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The 4 Subspace Problem

Bachelor's project in Mathematical Sciences
Supervisor: Sverre Olaf Smalø

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Contents

Introduction

Let V be a vector space and $V_i \subseteq V$ be a subspace $\forall i \in \{1, ..., n\}$ for some $n \in \mathbb{N}$. Then the system $(V; V_1, ..., V_n)$ is said to be **decomposable** $\Leftrightarrow$ there exist non-trivial vector spaces W' and W'' such that

1. $V = W' \oplus W''$
2. $V_i = (V_i \cap W') \oplus (V_i \cap W'') \; \forall i \in \{1, ..., n\}$.

Summary

The 4 subspace problem is the issue of finding all indecomposable systems consisting of a vector space with four subspaces. This paper translates the notion of systems into subspace representations of star quivers and finds a sequence with indecomposable representations as its elements, as well as a few representations which are not in the sequence, but are related to it nonetheless.

1 Representations and decomposability

As a first step on our journey towards the desired sequence we described in the summary, we substitute the somewhat ambiguous "systems" with something called "representations of quivers". To explain what that means, we first define what a quiver is.

Definition 1.1. A **quiver** $\Gamma = (\Gamma_0, \Gamma_1)$ is a directed graph where

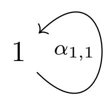
1. $\Gamma_0 = \{\text{vertices}\}$
2. $\Gamma_1 = \{\text{arrows}\}$.

△

Remark 1.1. It is common to represent the vertices of a quiver by natural numbers, so that $\Gamma_0 = \{1, ..., n\}$, where n is the number of vertices of the quiver. Let $\alpha \in \Gamma_1$ such that $\alpha : i \to j$ for some $i, j \in \Gamma_0$. We denote $\alpha = \alpha_{j,i}$, and we use this as the standard notation for arrows. △

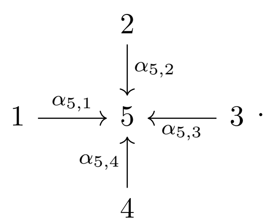
Example 1.1. These are some examples of how quivers can be illustrated.

1. $\Gamma = (\{1, 2\}, \{\alpha_{2,1}\})$ can be illustrated as $1 \xrightarrow{\alpha_{2,1}} 2$.

2. $\Gamma = (\{1\}, \{\alpha_{1,1}\})$ can be illustrated as 

.

3. $\Gamma = (\{1, 2, 3, 4, 5\}, \{\alpha_{5,1}, \alpha_{5,2}, \alpha_{5,3}, \alpha_{5,4}\})$ can be illustrated as



.

△

Remark 1.2. The quiver $(\{1, 2, 3, 4, 5\}, \{\alpha_{5,1}, \alpha_{5,2}, \alpha_{5,3}, \alpha_{5,4}\})$ in Example 1.1.3 is going to be used several times on our path. We therefore give it the name $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$. Every mention of $\mathcal{Q}$, $\mathcal{Q}_0$ or $\mathcal{Q}_1$ from now on will implicitly refer to this remark. △

$\mathcal{Q}$ is a certain kind of quiver that is important to us, which we give a unique name and describe in the following definition.

Definition 1.2. A **star quiver** $\Gamma^* = (\Gamma_0^*, \Gamma_1^*)$ is a quiver with $n \in \mathbb{N}$ vertices and $n - 1$ arrows such that $\exists! \alpha_{n,i} \in \Gamma_1^*$ for each $i \in \Gamma_0^* \setminus \{n\}$. △

We now define the concept that will replace systems, namely representations of quivers.

Definition 1.3. A **representation** (V, f) of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ over a field k is a collection $V = \{V(i)\}_{i=1}^n$ of vector spaces over k and a collection f of k-linear maps such that for every arrow $\alpha_{j,i} \in \Gamma_1$ there exists a unique map $f_{\alpha_{j,i}} \in f$ where $f_{\alpha_{j,i}} : V(i) \to V(j)$. △

Example 1.2. Let (V, f) be a representation of $\mathcal{Q}$ over some field k where $V(i) \subseteq V(5)$ and $f_{\alpha_{5,i}}$, which corresponds to $\alpha_{5,i} \in \mathcal{Q}_1$, is the inclusion map $f_{\alpha_{5,i}} : V(i) \to V(5)$ such that $v \mapsto v \; \forall v \in V(i)$, $\forall i \in \{1, 2, 3, 4\}$. We illustrate this representation as

$$\begin{array}{ccccc} & & V(2) & & \\ & & \big\downarrow & & \\ V(1) & \hookrightarrow & V(5) & \hookleftarrow & V(3) \\ & & \big\uparrow & & \\ & & V(4) & & \end{array} \;.$$

△

Remark 1.3. The representation in the example above is a special case of what we call subspace representation. We explain what that is next. △

Definition 1.4. A **subspace representation** is a representation (V, f) of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ over a field k where $V(i) \subseteq V(j)$ and f_α is injective $\forall \alpha_{j,i} \in \Gamma_1$. △

Remark 1.4. If the k-linear maps of a subspace representation are inclusions, it is sufficient to give the collection of vector spaces to define a subspace representation of a quiver. We can then denote f by $\hookrightarrow$. △

Now that know what representations of quivers are, we can use them instead of systems. Still, if we want to arrive at the sequence we desire, we will need to translate the concept of decomposability into the language of representations. This we do by defining the direct sum of representations, followed by the definition of decomposability.

Definition 1.5. Let $\{(V_r, f_r)\}_{r=1}^t$ be a collection of representations of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ over a field k for some $t \in \mathbb{N}$. The **direct sum** $\bigoplus_{r=1}^{t} (V_r, f_r)$ of these representations is a representation (V, f) of Γ over k where $V(i) =$

$\bigoplus\limits_{r=1}^{t} V_r(i)\ \forall i \in \Gamma_0$ and

$$f_{\alpha_{j,i}} = \bigoplus_{r=1}^{t} (f_r)_{\alpha_{j,i}} : V(i) \to V(j)\ \forall \alpha_{j,i} \in \Gamma_1.$$

$\triangle$

Remark 1.5. We say that a representation (V, f) over a field k is finite-dimensional over $k \Leftrightarrow$ each vector space in V is finite-dimensional over k. Then, if a collection $\{(V_r, f_r)\}_{r=1}^{t}$ of representations over k is finite-dimensional over k and $n \in \mathbb{N}$, their direct sum $(V, f) = \bigoplus\limits_{r=1}^{t} (V_r, f_r)$ is finite-dimensional over k because each element in V is a finite direct sum of finite-dimensional vector spaces. In this case,

$$f_{\alpha_{j,i}} := \begin{pmatrix} (f_1)_{\alpha_{j,i}} & 0 & \cdots & 0 \\ 0 & (f_2)_{\alpha_{j,i}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (f_t)_{\alpha_{j,i}} \end{pmatrix} \ \forall \alpha_{j,i} \in \Gamma_1.$$

Even if (V, f) is infinite-dimensional, it is still possible to use matrix notation for the k-linear maps, though the notation would only be symbolic in that case. $\triangle$

Example 1.3. Let (W', f') and (W'', f'') be two representations of $\mathcal{Q}$ that are finite-dimensional over some field k. Then $(W', f') \oplus (W'', f'')$ can be illustrated as

$$\begin{array}{ccccc} & & W'(2) & & \\ & & \big\downarrow f'_{\alpha_{5,2}} & & \\ W'(1) & \xrightarrow{f'_{\alpha_{5,1}}} & W'(5) & \xleftarrow[f'_{\alpha_{5,3}}]{} & W'(3) \\ & & f'_{\alpha_{5,4}} \big\uparrow & & \\ & & W'(4) & & \end{array} \quad \oplus \quad \begin{array}{ccccc} & & W''(2) & & \\ & & \big\downarrow f''_{\alpha_{5,2}} & & \\ W''(1) & \xrightarrow{f''_{\alpha_{5,1}}} & W''(5) & \xleftarrow[f''_{\alpha_{5,3}}]{} & W''(3) \\ & & f''_{\alpha_{5,4}} \big\uparrow & & \\ & & W''(4) & & \end{array}$$

$$
= \quad
\begin{array}{ccccc}
 & & W'(2)\oplus W''(2) & & \\
 & & \Big\downarrow \begin{pmatrix} f'_{\alpha_{5,2}} & 0 \\ 0 & f''_{\alpha_{5,2}} \end{pmatrix} & & \\
W'(1)\oplus W''(1) & \xrightarrow{\begin{pmatrix} f'_{\alpha_{5,1}} & 0 \\ 0 & f''_{\alpha_{5,1}} \end{pmatrix}} & W'(5)\oplus W''(5) & \xleftarrow[\begin{pmatrix} f'_{\alpha_{5,3}} & 0 \\ 0 & f''_{\alpha_{5,3}} \end{pmatrix}]{} & W'(3)\oplus W''(3) \\
 & & \begin{pmatrix} f'_{\alpha_{5,4}} & 0 \\ 0 & f''_{\alpha_{5,4}} \end{pmatrix} \Big\uparrow & & \\
 & & W'(4)\oplus W''(4) & &
\end{array} .
$$

$\triangle$

Definition 1.6. A representation (V, f) of a quiver Γ over a field k is **decomposable** $\Leftrightarrow$ there exist non-trivial representations (W', f') and (W'', f'') of Γ over k such that $(V, f) = (W', f') \oplus (W'', f'')$. $\triangle$

Remark 1.6. The trivial representation (V, f) of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ is the representation where $V(i) = 0 \ \forall i \in \Gamma_0$ and $f_{\alpha_{j,i}} = 0 \ \forall \alpha_{j,i} \in \Gamma_1$. Note that $V(i) = 0 \ \forall i \in \Gamma_0$ necessarily implies $f_{\alpha_{j,i}} = 0 \ \forall \alpha_{j,i} \in \Gamma_1$. $\triangle$

Theorem 1.1. *Let (V, f) be a subspace representation of a star quiver $\Gamma^* = (\Gamma_0^*, \Gamma_1^*)$ with n vertices. Then*
(V, f) is decomposable $\Leftrightarrow$ the system $\big(V(n); V(1), ..., V(n-1)\big)$ is decomposable.

Proof. Before we prove the implications, we should verify that $V(i) \subseteq V(n)$ $\forall i \in \Gamma_0^*$, so that it is actually possible for $\big(V(n); V(1), ..., V(n-1)\big)$ to be decomposable.
First of all, Γ^* is a star quiver, so $\exists \alpha_{n,i} \in \Gamma_1^* \ \forall i \in \Gamma_0^*$. Then, because (V, f) is a subspace representation, $V(i) \subseteq V(n) \ \forall i \in \Gamma_0^*$.

($\Rightarrow$) (V, f) is decomposable $\Rightarrow$ there exist non-trivial representations (W', f') and (W'', f'') such that $(V, f) = (W', f') \oplus (W'', f'')$. Let us investigate if (W', f') and (W'', f'') are subspace representations.
Let $\alpha_{n,i} \in \Gamma_1$. Then

$$V(i) = W'(i) \oplus W''(i) \subseteq W'(n) \oplus W''(n) = V(n),$$

and $f_{\alpha_{j,i}} = f'_{\alpha_{j,i}} \oplus f''_{\alpha_{j,i}}$ where $f_{\alpha_{j,i}} : W'(i) \to W'(n)$ and $f''_{\alpha_{j,i}} : W''(i) \to W''(n)$. $f_{\alpha_{j,i}}$ is injective, so $f'_{\alpha_{j,i}}$ and $f''_{\alpha_{j,i}}$ are also injective.
Thus, since $W'(i) \subseteq W'(n) \oplus W''(n)$ and $W''(i) \subseteq W'(n) \oplus W''(n)$,

$W'(i) \subseteq W'(n)$ and $W''(i) \subseteq W''(n)$ $\forall i \in \Gamma_0^*$, and all the k-linear maps in f' and f'' are inclusions, so (W', f') and (W'', f'') are subspace representations. Having shown this, we can show that

$$V(i) = \big[V(i) \cap W'(n)\big] \oplus \big[V(i) \cap W''(n)\big] \ \forall i \in \Gamma_0^*.$$

We have that

$$V(i) \cap W'(n) = \big[W'(i) \oplus W''(i)\big] \cap W'(n)$$

$$= \big[W'(i) \cap W'(n)\big] \oplus \big[W''(i) \cap W'(n)\big] = \big[W'(i) \cap W'(n)\big] \oplus 0$$

$$= W'(i) \cap W'(n) = W'(i)$$

and dually

$$W''(i) = V(i) \cap W''(n)$$

since $W'(i) \subseteq W'(n)$, $W''(i) \subseteq W''(n)$, $V(i) = W'(i) \oplus W''(i)$ when $i \in \Gamma_0^*$. Thus

$$V(i) = W'(i) \oplus W''(i) = \big[V(i) \cap W'(n)\big] \oplus \big[V(i) \cap W''(n)\big] \ \forall i \in \Gamma_0^*.$$

This coupled with the fact that $V(n) = W'(n) \oplus W''(n)$ implies that the system $\big(V(n); V(1), ..., V(n-1)\big)$ is decomposable.

($\Leftarrow$) Suppose the system $\big(V(n); V(1), ..., V(n-1)\big)$ is decomposable. Then there exist non-trivial vector spaces W' and W'' such that

$$V(n) = W' \oplus W''$$

and

$$V(i) = \big[V(i) \cap W'\big] \oplus \big[V(i) \cap W''\big] \ \forall i \in \Gamma_0^*.$$

Thus we can construct representations (U', f') and (U'', f'') of Γ^* over k where

$$U' = \{W'\} \cup \{V(i) \cap W'\}_{i=1}^{n-1},$$

$$U'' = \{W''\} \cup \{V(i) \cap W''\}_{i=1}^{n-1},$$

$$f'_{\alpha_{n,i}}(u') = f_{\alpha_{n,i}}(u') \ \forall u' \in U'(i),$$

$$f''_{\alpha_{n,i}}(u'') = f_{\alpha_{n,i}}(u'') \ \forall u'' \in U''(i),$$

where $\alpha_{n,i} \in \Gamma_1^*$. To clarify, we let $U'(n) = W'$ and $U''(n) = W''$, and $U'(i) = V(i) \cap W'$ and $U''(i) = V(i) \cap W''$ when $i \in \Gamma_0^*$. Hence f' and

f'' are collections of injective maps. $W' \oplus W'' = V(n)$ and $\big[V(i) \cap W'\big] \oplus \big[V(i) \cap W''\big] = V(i)\ \forall i \in \Gamma_0^*$ imply

$$\{W' \oplus W''\} \cup \big\{\big[V(i) \cap W'\big] \oplus \big[V(i) \cap W''\big]\big\}_{i=1}^{n-1} = \{V(i)\}_{i=1}^{n} = V.$$

Now let $\alpha_{n,i} \in \Gamma_1^*$ and $v \in V(i)$. Then, since $V(i) = \big[V(i) \cap W'\big] \oplus \big[V(i) \cap W''\big]$, there are elements $w' \in V(i) \cap W'$ and $w'' \in V_i \cap W''$ such that $v = w' + w''$. Hence

$$(f'_{\alpha_{n,i}} \oplus f''_{\alpha_{n,i}})(v) = (f'_{\alpha_{n,i}} \oplus f''_{\alpha_{n,i}})(w' + w'') = f'_{\alpha_{n,i}}(w') + f''_{\alpha_{n,i}}(w'')$$

$$= f_{\alpha_{n,i}}(w') + f_{\alpha_{n,i}}(w'') = f_{\alpha_{n,i}}(w' + w'') = f_{\alpha_{n,i}}(v),$$

so $f_{\alpha_{n,i}} = f'_{\alpha_{n,i}} \oplus f''_{\alpha_{n,i}}$.
Thus $(U', f') \oplus (U'', f'') = (V, f)$, so (V, f) is decomposable.

Hence (V, f) is decomposable $\Leftrightarrow \big(V(n); V(1), ..., V(n-1)\big)$ is decomposable, which proves the theorem. □

Now we have the equivalence we wanted between certain representations and systems. Thus we can restate the 4 subspace problem into the problem of finding all indecomposable subspace representations of the star quiver $\mathcal{Q}$. Our next step towards the sequence of indecomposables is to find an equivalent way to define decomposability of representations using endomorphism rings. To do this, we first define homomorphisms of representations, which endomorphisms are a special case of.

