Even Aslaksen

Monads, Algebras and Descent Theory

Bachelor's project in Mathematical Sciences
Supervisor: Drew Heard
May 2021

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NTNU
Norwegian University of Science and Technology
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Abstract

This bachelor's thesis is an exposition of the articles Facets of Descent I and III by G. Janelidze and W. Tholen on descent theory. We start by studying the preliminary theory of monads and the Barr–Beck theorem. Then, we develop the monadic approach to descent theory and the approach via Grothendieck fibrations. Finally, we give an elementary approach to the classical descent problem for modules and algebras.

Contents

1 Introduction

In the abstract context of fibered categories, Grothendieck [Gro59], [Gro70] developed descent theory. (An exposition on Grothendieck's work in English is [Vis07].) The general problem of descent is *base change* and how to compensate for the loss of information by determining the *(effective) descent morphisms*. For commutative rings, the classical question is as follows:

Question 1.1. Let $p\colon R \to S$ be a homomorphism of commutative rings. Given a S-module N, what *data* on N determines an R-module M, together with an isomorphism of S-modules $M \otimes_R S \cong N$?

Reformulated in the language of monads, Grothendieck [Gro59] answered Question 1.1 with the following theorem:

Theorem (Theorem 4.1)**.** For a homomorphism $p\colon R \to S$ of commutative rings, the extension-of-scalars functor $S \otimes_R (-)\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ is comonadic whenever p makes S a faithfully flat R-module.

Janelidze and Tholen [JT04] gave a stronger answer than Grothendieck to Question 1.1 via a significant contribution of Mesablishvili [Mes00]:

Theorem (Theorem 4.15)**.** A homomorphism $p\colon R \to S$ of commutative rings is an effective descent morphism if and only if it is a *pure* morphism of R-modules.

The authors of [JT04] note the obvious monadic connection between Grothendieck's theorem by applying the Barr–Beck theorem. However, monads were not "popular" at the time. Although Bénabou and Roubaud [BR70] explicitly described the monadic approach, neither Grothendieck nor anyone else at his school used the approach.

A monad is by itself a natural construction that has had many names throughout history. After their period as a "standard construction," they were named a *triple* which is not very explicit but allowed them to be studied. Specifically, Huber [Hub61] first discovered that every adjoint pair gives rise to a monad (proved here as Theorem 2.3). Then, both Kleisli [Kle65] and Eilenberg–Moore [EM65] proved the converse independently (proved here as Theorem 2.7). Sometime after, the breakthrough for monads was the Barr–Beck theorem [Bec67]. In particular, the Barr–Beck theorem was very useful because it was "easy" to give variations of the theorem. A graduate student W. Butler (unpublished) established 64 theorems on adjoint pairs and monads. Later, J. Power [Pow72] published these results in his doctoral thesis.

As alluded to, the Barr–Beck theorem gave access to a stronger result than Grothendieck's Theorem 4.1. Indeed, the theory developed by Janelidze and Tholen [JT94] bases itself on the sheaf theoretic connection of Grothendieck's descent and monadic descent. Moreover, applying the Barr–Beck theorem to the adjoint situation between base-extension and base-restriction of topological bundles inspired the generalization of determining descent morphisms of arbitrary categories with pullbacks and coequalizers. There is a complete characterization of topological descent by one of the authors in [RT94].

The surprising discovery in [JT94] is that all the definitions of the categorical approach of monads survived the abstraction to (bi)fibered categories. In other words, the Barr–Beck theorem remained important for characterizing descent morphisms. Specifically, the definitions of monadic descent survived within a bifibered category over a category with pullbacks that satisfied the Beck–Chevalley condition. This discovery within (bi)fibered categories gave rise to the stronger Theorem 4.15.

Descent theory has become very useful in the modern era, in particular, within homotopy theory and algebraic geometry. For example, Lurie [Lur11] has given many "higher" categorical descent results, and he recently gave an extension of the Barr–Beck theorem to stable ∞-categories [Lur17, Theorem 4.7.3.5].

1.1 Contents

In the context of this thesis, we will develop descent theory following the works of [JT94] and [JT04]; that is, monadic descent, descent theory with respect to fibrations and descent for rings and algebras as a special case. We will not follow the works of Grothendieck.

The first section follows [BW85] with a couple of detours within [JT04]. We first define and fix some notation before we study the properties of a monad. The constructions of Eilenberg–Moore is of particular importance because it introduces the canonical comparison functor to the category of Eilenberg–Moore algebras. We dedicate an entire "bonus" subsection to emphasize monads as generalized rings based on the example of a monad on the category of Abelian groups. Next, we recall a few results of coequalizers and develop the necessary tools to prove the Barr–Beck theorem. Finally, we give the Barr–Beck theorem, which states criteria for whenever a functor is monadic, or rather, whenever the comparison functor is an equivalence of categories. Then, we give a few variations of the theorem that will be useful for our final section on descent for modules and algebras.

The second section introduces monadic descent theory and descent theory with respect to fibrations directly following [JT94]. We start in the subsection of monadic descent theory, where we state the question of descent in terms of a slice category and attempt to gain intuition from topological descent as we move forward. We define the category of descent data and deduce that the pullback functor can be lifted to a comparison functor (in the sense of Eilenberg–Moore) to the descent category. In particular, we characterize when this comparison functor is an equivalence of categories. We conclude monadic descent theory by a methodology on how to approach "difficult" categories and an interesting corollary to our work on torsion-free abelian groups. In the subsection on descent theory with respect to fibrations, we start by carefully developing the notions of a (Grothendieck) fibration. Then, we define the category of descent data with respect to a fibration. In a similar manner to monadic descent, we determine a comparison functor to the new descent category. Finally, we observe a bijective correspondence between the descent data for monadic descent and descent with respect to a (bi)fibration. Specifically, the category of descent data with respect to a (bi)fibration is isomorphic to the category of Eilenberg–Moore algebras if the (bi)fibration satisfies the Beck–Chevalley condition. Therefore, the effective descent morphisms with respect to a (bi)fibration are precisely those of monadic descent.

The third section considers a special case of descent theory: descent for rings and algebras, following [JT04]. First, we define the extension-of-scalars functor and the restriction-of-scalars functor. Then, we determine that the extension-of-scalars functor is comonadic whenever the underlying ring homomorphism is a *pure* morphism of both bimodules and modules by applying the Barr–Beck theorem and using some "homological algebra methods." Afterward, for a ring homomorphism of commutative rings, we find that the extension-of-scalars functor for modules is comonadic if and only if the induced extension-of-scalars functor for algebras is comonadic. In other words, the results can be further generalized to various types of algebras. Finally, by observing that a bifibration of the category of all modules of commutative rings over the category of commutative rings satisfies the Beck–Chevalley condition, we conclude with Theorem 4.14 that characterizes descent for the various types of algebras where Theorem 4.15 is a special case.

In the appendix, there is a recollection of the categories considered in this thesis.

1.2 Prerequisites

The thesis assumes the reader is well-acquainted with category theory; functors, natural transformations, equivalences of categories. Furthermore, the reader should be familiar with basic homological algebra, such as the Hom-tensor adjunction and exact sequences.

1.3 Conventions

- For any pair of categories $\mathcal{C}$ and $\mathcal{D}$, denote by $\mathcal{C}^{\mathcal{D}}$ the functor category whose objects are functors $\mathcal{D} \to \mathcal{C}$, and whose morphisms are natural transformations.
- The category Cat is the (2)-category of locally small categories.
- A projection map is a map denoted $\mathrm{pr}_i \colon \prod_k X_k \to X_i$. The index i denotes which "component" (of the object $\prod_k X_k$) the map is *projecting*. In case where there are two indices i and j, the projection map is projecting the product of those objects with indices i and j; that is, $\mathrm{pr}_{i,j} \colon \prod_k X_k \to X_i \times X_j$. In particular, $\mathrm{pr}_{i,j} \neq \mathrm{pr}_{j,i}$ where the latter is a map $\mathrm{pr}_{j,i} \colon \prod_k X_k \to X_j \times X_i$.
- A ring is always unital unless we specify it without a unit.
- A module will always mean a left module unless we specify it as a right module.
- For a ring R, the category of left R-modules is denoted by ${}_R\mathsf{Mod}$, and the category of right R-modules is denoted by Mod_R.

1.4 Acknowledgements

I wish to thank my supervisor Drew Heard for his patient guidance and advice throughout the writing of this thesis. In particular, I would like to express my appreciation for him sharing his time so generously.

2 Monadicity

A monad is an abstraction of algebraic structures that occur very naturally in algebraic contexts. In light of this, the first subsection will define and explore why they appear so naturally. Moreover, it will attempt to shed light on why they are of interest in algebraic contexts.

2.1 Monads

Definition 2.1. A *monad* on a category $\mathcal{C}$ consists of
- an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$;
- a *unit* natural transformation $\eta\colon \mathrm{id}_{\mathcal{C}} \to T$;
- a *multiplication* natural transformation $\mu\colon T^2 \to T$;

so that the diagrams

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T, \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} T & \xrightarrow{T\eta} & T^2 & \xleftarrow{\eta T} & T \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu} & \diagup\!\!\diagup & \\ & & T & & \end{array}$$

commute in $\mathcal{C}^{\mathcal{C}}$. Denote by $\mathbb{T} = (T, \eta, \mu)$ the monad on $\mathcal{C}$.

This definition is similar to that of monoids, unital rings, and $\Bbbk$-algebras. Indeed, they are all examples of monoids in a monoidal category. Although, this observation is far more abstract than what is needed to appreciate monads.

Example 2.2.
1. Let M be a monoid and define $T\colon \mathsf{Set} \to \mathsf{Set}$ by $X \mapsto M \times X$. Let $\eta_X\colon X \to M \times X$ take $x \mapsto (\mathrm{id}_M, x)$ and $\mu_X\colon M \times M \times X \to M \times X$ take $(m, n, x) \mapsto (mn, x)$. Then $\mathbb{T} = (T, \eta, \mu)$ defines a monad on Set.
2. In a similar manner, let R be a commutative ring and A an associative unitary R-algebra. Then there is a monad on the category Mod_R of R-modules taking $M \mapsto A \otimes M$.

These examples of monads share some similarities to an adjunction. Specifically, the unit and multiplication map of the monad is similar to the unit and counit of an adjunction. This observation led Huber to suspect and prove:

Theorem 2.3. [BW85, Chapter 3.1, Theorem 1] Let $U\colon \mathcal{B} \to \mathcal{C}$ have a left adjoint $F\colon \mathcal{C} \to \mathcal{B}$ with adjunction unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to UF$ and counit $\varepsilon\colon FU \to \mathrm{id}_{\mathcal{D}}$. Then $\mathbb{T} = (UF, \eta, U\varepsilon F)$ is a monad on $\mathcal{C}$.

Proof. By the triangle identities of the adjunction, the unit evidently remains the unit of the monad. To check that $U\varepsilon F$ satisfies the multiplication map, first note that ε is natural. Then, the diagram

$$\begin{array}{ccc} FUX & \xrightarrow{\varepsilon_X} & X \\ {\scriptstyle FUf}\downarrow & & \downarrow{\scriptstyle f} \\ FUY & \xrightarrow[\varepsilon_Y]{} & Y \end{array}$$

commutes. Choose $X = FUY$ and $f = \varepsilon_Y$, and apply U to the diagram, then $U\varepsilon F$ satisfies the multiplication map. □

This theorem gives us access to a large family of monads by considering the family of "free" and "forgetful" adjunctions (see [Rie16, Example 5.1.4 and 5.1.5] for more details).

Example 2.4.
1. The free-forgetful adjunction between Set and ${}_R\mathsf{Mod}$ induces the *free R-module monad* $R[-]\colon \mathsf{Set} \to \mathsf{Set}$ given by the set $R[A]$ of finite formal R-linear combinations of elements of A. Two special cases of this monad are the *free abelian group monad* and the *free vector space monad*.
2. The free-forgetful adjunction between Set and Grp induces the *free group monad* that sends a set A to the set $F(A)$ of finite words in the letters $a \in A$ together with formal inverses a^{-1}.
3. The *Giry monad* on the category Meas of measurable spaces sends a measurable space A to the probability measures on A.

There is also the dual of the monad.

Definition 2.5. A *comonad* on a category $\mathcal{B}$ is a monad on $\mathcal{B}^{\mathrm{op}}$. Explicitly, a *comonad* on a category $\mathcal{B}$ consists of

- an endofunctor $G\colon \mathcal{B} \to \mathcal{B}$;
- a *counit* natural transformation $\varepsilon\colon G \to \mathrm{id}_{\mathcal{B}}$;
- a *comultiplication* natural transformation $\delta\colon G \to G^2$;

so that the diagrams

$$\begin{array}{ccc} G^3 & \xleftarrow{G\delta} & G^2 \\ {\scriptstyle \delta G}\uparrow & & \uparrow{\scriptstyle \delta} \\ G^2 & \xleftarrow[\delta]{} & G, \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} G & \xleftarrow{G\varepsilon} & G^2 & \xrightarrow{\varepsilon G} & G \\ & \diagdown\!\!\diagdown & \uparrow{\scriptstyle \delta} & \diagup\!\!\diagup & \\ & & G & & \end{array}$$

commute in $\mathcal{B}^{\mathcal{B}}$. Denote by $\mathbb{G} = (G, \varepsilon, \delta)$ the comonad on $\mathcal{B}$.

Explicitly, the dual statement of Huber is also true.

Proposition 2.6. [BW85, Chapter 3.1, Proposition 2] Let $U\colon \mathcal{B} \to \mathcal{C}$ have a left adjoint $F\colon \mathcal{C} \to \mathcal{B}$ with adjunction unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to UF$ and counit $\varepsilon\colon FU \to \mathrm{id}_{\mathcal{B}}$. Then $\mathbb{G} = (FU, \varepsilon, F\eta U)$ defines a comonad on $\mathcal{C}$.

The converse statements are true too.

Theorem 2.7. [BW85, Chapter 3.2, Theorem 1] Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. Then there is a category $\mathcal{B}$ and an adjoint pair $F\colon \mathcal{C} \to \mathcal{B}$, $U\colon \mathcal{B} \to \mathcal{C}$ where F is left adjoint to U such that $T = UF$, $\eta\colon \mathrm{id}_{\mathcal{C}} \to UF = T$ is the unit and multiplication $\mu = U\varepsilon F$ where ε is the counit of the adjunction.

Dually, let $\mathbb{G} = (G, \varepsilon, \delta)$ be a comonad on $\mathcal{B}$. Then there is a category $\mathcal{C}$ and an adjoint pair $F\colon \mathcal{C} \to \mathcal{B}$, $U\colon \mathcal{B} \to \mathcal{C}$ where F is left adjoint to U such that $G = FU$, $\varepsilon\colon G = FU \to \mathrm{id}_{\mathcal{C}}$ is the counit and comultiplication $\delta = F\eta U$ where η is the unit of the adjunction.

Remark 2.8. We give a proof of Theorem 2.7 later by using the Eilenberg–Moore constructions, but we prefer to introduce the construction separately first. There is an alternative proof by Kleisli [Kle65], or, as further reading in [BW85, Chapter 3.2, Theorem 1].

The following are the constructions by Eilenberg–Moore [EM65].

Definition 2.9. Consider a monad $\mathbb{T} = (T, \eta, \mu)$ on $\mathcal{C}$. A $\mathbb{T}$-*algebra* is a pair (A, α) where A is an object of $\mathcal{C}$ and $\alpha\colon TA \to A$ is a morphism of $\mathcal{C}$ such that the following diagrams

$$\begin{array}{ccc} T^2A & \xrightarrow{T\alpha} & TA \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \alpha} \\ TA & \xrightarrow[\alpha]{} & A, \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \alpha} \\ & & A \end{array}$$

commute. The morphism α is called the *structure map* of the algebra. Often the category $\mathcal{C}^{\mathbb{T}}$ is referred to as the *category of Eilenberg–Moore algebras*.

