma_{j=0}^{\infty} a_j(w) z^j$ which is holomorphic in $\Delta \times \Delta$. Consequently, if $f \in H(\Omega)$ then there exists a function $F \in H(\Delta^2)$ such that $F \mid_\Omega = f$.

The definition of a peak function is as follows

Definition 3. *Let Ω be a domain in $\mathbb{C}^n$ and $p \in \partial\Omega$ be a point. If there exists a function $f \in H(\Omega) \cap C(\overline{\Omega})$ such that $f(p) = 1$ and $\mid f(z) \mid< 1$ if $z \in \overline{\Omega} \setminus \{p\}$, then f is a peak function and p is a peak point.*

In order to illustrate the importance of the choice of Ω regarding peak functions, consider two examples.

Positive example: Recall the definition of strict convexity for a set. All points on a line connecting two points in the set, also must be in the set. So consider Ω as a strict convex domain of $\mathbb{C}^n$. Then every point $p \in \partial\Omega$ is a peak point.

Denote $p = (\xi_1, ..., \xi_n) = (u_1 + iv_1, ..., u_n + iv_n)$ and an arbitrary point $z \in \Omega$ as $z = (z_1, ..., z_n) = (x_1 + iy_1, ..., x_n + iy_n)$. Consider the function $L(x_1, y_1, ..., x_n, y_n) = \Sigma_{j=1}^n a_j(x_j - u_j) + b_j(y_j - v_j)$ in $\overline{\Omega}$, where $a_j, b_j \in \mathbb{R}$. Since Ω is strictly convex we find that there exists $a_j, b_j \in \mathbb{R}$ such that $L(x_1, y_1, ..., x_n, y_n) < 0$ in $\overline{\Omega} \setminus \{p\}$ while $L(p) = 0$. Hence by the function $h(z_1, ..., z_n) \equiv \Sigma_{j=1}^n (a_j - ib_j)(z_j - \xi_j)$ and $z_j - \xi_j = x_j - u_j + i(y_j - v_j)$ some swift calculations show that $\operatorname{Re} h(z_1, ..., z_n) = L(x_1, y_1, ..., x_n, y_n) < 0$. Finally look at $f(z) \equiv e^{h(z)}$ which then has the properties $f(p) = 1$ and $\mid f(z) \mid = e^{\operatorname{Re} h(z)} < 1$ when $z \in \overline{\Omega} \setminus \{p\}$. Thus f is a peak function for the point p.

Thus the open ball $\mathbf{B}^n$ have only peak points on its boundary as it is strictly convex. Since strict convexity implies pseudoconvexity, it follows that the open ball is pseudoconvex.

Negative example: Consider $\Delta^2 = \Delta(0,1) \times \Delta(0,1)$ and choose a point $p \in \partial\Delta^2$. Assume that $p = (0,1)$ and that there exists a function $f \in H(\Delta^2) \cap C(\overline{\Delta}^2)$ such that $f(0,1) = 1$ and $\mid f(z,w) \mid < 1$ in $\overline{\Delta}^2 \setminus (0,1)$. This is a contradiction however, as we will see.

By these assumptions we have for an $a \in \mathbb{R}^+$ that $\mid f(z,1) \mid \leq a < 1$ when $\mid z \mid = 1/2$. Then consider the family of discs $\overline{\Delta}_t := \overline{\Delta}(0, 1/2) \times \{t\}$ where $t \in [1-\delta, 1]$ for a $0 < \delta \ll 1$. Since $z \mapsto f(z,t)$ is analytic with $z \in \overline{\Delta}(0,1/2)$ we may look at $f \mid_{\overline{\Delta}_t}$ and let $t \to 1$. As t approaches one, we find that $f(0,t) \to 1$ and $\mid f(z,t) \mid$ goes to a value $\leq a$ if $\mid z \mid = 1/2$. Since we are looking at $\overline{\Delta}_t$ and $z \mapsto f(z,t)$ being analytic, we find $\mid f(0,t) \mid > \mid f(z,t) \mid$ where $z \in \partial\Delta(0,1/2)$ and t close to 1. This is a contradiction to the maximum principle because $(0,t)$ is an inner point, and not on the boundary of $\overline{\Delta}_t$. In order to avoid contradiction of the maximum principle we must have $f(z,1) = 1$ on $\overline{\Delta}_1$, which then contradict the assumption of p being a peak point.

The unit polydisc is also pseudoconvex.

From BEDFORD-FORNÆSS 1978[1] we study the case $\Omega = \{(z,w) \mid \operatorname{Re} w + P_{2k}(z, \overline{z}) < 0\}$ where P_{2k} subharmonic, homogeneous and not harmonic. It may also be stated that when P_{2k} satisfy these properties, it follows that Ω is pseudoconvex. Thus this bachelor thesis will study the properties of P_{2k} from BEDFORD-FORNÆSS 1978[1].

2. Tasks

This section is included as preparation for the relevant theory.