Definition 1.7. Let (V, f) and (V', f') be representations of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ with $n \in \mathbb{N}$ vertices over a field k. A **homomorphism of representations** between (V, f) and (V', f') is a collection h of n k-linear maps $h(i) : V(i) \to V'(i)$ such that the following diagram commutes $\forall \alpha_{j,i} \in \Gamma_1$.

$$\begin{array}{ccc} V(i) & \xrightarrow{f_{\alpha_{j,i}}} & V(j) \\ \downarrow{\scriptstyle h(i)} & & \downarrow{\scriptstyle h(j)} \\ V'(i) & \xrightarrow{f'_{\alpha_{j,i}}} & V'(j) \end{array}$$

That is, $\Big(f'_{\alpha_{j,i}} \circ h(i)\Big)(v_i) = \big(h(j) \circ f_{\alpha_{j,i}}\big)(v_i)\ \forall v_i \in V(i)$. △

Definition 1.8. Let (V, f) and (V', f') be representations of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ with $n \in \mathbb{N}$ vertices over a field k. (V, f) is a **subrepresentation** of $(V', f') \Leftrightarrow V(i) \subseteq V'(i)\ \forall i \in \Gamma_0$ and $f_{\alpha_{j,i}} = f'_{\alpha_{j,i}}|_{V(i)}\ \forall \alpha_{j,i} \in \Gamma_1$. △

Remark 1.7. If (V, f) is a subrepresentation of (V', f'), then there exists a collection of maps $h : (V, f) \to (V', f')$ such that $\big[h(i)\big](v) = v \ \forall v \in V(i)$ $\forall i \in \Gamma_0$.
Let $\alpha_{j,i} \in \Gamma_1$ and $v \in V(i)$. Then

$$\big[f'_{\alpha_{j,i}} \circ h(i)\big](v) = f'_{\alpha_{j,i}}\Big(\big[h(i)\big](v)\Big) = f'_{\alpha_{j,i}}(v)$$

$$= f'_{\alpha_{j,i}}|_{V(i)}(v) = f_{\alpha_{j,i}}(v) = \big[h(j)\big]\big[f_{\alpha_{j,i}}(v)\big] = \big[h(j) \circ f_{\alpha_{j,i}}\big](v),$$

so h is a homomorphism.
h is called an inclusion homomorphism. △

Definition 1.9. An **endomorphism on a representation** is a homomorphism of representations between a representation (V, f) and itself. △

Theorem 1.2. *Let (V, f) be a representation of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ with n vertices over a field k. Then the set of endomorphisms on (V, f), denoted by* $\mathrm{End}(V, f)$*, form a ring under homomorphism addition and composition.*

Proof. Addition of homomorphisms is defined such that the sum $h_1 + h_2$ of two homomorphisms h_1 and h_2 between two representations (V', f') and (V'', f'') of the same quiver satisfies

$$\big[(h_1 + h_2)(i)\big](x) = \big[h_1(i)\big](x) + \big[h_2(i)\big](x) \ \forall x \in V'(i) \ \forall i \in \Gamma_0.$$

Homomorphism composition is defined such that the composition $h_1 \circ h_2$ of two homomorphisms $h_2 : (V'', f'') \to (V''', f''')$ and $h_1 : (V', f') \to (V, f)$, where (V', f'), (V'', f'') and (V''', f''') are three representations of the same quiver, satisfies

$$\big[(h_1 \circ h_2)(i)\big](x) = \Big(\big[h_1(i)\big] \circ \big[h_2(i)\big]\Big)(x) \ \forall x \in V'(i) \ \forall i \in \Gamma_0.$$

To prove that $\mathrm{End}(V, f)$ is a ring, we prove that

1. $\mathrm{End}(V, f)$ is closed under addition and composition
2. $\mathrm{End}(V, f)$ is an abelian group under addition
3. $\mathrm{End}(V, f)$ is a monoid, i.e. composition is associative and there is an identity under composition in $\mathrm{End}(V, f)$ and
4. composition is distributive.

Let's get to it.

1. Let $h_1, h_2 \in \operatorname{End}(V, f)$.
   Then the following diagrams commute $\forall \alpha_{j,i} \in \Gamma_1$.

$$\begin{CD} V(i) @>{f_{\alpha_{j,i}}}>> V(j) \\ @V{h_1(i)}VV @VV{h_1(j)}V \\ V(i) @>{f_{\alpha_{j,i}}}>> V(j) \end{CD} \qquad \begin{CD} V(i) @>{f_{\alpha_{j,i}}}>> V(j) \\ @V{h_2(i)}VV @VV{h_2(j)}V \\ V(i) @>{f_{\alpha_{j,i}}}>> V(j) \end{CD}$$

   We show that these diagrams commute as well.

$$\begin{CD} V(i) @>{f_{\alpha_{j,i}}}>> V(j) \\ @V{(h_1+h_2)(i)}VV @VV{(h_1+h_2)(j)}V \\ V(i) @>{f_{\alpha_{j,i}}}>> V(j) \end{CD} \qquad \begin{CD} V(i) @>{f_{\alpha_{j,i}}}>> V(j) \\ @V{(h_1 \circ h_2)(i)}VV @VV{(h_1 \circ h_2)(j)}V \\ V(i) @>{f_{\alpha_{j,i}}}>> V(j) \end{CD}$$

   Let $x \in V(i)$.

   1.1. We show that the diagram to the left commutes.

$$\Big(\big[(h_1 + h_2)(j)\big] \circ f_{\alpha_{j,i}}\Big)(x) = \big[(h_1 + h_2)(j)\big]\big[f_{\alpha_{j,i}}(x)\big]$$

$$= \big[h_1(j)\big]\big[f_{\alpha_{j,i}}(x)\big] + \big[h_2(j)\big]\big[f_{\alpha_{j,i}}(x)\big]$$

$$= \Big(\big[h_1(j)\big] \circ f_{\alpha_{j,i}}\Big)(x) + \Big(\big[h_2(j)\big] \circ f_{\alpha_{j,i}}\Big)(x)$$

$$= \Big(f_{\alpha_{j,i}} \circ \big[h_1(i)\big]\Big)(x) + \Big(f_{\alpha_{j,i}} \circ \big[h_2(i)\big]\Big)(x)$$

$$= f_{\alpha_{j,i}}\Big(\big[h_1(i)\big](x)\Big) + f_{\alpha_{j,i}}\Big(\big[h_2(i)\big](x)\Big)$$

$$= f_{\alpha_{j,i}}\Big(\big[h_1(i)\big](x) + \big[h_2(i)\big](x)\Big)$$

$$= f_{\alpha_{j,i}}\Big(\big[(h_1 + h_2)(i)\big](x)\Big) = \Big(f_{\alpha_{j,i}} \circ \big[(h_1 + h_2)(i)\big]\Big)(x).$$

   Thus the diagram

$$\begin{CD} V(i) @>{f_{\alpha_{j,i}}}>> V(j) \\ @V{(h_1+h_2)(i)}VV @VV{(h_1+h_2)(j)}V \\ V(i) @>{f_{\alpha_{j,i}}}>> V(j) \end{CD}$$

   commutes, so $\operatorname{End}(V, f)$ is closed under addition.

1.2. We show that the diagram to the right commutes.

$$\Big(\big[(h_1 \circ h_2)(j)\big] \circ f_{\alpha_{j,i}}\Big)(x) = \Big[\Big(\big[h_1(j)\big] \circ \big[h_2(j)\big]\Big) \circ f_{\alpha_{j,i}}\Big](x)$$

$$= \Big[\big[h_1(j)\big] \circ \Big(\big[h_2(j)\big] \circ f_{\alpha_{j,i}}\Big)\Big](x) = \Big[\big[h_1(j)\big] \circ \Big(f_{\alpha_{j,i}} \circ \big[h_2(i)\big]\Big)\Big](x)$$

$$= \Big[\Big(\big[h_1(j)\big] \circ f_{\alpha_{j,i}}\Big) \circ \big[h_2(i)\big]\Big](x) = \Big[\Big(f_{\alpha_{j,i}} \circ \big[h_1(i)\big]\Big) \circ \big[h_2(i)\big]\Big](x)$$

$$= \Big[f_{\alpha_{j,i}} \circ \Big(\big[h_1(i)\big] \circ \big[h_2(i)\big]\Big)\Big](x) = \Big(f_{\alpha_{j,i}} \circ \big[(h_1 \circ h_2)(i)\big]\Big)(x).$$

Thus the diagram

$$\begin{array}{ccc} V(i) & \xrightarrow{f_{\alpha_{j,i}}} & V(j) \\ \big\downarrow{\scriptstyle (h_1 \circ h_2)(i)} & & \big\downarrow{\scriptstyle (h_1 \circ h_2)(j)} \\ V(i) & \xrightarrow{f_{\alpha_{j,i}}} & V(j) \end{array}$$

commutes, so $\mathrm{End}(V, f)$ is closed under composition.

Hence $\mathrm{End}(V, f)$ is closed under addition and composition.

2. To get that $\mathrm{End}(V, f)$ is an abelian group under addition, we need to show that addition is associative, $\mathrm{End}(V, f)$ has an additive identity, every element in $\mathrm{End}(V, f)$ has an additive inverse and addition is commutative.

2.1. Let $h_1, h_2, h_3 \in \mathrm{End}(V, f)$, $i \in \Gamma_0$ and $x \in V(i)$.

$$\Big(\big[(h_1 + h_2) + h_3\big](i)\Big)(x) = \big[(h_1 + h_2)(i)\big](x) + \big[h_3(i)\big](x)$$

$$= \Big(\big[h_1(i)\big](x) + \big[h_2(i)\big](x)\Big) + \big[h_3(i)\big](x)$$

$$= \big[h_1(i)\big](x) + \Big(\big[h_2(i)\big](x) + \big[h_3(i)\big](x)\Big)$$

$$= \big[h_1(i)\big](x) + \big[(h_2 + h_3)(i)\big](x) = \Big(\big[h_1 + (h_2 + h_3)\big](i)\Big)(x),$$

which is possible since addition is associative in vector spaces. Thus $(h_1 + h_2) + h_3 = h_1 + (h_2 + h_3)$, so addition is associative.

2.2. Let $h \in \text{End}(V, f)$.
Consider the collection 0_{hom} of k-linear maps such that

$$\big[0_{\text{hom}}(i)\big](x) = 0 \ \forall x \in V(i) \ \forall i \in \Gamma_0.$$

$0_{\text{hom}} \in \text{End}(V, f)$ since every vector space has an additive identity. Then

$$\big[(h + 0_{\text{hom}})(i)\big](x) = \big[h(i)\big](x) + \big[0_{\text{hom}}(i)\big](x) = \big[h(i)\big](x) + 0$$
$$= \big[h(i)\big](x)$$
$$= 0 + \big[h(i)\big](x) = \big[0_{\text{hom}}(i)\big](x) + \big[h(i)\big](x) = \big[(0_{\text{hom}} + h)(i)\big](x).$$

Thus $h + 0_{hom} = h = 0_{\text{hom}} + h$, so 0_{hom} is an additive identity in $\text{End}(V, f)$.

2.3. Let $h \in \text{End}(V, f)$.
Consider the collection $-h$ of k-linear maps such that

$$\big[(-h)(i)\big](x) = -\big[h(i)\big](x) \ \forall x \in V(i) \ \forall i \in \Gamma_0.$$

$-h \in \text{End}(V, f)$ since every vector space has an additive inverse for each element in it. Then

$$\Big(\big[h + (-h)\big](i)\Big)(x) = \big[h(i)\big](x) + \big[(-h)(i)\big](x)$$
$$= \big[h(i)\big](x) + \Big(-\big[h(i)\big](x)\Big)$$
$$= 0$$
$$= -\big[h(i)\big](x) + \big[h(i)\big](x)$$
$$= \big[(-h)(i)\big](x) + \big[h(i)\big](x) = \big[(-h + h)(i)\big](x).$$

Thus $h + (-h) = -h + h$, so $-h$ is an additive inverse of h in $\text{End}(V, f)$.

2.4. Let $h_1, h_2 \in \text{End}(V, f)$ and $x \in V(i)$ for some $i \in \Gamma_0$. Then

$$\big[(h_1 + h_2)(i)\big](x) = \big[h_1(i)\big](x) + \big[h_2(i)\big](x)$$
$$= \big[h_2(i)\big](x) + \big[h_1(i)\big](x) = \big[(h_2 + h_1)(i)\big](x),$$

since $\big[h_1(i)\big](x), \big[h_2(i)\big](x) \in V(i)$ and vector spaces are abelian groups under addition.
Thus $h_1 + h_2 = h_2 + h_1$ addition is commutative in $\text{End}(V, f)$.

Hence $\mathrm{End}(V, f)$ is an abelian group under addition.

3. Now we show that $\mathrm{End}(V, f)$ is a monoid under composition.

   3.1. We first show that composition is associative.
   Let $h_1, h_2, h_3 \in \mathrm{End}(V, f)$ and $x \in V(i)$ for some $i \in \Gamma_0$. Then

   $$\Big(\big[(h_1 \circ h_2) \circ h_3\big](i)\Big)(x) = \big[(h_1 \circ h_2)(i) \circ (h_3)(i)\big](x)$$
   $$= \Big(\big[h_1(i) \circ h_2(i)\big] \circ h_3(i)\Big)(x) = \Big(h_1(i) \circ \big[h_2(i) \circ h_3(i)\big]\Big)(x)$$
   $$= \big[h_1(i) \circ (h_2 \circ h_3)(i)\big](x) = \Big(\big[h_1 \circ (h_2 \circ h_3)\big](i)\Big)(x),$$

   which is possible since composition of linear transformations is associative.
   Thus, $(h_1 \circ h_2) \circ h_3 = h_1 \circ (h_2 \circ h_3)$, so composition of endomorphisms is associative.

   3.2. Now we show that there is a compositional identity in $\mathrm{End}(V, f)$.
   Let $h \in \mathrm{End}(V, f)$.
   Consider the collection id_V of fixed maps such that

   $$\big[\mathrm{id}_V(i)\big](x) = x \ \forall x \in V(i) \ \forall i \in \Gamma_0.$$

   $\mathrm{id}_V \in \mathrm{End}(V, f)$, since every vector space has a fixed linear transformation for each element in the vector space. Then

   $$\big[(h \circ \mathrm{id}_V)(i)\big](x)$$
   $$= \big[h(i) \circ \mathrm{id}_V(i)\big](x) = \big[h(i)\big]\Big(\big[\mathrm{id}_V(i)\big](x)\Big)$$
   $$= \big[h(i)\big](x)$$
   $$= \big[\mathrm{id}_V(i)\big]\Big(\big[h(i)\big](x)\Big) = \big[\mathrm{id}_V(i) \circ h(i)\big](x)$$
   $$= \big[(\mathrm{id}_V \circ h)(i)\big](x).$$

   Thus $h \circ \mathrm{id}_V = \mathrm{id}_V \circ h$, so $\mathrm{End}(V, f)$ has a compositional identity. Note that this point can be omitted if we use the definition of a ring that does not require a multiplicative identity.

Hence $\mathrm{End}(V, f)$ is a monoid under composition.

4. Now we show that composition is distributive by proving that both of the distribute laws hold.
Let $h_1, h_2, h_3 \in \text{End}(V, f)$ and $x \in V(i)$ for some $i \in \Gamma_0$.

4.1. We prove the left distributive law.

$$\Big(\big[h_1 \circ (h_2 + h_3)\big](i)\Big)(x) = \big[h_1(i) \circ (h_2 + h_3)(i)\big](x)$$

$$= \big[h_1(i)\big]\Big(\big[(h_2 + h_3)(i)\big](x)\Big) = \big[h_1(i)\big]\Big(\big[h_2(i)\big](x) + \big[h_3(i)\big](x)\Big)$$

$$= \big[h_1(i)\big]\Big(\big[h_2(i)\big](x)\Big) + \big[h_1(i)\big]\Big(\big[h_3(i)\big](x)\Big)$$

$$= \big[h_1(i) \circ h_2(i)\big](x) + \big[h_1(i) \circ h_3(i)\big](x)$$

$$= \big[(h_1 \circ h_2)(i)\big](x) + \big[(h_1 \circ h_3)(i)\big](x),$$

which is possible since composition of linear transformations is distributive.
Thus $h_1 \circ (h_2 + h_3) = h_1 \circ h_2 + h_1 \circ h_3$, so the left distributive law holds.

4.2 We prove the right distributive law.

$$\Big(\big[(h_2 + h_3) \circ h_1\big](i)\Big)(x) = \big[(h_2 + h_3)(i) \circ h_1(i)\big](x)$$

$$= \big[(h_2 + h_3)(i)\big]\Big(\big[h_1(i)\big](x)\Big)$$

$$= \big[h_2(i)\big]\Big(\big[h_1(i)\big](x)\Big) + \big[h_3(i)\big]\Big(\big[h_1(i)\big](x)\Big)$$

$$= \big[h_2(i) \circ h_1(i)\big](x) + \big[h_3(i) \circ h_1(i)\big](x)$$

$$= \big[(h_2 \circ h_1)(i)\big](x) + \big[(h_3 \circ h_1)(i)\big](x).$$

Thus $(h_2 + h_3) \circ h_1 = h_2 \circ h_1 + h_3 \circ h_1$, so the right distributive law holds.

Hence composition is distributive.

Thus $\text{End}(V, f)$ is a ring under homomorphism addition and composition. □

With this result in our toolbox, we go on to state the next theorem, which is about a condition that implies indecomposability of representations.

