Max Lunde Hauge

# Cantor Minimal Systems

Bachelor's project in Mathematical Sciences
Supervisor: Eduardo Ortega Esparza

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NTNU
Norwegian University of Science and Technology
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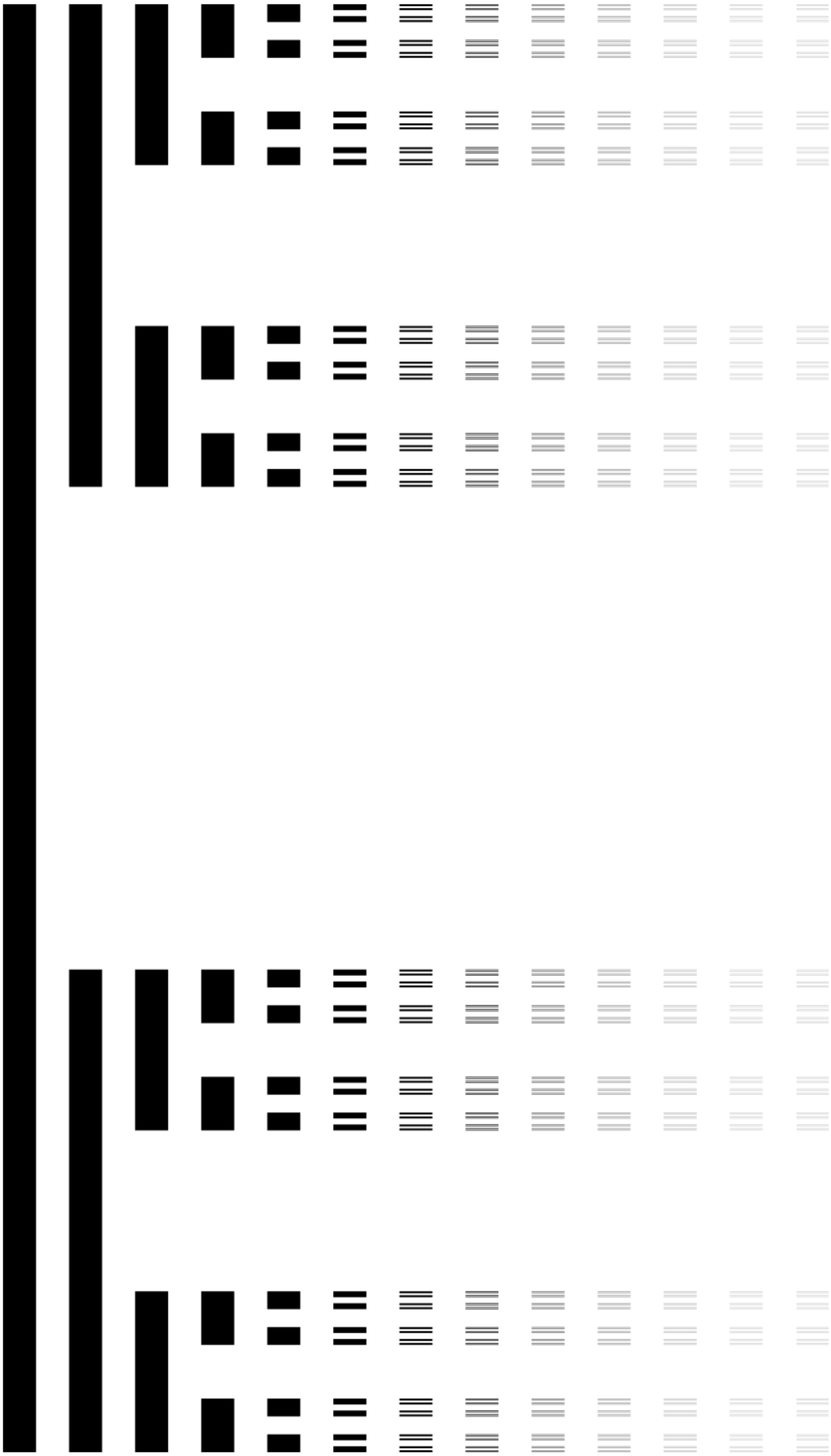
## Abstract

In this thesis, we consider the construction of the Cantor set with its unique mathematical properties, together with different equivalent representations of the set in both metric spaces and general topological spaces. Last, we define the general way we may construct automorphisms on the Cantor set using the notion of Brattelie diagrams.

*Even a simple rule can generate complexity*



# Introduction

The main object of study in this thesis is a well the Cantor set, named after the German mathematician George Cantor. In 1883 he gave a general definition for the set, as an example of a subset of the real line, with what may seem like counterintuitive properties. It is these properties that will be of study, and as we'll see, actually characterize the set itself. We will look at different, although equivalent, representations of the Cantor set, with the primary goal of constructing automophsims on the Cantor set. In the abstract theory of analysis it is often used as an example to illustrate the various mathematical definitions, that includes one of the main sources for this thesis, Abbott (2015).

In addition to being a well-studied object in the aforementioned field, the Cantor set finds its way into many other fields as well. In the study of dynamical systems, one can find that the fractal geometry, the self-similarity or pattern, as we may all call it, actually can describe chaotic systems of the real world. By describe, we mean that they "converge" to the geometry described by a fractal. In the case the Cantor set, or in most cases, its graphical representation named the *Devil's staircase*, which has been shown to describe a wide range of phenomena. Everything from earthquakes (Chen et al., 2020) to the distribution of the galaxies in the universe (Choudhury et al., 2019). In some sense, there is truth to the statement that the Cantor set can be reckoned as the discretization of the very space we occupy. It is the "distribution" of the stardust of the universe. Then, as the evolution of the space in time can be reflected by the automorphism defined on the Cantor set, one can, in theory, describe all instances of a dynamical system. Which is to say, we can predict to a certain degree of accuracy the future of the system.

Furthermore, the Cantor set is one of the many ways one can discretize the real line. This is important, as we will later observe that binary sequences can represent the elements of the set. Though they are of infinite length, this is an obvious connection to the way computers work. However, they do not work in the continuum, as everything about a computer has to be finite or discretized, and somehow the Cantor set is the best way to do this. The mathematics underlying how one can approximate elements of the Cantor set with finite sequences of $0$'s and $1$'s, is the same underlying how the computer deals with the infinite numbers of the real line. Take the number $\pi = 3.14\ldots$. It is not convenient nor efficient to store all the digits of $\pi$ on the computer. Instead, the number can be represented as a convergent geometric series, the same for which we will use to describe elements of the Cantor set.

Now before we begin, a special mention is to be made of a fellow *algebro* in the *house of two rooms*, Jørn Olav Jensen for great laughs, valuable feedback, and clearing up a few things in the thesis, and topology in general.

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# 1 Cantor spaces $\mathscr{C}$

## 1.1 The Cantor set $C$

We begin with a fascinating mathematical construction on the real number line $\mathbb{R}$. This unique set of numbers, formed as a subset of the real line, is a helpful tool for extending the intuition and understanding of mathematical properties and definitions of both the real line and its subsets.

One category of examples of the Cantor set $C$, is any closed subset of a closed interval, formed by removing open intervals in a specific manner. While there are many ways to construct a cantor set this way, the most used modern construction is called the *Cantor ternary set*, and it is defined on the closed interval between 0 and 1 of the real numbers, referred to as the unit interval. As the name suggests, this set is constructed via iteratively deleting the open middle third of the line segment *ad infinitum*. It is the closed subsets on the interval that are not removed that form the Cantor ternary set $C$.

*Remark* 1.1. Take note of the construction of the ternary set. While it is the most used example, one is not limited to removing middle thirds. In fact, removing any fixed quantity from the original line segment will yield a Cantor set.

**Figure 1.1:** A graphical illustration of first six iterations of the Cantor ternary set. The black "bars" represents the segments that eventually form the cantor ternary set.

**Definition 1.2.** Given the *closed* interval $[0, 1] \subset \mathbb{R}$. We construct the Cantor ternary set $C$ by removing the *open* middle third interval *ad infinitum*. In other words, through iteration from $C_0 = [0, 1]$ one can recursively construct each subsequent closed set $C_n \subset C$ for a natural number $n \geq 1$.

$$C_0 = [0, 1], \tag{1.1}$$

$$C_1 = C_0 \backslash \left( \frac{1}{3}, \frac{2}{3} \right) = \left[ 0, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, 1 \right], \tag{1.2}$$

$$\vdots \tag{1.3}$$

$$C_n = \frac{C_{n-1}}{3} \cup \left( \frac{2}{3} + \frac{C_{n-1}}{3} \right). \tag{1.4}$$

From this, we have that the intersection of the intervals that are not removed is what results in the Cantor ternary set.

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Observe that each $C_n$ yield $2^n$ closed intervals, all with length (Lebesgue measure) $1/3^n$. Figure 1.1 illustrates the first 6 iterations.

### 1.1.1 Elements of the Cantor Set

Since the interval removed is always internal to the current interval, the endpoints of the current interval are never removed in any of the consecutive eliminations. Take the endpoints $0, 1 \in C$ as an example. Observe that they are never removed in any of the consecutive eliminations of the middle third interval and are therefore an element of $C$. This is also true for the next set of endpoints $\frac{1}{3}, \frac{2}{3} \in C$. The endpoints are essential to highlight, as we can via a sum of the endpoints running from left to right, approximate every element in $C$, which motivates the following definition and lemma.

**Definition 1.3.** An arbitrary finite sequence of numbers $(a_0 \ldots a_n)$ of length $n \in \mathbb{N}$, that for simplicity is noted as $(a_n)$ is a *binary sequence* if each element $a_n$ can take only on two distinct values, either $0$ or $1$. A finite binary sequence $(a_n)$ of length $n$ is noted as the following element from the set of all finite binary sequences of length $n$ by $a_n \in \{0, 1\}^n$. An infinite binary sequence $(a_n)$ is noted as the following element from the set of all infinite binary sequences by $(a_n) \in \{0, 1\}^{\mathbb{N}}$.

*Remark* 1.4. Each removal leaves behind two separated intervals. By labelling the interval to the left $0$ and the interval to the right $1$ we can use this to our advantage when formulating the following lemma.

Before we do that however, we must bring to mind a few critical results from calculus on the topic of geometric series, in order to prove the lemma. If $|r| < 1$ for an infinite geometric series, then the series converges to the following

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}. \tag{1.5}$$

If $r \neq 1$ for a finite geometric series, then the series converges to the following

$$\sum_{k=0}^{n} ar^k = a\left(\frac{1-r^{n+1}}{1-r}\right). \tag{1.6}$$

**Lemma 1.5.** *Every element $x \in C$ can be written as a geometric series of the form*

$$\sum_{n=1}^{\infty} \frac{2a_n}{3^n} \quad \textit{for } a_n \in \{0, 1\}.$$

*Proof.* Let $m \in \mathbb{N}$, $m \geq 1$, and $C_m$ be the $m$-th set after a *finite* number of iterations $m$. If we sum from the left, and always sum two times the fraction we removed, then the geometric series will always converge on the left endpoint of any interval in $C_m$ (Recall that $C_m$ is a union of intervals). In other words it will be a finite geometric series up to index $m$. However, to approximate any element $x \in C$ we need the sum of all the subsequent intervals *ad infinitum*. In other words this will be an infinite geometric series. We dub the finite geometric series as the *head*, and the infinite geometric series as the *tail*.

$$\underbrace{\sum_{n=1}^{m} \frac{2a_n}{3^n}}_{Head} + \underbrace{\sum_{n=m+1}^{\infty} \frac{2a_n}{3^n}}_{Tail} \quad \text{for } a_n \in \{0,1\}\,. \tag{1.7}$$

It can easily be verified that the head converges to left endpoint of any interval. The interesting series is the tail. If we can approximate the right interval endpoint, then we know we can approximate any element on this interval by changing the values for $a_n$. For the series to approximate the right element, then $a_n = 1$ for all $n$. By using both equation (1.5) and equation (1.6) we can show that it converges to the following element.

$$\begin{aligned}
\sum_{n=m+1}^{\infty} \frac{2a_n}{3^n} &\leq 2\left(\sum_{n=m+1}^{\infty} \frac{1}{3^n}\right) = 2\left(\sum_{n=0}^{\infty} \frac{1}{3^n} - \sum_{n=0}^{m} \frac{1}{3^n}\right), \\
&\leq 2\left(\frac{1}{1-\frac{1}{3}} - \frac{1-\left(\frac{1}{3}\right)^{m+1}}{1-\frac{1}{3}}\right) = 2\left(\frac{1+(-1)+\left(\frac{1}{3}\right)^{m+1}}{\frac{2}{3}}\right), \\
&\leq 3\left(\frac{1}{3}\right)^{m+1} = \frac{1}{3^m}.
\end{aligned}$$

Which is exactly the length (Lebesgue measure) of each interval in $C_m$! Since we can approximate $x$ with the both head and the tail we must conclude that $x \in C_m$ for all $m$, and thus also belong in $C = \bigcap_{m=1}^{\infty} C_m$. By combining the head and the tail we get the following

$$\sum_{n=1}^{\infty} \frac{2a_n}{3^n} \quad \text{for } a_n \in \{0,1\}\,.$$

□

*Remark* 1.6. If we let $a_n = 0$ for every $n$, then the geometric series sum to 0, and if we let $a_n = 1$ for every $n$, then the geometric series converges to 1. Thus any sum corresponding to an element of the Cantor space is bounded by these values, as well as demonstrating that both $0, 1 \in C$ as expected. Additionally, it follows that the Cantor space is non-empty, as by its very construction must at least include the endpoints on the interval on which it is constructed.

$$\sum_{n=1}^{\infty} \frac{2\cdot 0}{3^n} = 0, \qquad\qquad \sum_{n=1}^{\infty} \frac{2\cdot 1}{3^n} = 2\cdot\frac{\frac{1}{3}}{1-\frac{1}{3}} = 1.$$

*Remark* 1.7. Observe from the proof that if given $x \in C$, it will have a unique binary sequence that can be viewed as a "set of directions" on how to find $x$ within $C$. The sum of the endpoints by lemma 1.5 will converge to $x$ if one follows its unique binary sequence. Note that two elements of the Cantor space cannot have the same binary sequence unless they are equal.