Definition 2.10. Consider a monad $\mathbb{T} = (T, \eta, \mu)$ on $\mathcal{C}$. The *category of* $\mathbb{T}$-*algebras* is denoted by $\mathcal{C}^{\mathbb{T}}$, and consists of

(i) objects that are $\mathbb{T}$-algebras;
(ii) morphisms $f\colon (A, \alpha) \to (B, \beta)$ is a morphism $f\colon A \to B$ in $\mathcal{C}$ that are $\mathbb{T}$-linear; that is, the following diagram

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[f]{} & B \end{array}$$

commutes.

Definition 2.11. Consider a monad $\mathbb{T} = (T, \eta, \mu)$ on $\mathcal{C}$. The *free-algebra functor* $F^{\mathbb{T}}\colon \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$ is given by

$$F^{\mathbb{T}}A := (TA, \mu_A), \quad F^{\mathbb{T}}(f\colon A \to B) := (Tf\colon TA \to TB)$$

for an object $A \in \mathcal{C}$ and a morphism $f\colon A \to B$ in $\mathcal{C}$. It is left adjoint to the *forgetful functor* $U^{\mathbb{T}}\colon \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$, which forgets the structure map. This adjunction is often referred to as the *Eilenberg–Moore adjunction*. The unit of the adjunction is $\eta\colon \mathrm{id}_{\mathcal{C}} \to U^{\mathbb{T}}F^{\mathbb{T}} = T$ is the unit of the monad $\mathbb{T}$, whereas the counit $\varepsilon^{\mathbb{T}}\colon F^{\mathbb{T}}U^{\mathbb{T}} \to \mathrm{id}_{\mathcal{C}^{\mathbb{T}}}$ is given for every $\mathbb{T}$-algebra (A, a) by $\varepsilon^{\mathbb{T}}_{(A,a)} = a$.

The dual of this construction is as follows.

Definition 2.12. Consider a comonad $\mathbb{G} = (G, \varepsilon, \delta)$ on $\mathcal{C}$. A *$\mathbb{G}$-coalgebra* is a pair (A, α) where A is an object of $\mathcal{C}$ and $\alpha \colon A \to GA$ is a morphism of $\mathcal{C}$ such that the following diagrams

$$\begin{array}{ccc} T^2A & \xleftarrow{G\alpha} & TA \\ {\scriptstyle \delta_A}\uparrow & & \uparrow{\scriptstyle \alpha} \\ TA & \xleftarrow[\alpha]{} & A, \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xleftarrow{\varepsilon_A} & TA \\ & \diagdown\!\!\diagdown & \uparrow{\scriptstyle \alpha} \\ & & A \end{array}$$

commute. The arrow α is called the *(co)structure map* of the coalgebra.

Definition 2.13. Consider a comonad $\mathbb{G} = (G, \varepsilon, \delta)$ on $\mathcal{C}$. The *category of $\mathbb{G}$-coalgebras* is denoted by $\mathcal{C}^{\mathbb{G}}$. It consists of

(i) objects that are $\mathbb{G}$-algebras;
(ii) morphisms $f \colon (A, \alpha) \to (B, \beta)$ is a morphism $f \colon A \to B$ in $\mathcal{C}$ that are $\mathbb{G}$-linear; that is the following diagram

$$\begin{array}{ccc} GA & \xrightarrow{Gf} & GB \\ {\scriptstyle \alpha}\uparrow & & \uparrow{\scriptstyle \beta} \\ A & \xrightarrow[f]{} & B \end{array}$$

commutes.

Definition 2.14. Consider a comonad $\mathbb{G} = (G, \varepsilon, \delta)$ on $\mathcal{C}$. The *free-coalgebra functor* $F^{\mathbb{G}} \colon \mathcal{C} \to \mathcal{C}^{\mathbb{G}}$ is given by

$$F^{\mathbb{G}}A := (GA, \delta_A), \quad F^{\mathbb{G}}(f \colon A \to B) := (Gf \colon GA \to GB)$$

for an object $A \in \mathcal{C}$ and a morphisms $f \colon A \to B$ in $\mathcal{C}$. It is left adjoint to the *forgetful functor* $U^{\mathbb{G}} \colon \mathcal{C}^{\mathbb{G}} \to \mathcal{C}$, which forgets the (co)structure map. This adjunction is often referred to as the *co-Eilenberg–Moore adjunction.* The counit of the adjunction $\varepsilon \colon U^{\mathbb{G}}F^{\mathbb{G}} \to \mathrm{id}_{\mathcal{C}}$ is the counit of $\mathbb{G}$, whereas the unit $\eta^{\mathbb{G}} \colon \mathrm{id}_{\mathcal{C}^{\mathbb{G}}} \to F^{\mathbb{G}}U^{\mathbb{G}}$ is given for every $\mathbb{G}$-coalgebra (A, α) by $\eta^{\mathbb{G}}_{(A,\alpha)} = \alpha$.

Proof of Theorem 2.7. It is sufficient to prove the Eilenberg–Moore adjunction by considering the map

$$\varphi \colon \mathrm{Hom}_{\mathcal{C}^{\mathbb{T}}}((UFC, \mu_C), (C^{'}, \gamma^{'})) \to \mathrm{Hom}_{\mathcal{C}}(C, C^{'})$$

of Hom-sets. Clearly φ maps a morphism $h \colon UFC \to C^{'}$ of algebras to $h \circ \eta_C$, and the inverse φ^{-1} maps a morphism $g \colon C \to C^{'}$ to $\gamma^{'} \circ UFg$. Therefore φ is necessarily an isomorphism. It remains to check that the unit and counit form the monad, however by construction, this follows from Theorem 2.3. □

To summarize, there is an adjunction if and only if there is a (co)monad. This result is much more important than what it appears to be. When studying the relationship between two categories, a first step could be to check if there is an adjunction. Thus, in such cases, the monad appears quite naturally. In other words, if there is an adjunction, or a monad, the next question to consider is:

Question 2.15. How can a monad characterize the relationship between two categories?

The construction of the category Eilenberg–Moore algebras gives an answer to Question 2.15 by the following definition.

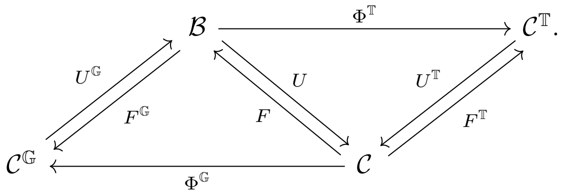
Definition 2.16. Consider the adjunction $F \colon \mathcal{C} \to \mathcal{B}$, $U \colon \mathcal{B} \to \mathcal{C}$ where F is left adjoint to U. The *Eilenberg–Moore comparison functor* is the functor $\Phi \colon \mathcal{B} \to \mathcal{C}^{\mathbb{T}}$ given by

$$\Phi^{\mathbb{T}}(B) := (UB, U\varepsilon_B), \quad \Phi(f \colon A \to B) := (Uf \colon UA \to UB).$$

Dually, there is a similar co-comparison functor,

$$\Phi^{\mathbb{G}}(B) := (FB, F\eta_B), \quad \Phi^{\mathbb{G}}(f \colon A \to B) := (Ff \colon FA \to FB).$$

Diagrammatically,



Perhaps surprisingly, the pursuit of this thesis is to determine when this comparison functor is full and faithful or an equivalence (or even an isomorphism) by only studying the adjunction. The usefulness is if, for example, the comparison functor an isomorphism of categories, then the category of the domain in the right adjoint has an algebra structure as described above, and many results can be derived thereof.

2.2 Monads as Generalized rings

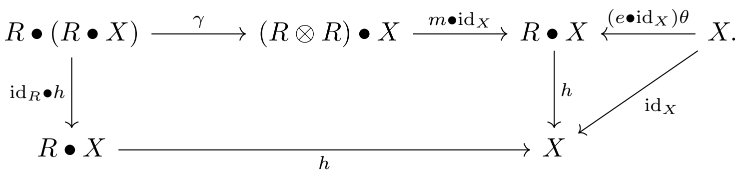
The following subsection aims to generalize the observations of Example 2.19. In a similar manner to [JT04], the reader who is not familiar with monoidal categories can safely skip forward to Example 2.19. Alternatively, see [Sch03, Sections 2.1 and 2.2] for a thorough introduction to what they call *(associative) bifunctors or functors of $\mathcal{C}$-categories*, and what we will call *lax actions* to keep the notation consistent with [JT04].

Definition 2.17. Let $\mathcal{C}$ be a monoidal category, and let $\mathcal{X}$ be any category be equipped with a *lax action*. A *lax action* is a functor

$$\bullet\colon \mathcal{C}\times\mathcal{X}\to\mathcal{X}$$

satisfying the usual action axioms up to specified natural morphisms $\gamma\colon A\bullet(B\bullet X)\to(A\otimes B)\bullet X$ and $\theta\colon X\to \mathbb{1}\bullet X$ (where $\mathbb{1}$ is the identity object for the tensor product $\otimes$ in $\mathcal{C}$) satisfying suitable coherence conditions. The lax action is called (*strict*) *strong* if γ and θ are (isomorphisms) identity morphisms.

Definition 2.18. Given a lax action $\bullet\colon \mathcal{C}\times\mathcal{X}\to\mathcal{X}$ of a category $\mathcal{X}$ acting on a monoidal category $\mathcal{C}$, and a monoid $R=(R,e,m)$ in $\mathcal{C}$, denote by $\mathcal{X}^R$ the category of R-actions in $\mathcal{X}$ whose objects (X,h) are pairs where X is an object in $\mathcal{X}$ and $h\colon R\bullet X\to X$ is a morphism making the diagram



(1)

commute in $\mathcal{X}$.

The idea is to generalize the monoid actions of a monoidal category; then, it will follow that rings and modules could be considered a special case of monads and algebras.

Table 1 is a collection of standard examples of this process, and the table originates from [BJK05, Examples 2.2].

The examples of Table 1 show that algebras over monads are monoid actions. However, the converse is also true:

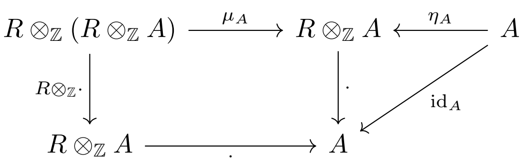
(i) a lax action $\mathcal{C}\times\mathcal{X}\to\mathcal{X}$ of a monoidal category $\mathcal{C}$ can be presented as a monoidal functor $\mathcal{C}\to\mathsf{End}(\mathcal{X})$;
(ii) a monoid R in $\mathcal{C}$ can be presented as a monoidal functor $1\to\mathcal{C}$;
(iii) the composite $1\to\mathcal{C}\to\mathsf{End}(\mathcal{X})$ determines a monoid in $\mathsf{End}(\mathcal{X})$, and hence a monad on $\mathcal{X}$ by row (f);
(iv) the algebras over that monad are the same R-actions in $\mathcal{X}$ with respect to the lax action $\mathcal{C}\times\mathcal{X}\to\mathcal{X}$.

This "logical equivalence" between monoids in monoidal categories and monads makes it possible to present descent theory for modules in Section 4 in either the language, "monoidal" or "monadic." To no surprise, we opted for the latter.

Example 2.19. Every ring R determines a monad on the category Ab of abelian groups, whose algebras are R-modules. This observation can be deduced from the Table 1, or directly: An algebra is an abelian group A equipped with a homomorphism $\cdot\colon R\otimes_{\mathbb{Z}}A\to A$ satisfying a pair of axioms. By the universal property of the tensor product, this homomorphism encodes a $\mathbb{Z}$-bilinear map $(r,a)\mapsto r\cdot a\colon R\times A\to A$ which defines "scalar multiplication." The commutative diagram

Table 1: Standard examples of monoid actions and their algebra structure.

	$\mathcal{C} = (\mathcal{C}, \otimes)$	$\mathcal{X}$	$A \bullet X$	monoids in $\mathcal{C}$	R-actions in $\mathcal{X}$
(a)	(Sets, $\times$)	$\mathcal{X} = \mathcal{C}$	$A \times X$	ordinary monoids	R-sets
(b)	(Topological spaces, $\times$)	$\mathcal{X} = \mathcal{C}$	$A \times X$	topological monoids	topological spaces equipped with a continuous R-action
(c)	(Abelian groups, $\otimes$)	$\mathcal{X} = \mathcal{C}$	$A \otimes X$	rings	R-modules
(d)	(K-modules, $\otimes_K$), where K is a commutative ring	$\mathcal{X} = \mathcal{C}$	$A \otimes_K X$	(associative) K-algebras	R-modules
(e)	(Abelian monoids, $\otimes$)	$\mathcal{X} = \mathcal{C}$	$A \otimes X$	semi-rings	R-semimodules
(f)	($\mathsf{End}(\mathcal{X})$, $\circ$), the category of endofunctors of an arbitrary category $\mathcal{X}$; $\circ$ is the composition of endofunctors	$\mathcal{X} = \mathcal{X}$	$A(X)$	monads on $\mathcal{X}$	R-algebras
(g)	($\mathcal{C}$, $+$), where $\mathcal{C}$ is an arbitrary category with finite coproducts	$\mathcal{X} = \mathcal{C}$	$A + X$	every object in $\mathcal{C}$ has a unique monoid structure	pairs (X, h) where $h\colon R \to X$ is a morphism in $\mathcal{C}$



ensures that $1 \cdot a = a$ and $r \cdot (r' \cdot a) = (rr') \cdot a$ (associative and unital). Therefore, the algebra for the monad $R \otimes_{\mathbb{Z}} -$ on Ab is precisely an R-module.

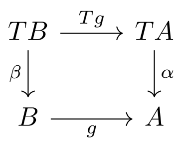
2.3 Monadicity

Definition 2.20. A functor $U\colon \mathcal{B} \to \mathcal{C}$ with left adjoint is *(premonadic) monadic* if the Eilenberg–Moore comparison functor is (fully faithful) an equivalence of categories.

This definition is precisely what monadic descent theory characterizes. Therefore, some authors might say U is of *descent type* if U is premonadic, and of *effective descent type* if U is monadic. First, note the simple result from its definition.

Proposition 2.21. [BW85, Chapter 3.3, Proposition 1] Any monadic functor reflects isomorphisms.

Proof. Since equivalences of categories reflect isomorphisms, it is sufficient to show that for any monad $\mathbb{T}$ on a category $\mathcal{C}$, the underlying functor $U^{\mathbb{T}}\colon \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ reflects isomorphisms. Suppose $f\colon (A, \alpha) \to (B, \beta)$ is such that $f\colon A \to B$ is an isomorphism in $\mathcal{C}$. Let $g = f^{-1}$, then we must show that the diagram



commutes. A diagram chase gives:

$$\alpha \circ Tg = g \circ f \circ \alpha \circ Tg = g \circ \beta \circ Tf \circ Tg = g \circ \beta \circ T(\mathrm{id}_B) = g \circ \beta. \qquad \square$$

To state the Barr–Beck theorem and its variations, we must recall some background information.

Definition 2.22. A *parallel pair* $f, g\colon A \rightrightarrows B$ in a category, is a pair of morphisms that share the same domain and codomain. The parallel pair (f, g) is said to be *contractible* or *split* if there is a morphism $t\colon B \to A$ such that $f \circ t = \mathrm{id}_B$, and $g \circ t \circ f = g \circ t \circ g$.

Definition 2.23. A coequalizer is *contractible* if it consists of morphisms and objects as in the diagram

$$\begin{array}{ccccc} A & \overset{f}{\underset{g}{\rightrightarrows}} & B & \xrightarrow{h} & C \\ & \xleftarrow[t]{} & & \xleftarrow[s]{} & \end{array}$$

for which
(i) $f \circ t = \mathrm{id}_B$,
(ii) $g \circ t = s \circ h$,
(iii) $h \circ s = \mathrm{id}_C$, and
(iv) $h \circ f = h \circ g$.

Eventually, we can state that any Eilenberg–Moore algebra is a coequalizer of a parallel pair which becomes contractible upon applying $U^{\mathbb{T}}$.

Proposition 2.24. [BW85, Chapter 3.3, Proposition 2]
(i) Any contractible coequalizer is a coequalizer.
(ii) A contractible coequalizer is an *absolute colimit*; that is, it is preserved by every functor.
(iii) If there is a coequalizer of a contractible parallel pair, then it is necessarily a contractible coequalizer.