Theorem 1.3. *Let (V, f) be a representation of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ over a field k. Then*
$\mathrm{End}(V, f) \cong k \Rightarrow (V, f)$ *is indecomposable.*

Proof. We first find a subfield of $\mathrm{End}(V, f)$ that is isomorphic to k. This will aid us in proving the theorem.
Let $a \in k$ and define $h_a : (V, f) \to (V, f)$ such that

$$\big[h_a(i)\big](v) = av \;\forall v \in V(i) \;\forall i \in \Gamma_0.$$

Every $f_{\alpha_{j,i}}$ is k-linear, that is,

$$f_{\alpha_{j,i}}(\lambda x) = \lambda f_{\alpha_{j,i}}(x) \;\forall \lambda \in k, x \in V(i), \alpha_{j,i} \in \Gamma_1,$$

so then

$$\big[h_a(j) \circ f_{\alpha_{j,i}}\big](v) = \big[h_a(j)\big]\big[f_{\alpha_{j,i}}(v)\big] = a\big[f_{\alpha_{j,i}}(v)\big] = f_{\alpha_{j,i}}(av)$$

$$= f_{\alpha_{j,i}}\Big(\big[h_a(i)\big](v)\Big) = \big[f_{\alpha_{j,i}} \circ h_a(i)\big](v).$$

Thus h_a is an endomorphism on (V, f). Now define

$$k(V, f) = \big\{h_a \in \mathrm{End}(V, f)\big|[h_a(i)](v) = av \;\forall v \in V(i), i \in \Gamma_0 \, a \in k\big\}.$$

Hence, there is a natural bijection $\Phi : k(V, f) \to k$ such that $h_a \mapsto a \;\forall a \in k$.
Furthermore, if $b \in k$ and $v \in V(i)$ for some $i \in \Gamma_0$, then

$$\big[(h_a+h_b)(i)\big](v) = \big[h_a(i)\big](v)+\big[h_b(i)\big](v) = av+bv = (a+b)v = \big[h_{a+b}(i)\big](v)$$

and

$$\big[(h_a \circ h_b)(i)\big](v) = \big[h_a(i) \circ h_b(i)\big](v) = \big[h_a(i)\big]\Big(\big[h_b(i)\big](v)\Big) = \big[h_a(i)\big](bv)$$

$$= a(bv) = (ab)v = \big[h_{ab}(i)\big](v).$$

Thus $\Phi(h_a + h_b) = \Phi(h_{a+b}) = a + b = \Phi(h_a) + \Phi(h_b)$
and $\Phi(h_a \circ h_b) = \Phi(h_{ab}) = ab = \Phi(h_a)\Phi(h_b)$,
so Φ is a ring homomorphism and therefore an isomorphism.
Hence $k(V, f) \cong k$.
Having shown this, we are finally ready to prove the statement of the theorem.
Suppose $\mathrm{End}(V, f) \cong k$. Then $\mathrm{End}(V, f) = k(V, f)$ since $k(V, f) \subseteq \mathrm{End}(V, f)$.
Let $\alpha_{j,i} \in \Gamma_1$ and suppose there are representations (V', f') and (V'', f'')

such that $(V,f) = (V',f') \oplus (V'',f'')$. For every $v \in V(i)$ there are unique vectors $v' \in V'(i)$ and $v'' \in V''(i)$ such that $v = v' + v''$. Define a collection of maps $h_{V'} : (V,f) \to (V,f)$ such that $v \mapsto h(v')$ for each $v \in V(i)$. It is clear that $\big[h_{V'}(i)\big]\big[V(i)\big] = V'(i)$ and $\ker\big[h_{V'}(i)\big] = V''(i)\ \forall i \in \Gamma_0$. We also have

$$\begin{aligned}\big[h_{V'}(i)\big]\big[\lambda(v_1+v_2)\big] &= \big[h_{V'}(i)\big](\lambda v_1 + \lambda v_2)\\ &= (\lambda v_1 + \lambda v_2)' = (\lambda v_1)' + (\lambda v_2)' = \lambda v_1' + \lambda v_2'\\ &= \lambda\big[h_{V'}(i)\big](v_1) + \lambda\big[h_{V'}(i)\big](v_2)\\ &= \lambda\Big(\big[h_{V'}(i)\big](v_1) + \big[h_{V'}(i)\big](v_2)\Big)\ \forall v_1, v_2 \in V(i), \lambda \in k,\end{aligned}$$

so $h_{V'}(i)$ is k-linear. Then

$$\begin{aligned}\big[h_{V'}(j) \circ f_{\alpha_{j,i}}\big](v) &= \big[h_{V'}(j)\big]\big[f_{\alpha_{j,i}}(v)\big] = \big[h_{V'}(j)\big]\Big(\big[f'_{\alpha_{j,i}} \oplus f''_{\alpha_{j,i}}\big](v)\Big)\\ &= \big[h_{V'}(j)\big]\big[f'_{\alpha_{j,i}}(v') \oplus f''_{\alpha_{j,i}}(v'')\big]\\ &= \big[h_{V'}(j)\big]\big[f'_{\alpha_{j,i}}(v')\big] \oplus \big[h_{V'}(j)\big]\big[f''_{\alpha_{j,i}}(v'')\big]\\ &= \big[h_{V'}(j)\big]\big[f'_{\alpha_{j,i}}(v')\big] \oplus 0 = \big[h_{V'}(j)\big]\big[f'_{\alpha_{j,i}}(v')\big] = f'_{\alpha_{j,i}}(v')\\ &= f'_{\alpha_{j,i}}\Big(\big[h_{V'}(i)\big](v')\Big) = f'_{\alpha_{j,i}}\Big(\big[h_{V'}(i)\big](v')\Big) \oplus 0\\ &= f'_{\alpha_{j,i}}\Big(\big[h_{V'}(i)\big](v')\Big) \oplus f''_{\alpha_{j,i}}(0)\\ &= f'_{\alpha_{j,i}}\Big(\big[h_{V'}(i)\big](v')\Big) \oplus f''_{\alpha_{j,i}}\Big(\big[h_{V'}(i)\big](v'')\Big)\\ &= (f'_{\alpha_{j,i}} \oplus f''_{\alpha_{j,i}})\Big(\big[h_{V'}(i)\big](v)\Big) = f_{\alpha_{j,i}}\Big(\big[h_{V'}(i)\big](v)\Big) = \big[f_{\alpha_{j,i}} \circ h_{V'}(i)\big](v),\end{aligned}$$

so $h_{V'}$ is a homomorphism of representations, and since $\big[h_{V'}(i)\big] : V(i) \to V(i)$, we see that $h_{V'}$ is an endomorphism on (V,f). Then $h_{V'} \in \mathrm{End}(V,f) = k(V,f)$, so $h_{V'} = h_a$ for some $a \in k$. If $a = 0$, then $\big[h_{V'}(i)\big]\big[V(i)\big] = 0$, so $V'(i) = 0$ and $V''(i) = V(i)\ \forall i \in \Gamma_0$, which results in (V',f') being the trivial representation. If $a \neq 0$, then $h_{V'}(i)$ is injective since $av_1 = av_2 \Rightarrow a_1 = a_2$ for $a \in k, a_1, a_2 \in V(i)$, so $\ker\big[h_{V'}(i)\big] = 0$. Then $V''(i) = 0$ and $V'(i) = V(i)\ \forall i \in \Gamma_0$, which yields (V'',f'') being the trivial representation. In either case, (V,f) is indecomposable. □

The result above is what we will use to determine which representations are indecomposable.

Theorem 1.4. *Let (V, f) be a finite-dimensional representation of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ over a field k. Then (V, f) is isomorphic to a representation (V', f') of Γ over k where $V' = \{k^{n_i}\}_{i \in \Gamma_0}$ and $n_i = \dim_k[V(i)]$ $\forall i \in \Gamma_0$.*

Proof. Let $B = \{\beta_1, ..., \beta_{n_i}\}$ be a basis of $V(i)$. Then an arbitrary element $a \in V(i)$ can be expressed as $a = \sum_{m=1}^{n_i} a_m \beta_m$. Let $\alpha_{j,i} \in \Gamma_1$ and and define

$$f'_{\alpha_{j,i}}(a_1, a_2, ..., a_{n_i}) = \big(f_{\alpha_{j,i}}(a_1), f_{\alpha_{j,i}}(a_2), ..., f_{\alpha_{j,i}}(a_{\min\{n_i,n_j\}}\}, 0, ..., 0)\big),$$

where the number of zeros after $f_{\alpha_{j,i}}(a_{\min\{n_i,n_j\}}\}$ equals $n_j - n_i$ if $n_j \geq n_i$ and 0 otherwise.
Then define $h : (V, f) \to (V', f')$ such that
$\big[h(i)\big](a) = \big[h(i)\big]\left(\sum_{m=1}^{n_i} a_m \beta_m\right) = (a_1, a_2, ..., a_{n_i})$ $\forall i \in \Gamma$. Thus

$$\big[h(i)\big](\beta_1) = (1, 0, ..., 0),$$
$$\big[h(i)\big](\beta_2) = (0, 1, ..., 0),$$
$$\vdots$$
$$\big[h(i)\big](\beta_{n_i}) = (0, ..., 0, 1),$$

so $h(i)$ sends B to a basis of k^{n_i}, hence $h(i)$ is an bijective k-linear map.
Now we show that h is a homomorphism.

$$\Big(f'_{\alpha_{j,i}} \circ \big[h(i)\big]\Big)\left(\sum_{m=1}^{n_i} a_m\beta_m\right) = f'_{\alpha_{j,i}}\left(\big[h(i)\big]\left[\sum_{m=1}^{n_i} a_m\beta_m\right]\right)$$
$$= f'_{\alpha_{j,i}}(a_1, a_2, ..., a_{n_i}) = \big(f_{\alpha_{j,i}}(a_1), f_{\alpha_{j,i}}(a_2), ..., f_{\alpha_{j,i}}(a_{\min\{n_i,n_j\}}), 0, ..., 0\big)$$
$$= \big[h(j)\big]\left(\sum_{m=1}^{\min\{n_i,n_j\}} f_{\alpha_{j,i}}(a_m)\beta_m\right) = \big[h(j)\big]\left[f_{\alpha_{j,i}}\left(\sum_{m=1}^{n_i} a_m\beta_m\right)\right]$$
$$= \Big(\big[h(j)\big] \circ f'_{\alpha_{j,i}}\Big)\left(\sum_{m=1}^{n_i} a_m\beta_m\right).$$

Thus h is a homomorphism, and since it is bijective, it is also an isomorphism.
Hence $(V, f) \cong (V', f')$. ∎

Remark 1.8. In the case where (V, f) is a subspace representation we get that for any $\alpha_{j,i} \in \Gamma_1$,

$$f'_{\alpha_{j,i}}(a_1, a_2, ..., a_{n_i}) = (f_{\alpha_{j,i}}(a_1), f_{\alpha_{j,i}}(a_2), ..., f_{\alpha_{j,i}}(a_{n_i}), 0, ..., 0),$$

where the number of zeros after a_{n_i} is equal to $n_j - n_i$, since $f_{\alpha_{j,i}}$ is injective. Then $f'_{\alpha_{j,i}}$ is injective as well. Thus (V', f') is a subspace representation. $\triangle$

The theorem above constrains the amount of representations we need to look at when we want to find indecomposable representations. Now we only need to look at representations whose vector spaces are powers of k.
We conclude this section with investigating the endomorphism rings of a few representations. Perhaps some of them are indecomposable?

Example 1.4. We want to find the endomorphism rings of the following subspace representations of $\mathcal{Q}$ over some field k.

$$\mathcal{V}_{0,5} = (\{0, 0, 0, 0, k\}, f_{0,5}),$$

$$\mathcal{V}_{1,1} = (\{k, 0, 0, 0, k\}, f_{1,1}),$$
$$\mathcal{V}_{1,2} = (\{0, k, 0, 0, k\}, f_{1,2}),$$
$$\mathcal{V}_{1,3} = (\{0, 0, k, 0, k\}, f_{1,3}),$$
$$\mathcal{V}_{1,4} = (\{0, 0, 0, k, k\}, f_{1,4}).$$

We also visualize them, so it is easier to see what we are dealing with.

$$\mathcal{V}_{0,5} = \begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & k & \longleftarrow & 0 \\ & & \uparrow & & \\ & & 0 & & \end{array}$$

$$\mathcal{V}_{1,1} = \begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ k & \longrightarrow & k & \longleftarrow & 0 \\ & & \uparrow & & \\ & & 0 & & \end{array}, \quad \mathcal{V}_{1,2} = \begin{array}{ccccc} & & k & & \\ & & \downarrow & & \\ 0 & \longrightarrow & k & \longleftarrow & 0 \\ & & \uparrow & & \\ & & 0 & & \end{array}$$

$$\mathcal{V}_{1,3} = \begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & k & \longleftarrow & k \\ & & \uparrow & & \\ & & 0 & & \end{array}, \mathcal{V}_{1,4} = \begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & k & \longleftarrow & 0 \\ & & \uparrow & & \\ & & k & & \end{array}$$

1. Now we find $\text{End}(\mathcal{V}_{0,5})$. Let $h \in \text{End}(\mathcal{V}_{0,5})$. Then h satisfies $\big[h(5) \circ (f_{0,5})_{\alpha_{5,i}}\big](v) = \big[(f_{0,5})_{\alpha_{5,i}} \circ h(i)\big](v)\ \forall v \in 0, i \in \{1,2,3,4\}$, and since $(f_{0,5})_{\alpha_{5,i}}$ are maps whose domain are 0, the equation yields $0 = 0$, which means that these maps do not constrain h. If $i \in \{1,2,3,4\}$, then $\big[h(i)\big](v) = 0\ v \in 0$, meaning that this does not constrain h either. If $i = 5$, then $\big[h(i)\big](v) = av$ for some $a \in k$. Thus $h = h_a$ for some $a \in k$. Hence $\text{End}(\mathcal{V}_{0,5}) \cong k$.

2. To find the endomorphism rings of $\mathcal{V}_{1,1}$, $\mathcal{V}_{1,2}$, $\mathcal{V}_{1,3}$ and $\mathcal{V}_{1,4}$, we only look at $\mathcal{V}_{1,1}$, since the others only are "rotations" of $\mathcal{V}_{1,1}$ which will yield the same endomorphism rings as $\mathcal{V}_{1,1}$. Let $h \in \text{End}(\mathcal{V}_{1,1})$. If $i \in \{2,3,4\}$, then $\big[h(5) \circ (f_{1,1})_{\alpha_{5,i}}\big](v) = \big[(f_{1,1})_{\alpha_{5,i}} \circ h(i)\big](v)\ \forall v \in 0$ yields $0 = 0$. If $i = 1$, it yields $acv = cbv$ where $(f_{1,1})_{\alpha_{5,1}}(v) = cv$ for some $c \in k$ such that $c \neq 0$, $\big[h(1)\big](v) = av$ and $\big[h(5)\big](v) = bv$ for $a, b \in k$, so $av = bv$ which gives $a = b$. Then $h = h_a$ for some $a \in k$. Thus $\text{End}(\mathcal{V}_{1,1}) \cong k$ and dually we get $\text{End}(\mathcal{V}_{1,2}) \cong \text{End}(\mathcal{V}_{1,3}) \cong \text{End}(\mathcal{V}_{1,4}) \cong k$.

Hence $\mathcal{V}_{0,5}$, $\mathcal{V}_{1,1}$, $\mathcal{V}_{1,2}$, $\mathcal{V}_{1,3}$ and $\mathcal{V}_{1,4}$ are all indecomposable. △

Remark 1.9. We continue using the notation $\mathcal{V}_{0,5}$, $\mathcal{V}_{1,1}$, $\mathcal{V}_{1,2}$, $\mathcal{V}_{1,3}$ and $\mathcal{V}_{1,4}$ for the representations in the example above as we proceed. △

2 Kernels, cokernels and dimension vectors

In this section, we define a few terms and state some facts which will be helpful when constructing the sequence we are building towards. The central concepts are kernels, cokernels and dimension vectors of representations.

Definition 2.1. Let $h : (V, f) \to (V', f)$ be a homomorphism between representations of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ over a field k. The **kernel** of h is a tuple $\ker(h) = \big[\ker_{\text{ob}}(h), \ker_{\text{hom}}(h)\big]$ where

1. $\ker_{\text{ob}}(h) = (V'', f'')$ consists of a collection V'' of sets $V''(i)$ such that

$$V''(i) = \ker\big[h(i)\big]\ \forall i \in \Gamma_0$$

and a collection f'' of maps $f''_{\alpha_{j,i}}$ such that

$$f''_{\alpha_{j,i}}(v) = f_{\alpha_{j,i}}(v)\ \forall v \in V''(i), \alpha_{j,i} \in \Gamma_1.$$

2. $\ker_{\text{hom}}(h) : \ker_{\text{ob}}(h) \to (V, f)$ is a collection of maps $\big[\ker_{\text{hom}}(h)\big](i) : \big[(\ker_{\text{ob}})_0(h)\big](i) \to V(i)$ such that

$$\Big(\big[\ker_{\text{hom}}(h)\big](i)\Big)(v) = v \forall v \in \big[(\ker_{\text{ob}})_0(h)\big](i), i \in \Gamma_0.$$

△

Remark 2.1. We show that $\ker_{\text{ob}}(h)$ is a representation over k and $\ker_{\text{hom}}(h)$ is a homomorphism between representations.

