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Existence of Fatou Components in Two Complex Variables

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Sammendrag

I denne oppgaven viser vi at det eksisterer holomorfe funksjoner i $\mathbb{C}^2$ som har en invariant, ikke-rekkurent Fatou komponent, som er tiltrekkende. Vi viser og at denne komponenten er sammenhengende, men ikke enkeltsammenhengende.

Abstract

In this thesis we show that there exists holomorphic functions of $\mathbb{C}^2$ having an invariant, non-recurrent Fatou component which is attracting. We also show that the component is connected, but not simply connnected.

Contents

1 Introduction

1.1 Preliminary Definitions

Complex dynamics studies iterations of complex valued function in $\mathbb{C}^n$. When F is a function of several complex variables, the study of the behavior of its iterates gives rise to the *Fatou* and *Julia* sets. To properly define these we will first define what it means to be a *normal family* of function.

Definition 1.1. *Let $U \subseteq \mathbb{C}^n$ and let $\mathcal{F}$ be a family of holomorphic functions $f : U \longrightarrow \mathbb{C}^n$. The family is normal if for every sequence of functions, there is a subsequence which converges uniformly on compact subsets of U.*

We will denote the iterates of functions as follows:

$$f^k = f \circ f^{k-1}, \qquad f^0 = \mathrm{Id}.$$

Now we can properly define Fatou and Julia sets.

Definition 1.2. *A point $p \in \mathbb{C}^n$ belongs to the Fatou set, if there is a neighborhood U of p so that the family of iterates of $\{f^k|_U\}$ is normal. The Julia set is the complement of the Fatou set.*

A Fatou component is a connected subset of the Fatou set. We are after a Fatou component which is not simply connected. Furthermore, a Fatou component W is said to invariant if $F(W) = W$. It's attracting towards a fixed point, if there exists a point $p \in \overline{W}$ so that $\lim_{n\to\infty} F^n(p) = p$ for all $z \in W$. We say that if $p \in \partial W$, the component is non-recurrent.

The goal of this text will be to show existence of such a domain in $\mathbb{C}^2$, however the techniques used can further be generalized to prove existence of Fatou components in $\mathbb{C}^n$.

1.2 Outline of the Text

Through iteration of a germ of a biholomorphism we will use several tools to come to the desired conclusion.

Choosing a suitable function we will first classify how it behaves through iterations, namely find the domain in which we have convergence towards our fix point. Simultaneously we will be studying rate of convergence and behaviour of iterates. Choosing a suitable domain near the fixed point in the boundary will allow us to classify wherein the iterates converge.

Afterwards we will construct, through the local basin of attraction, an open set which will in the end be the Fatou component and show look at its topology, namely that it is not simply connected.

Lastly before the terminal proof, we have a result thanks to Pöchel[2] regarding the divisors of our constant λ. This will allow us to construct new coordinates for our function, to set up our proof. We also consider hyperbolic distance via the Kobayashi metric to estimate distances close to the fixed point.

2 The Existence of Fatou Components in $\mathbb{C}^2$

2.1 The Local Basin of Attraction

Let

$$B := \{(z,w) : -\frac{\pi}{8} < \arg(zw) < \frac{\pi}{8}, |zw| < \epsilon,\ |z|^{100} < |zw|^3, |w|^{100} < |zw|^3\}. \tag{1}$$

We will show that this set is a local basin of attraction to the origin of the function

$$F(z,w) = \big(z\lambda(1-\frac{1}{2}zw), w\overline{\lambda}(1-\frac{1}{2}zw)\big) + \mathcal{O}(||(z,w)||^{100}). \tag{2}$$

This means that repeated iterations of points in our set, will converge towards the origin. To this end we start with a result:

Lemma 2.1. *Let $\tilde{F} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be defined by*

$$(z,w) \longrightarrow (z\lambda(1-\frac{1}{2}zw), w\overline{\lambda}(1-\frac{1}{2}zw)) + \mathcal{O}((zw)^3) \tag{3}$$

where $\lambda = e^{2\pi i r}, r \in \mathbb{R} \setminus \mathbb{Q}$ and let

$$B := \{(z,w) : -\frac{\pi}{8} < \arg(zw) < \frac{\pi}{8}, |zw| < \epsilon,\ |z|^{100} < |zw|^3, |w|^{100} < |zw|^3\}, \tag{4}$$

then $\tilde{F}(B) \subset B$.