• ”Show that the laplacian Δ operator in polar coordinates is $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$”.
Given that $u = u(x, y)$ and $x = rcos\theta$ and $y = rsin\theta$, it follows that

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_x cos\theta + u_y sin\theta \\
\Rightarrow \frac{\partial^2 u}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) = u_{xx}cos^2\theta + u_{yy}sin^2\theta + u_{xy}cos\theta sin\theta + u_{yx}cos\theta sin\theta
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = r\left(u_y cos\theta - u_x sin\theta\right) \\
\Rightarrow \frac{\partial^2 u}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = r\left\{\cos\theta\frac{\partial}{\partial \theta}\left(u_y\right) - u_y sin\theta - \left[\sin\theta\frac{\partial}{\partial \theta}\left(u_x\right) + u_x cos\theta\right]\right\},
\end{aligned}$$

where the last equation is derived by using the formula for derivative of products. Calculating the second derivative of both formulas are done by using the formula for the first derivative in a recursive manner. Therefore the rightmost side of the last equation is written

$$\begin{aligned}
\frac{\partial^2 u}{\partial \theta^2} &= r^2\cos(\theta)\left(u_{yy}\cos(\theta) - u_{xy}\sin(\theta)\right) - ru_y\sin(\theta) - r^2\sin(\theta)\left(u_{yx}\cos(\theta) - u_{xx}\sin(\theta)\right) - u_x\cos(\theta) \\
&= -ru_x cos\theta - ru_y sin\theta + r^2\left(u_{xx}sin^2\theta + u_{yy}cos^2\theta - u_{xy}cos\theta sin\theta - u_{yx}cos\theta sin\theta\right).
\end{aligned}$$

That is, after subtracting terms and using the identity $cos^2\theta + sin^2\theta = 1$ the following solves the problem at hand:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

3. Remarks on subharmonic functions

Let $P_{2k}(z) = \Sigma_{j=0}^{2k} a_j z^j \overline{z}^{2k-j}$ be a real-valued polynomial of degree $2k$ defined on complex numbers, where the coefficients may be complex. It can be seen that $P_{2k}(tz) = t^{2k} P_{2k}(z)$ for real number t. Which means that P_{2k} is homogeneous.

Denote $z =\mid z \mid e^{i\theta}$ and by swift calculation we find $P_{2k}(z) =\mid z \mid^{2k} \Sigma_{j=0}^{2k} a_j e^{2i\theta(j-k)}$, so define $g(\theta)$ by $P_{2k}(z) =\mid z \mid^{2k} g(\theta)$. Homogeneity is important because it allows us to write P_{2k} this way.
For a complex number ω the expression $\omega + \overline{\omega}$ is real, and since P_{2k} is real-valued it follows that for every term in $g(\theta)$, the complex conjugate of the term in also a term in the sum of $g(\theta)$. Writing the coefficients in polar coordinates

$$g(\theta) = \Sigma_{j=0}^{2k} \mid a_j \mid e^{i[2\theta(j-k)+\theta_j]}, \tag{1}$$

we find that $g(\theta)$ is a linear combination of cosines. As such $g(\theta)$ has to be a wave function by the superposition principle.

Since $g(\theta)$ is a trigonometric polynomial it follows that $g(\theta)$ has a final number of roots. Also, every ray emanating from the origin in the complex plane corresponds to one value of θ.
This means that $\{z \mid P_{2k}(z) < 0\}$ consists of finite number of disjoint sectors that correspond to finite number of disjoint segments $I_1, ..., I_s$ of θ-values in the unit circle. Similarly with $\{z \mid P_{2k} > 0\}$ and segments $J_1, ..., J_t$.

Assume P_{2k} is subharmonic, that is $\Delta P_{2k} \geq 0$. Also assume that ΔP_{2k} is not identically zero. By using the polar Laplacian with $\mid z \mid= r$ we find:

$$\Delta P_{2k} = 2k(2k-1)r^{2(k-1)}g(\theta) + 2kr^{2(k-1)}g(\theta) + r^{2(k-1)}g''(\theta) = r^{2(k-1)}\left(4k^2 g(\theta) + g_{\theta\theta}\right).$$

Thus

$$(2k)^2 g + g_{\theta\theta} =\mid z \mid^{-2(k-1)} \Delta P_{2k} > 0, \tag{2}$$

except at isolated points.

Lemma 3.1. *Let $P_{2k}(z) =\mid z \mid^{2k} g(\theta)$ be a real-valued polynomial on complex numbers of degree $2k$ that is homogeneous and subharmonic, but not harmonic.*
If $g(\theta_0) = g'(\theta_0) = 0$ and $g \not\equiv 0$, then g vanishes to even order at θ_0, and there exists integer $m > 0$ such that $g^{(j)}(\theta_0) = 0, 1 \leq j \leq 2m-1$ and $g^{(2m)}(\theta_0) > 0$. It follows that if $\tilde{\theta}$ is such that $g(\tilde{\theta}) = 0$ then $\tilde{\theta}$ is the endpoint of at least one J-interval. Consequently the number of J-intervals is greater than or equal to the number of I-intervals, that is $t \geq s$.

Proof. By previous paragraphs and equation 1 it is clear that $g(\theta)$ is a real analytic function. By Taylor series expansion locally about a point θ_0 we get $g(\theta) = \Sigma_{j=l}^{\infty} \frac{g^{(j)}(\theta_0)}{j!}(\theta - \theta_0)^j$ where $l \geq 2$ by the hypothesis. That is $g(\theta) = C(\theta-\theta_0)^l + \mathcal{O}((\theta-\theta_0)^{l+1})$ for some constant C. This proof relies on the fact that g is not identically zero in an interval, how small it may ever be.

Sublemma 1. *The function g is not identically zero in an interval, how small it may ever be.*

Proof. If $g = 0$ for all θ in $(\tilde{\theta} - \epsilon, \tilde{\theta} + \epsilon)$ then $P_{2k} =\mid z \mid^{2k} g(\theta)$ is zero in a small open domain U in $\mathbb{C}$, and since P_{2k} is real analytic in a domain containing U it follows that $P_{2k} \equiv 0$. Which is a contradiction since the polynomial is not harmonic. Every zero of g will therefore be isolated. □

By equation 2 it follows

$$(2k)^2 g(\theta) + g''(\theta) = (2k)^2(\theta-\theta_0)^l + \mathcal{O}((\theta-\theta_0)^{l+1}) + Cl(l-1)(\theta-\theta_0)^{l-2} + \mathcal{O}((\theta-\theta_0)^{l-1})$$
$$= Cl(l-1)(\theta-\theta_0)^{l-2} + \mathcal{O}((\theta-\theta_0)^{l-1}) \geq 0.$$

If $C < 0$ the value of left hand side of equation 2 will be negative for some θ, so we must have $C > 0$. Similarly if $l-2$ is odd then the left hand side will be negative for values of θ close to θ_0, so $l-2$ and thus l must be even.
Denote $l = 2m$ for an integer $m > 0$. It must be that $g^{(j)}(\theta_0) = 0$ for $1 \leq j \leq 2m-1$ and $g^{(2m)}(\theta_0) > 0$.