**Example 1.8.** Let us do an example on the interval highlighted in figure 1.2. We use the geometric series lemma 1.5, and split it so that the first sum approximates the endpoint on the left of this interval. To reach this interval we need to let $n = 4$, and we want the contribution from the first 2/3, and the latter 2/9 as marked with green and yellow arrows. In other words we want $a_1 = 1$, $a_2 = 0$, $a_3 = 1$ and $a_4 = 0$ in our geometric series.

$$\sum_{n=1}^{4} \frac{2a_n}{3^n} = \frac{2 \cdot 1}{3^1} + \frac{2 \cdot 0}{3^2} + \frac{2 \cdot 1}{3^3} + \frac{2 \cdot 0}{3^4} = \frac{20}{27}.$$

Now for the sum to actually approximate an element of the Cantor set, it must run to infinity. In other words, the tail of lemma 1.5 after the split is what must converge to all elements of the Cantor set within this interval. If the series is still to converge on the endpoint on the left side we calculated above, we must have that every $a_n$ after $n = 4$ is equal to *zero*, as to get no contributions from the later fractions. If we want the sum to converge on the endpoint on the right side on this interval, then we need the contribution from every fraction after $n = 4$, and thus every $a_n$ after $n = 4$ must equal to *one*. As we showed in the proof of lemma 1.5, we know that the tail will converge to the length of this interval, which is $1/3^4$.

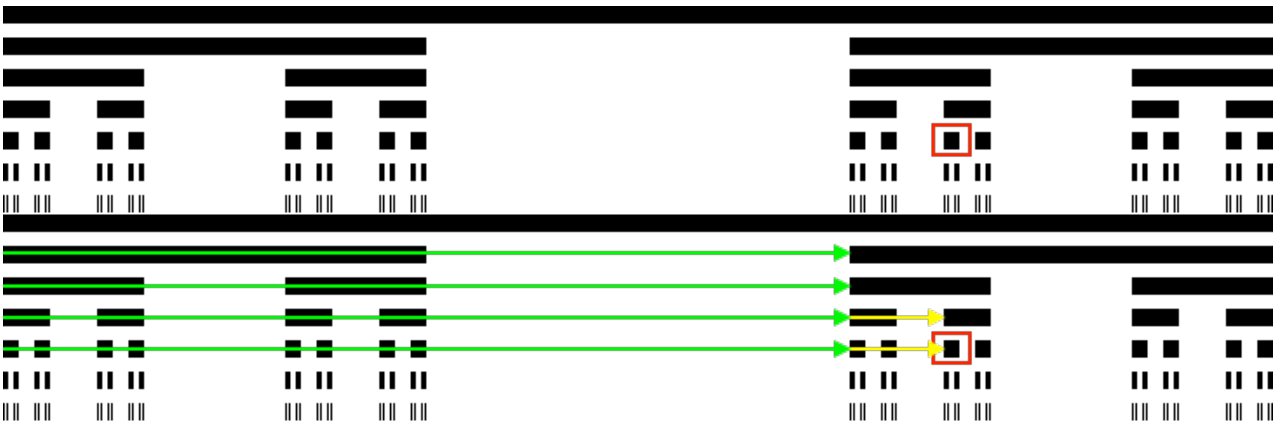


**Figure 1.2:** Illustration of the head in example 1.8.

### 1.1.2 Length of the Cantor Set

As $C \subseteq [0, 1] \subseteq \mathbb{R}$ it is Lebesgue-measurable. That is to say, we have one way of measuring lengths on subsets of $\mathbb{R}$.

**Definition 1.9.** For a subset $S \subseteq \mathbb{R}$, the *Lebesgue measure* $\ell$ of the interval $I = [a, b]$ (or $I = (a, b)$) gives the length of $I$ as

$$\ell(I) = b - a.$$

It sounds reasonable that if we sum the Lebesgue measure of all the open intervals that are removed, we can find the Lebesgue measure of the closed intervals that are left. In other words, we can measure the length of $C$. Recall that each iteration in equation (1.4) yields $2^n$ closed intervals of Lebesgue measure $1/3^n$. The same holds true for the intervals we remove, and we have the following geometric series

$$\frac{1}{3} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{27} + 8 \cdot \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n}.$$

From equation (1.5) we know that the geometric series converges to the following

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \sum_{n=1}^{\infty} 2^{n-1} \cdot 3^{-n} = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1.$$

In other words, the Lebesgue measure of the intervals removed is 1. Recall that $C$ is defined on the closed unit interval, which has Lebesgue measure 1. Now, subtracting the Lebesgue measure of the intervals removed yields the Lebesgue measure of $C$. Henceforth, the Cantor set is of *zero length.* A strange result, for which the preceding arguments is a proof for the following lemma.

**Lemma 1.10.** *The Cantor set has Lebesgue measure zero.*

### 1.1.3 Counting the Elements of the Cantor Set

By removing intervals not containing their endpoints (open intervals), one could come to the misleading conclusion that the only elements of the Cantor set are the endpoints themselves, as the removal is solely happening to the internal of an interval. However, while it is true that the Cantor set contains no intervals of any kind, it can be helpful to think of what is left as arbitrarily small sized intervals. What we are suggesting here is that $C$ must contain elements that are not endpoints. To clarify this, notice that all of the interval endpoints are rational numbers. If the Cantor set only contained the interval endpoints, then the set would be countable as the rational numbers are countable. Thus, if the Cantor set contains elements that are not endpoints, is it still countable?

**Lemma 1.11.** *The Cantor set is uncountable.*

*Proof.* Recall that any element of $C$ has a corresponding (infinite) binary sequence $(a_n) \in \{0,1\}^{\mathbb{N}}$. Suppose that we have a numbered list $L$ with all possible binary sequences in $C$. In other words, let $L = C$, and assume that $L$ is countable.

$$\begin{array}{cc} 1 & (a_n) \\ 2 & (b_n) \\ & \vdots \\ \mathbb{N} & (x_n) \end{array}$$

Can we form a new binary sequence that is not listed in $L$? If true, then we have a contradiction on the assumption that we had every element listed in $L$ to begin with. Let $(\alpha_n)$ be a new binary sequence not already listed in $L$. Furthermore, let the binary sequence be constructed in the following specific way. Take the *first* binary sequence $(a_n) \in L$. The value of the *first* position $\alpha_0 \in (\alpha_n)$ is determined by the value of the *first* position $a_0 \in (a_n)$.

$$\alpha_0 = \begin{cases} 1 & \text{if } a_0 = 0, \\ 0 & \text{if } a_0 = 1. \end{cases}$$

This insures that $\alpha_0 \neq a_0$, and thus $(\alpha_n) \neq (a_n)$. Now, take the *second* binary sequence $(b_n) \in L$, and determine the value of the *second* position $\alpha_1 \in (\alpha_n)$ by the value of the *second* position $b_0 \in (b_n)$.

$$\alpha_1 = \begin{cases} 1 & \text{if } b_1 = 0, \\ 0 & \text{if } b_1 = 1. \end{cases}$$

This insures us that $\alpha_1 \neq b_1$ and thus $(\alpha_n) \neq (b_n)$. Do this for every binary sequence $(x_n) \in L$. The resulting binary sequence $(\alpha_n)$ will for all $(x_n) \in L$ never be listed in $L$. By the specific manner of constructing $(\alpha_n)$ we have insured that it will always be different to every binary sequence already listed in $L$, no matter how long or "detailed" $L$ is. That is to say, if we added $(\alpha_n)$ to $L$, then we could construct another sequence $(\beta_n) \notin L$ in the same way. As we always can continue doing this, our list of binary sequences will never be complete. Thus we must conclude that every binary sequence cannot be listed in a numerated list, which is to say that they are uncountable. As every element in $C$ has a corresponding binary sequence, $C$ must be uncountable. □

The implications of lemma 1.11 are important. As mentioned, if $C$ only contained the interval endpoints, then our set would be countable. This result however, suggests that the Cantor set contains other points, and that there are uncountable many of these(!). To conclude, from the point of measuring the length of the Cantor set, it has the same Lebesgue measure as a single point. In terms of cardinality, it is uncountable.

### 1.1.4 Self-Similarity

| | Dimension | Scaling ×3 | # new copies | Magnification factor |
|---|---|---|---|---|
| Point | 0 | → | 1 | $3^0$ |
| Line | 1 | → | 3 | $3^1$ |
| Square | 2 | → | 9 | $3^2$ |
| Cube | 3 | → | 27 | $3^3$ |
| $C$ | $x$ | → | 2 | $3^x$ |

**Table 1.1:** Demonstrated in the table is one way we can find the dimension of $C$

Take a good look at the pattern in figure 1.1. Notice we have omitted any label of 0 and 1, which are the most apparent elements of $C \subset [0,1]$. Actually, we could have formed the Cantor ternary set on the closed interval $[0,3]$, where the only change would be a scaling of all elements of $C$ by a factor of three. Furthermore, observe in this case that the "original" Cantor ternary set is the left interval after the first removal. In other words, the Cantor ternary set is always equal to *two* copies of the original set if these two copies are shrunk by a factor of three and translated. The structure of the Cantor set under magnification is indistinguishable from the structure of the whole. This self-similar property means that it is equally valid to think of $C$ as a fractal. A fractal need not be intricate for it to be complex. The simple rule in which we form the Cantor set can, from one viewpoint, be the purest or prototype of all fractals as *one* single — simple — line — can become innumerable many — self-similar lines.

To make the link with fractals more evident, we will need a notion of dimension. Without giving any formal definition of dimension (of which there are several), there is a solid argument in the following. A point has dimension zero, a line segment has dimension one, a square has dimension two, and a cube has dimension three. Recall that the Cantor set, when scaled by a factor of three, resulted in two "new" copies of the original set. Observe in the following table 1.1 what happens when all of the aforementioned "sets" are scaled by the same factor.

Note that the dimension of the figure is always present in the exponent of the magnification factor. In other words, the dimension of the Cantor set should satisfy $2 = 3^x$, which with a simple logarithm yields $x = \frac{\log 2}{\log 3} \simeq 0.631$. The association behind the mathematical concept of *fractal* and the resulting *fractional dimension* above only further underlines this aspect of the Cantor set. However, it is worth mentioning that "fractal" was first used by Benoit Mandelbrot in 1975, while Cantor's original construction is, as mentioned in the introduction, almost 140 years old. However, the definition Cantor himself gave for $C$ was abstract and general, where the construction of the ternary set we have studied here, was only mentioned in passing as a short example (Edgar, 2004).

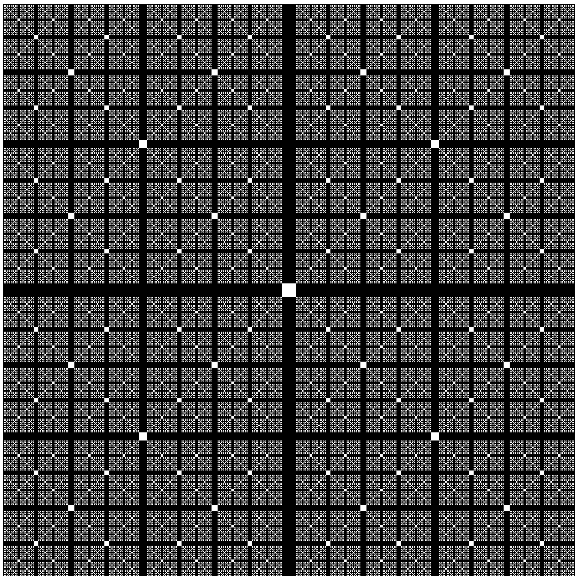


**Figure 1.3:** A illustration of the two dimensional Cantor set.

### 1.1.5 Alternate Representations

Another equally valid representation of the Cantor set builds directly on the notion of infinite binary sequences as directions to elements of the cantor set. In a binary sequence we have to possibilities for each position, by forming a map of the possibilities at each position we can construct the following infinite complete binary tree shown in figure 1.4. In other words, the infinite complete binary tree is "a map" of all possible binary sequences of $C$. Furthermore, observe the following geometric series for all the rightmost red nodes on the Cantor set in figure 1.4.

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \cdots = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{2 * 3^n} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{3^n}.$$

Using equation (1.5) we find that it converges to 1, which is exactly the right endpoint of the first interval $C_0$. By inducing our notion of selecting whether we can go right or left on each index $n$

we get the following geometric series

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{a_n}{3^n} \text{ for } a_n \in \{0, 1\}\,.$$

This geometric series coincides with lemma 1.5, in the sense that they both converge to the same elements for the same binary sequences, and is therefore an equally valid representation of the elements of $C$.

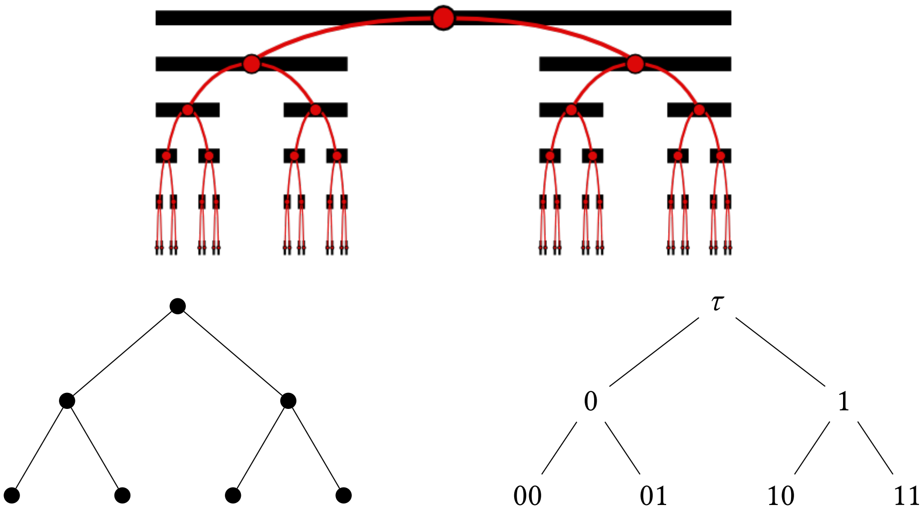



**Figure 1.4:** The development of the infinite complete binary tree associated with the Cantor ternary set.

As mentioned in remark 1.1, removing any fixed percentage in the same manner, will yield a Cantor set. Figure 1.5 illustrates what this set looks like, with its associated infinite complete tree.

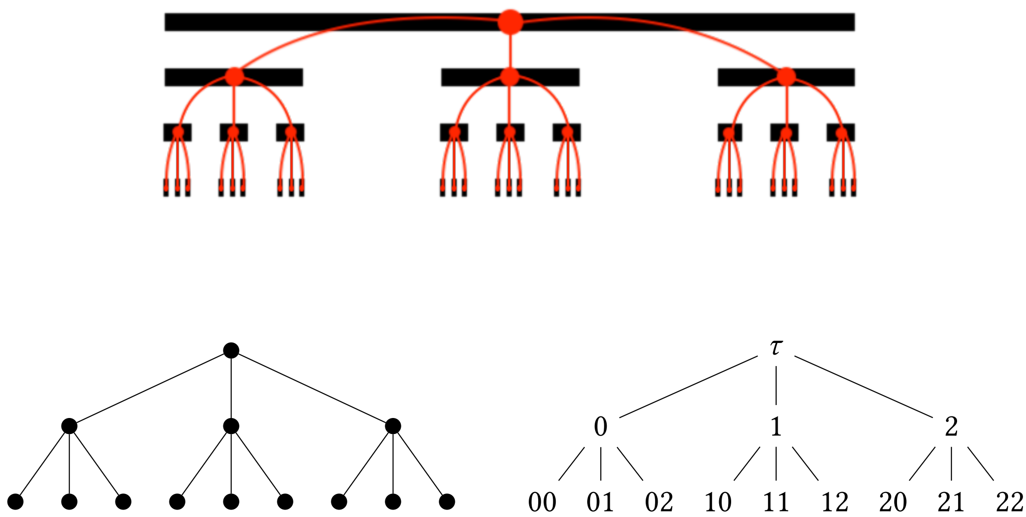



**Figure 1.5:** The development of the infinite complete tree associated with a different construction of the Cantor set.