Proof. (i) The induced map of the universal property is constructed by composing with the contraction, and its uniqueness follows since the coequalizer map is a split epimorphism.
(ii) Since a functor preserves composition and identities, and a coequalizer remains a coequalizer under the image of any functor, the statement follows.
(iii) By using the contractibility of the parallel pair and that the coequalizer map is an epimorphism, the claim follows immediately. □

In addition to being an absolute colimit, by a simple observation of retracts of contractible pairs, note the following corollary.

Corollary 2.25. [JT04, Corollary 1.3] Let $\mathcal{B}$ and $\mathcal{C}$ be any category, $\tau\colon \varphi \to \Psi$ a split epimorphism of functors $\varphi, \Psi\colon \mathcal{B} \to \mathcal{C}$, and (f, g) a pair of parallel morphisms in $\mathcal{B}$. Then if $(\varphi f, \varphi g)$ is contractible, then $(\Psi f, \Psi g)$ is contractible too.

Proof. Follows by using the commutative diagram

$$\begin{array}{ccc} \varphi A & \overset{\varphi f}{\underset{\varphi g}{\rightrightarrows}} & \varphi B \\ {\scriptstyle a}\uparrow\downarrow{\scriptstyle \tau(\varphi A)} & & {\scriptstyle b}\uparrow\downarrow{\scriptstyle \tau(\varphi B)} \\ \Psi A & \overset{\Psi f}{\underset{\Psi g}{\rightrightarrows}} & \Psi B. \end{array}$$

in $\mathcal{C}$. □

Definition 2.26. Consider a functor $U\colon \mathcal{B} \to \mathcal{C}$. A *$U$-contractible coequalizer pair* is a pair of morphisms $f, g\colon A \rightrightarrows B$ in $\mathcal{B}$ for which there exists a contractible coequalizer

$$\begin{array}{ccccc} UA & \overset{Uf}{\underset{Ug}{\rightrightarrows}} & UB & \xrightarrow{h} & C \\ & \xleftarrow[t]{} & & \xleftarrow[s]{} & \end{array}$$

in $\mathcal{C}$.

Proposition 2.27. [BW85, Chapter 3.3, Proposition 3] Let $U\colon \mathcal{B} \to \mathcal{C}$ be monadic, and $f, g\colon A \rightrightarrows B$ be a U-contractible coequalizer pair. Then the pair (f, g) has a coequalizer $h\colon B \to C$ in $\mathcal{B}$, and

$$UA \overset{Uf}{\underset{Ug}{\rightrightarrows}} UB \xrightarrow{Uh} UC$$

is a coequalizer in $\mathcal{C}$.

Proof. Let $\mathcal{B} = \mathcal{C}^{\mathbb{T}}$ for the induced $\mathbb{T} = (T, \eta, \mu)$, then consider a $U^{\mathbb{T}}$-contractible pair

$$(A, \alpha) \underset{g}{\overset{f}{\rightrightarrows}} (B, \beta)$$

in $\mathcal{C}^{\mathbb{T}}$ with contractible coequalizer

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{h}{\longrightarrow} C, \qquad t\colon B \to A, \quad s\colon C \to B$$

in $\mathcal{C}$. Also, consider the diagram

$$\begin{array}{ccccc}
T^2A & \underset{T^2g}{\overset{T^2f}{\rightrightarrows}} & T^2B & \overset{T^2h}{\twoheadrightarrow} & T^2C \\
{\scriptstyle T\alpha}\downdownarrows{\scriptstyle \mu_A} & & {\scriptstyle T\beta}\downdownarrows{\scriptstyle \mu_C} & & {\scriptstyle T\gamma}\downdownarrows{\scriptstyle \mu_{C''}} \\
TA & \underset{Tg}{\overset{Tf}{\rightrightarrows}} & TB & \overset{Th}{\twoheadrightarrow} & TC \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} & & {\scriptstyle \exists}\downarrow{\scriptstyle \gamma} \\
A & \underset{g}{\overset{f}{\rightrightarrows}} & B & \underset{h}{\twoheadrightarrow} & C
\end{array} \tag{2}$$

in $\mathcal{C}$. By Proposition 2.24(ii) all the rows of (2) are contractible coequalizers. The algebra structure maps α and β define the morphism γ; note that $h \circ \beta$ coequalizes the pair (Tf, Tg), thus, the universal property of coequalizers defines the map γ. The obvious commutative diagrams imply that (C, γ) is a $\mathbb{T}$-algebra by canceling the epimorphisms of the coequalizer maps. It follows by a similar argument that $h\colon (B, \beta) \to (C, \gamma)$ is a coequalizer of the pair $f, g\colon (A, \alpha) \rightrightarrows (B, \beta)$ in $\mathcal{C}^{\mathbb{T}}$. □

Remark 2.28. This proof has actually showed that any Eilenberg–Moore algebra is a coequalizer of a parallel pair which becomes contractible upon applying $U^{\mathbb{T}}$

Corollary 2.29. [BW85, Chapter 3.3, Proposition 4] Let $\mathbb{T}$ be a monad on a $\mathcal{C}$. Then for any (A, α) in $\mathcal{C}^{\mathbb{T}}$,

$$(T^2A, \mu_{TA}) \underset{TA}{\overset{\mu_A}{\rightrightarrows}} (TA, \mu_A) \tag{3}$$

is a U-contractible pair whose coequalizer in $\mathcal{C}^{\mathbb{T}}$ is (A, α).

Corollary 2.30. [BW85, Chapter 3.3, Corollary 5] Consider the adjunction

$$\mathcal{B} \underset{F}{\overset{U}{\leftrightarrows}} \mathcal{C}$$

where F is left adjoint to U, then for any object B in $\mathcal{B}$,

$$FUFUB \underset{FU\varepsilon_B}{\overset{\varepsilon_{FUB}}{\rightrightarrows}} FUB \tag{4}$$

is a U-contractible coequalizer pair.

These results have essentially brought about a connection between coequalizers, monadicity, and adjunctions. It not trivial that a functor preserves any coequalizer, although any functor preserves a contractible coequalizer. Yet, if the functor $U\colon \mathcal{B} \to \mathcal{C}$ is monadic, or even if there is an adjunction with left adjoint F, simply the existence of a U-contractible coequalizer pair implies there is a coequalizer in $\mathcal{B}$ (not necessarily contractible). In other words, there is a type of "lift" of coequalizers through U. Studying these lifts with the Eilenberg–Moore category and monadic functors will ultimately lead us to the Barr–Beck theorem.

The following definitions and lemmas precede the Barr–Beck theorem.

Definition 2.31. Let $f\colon A \to B$ be a morphism in a category.

(i) f is a *regular monomorphism* if it is the equalizer of some parallel pair of morphisms $B \rightrightarrows C$;

$$A \xrightarrow{f} B \rightrightarrows C$$

(ii) f is a *regular epimorphism* if it is the coequalizer of some parallel pair of morphisms $X \rightrightarrows A$;

$$X \rightrightarrows A \xrightarrow{f} B.$$

Lemma 2.32. [BW85, Chapter 3.3, Lemma 6 and Corollary 7] Consider the adjunction

$$\mathcal{B} \underset{U}{\overset{F}{\leftrightarrows}} \mathcal{C}$$

where F is left adjoint to U, and it induces a monad $\mathbb{T}$ on $\mathcal{C}$.

(i) The unit η_C is a regular monomorphism for all objects C of $\mathcal{C}$ if and only if the diagram

$$C \xrightarrow{\eta_C} TC \underset{T\eta_C}{\overset{\eta_{TC}}{\rightrightarrows}} T^2C \tag{5}$$

is an equalizer for every object C of $\mathcal{C}$.

(ii) Dually, the counit ε_B is a regular epimorphism for every object B of $\mathcal{B}$ if and only if the diagram

$$T^2B \underset{T\varepsilon_B}{\overset{\varepsilon_{TB}}{\rightrightarrows}} TB \xrightarrow{\varepsilon_B} B \tag{6}$$

is a coequalizer for every object B of $\mathcal{B}$.

Proof. We will prove (i). If (5) is an equalizer, then η_C is a regular monomorphism. Suppose that η_C is a regular monomorphism, then there is some equalizer

$$C \xrightarrow{\eta_C} TC \underset{g}{\overset{f}{\rightrightarrows}} Y$$

in $\mathcal{C}$. It is sufficient to check, by the universal property of an equalizer, that some morphism $w\colon X \to TC$ equalizes the pair (f, g) if and only if it equalizes the pair $(\eta_{TC}, T\eta_C)$. The "if" direction follows again by the universal property of the equalizer. In the "only if" direction, first note that the contractibility of the pair $(T\eta_{TC}, T^2\eta_C)$ implies $Tw\colon TX \to TC$ equalizes $(T\eta_{TC}, T^2\eta_C)$. Then by the naturality of η, the following diagram

$$\begin{array}{ccccc}
X & \xrightarrow{w} & TC & \underset{g}{\overset{f}{\rightrightarrows}} & Y \\
{\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_{TC}} & & \downarrow{\scriptstyle \eta_Y} \\
TX & \xrightarrow[Tw]{} & T^2C & \underset{Tg}{\overset{Tf}{\rightrightarrows}} & TY
\end{array}$$

commutes in $\mathcal{C}$. In particular, $f \circ w = g \circ w$ because η_Y is a (regular) monomorphism. □

Lemma 2.33. [BW85, Chapter 3.3, Lemma 8] For all objects B and $B^{'}$ of $\mathcal{B}$, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(FUB, B^{'}) \cong \mathrm{Hom}_{\mathcal{C}^{\mathbb{T}}}(\Phi FUB, \Phi B^{'})$$

of Hom-sets, where $\Phi\colon \mathcal{B} \to \mathcal{C}^{\mathbb{T}}$ is the comparison functor.

Proof. The statement follows the chain of isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{B}}(FUB, B^{'}) &\cong \mathrm{Hom}_{\mathcal{C}}(UB, UB^{'}) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(UB, U^{\mathbb{T}}(UB^{'}, U\varepsilon_{B'})) \\
&\cong \mathrm{Hom}_{\mathcal{C}^{\mathbb{T}}}(F^{\mathbb{T}}(UB), (UB^{'}, U\varepsilon_{B'})) \\
&\cong \mathrm{Hom}_{\mathcal{C}^{\mathbb{T}}}((TUB, \mu_{UB}), (UB^{'}, U\varepsilon_{B'})) \\
&\cong \mathrm{Hom}_{\mathcal{C}^{\mathbb{T}}}((\Phi FUB, \Phi B^{'})
\end{aligned}$$

□

In other words, this lemma says that the comparison functor Φ is full and faithful on morphisms from free objects. The next result by Beck follows this observation.

Theorem 2.34. [BW85, Chapter 3.3, Theorem 9] The comparison functor $\Phi\colon \mathcal{B} \to \mathcal{C}^{\mathbb{T}}$ is full and faithful if and only if ε_B is a regular epimorphism for all objects B of $\mathcal{B}$.

Equivalently, the right adjoint $U\colon \mathcal{B} \to \mathcal{C}$ is premonadic if and only if the counit ε_B of the adjunction is a regular epimorphism for all objects B in $\mathcal{B}$.

Proof. Since $U\varepsilon_B$ is a structure map of the algebra $(UB, U\varepsilon_B)$ in $\mathcal{C}^{\mathbb{T}}$. Further, by Corollary 2.29, $U\varepsilon_B$ is the coequalizer of parallel pair with domain in the image of $F^{\mathbb{T}}$, hence, in the image of Φ. Since $\Phi(\varepsilon_B) = U\varepsilon_B$, if Φ is full and faithful, then ε_B is a regular epimorphism (or rather the coequalizer of the underlying parallel pair).

Conversely, suppose ε_B is a regular epimorphism. If $f, g\colon B \rightrightarrows B'$ is a parallel pair in $\mathcal{B}$, and $Uf = Ug$, then $FUf = FUg$ and the diagram

$$\begin{array}{ccc} FUB & \overset{FUf}{\underset{FUg}{\rightrightarrows}} & FUB' \\ {\scriptstyle \varepsilon_B}\downarrow & & \downarrow{\scriptstyle \varepsilon_{B'}} \\ B & \overset{f}{\underset{g}{\rightrightarrows}} & B' \end{array}$$

commutes in $\mathcal{B}$. Since ε_B is an epimorphism, then $f = g$, and, hence, Φ is faithful. Moreover, by Corollary 2.30 and Lemma 2.32, (6) is a U-contractible coequalizer diagram. Since $U^{\mathbb{T}} \circ \Phi = U$, the image of (6) under Φ is a $U^{\mathbb{T}}$-contractible coequalizer diagram. Therefore, the following diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{B}}(B, B') & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(FUB, B') & \rightrightarrows & \mathrm{Hom}_{\mathcal{B}}(FUFUB, B') \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}^{\mathbb{T}}}(\Phi B, \Phi B') & \longrightarrow & \mathrm{Hom}_{\mathcal{C}^{\mathbb{T}}}(\Phi FUB, \Phi B') & \rightrightarrows & \mathrm{Hom}_{\mathcal{C}^{\mathbb{T}}}(\Phi FUFUB, \Phi B') \end{array}$$

commutes (in Set). The rows are equalizers of Hom-sets that are computed in their denoted categories, and the vertical morphisms are those induced by the functor Φ. In particular, by Lemma 2.33, the middle and the right vertical morphisms are isomorphisms. Thus, the left vertical morphism is an isomorphism too. Hence, Φ is faithful. □

2.4 The Barr–Beck theorem and variations

The Barr–Beck theorem is a powerful tool and it gives a precise criterion for monadicity. It applies naturally to contexts of monoidal categories acting on another category, but as will be shown in the next section, it is widely applicable in the cases of adjunctions.

Theorem 2.35. [BW85, Chapter 3.3, Theorem 10] The functor $U\colon \mathcal{B} \to \mathcal{C}$ is monadic if and only if

(i) U has a left adjoint F;
(ii) U reflects isomorphisms;
(iii) $\mathcal{B}$ has all coequalizers of U-contractible coequalizer pairs and U preserves them.

Proof. By Proposition 2.21 and Proposition 2.27 the "if" direction follows directly.

In the "only if" direction, we know that (4) is a U-contractible pair. Then by (iii) it has a coequalizer B'. Moreover, since ε is a natural transformation, ε_B coequalizes (4). Therefore there is a morphism $f\colon B' \to B$ making the diagram

$$\begin{array}{ccccc} FUFUB & \overset{\varepsilon_{FUB}}{\underset{FU\varepsilon_B}{\rightrightarrows}} & FUB & \longrightarrow & B' \\ & & & {\scriptstyle \varepsilon_B}\searrow & \downarrow{\scriptstyle f} \\ & & & & B \end{array}$$

commute in $\mathcal{B}$. Since $U\varepsilon_B$ is a coequalizer of the image of (4) under U, Uf is an isomorphism. Thus, f is an isomorphism, and ε_B is a regular epimorphism such that Φ is premonadic by Theorem 2.34.

Next, for any algebra (C, γ) in $\mathcal{C}^{\mathbb{T}}$, we must find an object B in $\mathcal{B}$ such that $\Phi(B) \cong (C, \gamma)$. Note that the image of the diagram

$$FUFC \overset{F\gamma}{\underset{\varepsilon_{FC}}{\rightrightarrows}} FC$$

under Φ is (3); that is, it is a U-contractible coequalizer. Therefore, by assumption, there is a coequalizer B of the pair $(F\gamma, \varepsilon_{FC})$ such that the diagram

$$UFUFC \overset{UF\gamma}{\underset{U\varepsilon_{FC}}{\rightrightarrows}} UFC \longrightarrow UB$$

is a coequalizer. By Corollary 2.29, this last diagram is the image under $U^{\mathbb{T}}$ of a coequalizer diagram in $\mathcal{C}^{\mathbb{T}}$ with coequalizer (C, γ). Since $U^{\mathbb{T}}$ reflects coequalizers, and $U^{\mathbb{T}} \circ \Phi = U$, we conclude that $\Phi(B) \cong (C, \gamma)$. □

There is an important version of the Barr–Beck theorem: the split monadicity theorem. We will generalize it for our purposes in the section on descent for rings and algebras.