Let $\tilde{\theta}$ be fixed such that $g(\tilde{\theta}) = 0$. Then, either $g'(\tilde{\theta}) = 0$ or $g'(\tilde{\theta}) \neq 0$.
If $g'(\tilde{\theta}) = 0$ the proof above applies and there exists a constant $\tilde{C}$ and integer $\tilde{m} > 0$ such that $g(\theta) = \tilde{C}(\theta-\tilde{\theta})^{2\tilde{m}} + \mathcal{O}((\theta-\tilde{\theta})^{2\tilde{m}+1})$ and $g(\theta) > 0$ for θ in an interval containing $\tilde{\theta}$ as an inner point. That is, two J-regions are neighbours.
If $g'(\tilde{\theta}) \neq 0$ it follows that g have opposite signs at opposite sides of $\tilde{\theta}$, for otherwise g would not be continuous. It follows that every I-region is neighboured on each side by J-regions. Consequently $\tilde{\theta}$ is an endpoint of at least one J-region, and since two of these J-regions may be neighbours we find $t \geq s$. □

Lemma 3.2. *Let $P_{2k}(z) =\mid z \mid^{2k} g(\theta)$ be a real valued polynomial on complex numbers of degree $2k$ that is homogeneous.*

If P_{2k} is subharmonic but not harmonic, then each interval I_m has length strictly less than $\pi/2k$ and each interval J_n has length strictly greater than $\pi/2k$.

Proof. The idea is to choose a function defined and continuous on I_m, J_n respectively, then taking the integral over said interval in order to derive an inequality involving the length of I_m, J_n respectively.

Since $I_m \neq \emptyset$ define $I_m = (\theta_0, \theta_0 + \delta)$ for some θ_0, where $\delta > 0$. This interval has length δ. It follows that $g(\theta) < 0$ for all θ in I_m, because $P_{2k} < 0$ for all θ in I_m by definition of I_m. Thus by equation 2 and $g < 0$ in I_m we have $4k^2g^2 + g''g \leq 0$ in I_m. By similar argumentation as in lemma 3.1 every zero of g is isolated since P_{2k} is not harmonic, so $4k^2g^2 + g''g < 0$ almost everywhere in I_m.

Let $h(\theta) = \arctan(g'(\theta)/2kg(\theta))$ on I_m be the function in question. If h is well defined in the endpoints of I_m it is necessarily a smooth function on I_m, because h is an arctan-function and g has no zeros in the interior of I_m. But by construction of I_m we have $\lim_{\theta\to\theta_0} g(\theta) = 0$, $\lim_{\theta\to\theta_0+\delta} g(\theta) = 0$. Since two I-intervals can not be neighbours it follows that $\lim_{\theta\to\theta_0} g'(\theta) < 0, \lim_{\theta\to\theta_0+\delta} g'(\theta) > 0$.
In other words $\lim_{\theta\to\theta_0} g'(\theta)/g(\theta) = -\infty$ and $\lim_{\theta\to\theta_0+\delta} g'(\theta)/g(\theta) = \infty$. Therefore $h : (\theta_0, \theta_0 + \delta) \to (-\pi/2, \pi/2)$ is smooth.

Calculation shows that

$$h'(\theta) = \frac{g''(\theta)g(\theta) - (g'(\theta))^2}{\left(1 + \left(\frac{g'(\theta)}{2kg(\theta)}\right)^2\right) 2kg(\theta)^2}.$$

From $4k^2g^2 + g''g < 0$ in I_m it follows that $g_{\theta\theta}g < -4k^2g^2$, hence

$$h'(\theta) < -\frac{(2kg(\theta))^2 + (g'(\theta))^2}{\left(1 + \left(\frac{g'(\theta)}{2kg(\theta)}\right)^2\right) 2kg(\theta)^2} = -2k \quad \text{almost everywhere in } I_m.$$

Thus $-\pi < \pi = \int_{h(\theta_0)}^{h(\theta_0+\delta)} dh = \int_{\theta_0}^{\theta_0+\delta} h'(\theta)d\theta < -2k\delta$, which yields $\delta < \pi/2k$.

Now look at J_n. The reasoning is similar as for I_m. Since $J_n \neq \emptyset$, define $J_n = (\theta_0, \theta_0 + \delta)$ for some θ_0. By definition of $J_n$we have $g(\theta) > 0$ for all θ in J_n. Now we need to make sure that J_n has both boundary points as zeroes of g. By lemma 3.1 two I-regions can not be neighbours.
If a J- and I-region are neighbours, they must have a point that separates them since they are disjoint. If the value of g is anything else than zero in this point, we have a contradiction with the continuity of g. So this point is a zero of g.
If two J-regions are neighbours, there is a point separating them since they are disjoint.

If the value of g at this point is greater than zero, we no longer have two J-regions but only one. If the value of g is below zero at this point, g will no longer be continuous. To avoid contradictions the point between these regions must be a zero of g.

