Henrik Snemyr Langesæter

# Projective Geometry

Bachelor's project in Mathematical Sciences
Supervisor: Sverre Olaf Smalø
January 2021

**Bachelor's project**

**NTNU**
Norwegian University of Science and Technology
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## 0.1 Abstract/Sammendrag

This paper's content is about projective geometry. In the paper the concept of a projective space of a vector space is presented with examples of such projective spaces. It look at maps from a projective space to itself, and prove that there exists an isomorphism between $Gr_{m,n}(\mathbb{F})$ and $Gr_{m,n-m}(\mathbb{F})$. At the end Plücker embedding is presented and wegde product is introduced.

Denne oppgaven handler om projektiv geometri. I oppgaven presenteres konseptet om et projektiv rom tilknyttet et vektorrom med eksempler på noen projektive rom. Man ser på avbildning fra et projektiv rom til seg selv, og beviser at det eksisterer en isomorfi mellom $Gr_{m,n}(\mathbb{F})$ og $Gr_{m,n-m}(\mathbb{F})$. Mot slutten av oppgaven blir Plücker Imbedding presentert og ytreprodukt blir introdusert.

# Chapter 1

# Projective geometry

## 1.1 Introduction to projective geometry

Projective geometry is the study of projections. In linear algebra a projective space of dimension $n$ over some field $\mathbb{F}$ is seen as the set of 1-dimensional subsets of the vectorspace $\mathbb{F}^{n+1}$. Where any $k$-dimensional subspaces in $\mathbb{F}^{n+1}$ represents a $k-1$ dimensional objects in the projective space. Another way of thinking of projective spaces, is as extensions of Euclidean spaces by adding "infinity points" that determines the direction of a line in the space. The weakness with this way of thinking is that it seperates the "infinity points" from the other points, but they really are "inseperable" from the other points.

**Definition 1** *Let $V$ be a $n$-dimensional vector space over a field $\mathbb{F}$, then the projective space given by $V$ is denoted by $\mathbb{P}(V)$ and consist of all the 1-dimensional subspaces of $V$. If $V = \mathbb{F}^n$ then the notation $\mathbb{P}^{n-1}(\mathbb{F})$ will be used.*

First, we will see the relation between projective geometry and projections. To do this I will define what a projection is.

**Definition 2** *Let $A$ be a set, and $B$ be a subset of $A$. Then we call the map $g : A \to B$ a projection if $g$ has the following property: $g \circ g = p$, which means that $g(g(a) = g(a)$ for all $a \in A$.*

If $V = \mathbb{F}^n$ for some field $\mathbb{F}$, Then the projection we will look at is the projection that maps all nonzero vectors in $V$, to a $n-1$ dimensional object in $V$. One way to do this is to let the $n-1$ dimensional object $A = A_{n-1} \cup A_{n-2} \cup ... \cup A_1 \cup A_0$ such that $A_i = \{(0, ..., 0, 1, x_{n-i+1}, ..., x_{n-1}, x_n) \mid x_i \in \mathbb{F}\}$ and map a point $a = (a_1, a_2, ..., a_n) \neq 0$ to the point in the intersection between $A$ and the line $l = \{(ka_1, ka_2, ..., ka_n) \mid k \in \mathbb{F}\}$, so if $a_1 \neq 0$

then $(a_1, a_2, ..., a_n)$ is mapped to $(1, a_1^{-1}a_2, a_1^{-1}a_3..., a_1^{-1}a_n)$. In the case that $a_1 = 0$, since $a \neq 0$, there must be an $k$ such that $a_k \neq 0$ and $a_m = 0$ for $m < i$ Then $a$ is mapped $a_i^k a$, Then this map is a projecting of $V$ on $A$ such that for any point $b \in A$, the set of vectors which is projected to $b$ is equal the nonzero part of the subspace generated by $b$, so this projection can be seen as projecting $\mathbb{F}^n$ onto the set of 1-dimensional subspaces of $V$

We will now look at some examples of projective lines and spaces over some finite fields, and the projective plane over $\mathbb{R}$ and the complex projective line.

## 1.2 Projective spaces over finite fields

First I want to define what homogenous coordinates is, as this is quite commonly used to name points in a projective geometry:

**Definition 3** *Let $V$ be a vector space over the field $\mathbb{F}$ with dimension $n$. Let $B$ be a set of ordered basis elements in $V$. Then homogenous coordinate $a = [a_1, a_2, ..., a_n], a \neq 0$ is the equivalence class such that $[a_1, a_2, ..., a_n] = [b_1, b_2, ..., b_n]$ only if there exist a $k \in \mathbb{F} - \{0\}$, such that $ka_i = b_i$ for $i = 1, 2, ..., n$. $[0, 0, ..., 0]$ is not a homogenous coordinate.*

Looking at the homogenous coordinates, it is clear that a homogenous coordinate represents the nonzero part of a 1-dimensional subspace in the vector space. Therefore it becomes quite useful when naming the points in a projective space, since a projective space is a set of 1-dimensional subspaces.

The projective line over $\mathbb{Z}_3$ can be seen as the set of 1-dimensional subspaces of the vectorspace $\mathbb{Z}_3 \times \mathbb{Z}_3$. The line contains four points. One can use homogenous coordinates to represent the different points, then $[1, 0], [1, 1], [1, 2], [0, 1]$ represents all the points in the projective line

Look at the vector space $V = \mathbb{Z}_2^3$. By mapping 1-dimensional subspaces of $V$ to points, 2-dimensional subspaces to line, one gets $\mathbb{P}_2(\mathbb{Z}_2)$ which can be illustrated by the the Fano plane shown below:

Since every 1-dimensional subspace of $V$ contains only one nonzero point, it is natural to name the points in the projective plane using the nonzero point of the corresponding 1-dimensional subspace of $V$, this is a consequence of that $V$ is a vector space over $\mathbb{Z}_2$ and $\mathbb{Z}_2$ contains only 2 elements and only 1 nonzero value. Since 1-dimensional subspaces over a vector space $U$ over a field $\mathbb{F}$ can be represented by a nonzero point $u$ in $U$. Since $f * u, f \in \mathbb{F}, f \neq 0$ represents the same subspace, one can say that the points $f * u$ and $u$ maps to the same point in the projective space given by $U$.