## 1.2 The Cantor Metric Spaces $C$

### 1.2.1 Metric Spaces

As we are now familiar with the Cantor set, we can progress towards forming the Cantor space $\mathscr{C}$. The Cantor space is, as we will see, essentially the Cantor set with additional structure added. The Cantor set is, as mentioned, a subset of the real line $\mathbb{R}$. On the real line we have a notion of length, which gives a length measure on the Cantor set as a subset of $\mathbb{R}$. The length measure is what we call a metric. However, if given a metric on a set, the metric must satisfy a few conditions if both the metric and the set are to be considered a metric space when viewed together. For more details on this section, the reader is referred to chapter 3 of Abbott (2015).

**Definition 1.12.** Let $X$ be a non-empty set, and let $d$ be a map $d : X \times X \to \mathbb{R}$ hereby referenced as the metric. A metric space $M$ is the set $X$ together with a metric $d$, noted $(X, d)$, where $d$ satisfies all the following conditions

- $d(x, y) \geqslant 0$ where $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$ for all $x, y \in X$,
- $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Example 1.13.** The real line $\mathbb{R}$ is an example of a metric space with the metric given by the absolute difference $d(a, b) = |a - b|$. It can easily be seen that $d$ satisfies the two first conditions. If given $a, b \in X$ the only case where $d(a, b) = 0$ is if $a = b$, otherwise $d(a, b) > 0$. Furthermore, the distance is symmetric for all $a, b \in \mathbb{R}$ due to the absolute value. The triangle inequality also holds as for all $a, b, c \in \mathbb{R}$.

Before we define the Cantor metric space and prove some of its distinctive properties, we need a better understanding of both the subsets of $\mathbb{R}$ themselves and their attributes. Nevertheless, what follows is one of the most important theorems of $\mathbb{R}$ going forward.

**Definition 1.14.** Given a metric a metric space $(X, d)$, a sequence $(a_k)_{k=0}^{\infty} \in X$ is a *Cauchy sequence* if there for any real number $r > 0$, exist $N \in \mathbb{N}$ such that for all positive integers $m, n \geqslant N$

$$d(a_n, a_m) < r.$$

**Definition 1.15.** Given a metric space $M = (X, d)$. The metric space is *complete* if every Cauchy sequence in $X$ converges to an element in $X$.

**Example 1.16.** A great example is the well-known result that the real line $\mathbb{R}$ is a complete metric space, as it can easily be shown that every Cauchy sequence on the real line converges to an element in $\mathbb{R}$

### 1.2.2 Open and Closed Sets

Let us take a moment to discuss open and closed sets, as they are fundamental in the construction going forward.

### Open sets

**Definition 1.17.** Given a metric space $(X, d)$. For a element $x \in X$ and a real number $\epsilon > 0$. The *open $\epsilon$-ball* centered at $x$ with radius $\epsilon$ is the subset of $X$

$$\mathrm{B}(x; \epsilon) = \{y \in X \mid d(x, y) < \epsilon\} . \tag{1.8}$$

**Example 1.18.** Given the metric space $(\mathbb{R}, d)$, then the open interval $(x - \epsilon; x + \epsilon)$ is the open ball centered around an element of the set $x$ with a radius of $\epsilon$.

**Definition 1.19.** Given a metric space $(X, d)$. A subset $S \subseteq X$ is called *open* if there for all elements $x \in S$ exist subsets of open $\epsilon$-balls $\mathrm{B}(x; \epsilon) \subseteq S$.

**Lemma 1.20.** *I: The* union *of any collection of open sets is open.*
*II: The* intersection *of a finite collection of open sets is open.*

*Proof.* I: Assume that we have a collection of open sets $\{A_\alpha\}$, where $\alpha \in \mathbb{N}$ is the number of sets in the collection, and let $U = \bigcup_{\alpha \in \mathbb{N}} A_\alpha$. Given an arbitrary element $a \in A_\alpha \subseteq U$, then, since we for all points $a \in U$ can produce an open $\epsilon$-ball $\mathrm{B} \subseteq A_\alpha \subseteq U$ centered at $a$, it can be seen that $B \subseteq U$, and thus $U$ is open.

II: Assume that we have a finite collection of open sets $\{A_0, \ldots, A_\beta\}$, where $\beta \in \mathbb{N}$ is the number of sets in the collection. Let $V = \bigcap_{k=0}^{\beta} A_k$, and given an arbitrary element $a \in U$. Observe that $a$ is a member of all the open sets $A_k$, and we can then produce an open $\epsilon$-ball $\mathrm{B} \subseteq A_k$ for all $0 \leq k \leq \beta$. We can form a set with all of these $\epsilon$-balls, and we must find the smallest ball. The reason for choosing the smallest ball is that it will be contained in all of the other balls of the set. In other words, $\epsilon = \min \epsilon_0, \ldots, \epsilon_\beta$, such that $\mathrm{B}(a, \epsilon) \subseteq \mathrm{B}(a, \epsilon_k)$. Hence, $\mathrm{B}(a, \epsilon) \subseteq V$ and $V$ is open. □

### Closed sets

It is equally valid to define closed sets, but first we need an understanding of limit points and isolated points.

**Definition 1.21.** Given a metric space $(X, d)$, and a subset $S \subseteq X$. Then, a point $x \in X$ is a *limit point* of $S$ if every open ball $\mathrm{B}(x; \epsilon)$ intersects $S$ at some point other than $x$.

**Definition 1.22.** Given a metric space $(X, d)$, and a subset $S \subseteq X$. Then, a point $x \in S$ is an *isolated point* of $S$ if it is not a limit point of $S$.

*Remark* 1.23. An isolated point is always a member of the set, while the same is not true for limit points.

**Definition 1.24.** Given a metric space $(X, d)$, an arbitrary element $x \in X$, and a real number $\epsilon > 0$. The *closed $\epsilon$-ball* centered at $x$ with radius $\epsilon$ is the subset of $X$

$$\overline{\mathrm{B}}(x; \epsilon) = \{y \in \mathbb{R} \mid d(x, y) \leqslant \epsilon\} . \tag{1.9}$$

**Lemma 1.25.** *Given a complete metric space $(X, d)$. A subset $S \subseteq X$ is* closed *if and only if every Cauchy sequence contained in $S$ converges to a limit $x$ in $S$. A closed subset of a complete metric space is a* complete *subset.*

*Proof.* Let $S$ be a closed subset of a complete metric space $M = (X, d)$. If we have a Cauchy sequence $(a_k)$ in $S$, then $(a_k)$ is also a Cauchy sequence in $X$ since $S \subseteq X$. As $M$ is complete, then the Cauchy sequence converges to a point $x \in X$. It also follows from lemma 1.25 that since $S$ is closed $S = \overline{S}$, then $x \in S$. It follows then that $S$ is complete. □

**Lemma 1.26.** *I: The* union *of a* $\overline{\textit{finite}}$ *collection of closed sets is closed.*
*II: The* intersection *of any collection of closed sets is closed.*

*Proof.* This proof uses some definitions that are to be defined just after. Given an index $I$ of some number of subsets $A_i \subseteq X$, the generalized versions of the well know De Morgan's laws are the following

$$\left(\bigcap_{i\in I} A_i\right)^c \equiv \bigcup_{i\in I} (A_i)^c \quad \text{and} \quad \left(\bigcup_{i\in I} A_i\right)^c \equiv \bigcap_{i\in I} (A_i)^c .$$

Observe that if we use lemma 1.20, take the complement of both statements, and then use De Morgan's laws, we have proved lemma 1.26. □

*Remark* 1.27. It is because of that latter part in the proof, that the notions of "finite" and "any collection" for union and intersection in lemma 1.26 are converse of the statements in lemma 1.20 for open sets.

**Example 1.28.** The Cantor set $C$ is closed since it is the intersection of a finite union of closed intervals. Furthermore, as $\mathbb{R}$ is complete, it follows from lemma 1.25 that the closed subset $C \subset \mathbb{R}$ is complete as well.

**Definition 1.29.** Given a complete metric space $(X, d)$. A subset $S \subseteq X$ is *clopen* if it is closed and open.

*Remark* 1.30. In standard English, it is impossible for a noun to be both open and closed simultaneously, they are antonyms. The mathematical view of open and closed however, is the opposite. The fact that a set is open does not exclude the fact that it can also be closed, and *vica verca*. It can very well happen that a set is both open and closed. Furthermore, if a set is not open, that does not imply that the set must be closed. It can be neither.

**Definition 1.31.** Given a metric space $(X, d)$, a subset $S \subseteq X$, and a set $L$ of all the limit points of $S$. The *closure* of $S$, noted as $\overline{S}$ is the following subset $\overline{S} \subseteq X$.

$$\overline{S} = S \cup L.$$

*Remark* 1.32. If $S$ is the open interval $(a, b)$, then $\overline{S} = (a, b) \cup L = [a, b]$. Thus for a closed interval, it is always the case that $\overline{S} = S$, and that $\overline{S}$ is always a closed set.

**Definition 1.33.** Given a complete metric space $(X, d)$, and a subset $S \subseteq X$. The *complement* of $S$, noted as $S^c$, is the following subset $S^c \subseteq X$.

$$S^c = \{a \in X \mid a \notin S\} .$$

**Lemma 1.34.** *I: A set $S$ is open if and only if its complement $S^c$ is closed.*
*II: Conversely, a set $S$ is closed if and only if its complement $S^c$ is open.*

*Proof.* I: Let $S$ be an open subset $S \subseteq X$. If given a limit point $x \in S^c$, then every open $\epsilon$-ball centered at $x$ will contain points of $S^c$. It cannot happen that $x \in S$ as that would imply there exists an $\epsilon$-ball $\mathrm{B} \subseteq O$. From remark 1.32 one can take note that a closed set contains it all limits points, and thus $x \in O^c$.

Conversely, if let $Q$ be a closed subset $Q \subseteq X$. Since $Q$ is closed, we know that it contains all its limit points, so if given a point $x \in Q^c$ we know that $x \notin Q$. For $Q^c$ to be open we must have an open $\epsilon$-ball centered at $x$ that does not intersect $Q$, such that $\mathrm{B} \subseteq Q^c$, thus $O^c$ is open.

II: Given $R \subseteq X$, observe that $(R^c)^c = R$. Applying the observation to I, yields the wanted results. □

**Theorem 1.35.** *Given a metric space $(X, d)$, and a subset $S \subseteq X$. Then, $S$ is* dense *in $X$ if and only if $\overline{S} = X$.*

**Definition 1.36.** Given a metric space $(X, d)$. The completion of $(X, d)$, noted as $(\overline{X}, \bar{d})$, is a complete metric space together with a function $f : X \to \overline{X}$ such that $\bar{d}(f(x), f(y)) = d(x, y)$ for all elements $x, y \in X$, and $f(X) = \{f(x) : x \in X\}$ is a dense subset of $\overline{X}$.

### 1.2.3 The Cantor Metric Space

The metric from the real line metric space $\mathbb{R}$ induces a metric on $C$ to form the Cantor metric space $(C, d)$. Following that, recall that every element $x \in C$ can be represented by the geometric series given in lemma 1.5, which results in the following definition. This is the beginning of what will be a new, more formal, definition for $C$.

**Definition 1.37.** The induced metric $d$ from $\mathbb{R}$ for two elements $a, b \in C$ is the map $d : C \times C \to \mathbb{R}$ where $d$ is the following

$$d(a, b) = d\left(\sum_{n=1}^{\infty} \frac{2a_n}{3^n}, \sum_{n=1}^{\infty} \frac{2b_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{2|a_n - b_n|}{3^n} \tag{1.10}$$

$$\text{where } a_n, b_n \in \{0, 1\}\,. \tag{1.11}$$

Let us emphasize an important characteristic of what the metric $d$ determines. Recall equation (1.8). If given a point $x \in C$, we can determine with the open ball $\mathrm{B}(x; \epsilon)$ which points $y \in C$ that are of distance $\epsilon$ of $x$. In other words, we can form an open set with all the points $y$ that are on the same $\epsilon$-length interval as $x$. Furthermore, recall lemma 1.5 and that the tail of the geometric series was bound by the value of $1/3^m$ where $m \in \mathbb{N}$ is the index of iteration. It follows from this that $\epsilon$ must be bounded by $\frac{1}{3^{n-1}} < \epsilon \leq \frac{1}{3^m}$, if the following lemma is to hold.

**Lemma 1.38.** *Given a real number $\epsilon > 0$, $n \in \mathbb{N}$, and a point $x \in C$. For the Cantor ternary set, the open ball centered at $x$ with radius $\epsilon = \frac{1}{3^n}$ determines all points $y \in C$ of distance $d(x, y) < \frac{1}{3^n}$ of $x$, if and only if the first $n$ coefficients of $x$ and $y$ coincide.*

*Proof.* Let $x \in C$, $n \in \mathbb{N}$, $\epsilon = \frac{1}{3^n}$, and $\mathrm{B}(x; \frac{1}{3^n}) = \left\{y \in C \mid d(x, y) < \frac{1}{3^n}\right\}$. Recall that $x = \sum_{n=0}^{\infty} \frac{2x_n}{3^n}$, and $y = \sum_{n=0}^{\infty} \frac{2y_n}{3^n}$, and observe the following.

$$\sum_{n=1}^{\infty} \frac{2|x_n - y_n|}{3^n} = \underbrace{\sum_{n=1}^{m} \frac{2|x_n - y_n|}{3^n}}_{Head} + \underbrace{\sum_{n=m+1}^{\infty} \frac{2|x_n - y_n|}{3^n}}_{Tail}.$$

We have already proven that the tail converges to $1/3^m$, the same value $\epsilon$ is bounded by. Furthermore, if the head sum to zero, then it does not matter what the tail converges to, as this will be less than the value we have chosen for $\epsilon = 1/3^m$. Observe that the power of $m$ for $\epsilon$ also determines the upper limit of the head. This is not incidental, as the only case for which the sum of the head is equal to zero is if the binary sequences coincide in these positions. If they do not, then the distance will be greater than $1/3^m$, and the points are not elements of the same open ball. The converse is also true. If the first $n$-coefficients of $x$ and $y$ coincide, then that means that the head will sum to zero. As the tail converges to $1/3^m$, then that means that they are elements of the same open ball. If the head does not sum to zero, then they are not elements of the same open ball.