Now we have shown that every two neighbouring regions has a zero that separates them. It remains to show that for every J_n we may choose to look at, it has neighbours on both sides. So consider the sequence of J- and I-regions that the graph of g consists of in the real plane. As g has a final number of roots, it also has a finite number of these J- and I-regions. This raises the question of what happens with the intervals at the "end" of the sequence of regions. Say one of these intervals at the end is an I-region. We then know that it has a finite length, so the I-region must have zeroes at both boundary points to avoid contradictions. Suppose then, that one of these intervals at the end is a J-region and call it $\tilde{J}$. Now we need to make sure that $\tilde{J}$ is in fact of finite length. By equation 1 it follows that g must have a period of 2π as it is a linear combination of cosines. Thus we may extend $g(\theta)$ so the variable θ may take all values of $\mathbb{R}$. Therefore $\tilde{J}$ may not have a length strictly greater than 2π. Consequently $\tilde{J}$ must have zeroes at both boundary points to avoid contradictions. So for any $J_n = (\theta_0, \theta_0 + \delta)$ we find $\lim_{\theta \to \theta_0} g(\theta) = 0$, $\lim_{\theta \to \theta_0 + \delta} g(\theta) = 0$ in J_n.

Since g is positive in J_n it follows that $\lim_{\theta \to \theta_0} g'(\theta) \geq 0$. Also $\lim_{\theta \to \theta_0 + \delta} g'(\theta) \leq 0$ because the neighbouring J- or I-region on right hand side of J_n gives a local minimum point or negative slope of g, respectively, in the point $\theta_0 + \delta$. Therefore, if we assume $g'(\theta) \neq 0$ in these boundary points the following claim is trivially true:

$$\lim_{\theta \to \theta_0} \frac{g\prime(\theta)}{g(\theta)} = \infty \quad \text{and} \lim_{\theta \to \theta_0 + \delta} \frac{g\prime(\theta)}{g(\theta)} = -\infty.$$

But this claim is also true otherwise. For if $g'(\theta) = 0$ in these points, lemma 3.1 states that there exists integer $m > 0$ such that $g^{(2m)}(\theta) > 0$ and $g^{(j)}(\theta) = 0, j < 2m$. Successive applications of L'Hopitals rule proves the claim true. Thus by the previous defined h we find $\lim_{\theta \to \theta_0} h(\theta) = \pi/2$ and $\lim_{\theta \to \theta_0 + \delta} h(\theta) = -\pi/2$ in J_n.

From equation 2 we have $4k^2 g^2 + g'' g \geq 0$ in J_n. Similarly as for I_m we then have $4k^2 g^2 + g'' g > 0$ for almost every point in J_n, so the inequality for $h'(\theta)$ is given as $h'(\theta) > -2k$ almost everywhere in J_n. Thus $-\pi = \int_{h(\theta_0)}^{h(\theta_0+\delta)} dh(\theta) = \int_{\theta_0}^{\theta_0+\delta} h'(\theta) d\theta > -2k\delta$, which yields $\delta > \pi/2k$ as desired. $\square$

Remark. *In order to motivate the choice of function h, consider the simplest version of $P_{2k}(z)$ as z^{2k}. A small perturbation of P_{2k} may then be written $z^{2k} + \epsilon \mid z^{2k} \mid$ for $\epsilon > 0$,* *so* $\operatorname{Re} z^{2k} + \epsilon \mid z^{2k} \mid = \mid z \mid^{2k} (\cos(2k\theta) + \epsilon)$ *which is analogue to* $P_{2k}(z) = \mid z \mid^{2k} g(\theta)$*. Let* $g(\theta) = \cos(2k\theta + \epsilon)$*. Calculation shows that* $\frac{g'(\theta)}{2kg(\theta)} = -\frac{\sin(2k\theta)}{\cos(2k\theta)+\epsilon}$ *which is almost a tangent function. Therefore let* $h(\theta) = \arctan\left(\frac{g'(\theta)}{2kg(\theta)}\right)$*.*

Corollary 3.2.1. *Let* $R(z,\eta) = \mid z \mid^{2k} G(\theta,\eta)$ *be a real-valued function on complex variables that is subharmonic and homogeneous in* z *of degree* $2k$*. Assume that for all* η *that* R *is not harmonic in* z *and the coefficients vary smoothly with* $\eta \in \mathbb{C}$*. Let* $R(z,0) = P_{2k}(z)$ *as the polynomial from previous lemmas, which gives* $G(\theta,0) = g(\theta)$*. Let* $I = (\theta_1, \theta_2)$ *be the angular sector in the unit circle that corresponds to the sector in* $\mathbb{C}$ *where* $R(z,0) = P_{2k}(z) < 0$*, i.e. where* $G(\theta,0) = g(\theta) < 0$*. Then* I *varies smoothly with* η *in the following sense: there exists* $\delta > 0$ *and smooth functions* $\theta_1(\eta), \theta_2(\eta)$ *defined near* $\eta = 0$ *such that* $\theta_1(0) = \theta_1, \theta_2(0) = \theta_2$ *and*

$$(\theta_1(\eta), \theta_2(\eta)) = \{\theta \in (\theta_1 - \delta, \theta_2 + \delta) \mid G(\theta,\eta) < 0\}. \tag{3}$$

Proof. By lemma 3.1 two I-intervals can not be neighbours, so it follows that $G_\theta(\theta_1, 0) < 0 < G_\theta(\theta_2, 0)$ since $G(\theta, 0) = g(\theta)$ is continuously differentiable. Thus by the Implicit Function Theorem there exists smooth $\theta_1(\eta), \theta_2(\eta)$ for η close to zero such that $\theta_j(0) = \theta_j$ and $G(\theta_j(\eta), \eta)$ is identically zero for $j = 1, 2$.