Theorem 2.36. [JT04, Theorem 2.2] The functor $U\colon \mathcal{B} \to \mathcal{C}$ is monadic if and only if
(i) U has a left adjoint F;
(ii) the counit of the adjunction $FU \to \mathrm{id}_A$ is a split epimorphism.

Proof. If the counit $\varepsilon\colon FU \to \mathrm{id}_A$ is a split epimorphism, then condition Theorem 2.35(iii) is trivial. Suppose the pair (Uf, Ug) is contractible, then so is (FUf, FUg), and then by Corollary 2.25 applied to ε, (f, g) is contractible too. Therefore, every functor preserves the coequalizer of the pair (f, g).

Additionally, Theorem 2.35(ii) becomes trivial because having $\zeta\colon \mathrm{id}_A \to FU$ with $\varepsilon\zeta = \mathrm{id}_A$ gives that if $U(f\colon A \to B)$ is invertible, then f is invertible with $f^{-1} = \varepsilon_A F(Uf)^{-1}\zeta_B$. □

Remark 2.37. The split monadicity theorem is actually derived from a more general theorem by Paré. However, we require a different type of generalization; suppose there is an additional functor H such that $H\varepsilon\colon HFU \to H$ is a split epimorphism. Then, we only require (Hf, Hg) to be contractible, and to conclude that U preserves the coequalizer of (f, g), there must be a "connection" between H and U. The following theorem formalizes this connection.

Theorem 2.38. [JT04, Theorem 2.3] A functor $U\colon \mathcal{B} \to \mathcal{C}$ is monadic if and only if
(i) U has a left adjoint F;
(ii) U reflects isomorphisms;
(iii) there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{U} & \mathcal{C} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H'} \\ \mathcal{X} & \xrightarrow[U']{} & \mathcal{Y} \end{array} \tag{7}$$

of functors such that;
(a) $H\varepsilon\colon HFU \to H$ is a split epimorphism;
(b) $\mathcal{B}$ has all coequalizers of U-contractible coequalizer pairs and H preserves them;
(c) H' reflects isomorphisms.

Proof. By Theorem 2.35 it suffices to check condition Theorem 2.35(iii) on U. Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C \tag{8}$$

be the coequalizer diagram of (f, g), and assume (Uf, Ug) is contractible. By (a) the image of (8) under H is a coequalizer diagram. Moreover, by Corollary 2.25 applied to $H\varepsilon$ with (Hf, Hg) as a retract of $(HFUf, HFUg)$, (Hf, Hg) is contractible and therefore a contractible coequalizer diagram. Since contractible coequalizer diagrams are absolute colimits, it must be that the image of (8) under $U'H$ is a coequalizer diagram. Additionally, since $U'H = H'U$, the same is true for the image under $H'U$. Next, consider the coequalizer diagram

$$UA \underset{Ug}{\overset{Uf}{\rightrightarrows}} UB \xrightarrow{p} Z \tag{9}$$

of (Uf, Ug). Since (Uf, Ug) is contractible, the image of (9) under H' is a coequalizer diagram. Moreover, since the image of (8) under $H'U$ is a coequalizer diagram, the image of the universal canonical morphism $H'\delta\colon Z \to UC$ making the diagram

$$\begin{array}{ccccc} H'UA & \underset{H'Ug}{\overset{H'Uf}{\rightrightarrows}} & H'UB & \xrightarrow{H'p} & H'Z \\ & & \downarrow{\scriptstyle H'h} & \swarrow{\scriptstyle H'\delta} & \\ & & H'UC & & \end{array}$$

commute is an isomorphism. Then since H' reflects isomorphisms, the image of (8) under U is a coequalizer diagram, as desired.

In the other direction, choose $H = U$ and $H' = U' = \mathrm{id}_{\mathcal{C}}$ and apply Theorem 2.35. □

Remark 2.39. From the "only if" direction of the proof, Theorem 2.38 contains Theorem 2.35. Additionally, it contains Theorem 2.36 by using $H = \mathrm{id}_{\mathcal{B}}$, $H^{'} = \mathrm{id}_{\mathcal{C}}$ and $U^{'} = U$. Finally, Theorem 2.38 could equivalently be considered as a corollary since conditions Theorem 2.38(i) and (iii) imply the condition Theorem 2.35(iii).

There is also a similar lemma to that of Theorem 2.38.

Lemma 2.40. [JT04, Lemma 2.5] A functor $U\colon \mathcal{B} \to \mathcal{C}$ is monadic if and only if
(i) U has a left adjoint F;
(ii) there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{U} & \mathcal{C} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H^{'}} \\ \mathcal{X} & \xrightarrow[U^{'}]{} & \mathcal{Y} \end{array}$$

of functors such that;
(a) $U^{'}$ is monadic;
(b) H preserves all coequalizers;
(c) $H^{'}$ reflects isomorphisms.

Proof. The proof is similar to Theorem 2.38. □

3 Descent Theory

These subsections will follow G. Janelidze and W. Tholen [JT94]. The first section considers the general question of descent in any category $\mathcal{C}$ and simultaneously treats the case of topological descent theory via Grothendieck's idea of descent. In fact, the story is better with topological intuition as its framework. The second subsection concerns itself with (Grothendieck) fibrations and how monadic descent theory survives the abstraction to bifibered categories. Indeed, the authors of [JT94] write that it is surprising that all the definitions survive the abstraction.

3.1 Monadic descent theory

Assume for the remainder of this section that the conditions of the following construction are true. Let $\mathcal{C}$ be a category with pullbacks, and let $\mathbb{E}$ be a class of morphisms in $\mathcal{C}$ closed under composition with isomorphisms. For an object B in $\mathcal{C}$, consider the full subcategory $\mathbb{E}(B)$ of the slice category $\mathcal{C}/B$ with objects in $\mathbb{E}$. The general aim of descent theory is to study the objects and morphisms of the category $\mathbb{E}(B)$ in terms of objects in the category $\mathbb{E}(E)$ which comes equipped with additional algebraic structure, so-called, *descent data*.

Example 3.1. If $\mathcal{C} = \mathsf{Top}$, then the fixed topological space B is the "base" space, and $\mathbb{E}$ is a class of continuous functions that are closed under composition with homeomorphisms. The full subcategory $\mathbb{E}(B)$ of Top/B is the category of $\mathbb{E}$-bundles over B. Moreover, E is the "extension" space of B. In other words, topological descent theory poses the following question:

Question 3.2. How and when can the $\mathbb{E}$-bundles over B be described by the $\mathbb{E}$-bundles over E?

Assume for the remainder of this section that $\mathbb{E}$ is stable under pullback along $p\colon E \to B$ (see Remark 3.9), and for any object (C, γ) in $\mathbb{E}(E)$ let

$$\begin{array}{ccc} E \times_B C & \xrightarrow{\mathrm{pr}_2} & C \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle p\circ\gamma} \\ E & \xrightarrow[p]{} & B \end{array} \tag{10}$$

be the pullback in $\mathcal{C}$ of $p \circ \gamma$ along p. (Stability means precisely that if $p \circ \gamma$ is in $\mathbb{E}$, then pr_1 is in $\mathbb{E}$ too.)

Example 3.3. If $\mathcal{C} = \mathsf{Top}$, let $p\colon E \to B$ be a continuous map, and let (C, γ) be an $\mathbb{E}$-bundle over E, then the fiber product is analogous to the pullback (10). Explicitly, the fiber product is given by

$$E \times_B C = \{(x, c) \colon p(x) = p \circ \gamma(c)\},$$

and it is considered a subspace of the topological product $E \times C$. Furthermore, there is additional structure related to the fiber product: for all points $x, y \in E$ with $p(x) = p(y)$ there is a *canonical embedding*

$$i_{x,y} \colon \gamma^{-1} \to E \times_B C, \quad c \mapsto (x, c),$$

where $E \times_B C$ is considered the join of the fibers $\gamma^{-1}y$ or rather the union of the subspaces $i_{x,y}(\gamma^{-1}y)$.

Definition 3.4. The category

$$\mathsf{Des}_{\mathbb{E}}(p)$$

of *descent data (relative to a morphism $p \colon E \to B$)* consists of:

(i) objects that are triples (C, γ, ξ), where (C, γ) is an object in $\mathbb{E}(E)$ and $\xi \colon E \times_B C \to C$ is a morphism in $\mathcal{C}$ such that the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\langle \gamma, \mathrm{id}_C \rangle} & E \times_B C \\ {\scriptstyle \mathrm{id}_C}\downarrow & \swarrow{\scriptstyle \xi} & \downarrow{\scriptstyle \mathrm{pr}_1} \\ C & \xrightarrow[\gamma]{} & E \end{array} \tag{11}$$

$$\begin{array}{ccc} E \times_B (E \times_B C) & \xrightarrow{\mathrm{id}_E \times_B \xi} & E \times_B C \\ {\scriptstyle \mathrm{id}_E \times_B \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle \xi} \\ E \times_B C & \xrightarrow[\xi]{} & C \end{array} \tag{12}$$

commute;

(ii) morphisms $h \colon (C, \gamma, \xi) \to (C', \gamma', \xi')$ are morphisms $h \colon (C, \gamma) \to (C', \gamma')$ in $\mathbb{E}(E)$ that are *compatible with the descent data*; that is, such that the diagram

$$\begin{array}{ccc} E \times_B C & \xrightarrow{\mathrm{id}_E \times_B h} & E \times_B C' \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \xi'} \\ C & \xrightarrow[h]{} & C' \end{array} \tag{13}$$

commutes.

Example 3.5. If $\mathcal{C} = \mathsf{Top}$, then the descent data for an $\mathbb{E}$-bundle (C, γ) over E (relative to the continuous map $p \colon E \to B$) is given by a family of continuous maps

$$\xi_{x,y} \colon \gamma^{-1}x \to \gamma^{-1}y$$

for points $x, y \in E$ with $p(x) = p(y)$ such that the following conditions hold:

(i) $xi_{x,x} = \mathrm{id}_{\gamma^{-1}x}$ for each $x \in E$,

(ii) $\xi_{x,z} = \xi_{y,z} \circ \xi_{x,y}$ for each $x, y, z \in E$ with $p(x) = p(y) = p(z)$

(iii) the unique map $\overline{\xi} \colon E \times_B C \to E \times_B C$, which makes all diagrams

$$\begin{array}{ccc} \gamma^{-1}x & \xrightarrow{\xi_{x,y}} & \gamma^{-1}y \\ {\scriptstyle i_{y,x}}\downarrow & & \downarrow{\scriptstyle i_{x,y}} \\ E \times_B C & \xrightarrow[\overline{\xi}]{} & E \times_B C \end{array}$$

commute, is continuous. Explicitly,

$$\overline{\xi}(y, c) = (x, \xi_{x,y}(c))$$

with $x = \gamma(c)$.

Conditions (i) and (ii) represent *functoriality* and (iii) represent *gluing*. Note that $\xi_{y,x} \circ \xi_{x,y} = \mathrm{id}_{\gamma^{-1}x}$, thus, $\xi_{x,y}$ is a homeomorphism. Thus, $\overline{\xi}^{-1} = \overline{\xi}$ and $\overline{\xi}$ is a homeomorphism too. Usually,

$$\overline{\xi} \colon E \times_B C \to E \times_B C$$

is referred to as descent data for a space (C, γ) over E.

Remark 3.6. There is nothing pathological when discussing $\overline{\xi}$ as the descent data, although it is defined as an algebra structure $\xi\colon E \times_B C \to C$ in the general sense of the category $\mathsf{Des}_{\mathbb{E}}(p)$ of descent data. As in [JT94], it is possible to explicitly define a bijective correspondence $(\xi \leftrightarrow \overline{\xi})$ between descent data ξ (as given an algebra structure) and the descent data $\overline{\xi}$ (as above).

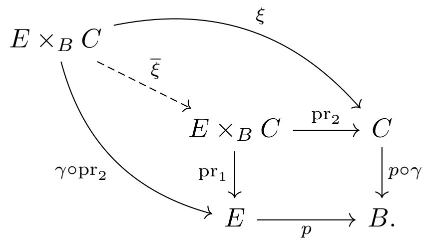
If we are given descent data $\overline{\xi}\colon E \times_B C \to E \times_B C$, define

$$\xi := \mathrm{pr}_2 \circ \overline{\xi}\colon E \times_B C \to C$$

to obtain descent data in terms of algebra structure. In the other direction, if there is descent data $\xi\colon E \times_B C \to C$, define

$$\overline{\xi} := \langle \gamma \circ \mathrm{pr}_2, \xi \rangle\colon E \times_B C \to E \times_B C,$$

as the morphism induced by the pair $(\gamma \circ \mathrm{pr}_2, \xi)$ in the pullback diagram



In particular, observe that $\mathrm{pr}_1 \circ \overline{\xi} \circ \overline{\xi} = \mathrm{pr}_1$, and with (12), $\mathrm{pr}_2 \circ \overline{\xi} \circ \overline{\xi} = \mathrm{pr}_2$. Thus, $\overline{\xi} \circ \overline{\xi} = \mathrm{id}_{E \times_B C}$, and $\overline{\xi}$ is an involution.

Example 3.7. Let $(U_i)_{i \in I}$ be an open cover of a base space B. Consider the induced map $p\colon E \to B$ of the topological sum $E = \coprod_{i \in I} U_i$ that is the identity map on each summand so that $E = \{(b, i)\colon b \in U_i\}$ with $p(b, i) = b$. Then, descent data for a space (C, γ) over E is given by maps

$$\xi_{(b,i),(b,j)}\colon \gamma^{-1}(b, i) \to \gamma^{-1}(b, j)$$

for $b \in U_i \cap U_j$. Gluing ξ along b give maps,

$$\xi_{i,j}\colon \gamma_i^{-1}(U_i \cap U_j) \to \gamma_j^{-1}(U_i \cap U_j)$$

for $i, j \in I$, and $\gamma_i\colon \gamma^{-1}(U_i) \to U_i$, is the restriction of γ with U_i considered a subspace of E.

Example 3.8. If $\mathcal{C} = \mathsf{Top}$, then $\mathbb{E}$-bundles over E are equipped with descent data for the objects $(C, \gamma, \overline{\xi})$ of the category $\mathsf{Des}_{\mathbb{E}}(p)$. A morphism $h\colon (C, \gamma, \overline{\xi}) \to (C', \gamma', \overline{\xi}')$ in $\mathsf{Des}_{\mathbb{E}}(p)$ is a morphism $h\colon (C, \gamma) \to (C', \gamma')$ of $\mathbb{E}$-bundles over E such that it is compatible with descent data:

$$h(\xi_{x,y}(c)) = \xi'_{x,y}(h(c)) \tag{14}$$

for each $x, y \in E$ with $p(x) = p(y)$ and $c \in \gamma^{-1}x$. Condition (14) is equivalent to

$$(\mathrm{id}_E \times_B h) \circ \overline{\xi} = \overline{\xi} \circ (\mathrm{id}_E \times_B h).$$

Remark 3.9. The central construction within these examples and definitions is the pullback. Indeed, note that the pullback of any object (A, α) in $\mathbb{E}(B)$

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\mathrm{pr}_2} & A \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B \end{array} \tag{15}$$

induces the object $(E \times_B A, \mathrm{pr}_1)$ in $\mathbb{E}(E)$ precisely if $\mathbb{E}$ is stable under pullback along p. Indeed assuming that $\mathbb{E}$ is stable under pullback will generalize this operation.

Definition 3.10. The functor

$$p^*\colon \mathbb{E}(B) \to \mathbb{E}(E), \quad (A, \alpha) \mapsto (E \times_B A, \mathrm{pr}_1)$$

(with respect to a morphism $p\colon E \to B$) is called the *pullback functor*. Moreover, it sends a morphism f to $\mathrm{id}_E \times_B f$.