□

### 1.2.4 Properties of the Cantor Metric Space

#### Compact

**Definition 1.39.** Given a metric space $(X, d)$, and a subset $S \subseteq X$. If there exists a real number $r > 0$, such that $d(a, b) < r$ for all $a, b \in S$, then $S$ is *bounded*. Furthermore, $(X, d)$ is a *bounded metric space*, if it is bounded as a subset of itself.

**Definition 1.40.** Given a complete metric space $(X, d)$, and a subset $S \subseteq X$. Then $S$ is *compact* if every sequence in $S$ has a converging subsequence in $S$.

*Remark* 1.41. From the the Heine–Borel theorem (Abbott, 2015) it has been shown that definition 1.40 is equivalent to the definition that $S \subseteq \mathbb{R}$ is compact if and only if $S$ is *closed* and *bounded*. (It follows from this that all compact subsets of $\mathbb{R}$ are closed and bounded.)

**Lemma 1.42.** *The Cantor metric space is compact.*

*Proof.* We showed in example 1.28 how $C$ by its very construction is closed and complete. Moreover, $C$ is also by its very construction bounded. Recall definition 1.39, and observe the following. The greatest distance between two elements in $C$ is equal to the distance between the endpoints on the closed interval $[a, b]$ it is constructed on. This result will always be less than a real number $b < |r|$. For the Cantor ternary set constructed on the unit interval $[0, 1]$, the distance between two elements will always be bounded by $d(a, b) \leqslant 1 < r$. As $C$ is closed and bounded, it follows from definition 1.40 that it is compact. □

*Remark* 1.43. Technically we can thus say that by the very construction of $C \subset [0, 1] \subset \mathbb{R}$, it is both closed and bounded, and is thus compact. □

#### Perfect

**Definition 1.44.** Given a metric space $(X, d)$, and a subset $S \subseteq X$. Then $S$ is *perfect* if it is closed and no point $a \in S$ is isolated.

*Remark* 1.45. Closed intervals $[c, d]$ with $c < d$, are the most evident classes of perfect sets.

**Lemma 1.46.** *The Cantor metric space is perfect.*

*Proof.* Let $x \in C_1$, where $C_1$ given below is one subset of the Cantor Ternary set equation (1.4).

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Observe that there exist an $x_1 \in C \cap C_1$ where $x_1 \neq x$, that still satisfies to be within the bound of $|x - x_1| \leq 1/3^1$. A look on figure 1.1 might further clarify this. As we iterate for each $n \in \mathbb{N}$ and calculate the corresponding $C_n$, there will always exist $x_n \in C \cap C_n$ where $x_n \neq x$, still satisfying to be within the bound of $|x - x_n| \leq 1/3^n$. In other words, $C$ cannot contain isolated points, and is thus perfect. □

#### Totally Disconnected

**Definition 1.47.** Given a complete metric space $(X, d)$, and two non-empty sets $S, Q \subseteq X$. Then $S, Q$ are *separated* if both $\overline{S} \cap Q$ and $S \cap \overline{Q}$ are empty.

*Remark* 1.48. The property of separated sets ensures us that the sets are non-empty and that they do not contain the limit points of each other (i.e. disjoint).

**Definition 1.49.** Given a metric space $(X, d)$. The set $R \subseteq X$ is *disconnected* if $R = S \cup Q$, where both $S$ and $Q$ are both non-empty and separated. If $R$ is not disconnected, it is *connected.* Furthermore, $R$ is *totally disconnected* if for any two distinct points $a, b \in R$, there exist separated sets where $a \in A$ and $b \in B$ such that $R = A \cup B$.

**Lemma 1.50.** *The Cantor metric space is totally disconnected.*

*Proof.* In the construction of $C$, there is always some interval left after the removal *ad infinitum.* However, this is only the case until we have the final resulting $C$, which has no intervals of any kind. In other words, if given any two distinct points $k, l \in C$ then there will always exist two separate sets where $k \in K$ and $l \in L$, where $K \cup L \subseteq C_p$, where $C_p \subseteq C_n$. □

*Remark* 1.51. What we are trying to emphasize here is that, if $k, l$ are not in the same interval, then it can easily be seen that they are totally disconnected as each $C_n$ is the union of two disjoint closed intervals. What the process of removal *ad infinitum* does, is that it insures that if $k, l$ were in the same interval in an arbitrary $C_n$, then they would still be totally disconnected because at some point the middle third in between them will be removed.

What is interesting, is that the proven properties of the Cantor metric space actually characterize the Cantor set itself. We conclude this chapter with what is the main definition of a Cantor set, and the definition we will use going forwards.

**Definition 1.52.** A Cantor set $C$ is any *non-empty, compact, perfect, totally disconnected metric space.*

### 1.2.5 Metric Spaces as Topological Spaces

Until now, we have worked with an analytical representation of the Cantor set. We are now going to transform our problem as a subspace of $\mathbb{R}$ into the realm of topological spaces. As will become apparent, we have been working with a topological space the whole time, as metric spaces are examples of topological spaces. On top of that, the open balls, which formed open sets of the Cantor metric space, are the open sets generating the topology on $C$. The resulting topological space is one representation of the Cantor space $\mathscr{C}$.

**Definition 1.53.** Given a set $X$, a *topology* $\mathcal{T}$, where $\mathcal{T}$ is a collection of subsets $S \subseteq X$ that are (by this definition) called open in $X$, satisfying the three following conditions.

- Both $X$, and $\emptyset$ must be in $\mathcal{T}$.
- For $U$ a sub-collection of $\mathcal{T}$, the union of all elements in $U$ must be in $\mathcal{T}$.

- For $U$ a *finite* sub-collection of $\mathcal{T}$, the intersection of all elements in $U$ must be in $\mathcal{T}$.

A set with a topology, noted as $(X, \mathcal{T})$, is called a *topological space.*

**Definition 1.54.** Given a set $X$. A *basis* $\mathcal{B}$ for a topology $\mathcal{T}$ on $X$ is the collection of subsets $B \subseteq X$ satisfying the two following conditions.

- For each element $x \in X$ there is a $B \in \mathcal{B}$ where $x \in B$.
- Let $u \in \mathcal{T}$. If $x \in u$ then there exists a $B \in \mathcal{B}$ where $x \in B$ and $B \subseteq u$.

**Lemma 1.55.** *Given a set $X$, and a basis $\mathcal{B}$ for the topology $\mathcal{T}$. Then collection of the union of all elements $B \in \mathcal{B}$ is the topology $\mathcal{T}$ on $X$.*

*Proof.* Observe that any element $B \in \mathcal{B}$ are an element of the topology $\mathcal{T}$. Let $U$ be a sub collection of $\mathcal{T}$, and choose for each element in $x \in U$ an element in $B \in \mathcal{B}$ such that $x \in B$. Now $U = \bigcup_{x \in U} B$, which is the union of elements of $\mathcal{B}$. □

**Example 1.56.** Given a metric space $(X, d)$. The collection of all open $\epsilon$-balls is a basis for the topology on $C$

$$\mathcal{B} = \{\mathrm{B}(x; \epsilon) \mid x \in X, \epsilon > 0\}.$$

#### Homeomorphisms

In order to compare different topological spaces, we need a notion of knowing whether we can form a structure preserving map between different topological spaces, and a notion of knowing whether topological spaces are the same, that is, having the same topological properties.

**Definition 1.57.** Given two topological spaces $X$ and $Y$. A function $f : X \to Y$ is *continuous* if for every open set $U \subset Y$ the preimage $f^{-1}(U)$ is an open set in $X$.

**Definition 1.58.** Given two topological spaces $X$ and $Y$. A map $f : X \to Y$ is a *homoemorphism* if $f$ is bijective, continuous, and with a continuous inverse. If there exists a homeomorphism $f$, we say that $X$ and $Y$ are *homeomorphic*, written as $X \cong Y$.

**Definition 1.59.** A map $f$ is an *automorphism* if $f$ is a homeomorphism of the space to itself.

It is important to emphasize here that if there exists a homeomorphism between two topological spaces, then they also "share" topological properties, in the sense that they are the same in both spaces. Recall the properties we proved for the Cantor metric space. As a metric space is a topological space, any other topological space that is homeomorphic to the Cantor set, must also have the same properties. As such, any topological space that is homeomorphic to the Cantor set is in fact what we will call a Cantor space $\mathscr{C}$.

## 1.3 The Cantor Spaces $\mathscr{C}$

Our description of the Cantor space is not especially elegant. Even though we have formed a topological space, the space has a analytic feel to it, as every element in the Cantor set is represented by the infinite geometric series. What is elegant about the Cantor set, is that it has a very combinitorial feel to it, and we can form a space that is simpler and more convenient for us going forward. Recall the infinite sequences of 0's and 1's. They were essential in order to guide us to an element of $C$. By considering the set $X = \{0, 1\}^{\mathbb{N}}$ of all infinite sequences of 0 and 1, we will show that the topology arising from the metric topology coincides with the product topology on $X$ where $\{0, 1\}$ is considered as a discrete space. This is the most obvious explicit example of the general case in which we consider the product topology on the set $X = \{0, \dots, p-1\}^{\mathbb{N}}$ where $p \geq 2$ a natural number. We will now go on to establish this as our new representation of the Cantor space $\mathscr{C}$.

### 1.3.1 Discrete Topology on Infinite Sets

**Definition 1.60.** Given a set $X$, and the collection $\mathcal{T}$ of subsets of $X$. If $\mathcal{T}$ contains all subsets of $X$, then $\mathcal{T}$ is the largest topology on $X$ and is referred to as the *discrete topology*. Any set equipped with a discrete topology is a *discrete topological space*. If $\mathcal{T}$ only contains the subsets $\emptyset, X \subseteq X$, then $\mathcal{T}$ is the smallest topology on $X$ and is referred to as the *indiscrete topology*. Any set equipped with a indiscrete topology is a *indiscrete topological space*.

*Remark* 1.61. Every point $x \in X$ is an open set in the discrete topology. Thus a discrete topological space is a space of separated isolated points.

*Remark* 1.62. If $X$ is a space with the discrete topology, then every map from $X$ to any other topological space is continuous.

The discrete topological space we are going do define is what we will call an alphabet going forward. Additionally, a sequence in the alphabet will be referred to as a word.

**Definition 1.63.** Given $p \in \mathbb{N}$, let the set $A = \{0, 1, 2, \dots, p-1\}$ be an *alphabet* with $n$-letters.
Let the set $A^{\mathbb{N}} = \{$All the words of infinite length of the alphabet $A\}$.
Given $m \in \mathbb{N}$, let the set $A^m = \{$All the words of finite length $m \in \mathbb{N}$ of the alphabet $A\}$.
Let the set $A^* = \bigcup_{m=1}^{\infty} A^m = \{$The union of all finite words of length $m \in \mathbb{N}$ of the alphabet $A\}$.

Furthermore, we define concatenation of two alphabets in the following way.

**Definition 1.64.** Let $x \in A^n$ and $y \in A^m$ be two finite words of length $n, m \in \mathbb{N}$. The concatenation of $x$ and $y$, denoted $x \circ y$ or for brevity as $xy$, is a word of length $n + m$.

$$xy = x \circ y \in A^{n+m}.$$

*Remark* 1.65. It is important to emphasize that in general $xy \neq yx$. The ordering is imperative.

**Example 1.66.** Given two finite words $x \in A^n$ and $y \in A^n$ such that $x = x_0x_1x_2 \dots x_n$ and $y = y_0y_1y_2 \dots y_m$. The concatenation of $x, y$ is in this case is either

$$xy = x_0 \dots x_n y_0 \dots y_m,$$
$$yx = y_0 \dots y_n x_0 \dots x_m.$$

If given an infinite word $z \in A^{\mathbb{N}}$ such that $z = z_0 z_1 \ldots z_i \ldots$, then the concatenation of $x, z$ is either

$$xz = x_0 \ldots x_n z_0 \ldots z_i \ldots,$$
$$zx = z_0 \ldots z_i \ldots \ldots \text{ at the "end" of an infinite long word } \ldots x_0 \ldots x_n.$$

**Definition 1.67.** If $x \in A^n$ and $y \in A^m$ are two finite words of length $n, m \in \mathbb{N}$ and $m > n$, we say that $x$ is a *prefix* of $y$, if there exists $z \in A^{m-n}$ such that $y = xz$.

*Remark* 1.68. If $x \in A^n$ a finite word of length $n$, and $y \in A^{\mathbb{N}}$ is an infinite word, then the concatenation is an infinite word $x \circ y \in A^{\mathbb{N}}$.

### 1.3.2 Product Topology on Infinite Sets

**Definition 1.69.** Given the following Cartesian product of topological spaces

$$X = \prod X_i.$$

The *product topology* on the space $X$ is generated by the following basis

$$\mathcal{B} = \left\{ \prod U_i \mid U_i \subseteq X_i \text{ is open, and } U_i = X_i \text{ for a finite number of indices } i \right\}.$$

*Remark* 1.70. The latter requirement on $U_i = X_i$ for a finite number of indices is critical when working with infinite Cartesian products. In such a case, it is not true that open sets are unions of open sets. We therefore let a finite number of sets be open, and let the rest be the whole space.

Let $A = \{0, \ldots, p-1\}$ be a topological space with the discrete topology, so every $x \in A$ is an open subset. Then in $A^{\mathbb{N}}$, the basis opens are what we will define as the cylinder sets. If given a finite word $\omega$, where $\omega \in A^*$ we define the cylinder set $\mathcal{Z}(\omega)$ as the following subset of $A^{\mathbb{N}}$.

$$\begin{aligned} \mathcal{Z}(\omega) &= \left\{ \omega \in A^*, x \in A^{\mathbb{N}} \mid \omega x \in A^{\mathbb{N}} \right\}, \\ &= \{\text{all infinite words with prefix } \omega\}. \end{aligned} \tag{1.12}$$

Observe that for $\mathcal{Z}(\omega)$ we let a finite number of letters be open, and let the rest be the whole alphabet, and that this coincides with the definition of the basis for the product topology. The product topology generated by the cylinder sets is the following

$$\prod^{\mathbb{N}} \{0, \ldots, p-1\} = A^{\mathbb{N}} = \{0, \ldots, p-1\}^{\mathbb{N}} \quad \text{where } p \geq 2.$$

*Remark* 1.71. The reason $p \geq 2$ a natural number, is to avoid the uninteresting case of words with with only "0"'s.