By continuity of g_θ and since $G_\theta(\theta_1, 0) < 0$ it follows that $G_\theta(\theta_1(\eta), \eta) < 0$ for η sufficiently small. Therefore for two points Θ_1, Θ_2 sufficiently close to $\theta_1(\eta)$, such that $\Theta_1 < \theta_1(\eta) < \Theta_2$, we find $G(\Theta_2, \eta) < 0 < G(\Theta_1, \eta)$. The sign of G is opposite in (Θ_1, η) and (Θ_2, η) because $(\theta_1(\eta), \eta)$ is a zero of G and $G_\theta(\theta_1(\eta), \eta) \neq 0$. A similar situation happens near $\theta_2(\eta)$.

Now, if equation 3 does not hold, there exists $\theta \in (\theta_1(\eta), \theta_2(\eta))$ such that $G(\theta, \eta) > 0$. By lemma 3.1 there must be a J-interval containing this θ or having θ as an boundary point. If θ is an boundary point of this interval, it follows that we may consider a point $\tilde{\theta}$ in the interior of non-empty open J such that $G(\tilde{\theta}, \eta) > 0$, so without loss of generality assume that $G(\theta, \eta) > 0$. That is $\theta \in J$. By lemma 3.2 each I-interval has length less than $\pi/2k$ and each J-interval has length greater than $\pi/2k$. Consequently the J-interval must contain $\theta_1(\eta)$ or $\theta_2(\eta)$ which is a contradiction. Thus equation 3 must hold. □

As a special case of the function $R(z, \eta)$ in the corollary we may consider the following lemma.

Lemma 3.3. *Let P_{2k} be a homogeneous suharmonic, but not harmonic, polynomial of degree $2k$. Write $P_{2k} =| z |^{2k} g(\theta)$ where $\theta = Argz$ and assume $g(\theta_0) = g'(\theta_0) = 0$ for a point θ_0. Let $R(z, \eta) = \operatorname{Re} \eta z^{2k} + P_{2k}(z) =| z |^{2k} G(\theta, \eta)$ for $\eta \in \mathbb{C}$.*

Then there exists a germ of a piecewise real analytic C^1 curve γ through the origin in the η-plane with the following properties:

1. *For each point $p \in \gamma$ there exists a point $\theta(p)$ close to θ_0 such that $G(\theta(p), p) = 0$ while $G(\theta, p) > 0$ whenever $0 <| \theta - \theta(p) |< \pi/4k$.*
2. *Locally about a point $p \in \gamma$ the curve γ will divide a neighbourhood into two sides γ^-, γ^+. For each point $p^- \in \gamma^-$ there exists $\theta_1(p^-), \theta_2(p^-)$ close to θ_0 such that $\theta_1(p^-) < \theta_2(p^-)$ and $G(\theta, p^-) < 0$ when $\theta_1(p^-) < \theta < \theta_2(p^-)$, and $G(\theta, p^-) > 0$ when $\theta \in \big(\theta_1(p^-) - \frac{\pi}{4k}, \theta_1(p^-)\big) \cup \big(\theta_2(p^-), \theta_2(p^-) + \frac{\pi}{4k}\big)$.*
   Moreover θ_1, θ_2 are smooth on γ^- and extend continuously to $\gamma^- \cup \gamma$. If $p^- \to p$, then $\theta_1(p^-) \to \theta(p)$ and $\theta_2(p^-) \to \theta(p)$.
3. *For each point in γ^+, denoted p^+, $G(\theta, p^+) > 0$ for all $\theta \in \big(\theta(p) - \frac{\pi}{4k}, \theta(p) + \frac{\pi}{4k}\big)$ where $\theta(p)$ is the value from property 2.*

Proof. If we write $\eta = re^{i\psi}$ and $z =| z | e^{i\theta}$, then $\operatorname{Re}(\eta z^{2k}) = \operatorname{Re}(r \mid z \mid^{2k} e^{i(\psi+2k\theta)}) =| z |^{2k} r\cos(\psi + 2k\theta)$. Thus $G(\theta, \eta) = r\cos(\psi + 2k\theta) + g(\theta)$ and it follows that $G(\theta, \eta)$ is real analytic because $g(\theta)$ is real analytic. If $\theta_0 \neq 0$ we may then look at the translation $\theta \mapsto \tilde{\theta} \equiv \theta - \theta_0$ which is well defined, and gives us $\tilde{\theta} = 0$ for $\theta = \theta_0$. Also, by this translation we find $\tilde{\psi} = \psi + 2k\theta_0$. Rename $\tilde{\theta}, \tilde{\psi}$ to θ, ψ and assume that $\theta_0 = 0$.

By the hypothesis $G_\theta(\theta_0, 0) = G(\theta_0, 0) = 0$ in the origin of the η-plane. In order to find the wanted curve γ in the η-plane that goes through the origin, look at the following system of equations:

$$\begin{cases} E_{(1)}: & G(\theta, re^{e\psi}) = r\cos(\psi + 2k\theta) + g(\theta) = 0 \\ E_{(2)}: & G_\theta(\theta, re^{i\psi}) = -2kr\sin(\psi + 2k\theta) + g_\theta(\theta) = 0 \end{cases}.$$

Which may be written

$$\begin{cases} E_{(1)}: & -r\cos(\psi + 2k\theta) = g(\theta) \\ E_{(2)}: & 2kr\sin(\psi + 2k\theta) = g_\theta(\theta) \end{cases}. \tag{4}$$

Ultimately these are two equations of three variables θ, r, η. By the Implicit Function Theorem we can find and solve r as a function of ψ. In order to find this relation we use the Implicit Function theorem again, but this time by looking at the equation resulting from dividing $E_{(1)}$ by $E_{(2)}$ where the variable r cancels. We thus find θ as a function of ψ which may be inserted into $E_{(1)}$ to find $r = r(\psi)$.