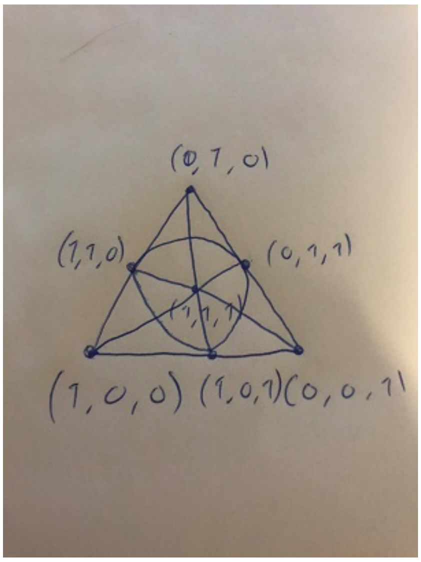


Figure 1.1: The projective plane over $\mathbb{Z}_2$, also known as the Fano plane.

Lets now look at the projective plane over $\mathbb{Z}_3$. As in last case this can be constrcuted by looking at the subspaces of $V = \mathbb{Z}_3^3$. In this case, since each 1-dimensional subspace contains 2 nonzero points. Using the natural basis for $V$, every 1-dimensional subspace can by represented by homogenous coordinates in $V$ such that if a 1-dimensional subspace in $V$ is generated by an point $a = (a_1, a_2, a_3)$, then the corresponding point in the projective plane is represented by the homogenous coordinates $[a_1, a_2, a_3]$. Under is an illustration of the the projective plane with homogenous coordinates:

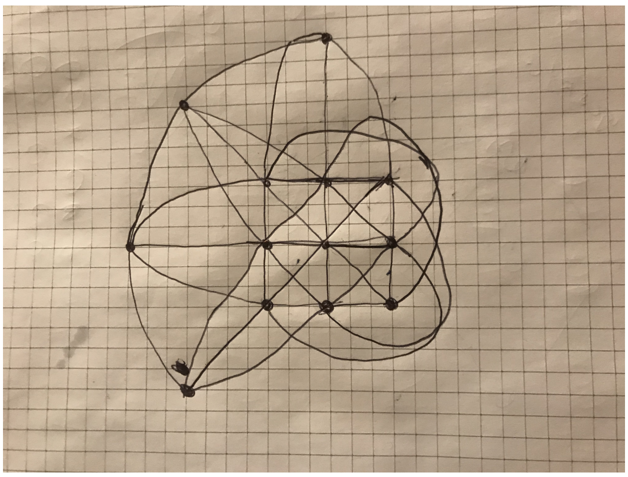

Figure 1.2: The projective plane over $\mathbb{Z}_3$

## 1.3 The real projective plane

Before we look at the real projective plane, we will look at the real projective line, $\mathbb{P}_1(\mathbb{R})$, which is again the set of all lines containing $(0,0)$ in the vector space $\mathbb{R}^2$. Now we can look at the intersection between these lines and the unit circle. Observe that a line intersects the circle at two points which are antipodal points, and that two different lines intersects in different points. Then we can say that the circle is an image of $\mathbb{P}_1(\mathbb{R})$, where an antipodal pair of points represents a point in the projective geometry. If we want that a point in the projetive geometry is represeenented by a unique point, if we denote the points by their angle, we just double the angle, such that the point which creates the angle $\alpha$ goes to the point which creates the point $2\alpha$. observe now if we have two antpodal points $\beta$ and $\pi + \beta$, then both goes to $2\beta$, and also if two points $\alpha$ and $\beta$ both goes to the same point, then either $\alpha = \beta$ or $\alpha$ and $\beta$ are antipodal points. then this is a representation of $\mathbb{P}_1(\mathbb{R})$.

To construct the real projective plane, it is possible to do the same as for the finite fields. An other way of obtaining it is extending the Euclidean plane such that every pair of lines intersect in exactly one point. It is done by adding a point in the "infinity" for every possible direction of a line in the euclidean plane. A line in this extend plane is then a line in the euclidean plane together with the corresponding "infinity point" or the line consisting of all infinity points. This plane is the projective plane of the real(Or isomorphic to it). It is important to see that there is no difference between a "normal" point and a "infinity" point.

## 1.4 The projective Complex line(The riemann sphere)

The projective complex line can be seen as the set of 1-dimensional subspaces of the vector space $\mathbb{C}^2$. It can also be imagined as the extension of the complex plane $(\mathbb{C})$. Since any 1-dimensional subspace of $\mathbb{C}^2$ is generated by a $v \in \mathbb{C}^2$ such that $v = (z_1, z_2)$ such that at least one of $z_i \neq 0$. Since $v$ and $k * v, k \in \mathbb{C} - \{0\}$ both generates the same subspace, one can use homogenous coordinates to name the points in the projective complex line. the points can then be named either $[1, z]$ or $[0, 1]$. If one thinks of the projective space as an extension of vector space of same dimension by adding "infinity" points, then the points on the form $[1, z]$ corresponces to $z$ in the complex number, while $[0, 1]$ represents the added infinity point. The projective complex line