Proof. Pick an $\epsilon > 0$, $(z,w) \in B$, and let $\tilde{F}(z,w) = (z_1, w_1)$. We can then evaluate the product

$$z_1 w_1 = zw|\lambda|(1-\frac{1}{2}zw)^2 + \mathcal{O}((zw)^3) = zw[(1-zw) + \mathcal{O}((zw)^2)]. \tag{5}$$

First we evaluate the modulus

$$|z_1 w_1| = |zw|\,|1 - zw + \mathcal{O}((zw)^2)| \tag{6}$$

$$\tag{7}$$

This expression is less than ϵ, whenever

$$|1 - zw + \mathcal{O}((zw)^2)|^2 < 1. \tag{8}$$

We can see this from

$$\begin{aligned}
|1 - zw + \mathcal{O}((zw)^2)|^2 &= 1 - 2\mathrm{Re}(zw + \mathcal{O}((zw)^2))) + |zw + \mathcal{O}((zw^2))|^2 \\
&= 1 - 2\mathrm{Re}(zw) - 2\mathrm{Re}(\mathcal{O}((zw)^2)) + |zw|^2\,|1 + \mathcal{O}((zw))|^2 \\
&\le 1 - 2\mathrm{Re}(zw) + C|zw|^2 \\
&= 1 - 2|zw|\cos(\arg(zw)) + C|zw|^2 \\
&\le 1 - 2|zw|\frac{1}{2} + C|zw|^2 \\
&\le 1 - |zw| + \frac{1}{2}|zw| = 1 - \frac{1}{2}|zw|
\end{aligned} \tag{9}$$

where we have chosen ϵ so that

$$0 < 1 - \frac{1}{2}|zw| < 1 \tag{10}$$

For the argument we do a coordinate change, $X := \frac{1}{zw}$, so that

$$X_1 = \frac{1}{\frac{1}{X}(1 - \frac{1}{X} + \mathcal{O}(\frac{1}{X^2}))} \tag{11}$$

$$= \frac{X}{1 - \frac{1}{X} + \mathcal{O}(\frac{1}{X^2})}. \tag{12}$$

and our region then changes to

$$W = \{X \in C : -\frac{\pi}{8} < \arg(X) < \frac{\pi}{8}, |X| > \frac{1}{\epsilon}\} \tag{13}$$

We recognize (12) as the sum of a geometric series, so we can write X_1 as

$$X \sum_{k=0}^{\infty} (\frac{1}{X} + \mathcal{O}(\frac{1}{X^2}))^k = 1 + X + \mathcal{O}(\frac{1}{X}) \tag{14}$$

thus we can notice that

$$|\arg(X_1)| = |\arctan(\frac{\mathrm{Im}(X_1)}{\mathrm{Re}(X_1)})| = |\arctan(\frac{\mathrm{Im}(X + \mathcal{O}(\frac{1}{X}))}{\mathrm{Re}(X + 1 + \mathcal{O}(\frac{1}{X}))})| \tag{15}$$

$$\leq |\arg(X)| < \frac{\pi}{8} \tag{16}$$

as X is large, making $\mathcal{O}(\frac{1}{X})$ negligible.

What remains is to show that

$$\begin{aligned} |z_1|^{100} &\leq |z_1 w_1|^3 \\ |w_1|^{100} &\leq |z_1 w_1|^3. \end{aligned} \tag{17}$$

We see this by

$$\begin{aligned} |z_1|^{100} &\leq (|z||1 - \frac{1}{2}zw| + \mathcal{O}(|zw|^3))^{100} \\ &\leq |z|^{100}(|1 - \frac{1}{2}zw| + \mathcal{O}(|zw|^2))^{100} \\ &\leq |zw|^3(|1 - \frac{1}{2}zw| + \mathcal{O}(|zw|^2))^{100} \\ &\leq |z_1 w_1|^3 \end{aligned} \tag{18}$$

as $|z|^{100} \leq |zw|^3$. We see by similar argument the same for $|w_1|$. This then shows that $\tilde{F}(B) \subset B$. $\square$

Now, this leads us to

Corollary 2.2. *Let $F : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be given by*

$$(z, w) \longrightarrow (z\lambda(1 - \frac{1}{2}zw), w\overline{\lambda}(1 - \frac{1}{2}zw)) + \mathcal{O}(||(z, w)||^{100}) \tag{19}$$

where $\lambda = e^{2\pi i r}, r \in \mathbb{R} \setminus \mathbb{Q}$, and let B be as in (4). Then $F(B) \subset B$.