By $E_{(1)}/E_{(2)}$: $\quad -1/\tan(\psi+2k\theta) = 2kg(\theta)/g_\theta(\theta)$ and the identity $\tan(x-\pi/2) = -1/\tan(x)$ we find for θ close to 0 that

$$\tan(\psi+2k\theta-\frac{\pi}{2}) = 2k\frac{g(\theta)}{g_\theta(\theta)} = 2k\frac{C\theta^{2m}+\mathcal{O}(\theta^{2m+1})}{2mC\theta^{2m-1}+\mathcal{O}(\theta^{2m})} = \frac{k}{m}\frac{\theta^{2m}+\mathcal{O}(\theta^{2m+1})}{\theta^{2m-1}+\mathcal{O}(\theta^{2m})} = \frac{k}{m}\frac{\theta+\mathcal{O}(\theta^2)}{1+\mathcal{O}(\theta)}.$$

Here the series expansion of g is given by the proof of lemma 3.1 where $C>0$ and m is a positive integer. Further simplification is done by noticing that

$$\frac{1}{1+\mathcal{O}(\theta)} = \frac{1}{1-\mathcal{O}(\theta)} = \Sigma_{j=0}^{\infty}[\mathcal{O}(\theta)]^j = 1+\Sigma_{j=1}^{\infty}\mathcal{O}(\theta^j),$$

so $\tan(\psi+2k\theta-\frac{\pi}{2}) = \frac{k}{m}(\theta+\mathcal{O}(\theta^2))$. At the same time the series expansion tangent states that $\tan(\psi+2k\theta-\frac{\pi}{2}) = \psi+2k\theta-\frac{\pi}{2}+\mathcal{O}((\psi+2k\theta-\frac{\pi}{2})^3)$ for ψ close to $\pi/2$. As it is easier to work with $\mathcal{O}((\psi+2k\theta-\frac{\pi}{2})^2)$ than $\mathcal{O}((\psi+2k\theta-\frac{\pi}{2})^3)$ and $\mathcal{O}((\psi+2k\theta-\frac{\pi}{2})^3) = \mathcal{O}((\psi+2k\theta-\frac{\pi}{2})^2)$, we then obtain after some rearrangement that $\psi-\frac{\pi}{2}+\mathcal{O}((\psi+2k\theta-\frac{\pi}{2})^2) = \frac{k-2km}{m}\theta+\mathcal{O}(\theta^2)$.

Notice that $(\psi+2k\theta-\frac{\pi}{2})^2 = (\psi-\frac{\pi}{2})^2+4(\psi-\frac{\pi}{2})k\theta+4k^2\theta^2$, and since we already know that $\theta=\theta(\psi)$ we may look at θ as function of $\psi-\pi/2$. As θ is supposed to vary with different choices of z we may also assume that θ is not a constant. Therefore $\mathcal{O}((\psi+2k\theta-\frac{\pi}{2})^2) = \mathcal{O}((\psi-\frac{\pi}{2})^2)$ and $\theta(\psi) = a(\psi-\frac{\pi}{2})+\mathcal{O}((\psi-\frac{\pi}{2})^2)$ where $a = \frac{m}{k-2km} < 0$.

From $E_{(1)}$: $\quad r = -g(\theta)/\cos(\psi+2k\theta)$ with the identity $\sin(x-\pi/2) = -\cos(x)$ it follows that

$$r = \frac{g(\theta)}{\sin\left(\psi+2k\theta(\psi)-\frac{\pi}{2}\right)} = \frac{g(\theta)}{\sin\left(\psi-\frac{\pi}{2}+2ka(\psi-\frac{\pi}{2})+\mathcal{O}((\psi-\frac{\pi}{2})^2)\right)} = \frac{g(\theta)}{\sin\left(\tilde{a}(\psi-\frac{\pi}{2})+\mathcal{O}((\psi-\frac{\pi}{2})^2)\right)},$$

where $\tilde{a} = 2ka+1 = \frac{2km}{k-2km}+1 = \frac{1}{(1/2m)-1}+1 < 0$. By the series expansion of sine and similar use of geometric series with big oh notation as before, we find

$$r = \frac{g(\theta)}{\tilde{a}(\psi-\frac{\pi}{2})+\mathcal{O}((\psi-\frac{\pi}{2})^2)} = \frac{g(\theta)}{\tilde{a}(\psi-\frac{\pi}{2})}\frac{1}{1-\mathcal{O}(\psi-\frac{\pi}{2})} = \frac{g(\theta)}{\tilde{a}(\psi-\frac{\pi}{2})}\left(1+\mathcal{O}(\psi-\frac{\pi}{2})\right).$$

Meanwhile, by using the binomial theorem for multiplying out $\theta(\psi)^{2m}, \theta(\psi)^{2m+1}$. By inspection we find the following expression

$$g(\theta) = C\theta(\psi)^{2m}+\mathcal{O}(\theta(\psi)^{2m+1}) = Ca^{2m}(\psi-\frac{\pi}{2})^{2m}+\mathcal{O}((\psi-\frac{\pi}{2})^{2m+1})+\mathcal{O}\left((\psi-\frac{\pi}{2})^{2m+1}+\mathcal{O}((\psi-\frac{\pi}{2})^{2m+2})\right).$$