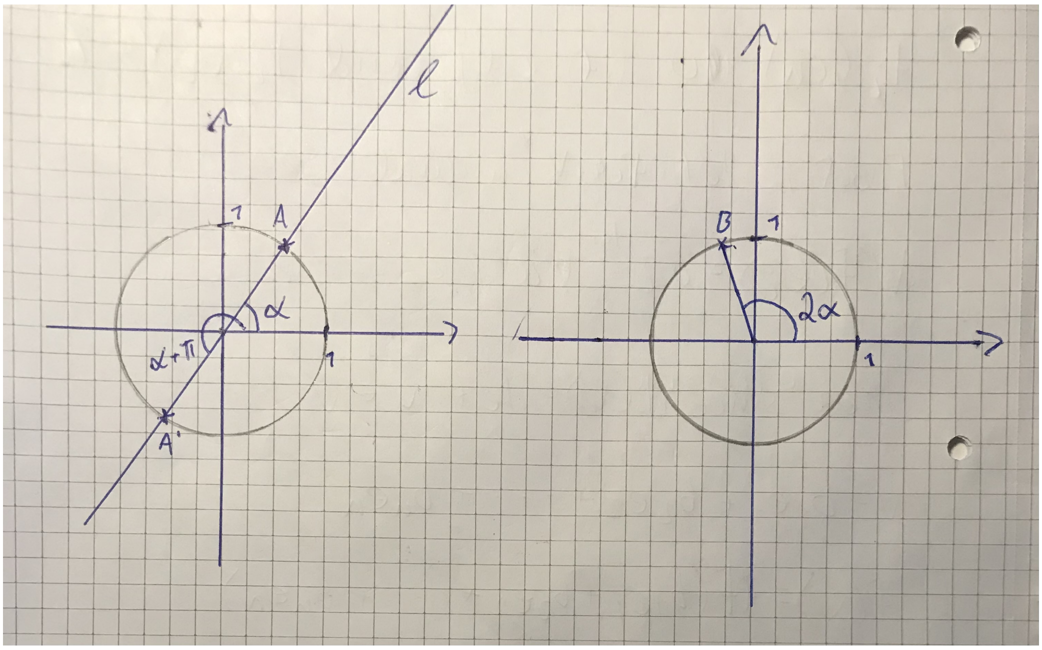


Figure 1.3: If $l$ represents a 1-dimensional subspace of $\mathbb{R}^2$, then $A$ and $A'$ arethe points where $l$ and the unit circle intersects and has the angle $\alpha$ and $\pi + \alpha$, they are antipodal points. By mapping $A$ to the point $B$ such that the angle given by $B$ is $2\alpha$ we get a mapping fro m the unit circle to itself which is surjective, and also $C$ and $C'$ maps to the same point $P$ then either $C = C'$ or $C$ and $C'$ are antipodal points. So $A'$ does also map to $B$. So $B$ represents the subspace of $\mathbb{R}^2$ given by $l$.

can also be represented by the unit sphere in $\mathbb{R}^3$. This is done by saying that $(0, 1, 0)$ is the infinity point. $(0, -1, 0)$ is 0. The complex number are placed such that if $[z, 1], z = a + bi$ then $[z, 1]$ is represented by the second intersection point between the line given by $(0, 0, 1)$ and $(a, b, 0)$, and the unit sphere.

## 1.5 Mappings on a projective space

First we want to introduce some notations.

**Definition 4** *Let $V$ be a $n$-dimensional vector space over a field $\mathbb{F}$. Then the set of linear maps from $V$ to itself that is bijective is called $GL(V)$. If $V = \mathbb{F}^n$ we use $GL_n(\mathbb{F})$ instead of $GL(V)$.*

$GL_n(\mathbb{F})$ is frequently seen as the set of invertible $n \times n$ matrixes over

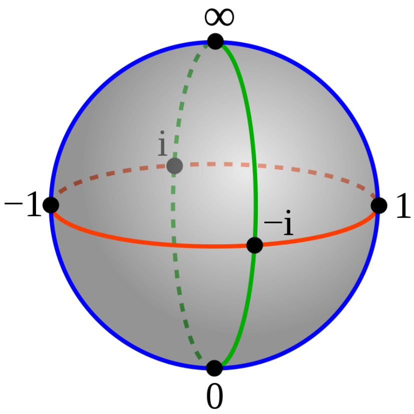


Figure 1.4: The Riemann sphere
Source: https://commons.wikimedia.org/wiki/File:RiemannKugel.svg

the field $\mathbb{F}$ since $\mathbb{F}^n$ has a natural basis. For a arbitrary $n$-dimensional vector space over $\mathbb{F}$, the matrix associated with an linear map $L$ over $V$ is dependent on a chosen basis.

**Definition 5** *Let $A$ be a set containing a finite number of elements. Then the number of elements in $A$ is denoted as $|A|$.*

Example: Let $A = \{1, 2, 3, 4, 7, 10\}$. Counting the number of elements in $A$, we get that $A$ contains 6 elements. Therefore $|A| = 6$

**Definition 6** *Let $A$ and $B$ be sets, and let $g$ be a function from $A$ to $B$;*

$$g : A \to B$$

*If $C \subset A$, we denote $\{g(x) \mid x \in C\}$ as $gC$ or $g(C)$*

Let us first look at the size of $GL_n\mathbb{F}$ if $\mathbb{F}$ is a finite field.

**Theorem 1** *Let $\mathbb{F}$ be a finite field, $|\mathbb{F}| = q = p^k$ where $p$ is a prime number, and $k$ a nonnegative integer. Then the following statement is true:*

$$|GL_n(\mathbb{F})| = \prod_{i=0}^{n-1} q^n - q^i$$

Proof: Since $GL_n(\mathbb{F})$ can be seen as the set of invertible $n \times n$ matrices over $\mathbb{F}$ and a square matrix is invertible if and only if the set of coloumvectors of the matrix are linearly independent. So if we want to construct a matrix of this form, for the first row we can choose any nonzero vector in $\mathbb{F}^n$, so the numbers of vectors we can choose from is $q^n - 1 = q^n - q^0$. for the next we can not choose a linear combination of the last one, which are $q$ vectors, so the numbers of vectors we can choose from is now $q^n - q$. Observe that if we have $i$ numbers of lineraly independent vectors. the number of vectors which can be writen as a linear combinations of these vectors is $q^i$. Therefore when choosing column number $m$ there are $q^n - q^{m-1}$ we can choose from. This gives the formula in the theorem.

Lets look at some examples. Let $\mathbb{F} = \mathbb{Z}_3$ and we are interested in finding the size of $GL_5(\mathbb{F})$. First $|\mathbb{F}| = 3$. Using the formula from the theorem we then get that:

$$|GL_5(\mathbb{F})| = \prod_{i=0}^{4} 3^5 - 3^i = 242 * 240 * 234 * 216 * 162 = 475566474240$$

So $GL_5(\mathbb{F})$ contains 475566474240 elements.

If we have a projective space $\mathbb{P}$, we want to look at automorphism of this $\mathbb{P}$. The mappings from $\mathbb{P}$ to itself such that structure is preserved. With that I mean that an $n$-dimensional object in $\mathbb{P}$ should map onto an $n$-dimensional object in $\mathbb{P}$

**Definition 7** *Let $\mathbb{F}$ be a field, and let $U = \mathbb{P}^n(\mathbb{F})$ be the $n$-dimensional projective space over $\mathbb{F}$. Then the set of mappings $L$ from $U$ into itself such that a $k$-dimensional object in $U$ is sent to a $k$-dimensional object we call $PGL_{n+1}(\mathbb{F})$*

If $\mathbb{P}$ is the set of 1-dimensional subspaces of a vector space $V$, it is clear that any $L \in GL(V)$ conserves structure also if one see $L$ as an mapping from $\mathbb{P}$ to itself.