Note 3.11. The object $p^*(A, \alpha)$ comes equipped with *canonical descent data*:

$$\mathrm{id}_E \times_B \mathrm{pr}_2 \colon E \times_B (E \times_B A) \to E \times_B A.$$

Thus, we can lift the pullback functor as follows.

Definition 3.12. The functor

$$\Phi^p_{\mathbb{E}} \colon \mathbb{E}(B) \to \mathsf{Des}_{\mathbb{E}}(p), \quad (A, \alpha) \mapsto (E \times_B A, \mathrm{pr}_1, \mathrm{id}_E \times_B \mathrm{pr}_2) \tag{16}$$

such that the diagram

$$\begin{array}{ccc} \mathbb{E}(B) & \xrightarrow{\Phi^p_{\mathbb{E}}} & \mathsf{Des}_{\mathbb{E}}(p) \\ & {\scriptstyle p^*}\searrow \quad \swarrow{\scriptstyle U^p} & \\ & \mathbb{E}(E) & \end{array}$$

commutes, where U^p is the obvious forgetful functor. The functor $\Phi^p_{\mathbb{E}}$ is called the *comparison functor*.

Example 3.13. If $\mathcal{C} = \mathsf{Top}$, then the $\mathbb{E}$-bundle $p^*(A, \alpha)$ (over E) comes equipped with canonical descent data:

$$\varphi_{x,y} \colon \mathrm{pr}_1^{-1} x \to \mathrm{pr}_1^{-1} y$$

for $x, y \in E$. Thus, $\overline{\varphi} \colon E \times_B (E \times_B A) \to E \times_B (E \times_B A)$ is the involution $(y, (x, a)) \mapsto (x, (y, a))$.

Hence, the comparison functor

$$\Phi^p_{\mathbb{E}} \colon \mathbb{E}(B) \to \mathsf{Des}_{\mathbb{E}}(p)$$

is given by $(A, \alpha) \mapsto (E \times_B A, \mathrm{pr}_1, \overline{\varphi})$.

The comparison functor allows us to characterize the descent question as follows.

Definition 3.14. Let $\mathcal{C}$ be a category with pullback and let $p \colon E \to B$ be a morphism in $\mathcal{C}$. Let $\mathbb{E}$ be a class of morphisms in $\mathcal{C}$ closed under composition with isomorphisms and stable under pullback along p. The morphism p is said to be $\mathbb{E}$-*descent* if $\Phi^p_{\mathbb{E}}$ is full and faithful, and it is an *effective* $\mathbb{E}$-*descent morphism* if $\Phi^p_{\mathbb{E}}$ is an equivalence of categories.

This definition says that a morphism $p \colon E \to B$ is $\mathbb{E}$-descent if any morphism $f \colon (A, \alpha) \to (A', \alpha')$ is completely described by morphisms

$$h \colon (E \times_B A, \mathrm{pr}_1, \mathrm{id}_E \times_B \mathrm{pr}_2) \to (E \times_B A', \mathrm{pr}_1', \mathrm{id}_E \times_B \mathrm{pr}_2')$$

such that $h = \mathrm{id}_E \times_B f$ (compatible with descent data). A morphism $p \colon E \to B$ is effective $\mathbb{E}$-descent if, in addition, up to isomorphism, objects (C, γ, ξ) in $\mathsf{Des}_{\mathbb{E}}(p)$ are of the form $(E \times_B A, \mathrm{pr}_1, \mathrm{id}_E \times_B \mathrm{pr}_2)$.

However, there is a caveat to this definition. The fundamental application of monads is only available if there exists a left adjoint to p^*. It follows that the adjunction induces a monad on the category $\mathbb{E}(E)$, and the Barr–Beck theorem will be applicable. The following condition will ensure the existence of such a left adjoint.

Lemma 3.15. If the class $\mathbb{E}$ is *stable under composition with* p *from the left*; that is, if $\gamma \in \mathbb{E}$, then $p \circ \gamma \in \mathbb{E}$. Then the functor $p^* \colon \mathbb{E}(B) \to \mathbb{E}(E)$ has a left adjoint

$$p_! \colon \mathbb{E}(E) \to \mathbb{E}(B), \quad (C, \gamma) \mapsto (C, p \circ \gamma).$$

Therefore, assume that the class $\mathbb{E}$ is stable under composition with p from the left. It follows that the pair of adjoint functors $p_! \dashv p^*$ induces a monad $\mathbb{T}^p$ on $\mathbb{E}(E)$, and the Eilenberg–Moore category $\mathbb{E}(E)^{\mathbb{T}^p}$ is exactly by construction the category $\mathsf{Des}_{\mathbb{E}}(p)$ of descent data. Hence, if the reader did not already guess it, the following proposition follows under this assumption.

Proposition 3.16. If $\mathbb{E}$ is closed under composition with p from the left, then $\mathsf{Des}_{\mathbb{E}}(p)$ is exactly the Eilenberg–Moore category of the monad induced by the adjunction $p_! \dashv p^*$, and p is an (effective) $\mathbb{E}$-descent morphism if and only if p^* is premonadic (monadic).

Remark 3.17. If the assumption that $\mathbb{E}$ is closed under composition with p from the left is removed, Proposition 3.16 might fail. That is, there are classes $\mathbb{E}$ of morphisms in $\mathcal{C}$ such that p is not effective for $\mathbb{E}$-descent, although the functor p^* is monadic. Note the following example.

If $\mathcal{C} = \mathsf{Top}$, consider the class $\mathbb{E}_c$ of closed-subspace embeddings in Top. Let X be a set with two points, let E be the space with the discrete topology on X, let B be the space with the indiscrete topology, and let $p\colon E \to B$ be the identity map. Note that up to categorial equivalence, $\mathbb{E}_c(E)$ is the partially ordered powerset of X, and $\mathbb{E}_c(B)$ is the 2-element chain. By inspection, the category $\mathsf{Des}_{\mathbb{E}}(p)$ is equivalent to $\mathbb{E}_c(E)$, and hence, not equivalent to $\mathbb{E}_c(B)$, despite p^* being monadic.

This proposition gives many opportunities to exploit the various variations of the Barr–Beck theorem. The first step is to determine which coequalizers must exist in the category $\mathcal{C}$ as this is an often overlooked assumption of the Barr–Beck theorem. Thus, we give the following definition:

Definition 3.18. Denote by $\mathbb{E}^*(p)$ the class of all morphisms which are pullbacks of p along a morphism in $\mathbb{E}$; that is all morphism pr_2 of every pullback diagram (15) with $\alpha \in \mathbb{E}$, and the composites with isomorphisms. The morphism p is an $\mathbb{E}$-*universal regular epimorphism* if the class of regular epimorphisms contains $\mathbb{E}^*(p)$.

Therefore, assume, in addition, that $\mathcal{C}$ has coequalizers of parallel pairs of morphisms in $\mathbb{E}^*(p)$.

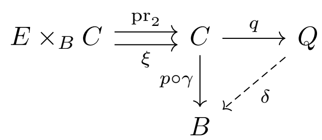
Proposition 3.19. The morphism p is an $\mathbb{E}$-descent morphism of $\mathcal{C}$ if and only if p is an $\mathbb{E}$-universal regular epimorphism of $\mathcal{C}$. The $\mathbb{E}$-descent morphism p is effective, if $\mathbb{E}$ is right cancellable with respect to those regular epimorphisms of $\mathcal{C}$ which are coequalizers of the morphisms in $\mathbb{E}^*(p)$ over B, and if these coequalizers are stable under pullback along p.

Proof. By Theorem 2.34, the the functor Φ^p is full and faithful if and only if the counits of the adjunction $p_! \dashv p^*$ are regular epimorphisms in $\mathbb{E}(B)$. Yet, the counits are given by the projections pr_2 in a pullback diagram (15) with $\alpha \in \mathbb{E}$. This proves the first assertion (with Remark 3.20 in mind).

For the second assertion, for every (C, γ, ξ) in $\mathsf{Des}_{\mathbb{E}}(p)$ construct the coequalizer

$$E \times_B C \underset{\xi}{\overset{\mathrm{pr}_2}{\rightrightarrows}} C \xrightarrow{\;q\;} Q. \tag{17}$$

Also note that both pr_2 and $\xi = \mathrm{pr}_2 \circ \overline{\xi}$ belong to $\mathbb{E}^*(p)$. By the universal property of the coequalizer, there exists a unique morphism $\delta\colon Q \to B$



such that $\delta \circ q = p \circ \gamma$. Thus, by the right cancellability of $\mathbb{E}$, δ lives in $\mathbb{E}(B)$. Therefore, define

$$\Psi^p_{\mathbb{E}}\colon \mathsf{Des}_{\mathbb{E}}(p) \to \mathbb{E}(B), \quad (C, \gamma, \xi) \mapsto (Q, \delta)$$

as the left adjoint to $\Phi^p_{\mathbb{E}}$. Finally, by the Barr–Beck theorem, the unit of the adjunction $\Psi^p_{\mathbb{E}} \dashv \Phi^p_{\mathbb{E}}$ is an isomorphism if and only if p^* preserves the coequalizer (17). □

Remark 3.20. There is a nontrivial technical detail omitted in this proof. Specifically, we have not shown that the treatment of the regular epimorphisms in $\mathbb{E}(B)$ are the same as those in $\mathcal{C}$. This statement follows [JT94, Theorem 3.6], but we will not investigate descent theory with respect to fibrations in such detail.

Note 3.21. Further characterizations could be made for $\mathcal{C} = \mathsf{Top}$. By constructing a left adjoint similarly to the proof of Proposition 3.19, it is possible to show that the class of regular epimorphisms coincides with the class of quotient maps [JT94, Corollary 1.8]. However, characterizing topological descent maps has served its purpose. Yet, the key theorems of descent theory of topological spaces are [JT94, Proposition 1.6, and Theorem 1.10]. Moreover, for those interested, W. Tholen with J. Reiterman [RT94] wrote up a complete characterization of effective descent of topological spaces.

In the literature, if $\mathbb{E}$ is the class of all morphisms in $\mathcal{C}$, then one speaks of *(effective) global-descent*. Indeed, in this case, it is trivial that the class closed under composition with p from the left. Therefore, Proposition 3.19 has the following corollary:

Corollary 3.22. A morphism p is a global-descent morphism if and only if p is a universal regular epimorphism. A global-descent morphism $p\colon E \to B$ is an effective global-descent morphism if the coequalizer of every parallel pair of universal regular epimorphisms over B exists and is stable under pullback along p.

Remark 3.23. The preservation of coequalizers is guaranteed if p^* has a *right* adjoint.

This corollary says that if our category with pullbacks and coequalizers is locally cartesian closed, then the effective descent morphisms are exactly the (necessarily universal) regular epimorphisms. In particular, it allows us to develop a workaround to study (effective) descent morphisms in an arbitrary category (that may not satisfy all the stated conditions).

Consider two classes $\mathbb{E}_0$ and $\mathbb{E}_1$ of morphisms in a category with pullbacks, both stable under pullback along $p\colon E \to B$ and under composition with isomorphism. Assume that $E_0 \subseteq E_1$.

Proposition 3.24. $\mathbb{E}_1$-descent for p implies $\mathbb{E}_0$-descent for p. The effective $\mathbb{E}_1$-descent morphism p is an effective $\mathbb{E}_0$-descent morphism if and only if the following conditions hold: for every pullback diagram

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\mathrm{pr}_2} & A \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B, \end{array}$$

$\mathrm{pr}_1 \in \mathbb{E}_0$ and $\alpha \in \mathbb{E}_1$ implies $\alpha \in \mathbb{E}_0$.

Proof. The first assertion follows that $\Phi^p_{\mathbb{E}_0} : \mathbb{E}_0(B) \to \mathsf{Des}_{\mathbb{E}_0}(p)$ is just the restriction of $\Phi^p_{\mathbb{E}_1}$.

For the other assertion, let p be an effective $\mathbb{E}_1$-descent morphism, then it suffices to show that $\Phi^p_{\mathbb{E}_0}$ is an equivalence of categories. By the first assertion, the functor $\Phi^p_{\mathbb{E}_0}$ is full and faithful. Since p is effective, for every object (C, γ, ξ) in $\mathsf{Des}_{\mathbb{E}_0}(p)$, there is an object (A, α) in $\mathbb{E}_1(B)$ such that $\Phi^p_{\mathbb{E}_1}(A, \alpha) \cong (C, \gamma, \xi)$. Hence, $p^*(A, \alpha) \cong (C, \gamma)$ in $\mathbb{E}_0(E)$ implies (A, α) in $\mathbb{E}_0(B)$, and $\Phi^p_{\mathbb{E}_0}(A, \alpha) \cong (C, \gamma, \xi)$.

In the other direction, $\Phi^p_{\mathbb{E}_0}$ must be an equivalence of categories. If from in the pullback diagram, $\mathrm{pr}_1 \in \mathbb{E}_0$ with $(A, \alpha) \in \mathbb{E}_1(B)$, then $\Phi^p_{\mathbb{E}_1}(A, \alpha)$ is in $\mathsf{Des}_{\mathbb{E}_0}(p)$. Hence, there is an object (A', α') in $\mathbb{E}_0(B)$ such that

$$\Phi^p_{\mathbb{E}_1}(A, \alpha) \cong \Phi^p_{\mathbb{E}_0}(A', \alpha') \cong \Phi^p_{\mathbb{E}_1}(A', \alpha'),$$

and it follows $(A, \alpha) \cong (A', \alpha')$ in $\mathbb{E}_0(B)$. □

This proposition gives us direct access for the next two cases:

Corollary 3.25. (i) For a $\mathcal{C}$ with pullbacks and $\mathbb{E}$ stable under pullback along the effective global-descent morphism p of $\mathcal{C}$, p is an effective $\mathbb{E}$-descent morphism if and only if in every pullback square

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\mathrm{pr}_2} & A \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B, \end{array}$$

$\mathrm{pr}_1 \in \mathbb{E}$ implies $\alpha \in \mathbb{E}$.

(ii) For $\mathcal{D}$ with pullbacks and $\mathcal{C}$ a full subcategory closed under pullbacks in $\mathcal{D}$, a morphism p of $\mathcal{C}$ which is an effective global-descent morphism in $\mathcal{D}$ is also an effective global-descent morphism in $\mathcal{C}$ if and only if in every pullback square

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\mathrm{pr}_2} & A \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B, \end{array}$$

of $\mathcal{D}$, $E \times_B A$ in $\mathcal{C}$ implies A in $\mathcal{C}$.

The first part Corollary 3.25(i) is useful when the effective global-descent morphism in a category $\mathcal{C}$ is known, and the effective descent morphism with respect to some subclass $\mathbb{E}$ of morphisms is of interest.

The second part Corollary 3.25(ii) is useful when our category $\mathcal{C}$ is embedded in a larger category $\mathcal{D}$ where effective descent morphisms are easily characterized. Specifically, if $\mathcal{D}$ is locally cartesian closed or exact, see for example [JT94, Theorem 2.5] and [JT94, Corollary 2.5]. In other words, descent theory is easier in these types of categories, and the workaround is to fully embed the "difficult" category into these "nice" categories, and interpret the results thereof. Applications of these procedures are in the following examples.

Example 3.26. The following are examples of regular categories in which regular epimorphisms may fail to be effective for descent.

(i) Consider the full subcategory $\mathcal{C}$ of semigroups which have at most one idempotent element. It is closed under products and sub-semigroups. Hence, it is finitely complete and has (regular-epimorphism, monomorphism)-factorizations, and the regular epimorphisms are, as in the category $\mathcal{D}$ of all semigroups, stable under pullback. That is, $\mathcal{C}$ is a regular category. The regular epimorphism $p\colon E \to 1$ where E is a non-empty semigroup without any idempotents is an effective descent morphism of $\mathcal{D}$, but by Corollary 3.25, not of $\mathcal{C}$.