1. Since $V''(i) = \ker\big[h(i)\big] \subseteq V(i)\ \forall i \in \Gamma_0$, $V''(i)$ is a vector space over k. Then V'' is a collection of vector spaces over k.
   $f''_{\alpha_{j,i}}(v) = f_{\alpha_{j,i}}(v)\ \forall v \in V''(i)\ \forall \alpha_{j,i} \in \Gamma_1 \Rightarrow f''_{\alpha_{j,i}} = f_{\alpha_{j,i}}|_{V''(i)}$, so $f''_{\alpha_{j,i}}$ is k-linear.
   We also need to show that $f''_{\alpha_{j,i}} : V''(i) \to V''(j)$. Let $v \in V''(i)$. Then $\big[h(i)\big](v) = 0$ and

   $$\big[h(j)\big]\big[f''_{\alpha_{j,i}}(v)\big] = \big[h(j)\big]\big[f_{\alpha_{j,i}}(v)\big] = \big[h(j) \circ f_{\alpha_{j,i}}\big](v) = \big[f_{\alpha_{j,i}} \circ h(i)\big](v)$$

   $$= f_{\alpha_{j,i}}\Big(\big[h(i)\big](v)\Big) = f_{\alpha_{j,i}}(0) = 0.$$

   Thus $f''_{\alpha_{j,i}}(v) \in V''(j)$, so $f''_{\alpha_{j,i}} : V''(i) \to V''(j)$.
   Hence $\ker_{\text{ob}}(h)$ is a representation over k.
   From the arguments above we also obtain that $\ker_{\text{ob}}(h)$ is a subrepresentation of (V, f).

2. We see that $\ker_{\text{hom}}(h)$ is a collection of inclusion maps. It is then a homomorphism.

△

Remark 2.2. We typically denote $\ker_{\text{ob}}(h) = \big[(\ker_{\text{ob}})_0(h), (\ker_{\text{ob}})_1(h)\big]$ such that $\big[(\ker_{\text{ob}})_0(h)\big](i)$ is the vector space corresponding to vertex i $\forall i \in \Gamma_0$ and $\big[(\ker_{\text{ob}})_1(h)\big]_{\alpha_{j,i}}$ is the k-linear map corresponding to arrow $\alpha_{j,i}\ \forall \alpha_{j,i} \in \Gamma_1$. △

Definition 2.2. Let g be an R-homomorphism between R-modules M and N where R is a ring.
The **cokernel** of g is a tuple $\operatorname{cok}(g) = \big(\operatorname{cok}_{\text{ob}}(g), \operatorname{cok}_{\text{hom}}(g)\big)$ such that

1. $\operatorname{cok}_{\text{ob}}(g)$ is the quotient module $N/g(M)$ over R,
2. $\operatorname{cok}_{\text{hom}}(g)\big)$ is the R-homomorphism $\operatorname{cok}_{\text{hom}}(g)\big) : N \to N/g(M)$ such that $\operatorname{cok}_{\text{hom}}(g)(n) = n + g(M)\ \forall n \in N$.

$\triangle$

This definition is not that interesting to us by itself. We really just need it in order to define cokernels of homomorphisms of representeations, which is what we do next.

Definition 2.3. Let h be a homomorphism between two representations (V, f) and (V', f') of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ with n vertices over a field k. The **cokernel** of h is a tuple $\operatorname{cok}(h) = \big(\operatorname{cok}_{\text{ob}}(h), \operatorname{cok}_{\text{hom}}(h)\big)$ where

1. $\operatorname{cok}_{\text{ob}}(h) = (V', f')/h(V, f) = (V'', f'')$ is a collection V'' of sets $V''(i)$ such that
$$V''(i) = \frac{V'(i)}{\big[h(i)\big]\big[V(i)\big]}\ \forall i \in \Gamma_0$$
and a collection f'' of maps $f''_{\alpha_{j,i}}$ such that
$$f''_{\alpha_{j,i}}\Big(v' + \big[h(i)\big]\big[V(i)\big]\Big) = f'_{\alpha_{j,i}}(v') + \big[h(j)\big]\big[V(j)\big]$$
$$\forall v' \in V'(i), \alpha_{j,i} \in \Gamma_1,$$
2. $\operatorname{cok}_{\text{hom}}(h) : (V', f') \to (V'', f'')$ is a collection of maps $\big[\operatorname{cok}_{\text{hom}}(h)\big](i) : V'(i) \to V''(i)$ such that
$$\Big(\big[\operatorname{cok}_{\text{hom}}(h)\big](i)\Big)(v') = \Big(\operatorname{cok}_{\text{hom}}\big[h(i)\big]\Big)(v') = v' + \big[h(i)\big]\big[V(i)\big]$$
$$\forall v' \in V'(i), i \in \Gamma_0.$$

$\triangle$

Remark 2.3. We can show that $\operatorname{cok}_{\text{ob}}(h)$ is a representation and $\operatorname{cok}_{\text{hom}}(h)$ is a homomorphism.

1. We recognize that $V''(i) = \text{cok}_{\text{ob}}\big[h(i)\big]$, which by definition is a k-module, i.e. a vector space over k, since $h(i)$ is k-linear. Thus V'' is a collection of vector spaces over k. Furthermore, if $v' \in V'(i)$, then $v' + \big[h(i)\big]\big[V(i)\big] \in V''(i)$ and for each $v'' \in V''(i)$ $\exists v' \in V'(i)$ such that $v'' = v' + \big[h(i)\big]\big[V(i)\big]$ $\forall i \in \Gamma_0$. Thus the domain of $f''_{\alpha_{j,i}}$ is $V''(i)$. We also have that for any $v' \in V'(i)$, $f'_{\alpha_{j,i}}(v') \in V'(j)$, so

$$f''_{\alpha_{j,i}}\Big(v' + \big[h(i)\big]\big[V(i)\big]\Big) = f'_{\alpha_{j,i}}(v') + \big[h(j)\big]\big[V(j)\big] \in V''(j).$$

   Thus $f''_{\alpha_{j,i}} : V''(i) \to V''(j)$. Additionally, since $f'_{\alpha_{j,i}}$ is k-linear, $f''_{\alpha_{j,i}}$ is k-linear as well.
   Hence $\text{cok}_{\text{ob}}(h)$ is a representation of Γ over k.

2. We show that $f'' \circ \text{cok}_{\text{hom}}(h) = \text{cok}_{\text{hom}}(h) \circ f'$. k-linearity follows from the way addition and scalar multiplication are defined in quotient modules.
   Let $\alpha_{j,i} \in \Gamma_1$ and $v' \in V'(i)$. Then

$$\Big(f''_{\alpha_{j,i}} \circ \big[\text{cok}_{\text{hom}}(h)\big](i)\Big)(v') = f''_{\alpha_{j,i}}\bigg[\Big(\big[\text{cok}_{\text{hom}}(h)\big](i)\Big)(v')\bigg]$$
$$= f''_{\alpha_{j,i}}\Big(v' + \big[h(i)\big]\big[V(i)\big]\Big) = f'_{\alpha_{j,i}}(v') + \big[h(j)\big]\big[V(j)\big]$$
$$= \Big(\big[\text{cok}_{\text{hom}}(h)\big](i)\Big)\big[f'_{\alpha_{j,i}}(v')\big] = \Big(\big[\text{cok}_{\text{hom}}(h)\big](i) \circ f'_{\alpha_{j,i}}\Big)(v').$$

   Thus $\text{cok}_{\text{hom}}(h)$ is a homomorphism.

$\triangle$

Remark 2.4. We will usually denote $\text{cok}_{\text{ob}}(h) = \big[(\text{cok}_{\text{ob}})_0(h), (\text{cok}_{\text{ob}})_1(h)\big]$ such that $\big[(\text{cok}_{\text{ob}})_0(h)\big](i)$ is the vector space corresponding to vertex i $\forall i \in \Gamma_0$ and $\big[(\text{cok}_{\text{ob}})_1(h)\big]_{\alpha_{j,i}}$ is the k-linear map corresponding to arrow $\alpha_{j,i}$ $\forall \alpha_{j,i} \in \Gamma_1$. $\triangle$

Theorem 2.1. *Let $h_1 : (V,f) \to (V',f')$, $h_2 : (W,g) \to (W',g')$, $L_2 : (V,f) \to (W,g)$ and $L_3 : (V',f') \to (W',g')$ be four homomorphisms between representations of a common quiver $\Gamma = (\Gamma_0, \Gamma_1)$ over a field k such that that the diagram*

$$\begin{array}{ccc} (V,f) & \xrightarrow{h_1} & (V',f') \\ \big\downarrow{\scriptstyle L_2} & & \big\downarrow{\scriptstyle L_3} \\ (W,g) & \xrightarrow{h_2} & (W',g') \end{array}$$

commutes. Then two homomorphisms

$$L_1 : \ker_{\text{ob}}(h_1) \to \ker_{\text{ob}}(h_2)$$

and

$$L_4 : \text{cok}_{\text{ob}}(h_1) \to \text{cok}_{\text{ob}}(h_2)$$

are induced such that the diagram

$$\begin{array}{ccccccc}
\ker_{\text{ob}}(h_1) & \xrightarrow{\ker_{\text{hom}}(h_1)} & (V,f) & \xrightarrow{h_1} & (V',f') & \xrightarrow{\text{cok}_{\text{hom}}(h_1)} & \text{cok}_{\text{ob}}(h_1) \\
\downarrow L_1 & & \downarrow L_2 & & \downarrow L_3 & & \downarrow L_4 \\
\ker_{\text{ob}}(h_2) & \xrightarrow{\ker_{\text{hom}}(h_2)} & (W,g) & \xrightarrow{h_2} & (W',g') & \xrightarrow{\text{cok}_{\text{hom}}(h_2)} & \text{cok}_{\text{ob}}(h_2)
\end{array}$$

commutes.

Proof. We first show the kernel homomorphism and then the cokernel homomorphism.

1. Define L_1, a collection of maps such that

$$\big[L_1(i)\big](v) = \big[L_2(i)\big](v)\ \forall v \in \ker\big[h_1(i)\big], \forall i \in \Gamma_0.$$

Since $\ker_{\text{hom}}(h_1)$ and $\ker_{\text{hom}}(h_2)$ are inclusion, the diagram without cokernels commutes. Is the codomain of L_1 really $\ker_{\text{ob}}(h_2)$? Suppose $i \in \Gamma_0$ and $v \in \ker\big[h(i)\big]$. Then

$$\big[h_2(i) \circ L_1(i)\big](v) = \big[h_2(i)\big]\Big(\big[L_1(i)\big](v)\Big) = \big[h_2(i)\big]\Big(\big[L_2(i)\big](v)\Big)$$

$$\big[h_2(i) \circ L_2(i)\big](v) = \big[L_3(i) \circ h_1(i)\big](v) = \big[L_3(i)\big]\Big(\big[h_1(i)\big](v)\Big)$$
$$= \big[L_3(i)\big](0) = 0$$
$$\Rightarrow \big[L_1(i)\big](v) \in \ker\big[h_2(i)\big].$$

Thus L_1 is well-defined.
Now we show that L_1 is a homomorphism.
Let $\alpha_{j,i} \in \Gamma_1$ and $v \in \ker_{\text{ob}}(h_1)$. Then

$$\Big(\big[(\ker_{\text{hom}})_1(h_2)\big]_{\alpha_{j,i}} \circ \big[L_1(i)\big]\Big)(v) = \big[(\ker_{\text{hom}})_1(h_2)\big]_{\alpha_{j,i}}\Big(\big[L_1(i)\big](v)\Big)$$
$$= \big[(\ker_{\text{hom}})_1(h_2)\big]_{\alpha_{j,i}}\Big(\big[L_2(i)\big](v)\Big) = g_{\alpha_{j,i}}\Big(\big[L_2(i)\big](v)\Big)$$
$$= \big[L_2(i)\big]\big[f_{\alpha_{j,i}}(v)\big] = \big[L_2(i)\big]\Big(\big[(\ker_{\text{hom}})_1(h_2)\big]_{\alpha_{j,i}}(v)\Big)$$
$$= \big[L_1(i)\big]\Big(\big[(\ker_{\text{hom}})_1(h_2)\big]_{\alpha_{j,i}}(v)\Big) = \Big(L_1(i) \circ \big[(\ker_{\text{hom}})_1(h_2)\big]_{\alpha_{j,i}}\Big)(v).$$

Hence L_1 is a homomorphism.

2. Define L_4, a collection of maps such that

$$\big[L_4(i)\big]\Big(v + \big[h_1(i)\big]\big[V(i)\big]\Big)$$
$$= \big[L_3(i)\big](v) + \big[h_2(i)\big]\big[W'(i)\big] \; \forall v \in V'(i) \; \forall i \in \Gamma_0.$$

We see that the digram above commutes by the definition of L_4.
Is L_4 well-defined? Let $i \in \Gamma_0$ and $v' \in \big[h_1(i)\big]\big[V(i)\big]$. Then $\exists v \in V(i)$ such that $v' = \big[h_1(i)\big](v)$, so

$$L_3(v') = \big[L_3(i)\big]\Big(\big[h_1(i)\big](v)\Big) = \big[h_3(i)\big]\Big(\big[L_2(i)\big](v)\Big) \in \big[h_2(i)\big]\big[W(i)\big].$$

Thus, since h_1, h_2, L_2 and L_3 are well-defined, it follows that L_4 is well-defined.

We prove that L_4 is a homomorphism. Let $\alpha_{j,i} \in \Gamma_1$.

$$\Big(\big[(\mathrm{cok}_{\mathrm{ob}})_1(h_2)\big]_{\alpha_{j,i}} \circ \big[L_4(i)\big]\Big)\Big(v + \big[h_1(i)\big]\big[V(i)\big]\Big)$$
$$= \big[(\mathrm{cok}_{\mathrm{ob}})_1(h_2)\big]_{\alpha_{j,i}}\Big[\big[L_4(i)\big]\Big(v + \big[h_1(i)\big]\big[V(i)\big]\Big)\Big]$$
$$= \big[(\mathrm{cok}_{\mathrm{ob}})_1(h_2)\big]_{\alpha_{j,i}}\Big(\big[L_3(i)\big](v) + \big[h_2(i)\big]\big[W'(i)\big]\Big)$$
$$= g''_{\alpha_{j,i}}\Big(\big[L_3(i)\big](v)\Big) + \big[h_2(j)\big]\big[W'(j)\big]$$
$$= \Big(g''_{\alpha_{j,i}} \circ \big[L_3(i)\big]\Big)(v) + \big[h_2(j)\big]\big[W'(j)\big]$$
$$= \Big(\big[L_3(j)\big] \circ g''_{\alpha_{j,i}}\Big)(v) + \big[h_2(j)\big]\big[W'(j)\big]$$
$$= \big[L_3(j)\big]\big[g''_{\alpha_{j,i}}(v)\big] + \big[h_2(j)\big]\big[W'(j)\big]$$
$$= \big[L_4(j)\big]\Big(g''_{\alpha_{j,i}}(v) + \big[h_2(j)\big]\big[W'(j)\big]\Big)$$
$$= \big[L_4(j)\big]\Big[\big[(\mathrm{cok}_{\mathrm{ob}})_1(h_2)\big]_{\alpha_{j,i}}\Big(v + \big[h_1(i)\big]\big[V(i)\big]\Big)\Big]$$
$$= \Big(\big[L_4(j)\big] \circ \big[(\mathrm{cok}_{\mathrm{ob}})_1(h_2)\big]_{\alpha_{j,i}}\Big)\Big(v + \big[h_1(i)\big]\big[V(i)\big]\Big).$$

Thus L_4 is a homomorphism.