**Theorem 2** *Let $\mathbb{F}$ be a finite field, $|\mathbb{F}| = q = p^k$ where $p$ is a prime. Then $|PGL_n(\mathbb{F})| = \frac{|GL_n(\mathbb{F})|}{q-1}$ where $PGL_n(\mathbb{F})$ is the set of mappings from $\mathbb{P}^{n-1}(\mathbb{F})$ to itself, where the map is bijective and conserves the structure.*

Proof: As any $L \in GL_n(\mathbb{F})$ conserves subspace structure in the vector space $V = \mathbb{F}^n$, it will also conserve structure in $\mathbb{P}_{n-1}(\mathbb{F})$. It is also clear that $GL_n(\mathbb{F})$ contains any mapping with this property. We are interested to know which mappings which works as the identity on $\mathbb{P}_{n-1}(\mathbb{F})$. Let $L \in GL_n(\mathbb{F})$ be a mapping with this property, then we have that $Lv = u = kv$, for all $v \in V$, where $k \in \mathbb{F}, k \neq 0$, since the subspace generated by $v$ and $u$ must be the same. So this implies that $k * id, k \neq 0$ in $GL_n(\mathbb{F})$ is the only maps which works as the identity on $\mathbb{P}^{n-1}(\mathbb{F})$, and there exist $|\mathbb{F}| - 1 = q - 1$ such maps. So $|PGL_n(\mathbb{F})| = \frac{|GL_n(\mathbb{F})|}{q-1}$

**Theorem 3** *Let $U = \mathbb{P}^1(\mathbb{F})$ be a 1-dimensional projective space over a finite field $\mathbb{F}$ such that $|F| = q = p^k$ then any map $L \in PGL_2(\mathbb{F})$ can be constructed by taking 3 distinct points in $U$ and choose 3 distinct points to map them to. There is no restriction to which points one can map to. So the size of $PGL_2(\mathbb{F})$ is:*

$$|PGL_2(\mathbb{F})| = (q+1)q(q-1)$$

Proof: Let $[a,b],[c,d] \in \mathbb{P}^1(\mathbb{F})$, such that $[a,b] \neq [c,d]$. Then there exists a mapping $L \in PGL_2(\mathbb{F})$ which has the property that $L([1,0]) = [a,b]$ and $L([0,1]) = [c,d]$. Let $L' \in GL_2(\mathbb{F})$ be a linear map such that $L'A = L(A)$ where $A$ is a 1-dimensional subspace of $\mathbb{F}^2$, which also represents a point in $\mathbb{P}^1(\mathbb{F})$. Then we know that $L'(1,0) = k_1(a,b)$ and $L(0,1) = k_2(c,d)$ for some $k_1, k_2 \in \mathbb{F} - \{0\}$. Then we get that

$$L(1,1) = L(1,0) + L(0,1) = k_1(a,b) + k_2(c,d)$$

, And $L([1,1]) = [k_1a + k_2c, k_1b + k_2d]$. Observe that depending on $L$, $[k_1a + k_2c, k_1b + k_2d]$ can be any point in $\mathbb{P}^1(\mathbb{F})$ except $[a,b]$ and $[c,d]$, so we can atleast find a map $L \in PGL_2(\mathbb{F})$ such that $[1,0],[0,1]$ and $[1,1]$ can be mapped to any selection of three different points, This gives us that there are at least $(q+1)q(q-1)$ different mappings. Given that the first theorem gives us that there are only $(q+1)q(q-1)$ different mappings, we have that the map is determined by the mapping of three seperate points.

**Definition 8** *Let $\mathbb{F}$ be a field and let $V$ be a vector space over $\mathbb{F}$, then the set of all linear maps of the form $V \to \mathbb{F}$ is called the dual space of $V$ and is denoted $V^*$.*

**Theorem 4** *Let $\mathbb{F}$ be a field, and let $V$ be an $n$-dimensional vector space over $\mathbb{F}$ with an ordered basis $B = \{e_1, e_2, ..., e_n\}$. Let $B' = \{f_1, f_2, ..., f_n\}$, $f_i \in V^*$, where $f_i(e_j)$ is 0 if $i \neq j$ and 1 if $i = j$. Then $B'$ is a basis for $V^*$ and the linear map $L : V \to V^*$ given by $L(e_i) = f_i$ is an isomorphy from $V$ to $V^*$*

Proof: First we show that $B'$ is a basis for $V^*$. Let $f \in V^*$. Look at the values of $f(e_i) = d_i$. Let $v \in V$, Then $v = \sum_{i=1}^n a_ie_i$ for some $a_i \in \mathbb{F}$ so

$$f(v) = f(\sum_{i=1}^n a_ie_i) = \sum_{i=1}^n f(a_ie_i) = \sum_{i=1}^n a_if(e_i) = \sum_{i=1}^n a_id_i$$

This implies that $f = \sum_{i=1}^n d_if_i$, so $B'$ generates $V^*$. It is also clear that $B'$ is linear independent since if $f' = \sum_{i=1}^n b_if_i$ such that for at least one $i, b_i \neq 0$. Then $f(e_i) = b_i \neq 0$ so $f \neq 0$ and $B'$ is linearly independent and therefore a basis.

**Definition 9** *Let $\mathbb{F}$ be a field and $V$ a vector space over $\mathbb{F}$. Then the set containing all $k$-dimensional subspaces of $V$ is denoted with $Gr_k(V)$ and is called the the grassmanian of $k$-dimensional subspace of $V$. If $V = \mathbb{F}^n$ we usually denote this by $Gr_{k,n}(\mathbb{F})$*

If $V$ is a finite dimensional with dimension $n$, we want to show that it is possible to construct an isomorphy between $Gr_k(V)$ and $Gr_{n-k}(V)$. But first I will prove some statements that will help us.