Finally, with $g(\psi) = Ca^{2m}(\psi - \frac{\pi}{2})^{2m} + \mathcal{O}((\psi - \frac{\pi}{2})^{2m+1})$ we may write

$$r(\psi) = \frac{Ca^{2m}(\psi - \frac{\pi}{2})^{2m} + \mathcal{O}((\psi - \frac{\pi}{2})^{2m+1})}{\tilde{a}(\psi - \frac{\pi}{2})}\Big[1 + \mathcal{O}(\psi - \frac{\pi}{2})\Big] = \alpha(\psi - \frac{\pi}{2})^{2m-1} + \mathcal{O}((\psi - \frac{\pi}{2})^{2m}), \tag{5}$$

where $\alpha = Ca^{2m}/\tilde{a} < 0$ and $r - \alpha(\psi - \pi/2)^{2m-1}$ is real analytic. Since the situation is local , this parametrization of a real analytic curve is valid for $0 < r \ll 1$ and $\psi \in [\pi/2 - \delta, \pi/2 + \delta]$ for some $\delta > 0$. The curve can be extended to the origin where $\mathrm{Re}(\eta) = 0$ is the tangent line. A similar curve is obtained for $0 < r \ll 1$, $\psi \in [-\pi/2 - \delta, -\pi/2 + \delta]$ for some $\delta > 0$. Thus we have a piecewise real analytic C^1 curve γ through the origin.

1) Now that we have shown the existence of γ, we know by the Implicit Function theorem and equation 4 that any fixed p on γ will give the function $\theta = \theta(p)$ close to $\theta_0 = 0$ such that $G(\theta(p), p) = 0$. This is done by looking at $E_{(1)}$. Similarly we find $G_\theta(\theta(p), p) = 0$ by $E_{(2)}$. The function G now satisfies the conditions for lemma 3.1 in the z-plane. As explained in the last paragraph of the proof of lemma 3.1 we thus have $G(\theta, p) > 0$ for θ close to $\theta(p)$, where $\theta < \theta(p)$ and/or $\theta > \theta(p)$. Where g is linear combination of trigonometric functions, G is also linear combinations of trigonometric functions. Consequently G will have both I and J-regions. Thus we have adjacent J-regions of G on each side of $(\theta(p), p)$. They must have length strictly greater than $\pi/2k$ by lemma 3.2. Ultimately this means that $G(\theta, p) > 0$ whenever $0 <\mid \theta - \theta(p) \mid< \pi/2k$, which implies $G(\theta, p) > 0$ whenever $0 <\mid \theta - \theta(p) \mid< \pi/4k$. Thus, as we could without loss of generality assume that $\theta_0 = 0$, we find that property 1 of the lemma has been proven.

2) Consider the η-plane where the first axis is the real part of η and the second axis is the imaginary part. The formula for $r(\psi)$ given above is not defined for $\psi > \pi/2$ as this will give us $r < 0$ which is absurd. Therefore look at the first quadrant. By inspection, differentiating $r(\psi)$ once shows us that the growth rate is negative, and thus that $r(\psi)$ decreases. While differentiating $r(\psi)$ twice shows us that the only local inflection point is the origin. Thus for the neighbourhood where the curve γ is locally defined, γ will separate the set of complex points in the neighbourhood into two connected components. One on the left hand side and one on the right hand side. Denote these by γ^- and γ^+ respectively.

Choose the point $p^- \in \gamma^-$ and keep it fixed. Rotate this point by the transformation $p^- \to p \equiv p^- e^{i\alpha}$ where $\alpha < 0$ such that $p \in \gamma$. As before we may apply the implicit function theorem for a point on γ and find that $\theta = \theta(q)$ for points q in a neighbourhood of p. By definition, any point in γ^- is close to γ. Consequently a fixed p^- ensures that $\theta = \theta(p)$ is fixed. Denote $p^- = r_0 e^{i\psi_0}$. We find $G_\psi \mid_{(\theta(p),p)} = -r_0 \sin(\psi_0 - 2k\theta) < 0$ and $G(\theta, p) = 0$. Thus we may look at the transformation $p \to e^{-i\alpha} p = p^-$ and find that

$G(\theta(p^-), p^-) < 0$.

Thus for all $p^- \in \gamma^-$ there exists a θ such that $G(\theta(p), p^-) < 0$. Consequently there must exist an I interval of the θ variable as G is real analytic. Denote $I = (\theta_1, \theta_2)$. Since all conditions of corollary 3.2.1 are met we thus have $\theta_1(p^-), \theta_2(p^-)$ both close to 0 such that $\theta_1(p^-) < \theta_2(p^-)$ and $G(\theta, p^-) < 0$ whenever $\theta_1(p^-) < \theta < \theta_2(p^-)$. By lemma 3.1 this I-region can not have another I-region as neighbour, so lemma 3.2 gives us $G(\theta, p^-) > 0$ for when $\theta \in \big(\theta_1(p^-) - \frac{\pi}{2k}, \theta_1(p^-)\big) \cup \big(\theta_2(p^-), \theta_2(p^-) + \frac{\pi}{2k}\big)$. Consequently $G(\theta, p^-) > 0$ for $\theta \in \big(\theta_1(p^-) - \frac{\pi}{4k}, \theta_1(p^-)\big) \cup \big(\theta_2(p^-), \theta_2(p^-) + \frac{\pi}{4k}\big)$.

Corollary 3.2.1 also states that these θ_1, θ_2 are smooth on γ^-, and as γ^- is a connected component in the η-plane it follows that θ_1, θ_2 can be extended continuously to the boundary γ of γ^-. That is, θ_1, θ_2 can be extended continuously to $\gamma^- \cup \gamma$. Now we may investigate what happens when we let $p^- \to p \in \gamma$. As $p \in \gamma$ the equation $E_{(1)}$ is valid, and considering $p^- \to p$ we find that $\theta_1(p^-), \theta_2(p^-)$ will change in a smooth manner to the point where $E_{(1)}$ applies. In other words $\mid r\cos(\psi + 2k\theta_i) + g(\theta_i) \mid \to 0$ for $i = 1, 2$. Consequently the points θ_1, θ_2 will coincide as the I-region shrinks and we find $\theta_i(p^-) \to \theta(p)$ for $i = 1, 2$. As we have assumed without loss of generality that $\theta_0 = 0$, we find that property 2 from the lemma has been proven.