#### Union and Intersection of Cylinder Sets

**Lemma 1.72.** *Given two distinct cylinder sets $\mathcal{Z}(\omega)$ and $\mathcal{Z}(\eta)$, with $\omega$ and $\eta$ being two arbitrary finite words not necessarily of the same length $n$, and where $y$ the leftover binary sequence if one word is* *contained in the other. The* union *of* $\mathcal{Z}(\omega)$ *and* $\mathcal{Z}(\eta)$ *is one of the following four possibilities.*

$$\mathcal{Z}(\omega) \cup \mathcal{Z}(\eta) = \begin{cases} \mathcal{Z}(\omega) & \text{if } \eta = \omega y, \\ \mathcal{Z}(\eta) & \text{if } \omega = \eta y, \\ \mathcal{Z}(\omega_0 \dots \omega_{n-1}) & \text{iff } \eta_n \neq \omega_n \text{ where both are of same length } n, \\ \mathcal{Z}(\omega) \cup \mathcal{Z}(\eta) & \text{otherwise.} \end{cases}$$

*Proof.* I: Let $\omega = \omega_0 \dots \omega_n$ and $\eta = \eta_0 \dots \eta_m$ be two finite words of length $n$ and $m$, and let $n \neq m$. If $\omega_i \neq \eta_i$ for all indices $0 \leq i \leq \min\{n, m\}$, then the cylinder sets have no coinciding words, which is to say that either of the cylinder sets are contained in the other, and thus the union cannot be shortened.

II: If this is not the case observe the following. If $\omega$ is a prefix for $\eta$, then we have that $\eta = \omega y$, where $y$ is the leftover sequence. Now $\mathcal{Z}(\eta) \subseteq \mathcal{Z}(\omega)$, and as such $\mathcal{Z}(\omega) = \mathcal{Z}(\omega) \cup \mathcal{Z}(\eta)$. The converse, if $\eta$ is a prefix of $\omega$ yields the same result.

III: The last case we need to check is if $n = m$, and where the first $n-1$ positions coincide, and then at $n$ is different. Then we have the case that $\omega = \eta_0 \dots \eta n - 1\omega_n$ and that $\eta = \omega_0 \dots \omega n - 1\eta_n$. Then the union will be the following cylinder set $\mathcal{Z}(\eta_0 \dots \eta n - 1)$. □

**Lemma 1.73.** *Given two arbitrary cylinder sets* $\mathcal{Z}(\omega)$ *and* $\mathcal{Z}(\eta)$*, with* $\omega$ *and* $\eta$ *being two finite words not necessarily of the same length n, and where y the leftover binary sequence if one word is contained in the other. The* intersection *i of* $\mathcal{Z}(\omega)$ *and* $\mathcal{Z}(\eta)$ *is one of the following three possibilities.*

$$i = \mathcal{Z}(\omega) \cap \mathcal{Z}(\eta) = \begin{cases} \mathcal{Z}(\eta) & \text{if } \omega \subseteq \eta \quad \text{i.e iff } \eta = \omega y, \\ \mathcal{Z}(\omega) & \text{if } \eta \subseteq \omega \quad \text{i.e iff } \omega = \eta y, \\ \emptyset. \end{cases}$$

*Proof.* I: Let $\omega = \omega_0 \dots \omega_n$ and $\eta = \eta_0 \dots \eta_m$ be two finite words of length $n$ and $m$, and let $n \neq m$. If $\omega_i \neq \eta_i$ for all indices $0 \leq i \leq \min\{n, m\}$, then the cylinder sets have no coinciding words which is to say that either of the cylinder sets is contained in the other, so the intersection must be empty.

II: The intersection is not empty only if either $\omega$ is a prefix for $\eta$ such that $\eta = \omega y$, or $\eta$ is a prefix for $\omega$ such that $\omega = \eta y$, where $y$ is the leftover binary sequence. This is equivalent to the cylinder set $\mathcal{Z}(\eta) \subseteq \mathcal{Z}(\omega)$ or $\mathcal{Z}(\omega) \subseteq \mathcal{Z}(\eta)$, respectively. As such $\mathcal{Z}(\omega) \cap \mathcal{Z}(\eta)$ is either $\mathcal{Z}(\omega)$ or $\mathcal{Z}(\eta)$, respectively. □

**Lemma 1.74.** *Cylinder sets are clopen.*

*Proof.* Recall that the complement of an open set is a closed set (lemma 1.34). Observe that if a finite word $\omega \in \{0, \dots, p-1\}^{\mathbb{N}}$, then the complement is a union of cylinder sets. As such the cylinder sets must also be closed. □

### Homeomorphic spaces

While not entirely obvious, our space $\{0, \dots, p-1\}^{\mathbb{N}}$ is in fact an equally valid representation of a Cantor space $\mathscr{C}$, as $C$ with the topology generated by the open $\epsilon$-balls, i.e. $\{0, \dots, p-1\}^{\mathbb{N}} \cong \mathscr{C}$.

**Lemma 1.75.** *The map* $j\colon \{0, \dots, p-1\}^{\mathbb{N}} \to \mathscr{C}$ *is an homeomorphism of topological spaces, where* $j$ *maps a cylinder set to its equivalent* $\epsilon$*-ball.*

While we will not prove that $j$ is a homeomorphism in the general case, we will instead take on a specific case to explicitly formulate it in the next section.

### 1.3.3 The Discrete 2-Point Space $\{0,1\}^{\mathbb{N}}$

Let us study the particular case of $p = 2$, that is an alphabet $A$ only consisting of two letters, and explicitly construct the homeomorphism $j$ between the two representations of the Cantor space.

$$\prod^{\mathbb{N}} \{0,1\} = \{0,1\}^{\mathbb{N}} .$$

This is the most interesting characterization for us, as the homeomorphism comes almost naturally when considering discrete spaces of 2 points and the Cantor ternary set. This from the fact that we have already shown that each element in the Cantor set has a unique infinite binary sequences to represent it. Or rather, the geometric sum on the infinite binary sequence to represent it. Let us first do some calculative examples of both the union and intersection of cylinder sets in the discrete 2-point space.

**Example 1.76.** A selection of calculative examples of finding the union in a special case for $n = 2$ of the alphabet $A$. Given two cylinder sets $\mathcal{Z}(\omega)$ and $\mathcal{Z}(\eta)$ the following examples of unions hold. A look on figure 1.6 might help the reader to verify the examples below.

$$\begin{aligned}
\mathcal{Z}(00) \cup \mathcal{Z}(01) &= \mathcal{Z}(0), \\
\mathcal{Z}(00) \cup \mathcal{Z}(0010) &= \mathcal{Z}(00), \\
\mathcal{Z}(10) \cup \mathcal{Z}(110) &= \mathcal{Z}(10) \cup \mathcal{Z}(110), \\
\mathcal{Z}(01) \cup \mathcal{Z}(10) &= \mathcal{Z}(01) \cup \mathcal{Z}(10).
\end{aligned}$$

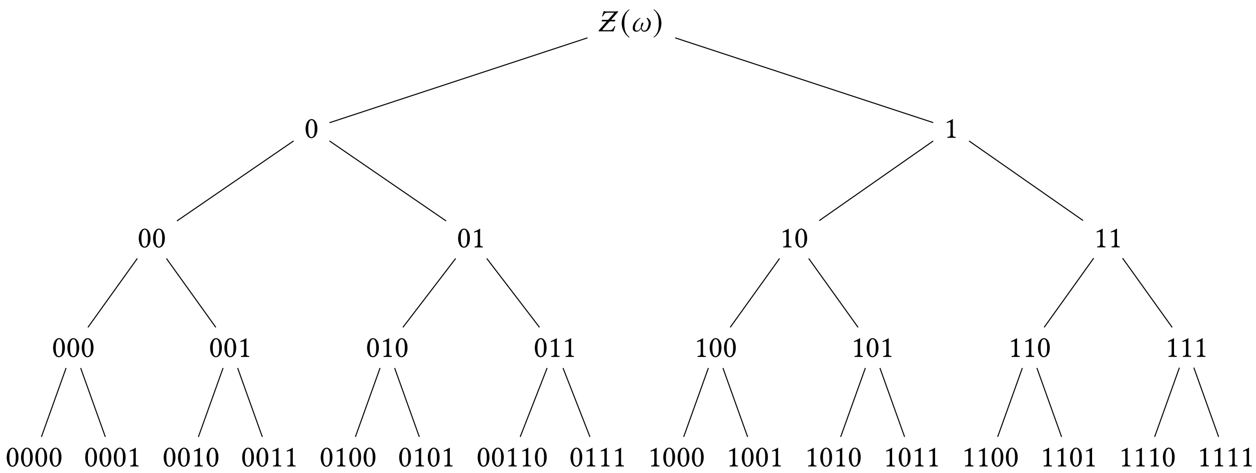



**Figure 1.6:** A binary tree of all cylinder sets of words $\omega$ up to length $4$.

**Example 1.77.** A selection of calculative examples of finding the intersection in a special case for $p = 2$ of the alphabet $A$. Given two cylinder sets $\mathcal{Z}(\omega)$ and $\mathcal{Z}(\eta)$ the following examples of intersection hold. A look on figure 1.6 might help the reader to verify the examples below.

$$\begin{aligned}
\mathcal{Z}(00) \cap \mathcal{Z}(01) &= \emptyset, \\
\mathcal{Z}(00) \cap \mathcal{Z}(0010) &= \mathcal{Z}(0010), \\
\mathcal{Z}(10) \cap \mathcal{Z}(110) &= \emptyset, \\
\mathcal{Z}(01) \cap \mathcal{Z}(10) &= \emptyset.
\end{aligned}$$

**Example 1.78.** Given a finite word $\omega = 01$, then the complement is the following union of cylinder sets

$$\overline{Z(01)} = Z(00) \cup Z(1).$$

A look on figure 1.6 might further verify this.

### Homeomorphic spaces

**Lemma 1.79.** *Given the following map $j\colon \{0,1\}^{\mathbb{N}} \to \mathscr{C}$. Then $j$ is an homeomorphism of topological spaces. Furthermore, $j(\eta)\colon Z(\eta_0 \dots \eta_m) \to \mathrm{B}(\sum_{i=0}^{\infty} \frac{2\eta_i}{3^i}, 1/3^m)$.*

*Proof.* The map $j$ is clearly bijective. Additionally, if given a cylinder set $Z(\omega) \subseteq \{0,1\}^{\mathbb{N}}$ of an infinite word with the finite prefix of length $m \in \mathbb{N}$, noted $\omega = \omega_0 \dots \omega_m$, observe the following

$$j\left(Z(\omega)\right) = \left\{ \sum_{n=0}^{\infty} \frac{2x_n}{3^n} : x_i = \omega_i,\ \forall\, 0 \le i \le m \right\} = \mathrm{B}\left( \sum_{i=0}^{\infty} \frac{2\omega_i}{3^i}; \frac{1}{3^m} \right).$$

Now, given an element $x \in \widehat{\mathbb{Z}}_p$ and $m > 0$, we have that

$$j^{-1}\left( \mathrm{B}\left( \sum_{i=0}^{\infty} \frac{2x_i}{3^i}; \frac{1}{3^m} \right) \right) = Z\left(\omega_0 \dots \omega_m\right).$$

Then the preimage $j^{-1}$ of any open ball $\mathrm{B} \subseteq \widehat{\mathbb{Z}}_p$ is a clopen cylinder set $Z(x) \subseteq \{0,1\}^{\mathbb{N}}$, thus $h$ is continuous. The converse, the preimage of the inverse $\left(j^{-1}\right)^{-1} = j$ of any cylinder set $Z(x)$ is an open ball B, thus $j^{-1}$ is continuous. As $j$ is bijective, continuous, and $j^{-1}$ is continuous, it follows then from definition 1.58 that $j$ is a homeomorphism, and that $\{0,1\}^{\mathbb{N}} \cong \mathscr{C}$. □

Let us end this chapter with the following example on the similarities between the open sets of the Cantor metric space and open sets of the discrete product topology.

**Example 1.80.** Given a binary sequence $x \in \{0,1\}^{\mathbb{N}}$, and the following cylinder set and open balls defined around the element.

$$\mathrm{B}\left(j(x); 1/3^n\right) \cong Z(x),$$
$$\{y \in C \mid d(x,y) < 1/3^n\} \cong \left\{x \in A^*, y \in A^{\mathbb{N}} \mid xy \in A^{\mathbb{N}}\right\}.$$

Let the prefix be given as $x = 01110$. The following example illustrates the correspondence between open balls and cylinder sets.

$$\mathrm{B}(j(x); 1/3^3) \cong Z(011),$$
$$\mathrm{B}(j(x); 1/3^5) \cong Z(01110).$$

and also that

$$\mathrm{B}(j(x); 1/3^5) \subseteq \mathrm{B}(j(x); 1/3^3),$$
$$Z(01110) \subseteq Z(011).$$

# 2 Automorphisms on the Cantor Space

We want to construct structure preserving maps on the Cantor space. More specifically, we want to construct a set of automorphisms that we name ”odometers” or ”adding machines”. The reason for the naming will become apparent later on, and we focus first on the following. The name ”adding machine”, emphasizes a problem apparent with the Cantor space $\{0, \ldots, p-1\}^{\mathbb{N}}$ for $p \geq 2$ we are working in. As the name obviously suggests, the odometer performs the binary operation of addition of two elements. However, we have the inconvenience of having no notion of addition in our current topological space. Therefore, we must create a homeomorphism over to a more convenient topological space where the binary operation of addition is defined, that is, the topological space of $p$-adic integers $\widehat{\mathbb{Z}}_p$.

## 2.1 Topological Groups

**Definition 2.1.** Given a set $G$. If $G$ is closed under a binary operation $*$ and satisfies the following conditions, then $G$ forms a *group* that we note $\langle G, * \rangle$.

- Associativity: $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$.
- Identity element: There is an element $e \in G$ such that $e * x = x * e$ for all $x \in G$.
- Inverse: There is for each $x \in G$ an inverse $a' \in G$ such that $a * a' = a' * a = e$.