□

Theorem 2.2. *Let $h_1 : (V, f) \to (V', f')$ and $h_2 : (V', f') \to (V'', f'')$ be two homomorphisms between three representations of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ over a field k, where h_1 is injective and h_2 is surjective. Then*

$$\big[(V, f), h_1\big] \cong \ker(h_2) \Leftrightarrow \big[(V'', f''), h_2\big] \cong \operatorname{cok}(h_1).$$

Proof. Suppose $\big[(V, f), h_1\big] \cong \ker(h_2)$ and let $i \in \Gamma_0$. Then $\big[h_1(i)\big]\big[V(i)\big] = V(i) = \ker\big[h_2(i)\big]$. Thus, by the fundamental theorem of homomorphisms and h_2 being surjective,

$$V''(i) \cong \frac{V'(i)}{V(i)}.$$

Let $\alpha_{j,i} \in \Gamma_1$ and $v'' \in V''(i)$. Then $\exists v' \in V'(i)$ such that $v'' = \big[h_2(i)\big](v')$. Then

$$f''_{\alpha_{j,i}}(v'') = f''_{\alpha_{j,i}}\Big(\big[h_2(i)\big](v')\Big) = \big[h_2(i)\big]\big[f'_{\alpha_{j,i}}(v')\big],$$

so if $h_2 = \operatorname{cok}_{\text{hom}}(h_1)$, then f'' satisfies the property of the collection of maps in $\ker_{ob}(h_2)$. Let $i \in \Gamma_0$ and $v' \in V'(i)$. Then

$$\big[h_2(i)\big](v') \in V''(i) \cong \frac{V'(i)}{V(i)},$$

so h_2 behaves like the cokernel homomorphism of h_1.
Now suppose $\big[(V'', f''), h_2\big] \cong \operatorname{cok}(h_1)$ and let $i \in \Gamma_0$.

$$V''(i) = \frac{V'(i)}{\big[h_1(i)\big]\big[V(i)\big]},$$

so $V(i) \cong \ker\big[h_2\big]$. Let $\alpha_{j,i} \in \Gamma_1$ and $v \in V(i)$. Since h_1 is injective, $h_1(v) = 0 \Leftrightarrow v = 0$. Then

$$\big[h_1(j)\big]\big[f_{\alpha_{j,i}}(v)\big] = f'_{\alpha_{j,i}}\Big(\big[h(i)\big](v)\Big),$$

and if $h_1 = \ker_{\text{hom}}(h_2)$, h_1 will be an inclusion, which will give us our desired result.
$V(i) \cong \ker\big[h_2\big]$, so h_1 acts like $\ker_{\text{hom}}(h_2)$.
This proves the theorem. □

Sequences like the one in the theorem above are called short exact.

Definition 2.4. Let (V, f) be a finite-dimensional representation of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ with $n \in \mathbb{N}$ vertices over a field k. The **dimension vector** of

(V, f) over k is the n-tuple

$$\dim_k(V,f) = \begin{bmatrix} \dim_k\big[V(1)\big] \\ \dim_k\big[V(2)\big] \\ \vdots \\ \dim_k\big[V(n)\big] \end{bmatrix} \in \mathbb{Z}^n.$$

△

Theorem 2.3. *Let h be a homomorphism between finite-dimensional representations (V, f) and (V', f') of a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ with $n \in \mathbb{N}$ vertices over k. Then*

1. $\dim_k\big[\ker_{\text{ob}}(h)\big] = \dim_k(V,f) - \dim_k\big[h(V,f)\big]$,
2. $\dim_k\big[\text{cok}_{\text{ob}}(h)\big] = \dim_k(V',f') - \dim_k\big[h(V,f)\big]$.

Proof. We first verify that $h(V,f) = \big(h(V), h(f)\big)$ is a finite-dimensional representation over k.
$h(V) = \{\big[h(i)\big]\big[V(i)\big]\}_{i\in\Gamma_0}$ is a collection of vector spaces. Let $\alpha_{j,i} \in \Gamma_1$ and consider the restriction map $f'_{\alpha_{j,i}}|_{[h(i)][V(i)]}$ such that

$$f'_{\alpha_{j,i}}|_{[h(i)][V(i)]}(v') = f'_{\alpha_{j,i}}(v') \;\forall v' \in \big[h(i)\big]\big[V(i)\big].$$

We want this to be a map in $h(f)$ such that $h(V,f)$ can be a representation, and on the surface it might seem like it automatically is, but we should reassure ourselves that

$$f'_{\alpha_{j,i}}|_{[h(i)][V(i)]} : \big[h(i)\big]\big[V(i)\big] \to \big[h(j)\big]\big[V(j)\big].$$

Since h is a homomorphism,

$$f'_{\alpha_{j,i}}|_{[h(i)][V(i)]}\Big(\big[h(i)\big]\big[V(i)\big]\Big) = f'_{\alpha_{j,i}}\Big(\big[h(i)\big]\big[V(i)\big]\Big) = \big[f'_{\alpha_{j,i}} \circ h(i)\big]\big[V(i)\big]$$

$$= \big[h(j) \circ f_{\alpha_{j,i}}\big]\big[V(i)\big] = \big[h(j)\big]\Big(f_{\alpha_{j,i}}\big[V(i)\big]\Big) \subseteq \big[h(j)\big]\big[V(i)\big].$$

Thus, if $f'_{\alpha_{j,i}}|_{[h(i)][V(i)]} \in h(f)$, $h(V,f) = \big(h(V), h(f)\big)$ is a representation. Next, $\big[h(i)\big]\big[V(i)\big] \subset V'(i)$ implies $\big[h(i)\big]\big[V(i)\big]$ is finite-dimensional since by assumption $V'(i)$ is finite-dimensional $\forall i \in \Gamma_0$. Thus $h(V,f)$ is finite-dimensional. Now we go on to prove the theorem.
Let $i \in \Gamma_0$.

1. We first find $\dim_k\big[\ker_{\text{ob}}(h)\big]$.

$$\dim_k\Big(\big[\ker_{\text{ob}}(h)\big](i)\Big) = \dim_k\Big(\ker\big[h(i)\big]\Big)$$

$$= \dim_k\big[V(i)\big] - \dim_k\Big(\big[h(i)\big]\big[V(i)\big]\Big) = \dim_k\big[V(i)\big] - \dim_k\Big(\big[h(V)\big](i)\Big)$$

$$\Rightarrow \dim_k\big[\ker_{\text{ob}}(h)\big] = \dim_k(V, f) - \dim_k\big[h(V, f)\big].$$

2. Now we find $\dim_k\big[\text{cok}_{\text{ob}}(h)\big]$.

$$\dim_k\Big(\big[(\text{cok}_{\text{ob}})_0(h)\big](i)\Big) = \dim_k\left(\frac{V'(i)}{\big[h(i)\big]\big[V(i)\big]}\right)$$

$$= \dim_k\big[V'(i)\big] - \dim_k\Big(\big[h(i)\big]\big[V(i)\big]\Big) = \dim_k\big[V'(i)\big] - \dim_k\Big(\big[h(V)\big](i)\Big)$$

$$\Rightarrow \dim_k\big[\text{cok}_{\text{ob}}(h)\big] = \dim_k(V', f') - \dim_k\big[h(V, f)\big].$$

This concludes the proof. □

3 A sequence of indecomposable representations

In this section, we define a sequence of representations of $\mathcal{Q}$ over some arbitrary common field k. The sequence is to be derived by means of kernels and cokernels, and when it is determined, we show that (almost) all elements of the sequence are indecomposable.

We describe a sequence of representations, which will be derived using cokernels. We consider $\mathcal{Q}$ and the representations $\mathcal{V}_{0,5}$, $\mathcal{V}_{1,1}$, $\mathcal{V}_{1,2}$, $\mathcal{V}_{1,3}$ and $\mathcal{V}_{1,4}$. Let $H_{i,5} : \mathcal{V}_{i,5} \to \mathcal{V}_{i+1,1} \oplus \mathcal{V}_{i+1,2} \oplus \mathcal{V}_{i+1,3} \oplus \mathcal{V}_{i+1,4}$ and $h_{i,j} : \mathcal{V}_{i,j} \to \mathcal{V}_{i,5}$ be homomorphisms of representations, and define the sequence $\mathcal{V}$ such that

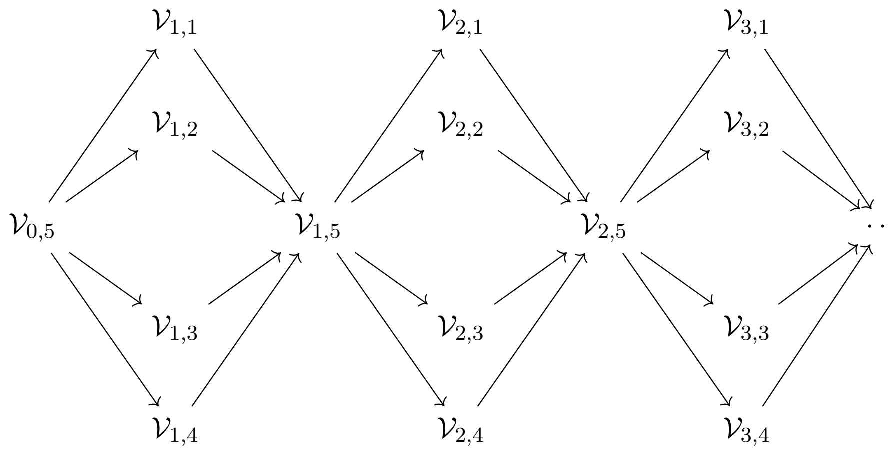
$$\mathcal{V}_{i,5} = \text{cok}_{\text{ob}}(H_{i,5})$$

and

$$\mathcal{V}_{i,j} = \text{cok}_{\text{ob}}(h_{i,j}), j \in \{1, 2, 3, 4\},$$

are the elements of $\mathcal{V}$ $\forall i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, assuming the inductor maps, i.e. the maps we induce the sequence from, are injective. Additionally, for future use we write $\mathcal{V}_{i,j} = \big[(\mathcal{V}_{i,j})_0, (\mathcal{V}_{i,j})_1\big]$ $\forall i \in \mathbb{N}, j \in \{1, 2, 3, 4, 5\}$, and also

$\mathcal{V}_{0,5} = \big[(\mathcal{V}_{0,5})_0, (\mathcal{V}_{0,5})_1\big]$.
To make things a bit clearer, we visualize the sequence.



Say we want to find out what the representations in this sequence look like. To do this we use the dimension vectors of the representations. Note that the maps $H_{m,5}$ and $h_{m,n}$ are injective $\forall m \in \mathbb{N}, n \in \{1,2,3,4\}$.

$$\dim_k(\mathcal{V}_{0,5}) = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{1,1}) = \begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix}, \dim_k(\mathcal{V}_{1,2}) = \begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{1,3}) = \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix}, \dim_k(\mathcal{V}_{1,4}) = \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}.$$

$$\Rightarrow \dim_k(\mathcal{V}_{1,5}) = \dim_k\left(\bigoplus_{j=1}^{4} \mathcal{V}_{1,j}\right) - \dim_k(\mathcal{V}_{0,5}) = \sum_{j=1}^{4} \dim_k(\mathcal{V}_{1,j}) - \dim_k(\mathcal{V}_{0,5})$$

$$= \begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix} - \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\\3\end{bmatrix}$$

$$\dim_k(\mathcal{V}_{2,1}) = \dim_k(\mathcal{V}_{1,5}) - \dim_k(\mathcal{V}_{1,1}) = \begin{bmatrix}1\\1\\1\\1\\3\end{bmatrix} - \begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{2,2}) = \dim_k(\mathcal{V}_{1,5}) - \dim_k(\mathcal{V}_{1,2}) = \begin{bmatrix}1\\1\\1\\1\\3\end{bmatrix} - \begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{2,3}) = \dim_k(\mathcal{V}_{1,5}) - \dim_k(\mathcal{V}_{1,3}) = \begin{bmatrix}1\\1\\1\\1\\3\end{bmatrix} - \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{2,4}) = \dim_k(\mathcal{V}_{1,5}) - \dim_k(\mathcal{V}_{1,4}) = \begin{bmatrix}1\\1\\1\\1\\3\end{bmatrix} - \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\0\\2\end{bmatrix}$$

Continuing the process a few more times, we obtain

$$\dim_k(\mathcal{V}_{2,5}) = \begin{bmatrix}2\\2\\2\\2\\5\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{3,1}) = \begin{bmatrix}2\\1\\1\\1\\3\end{bmatrix}, \dim_k(\mathcal{V}_{3,2}) = \begin{bmatrix}1\\2\\1\\1\\3\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{3,3}) = \begin{bmatrix}1\\1\\2\\1\\3\end{bmatrix}, \dim_k(\mathcal{V}_{3,4}) = \begin{bmatrix}1\\1\\1\\2\\3\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{3,5}) = \begin{bmatrix}3\\3\\3\\3\\7\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{4,1}) = \begin{bmatrix}1\\2\\2\\2\\4\end{bmatrix}, \dim_k(\mathcal{V}_{4,2}) = \begin{bmatrix}2\\1\\2\\2\\4\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{4,3}) = \begin{bmatrix}2\\2\\1\\2\\4\end{bmatrix}, \dim_k(\mathcal{V}_{4,4}) = \begin{bmatrix}2\\2\\2\\1\\4\end{bmatrix}.$$

Remark 3.1. Notice that

$$\begin{aligned}\dim_k(\mathcal{V}_{1,5}) - \dim_k(\mathcal{V}_{0,5}) &= \begin{bmatrix}1\\1\\1\\1\\3\end{bmatrix} - \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}\\ &= \dim_k(\mathcal{V}_{2,5}) - \dim_k(\mathcal{V}_{1,5}),\\ \dim_k(\mathcal{V}_{3,1}) - \dim_k(\mathcal{V}_{1,1}) &= \begin{bmatrix}2\\1\\1\\1\\3\end{bmatrix} - \begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}\\ &= \dim_k(\mathcal{V}_{3,2}) - \dim_k(\mathcal{V}_{1,2})\\ &= \dim_k(\mathcal{V}_{3,3}) - \dim_k(\mathcal{V}_{1,3})\\ &= \dim_k(\mathcal{V}_{3,4}) - \dim_k(\mathcal{V}_{1,4}).\end{aligned}$$

$$\dim_k(\mathcal{V}_{4,1}) - \dim_k(\mathcal{V}_{2,1}) = \begin{bmatrix}1\\2\\2\\2\\4\end{bmatrix} - \begin{bmatrix}0\\1\\1\\1\\2\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}$$

$$= \dim_k(\mathcal{V}_{4,2}) - \dim_k(\mathcal{V}_{2,2})$$

$$= \dim_k(\mathcal{V}_{4,3}) - \dim_k(\mathcal{V}_{2,3})$$

$$= \dim_k(\mathcal{V}_{4,4}) - \dim_k(\mathcal{V}_{2,4}),$$

We should then investigate if

$$\dim_k(\mathcal{V}_{i,5}) - \dim_k(\mathcal{V}_{i-1,5}) = \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} \quad \forall i \in \mathbb{N}$$

and if

$$\dim_k(\mathcal{V}_{i+2,j}) - \dim_k(\mathcal{V}_{i,j}) = \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} \quad \forall i \in \mathbb{N}, j \in \{1,2,3,4\}.$$