Proof. Write the product

$$z_1 w_1 = zw(1 - \frac{1}{2}zw)^2 + \mathcal{O}(z^{100}, w^{100}, (zw)^{100}) = zw(1 - \frac{1}{2}zw)^2 + \mathcal{O}(|zw|^3)) \tag{20}$$

and then the result follows from lemma 2.1. □

Knowing that F is B-invariant, we can further tackle looking at repeated iteration of F. In particular we will now show that

$$F^n(z, w) \longrightarrow (0, 0) \tag{21}$$

as $n \longrightarrow \infty$. First, however, we state

Lemma 2.3. *Let F be as before, and set $(z_n, w_n) = F^n(z, w)$, then $(z_n w_n) \longrightarrow 0$ as $n \longrightarrow 0$. Furthermore, $|z_n w_n| \sim \frac{1}{n}$ for all $n \geq n_0$.*

Proof. From (17) we have by induction

$$|z_n|^{100} \leq |z_n w_n|^3 \tag{22}$$

$$|w_n|^{100} \leq |z_n w_n|^3. \tag{23}$$

Using this and (9), we write

$$|z_{n+1} w_{n+1}| = |z_n w_n|(1 - |z_n w_n| + \mathcal{O}(|z_n w_n|^2)) \tag{24}$$

$$\leq |z_n w_n|(1 - \frac{1}{2}|z_n w_n|). \tag{25}$$

This sequence is monotone non-increasing and bounded below by 0. Thus we know there exists a limit point, $|(zw)^*|$. This point must satisfy

$$|(zw)^*| \leq |(zw)^*|(1 - \frac{1}{2}|(zw)^*|), \tag{26}$$

which implies the point must be 0.

The rate of convergence we find by utilizing the same procedure and variable change as in (12). This will then give us

$$X_1 = \frac{1}{\frac{1}{X}(1 - \frac{1}{X} + \mathcal{O}(\frac{1}{X^2}))} \tag{27}$$

$$= 1 + X + \mathcal{O}(\frac{1}{X}) \tag{28}$$

and then by iteration

$$X_{n+1} = 1 + X_n + \mathcal{O}(\frac{1}{X_n}) \tag{29}$$
$$= [X_{n-1} + 1 + \mathcal{O}(\frac{1}{X_{n-1}})] + 1 + \mathcal{O}(\frac{1}{X_n}) \tag{30}$$
$$= X_{n-1} + 2 + \mathcal{O}(\frac{1}{X_{n-1}}) + \mathcal{O}(\frac{1}{X_n}) \tag{31}$$

Continuing this process gives

$$\begin{aligned} X_{n+1} &= X + (n+1) + \mathcal{O}(\frac{1}{X}) + \mathcal{O}(\frac{1}{X_1}) + ... + \mathcal{O}(\frac{1}{X_n}) \\ &= X + (n+1) + \mathcal{O}(\frac{1}{X}) + \sum_{j=1}^{n} \mathcal{O}(\frac{1}{X_j}) \end{aligned} \tag{32}$$

.

From (32) we can recognize that

$$\mathrm{Re}(X_k) = \mathrm{Re}(X) + k + \mathrm{Re}(\mathcal{O}(\frac{1}{X}) + \sum_{j=1}^{k} \mathcal{O}(\frac{1}{X_j})) \tag{33}$$
$$\geq \mathrm{Re}(X) + \frac{1}{2}k \tag{34}$$

as $\mathrm{Re}(\mathcal{O}(\frac{1}{X}) + \sum_{j=1}^{k} \mathcal{O}(\frac{1}{X_j}))$ is strictly positive.

$$|\frac{1}{\mathrm{Re}(X_k)}| \leq \frac{1}{\mathrm{Re}(X) + \frac{1}{2}k} \leq \frac{1}{\frac{1}{\epsilon} + \frac{1}{2}k} \tag{35}$$

then gives further

$$|\frac{C}{X_k}| \leq \frac{|C|}{\frac{1}{\epsilon} + \frac{1}{2}k} \tag{36}$$
$$\leq |C| \Big(\frac{1}{\frac{1}{k\epsilon} + \frac{1}{2}} \Big) \frac{1}{k} \tag{37}$$
$$\leq 2|C| \frac{1}{k}. \tag{38}$$