**Theorem 5** *Let $\mathbb{F}$ be a field and $V$ an $n$-dimensional vector space over $\mathbb{F}$. let $L, L' \in V^* - \{0\}$, then $\ker(L) = \ker(L')$ if and only if $L = aL'$ where $a \in \mathbb{F} - \{0\}$*

Proof of " $\Longleftarrow$ ": Assume $L = aL', a \neq 0$. Then if $v \in \ker(L)$ we get this: $L'v = a^{-1}Lv = a^{-1} * 0 = 0$ so $v \in \ker(V')$. If $u \in \ker(V')$ then $Lv = aL'v = a * 0 = 0$, so $u \in \ker(L)$. This implies that $\ker(V) = \ker(V')$.

Proof of " $\Longrightarrow$ ": Assume $\ker(V) = \ker(V')$, Let $B = \{b_1, b_2, ..., b_{n-1}\}$ be a basis for $\ker(V)$. choose $u \in V - \ker(V)$, then $B \cup \{u\}$ is a basis for $V$. Since $u \notin \ker(V)$ we have that $Lu = k$ and $L'u = k'$ where $k \neq 0$ and $k' \neq 0$. Let $v \in V$, then $v = cu + \sum_{i=1}^{n-1} c_i b_i$ so $Lv = ak = akk'^{-1}k' = kk'^{-1}L'v$. This implies that $L = aL'$ where $a = kk'^{-1}$

**Theorem 6** *Let $\mathbb{F}$ be a field, and $V$ an $n$-dimensional vector space over $\mathbb{F}$. Let $A = \{f_1, f_2, ..., f_k \mid f_i \in V^* - \{0\}\}$ with $k < n$ such that $A$ is a lineary independent set. Then the intersection of the kernels of the maps in $A$ is a subspace of $V$ with dimension $n - k$.*

$$U = \bigcap_{i=1}^{k} \ker(f_i)$$

*and $dim(U) = n - k$*

Proof: We prove this by induction. If $k = 1$, so $A = \{f_1\}$.Since $f_1$ is a nonzero linear map from $V$ to $\mathbb{F}$, we get that $\ker(f_1)$ has dimension $n - 1$.

Now assume that we know that the theorem is true for $k = m, m < n$, let $A = \{f_1, f_2, ..., f_{m+1} \mid f_i \in V^* - \{0\}\}$ be a lineary independent set. Since $A - \{f_{m+1}\}$ also is a lineary independent set, containing m different mappings, we get that $U = \cap_{i=1}^{m} \ker(f_i)$ is an $(n - m)$-dimensional subset of $V$. Since $\ker(f_{m+1})$ is a $n - 1$ dimensional space, we get that $W = U \cup \ker(f_{m+1})$ is a subspace of $V$ with dimension $n - m$ or $n - (m + 1)$. Assume that $W$ has dimension $n - m$. This implies that $W = U$, so $U \subset \ker(f_{m+1})$.

Now look at span$(A)$. From that $A$ is a set of $m+1$ linearly independent elements, we get that span$(A)$ is a $m+1$ dimensional subspace of $V^*$. Since for any $g \in \text{span}(A), W \subset \ker(g)$. So the kernel of $g$ can be generated by a basis $B$ for $W$ together with $m-1$ linearly independent elements in $V$. Given that for $h, h' \in V^*, \ker(h) = \ker(h')$ implies that $h = ah'$ we get that span$(A)$ has dimension $m$ which is a contradiction. Therefore the assumtion that $W$ is a $n-m$ dimensional subspace is incorrect, so $W$ is a $n-(m+1)$ dimensional subspace of $V$. This concludes the induction proof.

**Theorem 7** *Let $\mathbb{F}$ be a field and $V$ an $n$-dimensional vector space of $\mathbb{F}$. Let $W$ be a subspace of $V^*$ where $B_1 = \{f_1, f_2, ..., f_k\}$ and $B_2 = \{g_1, g_2, ..., g_k\}$ both are basises for $W$. Then the following is true:*

$$\bigcap_{i=1}^{k} \ker(f_i) = \bigcap_{i=1}^{k} \ker(g_i) = \bigcap_{f \in W} \ker(f)$$

Proof: Since $B_1$ is a basis for $W$ and $B_2$ is a subset of $W$, we get that

$$g_i = \sum_{j=1}^{k} a_{i,j} f_j$$

So if $U = \cap_{i=1}^{k} \ker(f_i), U' = \cap_{i=1}^{k} \ker(g_i)$, and $v \in U$ then:

$$g_i(v) = \sum_{j=1}^{k} a_{i,j} f_j = 0$$

So $U \subset ker(g_i)$ for all $i = 1, 2, ..., k$. But this implies $U \subset U'$, but since both $U$ and $U'$ are subspaces of $V$ with dimension $n-k$, we get that $U = U'$

**Theorem 8** *Let $\mathbb{F}$ be a field, and $V$ an $n$-dimensional vector space over $\mathbb{F}$ with a basis $B$. Let $W$ be a subspace of $V$, where $B_1 = \{u_1, u_2, ..., u_k\}$ and $B_2 = \{v_1, v_2, ..., v_k\}$ both are basises of $W$. Then the following statement is true:*

$$\bigcap_{i=1}^{k} \ker(Lu_i) = \bigcap_{i=1}^{k} \ker(Lv_i)$$

*where $L$ is the isomorphism between $V$ and $V^*$ given by the basis $B$*

Proof: Since $B_1$ and $B_2$ both are are basises for the same subspace $W$ in $V$ we get that the sets $\{Lu_1, Lu_2, ..., Lu_k\}$ and $\{Lv_1, Lv_2, ..., Lv_k\}$ both are basises for the same subspace $W'$ of $V^*$. Last theorem then gives us that:

$$\bigcap_{i=1}^{k} \ker(Lu_i) = \bigcap_{i=1}^{k} \ker(Lv_i)$$

**Theorem 9** *Let $\mathbb{F}$ be a field, and $V$ an $n$-dimensional vector space over $\mathbb{F}$. Then there exists an isomorphism between $Gr_k(V)$ and $Gr_{n-k}(V)$. Given a basis $B$ for $V$ such an isomorphism can be constructed.*