These statements together with the fact that we already know what the first few elements of $\mathcal{V}$ are, are equivalent to saying that

$$\dim_k(\mathcal{V}_{l,5}) = \dim_k(\mathcal{V}_{0,5}) + l \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = \begin{bmatrix}l\\l\\l\\l\\2l+1\end{bmatrix} \quad \forall l \in \mathbb{N}, \tag{1}$$

$$\dim_k(\mathcal{V}_{2m+1,1}) = \dim_k(\mathcal{V}_{1,1}) + m \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = \begin{bmatrix}m+1\\m\\m\\m\\2m+1\end{bmatrix}, \tag{2}$$

$$\dim_k(\mathcal{V}_{2m+2,1}) = \dim_k(\mathcal{V}_{2,1}) + m \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} m \\ m+1 \\ m+1 \\ m+1 \\ 2m+2 \end{bmatrix}, \tag{3}$$

$$\dim_k(\mathcal{V}_{2m+1,2}) = \dim_k(\mathcal{V}_{1,2}) + m \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} m \\ m+1 \\ m \\ m \\ 2m+1 \end{bmatrix}, \tag{4}$$

$$\dim_k(\mathcal{V}_{2m+2,2}) = \dim_k(\mathcal{V}_{2,2}) + m \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} m+1 \\ m \\ m+1 \\ m+1 \\ 2m+2 \end{bmatrix}, \tag{5}$$

$$\dim_k(\mathcal{V}_{2m+1,3}) = \dim_k(\mathcal{V}_{1,3}) + m \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} m \\ m \\ m+1 \\ m \\ 2m+1 \end{bmatrix}, \tag{6}$$

$$\dim_k(\mathcal{V}_{2m+2,3}) = \dim_k(\mathcal{V}_{2,3}) + m \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} m+1 \\ m+1 \\ m \\ m+1 \\ 2m+2 \end{bmatrix}, \tag{7}$$

$$\dim_k(\mathcal{V}_{2m+1,4}) = \dim_k(\mathcal{V}_{1,4}) + m \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} m \\ m \\ m \\ m+1 \\ 2m+1 \end{bmatrix}, \tag{8}$$

$$\dim_k(\mathcal{V}_{2m+2,4}) = \dim_k(\mathcal{V}_{2,4}) + m \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} m+1 \\ m+1 \\ m+1 \\ m \\ 2m+2 \end{bmatrix}, \tag{9}$$

$$\forall m \in \mathbb{N}_0.$$

△

Proof. Suppose $l = 2m+1$ and that these statements hold for all $n \in \mathbb{N}_0$ such that $n \leq l$. Then

$$\begin{aligned}
&\dim_k(\mathcal{V}_{2m+2,5}) \\
&= \dim_k\big[\mathrm{cok}_{\mathrm{ob}}(\mathcal{V}_{2m+1,5} \hookrightarrow \mathcal{V}_{2m+2,1} \oplus \mathcal{V}_{2m+2,2} \oplus \mathcal{V}_{2m+2,3} \oplus \mathcal{V}_{2m+2,4})\big] \\
&= \dim_k(\mathcal{V}_{2m+2,1} \oplus \mathcal{V}_{2m+2,2} \oplus \mathcal{V}_{2m+2,3} \oplus \mathcal{V}_{2m+2,4}) - \dim_k(\mathcal{V}_{2m+1,5}) \\
&= \dim_k(\mathcal{V}_{2m+2,1}) + \dim_k(\mathcal{V}_{2m+2,2}) + \dim_k(\mathcal{V}_{2m+2,3}) + \dim_k(\mathcal{V}_{2m+2,4}) \\
&\quad -\dim_k(\mathcal{V}_{2m+1,5}) \\
&= \begin{bmatrix} m \\ m+1 \\ m+1 \\ m+1 \\ 2m+2 \end{bmatrix} + \begin{bmatrix} m+1 \\ m \\ m+1 \\ m+1 \\ 2m+2 \end{bmatrix} + \begin{bmatrix} m+1 \\ m+1 \\ m \\ m+1 \\ 2m+2 \end{bmatrix} + \begin{bmatrix} m+1 \\ m+1 \\ m+1 \\ m \\ 2m+2 \end{bmatrix} - \begin{bmatrix} 2m+1 \\ 2m+1 \\ 2m+1 \\ 2m+1 \\ 4m+3 \end{bmatrix} \\
&= \begin{bmatrix} 2m+2 \\ 2m+2 \\ 2m+2 \\ 2m+2 \\ 2(2m+2)+1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \dim_k(\mathcal{V}_{2m+3,1}) &= \dim_k\big[\mathrm{cok}_{\mathrm{ob}}(\mathcal{V}_{2m+2,1} \hookrightarrow \mathcal{V}_{2m+2,5})\big] \\
&= \dim_k(\mathcal{V}_{2m+2,5}) - \dim_k(\mathcal{V}_{2m+2,1}) \\
&= \begin{bmatrix} 2m+2 \\ 2m+2 \\ 2m+2 \\ 2m+2 \\ 2(2m+2)+1 \end{bmatrix} - \begin{bmatrix} m \\ m+1 \\ m+1 \\ m+1 \\ 2m+2 \end{bmatrix} = \begin{bmatrix} m+2 \\ m+1 \\ m+1 \\ m+1 \\ 2m+2+1 \end{bmatrix},
\end{aligned}$$

and we can in a similar way show that

$$\dim_k(\mathcal{V}_{2m+3,2}) = \begin{bmatrix} m+1 \\ m+2 \\ m+1 \\ m+1 \\ 2m+2+1 \end{bmatrix}, \dim_k(\mathcal{V}_{2m+3,3}) = \begin{bmatrix} m+1 \\ m+1 \\ m+2 \\ m+1 \\ 2m+2+1 \end{bmatrix},$$

$$\dim_k(\mathcal{V}_{2m+3,4}) = \begin{bmatrix} m+1 \\ m+1 \\ m+1 \\ m+2 \\ 2m+2+1 \end{bmatrix}.$$

This again implies that

$$\dim_k(\mathcal{V}_{2m+3,5}) = \sum_{j=1}^{4} \dim_k(\mathcal{V}_{2m+3,j}) - \dim_k(\mathcal{V}_{2m+2,5})$$

$$= \begin{bmatrix} m+2 \\ m+1 \\ m+1 \\ m+1 \\ 2m+3 \end{bmatrix} + \begin{bmatrix} m+1 \\ m+2 \\ m+1 \\ m+1 \\ 2m+3 \end{bmatrix} + \begin{bmatrix} m+1 \\ m+1 \\ m+2 \\ m+1 \\ 2m+3 \end{bmatrix} + \begin{bmatrix} m+1 \\ m+1 \\ m+1 \\ m+2 \\ 2m+3 \end{bmatrix}$$

$$- \begin{bmatrix} 2m+2 \\ 2m+2 \\ 2m+2 \\ 2m+2 \\ 4m+5 \end{bmatrix} = \begin{bmatrix} 2m+3 \\ 2m+3 \\ 2m+3 \\ 2m+3 \\ 2(2m+3)+1 \end{bmatrix}$$

$$\Rightarrow \dim_k(\mathcal{V}_{2m+4,1}) = \dim_k(\mathcal{V}_{2m+3,5}) - \dim_k(\mathcal{V}_{2m+3,1})$$

$$= \begin{bmatrix} 2m+3 \\ 2m+3 \\ 2m+3 \\ 2m+3 \\ 4m+7 \end{bmatrix} - \begin{bmatrix} m+2 \\ m+1 \\ m+1 \\ m+1 \\ 2m+2+1 \end{bmatrix} = \begin{bmatrix} m+1 \\ m+2 \\ m+2 \\ m+2 \\ 2m+2+2 \end{bmatrix},$$

and we can in a similar way show that

$$\dim_k(\mathcal{V}_{2m+4,2}) = \begin{bmatrix} m+2 \\ m+1 \\ m+2 \\ m+2 \\ 2m+2+2 \end{bmatrix}, \dim_k(\mathcal{V}_{2m+4,3}) = \begin{bmatrix} m+2 \\ m+2 \\ m+1 \\ m+2 \\ 2m+2+2 \end{bmatrix},$$

$$\dim_k(\mathcal{V}_{2m+4,4}) = \begin{bmatrix} m+2 \\ m+2 \\ m+2 \\ m+1 \\ 2m+2+2 \end{bmatrix}.$$

Thus the equations (1), ..., (9) we wanted to show, hold by strong induction on m, since we already have shown all the necessary base cases. $\square$

Hence, we now have an explicit form for all the elements of $\mathcal{V}$.

After some meandering, we will finally show that the elements of $\mathcal{V}$ are indecomposable. But first, some notation. When a map is just multiplication of some constant scalar a in some field k, we denote the map by a. For example, we denote inclusions by 1.

Remark 3.2. To show indecomposability, first specify how we construct $\mathcal{V}$ a little more carefully. In particular, we define the injective inductor maps. We first define the inclusion maps $H_{0,j} : \mathcal{V}_{0,5} \to \mathcal{V}_{1,j}$ $\forall j \in \{1,2,3,4\}$ such that $\big[H_{0,j}(i)\big](v) = v$ $\forall v \in (\mathcal{V}_{0,5})_0(i)$ $\forall i \in \mathcal{Q}_0$. Consider $h_2 \in \operatorname{End}(\mathcal{V}_{1,j})$, $j \in \{1,2,3,4\}$. We have shown that $\operatorname{End}(\mathcal{V}_{1,j}) \cong k$, so $h_2 = (h_2)_a$, which is scalar multiplication of with $a \in k$. Since $\operatorname{End}(\mathcal{V}_{0,5}) \cong k$, there is an endomorphism $h_1 = (h_1)_a$ on $\mathcal{V}_{0,5}$ that is also scalar multiplication with a. Then

$$\big[h_{0,j} \circ (h_1)_a\big](v) = av = \big[(h_2)_a \circ h_{0,j}\big](v).$$

Now we conjoin the inclusion maps $H_{0,j}$ from above into $H_{0,5} : \mathcal{V}_{0,5} \to \mathcal{V}_{1,1} \oplus \mathcal{V}_{1,2} \oplus \mathcal{V}_{1,3} \oplus \mathcal{V}_{1,4}$ such that $\big[H_{0,5}(i)\big](v) = (v,v,v,v)$ $\forall v \in (\mathcal{V}_{0,5})_0(i)$ $\forall i \in \mathcal{Q}_0$. From what we said above this, the endomorphisms on $\mathcal{V}_{1,1} \oplus \mathcal{V}_{1,2} \oplus \mathcal{V}_{1,3} \oplus \mathcal{V}_{1,4}$ are on the form of

$$h = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix},$$

where $a_1, a_2, a_3, a_4 \in k$. We want a scalar multiplication endomorphism a on $\mathcal{V}_{0,5}$) such that $a \in k$.

$$(h \circ H_{0,5})(v) = (H_{0,5} \circ h_a)(v) \ \forall v \in (\mathcal{V}_{0,5})_0(i) \ \forall i \in \mathcal{Q}_0.$$

$$(h \circ H_{0,5})(v) = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix} \begin{bmatrix} v \\ v \\ v \\ v \end{bmatrix} = \begin{bmatrix} a_1 v \\ a_2 v \\ a_3 v \\ a_4 v \end{bmatrix},$$

$$(H_{0,5} \circ a)(v) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} av = \begin{bmatrix} av \\ av \\ av \\ av \end{bmatrix}$$

$$\Rightarrow a = a_1 = a_2 = a_3 = a_4.$$

We now define the rest of the inductor maps.
Let $h_{m,5} : \mathcal{V}_{m,1} \oplus \mathcal{V}_{m,2} \oplus \mathcal{V}_{m,3} \oplus \mathcal{V}_{m,4} \to \mathcal{V}_{m,5}$ such that

$$h_{m,5} = \begin{bmatrix} h_{m,1} & h_{m,2} & h_{m,3} & h_{m,4} \end{bmatrix} = \text{cok}_{\text{hom}}(H_{m-1,5}),$$

$h_{m,n} : \mathcal{V}_{m,n} \to \mathcal{V}_{m,5}$ $\forall n \in \{1, 2, 3, 4\}$,
where, unless $m = 1$, $H_{m-1,5} : \mathcal{V}_{m-1,5} \to \mathcal{V}_{m,1} \oplus \mathcal{V}_{m,2} \oplus \mathcal{V}_{m,3} \oplus \mathcal{V}_{m,4}$ where

$$H_{m-1,5} = \begin{bmatrix} H_{m-1,1} \\ H_{m-1,2} \\ H_{m-1,3} \\ H_{m-1,4} \end{bmatrix},$$

in which $H_{m-1,n} : \mathcal{V}_{m-1,5} \to \mathcal{V}_{m,n}$ is a map satisfying

$$H_{m-1,n} = \text{cok}_{\text{hom}}(h_{m-1,n}) \ \forall n \in \{1, 2, 3, 4\} \ \forall m \in \mathbb{N}.$$

We now claim that all representations in $\mathcal{V}$ are indecomposable. $\triangle$

Proof. Let $m \in \mathbb{N}$ and suppose the endomorphism rings of $\mathcal{V}_{m-1,5}$, $\mathcal{V}_{m,1}$, $\mathcal{V}_{m,2}$, $\mathcal{V}_{m,3}$ and $\mathcal{V}_{m,4}$ are isomorphic to k. Let $h \in \text{End}(\mathcal{V}_{m,5})$. Then, since $\mathcal{V}_{m,5}) = \text{cok}_{\text{ob}}(H_{m-1,5})$, $\exists h_1 \in \text{End}(\mathcal{V}_{m,1} \oplus \mathcal{V}_{m,2} \oplus \mathcal{V}_{m,3} \oplus \mathcal{V}_{m,4})$ such that $h_{m,5} \circ h_1 = h \circ h_{m,5}$. $\text{End}(\mathcal{V}_{m,1} \oplus \mathcal{V}_{m,2} \oplus \mathcal{V}_{m,3} \oplus \mathcal{V}_{m,4}) \cong k^4$ implies

$$h_1 = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix}, a_1, a_2, a_3, a_4 \in k.$$

If $m = 1$, then $\exists a \in k$ such that

$$h_1 = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}.$$

Then

$$h_{m,5} \circ h_1 = h_{m,5} \circ aI_4 = ah_{m,5} = h \circ h_{m,5},$$

so $h = a$ since $h_{m,5}$ is a cokernel mapping $\Rightarrow \text{End}(\mathcal{V}_{m,5}) \cong k$.
If $m \geq 2$, then $\mathcal{V}_{m,n} = \text{cok}_{\text{ob}}(h_{m-1,n})$ $n \in \{1, 2, 3, 4\}$. Then for any $h_2 \in \text{End}\left(\bigoplus_{n=1}^{4} \mathcal{V}_{m,n}\right)$, $\exists a \in \text{End}(\mathcal{V}_{m-1,5})$ such that $H_{m-1,n} \circ a = h_2 \circ H_{m-1,n}$ $\Rightarrow h_2 = a$ $\forall n \in \{1, 2, 3, 4\}$. Then $h_1 = aI_4$.

In either case, $h = a$, implying that $\mathrm{End}(\mathcal{V}_{m,5}) \cong k$.
Now suppose the endomorphism ring of $\mathcal{V}_{m,5}$ is isomorphic to k. Let $h \in \mathrm{End}(\mathcal{V}_{m+1,n}) \forall n \in \{1,2,3,4\}$. Since $\mathcal{V}_{m+1,n} = \mathrm{cok}_{\mathrm{ob}}(h_{m,n})$, $\exists h_1 \in \mathrm{End}(\mathcal{V}_{m,n})$ such that $H_{m,n} \circ h_1 = h \circ H_{m,n}$. $\mathrm{End}(\mathcal{V}_{m,5}) \cong k$, so $h_1 = a$ for some $a \in k$. Then $H_{m,n} \circ a = aH_{m,n} = h \circ H_{m,n} \Rightarrow h = a$.
Thus, by induction, all elements of $\mathcal{V}$ are indecomposable. □

Moving forward, we find a $\mathbb{Z}$-linear map related to $\mathcal{V}$ as motivation for finding more indecomposables.

Remark 3.3. The particular map we want, is $L : \mathbb{Z}^5 \to \mathbb{Z}^5$ such that

$$L\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\\3\end{bmatrix}, L\begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\\1\\2\end{bmatrix}, L\begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\\1\\2\end{bmatrix}, L\begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\\1\\2\end{bmatrix}, L\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\0\\2\end{bmatrix}.$$

Then

$$L\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} = L\left(\begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix} - \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\1\\1\\2\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\3\end{bmatrix} = \begin{bmatrix}-1\\0\\0\\0\\-1\end{bmatrix},$$

$$L\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} = L\left(\begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} - \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\1\\1\\2\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\3\end{bmatrix} = \begin{bmatrix}0\\-1\\0\\0\\-1\end{bmatrix},$$

$$L\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} = L\left(\begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} - \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\\0\\1\\2\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\3\end{bmatrix} = \begin{bmatrix}0\\0\\-1\\0\\-1\end{bmatrix},$$

$$L\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix} = L\left(\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix} - \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\\1\\0\\2\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\-1\\-1\end{bmatrix}.$$

Thus

$$L = L \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & -1 & 3 \end{bmatrix}.$$

△

Remark 3.4. Now notice that

$$L \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow L^n \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \forall n \in \mathbb{N}.$$

We have that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = L \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

meaning

$$L^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = L \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow L^n \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + n \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \dim_k(\mathrm{V}_{n,5}) \ \forall n \in \mathbb{N}.$$

We also have that

$$L^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = L \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

and similarly

$$L^2 \begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2\\1\\1\\3\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$L^2 \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\2\\1\\3\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$L^2 \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\2\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

in addition to

$$L^2 \begin{bmatrix}0\\1\\1\\1\\2\end{bmatrix} = L \begin{bmatrix}2\\1\\1\\1\\3\end{bmatrix} = \begin{bmatrix}1\\2\\2\\2\\4\end{bmatrix} = \begin{bmatrix}0\\1\\1\\1\\2\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

and

$$L^2 \begin{bmatrix}1\\0\\1\\1\\2\end{bmatrix} = \begin{bmatrix}2\\1\\2\\2\\4\end{bmatrix} = \begin{bmatrix}1\\0\\1\\1\\2\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$L^2 \begin{bmatrix}1\\1\\0\\1\\2\end{bmatrix} = \begin{bmatrix}2\\2\\1\\2\\4\end{bmatrix} = \begin{bmatrix}1\\1\\0\\1\\2\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$L^2 \begin{bmatrix}1\\1\\1\\0\\2\end{bmatrix} = \begin{bmatrix}2\\2\\2\\1\\4\end{bmatrix} = \begin{bmatrix}1\\1\\1\\0\\2\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}.$$

These equations then yield the following equations.

$$L^{2m}\begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix} + m\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = \dim_k(\mathcal{V}_{2m+1,1}),$$

$$L^{2m+1}\begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\\1\\2\end{bmatrix} + m\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = \dim_k(\mathcal{V}_{2m+2,1}),$$

$$L^{2m}\begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} + m\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = \dim_k(\mathcal{V}_{2m+1,2}),$$

$$L^{2m+1}\begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\\1\\2\end{bmatrix} + m\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = \dim_k(\mathcal{V}_{2m+2,2}),$$

$$L^{2m}\begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} + m\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = \dim_k(\mathcal{V}_{2m+1,3}),$$

$$L^{2m+1}\begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\\1\\2\end{bmatrix} + m\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = \dim_k(\mathcal{V}_{2m+2,3}),$$

$$L^{2m}\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix} + m\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = \dim_k(\mathcal{V}_{2m+1,4}),$$

$$L^{2m+1}\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\0\\2\end{bmatrix} + m\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = \dim_k(\mathcal{V}_{2m+2,4}),$$

$$\forall m \in \mathbb{N}_0.$$

Thus we see that there is a strong relationship between L and $\mathcal{V}$ as the dimension vector of every element in $\mathcal{V}$ can be determined by applying L to

$$\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} \text{ or } \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}$$

some number of times. This is one of the reasons why we're interested in L.
△

What we do next, is to gain some motivation for defining a new sequence of indecomposable representations. This sequence has strong ties to $\mathcal{V}$, as its elements could almost be obtained if we began with the same representations as we started with in $\mathcal{V}$ and instead took kernels instead of cokernels. Hopefully all of this will become clearer as we proceed.