By definition

$$\mathcal{O}(\frac{1}{X_k}) = \frac{|C|}{X_k} + \mathcal{O}(\frac{1}{X_k^2}) \tag{39}$$

This holds for all k, and now putting it into the series in (32),

$$\sum_{j=1}^{n} \mathcal{O}(\frac{1}{X_j}) \leq \sum_{j=1}^{n} \frac{2|C|}{j} + \mathcal{O}(\frac{1}{X_j^2}) = \mathcal{O}(\log(n)) \tag{40}$$

as the harmonic series is of $\mathcal{O}(\log(n))$. Now

$$X_{n+1} = X + (n+1) + \mathcal{O}(\log(n)) \tag{41}$$

$$= (n+1)[\frac{X}{n+1} + 1 + \frac{\mathcal{O}(\log(n))}{n+1}] \tag{42}$$

which then implies that as $n \longrightarrow \infty$ we get that

$$\frac{X}{n+1} \longrightarrow 0 \tag{43}$$

$$\frac{\mathcal{O}(\log(n))}{n+1} \longrightarrow 0. \tag{44}$$

This then yields, for all $n \geq n_0$, that

$$X_n \sim n \tag{45}$$

and now we see that

$$z_n w_n \sim \frac{1}{n}. \tag{46}$$

□

Now looking at the transform in each variable, we state

Proposition 2.4. *Let F be as before and set $(z_n, w_n) = F^n(z, w)$, then $(z_n, w_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and $|z_n| \sim |w_n| \sim \frac{1}{\sqrt{n}}$.*

Proof. Looking at the transform in each variable we have

$$z \longrightarrow z\lambda(1 - \frac{1}{2}zw + \mathcal{O}((zw)^2) + \mathcal{O}(||(z,w)||^M) \tag{47}$$

$$w \longrightarrow w\overline{\lambda}(1 - \frac{1}{2}zw + \mathcal{O}((zw)^2) + \mathcal{O}(||(z,w)||^M). \tag{48}$$

We can, in B, write

$$z \longrightarrow z\lambda(\,1 - \frac{1}{2}zw + \mathcal{O}((zw)^2) + \mathcal{O}((zw)^3)\,) \tag{49}$$

$$= z\lambda(\,1 - \frac{1}{2}zw + \mathcal{O}((zw)^2)\,) \tag{50}$$

$$w \longrightarrow w\overline{\lambda}(\,1 - \frac{1}{2}zw + \mathcal{O}((zw)^2)\,) \tag{51}$$

We see that the logarithm is well defined in our region so

$$\log(1 - \frac{1}{2}zw + \mathcal{O}((zw)^2)) = -\frac{1}{2}(zw) + \mathcal{O}((zw)^2) + \frac{1}{2}(-\frac{1}{2}(zw) + \mathcal{O}((zw)^2))^2 + \tag{52}$$

$$... + \frac{1}{j}(-\frac{1}{2}(zw) + \mathcal{O}((zw)^2))^j + ... \tag{53}$$

$$= -\frac{1}{2}zw + \mathcal{O}((zw)^2) \tag{54}$$

by using the Taylor series expansion of $\log(1+x)$, where $x = -\frac{1}{2}zw + \mathcal{O}((zw)^2)$. Then we see that

$$(1 - \frac{1}{2}zw + \mathcal{O}((zw)^2)) = e^{\log(1-\frac{1}{2}zw+\mathcal{O}((zw)^2))} \tag{55}$$

$$= e^{-\frac{1}{2}zw+\mathcal{O}((zw)^2)}, \tag{56}$$

so the transforms then look like

$$z_{n+1} = z_n\lambda(e^{-\frac{1}{2}z_nw_n+\mathcal{O}((z_nw_n)^2)}) \tag{57}$$

$$w_{n+1} = w_n\overline{\lambda}(e^{-\frac{1}{2}z_nw_n+\mathcal{O}((z_nw_n)^2)}). \tag{58}$$