(ii) Consider $\mathcal{C}$ as the regular-epi-reflective subcategory of the category Ab of abelian groups. Assume for a fixed natural number $n \geq 2$ that the abelian groups satisfy the property

$$n^2 x = 0 \implies nx = 0. \tag{18}$$

The regular epimorphism $p\colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ (in Ab) fails to be effective for descent in $\mathcal{C}$. Namely, let $A = \mathbb{Z}/n^2\mathbb{Z}$, $B = \mathbb{Z}/n\mathbb{Z}$ and $P = \mathbb{Z} \times_B A$, then consider the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{\mathrm{pr}_2} & A \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathbb{Z} & \xrightarrow[p]{} & B. \end{array}$$

Then P satisfies (18), but A does not. In fact, for $(a, b) \in P$ with $n^2(a, b) = 0$, then $a = 0$, and therefore $\alpha(b) = p(a) = 0$. Hence, $b = nk + n^2\mathbb{Z}$ with $k \in \mathbb{Z}$, and thus, $nb = 0$ in A.

Now that we have seen a couple of ways the workaround does not work, there are cases where you receive something useful.

Corollary 3.27. In the category $\mathcal{C}$ of torsion-free abelian groups, every surjective homomorphism is an effective descent morphism.

Proof. Let B, E and $E \times_B A$ be torsion free, and consider the pullback diagram

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\mathrm{pr}_2} & A \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B. \end{array}$$

For any $a \in A$ such that $na = 0$ and $n \neq 0$. It follows, $n\alpha(a) = 0$, and since B is torsion-free, $\alpha(a) = 0$. Hence, $(0, a) \in E \times_B A$, and $n(0, a) = 0$. Finally, since $E \times_B A$ is torsion-free, it must be that $(0, a) = 0$, and thus $a = 0$. □

Note 3.28. A step further follows by noting that all these results have exclusively considered the adjunction

$$p_! \dashv p^* \colon \mathbb{E}(B) \to \mathbb{E}(E).$$

However, there are cases where the left adjoint to p^* is not given by composition with p from the left. Indeed, one such workaround is by constructing a factorization system $(\mathbb{D}, \mathbb{E})$; that is, every morphism f factors as αq with $q \in \mathbb{D}$ and $\alpha \in \mathbb{E}$, satisfying certain criteria. Then, the construction of the left adjoint is completed by factorizing $p \circ \gamma$, then extending our previous constructions using the factorization system. Yet, this note ends the journey of monadic descent theory in this thesis.

3.2 Descent theory with respect to fibrations

Before we state many definitions, it is worth noting that fibrations are often thought of as "nice" projections, and an abstraction of the pullback. This intuition is clearer given the examples given after the definitions.

Also, have in mind that our goal with this section is to show how the Beck–Chevalley condition applies in the context of bifibered categories, and not to get stuck in the technicalities that comes with this abstraction. In particular, this discovery will neatly tie the next section on monadic descent theory for rings and algebras to the first section on monadic descent theory.

Let $P\colon \mathcal{E} \to \mathcal{C}$ be an arbitrary functor, and let $p\colon E \to B$ be a morphism in $\mathcal{C}$.

Definition 3.29. The *fiber* $\mathcal{E}(B) := P^{-1}(B)$ *of* P *at* B is the (non-full) subcategory of $\mathcal{E}$ whose objects are in $P^{-1}(B)$ (that is, those objects A in $\mathcal{E}$ with $PA = B$) and whose morphisms $f\colon A \to A^{'}$ satisfy $P(f) = \mathrm{id}_B$.

Definition 3.30. Let A be an object in $\mathcal{E}(B)$, a pair (C, c) with $C \in \mathcal{E}(E)$ and $c\colon C \to A$ a morphism in $\mathcal{E}$ with $Pc = p$ is called a *P-lifting of p at A*.

Definition 3.31. A morphism $c\colon C \to A$ in $\mathcal{E}$ is called *P-cartesian* if it is a terminal P-lifting of Pc at A; that is for any morphism $d\colon D \to A$ in $\mathcal{E}$, and any $q\colon PD \to PC$ with $Pc \circ q = Pd$, there is a unique morphism $g\colon D \to C$ with $c \circ g = d$ and $Pg = q$,

$$\begin{array}{ccc} D & \overset{\exists ! g}{\dashrightarrow} & C \\ & \searrow^{d} & \downarrow c \\ & & A \end{array} \quad \longmapsto \quad \begin{array}{ccc} PD & \xrightarrow{q} & PC \\ & \searrow^{Pd} & \downarrow Pc \\ & & PA. \end{array}$$

Definition 3.32. A functor $P\colon \mathcal{E} \to \mathcal{C}$ is called a *(Grothendieck) fibration* if for any morphism $p\colon E \to B$ in $\mathcal{C}$ and every object $\mathcal{E}(B)$ there is a cartesian lifting $(p^*X, \vartheta_p X)$ over p at X.

Definition 3.33. The *inverse-image functor along p*

$$p^*\colon \mathcal{E}(B) \to \mathcal{E}(E)$$

assigns the cartesian lifting of p at A. Explicitly, for every object A in $\mathcal{E}(B)$ there is a cartesian lifting $(p^*A, \vartheta_p A)$ of p at A

$$\begin{array}{ccc} p^*A & \overset{\vartheta_p A}{\dashrightarrow} & A \\ & \Downarrow P & \\ E & \xrightarrow[p]{} & B. \end{array}$$

In other words, it assigns the domain of p in the *cleavage* (above p) whose codomain is A. The *cleavage* is the functor

$$\vartheta_p\colon J_E \circ p^* \to J_B$$

where $J_E\colon \mathcal{E}(E) \to \mathcal{E}$ and $J_B\colon \mathcal{E}(B) \to \mathcal{E}$ are the inclusion functors, with $P\vartheta_p$ as the constant natural transformation (given by $P\vartheta_p = \Delta p\colon \Delta E \to \Delta B$).

Definition 3.34. A functor $P\colon \mathcal{E} \to \mathcal{C}$ is a *(cloven) fibration* if every morphism $p\colon E \to B$ admits a (specified) cartesian lifting at every object A in $\mathcal{E}(B)$.

Remark 3.35. The *inverse-image functor* $(-)^*\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}$

$$\begin{array}{ccc} B & \longmapsto & \mathcal{E}(B) \\ {\scriptstyle p}\uparrow & & \downarrow{\scriptstyle p^*} \\ E & \longmapsto & \mathcal{E}(E) \end{array}$$

defines a pseudo-functor since there are uniquely determined natural equivalences

$$i_B\colon \mathrm{id}_{\mathcal{E}(B)} \to (\mathrm{id}_B)^*, \quad \text{and} \quad j_{p,q}\colon q^*p^* \to (pq)^*$$

such that

$$\vartheta_{\mathrm{id}_B} \circ J_B i_B = \mathrm{id}_{J_B}, P(J_B \circ i_B) = \Delta \mathrm{id}_B,$$

and

$$\vartheta_{pq} \circ J_X \circ j_{p,q} = \vartheta_p \circ \vartheta_q p^*, P J_X j_{p,q} = \Delta \mathrm{id}_X$$

for all $p\colon E \to B$ and $q\colon X \to E$ in $\mathcal{C}$ by the definition of cartesian lifting.

In particular, this observation implies $(-)^*\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}$ is a $\mathcal{C}$-indexed category. Therefore, one could equivalently describe the descent problem in $\mathcal{C}$-indexed categories such as in [JT97].

Remark 3.36. The condition of the fibration being cloven could be thought of as a choice of a single cartesian lifting determined by the cleavage. Conversely, if one assumes the axiom of choice, then every fibration is cloven. The following examples illustrate this.

Example 3.37.

1. For every category $\mathcal{C}$, the identity functor $\mathrm{id}_{\mathcal{C}}$ is a fibration.
2. Let $\mathcal{C}$ be a category and let $\mathbb{E}$ be a class of morphisms in $\mathcal{C}$. The slice category $\mathbb{E}(C)$ for C in $\mathcal{C}$ as considered in the very beginning of Section 3 can be interpreted as the fibers of the codomain functor

   $$P_{\mathbb{E}} \colon \mathbb{E}^2 \to \mathcal{C},$$

   where $\mathbb{E}^2$ is the category whose objects are all morphisms in $\mathbb{E}$ and whose morphisms $(p^{'}, p) \colon \alpha^{'} \to \alpha$ with $\alpha^{'}, \alpha \in \mathbb{E}$ are commutative diagrams in $\mathcal{C}$:

   $$\begin{array}{ccc} \cdot & \xrightarrow{p^{'}} & \cdot \\ {\scriptstyle \alpha^{'}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \cdot & \xrightarrow[p]{} & \cdot \end{array}$$

   If the diagram above is a pullback, then it represents a $P_{\mathbb{E}}$-cartesian lifting morphism of $\mathbb{E}^2$. However, if the class $\mathbb{E}$ contains all isomorphisms of $\mathcal{C}$, then every $P_{\mathbb{E}}$-cartesian lifting morphism of $\mathcal{C}$ is given by a pullback diagram.
   Let $p \colon E \to B$ be a morphism in $\mathcal{C}$, and suppose that for every $\alpha \colon A \to B$ in $\mathbb{E}$ the pullback

   $$\begin{array}{ccc} E \times_B A & \xrightarrow{\mathrm{pr}_2} & A \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B. \end{array}$$

   exists in $\mathcal{C}$. Then, for every (A, α) in $\mathbb{E}(B)(= \mathbb{E}^2(B))$, a cartesian $P_{\mathbb{E}}$-lifting of p at (A, α) exists. Therefore, the inverse-image functor p^* and the cleavage ϑ_p are given by the choice of the pullback. Hence, if $\mathcal{C}$ is a category with (chosen) pullbacks, and if $\mathbb{E}$ is stable under pullback, then $P_{\mathbb{E}}$ is a (cloven) fibration.

For simplicity, only the necessary definitions and observations are stated.

Definition 3.38. Let $\mathcal{C}$ be a category with pullbacks, $p \colon E \to B$ a morphism in $\mathcal{C}$ and $P \colon \mathcal{E} \to \mathcal{C}$ a fibration. The *descent data* (C, ξ) *for* C *in* $\mathcal{E}(E)$ *(relative to* p*)* are given by certain morphisms $\hat{\xi} \colon p_1^* \to p_2^*$ in $\mathcal{E}(E \times_B E)$ such that the diagrams

$$\begin{array}{ccc} p_1^*C & \xrightarrow{\hat{\xi}} & p_2^*C \\ {\scriptstyle \delta_1}\nwarrow & & \swarrow{\scriptstyle \vartheta_{p_2}C} \\ & C & \end{array} \tag{19}$$

$$\begin{array}{ccccc} & (\mathrm{pr}_1)^*p_2^*C & \xrightarrow{j} & (\mathrm{pr}_2)^*p_1^*C & \\ {\scriptstyle (\mathrm{pr}_1)^*\hat{\xi}}\nearrow & & & & \searrow{\scriptstyle (\mathrm{pr}_2)^*\hat{\xi}} \\ (\mathrm{pr}_1)^*p_1^*C & & & & (\mathrm{pr}_2)^*p_2^*C \\ {\scriptstyle j_1^{-1}}\searrow & & & & \nearrow{\scriptstyle j_2} \\ & (\mathrm{pr})^*p_1^*C & \xrightarrow[(\mathrm{pr})^*\hat{\xi}]{} & (\mathrm{pr})^*p_2^*C. & \end{array} \tag{20}$$

commute in $\mathcal{C}$. The morphism $\delta_i \colon C \to p_i^*C$ is the unique morphism induced by the cartesian lifting $(p_i^*C, \vartheta_{p_i}C)$ with respect to the "diagonal" morphism $\delta \colon E \to E \times_B E$ for which $p_1 \circ \delta = p_2 \circ \delta = \mathrm{id}_E$; that is, δ_i is the unique morphism such that $P(\delta_i) = \delta$ and $\vartheta_{p_i}C \circ \delta_i = \mathrm{id}_C$. The morphism

$$\mathrm{pr} := \langle p_1\mathrm{pr}_1, p_2\mathrm{pr}_2 \rangle \colon (E \times_B E) \times_E (E \times_B E) \to E \times_B E$$

is the morphism induced by the pair $(p_1\mathrm{pr}_1, p_2\mathrm{pr}_2)$, where pr_1 and pr_2 are given by the following pullback diagram

$$\begin{array}{ccc} (E\times_B E)\times_E (E\times_B E) & \xrightarrow{\mathrm{pr}_2} & (E\times_B E) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ (E\times_B E) & \xrightarrow[p_2]{} & E. \end{array}$$

The canonical isomorphisms j and j_i arise from the identities

$$p_1\mathrm{pr}_2 = p_2\mathrm{pr}_1 \quad \text{and} \quad p_i\mathrm{pr} = p_i\mathrm{pr}_i$$

for $i = 1, 2$ as in Remark 3.35. In particular, $\hat{\xi}$ is an isomorphism (see [Pav90, Remark 44]), hence all morphisms in (20) are isomorphisms.

Definition 3.39. The category $\mathsf{Des}_{\mathcal{E}}(p)$ of descent data $(C, \hat{\xi})$ relative to p consists of

(i) descent data $(C, \hat{\xi})$ with C in $\mathcal{E}(E)$ and $\hat{\xi}\colon p_1^*C \to p_2^*C$ in $\mathcal{E}(E\times_B E)$ such that (19) and (20) commute;

(ii) morphisms $h\colon (C, \hat{\xi}) \to (C', \hat{\xi}')$ are morphisms $h\colon C \to C'$ in $\mathcal{E}(E)$ such that

$$\begin{array}{ccc} p_1^*C & \xrightarrow{p_1^*h} & p_1^*C' \\ {\scriptstyle \hat{\xi}}\downarrow & & \downarrow{\scriptstyle \hat{\xi}'} \\ p_2^*C & \xrightarrow[p_1^*h]{} & p_2^*C' \end{array} \tag{21}$$

commutes.

Note 3.40. For any $A \in \mathcal{E}(B)$, p^*A comes equipped with canonical descent data

$$\hat{\varphi} = (j^{-1}_{p,p_2}A)(j_{p,p_1}A)\colon p_1^*p^*A \to p_2^*p^*A.$$

Hence, p^* can be lifted to the comparison functor $\Phi^p\colon \mathcal{E}(B) \to \mathsf{Des}_{\mathcal{E}}(p)$ by $A \mapsto (p^*A, \hat{\varphi})$, which makes the diagram

$$\begin{array}{ccc} \mathcal{E}(B) & \xrightarrow{\Phi^p} & \mathsf{Des}_{\mathcal{E}}(p) \\ & {\scriptstyle p^*}\searrow \quad \swarrow{\scriptstyle U^p} & \\ & \mathcal{E}(E) & \end{array}$$

commute. (The functor U^p is the obvious forgetful functor.)

The next definition is in accordance with Grothendieck [Gro59].

Definition 3.41. Let $P\colon \mathcal{E} \to \mathcal{C}$ be a fibration with $\mathcal{C}$ a category with pullbacks. A morphism $p\colon E \to B$ in $\mathcal{C}$ is an *(effective) $\mathcal{E}$-descent morphism* if Φ^p is full and faithful (an equivalence of categories).

Definition 3.42. A functor $P\colon \mathcal{E} \to \mathcal{C}$ is a *(cloven) bifibration* is both P and $P^{\mathrm{op}}\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ are (cloven) fibrations.

Dually to the inverse image functor and the cleavage there are the following definitions.

Definition 3.43. The *direct image functor along p*

$$p_!\colon \mathcal{E}(E) \to \mathcal{E}(B)$$

and the *co-cleavage*

$$\delta_p\colon J_E \to J_B p_!$$

that act as the inverse image functor and cleavage to the fibration P^{op}.