Remark 3.5. We consider the matrix

$$M = \begin{bmatrix}0 & 1 & 1 & 1 & -1\\1 & 0 & 1 & 1 & -1\\1 & 1 & 0 & 1 & -1\\1 & 1 & 1 & 0 & -1\\1 & 1 & 1 & 1 & -1\end{bmatrix}$$

and see that

$$LM = \begin{bmatrix}-1 & 0 & 0 & 0 & 1\\0 & -1 & 0 & 0 & 1\\0 & 0 & -1 & 0 & 1\\0 & 0 & 0 & -1 & 1\\-1 & -1 & -1 & -1 & 3\end{bmatrix}\begin{bmatrix}0 & 1 & 1 & 1 & -1\\1 & 0 & 1 & 1 & -1\\1 & 1 & 0 & 1 & -1\\1 & 1 & 1 & 0 & -1\\1 & 1 & 1 & 1 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix}$$

$$= ML.$$

Thus M is the inverse of L. Next we get that

$$M\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = M\left(L\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}\right) = I_5\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}.$$

and

$$M\begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\0\\0\\0\end{bmatrix}, M\begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\0\\0\\0\end{bmatrix}, M\begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\-1\\0\\0\end{bmatrix}, M\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\-1\\0\end{bmatrix},$$

$$M\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\\-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}$$

$$\Rightarrow M^2\begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix} = M\begin{bmatrix}-1\\0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\-1\\-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$M^2\begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} = M\begin{bmatrix}0\\-1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\-1\\-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$M^2\begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} = M\begin{bmatrix}0\\0\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\-1\\0\\-1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$M^2\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix} = M\begin{bmatrix}0\\0\\0\\-1\\0\end{bmatrix} = \begin{bmatrix}-1\\-1\\-1\\0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$M^2 \begin{bmatrix}-1\\0\\0\\0\\0\end{bmatrix} = M \begin{bmatrix}0\\-1\\-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}-2\\-1\\-1\\-1\\-2\end{bmatrix} = \begin{bmatrix}-1\\0\\0\\0\\0\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$M^2 \begin{bmatrix}0\\-1\\0\\0\\0\end{bmatrix} = M \begin{bmatrix}-1\\0\\-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}-1\\-2\\-1\\-1\\-2\end{bmatrix} = \begin{bmatrix}0\\-1\\0\\0\\0\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$M^2 \begin{bmatrix}0\\0\\-1\\0\\0\end{bmatrix} = M \begin{bmatrix}-1\\-1\\0\\-1\\-1\end{bmatrix} = \begin{bmatrix}-1\\-1\\-2\\-1\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\-1\\0\\0\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$M^2 \begin{bmatrix}0\\0\\0\\-1\\0\end{bmatrix} = M \begin{bmatrix}-1\\-1\\-1\\0\\-1\end{bmatrix} = \begin{bmatrix}-1\\-1\\-1\\-2\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\-1\\0\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\Rightarrow$$

$$M^n \begin{bmatrix}-1\\-1\\-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}-1\\-1\\-1\\-1\\-1\end{bmatrix} - n \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} \quad \forall n \in \mathbb{N}_0$$

$$M^{2m} \begin{bmatrix}0\\-1\\-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}0\\-1\\-1\\-1\\-1\end{bmatrix} - m \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$M^{2m+1} \begin{bmatrix}0\\-1\\-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}0\\-1\\-1\\-1\\-1\end{bmatrix} - (m+1) \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$M^{2m} \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \\ -1 \end{bmatrix} - m \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

$$M^{2m+1} \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \\ -1 \end{bmatrix} - (m+1) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

$$M^{2m} \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ -1 \end{bmatrix} - m \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

$$M^{2m+1} \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ -1 \end{bmatrix} - (m+1) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

$$M^{2m} \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \\ -1 \end{bmatrix} - m \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

$$M^{2m+1} \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \\ -1 \end{bmatrix} - (m+1) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

$$\forall m \in \mathbb{N}_0.$$

△

We will later see that the equations above will be the negative of the dimension vectors of the elements of a sequence. We will now define that sequence.

Let

$$\mathcal{V}_{-1,5} = (\{k,k,k,k,k\}, f_{-1,5}),$$
$$\mathcal{V}_{-1,1} = (\{0,k,k,k,k\}, f_{-1,1}),$$
$$\mathcal{V}_{-1,2} = (\{k,0,k,k,k\}, f_{-1,2}),$$
$$\mathcal{V}_{-1,3} = (\{k,k,0,k,k\}, f_{-1,3})$$

and

$$\mathcal{V}_{-1,4} = (\{k,k,k,0,k\}, f_{-1,4})$$

be subspace representations of $\mathcal{Q}$. Define $h_{-1,n} : \mathcal{V}_{-1,n} \to \mathcal{V}_{-1,5}$ such that $h_{-1,n}$ is the inclusion map, i.e. $h_{-1,n} = 1 \ \forall n \in \{1,2,3,4\}$.
Let $h_{-1,5} : \mathcal{V}_{-1,1} \oplus \mathcal{V}_{-1,2} \oplus \mathcal{V}_{-1,3} \oplus \mathcal{V}_{-1,4} \to \mathcal{V}_{-1,5}$ be the map such that

$$h_{-1,5} = \begin{bmatrix} h_{-1,1} & h_{-1,2} & h_{-1,3} & h_{-1,4} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}.$$

Define $H_{-m,5} : \mathcal{V}_{-m,5} \to \mathcal{V}_{-m+1,1} \oplus \mathcal{V}_{-m+1,2} \oplus \mathcal{V}_{-m+1,3} \oplus \mathcal{V}_{-m+1,4}$ such that

$$H_{-m,5} = \begin{bmatrix} H_{-m,1} \\ H_{-m,2} \\ H_{-m,3} \\ H_{-m,4} \end{bmatrix} = \ker_{\text{hom}}(h_{-m+1,5}),$$

where, unless $m = 2$, $h_{-m+1,5} : \mathcal{V}_{-m+1,1} \oplus \mathcal{V}_{-m+1,2} \oplus \mathcal{V}_{-m+1,3} \oplus \mathcal{V}_{-m+1,4} \to \mathcal{V}_{-m+1,5}$ is a map such that

$$h_{-m+1,5} = \begin{bmatrix} h_{-m+1,1} & h_{-m+1,2} & h_{-m+1,3} & h_{-m+1,4} \end{bmatrix},$$

in which $h_{-m+1,n} : \mathcal{V}_{-m+1,n} \to \mathcal{V}_{-m+1,5}$ is a map satisfying

$$h_{-m+1,n} = \ker_{\text{hom}}(H_{-m+1,n}) \ \forall n \in \{1,2,3,4\},$$
$$\forall m \in \mathbb{N} \setminus \{1\}.$$

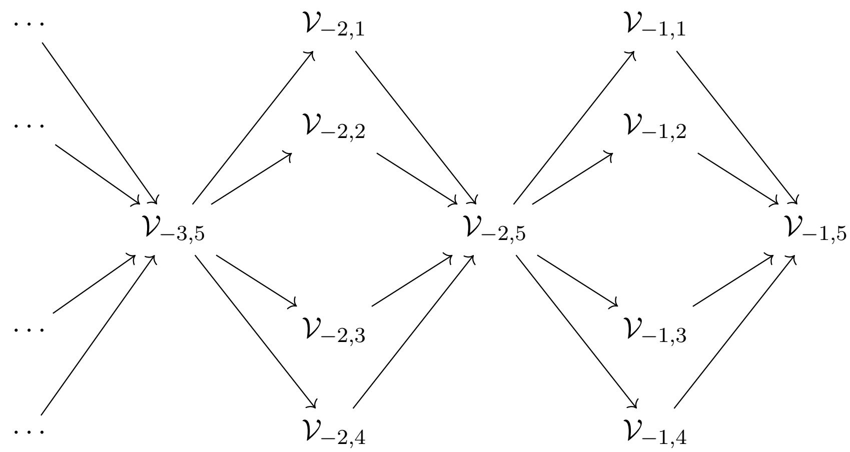
For this to make sense, we also define

$$\mathcal{V}_{-m,5} = \ker_{\text{ob}}(h_{-m+1,5})$$

and

$$\mathrm{V}_{-m,n} = \ker_{\text{ob}}(H_{-m,n}) \ \forall n \in \{1,2,3,4\},$$
$$\forall m \in \mathbb{N} \setminus \{1\}.$$

We can visualize the sequence as



Remark 3.6. $\mathrm{V}_{-m,n}$ is indecomposable $\forall m \in \mathbb{N}$ $\forall n \in \{1, 2, 3, 4, 5\}$. $\triangle$

Proof. We prove this by induction on m, and begin with showing that the endomorphism rings of $\mathcal{V}_{-1,1}$, $\mathcal{V}_{-1,2}$, $\mathcal{V}_{-1,3}$, $\mathcal{V}_{-1,4}$ and $\mathcal{V}_{-1,5}$ are isomorphic to k.
Let $h \in \mathrm{End}(\mathcal{V}_{-1,5})$ such that $h = \{a_1, a_2, a_3, a_4, a_5\}, a_1, a_2, a_3, a_4, a_5 \in k$. Let $f_{-1,5} = \{b_1, b_2, b_3, b_4\}, b_1, b_2, b_3, b_4 \in k$. The assumptions above are possible since the domain and co-domain of each map is k. Then, since h is a homomorphism, we obtain

$$a_5 b_1 = b_1 a_1 \Rightarrow b_1 a_5 = b_1 a_1 \Rightarrow a_5 = a_1,$$

$$a_5 b_2 = b_2 a_2 \Rightarrow b_2 a_5 = b_2 a_2 \Rightarrow a_5 = a_2,$$

$$a_5 b_3 = b_3 a_3 \Rightarrow b_3 a_5 = b_3 a_3 \Rightarrow a_5 = a_3,$$

$$a_5 b_4 = b_4 a_4 \Rightarrow b_4 a_5 = b_4 a_4 \Rightarrow a_5 = a_4,$$

since k a commutative division ring. Thus $h = a_5$, so $\mathrm{End}(\mathcal{V}_{-1,5}) \cong k$.
Let $h \in \mathrm{End}(\mathcal{V}_{-1,1})$ such that $h = \{0, a_2, a_3, a_4, a_5\}, a_2, a_3, a_4, a_5 \in k$. Let $f_{-1,1} = \{0, b_2, b_3, b_4\}, b_2, b_3, b_4 \in k$. The assumptions above are possible since the domain and co-domain of each map is k. Then, since h is a homomorphism, we obtain

$$a_5 0 = 0 \cdot 0 \Rightarrow 0 = 0,$$

$$a_5 b_2 = b_2 a_2 \Rightarrow b_2 a_5 = b_2 a_2 \Rightarrow a_5 = a_2,$$

$$a_5 b_3 = b_3 a_3 \Rightarrow b_3 a_5 = b_3 a_3 \Rightarrow a_5 = a_3,$$

$$a_5 b_4 = b_4 a_4 \Rightarrow b_4 a_5 = b_4 a_4 \Rightarrow a_5 = a_4.$$

Since $a_5 \cdot 0 = 0$, we then obtain that $h = a_5$, so $\mathrm{End}(\mathcal{V}_{-1,1}) \cong k$.
The proofs for $\mathcal{V}_{-1,2}$, $\mathcal{V}_{-1,3}$ and $\mathcal{V}_{-1,4}$ are similar to that of $\mathcal{V}_{-1,1}$, so we omit them.
Hence

$$\mathrm{End}(\mathcal{V}_{-1,2}) \cong \mathrm{End}(\mathcal{V}_{-1,3}) \cong \mathrm{End}(\mathcal{V}_{-1,4}) \cong k.$$

Now suppose $m \in \mathbb{N}$ and

$$\mathrm{End}(\mathcal{V}_{-m,1}) \cong \mathrm{End}(\mathcal{V}_{-m,2}) \cong \mathrm{End}(\mathcal{V}_{-m,3}) \cong \mathrm{End}(\mathcal{V}_{-m,4}) \cong \mathcal{V}_{-m,5}) \cong k.$$

Let $h \in \mathrm{End}(\mathcal{V}_{-m-1,5})$. Then, since $\mathcal{V}_{-m-1,5} = \ker_{\mathrm{ob}}(h_{-m,5})$,
$\exists h_1 \in \mathrm{End}(\mathcal{V}_{-m,1} \oplus \mathcal{V}_{-m,2} \oplus \mathcal{V}_{-m,3} \oplus \mathcal{V}_{-m,4})$ such that
$H_{-m-1,5} \circ h_1 = h \circ H_{-m-1,5}$. Since

$$\mathrm{End}(\mathcal{V}_{-m,1}) \cong \mathrm{End}(\mathcal{V}_{-m,2}) \cong \mathrm{End}(\mathcal{V}_{-m,3}) \cong \mathrm{End}(\mathcal{V}_{-m,4}) \cong k,$$

$$h_1 = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix}, a_1, a_2, a_3, a_4 \in k.$$

$h_{-m,5} = \mathrm{cok}_{\mathrm{hom}}(H_{-m,5})$, so $\exists a \in \mathrm{End}(\mathcal{V}_{-m,5})$ such that
$h_{-m,5} \circ h_1 = a \circ h_{-m,5}$

$$\Rightarrow \begin{bmatrix} h_{-m,1} \circ a_1 & h_{-m,2} \circ a_2 & h_{-m,3} \circ a_3 & h_{-m,4} \circ a_4 \end{bmatrix} =$$

$$\begin{bmatrix} a \circ h_{-m,1} & a \circ h_{-m,2} & a \circ h_{-m,3} & a \circ h_{-m,4} \end{bmatrix}$$

$$\Rightarrow$$

$$h_{-m,1} \circ a_1 = a \circ h_{-m,1} \Rightarrow a_1 = a,$$

$$h_{-m,2} \circ a_2 = a \circ h_{-m,2} \Rightarrow a_2 = a,$$

$$h_{-m,3} \circ a_3 = a \circ h_{-m,3} \Rightarrow a_3 = a,$$

$$h_{-m,4} \circ a_4 = a \circ h_{-m,4} \Rightarrow a_4 = a,$$

since $h_{-m,n}$ is a kernel homomorphism, i.e. an inclusion, $\forall n \in \{1, 2, 3, 4\}$. Thus $h_1 = a$, so $H_{-m-1,5} \circ a = h \circ H_{-m-1,5}$, and since $H_{-m,5}$ is a kernel homomorphism, we obtain $h = a$. Hence $\mathrm{End}(\mathcal{V}_{-m-1,5}) \cong k$.
Now let $h_2 \in \mathrm{End}(\mathcal{V}_{-m-1,n})$ for some $n \in \{1, 2, 3, 4\}$. Then, since $h_{-m-1,n}$ is a kernel homomorphism, $\exists a \in \mathrm{End}(\mathcal{V}_{-m-1,5}), a \in k$ such that $h_{-m-1,n} \circ a =$

$h_2 \circ h_{-m-1,n}$. We then get $a = h_2$, hence $\mathrm{End}(\mathcal{V}_{-m-1,n}) \cong k$.
Thus, by induction,

$$\mathrm{End}(\mathcal{V}_{-m,1}) \cong \mathrm{End}(\mathcal{V}_{-m,2}) \cong \mathrm{End}(\mathcal{V}_{-m,3}) \cong \mathrm{End}(\mathcal{V}_{-m,4}) \cong \mathcal{V}_{-m,5}) \cong k$$

$\forall m \in \mathbb{N}$, hence $\mathcal{V}_{-m,1}$, $\mathcal{V}_{-m,2}$, $\mathcal{V}_{-m,3}$, $\mathcal{V}_{-m,4}$, $\mathcal{V}_{-m,5}$ are indecomposable $\forall m \in \mathbb{N}$. □

Just as we did with the other sequence, we find explicit form of the dimensions vectors of all the representations.