Proof: Let $B = \{e_1, e_2, ..., e_n\}$ be a basis for $V$, then $B' = \{f_1, f_2, ..., f_n\}$, such that $f_i = Le_i$ where $L$ is the isomorphism between $V$ and $V^*$ induced by the basis $B$. Then $B'$ is a basis for $V^*$. Define the map $g : Gr_k(V) \to Gr_{n-k}(V)$ such that if $W \in Gr_k(V)$, then

$$g(W) = \bigcap_{f \in LW} \ker(f) = \bigcap_{f \in S'} \ker(f) = \bigcap_{v \in S} \ker(Lv) = U$$

where $S$ is any basis of the subspace $W$ and $S'$ is any basis for $LW = \{Lv \mid v \in W\}$. Since $W$ is a $k$-dimensional subspace of $V$ and $L$ is an isomorphism, we get that $LW$ is a $k$-dimensional subspace of $V^*$. This implies that $U$ is a $n-k$ dimensional subspace of $V$. So it is clear that $g$ is a map. It is enough to prove injectiveness of this map, since that would imply that there also exists an injective map from $Gr_{n-k}(V)$ to $Gr_k(V)$ and therefore $g$ is bijective, and an isomorphism. Let $W, W' \in Gr_k(V)$ such that $W \neq W'$. This implies that $LW \neq LW'$ so there exist an $f_0$ in $LW$ such that $f_0 \notin LW'$. So let $S'$ be a basis of $LW'$, then it is clear that $A = S' \cup \{f_0\}$ is a lineary independent set. It is also clear that

$$\bigcap_{f \in LW} \ker(f) \cap \bigcap_{f \in LW'} \ker(f) \subset \bigcap_{f \in A} \ker(f)$$

But $\bigcap_{f \in A} \ker(f)$ is a subset of $V$ with dimension $n-k-1$, so $\bigcap_{f \in LW} \ker(f) \cap \bigcap_{f \in LW'} \ker(f)$ has dimension less than $n-k$. So

$$\bigcap_{f \in LW} \ker(f) \neq \bigcap_{f \in LW'} \ker(f)$$

. So $g$ is bijective and therefore an isomorphism between $Gr_k(V)$ and $Gr_{n-k}(V)$.

**Theorem 10** *Let $\mathbb{F}$ be a finite field, where $|\mathbb{F}| = q = p^k$ where $p$ is prime, then the following is true:*

$$|Gr_{m,n}(\mathbb{F})| = \frac{|GL_n(\mathbb{F})|}{|GL_m(\mathbb{F})||GL_{m-n}(\mathbb{F})|q^{(n-m)m}}$$

Proof: Let $W, U \in Gr_{m,n}(\mathbb{F})$ then there exist $L \in GL_n(\mathbb{F})$ such that $L(W) = U$. We want to find out how many $L \in GL_n(\mathbb{F})$ which has this properties. It is equivalent to look at the number of $L \in GL_n(\mathbb{F})$ such that $L(W) = W$ since $W$ and $U$ are isomorphic. So let $W = \{(x_1, x_2, ..., x_m, 0, 0, ..., 0)^T \mid$

$x_i \in K, i = 0, 1, ..., m\}$ then if $L$ has this property then the corresponding $n \times n$ matrix $A_L$ is on this form:

$$\begin{pmatrix} B_1 & V \\ O & B_2 \end{pmatrix}$$

where $B_1$ is an invertable $m \times m$ matrix, $V$ is an arbitrary $m \times (n-m)$ matrix, $O$ is the $(n-m) \times m$ 0-matrix, and $B_2$ is an invertable $(n-m) \times (n-m)$ matrix. Then $B_1$ has $|GL_m(\mathbb{F})|$ alternatives, $V$ has $|\mathbb{F}|^{(n-m)m} = q^{(n-m)m}$ alternatives, and $B_2$ has $|GL_{n-m}(\mathbb{F})|$ alternatives. This gives us that there exist $|GL_m(\mathbb{F})||GL_{(n-m)}(\mathbb{F})|q^{(n-m)m} = Q$ amounts of such linear projections. So the size of $Gr_{m,n}(\mathbb{F})$ is equal to the size of $GL_n(\mathbb{F})$ divided by Q.

Observe that the formula implies that if $\mathbb{F}$ is a finite field, then $Gr_{k,n}(\mathbb{F}) = Gr_{n-k,n}(\mathbb{F})$. This is expected since we before has shown that there exist an isomorphy between $Gr_{k,n}(\mathbb{F})$ and $Gr_{n-k,n}(\mathbb{F})$, and since both of these set contains a finite number of elements, they must have the same number of elements

As an example, let us look at the field $\mathbb{Z}_2$ and the size of $Gr_{k,3}(\mathbb{Z}_2)$ for $k = 1, 2$ This is the same as to look at the number of points and lines in the projective plane over $\mathbb{Z}_2$, or the Fano's plane. The formula gives us this:

$$|Gr_{1,3}(\mathbb{Z}_2)| = \frac{|GL_3(\mathbb{Z}_2)|}{|GL_1(\mathbb{Z}_2)||GL_2(\mathbb{Z}_2)|2^2} = \frac{7 * 6 * 4}{1 * 3 * 2 * 2^2} = \frac{168}{24} = 7$$

So there is 7 points in $\mathbb{P}^2(\mathbb{Z}_2)$, which is the correct number of point. Since $|Gr_{2,3}(\mathbb{Z}_2)| = |Gr_{1,3}(\mathbb{Z}_2)| = 7$, The number of lines is also 7, counting the lines of the drawing of the Fano's plane coincide with this number.

## 1.6 Plücker Embedings

First we will look at the determinant of a matrix.

**Definition 10** *Let $\mathbb{F}$ be a field and $V = \mathbb{F}^n$ is an vector space, then if we look at an $n \times n$ matrix as a matrix containing $n$- column vectors in $\mathbb{F}^n$, then the determinant is the only mapping $f : V \times V \times ... \times V \to \mathbb{F}$ ($V$ is repeated $n$ times) that has the following properties: The determinant of the identity should be* 1*:*

$$f(e_1 \times e_2 \times ... \times e_n) = 1$$

*It is linear in each coordinate, for all $i = 1, 2, ..., n$*

$$f(v_1 \times v_2 \times ... \times av_i + bu_i \times ... \times v_n)$$

$$= af(v_1 \times v_2 \times ... \times v_i \times ... \times v_n) + bf(v_1 \times v_2 \times ... \times u_i \times ... \times v_n)$$