**Definition 2.2.** Let $G$ be a set on which a group structure and a topology are given such that the group operations of product and inverses are continuous functions. If so, $G$ is called a *topological group.*

**Definition 2.3.** Let $p \in \mathbb{N}$ be a prime number. We define

$$\begin{aligned} \mathbb{N}[p] &= \left\{x_0 p^0 \cdots + x_i p^i \mid i \in \mathbb{N} \text{ and } x_i \in \{0, \ldots, p-1\}\right\}, \\ &= \left\{\sum_{i=0}^{m} x_i P^i \mid x_i \in \{0, \ldots, p-1\}\right\}. \end{aligned}$$

Going forwards, we define a metric $d_p$ on $\mathbb{N}[p]$ such that

$$d_p(x, y) = d_p(a_0 + \cdots + a_n p^n, b_0 + \cdots + b_m p^m) = \frac{|a_k - b_k|}{p^k},$$

where $k$ is the minimal number such that $a_k \neq b_k$. The completion of $\mathbb{N}[p]$ with the metric $d_p$ is the topological group of $p$-adic integers noted as

$$\widehat{\mathbb{Z}}_p = \left\{\sum_{i=0}^{\infty} x_i P^i \mid x_i \in \{0, \ldots, p-1\}\right\}.$$

**Lemma 2.4.** *Given the following map $h\colon \{0,\dots,p-1\}^{\mathbb{N}} \to \widehat{\mathbb{Z}}_p$. Then $h$ is an homeomorphism of topological spaces. Furthermore, a prefix $a = a_0 \dots a_m$ of the space $\{0,\dots,p-1\}^{\mathbb{N}}$ with length $m$, then $h(a) = \sum_{i=0}^{\infty} a_i p^i$.*

*Proof.* The map $h$ is clearly bijective. Additionally, if given a cylinder set $\mathcal{Z}(\omega) \subseteq \{0,\dots,p-1\}^{\mathbb{N}}$ of an infinite word with the finite prefix of length $m \in \mathbb{N}$, noted $\omega = \omega_0 \dots \omega_m$, observe the following

$$h\left(\mathcal{Z}(\omega)\right) = \left\{ \sum_{n=0}^{\infty} x_n P^n : x_i = \omega_i,\ \forall\, 0 \le i \le m \right\} = \mathrm{B}\left( \sum_{i=0}^{\infty} \omega_i p^i; \frac{1}{p^m} \right).$$

Now, given an element $x \in \widehat{\mathbb{Z}}_p$ and $m > 0$, we have that

$$h^{-1}\left( \mathrm{B}\left( \sum_{i=0}^{\infty} x_i P^i; \frac{1}{p^m} \right) \right) = \mathcal{Z}\left(\omega_0 \dots \omega_m\right).$$

Then the preimage $h^{-1}$ of any open ball $\mathrm{B} \subseteq \widehat{\mathbb{Z}}_p$ is a clopen cylinder set $\mathcal{Z}(x) \subseteq \{0,\dots,p-1\}^{\mathbb{N}}$, thus $h$ is continuous. The converse, the preimage of the inverse $\left(h^{-1}\right)^{-1}$ of any cylinder set $\mathcal{Z}(x)$ is an open ball $\mathrm{B}$, thus $h^{-1}$ is continuous. As $h$ is bijective, continuous, and $h^{-1}$ is continuous, it follows then from definition 1.58 that $h$ is a homeomorphism, and that $\{0,\dots,p-1\}^{\mathbb{N}} \cong \widehat{\mathbb{Z}}_p$. □

## 2.2 The Generalized Odometer $\varphi$

Given $p \in \mathbb{N}$ a prime and $q \in \mathbb{N}$, we want to define a map $\varphi_q\colon \{0,\dots,p-1\}^{\mathbb{N}} \to \{0,\dots,p-1\}^{\mathbb{N}}$, and show that it is an automorphism.

**Definition 2.5.** Let "0" be the symbol that represents the *identity element* $e \in \widehat{\mathbb{Z}}_p$ under the binary operation of addition.

**Definition 2.6.** Given $a, b, 0, 1 \in \widehat{\mathbb{Z}}_p$, the binary operation of addition of $a + b = c$ is defined in the following recursive way for each position $n \in \mathbb{N}$ of $a$ and $b$.

$$\begin{aligned}
c_0 &= (x+y)_0 = x_0 + y_0 + \varepsilon_0 \mod p \ \text{ where } \varepsilon_0 = 0,\\
&\vdots\\
c_n &= (x+y)_n = x_n + y_n + \varepsilon_n \mod p \ \text{ where } \varepsilon_n = \begin{cases} 0 & \text{if } x_{n-1} + y_{n-1} + \varepsilon_{n-1} < p,\\ 1 & \text{if } x_{n-1} + y_{n-1} + \varepsilon_{n-1} = p. \end{cases}
\end{aligned}$$

**Definition 2.7.** Given an element $x \in \widehat{\mathbb{Z}}_p$, the *inverse* of $x$ is the element $-x$, where $-x \in \widehat{\mathbb{Z}}_p$ such that

$$x + (-x) = 0.$$

**Definition 2.8.** Given $q \in \mathbb{N}$ and a prime $p \in \mathbb{N}$, we define the binary operation of addition to be the following map $\phi_q\colon \widehat{\mathbb{Z}}_p \to \widehat{\mathbb{Z}}_p$ such that $\phi_q(x) = x + q$, and $\phi_q^{-1}(x) = x - q$, for all elements $x \in \widehat{\mathbb{Z}}_p$.

*Remark* 2.9. To note several consecutive additions of $\phi$, we let $n \in \mathbb{N}$ represent the number of times $\phi$ is composed with itself, and note this as

$$\phi_q^n = \underbrace{\phi_q \circ \ldots \circ \phi_q}_{n\text{-times}} = \phi_{np} \text{ where } np = \underbrace{p + \cdots + p}_{n\text{-times}}.$$

**Lemma 2.10.** *The binary operation of addition $\phi$ is an automorphism of $\widehat{\mathbb{Z}}_p$.*

*Proof.* It can clearly be seen that $\phi$ is bijective. Recall from definition 2.2 of a topological group, that the binary operation and the function mapping inverses are continuous functions. As such, $\phi$ is an automorphism of $\widehat{\mathbb{Z}}_p$. □

**Definition 2.11.** Given $q \in \mathbb{N}$ and a prime $p \in \mathbb{N}$. We define the *odometer* as the map $\varphi_q \colon \{0, \ldots, p-1\}^{\mathbb{N}} \to \{0, \ldots, p-1\}^{\mathbb{N}}$, where, for all elements $x \in \{0, \ldots, p-1\}^{\mathbb{N}}$, $\varphi$ is the following composition,

$$\varphi_q(x) = h^{-1}(\phi_q(h(x))).$$

Here $h : \{0, \ldots, p-1\}^{\mathbb{N}} \to \widehat{\mathbb{Z}}_p$, and $\phi_q \colon \widehat{\mathbb{Z}}_p \to \widehat{\mathbb{Z}}_p$ is the $q$-addition, both defined before.

**Lemma 2.12.** *Given three topological spaces $X, Y, Z$ and two functions $f \colon X \to Y$ and $g \colon Y \to Z$. If both $f$ and $g$ are continuous maps, then the* composition *$g \circ f \colon X \to Z$ is continuous.*

*Proof.* Let the subset $U \subseteq Z$ be open in $Z$. Now $(g \circ f)^{-1}(U) = \{\forall x \in X \mid g(f(x)) \in U\} = \left\{\forall x \in X \mid x \in f^{-1}(g^{-1}(U))\right\} = f^{-1}(g^{-1}(U))$. We know from continuity of $g$ and $f$ that $g^{-1}(U)$ is open in $Y$, and $f^{-1}(g^{-1}(U))$ is open in $X$, and thus $(g \circ f)$ is continuous. □

*Remark* 2.13. The implication of the proof is significant. If both $f$ and $g$ were to be homeomorphisms, then the composition $(g \circ f)$ is also a homeomorphism. This follows from the proof that $(g \circ f)$ is continuous, that the composition of two bijections is also a bijection, and that it can easily be proven that $(g \circ f)^{-1}$ is a composition of continuous maps, and is thus continuous.

**Lemma 2.14.** *The odometer $\varphi_q$ is an automorphism of $\{0, \ldots, p-1\}^{\mathbb{N}}$.*

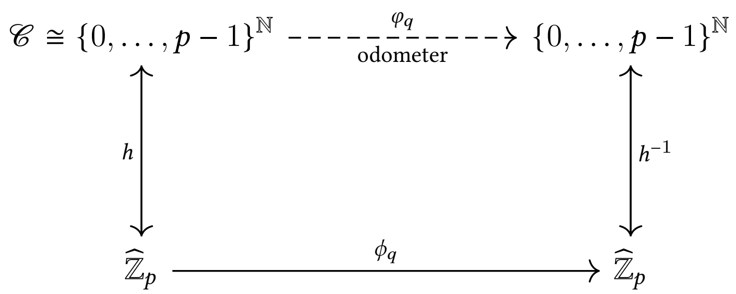



**Figure 2.1:** alt 1: A diagram of the generalized odometer $\varphi$. alt2: A diagram of the odometer $\varphi$

*Proof.* Given the odometer $\varphi = h^{-1}\phi h$. We have already proven that $h$ is a homeomorphism in lemma 2.4, and we know that the binary operation is an automorphism from definition 2.8. As $\varphi$ is a composition of homeomorphisms, it must also be a homeomorphism as already proven in lemma 2.12. □

## 2.3 The Adding Machine

We are now to demonstrate the specific case for $p = 2$ and $q = 1$, for which we will show the elegant form the odometer takes on, and give a few calculative examples.

**Definition 2.15.** Given $q = 1$, and $p = 2$. Now, the the binary operation of addition is the following map $\phi_1 : \widehat{\mathbb{Z}}_2 \to \widehat{\mathbb{Z}}_2$ such that $\phi_1(x) = x + 1$ and $\phi_1^{-1}(x) = x - 1$ for all elements $x \in \widehat{\mathbb{Z}}_2$.

*Remark* 2.16. The elements $1, -1 \in \widehat{\mathbb{Z}}_2$ are for $k \in \mathbb{N}$ the following

$$1 = 1 + 0 \cdot 2^1 + 0 \cdot 2^2 + \cdots = 1 + \sum_{n=1}^{\infty} 0 \cdot p^n \text{ and } -1 = 1 + 1 \cdot 2^1 + 1 \cdot 2^2 + \cdots = \sum_{n=0}^{\infty} 1 \cdot p^n.$$

*Remark* 2.17. Since we are adding and subtracting *one*, we can by composition of $\phi^n(x)$ add any number we want. The same is not not necessary true if $q \neq 1$.

**Example 2.18.** Let us do an example of addition in the 2-adic numbers, here showing that $-x$ is in fact the inverse of an arbitrary element $x \in \widehat{\mathbb{Z}}_2$. For ease of understanding, the corresponding sequence in $\{0,1\}^{\mathbb{N}}$ will be written on the right side of the equality sign.

$$\begin{aligned} x &= 1 + 2^1 + 2^2 + 2^3 + \cdots = (111111111111\ldots), \\ -x &= 1 + 0 + 0 + 0 + 0 + \cdots = (100000000000\ldots), \\ x + (-x) &= 0 + 0 + 0 + 0 + 0 + \cdots = (000000000000\ldots). \end{aligned}$$

As $0 \in \widehat{\mathbb{Z}}_2$ is in fact $0 = 0 + 0 \cdot 2^1 + 0 \cdot 2^2 + \ldots$ the addition performs as expected

**Example 2.19.** In this example we will do an illustrative example of the adding machine when we perform several consecutive additions. In figure 2.2 this can be seen for $\varphi^2$ and $\phi_q^2$.

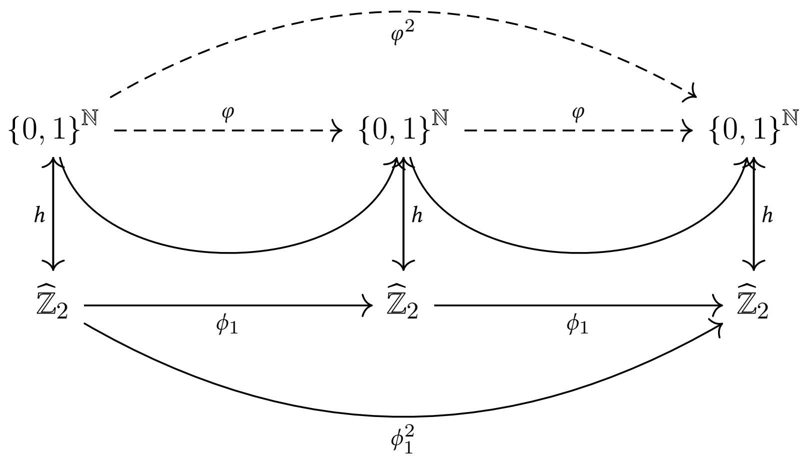



**Figure 2.2:** A diagram of several consecutive additions of $\varphi$ and $\phi$

This is however, a cumbersome way of applying the odometer a *single* time. As we are working with binary sequences, then doing addition should is as easy as 0+0 = 0, 0+1 = 1, and 1+1 = 0 with a "1" that carry over to the next addition in the sequence. Observe, that if we were to "carry out" the addition following the aforementioned rules for the binary sequences of $x$ and $-x$ in example 2.18, it would result in the exact same answer. We can with this clever observation redefine our odometer to the following uncomplicated, recursive formula. Compared to our previous definition, this is more elegant, as in addition to being much simpler, it clearly conveys the notion of self replication.

**Definition 2.20.** If $p = 2$ and $q = 1$, and given $\mathcal{Z}(\omega) \subseteq \{0,1\}^{\mathbb{N}}$ then we define $\varphi$ in the following recursive way on the prefix $\omega = \omega_i\omega' = \omega_0 \dots \omega_m$, where $\omega_i$ is an arbitrary element in the sequence, and $\omega'$ is the following positions after $\omega_i$.

$$\varphi(\omega_i\omega') = \begin{cases} 1\omega' & \text{if } \omega_{i-1} = 0, \\ 0\varphi(\omega') & \text{if } \omega_{i-1} = 1 \text{ for } 0 \leq i \leq n. \end{cases}$$

**Lemma 2.21.** *The recursive formula in definition 2.20 is equivalent to the original definition 2.11 for the odometer $\varphi$.*

*Proof.* Let $p = 2$ and $q = 1$, and $\mathcal{Z}(x) \subseteq \{0,1\}^{\mathbb{N}}$ is an arbitrary cylinder set with the following prefix of length $m$

$$\omega = \omega_0 \dots \omega_m = \omega_0\omega_1\omega'.$$

Let us first apply the original definition which would result in the following addition

$$\begin{aligned} \omega &= \omega_0 + \omega_1 \cdot 2^1 + \omega_2 \cdot 2^2 + \dots + \omega_m \cdot 2^m, \\ 1 &= 1 + 0 \cdot 2^1 + 0 \cdot 2^2 + \dots + 0 \cdot 2^m. \end{aligned}$$

Now, from definition 2.15 one can see that the addition of $\omega_0 + 1$ impacts the result of $\omega_1 + 0 + \epsilon$, from the "recursive formula" for $\epsilon$ for each position. Observe I: that if $\omega_0 = 1$ then the addition is equal to $0 = 2 \mod 2$, and we would have a carry over value of $1 = 2^1$ in the next addition of $\omega_1 + 0 + 2^1$. II: If the addition is equal to $1 \mod 2$ then we would be done, as $1 \in \widehat{\mathbb{Z}}_2$ only has a value other than $0$ (not $0 \in \widehat{\mathbb{Z}}_2$ !) in its first position.