Again performing the coordinate change $zw = \frac{1}{X}$, and iterating backwards from n, we get

$$z_n = z\lambda^n \exp\Big(-\frac{1}{2}(\frac{1}{X_n}) + \mathcal{O}(\frac{1}{X_n^2}) - \frac{1}{2}(\frac{1}{X_{n-1}}) + \mathcal{O}(\frac{1}{X_{n-1}^2}) - ... - \frac{1}{2}(\frac{1}{X}) + \mathcal{O}(\frac{1}{X^2})\Big) \tag{59}$$

$$w_n = w\overline{\lambda^n} \exp\Big(-\frac{1}{2}(\frac{1}{X_n}) + \mathcal{O}(\frac{1}{X_n^2}) - \frac{1}{2}(\frac{1}{X_{n-1}}) + \mathcal{O}(\frac{1}{X_{n-1}^2}) - ... - \frac{1}{2}(\frac{1}{X}) + \mathcal{O}(\frac{1}{X^2})\Big). \tag{60}$$

The exponential in each transform can be written

$$-\frac{1}{2}\Big(\frac{1}{X} + \sum_{j=1}^{n}\frac{1}{X_j}\Big) + \sum_{j=0}^{n}\mathcal{O}(\frac{1}{X_j^2}) \tag{61}$$

and from (40) in the previous lemma we have for some large j_0 that

$$-\frac{1}{2}\Big(\frac{1}{X} + \sum_{j=1}^{n}\frac{1}{X_j}\Big) = -\frac{1}{2}\log(n) + \frac{1}{2}\log(j_0) + G \tag{62}$$

where G is some bounded function of X_j, $\forall j < j_0$.

The sum

$$\sum_{j=0}^{n}\mathcal{O}(\frac{1}{X_j^2}) \sim \sum_{j}\frac{1}{j^2} \tag{63}$$

is also bounded. Hence we can write

$$z_n = z\, e^{-\frac{1}{2}\log(n)} e^{-2\pi inr} e^{G} \tag{64}$$

$$= z\big(\frac{1}{n}\big)^{\frac{1}{2}}\; e^{2\pi inr} e^{G} \tag{65}$$

$$w_n = w\big(\frac{1}{n}\big)^{\frac{1}{2}}\; e^{2\pi inr} e^{G} \tag{66}$$

which proves the claim. □

2.2 Topological Properties

We have classified the local basin of attraction, B. Further we call

$$\Omega = \bigcup_{k=0}^{\infty} F^{-k}(B) \tag{67}$$

the *global basin of attraction*. This turns out to be the sought after Fatou component, which we will see next. However first we have a topological property:

Proposition 2.5. Ω *is connected, but not simply connected.*

Proof. To start we show that B is not simply connected, so for the sake of a contradiction we assume B is simply connected.

Consider so the transform $\psi : B \longrightarrow \psi(B)$ given by

$$\psi(z, w) = (zw, w). \tag{68}$$

The transform is obviously surjective and holomorphic. Injectivity in the second variable is clear, and in the first variable we have

$$z_1 w_1 = z_2 w_2 \tag{69}$$

$$z_1 = z_2 \tag{70}$$

showing injectivity. The inverse is

$$\psi^{-1}(x, y) = (\frac{x}{y}, y) \tag{71}$$

and is holomorphic for all $y \neq 0$. So ψ is a biholomorphism. Pick thereafter an $r \in (0, \epsilon)$ and consider a path in B,

$$\gamma(t) = (re^{-it}, re^{it}) \tag{72}$$

In new coordinates we have

$$\psi(\gamma(t)) = (r^2, re^{it}) \tag{73}$$

which is a circle in the $\{r^2\} \times \mathbb{C}$ plane centered at $(r^2, 0)$. If B is simply connected, then we can contract the closed path to $(r^2, 0)$. However $(r^2, 0) \notin \psi(B)$. So B cannot be simply connected.

To then show that Ω is not simply connected, we again assume for the sake of contradiction that it is simply connected.