Example 3.44. The fibration $P_{\mathbb{E}}$ is a bifibration. It is a fibration by Example 3.37 (2), and to see that it is an opfibration, note that for any $p\colon E \to B$ in $\mathcal{C}$, and any object (C, γ) in $\mathbb{E}(E)(= E^2(E))$, an *opcartesian lifting* over p at x is given by

$$(\mathrm{id}_C, p)\colon \gamma \to p \circ \gamma$$

that is,

$$\begin{array}{ccc} C & \xrightarrow{\mathrm{id}_C} & C \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle p\circ\gamma} \\ E & \xrightarrow[p]{} & B \\ & \Downarrow P_{\mathbb{E}} & \\ E & \xrightarrow[p]{} & B \end{array}$$

Therefore if $P\colon \mathcal{E} \to \mathcal{C}$ is a bifibration, it gives rise the adjunction $p_! \dashv p^*$ as in the diagram

$$\mathcal{E}(B) \underset{p^*}{\overset{p_!}{\leftrightarrows}} \mathcal{E}(E).$$

Furthermore, the unit and counit of the adjunction are given by the (unique) natural transformations

$$\eta_p\colon \mathrm{id}_{\mathcal{E}(E)} \to p^*p_! \quad \text{and} \quad \varepsilon_p\colon p_!p^* \to \mathrm{id}_{\mathcal{E}(B)},$$

respectively, with $(\vartheta_p p_!)(J_E\eta_p) = \delta_p$ and $(J_B\varepsilon_p)(\delta_p p^*) = \vartheta_p$. Moreover, by considering the kernelpair (p_1, p_2) of p, there is the *Beck transformation*

$$\beta_p\colon (p_2)_!p_1^* \to p^*p_! \quad \text{with} \quad (J_E\beta_p)(\delta_{p_2}p_1^*) = (J_E\eta_p)\vartheta_{p_1}.$$

The bifibration P *satisfies the Beck–Chevalley condition for* p if β_p is a natural equivalence. Formally, the Beck–Chevalley condition is defined as:

Definition 3.45. Consider the commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & D \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle q} \\ E & \xrightarrow[p]{} & B \end{array} \tag{22}$$

in $\mathcal{C}$, then any bifibration $P\colon \mathcal{E} \to \mathcal{C}$ has the *Beck–Chevalley property* if the natural transformation

$$\beta\colon \varphi_!\psi^* \to q^*p_!$$

is an isomorphism for every pullback square (22) in $\mathcal{C}$.

For the bifibration $P = P_{\mathbb{E}}$, there is a bijective correspondence between the descent data $\hat{\xi}$ and the descent data ξ given in Definition 3.4 so that $\mathsf{Des}_{\mathbb{E}^2}(p)$ is isomorphic to $\mathsf{Des}_{\mathbb{E}}(p)$. Moreover, Proposition 3.16 gives criteria for when the category of descent data is the category of Eilenberg–Moore algebras. Further, if any bifibration $P\colon \mathcal{E} \to \mathcal{C}$ satisfies the Beck–Chevalley condition for p, then β_p is a natural equivalence, and therefore there is a bijective correspondence between $(\hat{\xi})^\sharp\colon (p_2)_!p_1^*C \to C$ and the algebra structure $\xi\colon p^*p_!C \to C$ obtained by the monad induced by $p_! \dashv p^*$:

$$\begin{array}{ccc} (p_2)_!p_1^*C & \xrightarrow{\beta_p C} & p^*p_!C \\ & {\scriptstyle(\hat{\xi})^\sharp}\searrow \quad \swarrow{\scriptstyle \xi} & \\ & C & \end{array}$$

where $(\hat{\xi})^\sharp = (\varepsilon_{p_2}C)((p_2)_!\hat{\xi})$. This (essentially) shows the bijective correspondences $(\hat{\xi} \leftrightarrow \xi)$ for any bifibration satisfying the Beck–Chevalley condition along p.

Theorem 3.46. [BR70] Let $P\colon \mathcal{E} \to \mathcal{C}$ be a bifibration with $\mathcal{C}$ a category with pullbacks. For a morphism $p\colon E \to B$ in $\mathcal{C}$ such that the Beck–Chevalley condition is satisfied for p, the category $\mathsf{Des}_{\mathcal{E}}(p)$ of descent data is isomorphic to the category of Eilenberg–Moore algebras of the monad induced by the adjunction $p_! \dashv p^*$. Therefore p is an (effective) $\mathcal{E}$-descent morphism if and only if p^* is premonadic (monadic).

There is an explicit proof of the above theorem by J. Bénabou and J. Roubaud [BR70] and Beck (unpublished). The above theorem corresponds to how the monadic characterization of descent translates to the abstract context of bifibered categories satisfying the Beck–Chevalley condition.

4 Monadic Descent Theory for Rings and Algebras

The goal of this section is to follow the development of [JT04] on the classical descent problem for modules and algebras. In particular, to characterize the analogue of the change-of-base functor for modules and algebras, the *extension-of-scalars functor*. Moreover, the descent theory with respect to fibrations gives rise to a stronger theorem than what could be characterized in a classical sense.

The classical theorem given by Grothendieck in [Gro59] is formulated as follows:

Theorem 4.1. For a homomorphism $p\colon R \to S$ of commutative rings, the extension-of-scalars functor $S \otimes_R (-)\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ is comonadic whenever p makes S a faithfully flat R-module.

However, at the time of Grothendieck's discovery, the obvious monadic connection was not present (recall that Grothendieck did consider the questions of descent in fibered categories), and in the later decades it was overlooked much because both the theory of monads and descent were not very "popular" independently of one another. That being said, by the works [BR70], and in the later decades [JT94], [JT97] and [JT04], there is a clear approach to the descent questions. In the case for modules and commutative rings, [JT04] found an even stronger result.

Theorem (Theorem 4.15)**.** For a homomorphism $p\colon R \to S$ of commutative rings, the extension-of-scalars functor $S \otimes_R (-)\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ is comonadic if and only if p is a pure morphism of R-modules (see Definition 4.7).

Notably, they derived this result from a more general and stronger result (Theorem 4.14) that applies to most types of algebras, unital or not, associative or not, commutative or not, Lie, Jordan, differential, etc. that satisfy a certain property.

Note 4.2. With the observations of Section 2.2 on monads as generalized rings it could be shown that the induced change-of-base functor for a morphism between monads mimics the situation for a morphism of rings. This result is found by explicitly stating the adjunction situation between the change-of-base functor and its left adjoint. Thus, one would attain a result in a monoidal category see [JT04, Theorem 4.1 and Corollary 4.2]. However, for our purposes this is an unnecessary detour of abstraction that can be skipped. Indeed, we will find a theorem that is not in general valid in an abstract monoidal category.

4.1 Comonadicity for ordinary modules

Definition 4.3. Let $p\colon R \to S$ be a morphism in the category Rng of rings. Define the functors

$$ {}_R\mathsf{Mod} \underset{e_p}{\overset{\rho_p}{\leftrightarrows}} {}_S\mathsf{Mod} $$

by

$$ \rho_p(M) = M \quad \text{and} \quad e_p(N) = S \otimes_R N $$

for $M \in {}_S\mathsf{Mod}$ and $N \in {}_R\mathsf{Mod}$. The functor ρ_p is called the *restriction-of-scalars functor* (by sending the action $\cdot_s$ to $\cdot_r$) and e_p is called the *extension-of-scalars functor*. These functors define an adjunction $\rho_p \dashv e_p$ with unit and counit given by

$$ \begin{aligned} \eta_N &: N \to S \otimes_R N, \quad n \mapsto 1 \otimes n \\ \varepsilon_M &: S \otimes_R M \to M, \quad m \otimes_R s \mapsto ms \end{aligned} $$

for all $M \in {}_S\mathsf{Mod}$ and $N \in {}_R\mathsf{Mod}$.

Remark 4.4. The adjunction of Definition 4.3 could equivalently be defined for any morphism $p\colon R \to S$ in any category of algebraic objects.

Recall first that $\mathbb{Q}/\mathbb{Z}$ is an *injective cogenerator* in the category Ab of abelian groups (see for example [KS06, Chapter 5.2]). If M is a left R-module, then it is a bimodule ${}_R\mathsf{Mod}_\mathbb{Z}$. Thus, the right R-module structure on $\mathrm{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$ is given by $hr\colon m \mapsto h(rm)$. Hence, there is a (representable) functor

$$ E\colon ({}_R\mathsf{Mod})^{\mathrm{op}} \to \mathsf{Mod}_R, \quad M \mapsto \mathrm{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}). $$

Note the following proposition as a corollary to [Rot09, Lemma 3.53].

Proposition 4.5. The functor $E\colon ({}_R\mathsf{Mod})^{\mathrm{op}} \to \mathsf{Mod}_R$ is exact.

Proof. Clearly E is an additive functor, then it suffices to show that if the sequence of left R-modules

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \tag{23}$$

is exact if and only if the sequence

$$C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^* \tag{24}$$

is exact, where $A^* = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ (similarly for B and C) and $E\alpha = \alpha^*$ (similarly for β).

In the one direction, if the sequence (23) is exact, then since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator, the contravariant functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ is exact. Therefore, the sequence (24) must be exact.

In the other direction, first let if the sequence (24) is exact, then we must first show that $\mathrm{im}\ \alpha \subseteq \ker \beta$. If $x \in A$ and $\alpha x \notin \ker \beta$ then since there are no nonzero x that can kill all of $\mathbb{Q}/\mathbb{Z}$, there is a map $f : C \to \mathbb{Q}/\mathbb{Z}$ with $f\beta\alpha(x) \neq 0$. Thus, $f \in C^*$ and $f\beta\alpha \neq 0$ contradicting the hypothesis that $\alpha^*\beta^* = 0$.

Next, we must show that $\ker \beta \subseteq \mathrm{im}\ \alpha$. Let $y \in \ker \beta$ with $y \notin \mathrm{im}\ \alpha$. Then since $y + \mathrm{im}\ \alpha$ is a nonzero element, there is a map $g : B/\mathrm{im}\ \alpha \to \mathbb{Q}/\mathbb{Z}$ with $g(y + \mathrm{im}\ \alpha) \neq 0$. If $v : B \to B/\mathrm{im}\ \alpha$ is the natural map, define $g' = gv \in B^*$. Note that $g'(y) = gv(y) = g(y + \mathrm{im}\ \alpha) \neq 0$. It follows, $g'(\mathrm{im}\ \alpha) = \{0\}$ such that $0 = g'\alpha = \alpha^*(g')$ and $g' \in \ker \alpha^* = \mathrm{im}\ \beta^*$. Thus, $g' = \beta^*(h)$ for some $h \in C^*$; that is, $g' = h\beta$. Hence, $g'(y) = h\beta(y)$ which is a contradiction since $g'(y) \neq 0$, while $h\beta(y) = 0$, because $y \in \ker \beta$. ☐

Lemma 4.6. For a ring homomorphism $p : R \to S$, the extension-of-scalars functor

$$S \otimes_R (-) : {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$$

is comonadic whenever it reflects isomorphisms and the map

$$\mathrm{Hom}_{\mathbb{Z}}(p, \mathbb{Q}/\mathbb{Z}) : \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}) \tag{25}$$

is a split epimorphism of (R, R)-bimodules.

Proof. The idea of the proof is to use Theorem 2.38 as follows:

- $\mathcal{B} = ({}_R\mathsf{Mod})^{\mathrm{op}}$, the opposite category of R-modules;
- $\mathcal{C} = ({}_S\mathsf{Mod})^{\mathrm{op}}$;
- $U : \mathcal{B} \to \mathcal{C}$ the dual of the extension-of-scalars functor $S \otimes_R (-) : {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$;
- $F : \mathcal{C} \to \mathcal{B}$ is thus the dual of the restriction of scalars functor;
- $\mathcal{X} = \mathsf{Mod}_R$, the category of right R-modules;
- $\mathcal{Y} = \mathsf{Ab}$, the category of abelian groups;
- $H : \mathcal{B} \to \mathcal{X}$ defined by $H(B) = \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ with the right R-module structure on $\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ defined by $(hr)(b) = h(rb)$; of course the only reason for using $\mathbb{Q}/\mathbb{Z}$ here is that it is an injective cogenerator in Ab;
- $U' : \mathcal{X} \to \mathcal{Y}$ defined by $U'(X) = \mathrm{Hom}_R(S, X)$ (considering $\mathrm{Hom}_R(S, X)$ just an an abelian group of course);
- $H' : \mathcal{C} \to \mathcal{Y}$ defined by $H'(C) = \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z})$.

Then observe:

1. For each R-module B, there are canonical isomorphisms using the Hom and $\otimes$ adjunction:

$$\mathrm{Hom}_R(S, \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(S \otimes_R B, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(B, \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z}))$$

   where the first Hom_R is used for the right R-module homomorphisms, while the second is used for the left ones, assuming that $\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ is considered as an R-module via $(rh)(s) = h(sr)$. In particular, the first isomorphism tells us that in this case diagram (7) commutes (up to an isomorphism).
2. Since the forgetful functor from the category of modules (over any ring) to Ab is monadic it reflects isomorphisms by Theorem 2.21. The forgetful functor is also exact[1], and by Proposition 4.5, the functors H and H' are exact. Since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in Ab, H and H' must necessarily reflect isomorphisms too.
3. If (25) is a split epimorphism of (R, R)-bimodules ($= R \otimes R^{\mathrm{op}}$-modules), then each

$$\mathrm{Hom}_{\mathbb{Z}}(\varepsilon_B, \mathbb{Q}/\mathbb{Z}) : \mathrm{Hom}_{\mathbb{Z}}(S \otimes_R B, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \tag{26}$$

[1] It has both a left adjoint and a right adjoint, therefore it is actually "stronger" than exact because it preserves all limits and colimits.

is a split epimorphism of right R-modules. Note that (up to natural isomorphism) (26) can be rewritten as

$$\mathrm{Hom}_{\mathbb{Z}}(p \otimes_R B, \mathbb{Q}/\mathbb{Z})\colon \mathrm{Hom}_{\mathbb{Z}}(S \otimes_R B, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(R \otimes_R B, \mathbb{Q}/\mathbb{Z})$$

and then as

$$\mathrm{Hom}_R(B, \mathrm{Hom}_{\mathbb{Z}}(p, \mathbb{Q}/\mathbb{Z}))\colon \mathrm{Hom}_R(B, \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z})) \to \mathrm{Hom}_R(B, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}))$$

and therefore the splitting (26) is natural in B.

Thus, whenever $S \otimes_R (-)$ reflects isomorphisms and (25) is a split epimorphism of (R, R)-bimodules, all assumptions of Theorem 2.38 are satisfied. □

In fact, there is a more precise formulation of the above lemma. First note the next definition.

Definition 4.7. A morphism $f\colon M \to M'$ in ${}_R\mathsf{Mod}$ is *pure* if for any N in ${}_R\mathsf{Mod}$,

$$\mathrm{id}_N \otimes_R f\colon N \otimes_R M \to N \otimes_R M'$$

is a monomorphism.

Then we obtain the (corrected) theorem of Caenepeel [Cae04] that was an adoption of the new arguments of Mesablishvili in [Mes00].

Theorem 4.8. For a ring homomorphism $p\colon R \to S$, consider the following conditions:

(i) p a pure morphism of (R, R)-bimodules;
(ii) the extension-of-scalars functor $S \otimes_R (-)\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ is comonadic;
(iii) p is a pure morphism of right R-modules.

Then, (i) $\implies$ (ii) $\implies$ (iii).

Proof. First observe that if the homomorphism (25) of Lemma 4.6 is a split epimorphism of (R, R)-bimodules, then by the Yoneda Lemma, this holds if and only if

$$\mathrm{Hom}_R(B, \mathrm{Hom}_{\mathbb{Z}}(p, \mathbb{Q}/\mathbb{Z}))\colon \mathrm{Hom}_{R\otimes R^{\mathrm{op}}}(B, \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z})) \to \mathrm{Hom}_{R\otimes R^{\mathrm{op}}}(B, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}))$$

is surjective for every (R, R)-bimodule B. Next, the canonical isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(B \otimes_{R\otimes R^{\mathrm{op}}} S, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{R\otimes R^{\mathrm{op}}}(B, \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z}))$$

says that the desired split property holds if and only if

$$B \otimes_{R\otimes R^{\mathrm{op}}} p\colon B \otimes_{R\otimes R^{\mathrm{op}}} R \to B \otimes_{R\otimes R^{\mathrm{op}}} S$$

is a monomorphism for every R module B; that is, if and only if $p\colon R \to S$ is a *pure* morphism of (R, R)-bimodules.