3) From property 2), if we let $p^- \to p$ we find that $G(\theta, p) > 0$ for $\theta \in (\theta(p) - \pi/4k, \theta(p)) \cup (\theta(p), \theta(p) + \pi/4k) = (\theta(p) - \pi/4k, \theta(p) + \pi/4k) \setminus \{\theta(p)\}$. At the point $\theta(p)$ we can not have $G(\theta(p), p) < 0$ as this would contradict continuity of G. Hence we have the situation as in property 1) where $G(\theta(p), p) = 0$. We may change the coordinates in the θ-plane by rotation such that $\theta(p) = 0$. Then consider $G_\psi(\theta, r, \psi) = -r\sin(\psi + 2k\theta)$ which states that $G_\psi(\theta, r, \psi) \mid_{\theta=\theta(p)=0} < 0$ where $\psi \in (\frac{\pi}{2} - \delta, \frac{\pi}{2})$ for a $\delta > 0$. As we go from p to a point p^+ we decrease the value of ψ and increase the value of G. Thus for all points $p^+ \in \gamma^+$ we know that there is a J-region of G of the variable θ, where we must have $\theta(p) = 0 \in J$.

Now that we have proven the existence of these J-intervals, it remains to investigate other points of the variable θ that may lie in them. This has to be done in a more stringent manner, since changing p to a point p^+ may change both variables r, ψ. For this purpose define the intervals $Int_\theta = (\theta(p) - \frac{\pi}{4k}, \theta(p) + \frac{\pi}{4k}) = (-\frac{\pi}{4k}, \frac{\pi}{4k})$, $Int_\Psi = (\frac{\pi}{2} - \delta, \frac{\pi}{2})$ and $D = \{(\theta, r, \psi) \mid \theta \in Int_\theta, \psi \in Int_\Psi, r \in (0, r_0)\}$, where $r_0 > 0$ has yet to be determined.

Now look at the set D in order to prove that $G > 0$ in D. Let $\psi \in Int_\Psi$ and denote $\psi = \frac{\pi}{2} - \tilde{\delta}$ where $0 < \tilde{\delta} < \delta$. We know that $g(\theta) > 0$ for $\theta \in Int_\theta$. Thus assume $\theta \in (-\pi/4k, \pi/4k)$ which corresponds to $2k\theta \in (-\pi/2, \pi/2)$. We find that $\psi + 2k\theta \in (-\tilde{\delta}, \pi - \tilde{\delta})$. Denote the cosine term in G as $r\cos(\psi + 2k\theta)$. The cosine term in G will be negative for $\psi + 2k\theta \in (\pi/2, \pi - \tilde{\delta})$. However, we may choose $r_0 \ll 1$ such that $G > 0$ in this specific interval since G and g is real analytic and $r \in \mathbb{R}$. In order to this we need $| r \cos(\psi + 2k\theta) | \leq r < g(\theta)$ for all $r < r_0$ and for all θ such that $\psi + 2k\theta \in (\pi/2, \pi - \tilde{\delta})$. By inverting $\psi + 2k\theta$ a quick calculation shows that this corresponds to $\theta \in (\tilde{\delta}/2k, \pi/4k)$. Hence it follows that $\min_{\theta \in (\tilde{\delta}/2k, \pi/4k)} g(\theta) > 0$ since $\tilde{\delta}/2k > \theta(p) = 0$. Thus choose $r_0 = \min_{\theta \in (\tilde{\delta}/2k, \pi/4k)} g(\theta)$ to be a fixed constant and we find that $G > 0$ in the interval $\psi + 2k\theta \in (\pi/2, \pi - \tilde{\delta})$. Rename the r_0 in the definition of D to the one being used here.

For $\psi + 2k\theta = \pi/2$ we find that the cosine term in G is zero, and thus G being equal to $g(\tilde{\delta}/2k) > 0$ since $\psi + 2k\theta = \pi/2$ corresponds to $\theta = \tilde{\delta}/2k > \theta(p) = 0$. The cosine term will be positive for $\psi + 2k\theta \in [0, \pi/2)$ and consequently $G > 0$. By the hypothesis it is assumed that $g(\theta(p)) = 0$, which correspond to the point $\psi + 2k\theta = \psi = \frac{\pi}{2} - \tilde{\delta} \in [0, \pi/2)$. This is not an issue because the cosine term of G is positive at this point. For the small interval $\psi + 2k\theta \in (-\tilde{\delta}, 0)$ we know that cosine is an even function and is therefore positive in this interval. In addition g is positive for all values θ in $(-\pi/4k, \pi/4k) = Int_\theta$, hence we find $G > 0$ for the part $\psi + 2k\theta \in (-\tilde{\delta}, 0)$. Thus we have found that $G(\theta, r, \psi) > 0$ in D.

For each point p^+ in γ^+ we have $\psi = \text{Arg}(p^+) \in Int_\Psi$. Since γ^+ lies in a neighbourhood of a point p on γ, we have $0 < r \ll 1$. We might as well assume that $r < r_0$. As $G > 0$ in D we conclude: For each point p^+ in γ^+, $G(\theta, p^+) > 0$ for all $\theta \in Int_\theta$. Ultimately as we have assumed that $\theta(p) = 0$ without loss of generality, we may conclude $G(\theta, p^+) > 0$ for all $\theta \in (\theta(p) - \frac{\pi}{4k}, \theta(p) + \frac{\pi}{4k})$ for each point $p^+ \in \gamma^+$. □

References

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