We note that $F^k(\gamma)$ is not contractible in B. Indeed by looking at the transform in the w variable, we have

$$w_1 = w\overline{\lambda}(1 - \frac{1}{2}zw + \mathcal{O}((zw)^2)) \tag{74}$$

in B. Now

$$|-\frac{1}{2}zw + \mathcal{O}((zw)^2)| \le \frac{1}{2}|zw| + |\mathcal{O}((zw)^2)| < |w|, \tag{75}$$

so we can apply Rouche's theorem [4] to conclude that w_1 and w have equally many zeros in the region enclosed by $\psi(\gamma)$ in the $r^2 \times \mathbb{C}$ plane. By the same argument, we can show that w_2 and w_1 have equally many zeros in the region, and w_3 and w_2 have equally many zeros in the region and so on. Inductively this then gives that w_k and w has equally many zeros in the region enclosed by $\psi(\gamma)$. This then shows that $F^k(\gamma)$ is not contractible in B for all k.

We construct the compact set

$$K = \bigcup_{s=0}^{1} \gamma_s(t) \subset \Omega \tag{76}$$

where $\gamma_1 = \gamma(t)$ and γ_0 is an arbitrary point in region enclosed by γ in the $r^2 \times \mathbb{C}$ plane. We know $F^k(z, w) \longrightarrow 0$ as $k \to \infty$, therefore we find an N so that

$$F^N(K) \subset B. \tag{77}$$

This would then imply that $F^N(\gamma_1)$ would be contractible in B, which is a contradiction. Thus Ω cannot be simply connected. □

2.3 The Final Results

To now show that Ω is the desired Fatou component, we will use a couple of tools: the Kobayashi metric and a theorem by Pöchel. These will allow us to set up nicely into the proof of the main result. So let $M \subset \mathbb{C}^n$ with $p \in M$. The *Kobayashi metric* is then given as:

$$k_M(p, \xi) := \inf \Big\{ \frac{1}{|c|} : \exists f : \Delta \longrightarrow M, f \text{ analytic}, f(0) = p, f'(0) = c\xi \Big\}. \tag{78}$$

Proposition 2.6. *Let $F : M \longrightarrow N$ be holomorphic with $M \subset \mathbb{C}^n, N \subset \mathbb{C}^k$. For $p \in M$, $\xi \in \mathbb{C}^n$ we have*

$$k_M(p, \xi) \ge k_N(F(p), F'(p)\xi) \tag{79}$$

Proof. Let $f = (f_1, ..., f_n) : \Delta \longrightarrow M$ be analytic with $f(0) = p$, then

$$(F \circ f) : \Delta \longrightarrow N \tag{80}$$

$$(F \circ f)(0) = F(p). \tag{81}$$

If $f'(0) = \xi$ then

$$(F \circ f)'(0) = F'(p)f'(0) = F'(p)\xi. \tag{82}$$

We then have

$$\inf\left\{\frac{1}{|c|} : \exists g : \Delta \longrightarrow N,\ g \text{ analytic},\ g(0) = F(p),\ g'(0) = cF'(p)\xi\right\} \tag{83}$$

$$\leq \inf\left\{\frac{1}{|c|} : \exists f : \Delta \longrightarrow M,\ f \text{ analytic},\ (F \circ f)(0) = F(p), (F \circ f)'(0) = cF'(p)\xi\right\} \tag{84}$$

$$= \inf\left\{\frac{1}{|c|} : \exists f : \Delta \longrightarrow M,\ f \text{ analytic},\ f(0) = p, f'(0) = c\xi\right\} \tag{85}$$

as

$$\begin{aligned}&\left\{\frac{1}{|c|} : \exists f : \Delta \longrightarrow M,\ f \text{ analytic},\ (F \circ f)(0) = F(p), (F \circ f)'(0) = cF'(p)\xi\right\} \\ &\quad \subseteq \left\{\frac{1}{|c|} : \exists g : \Delta \longrightarrow N,\ g \text{ analytic},\ g(0) = F(p),\ g'(0) = cF'(p)\xi\right\}.\end{aligned} \tag{86}$$

□

In particular the Kobayashi distance function is given by

$$D_K^M(\zeta, \zeta') = \inf\left\{\int_0^1 k_M(\gamma(t), \gamma'(t))dt\right\} \tag{87}$$

where the infimum is taken over all paths joining ζ to ζ'.