*It is alternating which means that:*

$$f(v_1, v_2, ..., v_n) = 0$$

*if* $\{v_1, v_2, ...v_n\}$ *is a lineary independent set.*

We will confirm that this mapping is unique and that it is the same as what we know as the determinant of a matrix. Assume we have a mapping $f : V \times V \times ... \times V \to \mathbb{F}$ ($V$ is repeated $n$ times). Let $A$ be an $n \times n$ matrix. From the properties we have that adding a scaled colomn to another column does not change the value of the function: $f(A) = f(A_{i,j})$ where $A_{i,j,a}$ is the matrix equal to $A$ everywhere except that column $i$ is equal to $v_i + av_j$. This opertation can be done by multipliying with the matrix $B$ from the right. $B = I + a\Delta_{i,j}$ where $\Delta_{i,j}$ is the matrix containing only zeros, except the element in the intersecting of row $i$ and column $j$, this element is equal to 1. The other operation we can do is to multiply column $i$ with $a \neq 0$ and column $j$ with $a^{-1}$, This can be achieved by multiplying the matrix $A$ with the matrix $C$ from the left, where $C$ is the diagonal matrix containing 1s in the diagonal except for the $i$th place which contains a $a$ and $j$th place which contains a $a^{-1}$. Observe that both of these types of matrixes has determinant 1 and therefore $\det(A) = \det(AB) = \det(AC)$. If $A$ is a matrix where the column vectors are lineraly independent, that $A$ is invertible. it is possible to find matrixes $B_i, i = 1, 2, ..., k$ such that $AB_1B_2...B_k$ is a diagonal matrix such that every diagonal element is 1 except for the last diagonal element. This last element is equal to $f(A)$ but also equal to $\det(A)$ so $f(A) = \det(A)$ if $A$ is invertible. If $A$ is not invertible then $f(A) = 0$, and since also $\det(A) = 0$ we have that $f = \det$. So det is the only mapping that has these properties.

**Theorem 11** *Let* $\mathbb{F}$ *be a field and* $A$ *an* $4 \times 4$ *matrix over* $\mathbb{F}$ *such that:*

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

*Then the following is true:*

$$\det(A) = \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \det\begin{pmatrix} c_3 & c_4 \\ d_3 & d_4 \end{pmatrix} - \det\begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} \det\begin{pmatrix} c_2 & c_4 \\ d_2 & d_4 \end{pmatrix}$$

$$+\det\begin{pmatrix}a_1 & a_4\\ b_1 & b_4\end{pmatrix}\det\begin{pmatrix}c_2 & c_3\\ d_2 & d_3\end{pmatrix}-\det\begin{pmatrix}a_2 & a_3\\ b_2 & b_3\end{pmatrix}\det\begin{pmatrix}c_1 & c_4\\ d_1 & d_4\end{pmatrix}$$

$$+\det\begin{pmatrix}a_2 & a_4\\ b_2 & b_4\end{pmatrix}\det\begin{pmatrix}c_1 & c_3\\ d_1 & d_3\end{pmatrix}-\det\begin{pmatrix}a_3 & a_4\\ b_3 & b_4\end{pmatrix}\det\begin{pmatrix}c_1 & c_2\\ d_1 & d_2\end{pmatrix}$$

Proof: we will show that this equation is true. If $A = I$ we get that the right side is equal to 1. The right side is also linear in a colom if the the other coloms are fixed. Last if we add a linear sum of the other colums to one of the colums not in the linear sum, the determinant does not change, which implies that if two coloms switches places the determinant changes to the additive inverse. so since the right side satisfies the these three conditions, we know that it is equal to the determinant.

Plücker embedings is a way of map a grassmanian set into a projective space. It is a mapping taking $Gr_{m,n}(\mathbb{F})$ into $\mathbb{P}^{\binom{n}{m}-1}(\mathbb{F})$. The mapping is defined in this way:

**Definition 11** *Let $V$ be a $n$-dimensional vector space over a finite field $\mathbb{F}$ where $|\mathbb{F}| = q = p^k$ where $p$ is prime. Then the map $L : Gr_{m,n}(V) \to p^{\binom{n}{m}-1}(F)$ such that for a $W \in Gr_{m,n}$ and $B'$ a basis of $W$ such that $B_i \in B, B_i = (b_{i,1}, b_{i,2}, ..., b_{i,n})$ and let the matrix $M$ be the $m \times n$ matrix where $m_{i,j} = b_{i,j}$ then let $L(W) = x_W$ where $x_W = [x_1, x_2, ..., x_{\binom{n}{m}}]$ where $x_1$ is equal to the determinant of the matrix one gets when removing the $n-m$ last colons of $M$, and continue.*

Now we want to look at an example of a Plücker embeding. Lets look at $Gr_{2,4}(\mathbb{F})$ for some field $\mathbb{F}$. Let $W \in Gr_{2,4}(\mathbb{F})$, and let $B = \{(a, b, c, d), (e, f, g, h)\}$ be a basis for $W$. Then the plücker embedding $L : Gr_{2,4}(V) \to \mathbb{P}^{\binom{4}{2}-1}(\mathbb{F})$ works like this on $W$:

$$L(W) = (det\begin{pmatrix}a & b\\ e & f\end{pmatrix}, det\begin{pmatrix}a & c\\ e & g\end{pmatrix}, det\begin{pmatrix}a & d\\ e & h\end{pmatrix}, det\begin{pmatrix}b & c\\ f & g\end{pmatrix}, det\begin{pmatrix}b & d\\ f & h\end{pmatrix}, det\begin{pmatrix}c & d\\ g & h\end{pmatrix})$$

$$= [d_{1,2}, d_{1,3}, d_{1,4}, d_{2,3}, d_{2,4}, d_{3,4}]$$

We are interesteded in which points $x \in \mathbb{P}^{\binom{4}{2}-1}(\mathbb{F})$ such that there exists $U \in Gr_{2,4}(\mathbb{F})$ where $LU = x$. In other words the image of $L$. Let $W \in Gr_{2,4}(\mathbb{F})$ such that $\{(a, b, c, d), (e, f, g, h)\}$ is a basis for $W$. Then look at the matrix:

$$A = \begin{pmatrix}a & b & c & d\\ e & f & g & h\\ 0 & b & c & d\\ 0 & f & g & h\end{pmatrix}$$