Recall that a right adjoint functor reflects isomorphisms if and only if all components of the counit of adjunction are extremal epimorphisms; that is, if the counit ε can be factorized as $\varepsilon = m \circ g$ where m is a monomorphism, then m is an isomorphism. Note that the extremal epimorphisms in $({}_R\mathsf{Mod})^{\mathrm{op}}$ are precisely the monomorphism in ${}_R\mathsf{Mod}$, and these components can be presented as $p \otimes_R B\colon R \otimes_R B \to S \otimes_R B$. It follows, the condition that the extension-of-scalars functor reflects isomorphisms in Lemma 4.6 is equivalent to p begin pure as a right R-module homomorphism implying the comonadicity of $S \otimes_R (-)$. Finally, comonadicity implies the reflection of isomorphisms. □

This result is more general than [Mes00] that only proved the case for commutative rings. In particular, we state [Mes00, Proposition 2 and 3] as the following corollary.

Corollary 4.9. For a homomorphism $p\colon R \to S$ of commutative rings, the following conditions are equivalent:

(i) p is a pure morphism of R-modules;
(ii) the extension-of-scalars functor $S \otimes_R (-)\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ is comonadic.

4.2 Comonadicity for algebras

Next is an important generalization from Lemma 2.40.

Theorem 4.10. For a homomorphism $p\colon R \to S$ of commutative rings, the extension-of-scalars functor

$$S \otimes_R (-)\colon\ {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$$

is comonadic if and only if the induced extension-of-scalars functor

$$S \otimes_R (-)\colon\ {}_R\mathsf{Alg} \to {}_S\mathsf{Alg}$$

is comonadic for any of the following kinds of algebras:

(i) arbitrary (not necessarily associative or commutative) algebras, with or without 1;
(ii) associative algebras, with or without 1;
(iii) (associative and) commutative algebras, with or without 1;
(iv) Lie algebras;
(v) Jordan algebras;
(vi) differential algebras.

Proof. In the one direction, for a morphism $p\colon R \to S$ of commutative rings, Lemma 2.40 says that the comonadicity of the extension-of-scalars functor for modules implies the same property for algebras. Indeed, consider the diagram

$$\begin{array}{ccc} ({}_R\mathsf{Alg})^{\mathrm{op}} & \xrightarrow{(S\otimes_R(-))^{\mathrm{op}}} & ({}_S\mathsf{Alg})^{\mathrm{op}} \\ \downarrow & & \downarrow \\ ({}_R\mathsf{Mod})^{\mathrm{op}} & \xrightarrow[(S\otimes_R(-))^{\mathrm{op}}]{} & ({}_S\mathsf{Mod})^{\mathrm{op}} \end{array}$$

where the vertical arrows are the (duals of the) forgetful functors from algebras to modules.

For example, a *Lie algebra* over a commutative ring R is an R-module L together with an R-bilinear map $[\_,\_]\colon L \times L \to L$ satisfying a pair of axioms. In this case, the forgetful functor forgets the R-bilinear map. Similarly, for all the other listed algebras.

In the other direction, apply Lemma 2.40 to the diagram

$$\begin{array}{ccc} ({}_R\mathsf{Mod})^{\mathrm{op}} & \xrightarrow{(S\otimes_R(-))^{\mathrm{op}}} & ({}_S\mathsf{Mod})^{\mathrm{op}} \\ \downarrow & & \downarrow \\ ({}_R\mathsf{Alg})^{\mathrm{op}} & \xrightarrow[(S\otimes_R(-))^{\mathrm{op}}]{} & ({}_S\mathsf{Alg})^{\mathrm{op}} \end{array}$$

where the vertical arrows are the functors carrying modules M to

1. M equipped with the zero multiplication if our algebras are not required to have 1;
2. the semidirect product of M with the ground ring (either R or S).

□

Remark 4.11. Theorem 4.10 could easily be extended from rings to monoids in a monoidal category. In particular, since commutativity is involved, those monoidal categories should at least be symmetric (or at least braided). For example, for a commutative ring R, a Lie algebra is an object in a symmetric monoidal R-linear category (satisfying a pair of axioms).

4.3 Comonadicity to (Co)descent

To tie comonadicity and (co)descent together, note the following definition and example.

Definition 4.12. Denote by $\mathcal{M}$ the *category of all modules* with objects as pairs (R, M) where R is a commutative ring and M is a module in ${}_R\mathsf{Mod}$, and morphisms as pairs $(f, \varphi)\colon (R, M) \to (S, N)$ where $f\colon R \to S$ is a morphism in CRng and $\varphi\colon M \to S \otimes_R N$ is a morphism in ${}_R\mathsf{Mod}$.

Example 4.13. The canonical forgetful functor $P\colon \mathcal{M} \to \mathsf{CRng}$ given by

$$(R, M) \mapsto R, \quad (f, \varphi) \mapsto f$$

is defines a bifibration

$$\begin{array}{ccc} (R, M) & \xrightarrow{(p,\varphi)} & (S, N) \\ & \Downarrow P & \\ R & \xrightarrow[p]{} & S = P(R, M). \end{array}$$

In fact, the inverse image functor is the pair $p^* = (p, e_p)\colon (R, {}_R\mathsf{Mod}) \to (S, {}_S\mathsf{Mod})$ and direct image functor is the pair $p_! = (p^{\mathrm{op}}, \rho_p)\colon (S, {}_S\mathsf{Mod}) \to (R, {}_R\mathsf{Mod})$ where $\rho_p \dashv e_p$ is the adjunction between restriction-of-scalars functor and the extension-of-scalars functor, respectively.

It is clear that the basic (bi)fibration over a category $\mathcal{C}$ with pullbacks satisfies the Beck–Chevalley condition. In other words, for every pullback square

$$\begin{array}{ccc} D & \xrightarrow{s} & E \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle q} \\ F & \xrightarrow[p]{} & B \end{array}$$

the canonical morphism between the two composites in the diagram

$$\begin{array}{ccc} \mathcal{C}/D & \xrightarrow{s_!} & \mathcal{C}/E \\ {\scriptstyle t^*}\uparrow & & \uparrow{\scriptstyle q^*} \\ \mathcal{C}/F & \xrightarrow[p_!]{} & \mathcal{C}/B \end{array}$$

is an isomorphism. Equivalently, for every pair (C, γ) in $\mathcal{C}/E$, there is an isomorphism

$$(E \times_B F) \times_F C \cong E \times_B C. \tag{27}$$

When $\mathcal{C}$ is the opposite category $\mathsf{CRng}^{\mathrm{op}}$ of commutative rings, then the isomorphism (27) becomes

$$(E \otimes_B F) \otimes_F C \cong E \otimes_B C,$$

which holds for every F-module C, and has nothing to do with any multiplication on C. Hence, the bifibration

$$\mathcal{M}^{\mathrm{op}} \to \mathsf{CRng}^{\mathrm{op}}$$

satisfies the Beck–Chevalley condition. The same holds true for any of the algebras listen in Theorem 4.10. Therefore, Theorem 4.10 becomes:

Theorem 4.14. A homomorphism $p\colon R \to S$ of commutative rings is an effective descent morphism with respect to any bifibration of modules or algebras from Theorem 4.10 if and only if the extension-of-scalars functor $S \otimes_R (-)\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ is comonadic.

Hence, Corollary 4.9 becomes:

Theorem 4.15. A homomorphism $p\colon R \to S$ of commutative rings is an effective descent morphism if and only if it is a pure morphism of R-modules.

Remark 4.16. The basic fibration for the opposite category of commutative rings is among those that occur in Theorem 4.14. Therefore, descent theory of modules and commutative rings is a special case of global-descent theory.

A Types of Categories

A.1 Comma Categories

Definition A.1. If $F\colon \mathcal{C} \to \mathcal{E}$ and $G\colon \mathcal{D} \to \mathcal{E}$ are functors, the *comma category* $F \downarrow G$ consists of the following datum:

- Objects are triples (c, d, α) where $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $\alpha\colon f(c) \to g(d)$ is a morphism in $\mathcal{E}$,
- Morphisms from (c_1, d_1, α_1) to (c_2, d_2, α_2) are pairs (β, γ), where $\beta\colon c_1 \to c_2$ and $\gamma\colon d_1 \to d_2$ are morphisms in $\mathcal{C}$ and $\mathcal{D}$, respectively, such that the diagram

$$\begin{array}{ccc} F(c_1) & \xrightarrow{F(\beta)} & F(c_2) \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ G(d_1) & \xrightarrow[G(\gamma)]{} & G(d_2) \end{array}$$

commutes.

- Composition of morphisms (β, γ) and $(\beta^{'}, \gamma^{'})$ is given on components by composition in $\mathcal{C}$ and $\mathcal{D}$; that is, the diagram

$$\begin{array}{ccccc} F(c_1) & \xrightarrow{F(\beta)} & F(c_2) & \xrightarrow{F(\beta^{'})} & F(c_3) \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} \\ G(d_1) & \xrightarrow[G(\gamma)]{} & G(d_2) & \xrightarrow[G(\gamma^{'})]{} & G(d_3) \end{array}$$

commutes.

There is an important special case of a comma category which will be useful for our discussion.

Definition A.2. Let $\mathcal{C}$ be a category. The *slice category* $\mathcal{C}/x$ of $\mathcal{C}$ over an object $x \in \mathcal{C}$ consists of the following datum:

- Objects are pairs (c, γ) for an object $c \in \mathcal{C}$ and morphism $\gamma \colon c \to x$ in $\mathcal{C}$,
- Morphisms $f \colon (c, \gamma) \to (c^{'}, \gamma^{'})$ in $\mathcal{C}/x$ is given by a morphism $f \colon c \to c^{'}$ in $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} c & \xrightarrow{f} & c^{'} \\ & {\scriptstyle \gamma}\searrow \quad \swarrow{\scriptstyle \gamma} & \\ & x & \end{array}$$

commutes.

Remark A.3. Explicitly, if F is the identity functor of $\mathcal{C}$ and G is the inclusion of an object $x \in \mathcal{C}$, then $F \downarrow G$ is the slice category $\mathcal{C}/x$.

A.2 Monoidal Categories

Definition A.4. A *monoidal category* is a category $\mathcal{C}$ consisting of the following datum:

- A functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product.
- An object $\mathbb{1} \in \mathcal{C}$ called the *tensor unit*.
- A natural isomorphism $a \colon ((-) \otimes (-)) \otimes (-) \xrightarrow{\simeq} (-) \otimes ((-) \otimes (-))$ with components of the form $a_{x,y,z} \colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ called the *associator*.
- A natural isomorphism $\lambda \colon (\mathbb{1} \otimes (-)) \xrightarrow{\simeq} (-)$ with components of the form $\lambda_x \colon \mathbb{1} \otimes x \to x$ called the *left unitor*.
- A natural isomorphism $\rho \colon (-) \otimes \mathbb{1} \xrightarrow{\simeq} (-)$ with components of the form $\rho_x \colon x \otimes \mathbb{1} \to x$ called the *right unitor*.

such that the diagrams commute, for all objects involved:

- The *triangle identity* (not the triangle identities of an adjunction):

$$\begin{array}{ccc} (x \otimes \mathbb{1}) \otimes y & \xrightarrow{a_{x,\mathbb{1},y}} & x \otimes (\mathbb{1} \otimes y) \\ & {\scriptstyle \rho_x \otimes \mathbb{1}_y}\searrow \quad \swarrow{\scriptstyle \mathbb{1}_x \otimes y} & \\ & x \otimes y & \end{array}$$

- The *pentagon identity*:

$$\begin{array}{ccccc} & & (w \otimes x) \otimes (y \otimes z) & & \\ & {\scriptstyle a_{w \otimes x, y, z}}\nearrow & & \searrow{\scriptstyle a_{w,x,y \otimes z}} & \\ ((w \otimes x) \otimes y) \otimes z & & & & w \otimes (x \otimes (y \otimes z)) \\ {\scriptstyle a_{w,x,y} \otimes \mathrm{id}_z}\downarrow & & & & \uparrow{\scriptstyle \mathrm{id}_w \otimes a_{x,y,z}} \\ (w \otimes (x \otimes y)) \otimes z & & \xrightarrow[a_{w, x \otimes y, z}]{} & & w \otimes ((x \otimes y) \otimes z) \end{array}$$

Definition A.5. A *braided monoidal category* is a monoidal category $\mathcal{C}$ equipped with a natural isomorphism

$$B_{x,y} \colon x \otimes y \to y \otimes x$$

called the *braiding*, such that the diagrams

$$\begin{array}{ccccc}
(x \otimes y) \otimes z & \xrightarrow{a_{x,y,z}} & x \otimes (y \otimes z) & \xrightarrow{B_{x,y\otimes z}} & (y \otimes z) \otimes x \\
\downarrow{\scriptstyle B_{x,y}\otimes \mathrm{id}_z} & & & & \downarrow{\scriptstyle a_{y,z,x}} \\
(y \otimes x) \otimes z & \xrightarrow{a_{y,x,z}} & y \otimes (x \otimes z) & \xrightarrow{\mathrm{id}_y \otimes B_{x,z}} & y \otimes (z \otimes x)
\end{array}$$

and

$$\begin{array}{ccccc}
x \otimes (y \otimes z) & \xrightarrow{a^{-1}_{x,y,z}} & (x \otimes y) \otimes z & \xrightarrow{B_{x\otimes y,z}} & z \otimes (x \otimes y) \\
\downarrow{\scriptstyle \mathrm{id}_x\otimes B_{y,z}} & & & & \downarrow{\scriptstyle a^{-1}_{z,x,y}} \\
x \otimes (z \otimes y) & \xrightarrow{a^{-1}_{x,z,y}} & (x \otimes z) \otimes y & \xrightarrow{B_{x,z} \otimes \mathrm{id}_y} & (z \otimes x) \otimes y
\end{array}$$

commute for all objects x, y, z, where $a_{x,y,z} \colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ denotes the components of the associator of $(\mathcal{C}, \otimes)$.

Definition A.6. A *symmetric monoidal category* is a braided monoidal category for which the braiding

$$B_{x,y} \colon x \otimes y \to y \otimes x$$

satisfies the condition

$$B_{y,x} \circ B_{x,y} = \mathrm{id}_{x\otimes y}$$

for all objects x, y.

A.3 Reflective subcategory

Definition A.7. A full subcategory $i \colon \mathcal{C} \hookrightarrow \mathcal{D}$ is *reflective* if the inclusion functor i has a left adjoint T:

$$(T \dashv i) \colon \mathcal{C} \underset{i}{\overset{T}{\leftrightarrows}} \mathcal{D}$$

The left adjoint T is called the *reflector*.

Then, we can define categories depending on the units the adjunction:

Definition A.8. Let $\mathcal{C}$ be a reflective category of $\mathcal{D}$ with

$$(T \dashv i) \colon \mathcal{C} \underset{i}{\overset{T}{\leftrightarrows}} \mathcal{D}$$

If the unit of the reflector is

1. a monomorphism, then $\mathcal{C}$ is a *mono-reflective category*
2. an epimorphism, then $\mathcal{C}$ is an *epi-reflective category*

A.4 Regular categories

Definition A.9. A category $\mathcal{C}$ is *regular* if

1. it is finitely complete; $\mathcal{C}$ admits all finite limits.
2. the kernel pair

$$\begin{array}{ccc}
d \times_c d & \xrightarrow{\mathrm{pr}_1} & d \\
{\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle f} \\
d & \xrightarrow[f]{} & c
\end{array}$$

   of any morphism $f \colon d \to c$ admits a coequalizer $d \times_c d \rightrightarrows d \to \mathrm{coeq}(\mathrm{pr}_1, \mathrm{pr}_2)$
3. regular epimorphisms are stable under pullback along any morphism.

Equivalently,

Definition A.10. A *regular category* is a finitely complete category with pullback-stable (regular-epimorphism, monomorphism)-factorizations; in the sense of the smallest monic through which a morphism factors.

Then, we can construct the following category:

Definition A.11. A *regular-epi-reflective category* is both regular and epi-reflective.

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