Now, let us apply the $\varphi$ in the recursive way. Observe that $\varphi(\omega_0\omega_1\omega') = 0\omega_1\omega'$ if $\omega_0 = 1$, and the recursive formula would apply one more time resulting in $\varphi(\omega_1\omega')$. Since applying the odometer again is the same as adding $1 = 2^1$ as we observed in I they must be equivalent. If $\varphi(\omega_0\omega_1\omega') = 1\omega_1\omega'$, then $\omega_0 = 0$, and we would be done for the exact same reason as II. In other words, the recursive formula is equivalent to the original definition. □

**Example 2.22.** Given $\omega$ be a finite word, and two infinite sequences $x, y \in \{0,1\}^{\mathbb{N}}$ such that $x = x_0x' = x_0x_1x''$ and $y = y_0y' = y_0x_1y''$, where the $x', y'$ and $x'', y''$ represent the leftover binary sequence.

We have the following general case for $\varphi^{-1}$, before we let $\omega = 01$. Recall that $\varphi^{-1}$ performs the addition of $-1 \in \widehat{\mathbb{Z}}_2$ which corresponds to the sequence $-1 = 1000\dots$ in $\{0,1\}^{\mathbb{N}}$.

$$\varphi^{-1}(\mathcal{Z}(\omega)) = \left\{x \in \{0,1\}^{\mathbb{N}} \mid \varphi(x) = \omega y\right\}.$$

Now, $\varphi^{-1}$ applied to $x = 1x'$ yields $\varphi(1x') = 0\varphi(x') = 0y'$. Furthermore, $\varphi^{-1}$ applied to $x = 10x''$ yields $\varphi(10x'') = 0\varphi(0x'') = 01\varphi(x'') = 01y''$. And we have that

$$\begin{aligned} \varphi(\mathcal{Z}(01)) &= \left\{x \in \{0,1\}^{\mathbb{N}} \mid \varphi(x) = 01y\right\}, \\ \varphi(\mathcal{Z}(01)) &= \mathcal{Z}(10). \end{aligned}$$

# 3 Cantor minimal systems $(\mathscr{C}, T)$

## 3.1 Minimality

The interesting property of the Odometer $\varphi$ is that we can iterate it for any $n \in \mathbb{Z}$. Owing to the fact that it is an automorphism on $\mathscr{C}$, the iterative property can be viewed as the spatial evolution of a point $x \in \mathscr{C}$ when we apply $\varphi$. The field of dynamical systems is in most cases concerned with such self-maps of spaces. The space can be a model for one configuration of the system of interest, and the self-map (in our case, an automorphism) is the evolution of the system in space or in time. It is here, convenient to think of time as a discrete thing rather than a continuous one.

*Remark* 3.1. In our case where $\varphi$ is an automorphism, we have a continuous inverse. A consequence of this is that our system is *reversible*, which is to say, without giving a formal definition, that we can travel backwards in "time".

We will restrict our attention to *minimal systems of the Cantor space.* That is to say, systems of a chaotic nature, where the automorphisms $\varphi\colon \mathscr{C} \to \mathscr{C}$ are minimal. While we will provide a formal definition later, the intuition behind minimality is that it guarantees that the system of interest has a certain amount of complexity. Minimality in the general sense, is a restriction on the self-map, such that no point $x \in \mathscr{C}$ is mapped onto itself by repeated iterations, and that $x$ can be everywhere in the space. However, and this is of utmost importance to emphasise, the intuitive definition of minimality in the sense that $x$ can be everywhere, can never happen for us. Take note that we are applying the odometer $\varphi$ a countable amount of times, as the power is indexed by $n \in \mathbb{Z}$. If it were true that $x$ could be everywhere, then it would suggest that $\mathscr{C}$ is countable. This cardinality problem we have proven can never happen (lemma 1.11). In spite of that, we have defined our topological space in such a way that its not necessary to be everywhere. This way of localising to a cylinder with a finite length prefix is the notion that we are going to use when we define minimality for $\mathscr{C}$. For more details on this section, the reader is referred to (Putnam, 2018).

*Remark* 3.2. It is also worth emphasizing, as it is as a consequence of the intuition on that a point can be everywhere in the space by applying $\varphi$ repeatedly, that it is impossible to study the system by dividing it into smaller parts. In our case, this would be that $x$ is moved as close as we want to everywhere, and therefore, a minimal system of the Cantor space is also irreducible.

**Definition 3.3.** A Cantor system is $(\mathscr{C}, T)$, where $\mathscr{C}$ is the Cantor space, with an automorphism $T : \mathscr{C} \to \mathscr{C}$.

**Definition 3.4.** Given a Cantor system $(\mathscr{C}, T)$. The *orbit* of an element $x \in \mathscr{C}$ is the following subset of $\mathscr{C}$,

$$\{T^n(x) \mid n \in \mathbb{Z}\}.$$

*Remark* 3.5. Take note of the wording. A nice analogy is that, as $T$ moves the point $x$ around in the space, one can view the resulting subset as the "final" orbit (i.e movement) of a planet $x$ in a solar system.

**Definition 3.6.** Given a Cantor system $(\mathscr{C}, T)$. A point $x \in \mathscr{C}$ is a *cyclic point*, if there exist an integer $n \neq 0$ such that $T^n(x) = x$.

**Definition 3.7.** Given a Cantor system $(\mathscr{C}, T)$. The orbit of an element $x \in \mathscr{C}$ is *dense*, if there for all $y \in \mathscr{C}$ and a real number $\epsilon > 0$, there exists $n \in \mathbb{Z}$ such that

$$d(T^n(x), y) < \epsilon.$$

*Remark* 3.8. The property of being dense guarantees you to be very very close or equal as seen in figure 3.1.

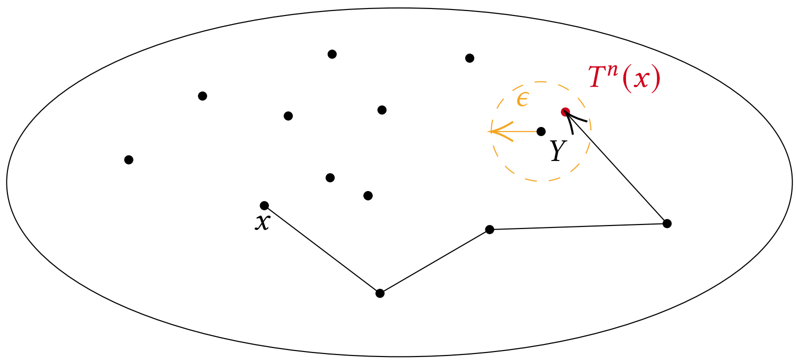



**Figure 3.1:** An illustration of a dense self-map $\mathrm{T^n x}$ from definition 3.7

**Definition 3.9.** Given a Cantor system $(\mathscr{C}, T)$. The automorphism $T$ is *minimal* if every orbit is dense.

*Remark* 3.10. As we are never to reach another point in the cantor space exactly, the best we can do is guarantee that we are within some radius $\epsilon$ of another element, which is to say that the prefix of $x$ coincides with the prefix of $y$ for a finite number of positions. What minimality guarantees us is that $x$ can be as close as we want to every other point in $\mathscr{C}$.

**Definition 3.11.** A *Cantor minimal system* $(\mathscr{C}, T)$, is a Cantor system, where $T : \mathscr{C} \to \mathscr{C}$ is a minimal automorphism.

Now, the interesting automorphisms for us are the odometer that we defined in the last chapter. What we are interested in knowing is when the odometers in general are minimal.

**Lemma 3.12.** *The generalised odometer $\varphi$ as stated in definition 2.11, is* minimal *if and only if $p$ and $q$ are relatively prime.*

*Proof.* Given $p, q \in \mathbb{N}$, two points $x, y \in \mathscr{C}$, and the automorphism $\varphi \colon \mathscr{C} \to \mathscr{C}$. It is well known from number theory that if $p$ and $q$ are to be relatively prime, then the $\gcd(p, q) = 1$. Furthermore, assume $\varphi$ to be minimal. That is, given $n \in \mathbb{N}$, when is $d(\varphi(x), y) < 1/3^n$, which we have shown is the same as asking when the first $n$ positions of the prefix of $x$ and $y$ coincide.

To show this, we will define the map $\Pi \colon \widehat{\mathbb{Z}}_p \to \mathbb{Z}/p^n\mathbb{Z}$, such that $\sum_{m=0}^{\infty} a_m p^m \to \sum_{m=0}^{n-1} a_m p^m \mod (2^n)$, where the latter is the a cyclic group. Observe that $\Pi$ is a surjective map, as we are killing everything greater or equal to $p^n$ due to the mod reduction in the cyclic group (as such elements in $\widehat{\mathbb{Z}}_p$ are going to have the same result in $\mathbb{Z}/p^n\mathbb{Z}$ after the reduction).

Observe from figure 3.2 that the two cases for addition should result in the same result. In other words, given $x \in \mathbb{Z}_p$, apply $\varphi x = x + q$, then apply $\Pi(x + q)$. This result should be equivalent to

$$\begin{array}{ccc} x \in \widehat{\mathbb{Z}}_p & \xrightarrow{\quad \Pi \quad} & \mathbb{Z}/p^n\mathbb{Z} \\ \Big\downarrow \varphi & & \Big\downarrow \varphi \\ \widehat{\mathbb{Z}}_p & \xrightarrow{\quad \Pi \quad} & \mathbb{Z}/p^n\mathbb{Z} \end{array}$$

**Figure 3.2:** A commutative diagram over addition of the odometer

applying $\Pi(x)$ first, and then apply $\varphi(\Pi(x)) = \Pi(x) + q$. If $\varphi$ is to be minimal, then it should be able to generate every element in $\widehat{\mathbb{Z}}_p$. If it is not able to generate every element, then $\varphi$ will not satisfy definition 3.7 of being dense, as there will be some "coefficients" $\varphi$ will never generate, i.e being close to. The only case where $\varphi$ is able to generate every element in $\widehat{\mathbb{Z}}_p$, is if $\gcd(p, q) = 1$. This is a well known fact from group theory, and as such, that is why the odometer is only minimal when $\gcd(p, q) = 1$ is satisfied. □

*Remark* 3.13. As will be highlighted in the example, the odometer has no cyclic points, yet it can still be viewed as an extension of cyclicity in the fact that the prefixes are "cyclic" (if viewed alone).

**Example 3.14.** We do the following example for $p = 2$ and $q = 3$ that have $\gcd(p, q) = 1$. Consider the point $x \in \{0, 1\}^{\mathbb{N}}$. We let $x$ consist of a finite prefix of length $m = 3$ such that $x = 100x'$. First, we are only interested in what is happening to the prefix of $x$. Observe the following for the odometer $\varphi^n \colon \{0, 1\}^{\mathbb{N}} \xrightarrow{+3} \{0, 1\}^{\mathbb{N}}$, where $n \in \mathbb{Z}$.

$$I : 100 \xrightarrow{+3} 001 \xrightarrow{+3} 111 \xrightarrow{+3} 010 \xrightarrow{+3} 101 \xrightarrow{+3} 000 \xrightarrow{+3} 110 \xrightarrow{+3} 011 \xrightarrow{+3} 100.$$

In the process of adding $3$, we are "visiting" every other possible combination of $0$ and $1$ of the $8$ in total, until $\varphi^8(100) = 100$, which is what we started with. In other words, when we only consider the prefix, we find that it is a "cyclic point". If we consider the whole word $x$, we would pass through every possible combination of $0$ and $1$, and have that $\varphi^n(x) \neq x$ for all $n \in \mathbb{N}$ except $n = 0$. In other words, $x$ would never be a cyclic point. Since we would pass through every possible combination of $0$ and $1$, we would also have that $\varphi$ is dense, as we for any radius $\epsilon$, would have that $\varphi^n(x)$ at some point would be "close" to a point $y \in \{0, 1\}^{\mathbb{N}}$. That is to say, that the first $k$ positions of both points would at some "time" $n$, coincide.

Now, let us consider the same prefix, but for $p = 2$ and $q = 2$ that have $\gcd(p, q) = 2$.

$$I : 100 \xrightarrow{+2} 110 \xrightarrow{+2} 101 \xrightarrow{+2} 111 \xrightarrow{+2} 100,$$
$$II : 010 \xrightarrow{+2} 001 \xrightarrow{+2} 011 \xrightarrow{+2} 000 \xrightarrow{+2} 010.$$

In the process of adding $2$, we can never "visit" every possible combination of $0$ and $1$. We need two different starting positions in order to demonstrate all $8$ combinations. The same observation for a cyclic point are the same as above. However, when considering the whole word $x$, then the property of a dense automorphism is not true, as we could find radius's $\epsilon$ that $x$ would never be within. That is to say, that the first $k$ positions of $\varphi^n(x)$ and $y$, never would coincide for any value of $n$.

*Remark* 3.15. Observe that if $\mathscr{C}, T$ has a cycle, then it cannot be minimal.

## 3.2 Bratteli–Vershik diagram $(V, E)$

In the latter chapter we highlighted one way in which we can construct automorphisms on the Cantor system $(C, T)$. However, there is actually another, more general way, to construct automorphisms to the cantor space. Throughout this thesis we have partly exploited the combinatorial aspect of the Cantor set in the notion that every element $x \in C$, can be represented with an infinite binary sequence. In order to construct all possible automorphisms, we are to transfer the Cantor dynamical system into a purely combinatorial object, and represent it as a *Bratteli diagram.*

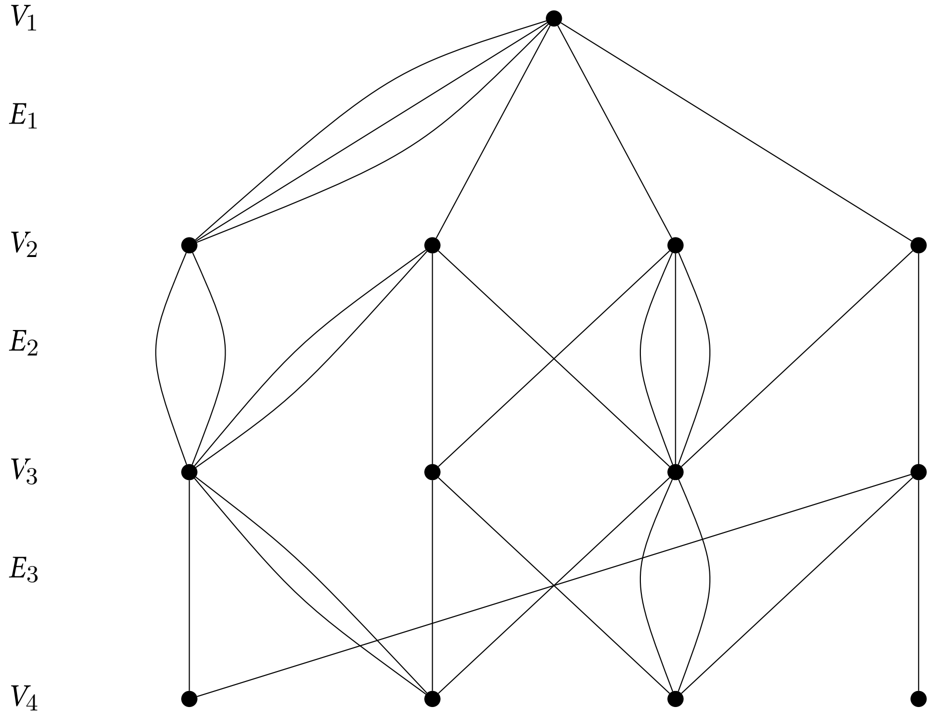



**Figure 3.3:** A Bratteli diagram

**Definition 3.16.** Given a natural number $n \geq 1$, a *Bratteli diagram* $(V, E)$ is a diagram with the following three properties. First, $V$ is an infinite sequence $V_0 \ldots V_n \ldots$ of finite, non-empty, pairwise disjoint (totally disconnected??) sets, that is $V_0 = \{v_0\}$, $V_1 = \{v_1, \ldots, \}$ etc, called vertices. Second, $E$ is an infinite sequence $E_0 \ldots E_n \ldots$ of finite, non-empty, pairwise disjoint sets, that is $E_0 = \{e_0, \ldots, \}$, $E_1 = \{e_1, \ldots, \}$ etc, called edges. Thirdly, it consists of two maps: A *source map* $s\colon E_n \longrightarrow V_{n-1}$, where $s^{-1}(\{v\})$ is non-empty for all $v$ in $\cup_{n\geq 0} V_n$. And a *range map* $r\colon E_n \longrightarrow V_n$, where $r^{-1}(\{v\})$ is non-empty for all $v$ in $\cup_{n\geq 1} V_n$.