Lemma 2.7. *Let $\Delta^* = \{\zeta \in \mathbb{C} : 0 < |\zeta| < 1\}$. For $p, q \in \Delta^*$ we have*

$$k_{\Delta^*}(p, q) \geq \Big|\log|\frac{\log|p|}{\log|q|}|\Big| \tag{88}$$

Proof. In the disk, the Kobayashi metric and the Poincare metric coincide [3], and then in the punctured disk we have

$$ds^2 = \frac{4}{|q|^2(\log(|q|^2)^2}|dq|^2 \tag{89}$$

Then we can evaluate

$$d(\gamma(0), \gamma(1)) = \int_0^1 \lambda_{\Delta^*}\big(\gamma(t)\big)\ \gamma'(t)dt \tag{90}$$

where $\gamma(t)$ is a path between p and q. So

$$\int_0^1 \frac{1}{|\gamma(t)| \log|\gamma(t)|} |\gamma'(t)| dt \geq \Big| \int_0^1 \frac{1}{\gamma(t)\log(\gamma(t))} \gamma'(t)dt \Big| \tag{91}$$

$$= \left| \log\big(\log(\gamma(t))\big)\Big|_0^1 \right| \tag{92}$$

and then

$$k_{\Delta^*}(p,q) \geq \Big| \log\Big|\frac{\log|p|}{\log|q|}\Big|\Big| \tag{93}$$

□

Lemma 2.8. *If F is as in* (2)*, and λ is Brjuno, then there exist a biholomorphism $G(z,w) = (z,w) + \mathcal{O}(||(z,w)||^l)$ at $(0,0)$ so that*

$$(G \circ F \circ G^{-1})(z,w) = (\lambda z + zwR_1(z,w), \overline{\lambda} w + zwR_2(z,w)) \tag{94}$$

where R_1, R_2 are germs of holomorphic functions at $(0,0)$.

Proof. As λ is Brjuno, the divisors $\lambda^k - \lambda$ and $\lambda^k - \overline{\lambda}$ are admissible in the sense of Pöchel[2] for all $k \geq 2$. So by theorem 1 in [2] there is, in a small disk around the origin, an injective holomorphic map $\phi_1 : \mathbb{D}_\delta \longrightarrow \mathbb{C}^2$, such that $\phi_1(0) = (0,0)$, $\phi_1' = (1,0)$ and

$$F(\phi_1(\zeta)) = \phi_1(\lambda\zeta) \qquad \forall \zeta \in \mathbb{D}_\delta. \tag{95}$$

As F is tangent to $\{w = 0\}$ up to order l. We can, thanks to[2], implicitly write $w = \psi_1(z)$ defining $\psi_1(\zeta) = \mathcal{O}(|\zeta|^l)$.

Similarly for $\overline{\lambda^k} - \lambda$ and $\overline{\lambda^k} - \overline{\lambda}$, we get the function $\psi_2(\zeta) = \mathcal{O}(|\zeta|^l)$. Thereafter we define $G(z,w) = (z - \psi_2(w), w - \psi_1(z)) = (z,w) + \mathcal{O}(||(z,w)||^l)$. This is a germ of a biholomorhpism at $(0,0)$ and $(G \circ F \circ G^{-1})$ takes the desired form. □

Now we are fully equipped to prove the result:

Theorem 2.9. *Let F be as in* (2) *and let Ω be as in* (67)*. Assume also that λ is Brjuno. Then Ω is the desired Fatou component.*

Proof. We know there exist a Fatou component containing Ω, so we assume for sake of a contradiction that there exists a connected set D so that

1. $\Omega \subset D$
2. $\Omega \neq D$
3. $q \in D \setminus \Omega \implies F^n(q) \longrightarrow 0.$

Now if $q \in D \setminus \Omega$, then $F^N(q) \notin B$ for any N. If this was the case then

$$q = F^{-N}(F^N(q)) \in F^{-N}(B) \subset \Omega, \tag{96}$$

so $F^N(q) \notin B$.