It is obvious that this matrix is not invertible, as the rows are not linearly independent. If we denote row $i$ as $r_i$, then $r_2 = \frac{e}{a}(r_1 - r_3) + r_4$. Therefore the determinant of $A$ must be 0. If we develope the determinant from the two first rows we get this:

$$\det(A) = \begin{pmatrix} a & b \\ e & f \end{pmatrix} \begin{pmatrix} c & d \\ g & h \end{pmatrix} - \begin{pmatrix} a & c \\ e & g \end{pmatrix} \begin{pmatrix} b & d \\ f & h \end{pmatrix} + \begin{pmatrix} a & d \\ e & h \end{pmatrix} \begin{pmatrix} b & c \\ f & g \end{pmatrix}$$
$$= d_{1,2}d_{3,4} - d_{1,3}d_{2,4} + d_{1,4}d_{2,3} = 0$$

So if $x$ is in the image of $L$, we have that $x = [x_1, x_2, x_3, x_4, x_5, x_6]$ must satisfy the following equation:

$$x_1x_6 - x_2x_5 + x_3x_4 = 0$$

If we let $\mathbb{F} = \mathbb{Z}_2$ and look at the plücker embeding $L$ mapping $Gr_{2,4}(\mathbb{Z}_2)$ into $\mathbb{P}^5$. First we will calculate the size of $Gr_{2,4}(\mathbb{Z}_2)$:

$$|Gr_{2,4}(\mathbb{Z}_2)| = \frac{|GL_4(\mathbb{Z}_2)|}{|GL_2(\mathbb{Z}_2)||GL_2(\mathbb{Z}_2)|2^4} = \frac{15 * 14 * 12 * 8}{3 * 2 * 3 * 2 * 2^4} = 35$$

Then we look at the number of points $x$ in $\mathbb{P}(\mathbb{Z}_2)^5, x = [x_1, x_2, x_3, x_4, x_5, x_6]$ such that the equation $x_1x_6 - x_2x_5 + x_3x_4 = 0$ holds. We know that if $x_1x_6 = 0$, we get that $x_2x_5 = x_3x_4$. $x_1x_6 = 0$ if either of $x_1$ or $x_6$ or both is 0. if $x_2x_5 = x_3x_4 = 1$ that is only if $x_2 = x_3 = x_4 = x_5 = 1$ and if $x_2x_5 = x_3x_4 = 0$, we have 9 different ways of choosing the values of $x_2, x_3, x_4, x_5$, but $x_i = 0$ for all $i$ is not valid so need to remove 1 possible way, so if $x_1x_6 = 0$ there are 29 ways to arrange the points. If $x_1x_6 = 1$ there are 6 solutions, and $29 + 6 = 35$. So there are an equal number of points in $\mathbb{P}^5(\mathbb{Z}_2)$ which satisfies the condition as the size of $Gr_{2,4}(\mathbb{Z}_2)$

In general, the image of an plücker embedding of $Gr_{k,m}(\mathbb{F})$, which is a subset of $\mathbb{P}^{\binom{n}{m}-1}(\mathbb{F})$, is the solution to several homogenous equations, as the one in the previous example.

**Definition 12** *Let $V$ be a finite-dimensional vectorspace with dimension $n$ and $x, y \in V$,, then the wegde product (using the symbol $\wedge$)*

$$x \wedge y = (d_{1,2}, d_{1,3}, ..., d_{1,n}, d_{2,3}, d_{2,3}, ..., d_{n-1,n})$$

*where* $d_{i,j} = x_iy_j - x_jy_i$

This operation can be exteded to work for a finite set of vectors

**Definition 13** *Let $V$ be a finite-dimensional vectorspace with dimension $n$ and $x_i \in V$ for $i \leq m$ be a set of $m$ vectors in $V$, then the wegde product*

$$\wedge_{i=1}^{m} x_i = \times_{i_1 < i_2 < \ldots < i_m} d_{i_1, i_2, \ldots, i_m} \tag{1.1}$$

*where $d_{i_1, i_2, \ldots, i_m}$ is the minor of the coloums $i_1, i_2, \ldots, i_m$ of the matrix $A$, where $A$ is the $m \times n$ matrix where row $i$ is $x_i$*

**Theorem 12** *Let $V$ be finite-dimensional vector space with dimension $n$, and $A = \{x_i \mid x_i \in V, i = 1, 2, \ldots, m\}$ be a set of $m$ vectors in $V$, then $\wedge_{i=1}^{m} x_i = 0$ if and only if $A$ is a lineary dependent set.*

From this theorem it is obvious that if a vegde product contains two of the same vectors, the product is equal to 0. Also if the vector space has dimension $n$, vegde product containing more than $n$ vectors must also be 0, since the vectors in the wegde product must be linearly dependent

**Definition 14** *Let $V$ be a $n$-dimensional vector space of a field $\mathbb{F}$. Then we define a algebra, $\mathbb{F} \times V \times V \wedge V \times V \wedge V \wedge V \times \ldots \times V \wedge V \wedge \ldots \wedge V$ with the natural vector space operation and*

$$(k_1, u_{1,1}, u_{2,1} \wedge u_{2,2}, u_{3,1} \wedge u_{3,2} \wedge u_{3,3} \times \ldots \times u_{n,1} \wedge u_{n,2} \wedge \ldots \wedge u_{n,n})$$

$$*(k_2, v_{1,1}, v_{2,1} \wedge v_{2,2}, v_{3,1} \wedge v_{3,2} \wedge v_{3,3} \times \ldots \times v_{n,1} \wedge v_{n,2} \wedge \ldots \wedge v_{n,n})$$

$$= (k_1 k_2, k_1 v_1 + k_2 u_1, k_1 u_{2,1} \wedge u_{2,2} + u_{1,1} \wedge v_{1,1}, \ldots,$$

$$k_1 \wedge_{i=1}^{m} v_{m,i} + \sum_{i=1}^{m} \wedge_{j=1}^{i} u_{i,j} \wedge_{j=1}^{m-i} v_{m-i,j} + k_2 \wedge_{i=1}^{m} u_{m,i}, \ldots,$$

$$k_1 \wedge_{i=1}^{n} v_{n,i} + \sum_{i=1}^{n} \wedge_{j=1}^{i} u_{i,j} \wedge_{j=1}^{n-i} v_{n-i,j} + k_2 \wedge_{i=1}^{n} u_{n,i})$$

NTNU

Norwegian University of
Science and Technology