*Remark* 3.17. It can help to think of a Bratteli diagram as a directed graph with no sinks, and no sources other than the single $v_0$. That is what the latter restrictions on the inverses of both the source and range maps insures.

**Definition 3.18.** Given a Bratteli diagram $(V, E)$, and $0 \leq m < n$. The set of all finite paths from $V_m$ to $V_n$, denoted as $P_{m,n}$, is the following

$$P_{m,n} = \{(p_{m+1}, p_{m+2}, \ldots, p_n) \mid p_i \in E_i, m < i \leq n, \text{ and } r(p_i) = s(p_{i+1}), m < i < n\}.$$

The set of all infinite paths beginning at $V_0 = \{v_0\}$, denoted as $p$, is the following

$$P_E = \{(p_1, p_2, \ldots) \mid p_k \in E_k, \text{ and } r(p_k) = s(p_{k+1}), k \geq 1\}.$$

*Remark* 3.19. It is worth emphasizing the latter requirement that the range map for a path $p_k$ *must* equal the source map of the next path $p_{k+1}$. That way the path consists of a single edge when traversing from one vertex to the next, and they are all connected.

However, not every Bratteli diagram can be used to find minimal systems of the Cantor space. Going forward we are going to consider a special type of Bratteli diagrams that are primitive. All examples are illustrated in figure 3.4.

**Definition 3.20.** Given $n \in \mathbb{N}$. Let $(E, V)$ be a Bratteli diagram, such that for all $n$ the $|v_n| = k$. We define the transition matrix at the $n$'th-level to be the following $M_n \in M_{k\times k}(\mathbb{N})$, where

$$M_n(i, j) = \text{ \# arrows in } E_n \text{ from } i \in V_{n-1} \text{ to } j \in V_n.$$

**Definition 3.21.** A Bratteli diagram is *stationary* if $M_n = M_m$ for all $n, m \in \mathbb{N}$. Furthermore, we let $M_n = M_E$.

**Definition 3.22.** Given a transition matrix $M_E$. Then $M_E$ is *primitive* if there exists $n \in \mathbb{N}$ such that the matrix $M_E^n = \underbrace{M_E \cdot \cdots \cdot M_E}_{n\text{-times}}$ has no zero entries.

**Example 3.23.** What follows are three examples of the transition matrix $M_E$, where the first two considered primitive, and the latter is not.

$$\underbrace{M_E = (2),}_{\text{for figure 3.4a}} \qquad \underbrace{M_E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},}_{\text{for figure 3.4c}} \qquad \underbrace{M_E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.}_{\text{for figure 3.4c}}$$

Observe that the latter one is an example of a bratteli diagram that only consists of two points.

**Definition 3.24.** A stationary Bratteli diagram is *simple* if $M_E$ is primitive and not equal to $(1)$.

Going forwards, we are now to define a topology on $P_E$ of a simple Bratteli diagram, such that if given a finite path $e_0 \ldots e_n$ that start at $v_0$, then we define the cylinder set such that the prefix $\omega = e_0 \ldots e_n$ and the corresponding cylinder set is the following clopen set

$$\begin{aligned} \mathcal{Z}(\omega) &= \{\omega x \in P_E \mid \omega = e_0 \ldots e_n, x \in P_E\}, \\ &= \{\text{all infinite paths with prefix } \omega\}. \end{aligned}$$

The topology on $P_E$ is the one that is generated by the cylinder sets, and as shown in (Putnam, 2018) this topological space actually satisfies what we had earlier; It's compact, totally disconnected, and does not contain any isolated points. In other words, $P_E$ is a Cantor space.

However to construct all an automorphism that is equivalent to what we had earlier, we must introduce an order to all the edges in the Bratteli diagram. The ordering is important as it allows us to compare paths on the diagram.

**Definition 3.25.** An *ordered* Bratteli diagram $(V, E, \geq)$ is a Bratteli diagram $(V, E)$ with order $\geq$ on $E$ such that two edges $e, e' \in E$ are comparable if and only if $r(e) = r(e')$. Additionally, two finite paths $\gamma = e_1 \dots e_n$ and $\rho = f_1 \dots f_n$ are comparable if $r(\gamma) = r(\rho)$ using the lexicographic order.

**Definition 3.26.** If $(V, E, \geq)$ is a simple ordered stationary Bratteli diagram, then there exist a unique $X_{max} = \alpha_0\alpha_1 \dots$ where $\alpha \in E_{max}$ and a unique $X_{min} = \beta_0\beta_1 \dots$ where $\beta \in E_{min}$.

**Definition 3.27.** Given a simple ordered stationary Bratteli diagram $(V, E, \geq)$ and a set of all the infinite paths $P_E$ of the diagram. We define the map $T\colon P_E \longrightarrow P_E$, named the *Vershik transformation*, to be the unique minimal path such that $T(X_{max}) = T(X_{min})$ and $T(x) = y(x_n{+}1)x_{n+1}$ where $x_n \in E_i$, and at least one $x_k \notin E_{max}$ where $k$ is the smallest such integer. Then $T(x) = (y_1y_2 \dots y_{k-1}(e_k + 1)e_{k+1} \dots)$, where $e_k + 1$ is the successor of the edge in the lexicographic ordering, and $f$ is the minimal path to $e_k + 1$.

**Example 3.28.** Given a sequence $x \in \{0, 1\}^{\mathbb{N}}$ where $x = 1110011x'$ and $x'$ is the leftover binary sequence. Now we apply $T$ on $x$, and get the following

$$T(x) = T(1110011x') = 000(0+1)\;011x' = 0001011x'.$$

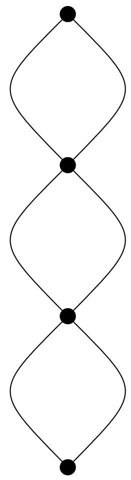
Observe that $k = 4$ is the smallest integer $e_k \notin E_{max}$. Now, given $q = 1$ (we still have $p = 2$ from $\{0, 1\}^{\mathbb{N}}$), and let the odometer $\varphi_1$ be applied to the prefix of $x$.

$$\varphi_1(1110011) = 0\varphi_1(110011) = 00\varphi_1(10011) = \cdots = 0001011.$$

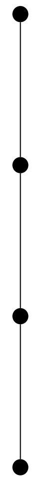
Which is the exact same sequence!

**Example 3.29.** Two other Vershik transformations are shown illustrative for figure 3.5, in figure 3.6. Observe that the section 3.2 show the most clear equivalence with $\{0, 1\}^{\mathbb{N}}$.

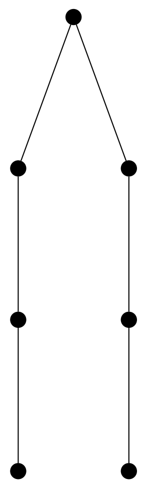
We conclude this thesis with what is the main result of this chapter. We have seen that by constructing equivalent simple ordered Bratteli diagrams to the discrete p-point product topology, we have a more general way to construct all the autmorphisms of the Cantor space. This equivalence also allows us to apply results from the field of Bratteli diagrams to the dynamical system of study, and *vica verca.*



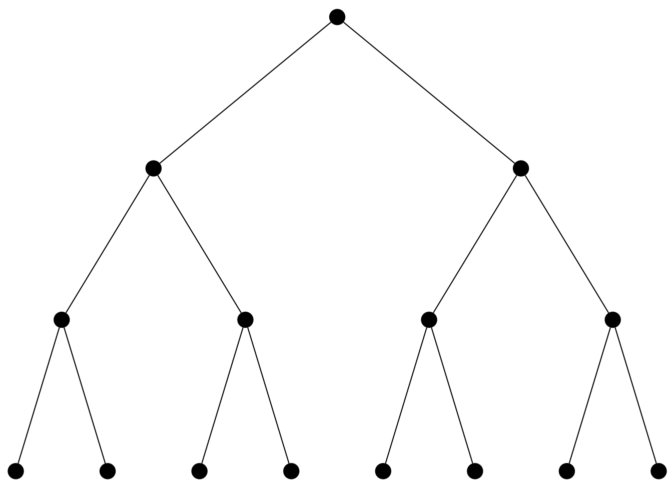
**(a)** Stationary — Primitive — Simple



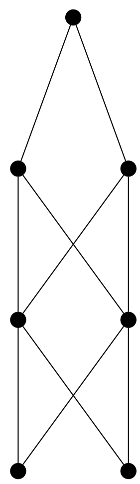
**(b)** Stationary — Primitive — Not simple



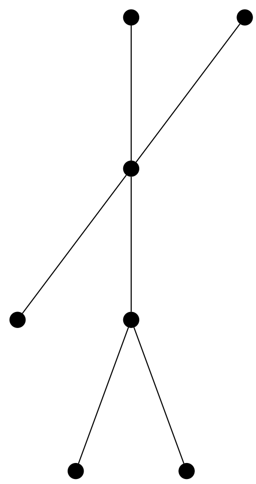
**(c)** Stationary — Not primitive — Not simple



**(d)** Not stationary — Not primitive — Not simple



**(e)** Stationary — Primitive — Simple



**(f)** Not a Bratteli diagram

**Figure 3.4:** Illustrated here are 5 Bratteli diagrams (a-e), and one non-Bratteli diagram (it has a sink and two sources). Observe that all Simple Bratteli diagrams. Furthermore, a Bratteli diagram should not be confused with a tree from graph theory, as they are two distinct mathematical objects. This is best exemplified in subfigure d) which, as we have seen, is a valid representation of the Cantor space if the illustration was of a binary tree. As a Bratteli diagram it is not a valid "representation" in the same sense. In our case, a Bratteli diagram is used to illustrate properties of Cantor minimal systems, and d) is not a diagram of a Cantor minimal system.

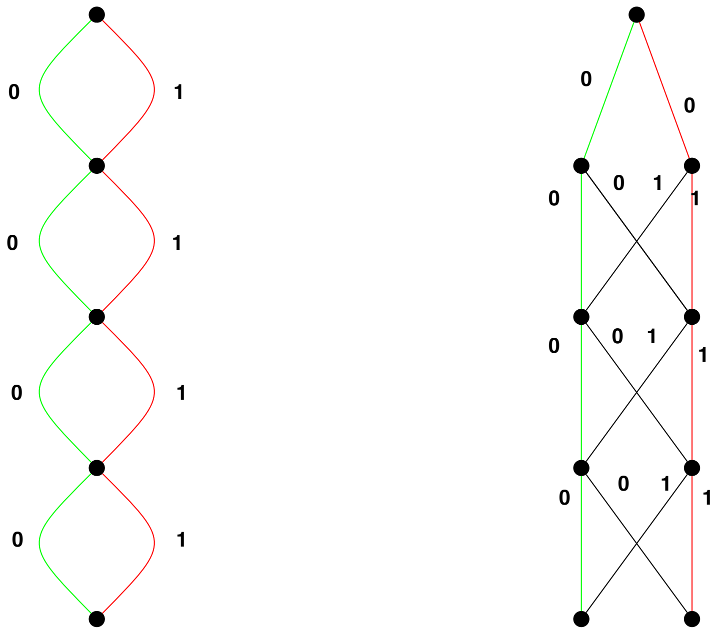



**Figure 3.5:** The maximal edges $\mathrm{E}_{\max}$ in red, and the minimal edges $\mathrm{E}_{\min}$ in green for two simple Bratteli diagrams

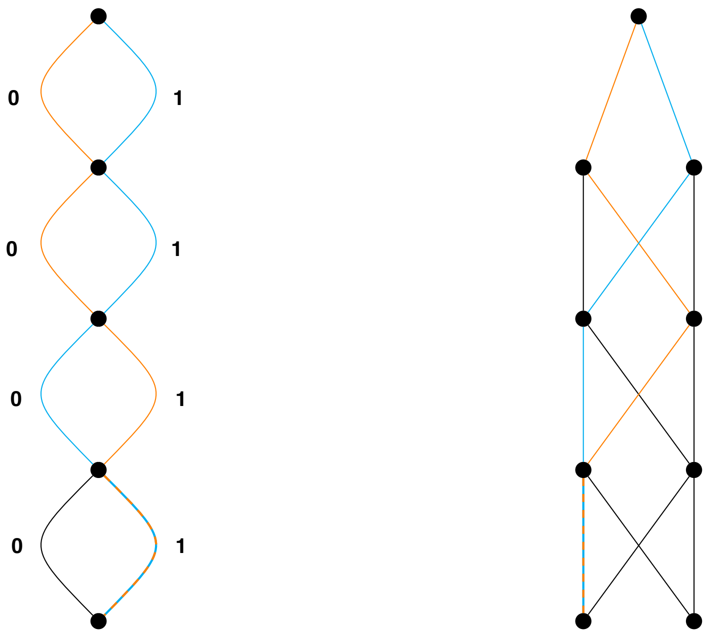



**Figure 3.6:** The Vershik transformation applied to the path in blue results in the orange path. In the edges where the paths overlap, the orange is dashed.

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