If $q \notin F^n(B)$ and $z_n w_n \longrightarrow 0$, then we must have that

$$|z| \geq |zw|^\alpha \text{ or } |w| \geq |zw|^\alpha \tag{97}$$

for all $\alpha \in (0, \frac{3}{100})$ and also

$$|z_n| \geq |z_n w_n|^\alpha \text{ or } |w_n| \geq |z_n w_n|^\alpha. \tag{98}$$

This can alternate between the two cases in the iterates. To work around this we choose a subsequence n_j so that for all n_j:

$$|z_{n_j}| \geq |z_{n_j} w_{n_j}|^\alpha \tag{99}$$

Then, as $n_j \longrightarrow \infty$,

$$\log\left|\frac{\log|z_{n_j}|}{\log|w_{n_j}|}\right| \not\longrightarrow 0 \tag{100}$$

because, from (98) we can compute

$$|z_{n_j}| > |z_{n_j}|^\alpha |w_{n_j}|^\alpha \tag{101}$$

$$\implies |z_{n_j}|^{\frac{1-\alpha}{\alpha}} > |w_{n_j}|. \tag{102}$$

This will then yield

$$\left|\frac{\log|z_{n_j}|}{\log|w_{n_j}|}\right| > \left|\frac{\log|z_{n_j}|}{\log|z_{n_j}|^{\frac{1-\alpha}{\alpha}}}\right| \tag{103}$$

$$= \left|\frac{\alpha}{1-\alpha}\right| \neq 0 \tag{104}$$

and

$$\left|\log\left|\frac{\log|z_{n_j}|}{\log|w_{n_j}|}\right|\right| > \left|\log\left|\frac{\alpha}{1-\alpha}\right|\right| = \log\left|\frac{1-\alpha}{\alpha}\right| > 0. \tag{105}$$

as $\frac{1-\alpha}{\alpha} > 1$.

As λ is Brjuno, lemma 2.8 holds. Thus we have an open neighborhood U of $(0,0)$ and a biholomorphism $G : U \longrightarrow G(U)$, so the coordinate change (94) holds for all $(z', w') \in G(U)$. In these coordinates it also holds that $D \cap U \subset \Delta^* \times \Delta^*$, as (94) is only a rotation on $\{z' = 0\}$ and $\{w' = 0\}$.

As D is connected, there exists points $p \in \Omega, q \in D \setminus \Omega$ so that

$$k_{U\cap D}(p,q) < \delta \tag{106}$$

for some small $\delta > 0$. Choose then $\delta < \frac{1}{100} \log |\frac{1-\alpha}{\alpha}|$. We also know that

$$k_D(F^N(p), F^N(q)) \leq k_{U\cap D}(p,q) < \delta \quad \forall N \in \mathbb{N} \tag{107}$$

from the property of Kobayashi metric. From the properties of iteration of F, there is a subsequence N_j such that

$$F^{N_j}(p),\ F^{N_j}(q) \in U \tag{108}$$

where $F^{N_j}(p) \in B$ and $F^{N_j}(q) \in D \setminus B$. Set

$$F^{N_j}(p) = (z_j, w_j) \tag{109}$$

$$F^{N_j}(q) = (x_j, y_j) \tag{110}$$

and from the triangle inequality obtain

$$k_{\Delta^*}(x_j, y_j) \leq k_{\Delta^*}(x_j, z_j) + k_{\Delta^*}(z_j, w_j) + k_{\Delta^*}(y_j, w_j). \tag{111}$$

We can further estimate

$$k_{\Delta^*}(x_j, z_j) = k_{\Delta^*}(\pi_1(F^{N_j}(p)), \pi_1(F^{N_j}(q))) \leq k_{D\cap U}(F^{N_j}(p), F^{N_j}(q)) < \delta \tag{112}$$

$$k_{\Delta^*}(y_j, w_j) = k_{\Delta^*}(\pi_2(F^{N_j}(p)), \pi_2(F^{N_j}(q))) \leq k_{D\cap U}(F^{N_j}(p), F^{N_j}(q)) < \delta \tag{113}$$

again by the Kobayashi property and projection functions $\pi_1,\ \pi_2$. Further $k_{\Delta^*}(z_j, w_j) \longrightarrow 0$, so we have for sufficiently large j that

$$k_{\Delta^*}(z_j, w_j) < \delta. \tag{114}$$

By then using an estimation from lemma 2.7 we can then see

$$\log\Big|\frac{1-\alpha}{\alpha}\Big| < \Big|\log\Big|\frac{\log|x_j|}{\log|y_j|}\Big|\Big| \leq k_{\Delta^*}(x_j, y_j) < 3\delta < \frac{3}{100}\log\Big|\frac{1-\alpha}{\alpha}\Big| \tag{115}$$

which is a contradiction as $F^{N_j}(q) \not\longrightarrow 0$.

□

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