Remark 3.7. We state some forms we suspect the representations to have.

$$\dim_k(\mathcal{V}_{-n-1,5}) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + n \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \forall n \in \mathbb{N}_0,$$

$$\dim_k(\mathcal{V}_{-2m-1,1}) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + m \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-2m-2,1}) = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + m \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-2m-1,2}) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + m \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-2m-2,2}) = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} + m \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-2m-1,3}) = \begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix} + m\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-2m-1,3}) = \begin{bmatrix}1\\1\\2\\1\\2\end{bmatrix} + m\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-2m-1,4}) = \begin{bmatrix}1\\1\\1\\0\\1\end{bmatrix} + m\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-2m-1,4}) = \begin{bmatrix}1\\1\\1\\2\\2\end{bmatrix} + m\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\forall m \in \mathbb{N}_0.$$

$\triangle$

Proof. We prove the assertions by induction on m.
For starters, we calculate enough representations to build a base case. When calculating, we use the fact that $h_{-m,5}$, $H_{-m,1}$, $H_{-m,2}$, $H_{-m,3}$ and $H_{-m,4}$ are inclusions.

$$\dim_k(\mathcal{V}_{-2,5}) = \sum_{i=1}^{4} \dim_k(\mathcal{V}_{-1,i}) - \dim_k(\mathcal{V}_{-1,5})$$

$$= \begin{bmatrix}0\\1\\1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\0\\1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\0\\1\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\\2\\3\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-2,1}) = \dim_k(\mathcal{V}_{-2,5}) - \dim_k(\mathcal{V}_{-1,1})$$

$$= \begin{bmatrix}2\\2\\2\\2\\3\end{bmatrix} - \begin{bmatrix}0\\1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\1\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-2,2}) = \dim_k(\mathcal{V}_{-2,5}) - \dim_k(\mathcal{V}_{-1,2})$$

$$= \begin{bmatrix}2\\2\\2\\2\\3\end{bmatrix} - \begin{bmatrix}1\\0\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\2\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-2,3}) = \dim_k(\mathcal{V}_{-2,5}) - \dim_k(\mathcal{V}_{-1,3})$$

$$= \begin{bmatrix}2\\2\\2\\2\\3\end{bmatrix} - \begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\2\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-2,4}) = \dim_k(\mathcal{V}_{-2,5}) - \dim_k(\mathcal{V}_{-1,4})$$

$$= \begin{bmatrix}2\\2\\2\\2\\3\end{bmatrix} - \begin{bmatrix}1\\1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\2\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-3,5}) = \begin{bmatrix}2\\1\\1\\1\\2\end{bmatrix} + \begin{bmatrix}1\\2\\1\\1\\2\end{bmatrix} + \begin{bmatrix}1\\1\\2\\1\\2\end{bmatrix} + \begin{bmatrix}1\\1\\1\\2\\2\end{bmatrix} - \begin{bmatrix}2\\2\\2\\2\\3\end{bmatrix} = \begin{bmatrix}3\\3\\3\\3\\5\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} + 2\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-3,1}) = \begin{bmatrix}3\\3\\3\\3\\5\end{bmatrix} - \begin{bmatrix}2\\1\\1\\1\\2\end{bmatrix} = \begin{bmatrix}1\\2\\2\\2\\3\end{bmatrix} = \begin{bmatrix}0\\1\\1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-3,2}) = \begin{bmatrix}3\\3\\3\\3\\5\end{bmatrix} - \begin{bmatrix}1\\2\\1\\1\\2\end{bmatrix} = \begin{bmatrix}2\\1\\2\\2\\3\end{bmatrix} = \begin{bmatrix}1\\0\\1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-3,3}) = \begin{bmatrix}3\\3\\3\\3\\5\end{bmatrix} - \begin{bmatrix}1\\1\\2\\1\\2\end{bmatrix} = \begin{bmatrix}2\\2\\1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-3,4}) = \begin{bmatrix}3\\3\\3\\3\\5\end{bmatrix} - \begin{bmatrix}1\\1\\1\\2\\2\end{bmatrix} = \begin{bmatrix}2\\2\\2\\1\\3\end{bmatrix} = \begin{bmatrix}1\\1\\1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\begin{aligned}\dim_k(\mathcal{V}_{-3,5}) &= \begin{bmatrix}1\\2\\2\\2\\3\end{bmatrix} + \begin{bmatrix}2\\1\\2\\2\\3\end{bmatrix} + \begin{bmatrix}2\\2\\1\\2\\3\end{bmatrix} + \begin{bmatrix}2\\2\\2\\1\\3\end{bmatrix} - \begin{bmatrix}3\\3\\3\\3\\5\end{bmatrix} = \begin{bmatrix}4\\4\\4\\4\\7\end{bmatrix}\\ &= \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} + 3\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},\end{aligned}$$

$$\dim_k(\mathcal{V}_{-4,1}) = \begin{bmatrix}4\\4\\4\\4\\7\end{bmatrix} - \begin{bmatrix}1\\2\\2\\2\\3\end{bmatrix} = \begin{bmatrix}3\\2\\2\\2\\4\end{bmatrix} = \begin{bmatrix}2\\1\\1\\1\\2\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-4,2}) = \begin{bmatrix}4\\4\\4\\4\\7\end{bmatrix} - \begin{bmatrix}2\\1\\2\\2\\3\end{bmatrix} = \begin{bmatrix}2\\3\\2\\2\\4\end{bmatrix} = \begin{bmatrix}1\\2\\1\\1\\2\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-4,3}) = \begin{bmatrix}4\\4\\4\\4\\7\end{bmatrix} - \begin{bmatrix}2\\2\\1\\2\\3\end{bmatrix} = \begin{bmatrix}2\\2\\3\\2\\4\end{bmatrix} = \begin{bmatrix}1\\1\\2\\1\\2\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-4,4}) = \begin{bmatrix}4\\4\\4\\4\\7\end{bmatrix} - \begin{bmatrix}2\\2\\2\\1\\3\end{bmatrix} = \begin{bmatrix}2\\2\\2\\3\\4\end{bmatrix} = \begin{bmatrix}1\\1\\1\\2\\2\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}.$$

Now suppose the suspected forms are true for all natural numbers lesser than or equal to $n = 2m + 1$. Then

$$\dim_k(\mathcal{V}_{-2m-2,5}) = \sum_{i=1}^{4} \dim_k(\mathcal{V}_{-2m-1,i}) - \dim_k(\mathcal{V}_{-2m-1,5})$$

$$= \begin{bmatrix}0\\1\\1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\0\\1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\\0\\1\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} + (4m - 2m)\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}$$

$$= \begin{bmatrix}2\\2\\2\\2\\3\end{bmatrix} + 2m\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} + (2m+1)\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}$$

$$\Rightarrow$$

$$\dim_k(\mathcal{V}_{-2m-2,1}) = \dim_k(\mathcal{V}_{-2m-2,5}) - \dim_k(\mathcal{V}_{-2m-1,1})$$

$$= \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} + (2m+1)\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} - \begin{bmatrix}0\\1\\1\\1\\1\end{bmatrix} - m\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} + (2m+1)\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}$$

$$= \begin{bmatrix}2\\1\\1\\1\\2\end{bmatrix} + m \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

and we can in a similar way show that $\dim_k(\mathcal{V}_{-2m-2,2}) = \begin{bmatrix}1\\2\\1\\1\\2\end{bmatrix} + m \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}$, $\dim_k(\mathcal{V}_{-2m-2,3}) = \begin{bmatrix}1\\1\\2\\1\\2\end{bmatrix} + m \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}$ and $\dim_k(\mathcal{V}_{-2m-2,4}) = \begin{bmatrix}1\\1\\1\\2\\2\end{bmatrix} + m \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}$. This implies

$$\dim_k(\mathcal{V}_{-2m-3,5}) = \begin{bmatrix}2\\1\\1\\1\\2\end{bmatrix} + \begin{bmatrix}1\\2\\1\\1\\2\end{bmatrix} + \begin{bmatrix}1\\1\\2\\1\\2\end{bmatrix} + \begin{bmatrix}1\\1\\1\\2\\2\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} + 4m \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} - (2m+1) \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}$$

$$= \begin{bmatrix}4\\4\\4\\4\\7\end{bmatrix} + (2m-1) \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} + (2m+2) \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}$$

$$\Rightarrow$$

$$\dim_k(\mathcal{V}_{-2m-3,1}) = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} + (2m+2) \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix} - \begin{bmatrix}2\\1\\1\\1\\2\end{bmatrix} + m \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\1\\1\\1\end{bmatrix} + (m+2) \begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

and we can in a similar way show that

$$\dim_k(\mathcal{V}_{-2m-3,2}) = \begin{bmatrix}1\\0\\1\\1\\1\end{bmatrix} + (m+2)\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

$$\dim_k(\mathcal{V}_{-2m-3,3}) = \begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix} + (m+2)\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix},$$

and

$$\dim_k(\mathcal{V}_{-2m-3,4}) = \begin{bmatrix}1\\1\\1\\0\\1\end{bmatrix} + (m+2)\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}.$$

By induction on m, the suspected forms are then true. □

Remark 3.8. Now we can remark a connection between $M = L^{-1}$ and the dimension vectors of our new sequence. The connection is shown below.

$$\dim_k(\mathcal{V}_{-n-1,5}) = M^n\big[\dim_k(\mathcal{V}_{-1,5})\big], \forall \in \mathbb{N}_0,$$
$$\dim_k(\mathcal{V}_{-2m-1,j}) = M^{2m}\big[\dim_k(\mathcal{V}_{-1,j})\big],$$
$$\dim_k(\mathcal{V}_{-2m-2,j}) = M^{2m}\big[\dim_k(\mathcal{V}_{-2,j})\big],$$
$$\forall j \in \{1,2,3,4\}\ \forall m \in \mathbb{N}_0.$$

△

We can now comfortably add the representations of the new sequence to $\mathcal{V}$. Then the indexing will almost fit perfectly. We are only missing four representations, namely $\mathcal{V}_{0,1}$, $\mathcal{V}_{0,2}$, $\mathcal{V}_{0,3}$ and $\mathcal{V}_{0,4}$. Earlier on we did some calculations which could give us an idea of what these representations should be. We found that

$$M\big[\dim_k(\mathcal{V}_{1,1})\big] = \begin{bmatrix}-1\\0\\0\\0\\0\end{bmatrix}, M\big[\dim_k(\mathcal{V}_{1,2})\big] = \begin{bmatrix}0\\-1\\0\\0\\0\end{bmatrix},$$

$$M\big[\dim_k(\mathcal{V}_{1,3})\big] = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, M\big[\dim_k(\mathcal{V}_{1,4})\big] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Then the missing representations should be

$$\mathcal{V}_{0,1} = (\{k, 0, 0, 0, 0\}, \{0, 0, 0, 0\}),$$

$$\mathcal{V}_{0,2} = (\{0, k, 0, 0, 0\}, \{0, 0, 0, 0\}),$$

$$\mathcal{V}_{0,3} = (\{0, 0, k, 0, 0\}, \{0, 0, 0, 0\}),$$

$$\mathcal{V}_{0,4} = (\{0, 0, 0, k, 0\}, \{0, 0, 0, 0\}).$$

However, none of these are subspace representations, so they cannot be part of the solution for the 4 subspace problem.

4 Exceptions

There is a subspace representation with which we have interacted quite a lot with, but know relatively little about. This representation is the one with dimension vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}.$$

We call this representation $\mathcal{W} = (\{k, k, k, k, k^2\}, f_{\mathcal{W}})$. Now we look at the subspace representations

$$\mathcal{W}_1 = (\{k, k, 0, 0, k\}, \{a_1, a_2, 0, 0\}), a_1, a_2 \in k,$$

$$\mathcal{W}_2 = (\{0, 0, k, k, k\}, \{0, 0, a_3, a_4\}), a_3, a_4 \in k,$$

$$\mathcal{W}_3 = (\{0, k, k, 0, k\}, \{0, a_2, a_3, 0\}), a_2, a_3 \in k,$$

$$\mathcal{W}_4 = (\{k, 0, 0, k, k\}, \{a_1, 0, 0, a_4\}), a_1, a_4 \in k,$$

$$\mathcal{W}_5 = (\{k, 0, k, 0, k\}, \{a_1, 0, a_3, 0\}), a_1, a_3 \in k,$$

$$\mathcal{W}_6 = (\{0, k, 0, k, k\}, \{0, a_2, 0, a_4\}), a_2, a_4 \in k.$$

We find the endomorphism rings of these.

1. Let $h \in \text{End}(\mathcal{W}_1)$. $h = \{h_1, h_2, 0, 0, h_5\}$,$h_1, h_2, h_5 \in k$. Then

$$h_5 \circ a_1 = a_1 \circ h_1 \Rightarrow h_5 = h_1,$$

$$h_5 \circ a_2 = a_2 \circ h_2 \Rightarrow h_5 = h_2,$$

$$h_5 \circ 0 = 0 \circ 0 \Rightarrow 0 = 0,$$

$$h_5 \circ 0 = 0 \circ 0 \Rightarrow 0 = 0,$$

which gives $h = h_5 \Rightarrow \text{End}(\mathcal{W}_1) \cong k$.

2. Let $h \in \text{End}(\mathcal{W}_2)$. $h = \{0, 0, h_3, h_4, h_5\}$,$h_3, h_4, h_5 \in k$. Then

$$h_5 \circ 0 = 0 \circ 0 \Rightarrow 0 = 0,$$

$$h_5 \circ 0 = 0 \circ 0 \Rightarrow 0 = 0,$$

$$h_5 \circ a_3 = a_3 \circ h_3 \Rightarrow h_5 = h_3,$$

$$h_5 \circ a_4 = a_4 \circ h_4 \Rightarrow h_5 = h_4,$$

which gives $h = h_5 \Rightarrow \text{End}(\mathcal{W}_2) \cong k$.

3. Let $h \in \text{End}(\mathcal{W}_3)$. $h = \{0, h_2, h_3, 0, h_5\}$,$h_2, h_3, h_5 \in k$. Then

$$h_5 \circ 0 = 0 \circ 0 \Rightarrow 0 = 0,$$

$$h_5 \circ a_2 = a_2 \circ h_2 \Rightarrow h_5 = h_2,$$

$$h_5 \circ a_3 = a_3 \circ h_3 \Rightarrow h_5 = h_3,$$

$$h_5 \circ 0 = 0 \circ 0 \Rightarrow 0 = 0,$$

which gives $h = h_5 \Rightarrow \text{End}(\mathcal{W}_3) \cong k$.

4. Let $h \in \text{End}(\mathcal{W}_4)$. $h = \{h_1, 0, 0, h_4, h_5\}$,$h_1, h_4, h_5 \in k$. Then

$$h_5 \circ a_1 = a_1 \circ h_1 \Rightarrow h_5 = h_1,$$

$$h_5 \circ 0 = 0 \circ 0 \Rightarrow 0 = 0,$$

$$h_5 \circ 0 = 0 \circ 0 \Rightarrow 0 = 0,$$

$$h_5 \circ a_4 = a_4 \circ h_4 \Rightarrow h_5 = h_4,$$

which gives $h = h_5 \Rightarrow \text{End}(\mathcal{W}_4) \cong k$.

5. Let $h \in \text{End}(\mathcal{W}_5)$. $h = \{h_1, 0, h_3, 0, h_5\}$,$h_1, h_3, h_5 \in k$. Then

$$h_5 \circ a_1 = a_1 \circ h_1 \Rightarrow h_5 = h_1,$$

$$h_5 \circ 0 = 0 \circ 0 \Rightarrow 0 = 0,$$

$$h_5 \circ a_3 = a_3 \circ h_3 \Rightarrow h_5 = h_3,$$

$$h_5 \circ 0 = 0 \circ 0 \Rightarrow 0 = 0,$$

which gives $h = h_5 \Rightarrow \text{End}(\mathcal{W}_5) \cong k$.

6. Let $h \in \text{End}(\mathcal{W}_6)$. $h = \{0, h_2, 0, h_4, h_5\}$,$h_2, h_4, h_5 \in k$. Then

$$h_5 \circ 0 = 0 \circ 0 \Rightarrow 0 = 0,$$

$$h_5 \circ a_2 = a_2 \circ h_2 \Rightarrow h_5 = h_2,$$

$$h_5 \circ 0 = 0 \circ 0 \Rightarrow 0 = 0,$$

$$h_5 \circ a_4 = a_4 \circ h_4 \Rightarrow h_5 = h_4,$$

which gives $h = h_5 \Rightarrow \text{End}(\mathcal{W}_6) \cong k$.

Thus all the representations above are indecomposable. If we take direct sums of pairs, we get

$$\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{W}_3 \oplus \mathcal{W}_4 = \mathcal{W}_5 \oplus \mathcal{W}_6 = \mathcal{W}.$$

Thus $\mathcal{W}$ can be expressed in several different ways as a direct sum of representations that are not in $\mathcal{V}$. These are also the only ones, since all other valid decompositions consisting of subspace representations would have to include elements of $\mathcal{V}$.

This concludes our exploration of the 4 subspace problem.

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